MATH 217A NOTES

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These notes were taken in Stanford's Math 217A class in Fall 2014, taught by Tian Yang. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to adebray@stanford.edu.

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Recall the two definitions from last time.[1]

Definition. M is a topological manifold of dimension n if there exist charts of coordinates $\{(U_\alpha, \phi_\alpha)\}$ such that

(1) $M = \bigcup_\alpha U_\alpha$, and
(2) for all α, U_α is open in M and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism, and its image is open in $\mathbb{R}^n$.

In addition M is smooth if for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ (corresponding to the region of intersection on M, but on subsets of $\mathbb{R}^n$) is C^∞ differentiable (i.e. smooth).

Definition. Let M and N be smooth manifolds (m- and n-dimensional, respectively); then, a function $f : M \to N$ is smooth if it is smooth locally; that is, f is smooth at p if there exists a neighborhood U of p and V of $f(p)$, with transition homeomorphisms

[1]...which I don't have notes from, so the definitions are reproduced here.

$\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$, then $f(U) \cap V$ is nonempty, so consider its preimage $X \subseteq \phi(U)$ and define $\psi \circ f \circ \phi^{-1} : X \to \psi(f(U) \cap V)$, which is a smooth map on $\mathbb{R}^m \to \mathbb{R}^n$. Then, f is called smooth if this map is smooth.

The idea is that it's hard to work with maps from manifolds, but we can locally understand them as maps between Euclidean spaces. Furthermore, we know that the smoothness of a function at p is independent of the coordinate neighborhood chosen, because of the smoothness restriction on M and N.

Definition. $f : M \to N$ is called smooth if it is smooth at each point.

To talk about tangent vectors, let's talk about them for $\mathbb{R}^n$ first. Thus, let $p \in \mathbb{R}^n$; then, V is a tangent vector at p if it is a tangent vector to some smooth curve $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $c(0) = p$, i.e. $c'(0) = v$.

Thus, we can consider the tangent vector as an equivalence class of curves at p, where two curves are equivalent if they agree on a small neighborhood of p (the germs of functions at p), and write the equivalence class as $v = \{c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \mid c(0) = p \text{ and } c'(0) = v\}$.

This is mostly reasonable to extend to manifolds, but how do we define two curves to be tangent on a manifold? The answer is to look locally.

Definition. Let $p \in M$; then, two curves $c_1, c_2 : (-\varepsilon, \varepsilon) \to M$, such that $c_1(0) = c_2(0) = p$ are equivalent if $\phi \circ c_1$ and $\phi \circ c_2$ are tangent to each other, for some chart (U, ϕ) containing p.

Since tangent curves are preserved by smooth maps, this equivalence relation is well-defined on smooth manifolds.

Definition. A tangent vector to a $p \in M$ is an equivalence class of smooth curves going through p.

Recall that if $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a smooth curve with $c(0) = p$, then the tangent vector is $c'(0) = V$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, then the directional derivative of g in the direction of V is $V(g) = D_V(g)$ (the first notation standard in differential topology and differential geometry) is defined to be

$$\left.\frac{\mathrm{d}(g \circ c)}{\mathrm{d}t}\right|_{t=0}.$$

If $c(t) = (x_1(t), \dots, x_n(t))$, then by the Chain Rule,

$$\begin{aligned} V(g) &= \frac{\mathrm{d}}{\mathrm{d}t} g((x_1(t), \dots, x_n(t)) \\ &= \sum_{i=1}^n x_i'(0) \left.\frac{\partial g}{\partial x_i}\right|_{t=0} \\ &= \left(\sum_i x_i'(0) \frac{\partial}{\partial x_i}\right)(g), \end{aligned}$$

i.e. the action of a first-order differential operator on g.

Unsurprisingly, we're going to extend this definition to manifolds.

Definition. Let M be a smooth n-dimensional manifold and $c : (-\varepsilon, \varepsilon) \to M$ be a smooth curve in M. Let D be the set of functions that are smooth at a $p \in M$. Then, the tangent vector (defined as a directional derivative) to c at a time $t = 0$ is the function $c'(0) : D \to \mathbb{R}$ defined by

$$c'(0)(g) = \left.\frac{\mathrm{d}(g \circ c)}{\mathrm{d}t}\right|_{t=0}.$$

Definition. Then, a tangent vector of M at a point p is a tangent vector to some smooth curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$.

Definition. The tangent space of M at p is the set of all tangent vectors at p, denoted T_pM.

This has a real vector space structure, given by

$$(aV_1 + bV_2)(g) = aV_1(g) + bV_2(g).$$

To see this, we just use a chart: this works in $\mathbb{R}^n$ (use a straight line to find a given tangent vector), and is preserved by smooth maps (take the preimage of that line). That is, since the tangent space to Euclidean space has a vector space structure, then so does the tangent space at a manifold; everything is borrowed from $\mathbb{R}^n$. In fact, since $T_x\mathbb{R}^n$ is n-dimensional, so is T_pM. More precisely, let $\phi : U \to \mathbb{R}^n$ be a chart, so that $\phi \circ c$ can be precisely expressed in coordinates: $(x_1(t), \dots, x_n(t))$, and then

$$c'(0)(g) = \left.\left(\sum x_i'(0) \frac{\partial}{\partial x_i}\right)\right|_{t=0} (g \circ \phi^{-1}).$$

The tangent vector corresponding to the differential operator $\frac{\partial}{\partial x_i}$ is the tangent vector to the coordinate curve $x_i(t)$, so we can use this notation. These basic operators,

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_{t=0}, \left.\frac{\partial}{\partial x_n}\right|_{t=0} \right\}$$

are a basis for M.

A smooth map on manifolds induces a map on their tangent spaces, called the tangent map.

Definition. Let $f : M \to N$ be a smooth mapping; for a $p \in M$, the tangent map $Df_p : T_pM \to T_{f(p)}N$ is defined as follows: using the first definition of the tangent vector, for any $v \in T_pM$, choose a $c : (-\varepsilon, \varepsilon) \to M$ be such that V is the equivalence class of c; then, $Df_p(v)$ is defined to be the equivalence class of $f \circ c$.

Alternatively, using v as a differential operator, $Df_p(v)(g) = v(f \circ g)$.

This last definition illustrates why Df is linear:

$$\begin{aligned} Df_p(aV_1 + bV_2)(g) &= (aV_1 + bV_2)(f \circ g) \\ &= aV_1(f \circ g) + bV_2(f \circ g) \\ &= aDF_p(V_1) + bDf_p(V_2). \end{aligned}$$

Now let's talk about the tangent bundle.

Definition. Let M be a smooth manifold; then, the tangent bundle of M is $TM = \{(p, v) : p \in M, v \in T_pM\}$.

Then, TM is a smooth manifold: let $\{(U_\alpha, \phi_\alpha)\}$ be a coordinate system for M and write $\phi(p) = (x_1(t), \dots, x_n(t))$. Each $V \in T_pM$ can be written as

$$V(p) = \sum_{i=1}^{n} u_i(p) \frac{\partial}{\partial x_i};$$

let $TV_\alpha = \{(p, v) \mid p \in V_\alpha, V \in T_pM\}$; then, the map $\psi_\alpha : TU_\alpha \to \mathbb{R}^{2n}$ sends $(p, V) \mapsto (x_1, \dots, x_n, U_1, \dots, U_n)$.

Then, $\{(TU_\alpha, \psi_\alpha)\}$ is a coordinate system for TM (which we have to show); certainly, TM is covered by all of the TU_α, since the U_α cover M. And since $\psi_\beta \circ \psi_\alpha^{-1} = (\psi_\beta \circ \psi_\alpha^{-1}, D(\psi_\beta \circ \psi_\alpha^{-1}))$, the former is smooth and the latter is linear, so we're done.

3. Immersions and Embeddings: 9/26/14

4. Riemannian Metrics: 9/29/14

Definition. A Riemannian metric on a smooth manifold M is an assignment of an inner product to each tangent space T_pM that varies smoothly in $p \in M$.

The inner product $\langle\ ,\ \rangle_p$ must be symmetric, positive-definite, and bilinear; that it is positive definite is probably the most important.

What does this smoothness mean? The stipulation is that for all vector fields X and Y, the function $f : M \to \mathbb{R}$ defined by $f(p) = \langle X(p), Y(p)\rangle_p$ is smooth.

In local coordinates (U, ϕ), where $\phi(p) = (x_1(p), \dots, x_n(p))$, we can define this metric in terms of

$$g_{ij}(x_1, \dots, x_n) = \left\langle D\phi^{-1}\left(\frac{\partial}{\partial x_i}\right), D\phi^{-1}\left(\frac{\partial}{\partial x_j}\right)\right\rangle_p.$$

This is called the coefficient of this inner product in the chosen local coordinates, and it is easy to see that these g_{ij} determine the metric.

Definition. A smooth manifold together with a Riemannian metric is called a Riemannian manifold.

Example 4.1. The simplest example is Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, g_{ij} = \delta_{ij})$.

Definition. A diffeomorphism $f : M \to N$ is called an isometry if for any $p \in M$ and $U, V \in T_pM$, $\langle U, V\rangle_p = \langle Df_p(U), Df_p(V)\rangle_{f(p)}$.

In other words, an isometry is a diffeomorphism that preserves the inner product. Since f is required to be a diffeomorphism, isometries can only exist between manifolds of the same dimension.

Definition. If $f : M \to N$ is any smooth mapping (not necessarily a diffeomorphism), it is called a local isometry at $p \in M$ if there exists an open neighborhood U of p such that $f|_U : U \to f(U)$ is a diffeomorphism and for all $V, W \in T_pM$, $\langle V, W\rangle_p = \langle Df_p(V), Df_p(W)\rangle_{f(p)}$.

This time, we don't require the dimensions to be the same, but we can see that $\dim M \leq \dim N$, because if not, some tangent vector must map to zero (which means the map can't preserve inner product, since its length changes).

Now we can give some more interesting examples. Recall the following definition:

Definition. A smooth map $f : M \to N$ is an immersion if for all p, $Df_p : T_pM \to T_{f(p)}N$ is injective.

Example 4.2 (Immersion manifolds). If N has a Riemannian metric $\langle \cdot, \cdot \rangle$, then an immersion $f : M \to N$ induces a Riemannian metric on M by $\langle U, V \rangle_p = \langle Df_p(U), Df_p(V) \rangle_{f(p)}$.

We need to show that this forms an inner product, though it suffices to show that for all $U \in T_pM$, $\langle U, U \rangle_p = \langle Df_p(U), f_p(U) \rangle_{f(p)} > 0$, which follows because Df_p is everywhere injective.

If $U \hookrightarrow \mathbb{E}^n$ is open, then it is a submanifold of Euclidean space.

It turns out all manifolds that are sufficiently topologically well-behaved admit Riemannian metrics.

Definition. The support of a smooth function $f : M \to \mathbb{R}$ is the closure of its nonzero points, denoted

$$\operatorname{supp} f = \overline{\{p \in M \mid f_\alpha(p) \neq 0\}}.$$

Definition. A partition of unity is a collection of open subsets $\{U_\alpha\}$ of M, where each U_α is homeomorphic to $\mathbb{R}^n$, and smooth functions $\{f_\alpha\}$ on M, such that:

(1) $M = \bigcup_\alpha U_\alpha$ (i.e. it's covered by them), and $\{U_\alpha\}$ is locally finite, i.e. for all p there exists a neighborhood W of p such that W intersects only finitely many U_α.
(2) f_α is supported in U_α, i.e. $\operatorname{supp} f_\alpha \subset U_\alpha$.
(3) For all $p \in M$, $\sum_\alpha f_\alpha(p) = 1$.

Intuitively speaking, a partition of unity allows one to glue things together smoothly, which will be useful in constructing a Riemannian metric.

Proposition 4.3. *If M is Hausdorff and has a countable basis, then M admits a partition of unity.*

Once this proposition is shown, then we can take a partition f_α of unity, and take g_α to be the Euclidean metric on each U_α; then, for all $p \in M$, we let the metric at p be $\sum_\alpha f_\alpha g_\alpha$.

The space of Riemannian metrics is a convex set; that is, for all metrics g_1 and g_2, a metric $g_t = tg_1 + (1-t)g_2$ is also a Riemannian metric, for $0 \le t \le 1$.

Recall from last time the following three definitions.

Definition. Let $f : M \to N$ be a smooth map between manifolds. Then, $q \in N$ is a regular value if for all $f \in f^{-1}(q)$, $Df_p : T_pM \to T_qN$ is surjective.

Definition. An immersion $f : M \to N$ is additionally called an embedding if $f : M \to f(M)$ is a homeomorphism with respect to the subspace topology of the image.

A typical example of an injective immersion which is not an embedding is the curve $y = sin(1/t)$, with the endpoint at $x = 1$ joined to the top of the interval $[0, 1]$. Then, an immersion of the unit interval into this interval is not an embedding, since the number of connected components of any neighborhood changes.

Definition. $M' \subset M$ is a submanifold of M if inclusion $i : M' \hookrightarrow M$ is an embedding.

Example 4.4. One well-known submanifold is the n-sphere,

$$\mathbb{S}^n = \left\{ (x_1, \ldots, x_{n+1}) \subseteq \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

Since it's a submanifold, then we get a Riemannian metric from $\mathbb{R}^n$ on the sphere.

Example 4.5. Another well-known and extremely important example is the hyperbolic metric

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\},$$

where

$$g_{ij}(x_1, \ldots, x_n) = \frac{\delta_{ij}}{x_n^2}.$$

There are several models for this geometry; this one is called the upper half-plane model. An alternate one, called the hyperboloid model, is defined as follows.

Let Minkowski space $\mathbb{E}^{n,1} = (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_{n,1})$, where the "inner product" (which is not actually an inner product), be given by

$$\langle x, y \rangle_{n,1} = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

The signature is $1, \ldots, 1, -1$, so it's not an inner product.

Now, we can define the hyperboloid model for hyperbolic space as

$$\mathcal{H}^n = \left\{ x \in \mathbb{E}^{n,1} \mid \langle x, x \rangle_{n,1} = -1, x_{n+1} > 0 \right\}.$$

There's something yet to show here, though: we need to show that $\langle \cdot, \cdot \rangle_{n+1}$ restricts to a Riemannian metric on $\mathcal{H}^n$, even though it isn't Riemannian on all of $\mathbb{E}^{n,1}$.

It suffices to show that for any tangent vector $u \in T_x\mathcal{H}^n$, then $\langle u, u\rangle_x > 0$. If this isn't the cae, then there's a $v \in T_x\mathcal{H}^n$ such that $\langle v, v\rangle_x \le 0$, and let $v = (v_1, \dots, v_{n+1})$ and $x = (x_1, \dots, x_{n+1})$. Since $\langle x, x\rangle = -1$, then

$$\sum_{i=1}^n x_i^2 + 1 = x_{n+1}^2.$$

Additionally, because $\langle v, v\rangle_{n,1} = \langle v, v\rangle_x \le 0$, then $\sum v_i^2 \le v_{n+1}^2$. Finally, we know that $\langle x, v\rangle_{n,1} = 0$; if it weren't, then there's a curve $c : (-\varepsilon, \varepsilon) \to \mathcal{H}^n$ such that $c(0) = x$ and $c'(0) = v$, so $\langle c(t), c(t)\rangle_{n,1} = -1$, but $2\langle c'(0), c(0)\rangle = 0$. Then, $\sum_1^n x_i v_i = x_{n+1}v_{n+1}$, so

$$\begin{aligned}\left(\sum_{i=1}^n v_i^2\right)\left(1 + \sum_{i=1}^n x_i^2\right) &\le v_{n+1}^2 x_{n+1}^2 \\ &= \left(\sum v_i^2 x_i^2\right) \\ &\le \left(\sum v_i^2\right)\left(\sum x_i^2\right),\end{aligned}$$

which contradicts what we saw earlier. Thus, this restriction is really positive definite!

It's also possible to compute this with isometries, by moving points around and using the fact that it's a regular value.

There is a projection π from $\mathcal{H}^n$ to the unit disc: a point p makes a line with $q = (0, 0, \dots, 0, -1)$, and $\pi(p)$ is defined to be the point where this intersects the hyperplane $\{x_n = 0\}$.

Exercise 1. Show that the above π is an isometry from $(\mathcal{H}^n, \langle\cdot,\cdot\rangle_{n,1})$ to $\mathbb{D}^n$ (i.e. unit disc) with the following metric:

$$g_{ij} = \frac{4\delta_{ij}}{\left(1 - \sum_{i=1}^n x_i^2\right)^2}.$$

This is known as the Poincaré ball model.

5. Indiana Jones and the Isometries of the Upper Half-Plane: 10/1/14

Given a Riemannian metric on a manifold, we can calculate the lengths of curves.

Definition. Let $c : (a, b) \to M$, where M is a Riemannian manifold. Then, the vector field $Dc\left(\frac{\partial}{\partial t}\right)$, denoted by $\frac{\mathrm{d}c}{\mathrm{d}t}$ or $c'(t)$, is the velocity field (or tangent field) of c.

Definition. The length of c is

$$\ell(c) = \int_a^b \left\langle \frac{\mathrm{d}c}{\mathrm{d}t}, \frac{\mathrm{d}c}{\mathrm{d}t}\right\rangle_{c(t)}^{1/2} \mathrm{d}t.$$

Example 5.1. For example, consider the hyperbolic plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with $g_{ij} = \delta_{ij}/y^2$, and let $c : (a, b) \to \mathbb{H}^2$ (with $a > 0$) be $c(t) = (0, t)$. The tangent field is $c'(t) = (0, 1)$, so

$$\begin{aligned}\ell(c) &= \int_a^b \langle(0,1),(0,1)\rangle_{(0,t)}^{1/2}\,\mathrm{d}t \\ &= \int_a^b \frac{1}{t}\,\mathrm{d}t = \ln\left(\frac{b}{a}\right).\end{aligned}$$

If $\widetilde{c} : (a.b) \to \mathbb{H}^2$ is any curve with endpoints $\widetilde{c}(a) = (0, a)$ and $\widetilde{c}(b) = (0, b)$, then write $\widetilde{c}(t) = (x(t), y(t))$; then, its length is

$$\begin{aligned}\ell(\widetilde{c}) &= \int_a^b \left\langle \frac{\mathrm{d}\widetilde{c}}{\mathrm{d}t}, \frac{\mathrm{d}\widetilde{c}}{\mathrm{d}t}\right\rangle_{\widetilde{c}(t)}^{1/2} \mathrm{d}t \\ &= \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,\mathrm{d}t \\ &\ge \int_a^b \frac{y'(t)}{y(t)} = \ln\left(\frac{b}{a}\right),\end{aligned}$$

with equality only when $x'(t) = 0$ and $y'(t) \ge 0$, i.e. $\widetilde{c} = c$ up to reparameterization. Thus, the y-axis is a geodesic (the locally shortest path between two points).

On the hyperbolic plane, vertical lines $x = a$ are geodesics, along with circular arcs centered on the x-axis. These can be obtained by solving the geodesic equation, but more intuitively, the image under an isometry of a geodesic must be a geodesic, so maybe we can get these as isometries of $\mathbb{H}^2$.

Here are some isometries of $\mathbb{H}^2$:

- Translation: $z \mapsto z + a$ for some $a \in \mathbb{R}$.
- Inversion: $z \mapsto -1/z$ (can explicitly check).
- Scalar multiplication: $f_\lambda : z \mapsto \lambda z$, where $\lambda > 0$.

If $z = (x, y) \in \mathbb{H}^2$ has tangent vector (a, b), $Df_\lambda(a, b) = (\lambda a, \lambda b)$, so $\langle (a, b), (a, b) \rangle_z = (a^2 + b^2)/y^2$, and $\langle (\lambda a, \lambda b), (\lambda b, \lambda b) \rangle_{\lambda z} = ((\lambda a)^2 + (\lambda b)^2)/\lambda^2 y^2$, so this cancels out and f_λ is indeed an isometry.

The circular arcs mentioned above are the images of vertical lines under inversions; thus, they are also geodesics.

Definition. If $a, b, c, d \in \mathbb{C}$, then

$$f(z) = \frac{az + b}{cz + d}$$

from $\mathbb{C} \to \mathbb{C}$ is called a Möbius transformation.

If $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, then f preserves the upper half-plane and in fact is an isometry, so the set of these Möbius transformations is a subgroup of $\mathrm{Iso}^+(\mathbb{H}^2)$, i.e. the group of orientation-preserving isometries of $\mathbb{H}^2$. All of the named isometries are examples of these Möbius transformations (i.e. $f_\lambda(z) = (\lambda^{1/2} z + 0)/(0x + \lambda^{-1/2})$).

There's a relationship between this Möbius subgroup and the projective special linear group $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$.

Proposition 5.2. *The map*

$$\left(z \mapsto \frac{az + b}{cz + d} \right) \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a group isomorphism from this Mòbius transformations group to $\mathrm{PSL}(2, \mathbb{R})$.

Theorem 5.3. *In fact,* $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Iso}^+(\mathbb{H}^2)$.

Proof sketch. For any $z \in \mathbb{H}^2$ and $v \in T_z\mathbb{H}^2$ with $\|v\| = 1$, there exists a Möbius transformation f such that $f(z) = i$ and $Df_z(v) = \frac{\partial}{\partial x}$. Then, by composing with the rotation matrix $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$, we can rotate the tangent vector wherever we want.

Thus, $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on the unit tangent vectors of $\mathbb{H}^2$.

Suppose $F \in \mathrm{Iso}^+(\mathbb{H}^2)$; let $z = F(u)$ and $v = DF_i\left(\frac{\partial}{\partial x}\right)$; then, there exists a Möbius transformation f such that $f(z) = i$ and $Df_z(v) = \frac{\partial}{\partial x}$. Then, $f \circ F$ is an orientation-preserving isometry that fixes i and $\frac{\partial}{\partial x}$, so it must also fix the tangent vector orthogonal to it, i.e. $\frac{\partial}{\partial y}$. In particular, $D(f \circ F)_i = \mathrm{id}_{T_i\mathbb{H}^2} : T_i\mathbb{H}^2 \to T_i\mathbb{H}^2$.

The next step requires the surjectivity of the exponential map, so let's talk about the exponential map.

Definition. Let M be a Riemannian manifold and $p \in M$; then, there is a map $\mathrm{Exp}_p : U \to M$, where $U \subset T_pM$ is a neighborhood of 0, given as follows: if $c : (-r, r) \to M$ is the geodesic such that $c(0) = p$, $c'(0) = v$, and $r > 1$, then $v \mapsto c'(|v|)$.

This relies on the fact that for any $p \in M$ and $v \in TpM$, there's a unique geodesic through p tangent to v, which we'll prove later. The idea is that for a given p and v, we go along the manifold along a geodesic for time $|v|$, and then that's $\mathrm{Exp}_p(v)$.

Definition. A Riemannian manifold M is called geodesically complete if every geodesic on M extends to infinity (i.e. infinite length, as calculated above).

For $\mathbb{H}^2$, $D\,\mathrm{Exp}_i : T_i\mathbb{H}^2 :\to \mathbb{H}^2$, i.e. $\mathbb{H}^2$ is geodesically complete. Furthermore, on $\mathbb{H}^2$, the exponential map at i (or anywhere, really) is surjective. With these, we may finish the proof.

If $f : M \to N$ is an isometry between manifolds, it's a tautology that the following diagram commutes:

$$\begin{array}{ccc} T_pM & \xrightarrow{Df_p} & T_{f(p)}N \\ \big\downarrow{\scriptstyle \exp_p} & & \big\downarrow{\scriptstyle \exp_{f(p)}} \\ M & \xrightarrow[f]{\sim} & N. \end{array}$$

Just think about what the definitions mean; however, it's rather false in the case where f isn't an isometry.

Plugging in $\mathbb{H}^2$, we get

$$\begin{array}{ccc} T_i\mathbb{H}^2 & \xrightarrow{\mathrm{id}} & T_i\mathbb{H}^2 \\ \big\downarrow{\scriptstyle \exp_i} & & \big\downarrow{\scriptstyle \exp_i} \\ \mathbb{H}^2 & \xrightarrow[f \circ F]{} & \mathbb{H}^2. \end{array}$$

Thus, we can pull back along $\exp_i$, and since the diagram commutes, then $f \circ F = \mathrm{id}_{\mathbb{H}^2}$, and $F = f^{-1} \in \mathrm{PSL}(2, \mathbb{R})$. ⊠

Given a Riemannian metric, we can not just define the lengths of curves, but also the volume of M.

Definition. Let M be a Riemannian manifold, and (U, ϕ) be a chart where $\phi(p) = (x_1(p), \ldots, x_n(p))$. Then, the volume element is

$$\mathrm{Vol}(x) = \sqrt{\det(g_{ij}(x)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n.$$

If in addition M has a finite partition of unity $\{(U_\alpha, f_\alpha)\}$, then its volume is defined as

$$\mathrm{Vol}(M) = \sum_\alpha \int_{U_\alpha} f_\alpha(x)\,\mathrm{Vol}(x).$$

This is more general than the volume form, since it works for some nonorientable manifolds; in fact, if M is orientable, there's a global volume form $\sqrt{\det g_{ij}}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}_n$. But this time, we need to be able to glue them together smoothly, which is where the partition of unity is useful.

For example, if one twists a $t \times 1$ rectangle into a Möbius band, its area is t. Not too weird.

You may be wondering at this point why we look at the square root of the determinant of the metric. The infinitesimal volume is given by the parallelogram (or a higher-dimensional generalization) with lines $\frac{\partial}{\partial x_i}$ in each direction. Take an orthonormal basis $\{e_1, \ldots, e_n\}$, i.e. $\langle e_i, e_j\rangle_x = \delta_{ij}$; then if $\frac{\partial}{\partial x_i} = \sum_{j=1}^n a_{ij}e_i$, then the volume of the parallelogram is $\det(a_{ij})$, and

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \sum_{k=1}^n a_{ik}a_{jk},$$

and thus $(g_{ij}) = (a_{ij})(a_{ij})^{\mathrm{T}}$ as matrices, which is where the square root comes from (the transpose disappears after the determinant); this is really calculating the volume form.

For $\mathbb{H}^2$, the volume form is $(\mathrm{d}x \wedge \mathrm{d}y)/y^2$.

Exercise 2. Show that if T is a triangle in $\mathbb{H}^2$ with angles α, β, and γ, then show that its area is $\pi - \alpha - \beta - \gamma$.

6. That's an Affine Connection You've Got There: 10/3/14

"…shame if something happened to it."

Today we'll discuss affine connections, which allow one to differentiate tangent vectors on arbitrary manifolds. This will allow us to define geodesics, parallel transport, and curvature on abstract manifolds. In this lecture, we don't require manifolds to have a Riemannian metric, but next time, we'll find the Levi-Civita connection given by symmetry and compatibility with the metric, which is unique.

Definition. Let M be a smooth manifold, and let $V(M)$ denote the set of vector fields on M, and $C^\infty(M)$ denote the set of smooth functions on M. Then, an affine connection is a mapping $\nabla : V(M) \times V(M) \to V(M)$, written $X, Y \mapsto \nabla_X Y$, such that:

(1) ∇ is $C^\infty(M)$-linear in the first argument: if $f, g \in C^\infty(M)$, then $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.
(2) ∇ is $\mathbb{R}$-linear in the second argument: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$.
(3) If $f \in C^\infty(M)$, then $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Given this, we can introduce the notion of a covariant derivative.

Proposition 6.1. *Let M be a smooth manifold and ∇ be an affine connection on M. Then, there is a unique correspondence which associates a vector field X along a smooth curve $c : I \to M$ to another vector field $\frac{\mathrm{D}X}{\mathrm{d}t}$ along c, which is called the covariant derivative of X along c such that:*

(1) For all $X, Y \in V(M)$, $\frac{\mathrm{D}(X+Y)}{\mathrm{d}t} = \frac{\mathrm{D}X}{\mathrm{d}t} + \frac{\mathrm{D}Y}{\mathrm{d}t}$.
(2) $\frac{\mathrm{D}(fX)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}X + f\frac{\mathrm{D}X}{\mathrm{d}t}$ (where $\frac{\mathrm{d}f}{\mathrm{d}t}$ is the derivative along c).
(3) If X is a restriction of a vector field X along M, then $\frac{\mathrm{D}X}{\mathrm{d}t} = \nabla_{\frac{\mathrm{d}c}{\mathrm{d}t}}X$.

Thus, a connection provides information about how to differentiate its second argument along the direction of the first vector field. Then, the covariant derivative is the derivative in the direction of the tangent vector field to the curve.

$\nabla_X Y$ only really depends on X at p, but depends on Y locally, so $\frac{\mathrm{d}c}{\mathrm{d}t}$ isn't really a vector field over the whole manifold in a unique way.

Proof of Proposition 6.1. Well, uh, let's work with coordinates (U, ϕ), where $\phi(p) = (x_1, \ldots, x_n)$, and

$$X = \sum_i a_i \frac{\partial}{\partial x_i}.$$

Suppose $\frac{\mathrm{D}X}{\mathrm{d}t}$ exists, and we'll show uniqueness; then, we'll go back and prove that it exists. Then, by (1) and (2),

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}a_j}{\mathrm{d}t}\frac{\partial}{\partial x_j} + \sum_j a_j \frac{\mathrm{D}(\frac{\mathrm{d}}{\mathrm{d}x_j})}{\mathrm{d}t}.$$

Let $c(t) = (x_1(t), \ldots, x_n(t))$. and by (3),

$$\frac{\mathrm{D}(\frac{\partial}{\partial x_j})}{\mathrm{d}t} = \nabla_{\frac{\mathrm{d}c}{\mathrm{d}t}} \frac{\partial}{\partial x_j} = \nabla_{\sum_j \frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$
$$= \sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}.$$

Substituting that back in, we get

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}a_j}{\mathrm{d}t} \frac{\partial}{\partial x_j} + \sum_{i,j} a_j \frac{\mathrm{d}x_i}{\mathrm{d}t} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}.$$

Thus, this is completely determined by things that have already been chosen, so it is unique. But we still need to show that it's globally defined; if (V, ψ) is another chart, then the uniqueness on $U \cap V$ ensures that the restriction of the covariant derivative on U or on V must match that on $U \cap V$, so it is well-defined.

Definition. The Christoffel symbols are the Γ_{ij}^k defined by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

They uniquely determine the connection.

Let $c : I \to M$ be a smooth curve and (U, ϕ) be a chart, in which $c(t) = (x_1(t), \ldots, x_n(t))$. Let $X = \sum_i a_i \frac{\partial}{\partial x_i}$ be a vector field; then, its covariant derivative, if it exists, must be

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}a_j}{\mathrm{d}t} \frac{\partial}{\partial x_j} + \sum_{i,j} a_j \frac{\mathrm{d}x_i}{\mathrm{d}t} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$
$$= \sum_k \left(\frac{\mathrm{d}a_k}{\mathrm{d}t} + \sum_{i,j} a_j \frac{\mathrm{d}x_i}{\mathrm{d}t} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}.$$

But this exists, and we've already shown that this satisfies the needed properties. ⊠

Example 6.2. We're now able to discuss an example of an affine connection, albeit not a very interesting one. Let $U \subset \mathbb{R}^n$ be open and $\Gamma_{ij}^k = 0$, Thus, $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$, so

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}a_i}{\mathrm{d}t} \frac{\partial}{\partial x_i} = \frac{\mathrm{d}X}{\mathrm{d}t}.$$

Well, that was silly... but now we have at least one example, and can use partitions of unity to show that an affine connection exists on well-behaved manifolds.

Proposition 6.3. *Let M be a Hausdorff manifold with a countable basis; then, M admits an affine connection.*

Proof. The conditions on M imply the existence of a partition of unity $\{(U_\alpha, f_\alpha)\}$; on each U_α, let ∇^α be the connection given by $\Gamma_{ij}^k = 0$. Then, define a connection $\nabla : V(M) \times V(M) \to V(M)$ on the whole manifold by letting

$$\nabla_X Y = \sum_\alpha f_\alpha \nabla_X^\alpha Y.$$

This is a good definition, but we need to verify the three conditions. (1) is pretty straightforward:

$$\nabla_{fX+gY} Z = \sum_\alpha f_\alpha \nabla_{fX+gY}^\alpha Z$$
$$= \sum_\alpha f_\alpha \left(f \nabla_X Z + g \nabla_Y Z \right)$$
$$= f \sum_\alpha f_\alpha \nabla_X^\alpha Z + g \sum_\alpha f_\alpha \nabla_Y^\alpha Z$$
$$= f \nabla_X Z + g \nabla_Y Z.$$

(2) is very similar, and is left as an exercise. But (3) is different: we need to verify the Leibniz rule.

$$\begin{aligned}\nabla_X(fY) &= \sum_\alpha f_\alpha \nabla_X^\alpha(fY)\\ &= \sum_\alpha f_\alpha\left(X(f)Y + f\nabla_X^\alpha Y\right)\\ &= \sum_\alpha f_\alpha X(f)Y + f\sum_\alpha \nabla_X^\alpha Y\\ &= X(f)Y + f\nabla_X Y.\end{aligned}$$

⊠

We can also use this to define a notion of parallel: in $\mathbb{R}^n$, a vector field is parallel if its derivative is zero. More generally, we will define a vector field on a manifold to be parallel when its covariant derivative is zero.

Definition. Let M be a smooth manifold and ∇ be an affine connection on M. Then, a vector field X along a smooth curve $c : I \to M$ if $\frac{\mathrm{D}X}{\mathrm{d}t} = 0$ for all $t \in I$.

Proposition 6.4. *Let $v_0 \in T_{c(t_0)}M$; then, there exists a unique parallel vector field X along c such that $X(c(t_0)) = v_0$.*

Proof. The proof amounts to finding and solving an ODE; if we can show that for any t the parallel transport from t_0 to t by ∇ exists and is unique, then we're done. In other words, it suffices to show that for any $t \in I$, $X(c(t))$ exists an is unique.

Since $c([t_0, t])$ is compact, we can cover it with finitely many charts $U_1, \dots, U_n$ covering $c([t_0, t])$; then, if we show existence and uniqueness on each piece, then we're done, as they agree on the chart intersections.

On U_i, if X exists, then it satisfies $\frac{\mathrm{D}X}{\mathrm{d}t} = 0$, i.e.

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \sum_k \left(\frac{\mathrm{d}A_k}{\mathrm{d}t} + \sum_{i,j} a_j \frac{\mathrm{d}x_i}{\mathrm{d}t}\Gamma_{ij}^k\right)\frac{\partial}{\partial x_k} = 0, \tag{1}$$

which is a system of linear ODEs, and thus has a unique solution for all time. ⊠

Definition. $c : I \to M$ is a geodesic if $\frac{\mathrm{d}c}{\mathrm{d}t}$ is parallel along c.

These end up satisfying a similar ODE to (1):

$$\frac{\mathrm{d}^2 x_k}{\mathrm{d}t^2} + \sum_{i,j}\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}x_j}{\mathrm{d}t}\Gamma_{ij}^k = 0$$

for all $k = 1, \dots, m$. Since this isn't linear, then there's a unique solution locally, but not globally.

7. Missed (Levi-Civita) Connections: 10/6/14

8. The Levi-Civita Connection and the Induced Metric: 10/8/14

"By the way, I'm teaching a course on ODEs this quarter, but I've never taken the course before..."

Proposition 8.1. *Let $M \subset N$ be a submanifold of a Riemannian manifold N, and suppose $X, Y \in V(M)$ extend to some $\overline{X}, \overline{Y} \in V(N)$. For all $p \in M$, let $\mathrm{Proj}_p : T_pN \to T_pM$ be the canonical projection, then, the connection $\nabla_X Y(p) = \mathrm{Proj}_p \overline{\nabla}_{\overline{X}}\overline{Y}$, where $\overline{\nabla}$ is the Levi-Civita connection on N, is the Levi-Civita connection on M in the induced metric.*

Proof. First, we must show that ∇ is a connection. The first two properties, that $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ and that $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$, follow from the fact that Proj_p is C^∞-linear.

For the third property, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, we want to calculate $\overline{\nabla}_{\overline{X}}(\bar{f}\overline{Y})$. First, does $\bar{f}$ exist? Sure; it exists in charts, and then we can get it globally with a partition of unity. Thus,

$$\begin{aligned}\overline{\nabla}_{\overline{X}}(\bar{f}\overline{Y}) &= \mathrm{Proj}_p(\overline{X}(\bar{f})\overline{Y} + \bar{f}\overline{\nabla}_{\overline{X}}\overline{Y})\\ &= X(f)Y + f\nabla_X Y,\end{aligned}$$

because the projection is C^∞-linear.

Thus, ∇ is a connection, but there's more to show so that it's the Levi-Civita. For symmetry, that $\nabla_X Y - \nabla_Y X = [X, Y]$, we have that $\mathrm{Proj}(\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}) = \mathrm{Proj}([\overline{X}, \overline{Y}])$, so we just want to show that projection commutes with the Lie bracket. But $[\overline{X}, \overline{Y}] \in T_pM$, so projection plays nicely with it.

Metric compatibility is about as easy to check; the key is that $\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\rangle = \langle \nabla_X Y, Z\rangle$, which is true because Z is in the tangent space, so this is $\langle \mathrm{Proj}\,\overline{\nabla}_{\overline{X}}\overline{Y}, Z\rangle$. ⊠

Remark. Note that if $f : M \to N$ is merely an immersion, the tangent map is injective, so one can still set up the same projection and much of this still works, albeit locally (locally, an immersion is an embedding).

This was pretty easy to verify, but is very useful. For example, suppose we have a surface $S \subset \mathbb{R}^3$ with the induced metric and a smooth curve $c : I \to S$. Let $X(t)$ be a vector field on S along c. Then, the covariant derivative is just the usual derivative, by the proposition:

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \mathrm{Proj}_{c(t)} \frac{\mathrm{d}X}{\mathrm{d}t},$$

where $\frac{\mathrm{d}X}{\mathrm{d}t}$ is the usual derivative. Additionally, X is parallel iff the covariant derivative is zero, which is true iff $\frac{\mathrm{d}X}{\mathrm{d}t}\big|_t \perp T_{c(t)}S$.

Along the unit sphere $S^2 \subset \mathbb{R}^3$, consider the great circle $c(t) = (\cos t, \sin t, 0)$; then, the above implies that $X(t) = (a\sin t, -a\cos t, 1 - a^2)$ is parallel, because (it's very easy to check)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (a\cos t, a\sin t, 0),$$

so at the point $c(t)$, this is parallel to $c(t)$, and thus perpendicular to $T_{c(t)}S^2$. In addition, $\frac{\mathrm{d}c}{\mathrm{d}t} = (-\sin t, \cos t, 0)$ is parallel; thus, c is a geodesic.

If instead $c(t) = ((1-a^2)\cos t, (1-a^2)\sin t, a)$, for $a \in (0,1)$, which is not a great circle, let v_0 be some tangent vector to $c(t)$, so $v_0 \in T_{c(t_0)}S^2$. If we want to find a parallel vector field X along c such that $X(t_0) = v_0$, a good trick to be aware of is to instead consider the cone C tangent to S^2 at c; thus, the tangent spaces of C and S are identical along c, and in particular, *if X is parallel along this curve on the cone, it is parallel along c on the sphere.*

But this cone can be cut open and developed, split along some axis and embedded into $\mathbb{R}^2$. This is an isometry with respect to the Euclidean metric. Now, given v_0, the parallel on $\mathbb{R}^2$ is easy to find (since we have actual parallel transport), and we can map back under the isometry.

The implication for one of our homework problems is that a surface of revolution has geodesics along the direction of revolution wherever the curve to be revolved has critical points, because these get turned into flat lines under the isometry.

Now, let's look at hyperbolic geometry. This cannot be embedded into Euclidean space (though it can be embedded into Minkowski space, which isn't so helpful since it's not Riemannian). Let's consider the model $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 midy > 0\}$ with the metric $g_{ij} = \delta_{ij}/y^2$.

Using the definition that geodesics locally minimize length, we found the geodesics (circular arcs centered on the x-axis and vertical lines), but didn't prove that they satisfied the geodesic equation, which will require calculating the Christoffel symbols

$$\Gamma_{ij}^{\ell} = \frac{1}{2}\sum_k \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) g^{k\ell}.$$

The metric and its inverse are

$$(g_{ij}) = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \qquad \text{and} \qquad (g^{ij}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

Thus, the Christoffel symbols can be laboriously calculated. So as not to waste your time, we get $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^2 = 0$, $\Gamma_{11}^2 = 1/y$, and $\Gamma_{12}^1 = \Gamma_{22}^2 = 1/y$.

Let $c(t) = (0,t)$, so $c'(t) = (0,1) = \frac{\partial}{\partial y}$. This isn't a geodesic, because we need to normalize! But let's see why:

$$\frac{\mathrm{D}c'(t)}{\mathrm{d}t} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \Gamma_{22}^1 \frac{\partial}{\partial x} + \Gamma_{22}^2 \frac{\partial}{\partial y} = -\frac{1}{y}\frac{\partial}{\partial y},$$

but this is nonzero. The point is, we need to parameterize the curve with respect to arc length, and $d(a,b) = \ln(b/a)$ for a, b on the y-axis, so when $c(t) = (0, e^t)$, $c'(t) = (0, e^t) = y\frac{\partial}{\partial y}$, so we get

$$\begin{aligned}
\frac{\mathrm{D}(c'(t))}{\mathrm{d}t} &= \nabla_{y\frac{\partial}{\partial y}} y\frac{\partial}{\partial y} \\
&= y\left(\nabla_{\frac{\partial}{\partial y}} y\frac{\partial}{\partial y}\right) \\
&= y\left(\frac{\partial}{\partial y} + y\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}\right) \\
&= 0.
\end{aligned}$$

So this curve is a geodesic. If we know the isometries of the hyperbolic plane, then we're done, as before, but we can do this explicitly.

Exercise 3. Show that the half-circles are geodesics in this manner. Once again, $c(t) = (\cos t, \sin t)$ is not a geodesic, but there is some $\theta(t)$ such that $(\cos\theta(t), \sin\theta(t))$ is a geodesic.

Consider $c(t) = (x, 1)$, a horizontal curve, and let $v_0 = \frac{\partial}{\partial y}$. We want to find a parallel vector field X along c such that $X(0) = v_0$. This boils down to solving an ODE (or maybe a system of ODEs in general): suppose

$$X(t) = a(t)\frac{\partial}{\partial x} + b(t)\frac{\partial}{\partial y},$$

so that

$$\frac{\mathrm{D}X}{\mathrm{d}t} = a'(t)\frac{\partial}{\partial x} + a(t)\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + b'(t)\frac{\partial}{\partial y} + b(t)\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y},$$

and we know the Christoffel symbols, and $y = 1$, so this simplifies to

$$= (a'(t) - b(t))\frac{\partial}{\partial x} + (b'(t) + a(t))\frac{\partial}{\partial y} = 0.$$

Neither the professor nor your correspondent claim to be very good at ODEs, but this system, i.e.

$$a'(t) - b(t) = 0$$
$$b'(t) + a(t) = 0,$$

we both know how to solve it and get rotation: $a(t) = \cos(t + c)$ and $b(t) = \sin(t + c)$. Pretty cool.

The notions of parallel transport, affine connection, and covariant derivative are all actually the same thing, which is occasionally useful to keep in mind.

Proposition 8.2. *Let M be a Riemannian manifold and $c : I \to M$, let $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ be the parallel transport map, sending $X_0 \mapsto X(t)$, where X is parallel along c with $X(t_0) = X_0$ and c is the integral curve along X.*[2] *Then, the connection*

$$\nabla_X Y(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \left(P^{-1}_{c,t_0,t} Y(c(t))\right)$$

is the Levi-Civita connection.

Proof. Let $e_1, \dots, e_n$ be an orthonormal basis of $T_{c(t_0)}M$, and let $e_i(t)$ be parallel along c such that $e_i(0) = e_i$. Then, $\{e_1(t), \dots, e_n(t)\}$ is orthonormal for all t, so write $Y = \sum a_i(t)e_i(t)$. Thus, since the $e_i(t)$ are parallel,

$$\begin{aligned}
\nabla_X Y = \frac{\mathrm{D}Y}{\mathrm{d}t} &= \sum_i \left(a_i'(t)e_i(t) + a_i(t)\frac{\mathrm{D}e_i(t)}{\mathrm{d}t}\right)\\
&= \sum_i a_i'(t_0)e_i\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum a_i(t)e_i\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \left(P^{-1}_{c,t_0,t}Y(c(t))\right).
\end{aligned}$$

⊠

This works for a general affine connection, though there it doesn't make sense to talk about an orthonormal basis, just any basis; then, existence and uniqueness guarantee the $e_i(t)$ still form a basis. Thus, the connection can be recovered from parallel transport.

9. Geodesics: 10/10/14

"Now, let's recall a theorem from ODE... which I don't remember how to prove."

Recall that if M is a Riemannian manifold and $\gamma : I \to M$ is a geodesic, then $\frac{\mathrm{D}(\gamma'(t))}{\mathrm{d}t} = 0$. In a chart (U, ϕ) with coordinates $\gamma(t) = (x_1(t), \dots, x_n(t))$, this looks like

$$\frac{\mathrm{D}(\gamma(t))}{\mathrm{d}t} = \sum_k \left(\frac{\mathrm{d}^2 k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}x_j}{\mathrm{d}t}\right)\frac{\partial}{\partial x_k} = 0.$$

That is, the geodesic satisfies the following second-order ODE:

$$\frac{\mathrm{d}^2 x_k}{\mathrm{d}t^2} + \sum_{i,j}\Gamma^k_{ij}\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}x_j}{\mathrm{d}t} = 0,$$

for $k = 1, \dots, n$. However, if we rewrite this as geodesic flow on the tangent bundle, which itself is a differentiable manifold, it becomes first-order, so that standard solutions for first-order ODEs can be used to solve it.

[2]This latter condition requires a condition from ODE theory, but that's not important right now.

Recall that if (U, ϕ) is a chart on M, with $\phi(p) = (x_1(p), \dots, x_n(p))$, then $U \times \mathbb{R}^n$ is a chart of TM, homomorphic to $TU = \{(p, v) \mid q \in U, v \in T_pM\}$. The homeomorphism is described by $(x_1, \dots, x_n, y_1, \dots, y_n) \leftrightarrow (p, v)$.

Any curve $\gamma : I \to M$ determines a curve $\overline{\gamma} : I \to TM$ given by $\overline{\gamma}(t) = (\gamma(t), \gamma'(t)) in T_{\gamma(t)}M \subset TM$. If $\overline{\gamma}(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))$ and γ is a geodesic, then the x_i and y_i should satisfy a system of differential equations, specifically

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = y_k \qquad \text{and} \qquad \frac{\mathrm{d}y_k}{\mathrm{d}t} = -\sum_{i,j} \Gamma_{ij}^k y_i y_j. \tag{2}$$

This is a first-order system, so from a course on ODEs, we know that there's always a unique local solution.

Now, let's consider the tangent vector field on this tangent bundle.

Proposition 9.1. *If M is a Riemannian manifold, there exists a unique vector field $G \in V(TM)$, whose trajectories are of the form $\overline{\gamma}(t) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is a geodesic on M.*

This vector field is called the geodesic vector field.

Proof. As always, we prove the uniqueness first, assuming the existence; then, this uniqueness will imply the existence.

Consider a chart (U, ϕ), and suppose $(x_1, \dots, x_n, y_1, \dots, y_n)$ is a coordinate. Let $\overline{\gamma}(t) = (\gamma(t), \gamma'(t)) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))$, and thus we must have $G(\overline{\gamma}(t)) = \overline{\gamma}'(t) = (\gamma'(t), \gamma''(t))$. Thus, each coordinate is either $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ or $\frac{\mathrm{d}y_i}{\mathrm{d}t}$. Plugging in (2), G must be of the form

$$\left(\dots, y_k(t), \dots, -\sum_{i,j} \Gamma_{ij}^k y_i(t) y_j(t), \dots\right).$$

Thus, G is unique, if it exists.

As tends to happen when we talk about geodesics, let's invoke a theorem from the theory of ODEs.

Theorem 9.2. *Suppose $U \subset \mathbb{R}^n$ is open, $X \in V(U)$, and $p \in U$. Then, there exists a neighborhood $U_0 \subset U$ of p, a $\delta > 0$, and a C^∞ mapping $\phi : (-\delta, \delta) \times U_0 \to M$ such that for all $q \in U_0$ and $t \in (-\delta, \delta)$, the curve $\gamma(t) = \phi(t, q)$ is the unique trajectory of X such that $\gamma(0) = q$ and $\gamma'(t) = X(\gamma(t))$.*

Now, we just apply this theorem to this vector field G. ⊠

Combining these two results together, there's a nice conclusion.

Corollary 9.3. *For all $p \in M$, there exists a neighborhood $V \in M$ of p, a $\delta > 0$, an $\varepsilon' > 0$, and a C^∞ mapping $\phi : (-\delta, \delta) \times \mathcal{U} \to M$, where*

$$\mathcal{U} = \{(q, v) \mid q \in V, v \in T_qM \text{ such that } |v| < \varepsilon'\},$$

such that for all $(q, v) \in \mathcal{U}$ and $t \in (-\delta, \delta)$, the curve $\gamma(t) = \phi(tq, v)$ is the unique geodesic such that $\gamma(0) = q$ and $\gamma'(0) = v$.

All that needs to be done to prove this is to project the curve from the tangent bundle to the manifold. Note that the original theorem doesn't guarantee a neighborhood of this form, but we can choose a smaller neighborhood of this form within the one provided by the theorem. Specifically, if $\pi : TM \to M$, given by $(p, v) \mapsto p$, then the curve we want is $\gamma(t) = \pi(\overline{\gamma}(t))$.

The point is, at any point and in any direction, if we don't go too far, there is a unique geodesic.

The next thing we need to show is that geodesics are homogeneous: if you go faster, they may exist for a shorter time. With this, we will be able to define the exponential map.

Lemma 9.4 (Homogeneity). *If the geodesic $\gamma(t) = \phi(t, q, v)$ (from Corollary 9.3) exists when $t \in (-\delta, \delta)$, then the geodesic $\phi(t, q, \alpha v)$, where $\alpha > 0$, exists for $t \in (-\delta/\alpha, \delta/\alpha)$, and $\phi(t, q, \alpha v) = \phi(\alpha t, q, v)$.*

Proof. Define $h : (-\delta/\alpha, \delta/\alpha) \to M$ by $h(t) = \phi(\alpha t, q, v)$, which is well-defined because $\alpha t \in (-\delta, \delta)$, and $h(0) = q$ and $h'(0) = \alpha\phi'(0, t, v) = \alpha b$. In general, $h'(t) = \alpha\phi'(\alpha t, p, v)$< so

$$\begin{aligned}\frac{\mathrm{D}(h'(t))}{\mathrm{d}t} &= \nabla_{h'(t)} h'(t) \\ &= \nabla_{\alpha\phi'(\alpha t, p, v)} \alpha\phi'(\alpha t, p, v) \\ &= \alpha^2 \nabla_{\phi'(\alpha t, p, v)} \phi'(\alpha t, p, v) \\ &= 0,\end{aligned}$$

because ϕ is a geodesic. ⊠

In essence, we are traveling faster for a shorter time. But it's about the journey, not the destination, right?

If the initial vector is shorter enough, we can travel for one second, and if not, we can travel more slowly such that it works eventually. This motivates the following result.

Proposition 9.5. *For all $p \in M$, there's a neighborhood $V \subset M$ of p, an $\varepsilon > 0$, and a C^∞ map $\phi : (-2,2) \times \mathcal{U} \to M$, where*

$$\mathcal{U} = \{(q, v) \mid q \in V, v \in T_q \text{ such that } |v| < \varepsilon\},$$

such that for all $(q, v) \in \mathcal{U}$ and $t \in (-2, 2)$, $\gamma(t) = \phi(t, q, v)$ is the unique geodesic such that $\gamma(0) = q$ and $\gamma'(0) = v$.

Proof. Let $\varepsilon = \delta\varepsilon'/2$, and use Lemma 9.4 ⊠

This seems silly (the statement is much longer than the proof, for example), but it's a nice uniformity result, and is useful for defining the exponential map.

Definition. The map $\exp : \mathcal{U} \to M$ given by $(q, v) \mapsto \phi(1, q, v)$. That is, for every point and direction, one travels along the geodesic in that direction for time 1. This map is called the exponential map on $\mathcal{U}$.

Definition. For any $q \in V$, the map $\exp_p : B_\varepsilon(0) \subset T_qM \to M$ given by $v \mapsto \exp(q, v)$ is known as the exponential map at q.

The first important feature of the exponential map is that it's a local diffeomorphism, because the tangent map is the identity: $\exp_q(0) = q$. We'll show this formally.

Lemma 9.6. $(D\exp_q)_0 : T_0(T_qM) = T_qM \to T_qM$ *is equal to* $\mathrm{id}_{T_qM} : T_qM \to TM$.

Proof. Let's stare at the definition: $(D\exp_q)_0(v)$ is the derivative of the geodesic in the direction of v, so it's $\frac{\mathrm{d}}{\mathrm{d}t}\phi(1, q, tv)$ (just the definition of the exponential map), but by the homgeneity of ϕ, this is $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t, q, v)$, which, since ϕ is a geodesic, is just v. ⊠

Corollary 9.7. *For any q, $\exp_q : B_\varepsilon(0) \to M$ is a homeomorphism onto an open subset of M.*

This follows because the tangent map is nondegenerate. In fact, since everything is smooth, it's even a diffeomorphism!

10. Geodesics are Locally Minimizing: 10/13/14

The goal of today's lecture is to show that geodesics are locally minimizing.

Definition. Let M be a Riemannian manifold. A curve $\gamma : [a, b] \to M$ is minimizing if $\ell(\gamma) \le \ell(c)$ for any piecewise smooth curve $c : [a, b] \to M$ such that $c(a) = \gamma(a)$ and $c(b) = \gamma(b)$.

So first, how local do we mean by locally? Remember that we defined the exponential map for any $p \in M$, i.e. there exists an $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \to M$ (where $B_\varepsilon(0) = \{v \in T_pM \mid |v| < \varepsilon\}$), with $\exp_p(v) = \gamma(1)$, where γ is the unique geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.

Proposition 10.1. *There exists a neighborhood $V \subset T_pM$ of 0 such that $\exp + p|_V$ is a diffeomorphism onto an open subset U of M.*

Since V is a neighborhood of 0, then U is a neighborhood of $\exp_p(0) = p$.

Definition. This neighborhood U is called a normal neighborhood of p. Specifically, a neighborhood U of p is called normal if $\exp_p : \exp_p^{-1}(U) \to U$ is a diffeomorphism.

The key is the bijective association between vectors in V and geodesics γ from p.

Definition. If $B_\varepsilon(0) \subset V$, then $\exp_p(B_\varepsilon(0))$, also written $B_\varepsilon(p)$, is called a normal ball centered at p.

In other words, the exponential map gives a natural choice of coordinates, locally.

Now, the main result today is that geodesics are length-minimizing in normal balls.

Proposition 10.2. *Let $p \in M$, U be a neighborhood of p, and $B \subset U$ be a normal ball centered at p. Let $\gamma : [0, 1] \to B$ be a geodesic segment with $\gamma(0) = p$; then, if $c : [0, 1] \to M$ is any piecewise smooth curve such that $c(0) = p$ and $c(1) = \gamma(1)$, then $\ell(\gamma) \le \ell(c)$ and if equality holds, then $c([0, 1]) = \gamma([0, 1])$ and c is monotonic (i.e. $c' \neq 0$).*

Proof. The proof will look somewhat similar to our discussion of geodesics in the upper half-plane.

First, assume $c([0, 1]) \subset B$ as well, and let $v \in T_pM$ be such that $\exp_p(v) = \gamma(1)$. Since $c([0, 1]) \subset B$, then there exists a $\bar{c} : [0, 1] \to B_\varepsilon(0)$ such that $c(t) = \exp_p(\bar{c}(t))$.

Let's introduce polar coordinates on this ball: write $\bar{c}(t) = r(t) \cdot w(t)$, where $r(t) \in \mathbb{R}$ and $w(t) \in T_pM$ such that $|w(t)| = 1$. Thus, $r(1) = |\bar{c}(1)| = |v| = \ell(\gamma)$.

For $w : [0, 1] \to T_pM$ such that $|w(t)|$ is constant (in this case, 1) define the map $f : [0, 1] \times [0, 1] \to M$ given by $f(r, t) = \exp_p(rw(t))$.

In this case, we have $c(t) = \exp_p(\bar{c}(t)) = \exp_p(r(t) \cdot w(t)) = f(r(t), t)$. Thus, by the Chain Rule,[3]

$$c'(t) = \frac{\partial f}{\partial r} \cdot r'(t) + \frac{\partial f}{\partial t}.$$

[3]Look how similar this seems to the hyperbolic case, which had $c'(t) = x'(t) + y'(t)$.

So first, we observe that $\frac{\partial f}{\partial r}$ is constant. Why is that?[4] Let $t \in (0,1)$; then,

$$\left|\frac{\partial f}{\partial r}t(r,t_0)\right| = \left|\frac{\mathrm{d}f}{\mathrm{d}r}\exp_p(rw(t_0))\right|.$$

But $\exp_p(rw(t_0)) = \gamma_{t_0}(r)$, the unique geodesic tangent to $w(t_0)$, so this is also equal to $|\gamma_{t_0}'(r)|$. But since a geodesic is a constant-speed curve, this is $|\gamma_{t_0}'(0)| = |w(t_0)| = 1$.

Now, if

$$|c'(t)|^2 = \left|\frac{\partial f}{\partial r}\cdot r'(t)\right|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \tag{3}$$

holds, then everything follows, because

$$|c'(t)|^2 = |r'(t)|^2 + \left|\frac{\partial f}{\partial r}\right|^2 \geq |r'(t)|^2,$$

so

$$\begin{aligned}\ell(c) &= \int_0^1 |c'(t)|\,\mathrm{d}t\\ &\geq \int_0^1 r'(t)\,\mathrm{d}t\\ &= r(1) - r(0) = \ell(\gamma).\end{aligned}$$

Additionally, equality holds iff $\frac{\partial f}{\partial t} = 0$ for all $(r(t),t)$ (which is enough).

At the beginning, we assumed that $c([0,1])$ is in the normal ball. What happens when this isn't the case? Then, consider the first point $t_1 \in [0,1]$ such that $c(t_1) \in \partial B$. (c may intersect ∂B more than once, so we just consider the first bit of the curve). Then, $\ell(c) \geq \ell_{[0,t_1]}(c)$, which is at least the radius of B, which is at least $|v| = \ell(\gamma)$ (since γ is a geodesic).

Now, we need to pin down (3), which comes from $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\right\rangle = 0$. This follows from the so-called Gauss Lemma.[5] The idea is that the exponential map doesn't always preserve the inner product, but does in some restricted cases.

Lemma 10.3 (Gauss). *Let $p \in M$ and $v \in T_pM$ be such that $\exp_p(v)$ is defined. If $w \in T_pM \approx T_v(T_pM)$, then*[6] $(D\exp_p)_V w \in T_{\exp_p(v)}M$*, and*

$$\left\langle (D\exp_p)_v w, (D\exp_p)_v v\right\rangle_{\exp_p(w)} = \langle w, v\rangle_p.$$

Assuming this lemma, then

$$\left\langle \frac{\partial f}{\partial p}, \frac{\partial f}{\partial t}\right\rangle_{c(t)} = \left\langle (D\exp_p)_{\bar c(t)}\bar c(t), (D\exp_p)_{\bar c(t)} r(t)w'(t)\right\rangle_{c(t)}.$$

This looks complicated, but follows from the definition of f. By the Gauss Lemma,

$$\begin{aligned}&= \langle r(t)w(t), r(t)w'(t)\rangle_p\\ &= r^2(t)\langle v(t), v'(t)\rangle_p = 0.\end{aligned}$$

⊠

Next time, we'll start by proving the Gauss Lemma.

11. The Gauss Lemma: 10/15/14

Recall that last time we were in the midst of proving Proposition 10.2, that if M is a Riemannian manifold, $p \in M$, and U is a normal neighborhood centered at p, (so that $\exp_p : \exp_p^{-1}(\mathcal{U}) \to M$ is a diffeomorphism onto $\mathcal{U}$), $R \subset \mathcal{U}$ is an open ball, and $B = \exp_p(B_\varepsilon(0))$, then a geodesic $\gamma : [0,1] \to B$ with $\gamma(0) = p$ is locally length-minimizing, in the sense described above.

We constructed a function $f : (0,1)\times(0,1) \to M$ given by $f(r,t) = \exp_p(rv(t))$, where $v(t) \in T_pM$ is such that $|v(t)| = 1$ (polar-like coordinates), and showed that, assuming the Gauss Lemma, the theorem holds.

This lemma states that the inner product is preserved by exp if one vector is along the geodesic. (In general, exp doesn't preserve the inner product.) Specifically, if $v \in T_pM$ such that $\exp_p(v)$ is defined, let $w \in T_pM \approx T_v(T_pM)$. Then, $\langle (D\exp_p)V(w), (D\exp_p)_v(v)\rangle_{\exp_p(v)} = \langle w, v\rangle_p$.

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[4] At this point in class, the fire alarm went off. Nobody left.

[5] There are at least three things called Gauss' Lemma: see `http://en.wikipedia.org/wiki/Gauss's_lemma`. We care about the one in Riemannian geometry, of course.

[6] The fire alarm stopped about here.

Proof of Lemma 10.3. Since $(D\exp_p)_v$ is linear, we can assume that $w \perp v$.

Consider $v(t) \subset T_pM$ such that $\langle v(t)\rangle = \langle v\rangle$, $v(0) = v$, and $v'(0) = w$. Since $\exp_p v$ is defined, then $\exp_p rv(t)$ is defined for $r \in (0,1)$ (since exp is still defined when we scale) and $t \in (-\varepsilon, \varepsilon)$ (since it's defined in a neighborhood). Define $f\colon (0,1)\times(-\varepsilon,\varepsilon) \to M$ by $f(r,t) = \exp_p(rv(t))$; then,

$$\langle (D\exp_p)_v(w), (D\exp_p)_v(v)\rangle_{\exp_p v} = \left\langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial r}(1,0)\right\rangle_{\exp_p v}. \tag{4}$$

This can be computed out from the expression for $v(t)$ in terms of r and v.

The goal is to show that this is zero, since $v \perp w$. First, we can show that

$$\left\langle \frac{\partial f}{\partial t}(r,t_0), \frac{\partial f}{\partial t}(r,t_0)\right\rangle$$

is constant for $t_0 \in (-\varepsilon,\varepsilon)$. Let $\gamma_{t_0}(r) = \exp_p(rv(t_0))$; then, this is true because when we differentiate with respect to r, the covariant derivative is metric-compatible, so

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\langle \frac{\partial f}{\partial t}, pfrft\right\rangle = \left\langle \frac{\mathrm{D}}{\mathrm{d}r}\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial r}\right\rangle + \left\langle \frac{\partial f}{\partial t}, \frac{\mathrm{D}}{\mathrm{d}r}\left(\frac{\partial f}{\partial r}\right)\right\rangle.$$

The second term is the covariant derivative of a vector along itself, which is necessarily zero. And by the symmetry of the metric,

$$= \left\langle \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial f}{\partial r}\right), \frac{\partial f}{\partial r}\right\rangle.$$

This merits more of an explanation: why is $\frac{\mathrm{D}}{\mathrm{d}r}\left(\frac{\partial f}{\partial t}\right) = \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial f}{\partial r}\right)$? This is ultimately because $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial t}$ are vector fields on $\mathrm{Im}(f)$, so

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}r}\left(\frac{\partial f}{\partial t}\right) - \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial f}{\partial r}\right) &= \nabla_{\frac{\partial f}{\partial r}}\frac{\partial f}{\partial t} - \nabla_{\frac{\partial f}{\partial t}}\frac{\partial f}{\partial r}\\ &= \left[\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\right] = Df\left(\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right]\right) = 0,\end{aligned}$$

where the fact that D commutes with the bracket comes from the first homework assignment.

Now that we've shown that (4) is constant, so if we can show it's zero anywhere, then it is zero everywhere. Let's start with

$$\begin{aligned}\frac{\partial f}{\partial t}(0,0) &= \lim_{r\to 0}\frac{\partial f}{\partial t}(r,0) = \lim_{r\to 0}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp_p rv(t)\\ &= \lim_{r\to 0}(D\exp_p)_{rv} rw = 0 = \langle w, v\rangle_p.\end{aligned}$$

Basically, as $r \to 0$, there's nowhere else this could go. rw is a vector field that is just w scaled by how far along v we are. Its image $D(\exp_p)_{tv}(tw)$ is called the Jacobi field: for each t, it projects down to a geodesic, and the curves for fixed r are the tangents to curves $rv(t)$. Thus, the Jacobi field can be thought of as an infinitesimal variation of geodesics. ⊠

So now we know that geodesics are locally minimal for arc length. It turns out the converse is also true.

Proposition 11.1. *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve such that $|\gamma'(t)|$ is constant (i.e. γ has constant speed) and $\ell(\gamma) \le \ell(c)$ for all piecewise smooth $c : [a,b] \to M$ such that $c(a) = \gamma(a)$ and $c(b) = \gamma(b)$; then, γ is a geodesic.*

The naïve idea is to pick a normal ball B and a geodesic $\widetilde{\gamma}$ with the same endpoints of γ, but since γ is minimizing, then by the previous proposition, it must be have the same image as $\widetilde{\gamma}$, which is nice. But this doesn't work on the center of B: it is smooth in either direction outwards, but maybe not at that center. Thus, we need to think some more.

Definition. For a $p \in M$, then W is a totally normal neighborhood of p if it is a normal neighborhood of any of its points.

Then, there's a theorem that means these exist around any p.

Theorem 11.2. *For any $p \in M$, there exists a normal neighborhood W and a $\delta > 0$ such that for any $q \in W$, $\exp_q : B_\delta(0) \to M$ is a diffeomorphism onto its image, and $W \subset \exp_q(B_\delta(0))$ is a totally normal neighborhood.*

Proof. Consider the map $F : (q,v) \mapsto (q, \exp_q(v))$ from $F : \mathcal{U} \subset TM \to M \times M$ (where $\mathcal{U}$ is a region on which this map makes sense). There are local coordinates in which this $DF = \begin{pmatrix} I & I \\ 0 & I\end{pmatrix}$ in block form, so there is some small neighborhood $\mathcal{U}'$ on which F is a diffeomorphism. Write $\mathcal{U}' = \{(q,v) \mid q \in V', v \in T_qM, |v| < \delta\}$ for some neighborhood V', and let W' be its image. Then, there's a $W \subset M$ such that $W \times W \subset W'$.

This W is exactly what we want, as $\{q\} \times W \subset F(\{q\} \times B_\delta(0))$. ⊠

Now, we can finish the proof of Proposition 11.1; we choose a *totally* normal neighborhood W this time, so now the entire part is a geodesic, since we can make it smooth for each $t \in W$.

12. The Hopf-Rinow Theorem: 10/17/14

Every Riemannian metric gives a structure of a metric space on a manifold; then, the Hopf-Rinow Theorem says that geodesic completeness is equivalent to completeness as a metric space.

Definition. A Riemannian manifold M is geodesically complete if for all $p \in M$, the exponential map $\exp_p$ is defined on the entire tangent space T_pM, i.e. every geodesic γ_t with $\gamma(0) = p$ is defined for all $t \in \mathbb{R}$.

Example 12.1. The three most common examples are Euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$, and hyperbolic space $\mathbb{H}^n$.

The unit disc $\mathbb{D}^n$, however, is not complete, because the geodesic through 0 in the direction $\frac{\partial}{\partial x}$ stops after time 1. A space with a hole is also not complete, e.g. $\mathbb{R}^2 \setminus \{0\}$.

Intuitively, a Riemannian manifold is geodesically complete if it has no holes and no boundary.

Definition. If M is a connected Riemannian manifold and $p, q \in M$, the distance between p and q is defined to be $d(p, q) = \inf_{\gamma\in\Gamma} \ell(\gamma)$, where Γ is the set of piecewise smooth curves connecting p and q.

First off, this definition makes sense: since M is connected, then it's path-connected, and the curve can be made smooth in local charts (since the path must be compact).

Proposition 12.2. *(M, d) is a metric space.*

Proof. This involves checking a bunch of obvious things: for the triangle inequality, if $p, q, r \in M$, then there are curves γ_1 joining p and q, and γ_2 joining q and r, so joining them produces a curve γ_{12} joining q and r, so $d(p, r) \le \ell(\gamma_{12}) = \ell(\gamma_1)+\ell(\gamma_2)$, and so forth. That it's symmetric follows by reversing the orientation of any curve, and that $d(p, q) = 0$ iff $p = q$ is because if $d(p, q) = 0$, then they must both lie in a normal neighborhood of p, where a geodesic can be drawn between them. ☒

Proposition 12.3. *The topology induced by d coincides with the topology of M.*

Proof. If γ connecting p and q is minimizing (so that it's a geodesic, as we saw last time), then $d(p, q) = \ell(\gamma)$, so when ε is sufficiently small, the metric ball $\overline{B}_\varepsilon(p) = \{q \in M \mid d(p, q) \le \varepsilon\}$ coincides with $B_\varepsilon(p)$, because in a normal neighborhood, geodesics are uniquely length-minimizing. ☒

Corollary 12.4. *For all $p_0 \in M$, the function d_{p_0} on M defined by $d_{p_0}(p) = d(p_0, p)$ is continuous.*

This is obvious because it's continuous with respect to the metric topology, but this is exactly the same as the original topology.

Recall that a metric space is complete if all Cauchy sequences on it converge.

Theorem 12.5 (Hopf-Rinow). *Let M be a Riemannian manifold and $p \in M$. Then, the following are equivalent:*

(1) M is geodesically complete.
(2) If $A \subset M$ is closed and bounded, then A is compact.
(3) (M, d) is complete in the sense of the metric.

In addition, any of these conditions implies that for all $p, q \in M$, there is a geodesic γ connecting p and q such that $\ell(\gamma) = d(p, q)$.

Note that the last condition is not true for manifolds in general, e.g. in $\mathbb{R}^2 \setminus \{0\}$, look at $p = (-1, 0)$ and $q = (1, 0)$. Then, $d(p, q) = 2$, as curves can approach the origin, but any curve between p and q with length 2 would go through the origin, which is disallowed.

Corollary 12.6. *Any compact manifold is complete, since it satisfies (2).*

Corollary 12.7. *Any closed submanifold of a complete manifold is complete in the induced metric (also by (2)).*

Proof of Theorem 12.5. First, why does (1) imply the existence of a geodesic with length equal to $d(p, q)$? Let $r = d(p., q)$ and $B_\delta(p)$ be a normal ball centered at p. Let $S = \partial B_\delta(p)$; since d_q is continuous on S and S is compact, then it has a minimum $x_0 \in S$, and since x_0 is on the boundary of the normal ball, then let $v \in T_pM$ be such that $x_0 = \exp_p \delta v$ and $|v| = 1$.

Then, we will want $q = \exp_p(rv)$, because then $\gamma(t) = \exp_p(tv)$ is the minimizing geodesic we are looking for. Let $A = \{t \in [0, r] \mid d(\gamma(t), q) = r - t\}$, and we want to show that $r \in A$. We'll do this by showing A is nonempty, closed, and open, and thus that it contains all of $[0, r]$.

First, $0 \in A$, since $d(\gamma(0), q) = d(p, q) = r - 0$. Also, it's closed, since both sides of $d(\gamma(t), q) = r - t$ are continuous in t (so the condition commutes with limits).

It's a little harder to show that it's open. We want to prove that if $t_0 \in A$, then for small δ, $t_0 + \delta' \in A$. At a given t_0, consider a normal ball $B_{\delta'}(\gamma(t_0))$ around $\gamma(t_0)$ and $S' = \partial B_{\delta'}(\gamma(t_0))$. Thus, d_q again has a minimum at x_0 on S', so it suffices to show that $x_0' = \gamma(t_0 + \delta')$, because if so, then $d(\gamma(t_0), q) = r - t_0$ and

$$d(\gamma(t_0), q) = \delta' + d(x_0', q) = \delta' + d(\gamma(t_0 + \delta'), q),$$

so $d(\gamma(t_0+\delta'), q) = r - (t_0+\delta')$, so $t_0+\delta' \in A$. Thus, all we need to do is show $x_0' = \gamma(t_0+\delta')$.

Now, $d(x_0') > d(p,q) - d(q,x_0') = r - (r-(t_0+\delta')) = t_0+\delta'$, and this distance is realized by the curve $\overline{p\gamma(t_0)q}$, and therefore, since this curve is length-minimizing, it must be a geodesic, and so $\gamma(t_0+\delta') = x_0'$.

Now, the rest of the proof is straightforward: to show (1) implies (2), let $A \subset M$ be closed and bounded, and $p \in A$. Then, since A is bounded, then $A \subset \overline{B}_r(p)$, and $\overline{\overline{B}_r(p)} \subset \overline{B_r(p)} = \exp_p(\overline{B_r(0)})$, which is compact (since $\exp_p$ is continuous and $\overline{B_r(0)}$ is compact); thus, A is a closed subset of a compact set, and thus is also compact.

(2) $\implies$ (3) is just point-set topology, so it's left as an exercise.

For (3) $\implies$ (2), choose a $p \in M$ and a geodesic γ such that $\gamma(0) = p$ and $\gamma(t)$ is determine for $t < t_0$, but not at t_0. Then, choose a Cauchy sequence $\{t_n\}$ converging to t, so thus $\{\gamma(t_n)\}$ is also Cauchy, because $d(\gamma(t_n), \gamma(t_n)) \le |t_n - t_m|$. Since M is metrically complete, then $\gamma(t_n) \to p_0 \in M$.

Let W be a totally normal neighborhood of p_0, and let's see what happens. Since W is totally normal, there's a unique geodesic γ_{mn} connecting $\gamma(t_n)$ and $\gamma(t_m)$, and it extends to and beyond p_0, so since it's unique, then γ also extends through t_n and t_m to t_0, which was a contradiction. ⊠

13. Curvature: 10/20/14

As suggested by the name, curvature measures how curved a space is; specifically, how different it appears locally from Euclidean space.

Definition. The curvature R of a Riemannian manifold M associates $X, Y \in V(M)$ to a mapping $R(X,Y) : V(M) \to V(M)$, which for a $Z \in M$ gives

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Example 13.1. If $M = \mathbb{R}^n$ with the metric $g_{ij} = \delta_{ij}$, let $X, Y, Z \in V(M)$, with $Z = (z_1(x), \dots, z_n(x))$. Then, $\nabla_X Z = (X(z_1), \dots, X(z_n))$, because $\Gamma_{ij}^k = 0$ for this metric. Thus, $\nabla_Y \nabla_X Z = (YX(z_1), \dots, YX(z_n))$ and $\nabla_X \nabla_Y Z = (XY(z_1), \dots, XY(z_n))$, so $\nabla_{[X,Y]} Z = ([X,Y](z_1), \dots, [X,Y](z_n))$. Thus, since $[X,Y] = XY - YX$, then the sum is $R(X,Y)Z = 0$.

In a general Riemannian manifold, consider a chart (U, ϕ), with local coordinates $(x_1, \dots, x_n)$. Let $X \in V(U)$. Then,

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) X &= \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} X - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} X + \nabla_{\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]} X \\ &= \left(\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}}\right) X. \end{aligned}$$

Thus, the curvature measures the infinitesimal noncommutativity of the covariant derivatives of X and Y.

The key property of curvature is that it is a tensor, i.e. it is C^∞ in each of its components. This is the main difference between the curvature and the connection.

Proposition 13.2. *R is $C^\infty(M)$-linear in X, Y, and Z, i.e.*

$$\begin{aligned} R(fX_1 + gX_2, Y)Z &= fR(X_1, Y)Z + gR(X_2, Y)Z \\ R(X, fY_1 + gY_2)Z &= fR(X, Y_1)Z + gR(X, Y_2)Z \\ R(X,Y)(fZ_1 + gZ_2) &= fR(X,Y)Z_1 + gR(X,Y)Z_2 \end{aligned}$$

where these vector fields are in $V(M)$.

Proof. This proof is highly repetitive, and everything follows from the properties of the connection and the bracket, so it's been abridged; specifically, we'll address the first and third equalities in the case of function multiplication. For the first one,

$$\begin{aligned} R(fX,Y)Z &= \nabla_T \nabla_{fX} Z - \nabla_{fX} \nabla_Y Z + \nabla_{fX,Y} Z \\ &= \nabla_Y (f \nabla_X Z) - f\nabla_X \nabla_Y Z + Y(f) \nabla_X Z + f\nabla_Y \nabla_X Z - f\nabla_X \nabla_Y Z \\ &= Y(f)\nabla_X Z + f\nabla_Y \nabla_X Z - f\nabla_X \nabla_Y Z + f\nabla_{[X,Y]} Z - Y(f)\nabla_X Z \\ &= fR(X,Y)Z. \end{aligned}$$

For the third equality,

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]}(fZ) \\ &= \nabla_Y (X(f)Z + f\nabla_X Z) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]}(fZ) \\ &= YX(f)Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y \nabla_X Z - XY(f)Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - f\nabla_X \nabla_Y Z + \nabla_{[X,Y]}(fZ) \\ &= YX(f)Z - XY(f)Z + f\nabla_Y \nabla_X Z - f\nabla_X \nabla_Y Z + [X,Y](f)Z + f\nabla_{[X,Y]} Z \\ &= fR(X,Y)Z. \end{aligned}$$

⊠

What's impressive is how everything follows from the definition, and in particular that it's a tensor.

Similarly to the Jacobi identity, there's a Bianchi[7] identity: if one rotates X, Y, and Z, the sum is zero.

Proposition 13.3 (Bianchi identity). *For all X, Y, and Z in $V(M)$, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.*

Proof. The proof follows from the symmetries of the Levi-Civita connection and the Jacobi identity.

We don't know anything else save the definition, so here goes.

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[X,Y]}Z.$$

The terms $\nabla_Y(\nabla_X Z - \nabla_Z X) = \nabla_Y[X,Z]$, so we can simplify this to

$$= \nabla_Y[Z,X] + \nabla_X[Z,Y] + \nabla_Z[Y,X] + \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[X,Z]}Y.$$

Again we use the symmetry of the Levi-Civita connection on Y and $[X,Z]$, and so forth.

$$= [Y,[Z,X]] + [X,[Z,Y]] + [Z,[Y,X]] = 0$$

by the Jacobi identity. ⊠

The consistent lesson is that once the definition is expanded out, the rest is just algebraic tricks.

Denote by $(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$, with $X,Y,Z,W \in V(M)$. This is a number for each choice of vector field, so this is a function on M, and satisfies a number of useful identities.

Proposition 13.4. *Let $X,Y,Z,W \in V(M)$. Then,*

(1) *$(\cdot,\cdot,\cdot,\cdot)$ is C^∞-linear in X, Y, Z, and W (since X, Y, and Z follow because R is a tensor, and W because we're taking an inner product).*
(2) *$(X,Y,Z,W) + (Y,Z,X,W) + (Z,X,Y,W) = 0$, from Proposition 13.3.*
(3) *$(X,Y,Z,W) = -(Y,X,Z,W)$ (proven by computing from the definition).*
(4) *$(X,Y,Z,W) = -(X,Y,W,Z)$.*
(5) *$(X,Y,Z,W) = (Z,W,X,Y)$.*

Proof. Several parts have easy proofs mentioned in the proposition statement, but (4) is trickier, so we'll provide a proof. It suffices to show that $(X,Y,Z,Z) = 0$, since then it can be applied to $(X,Y,Z+W,Z+W)$ to show the whole thing (by linearity). The key is to use the metric compatibility of the connection.

$$\begin{aligned}(X,Y,Z,Z) = \langle R(X,Y)Z,Z\rangle &= \langle \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z, Z\rangle \\ &= Y\langle\nabla_X Z, Z\rangle - \langle\nabla_X Z, \nabla_Y Z\rangle - X\langle\nabla_Y Z, Z\rangle + \langle\nabla_Y Z, \nabla_X Z\rangle + \langle\nabla_{[X,Y]}Z, Z\rangle \\ &= Y\langle\nabla_X Z, Z\rangle - X\langle\nabla_Y Z, Z\rangle + \langle\nabla_{[X,Y]}Z, Z\rangle \\ &= Y\langle\nabla_X Z, Z\rangle - X\langle\nabla_Y Z, Z\rangle + \frac{1}{2}[X,Y]\langle Z,Z\rangle,\end{aligned}$$

after expanding the last term and using the symmetry of the inner product. Next,

$$= \frac{1}{2}YX\langle Z,Z\rangle - \frac{1}{2}XZ\langle Z,Z\rangle + \frac{1}{2}[X,Y]\langle Z,Z\rangle = 0.$$

Then, (5) is a consequence of (2), (3), and (4). Specifically, from (2),

$$\begin{aligned}(X,Y,Z,W) + (Y,Z,X,W) + (Z,X,Y,W) &= 0\\ (Y,Z,W,X) + (Z,W,Y,X) + (W,Y,Z,X) &= 0\\ (Z,W,X,Y) + (W,X,Z,Y) + (X,Z,W,Y) &= 0\\ (W,X,Y,Z) + (X,Y,W,Z) + (Y,W,X,Z) &= 0.\end{aligned}$$

If one adds all of these together and invokes (3) and (4), things cancel, yielding

$$2(Z,X,Y,W) + 2(W,Y,X,Z) = 0.$$

Thus, $(Z,X,Y,W) = (W,Y,Z,X)$. ⊠

There's a big calculation for this in local coordinates, which we're not going to do in detail, but in local coordinates $(x_1,\ldots,x_n)$ let

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_\ell R_{ijk}^\ell \frac{\partial}{\partial x_\ell},$$

[7]There are a lot of Italians in Riemannian geometry, apparently partly because Riemann spent the last part of his life there.

then let $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $Y = \sum_j b_j \frac{\partial}{\partial x_j}$, and $Z = \sum_k c_k \frac{\partial}{\partial x_k}$, then R is C^∞-linear, so

$$R(X,Y)Z = \sum_\ell \left(\sum_{i,j,k} R_{ijk}^\ell a_i b_j c_k \right) \frac{\partial}{\partial x_\ell}.$$

We can write R_{ijk}^ℓ in terms of the Christoffel symbols, which is a lot of computation, so we'll just give the final result. Recall that they were defined as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Then, we have a big formula:

$$R_{ijk}^\ell = \sum_s \left(\Gamma_{ik}^s \Gamma_{js}^\ell - \Gamma_{jk}^s \Gamma_{is}^\ell \right) + \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} - \frac{\partial \Gamma_{jk}^\ell}{\partial x_i}. \tag{5}$$

This may or may not be useful in practice, but it tells us that the curvature depends only on the metric, and therefore is preserved by isometries.

The identities stated above can be rewritten in this coordinate-wise format.

Proposition 13.5.

(1) $R_{ijk\ell} + R_{jki}^\ell + R_{kij}^\ell = 0$.
(2) $R_{ijk}^\ell = -R_{jik}^\ell$.
(3) $R_{ijk}^\ell = -R_{ij\ell}^k$.
(4) $R_{ijk}^\ell = R_{k\ell i}^j$.

14. Sectional Curvature: 10/22/14

Recall that the curvature was defined as $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$, and its counterpart is $(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$. This operator $(,,,)$ is a tensor, i.e. $C^\infty(M)$-linear in each component. Specifically, $(X(p), Y(p), Z(p), W(p))$ only depends on $X(p)$, $Y(p)$, $Z(p)$, and $W(p)$, not on nearby values; in this respect, it is unlike the connection.

We proved several nice properties of these tensors, including Propositions 13.3 and 13.4, and also wrote down the cuvature in coordinates, as (5).

Definition. Let $\sigma \subset T_pM$ be a two-dimensional subspace spanned by $u, v \in T_pM$. Then, the sectional curvature of σ at p is

$$K(\sigma) = k(u,v) = \frac{(u,v,u,v)}{|u|^2|v|^2 - \langle u,v\rangle^2}. \tag{6}$$

Proposition 14.1. *This definition is well-founded, i.e. the sectional curvature is independent of the choice of u and v.*

Proof. Since $|u|^2|v|^2 - \langle u,v\rangle^2 = |u|^2|v|^2\sin^2\theta$, then $\sqrt{|u|^2|v|^2 - \langle u,v\rangle^2}$ is the area of the parallelogram determined by u and v.

To show this is independent of u and v, it suffices to show that $K(\sigma)$ is invariant under linear transformations, which are generated by $\{u,v\} \mapsto \{v,u\}$, $\{u,v\} \mapsto \{\lambda u, v\}$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and $\{u,v\} \mapsto \{u+v, v\}$.

These are all easy to check: (6) is symmetric in u and v, so the first one is OK; the second one multiplies the length of one side of the parallelogram by λ and therefore is canceled out on the top as λ^2/λ^2 (since v appears twice in the top), and so forth. ☒

Example 14.2. Let's look at $\mathbb{H}^2$, in which $g_{ij} = \delta_{ij}/y^2$. Recall $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = 1/y$, and $\Gamma_{12}^1 = \Gamma_{22}^2 = -1/y$. Choosing $u = \frac{\partial}{\partial x}$ and $v = \frac{\partial}{\partial y}$, we get that

$$\begin{aligned} K(T_pM) &= \frac{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}{\left|\frac{\partial}{\partial x}\right|^2 \left|\frac{\partial}{\partial y}\right|^2 - \left\langle \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right\rangle^2} \\ &= \frac{\left\langle R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle}{(1/y^4) - 0} \\ &= y^4 \left\langle R_{121}^1 \frac{\partial}{\partial x} + R_{121}^2 \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle \\ &= y^4 R_{121}^1. \end{aligned}$$

Plugging into (5) means an ugly and involved calculation which eventually gives $-1/y^4$, so the overall sectional curvature becomes $(-1/y^4)y^4 = -1$. Thus, $\mathbb{H}^2$ has constant sectional curvature -1.

Given the curvature tensor, we can calculate all sectional curvatures, but it turn out we can also go in the other direction: if one knows all of the sectional curvatures of all subspaces, then the original curvature tensor can be reconstructed.

Lemma 14.3. *Let V be an $\mathbb{R}$-vector space of dimension at least 2 and $\langle , \rangle$ be an inner product on V. Then, let R and R' be two trilinear mappings such that $(x, y, z, w) = \langle R(x, y, z), w\rangle$ and $(x, y, z, w)' = \langle R'(x, y, z), w\rangle$ and they satisfy the Bianchi identity and those in Proposition 13.4. Then, if the sectional curvatures induced by R and R' are the same for all 2-dimensional subspaces $\sigma \subset V$, then $R = R'$.*

The proof is purely algebraic; jut play around with the properties of the curvature tensor. The idea is, since $(x, y, x, y) = (x, y, x, y)'$, then apply this to $(x + y, y, x + y, z)$, showing after a few steps that $(x, y, z, y) = (x, y, z, y)'$. Doing it again with $(x, y + w, z, y + w)$ leads to things expanding and canceling, and so forth, leading to $(x, y, z, w) - (x, y, z, w)' = (y, z, x, w) - (y, z, x, w)'$, so permute this three times and use the Bianchi identity, so things cancel out. This is true for any w, so since the inner product is nondegenerate, then $R(x, y, z) = R'(x, y, z)$.

Geometrically, suppose $\sigma \subset T_pM$ is a two-dimensional subspace; then, $\exp_p(\sigma) \subset M$ is a surface. The Gaussian curvature of this surface is the sectional curvature of the space. As for Lemma 14.3, the intuition is that curvature is a tensor, so it only depends on the value of the vector fields at a point, which makes things look a little more reasonable.

Proposition 14.4. *If M is a Riemannian manifold and $p \in M$, then let $R' : (T_pM)^3 \to T_pM$ be a trilinear mapping given by*

$$\langle R'(X, Y, Z), W\rangle = \langle X, Y\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle$$

for all $X, Y, Z, W \in T_pM$. Then, M has constant sectional curvature k_0 at p iff $R = k_0R'$.

Proof. Let $(X, Y, Z, W)' = \langle R'(X, Y, Z), W\rangle$; then, this operator satisfies all of the properties in Proposition 13.4. If K' is the sectional curvature of k_0R', then for all two-dimensional subspaces σ, let $\sigma = \operatorname{span}\{X, Y\}$, so that

$$K'(\sigma) = \frac{K_0(X, Y, X, Y)'}{|X|^2|Y|^2 - \langle X.Y\rangle^2} = \frac{k_0(\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2)}{|X|^2|Y|^2 - \langle X, Y\rangle^2} = k_0.$$

Thus, since R has constant sectional curvature k_0, then by Lemma 14.3, $R = k_0R'$. ⊠

Corollary 14.5. *Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for T_pM. Then, R has constant sectional curvature at p iff $R_{ijk\ell} = k_0(\delta_{ik}\delta_{j\ell} - \delta_{j\ell}\delta_{ik})$, where $R_{ijk\ell} = (e_i, e_j, e_k, e_\ell)$.*

Now, we can use this to define Ricci curvature and scalar curvature. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of T_pM and $u = v_n$.

Definition. The Ricci curvature at p in the direction u is

$$\operatorname{Ric}_p(u) = \frac{1}{n-1}\sum_{i=1}^{n-1} K(v_i, u).$$

The scalar curvature is the average of the Ricci curvatures:

$$S(p) = \frac{1}{h}\sum_{i=1}^{n} \operatorname{Ric}_p(v_j) = \frac{1}{n(n-1)}\sum_{i,j} K(u_i, v_j).$$

Proposition 14.6. *The Ricci and scalar curvatures are independent of choice of basis.*

Proof. Let $Q : T_pM \times T_pM \to \mathbb{R}$ be given by

$$(u, v) \longmapsto (\operatorname{Tr}(w \mapsto R(u, w)v)).$$

Then, for any orthonormal basis $\{v_1, \dots, v_n\}$ with $v_n = u$, this means that

$$Q(u, u) = \sum_{i=1}^{n-1}\langle R(u, v_i)u, v_i\rangle = (n-1)\operatorname{Ric}_p(u),$$

but this means that $\operatorname{Ric}_p(u) = (1/(n-1))Q(u, u)$, and thus it depends only on Q and n, which means it is independent of coordinates.

Q induces a $K : T_pM \to T_pM$ by $\langle K(u), V\rangle = Q(u, v)$, and the trace of K is

$$\operatorname{Tr}(K) = \sum_{j=1}^{n}\langle K(v_i), v_i\rangle = \sum_{j=1}^{n} Q(v_i, v_i) = n(n-1)S(p),$$

so $S(p)$ depends ony on n and K, and thus is also independent of a choice of basis. ⊠

Definition. With Q as in the previous proof, $Q/(n-1) : T_pM \times T_pM \to \mathbb{R}$ is called the Ricci tensor.

"The people making all the loud noise are alumni, not undergrads. Right. Otherwise I might have told them to shut the fuck up!"

We've now finished the boring part of the course: the tools have been constructed, and now we can play with them, and see how they interact.

The sectional curvature $K(\sigma)$ measures how fast geodesics starting at p and tangent to σ spread apart. This intuition can be backed up with a geometric tool called the Jacobi field.

Recall that if $v \in T_pM$ is such that $\exp_p v$ is defined, then we can define an $f : (0, 1) \times (-\varepsilon, \varepsilon) \to M$ by $f(t, s) = \exp_p tv(s)$, where $v(s) \in T_pM$ with $v(0) = v$. This is a family of geodesics: when s is fixed, the curve is a geodesic $\gamma_s(t) = \exp_p tv(s)$. If $w = v'(0)$, $(D\exp_p)_v w = \frac{\partial f}{\partial s}(1, 0)$; this is because $f(1, s) = \exp_p v(s)$, and $v(s)$ is tangent to w, so $f(1, s)$ is tangent to $(D\exp_p)_v w$.

That's great, but why care about that vector? Because this measures the rate of spreading of $\{\gamma_s\}$. More generally, we can consider $(D\exp_p)_{tv} tw = \frac{\partial f}{\partial s}(t, 0)$. This is one of the Jacobi fields: at each point, it measures the rate of spread.

Proposition 15.1. *Let $J(t) = \frac{\partial f}{\partial s}(t, 0)$ along γ and R be the curvature tensor. Then, J satisfies the ODE*

$$J''(t) + R(\gamma'(t), J(t))\gamma'(t) = 0. \tag{7}$$

Definition. (7) is called the Jacobi equation, and a vector field $J(t)$ satisfying this equation is called a Jacobi field.

Lemma 15.2. *For all $v \in \operatorname{span}\left\{\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\}$,*

$$R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V = \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\mathrm{D}V}{\mathrm{d}s}\right) - \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\mathrm{D}V}{\mathrm{d}t}\right), \tag{8}$$

Proof. If $f(0, 1) \times (-\varepsilon, \varepsilon) \to M$, then as t_0 is fixed, $\alpha_{t_0} = f(t_0, s)$, and as s_0 is fixed, $\gamma_{s_0} = f(t, s_0)$. Then, $\frac{\mathrm{D}}{\mathrm{d}s}$ is the covariant derivative along α_{s_0}, and $\frac{\mathrm{D}}{\mathrm{d}t}$ is that along γ_{t_0}.

Now, the left-hand side of (8) is equal to

$$\nabla_{\frac{\partial f}{\partial t}}\nabla_{\frac{\partial f}{\partial s}}V - \nabla_{\frac{\partial f}{\partial s}}\nabla_{\frac{\partial f}{\partial t}}V + \nabla_{\left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right]}V.$$

The bracket is

$$\begin{aligned}\left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right] &= Df\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right)\\ &= 0\\ &= \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\mathrm{D}V}{\mathrm{d}s}\right) - \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\mathrm{D}V}{\mathrm{d}t}\right).\end{aligned}$$

Then, by the symmetry of γ, this is $\nabla_{\frac{\partial f}{\partial t}}\frac{\partial f}{\partial s} - \nabla_{\frac{\partial f}{\partial s}}\frac{\partial f}{\partial t}$, so (8) holds. ⊠

This formula tells us once again that the curvature is the commutator of the covariant derivatives.

Proof of Proposition 15.1. Since $\frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial t} = \gamma''(t) = 0$, then $\gamma(t) = \exp_p tv$ is a geodesic, and $\frac{\partial f}{\partial t} = \gamma'(t)$. Thus, by Lemma 15.2,

$$\begin{aligned}0 &= \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial t}\right) + \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial f}{\partial t}\right) - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}\\ &= \underbrace{\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial s}\right)}_{J''(t)} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}.\end{aligned}$$

⊠

This is a second-order ODE, which means that solutions locally exist, and are unique given values $J(0)$ and $J'(0)$. This suggests there should be $2n$ linearly independent solutions. This happens to be right!

Proposition 15.3. *Let $n = \dim(M)$; then, there are $2n$ linearly independent Jacobi fields along a geodesic γ.*

Proof. $J(t)$ is determined by $J(0)$ and $J'(0)$, which is fine, so let $\{e_1(t), \ldots, e_n(t)\}$ be parallel, orthonormal vector fields. Let

$$J(t) = \sum_i f_i(t)_i(t) \qquad \text{and} \qquad a_{ij}(t) = \langle R(\gamma'(t), e_i(t))\gamma'(t), e_j(t)\rangle.$$

Then,

$$J''(t) = \sum_i f_i''(t)e_i(t)$$

and

$$R(\gamma'(t), J(t))\gamma'(t) = \sum_{i,j} a_{ij}(t)f_j e_i(t).$$

Since
$$f_i''(t) + \sum_j a_{ij}(t) f_i(t) = 0,$$
for $i = 1, \ldots, n$, then these can be chosen to find $2n$ linearly independent Jacobi fields. ⊠

Remark. Two Jacobi fields are of particular interest, $\gamma'(t)$ and $t\gamma'(t)$. That these are Jacobi fields is because if $J_0(t) = \gamma'(t)$, then $J_0''(t) = 0$, and thus $R(\gamma'(t), J_0(t))\gamma'(t) = 0$. This field is the one with the initial conditions $J_0(0) = \gamma'(0)$ and $J_0'(0) = 0$.

If $J_1(t) = t\gamma'(t)$, then
$$J_1'(t) = \gamma'(t) + t\gamma''(t) = \gamma'(t),$$
so $J_1''(t) = \gamma''(t) = 0$. Thus,
$$R(\gamma'(t), J_1(t))\gamma'(t) = R(\gamma'(t), t\gamma'(t))\gamma'(t) = 0.$$

Suppose M is a manifold with constant sectional curvature. Then, what are its Jacobi fields? Suppose M has $K(\sigma) = k$ for all $p \in M$ and two-dimensional subspaces $\sigma \subset T_pM$. Let $\gamma : [0,1] \to M$ with $|\gamma'(0)| = 1$, and let $J(t)$ be a Jacobi field along γ such that $\langle J(t), \gamma'(t) \rangle = 0$ for all $t \in [0,1]$ and $J(0) = 0$. Then, the Jacobi equation reduces to
$$J''(t) + kJ(t) = 0,$$
because last time, we saw that since M has constant sectional curvature, then
$$\langle R(\gamma', J)\gamma', T \rangle = k(\langle \gamma', \gamma' \rangle \langle J, T \rangle - \langle \gamma', T \rangle \langle J, \gamma' \rangle) = k\langle J, T \rangle.$$
This is an ODE with constant coefficients which means I know how to solve it! Specifically, for all $w(t)$ parallel along γ with $\langle w(T), \gamma'(t) \rangle = 0$ and $|w(0)| = 1$, we have
$$J(t) = \begin{cases} \frac{\sin(t\sqrt{k})}{\sqrt{k}} w(t), & k > 0 \\ tw(t), & k = 0 \\ \frac{\sinh(t\sqrt{-k})}{\sqrt{-k}} w(t), & k < 0. \end{cases}$$
At the beginning of this class, we gave an explicit construction; next time, we'll show this is essentially the only way to do it, and do some other stuff relating to lengths of Jacobi fields, and conjugate points.

The idea for these measuring spreading is that the Taylor series for the length of J is
$$|J(t)| = t - \frac{K_p(0)}{6} t^3 + o(t^3),$$
when $J(0) = 0$ and $\sigma = \operatorname{span}\{J(0), J'(0)\}$, so this term is controlled by the sectional curvature.

16. Jacobi Fields Along Geodesics: 10/27/14

"Then, the coefficient of the fourth term is 8 over 4!, which is... 12? 24? Ok."

Recall that if M is Riemannian and $\gamma : [0, a] \to M$ is a geodesic, then a vector field J along γ is called a Jacobi field if it satisfies the Jacobi equation (7) for all $t \in [0, a]$. (7) is a second-order linear ODE, so it's determined by $J(0)$ and $J'(0)$, so for a given geodesic γ, there are $2n$ linearly independent Jacobi fields.

Recall that if $v \in T_pM$ is such that $\exp_p v$ is defined, then we can define $f : (0,1) \times (-\varepsilon, \varepsilon) \to M$ by $f(t,s) = \exp_t v(s)$, where $v(s) \in T_pM$ and $v(0) = v$. Let $\gamma(t) = \exp + ptv$ and $w = v'(0)$; then, $J(t) = (D\exp_p)_{tv} tw$ is a Jacobi field along γ with $J(0) = 0$, because $(D\exp_p)_0$ is linear, so it sends 0 to 0

For this Jacobi field, what's $J'(0)$? $J(t) = \frac{\partial f}{\partial s}(t, 0)$, so
$$\begin{aligned} J'(t) &= \frac{\mathrm{D}}{\mathrm{d}t}\left((D\exp_p)_{tv} tw\right) \\ &= \frac{\mathrm{D}}{\mathrm{d}t}\left(t(D\exp_p)_{tv} w\right) \\ &= (D\exp_p)_{tv} w + t\frac{\mathrm{D}}{\mathrm{d}t}\left((D\exp_p)_{tv} w\right). \end{aligned}$$
Thus, $J'(0) = (D\exp_p)_0 w = w$.

Proposition 16.1. *If $\gamma : [0, a] \to M$ is a geodesic, then the Jacobi field J along γ with $J(0) = 0$ and $J'(0) = w$ is $J(t) = (D\exp_p)_{t\gamma'(0)} tw$.*

Corollary 16.2. *Every Jacobi field J along γ with $J(0) = 0$ is of the form $J(t) = (D\exp_p)_{t\gamma'(0)} tJ'(0)$.*

But before this, let's return to the idea that Jacobi fields measure the spreading of geodesics.

Proposition 16.3. *Let $J(t) = (D\exp_p)_{tv} tw$ and $|w| = 1$. Then, the Taylor expansion of abs$J(t)^2$ at $t = 0$ is*
$$|J(t)|^2 = t^2 - \frac{1}{3}\langle R(V, W)V, W \rangle t^4 + O(t^4).$$

Proof. The proof is a calculation of derivatives. We know $J(0) = 0$ and $J'(0) = w$, so $\langle J, J\rangle(0) = 0$ and $\langle J, J\rangle'(0) = 2\langle J, J'\rangle(0) = 0$.

Keep calculating:

$$\langle J, J\rangle''(0) = 2\langle J', J'\rangle(0) + 2\langle J, J''\rangle(0) = 2$$
$$\langle J, J\rangle'''(0) = 6\langle J', J''\rangle(0) + 2\langle J, J'''\rangle(0) = 0.$$

The first term requires expanding out the Jacobi equation, at which point everything cancels. But the key is the fourth derivative:

$$\langle J, J\rangle^{(4)}(0) = 6\langle J'', J''\rangle(0) + 8\langle J', J'''\rangle(0) + 2\langle J, J^{(4)}\rangle(0).$$

We want to show that $J'''(0) = -R(\gamma'(0), J'(0))\gamma'(0)$. Let X be a vector field along γ, so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle R(\gamma', X)\gamma', J\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle R(\gamma', J)\gamma', X\rangle \\ &= \left\langle \frac{\mathrm{D}}{\mathrm{d}t}(R(\gamma', J)\gamma', X)\right\rangle + \left\langle R(\gamma', J)\gamma', \frac{\mathrm{D}W}{\mathrm{d}t}\right\rangle,\end{aligned}$$

but the second term is 0 at time $t = 0$.

$$\begin{aligned}\Longrightarrow \left\langle \frac{\mathrm{D}}{\mathrm{d}t}R(\gamma', J)\gamma', X\right\rangle &= \left\langle \frac{\mathrm{D}}{\mathrm{d}t}R(\gamma', X)\gamma', J\right\rangle + \langle R(\gamma', X)\gamma', J'\rangle. \\ &= \langle R(\gamma', J')\gamma', X\rangle\end{aligned}$$

when $t = 0$, and since X is arbitrary, then this is true for just the first argument in the inner products. Thus, $\langle J', J'''\rangle(0) = \langle J', R(\gamma', J')\gamma'\rangle = \langle R(V, W)V, W\rangle$. Thus, the coefficient is $8/4! = 3$. ☒

Corollary 16.4. *If $|v| = |w| = 1$, $v \perp w$, and $J(t)$ is as in Proposition 16.3, then*

$$|J(t)| = t - \frac{1}{6}K_p(\sigma)t^3 + o(t^3), \tag{9}$$

where $\sigma = \operatorname{span}(v, w)$ and $K_p(\sigma)$ is the sectional curvature.

Thus we see that if $K_p(\sigma)$ is negative, then the geodesics spread more than on T_pM, and if the curvature is positive, then the geodesics spread less than on the flat plane.

Example 16.5. We can actually use (9) to calculate the sectional curvature of some manifolds, such as the sphere $S^n \subset \mathbb{R}^{n+1}$.

For simplicity, we'll let $n = 2$. Let $v = \begin{pmatrix}1\\0\end{pmatrix}$ and $w = \begin{pmatrix}0\\1\end{pmatrix}$ within T_NS^2, where N denotes the north pole. Parameterize the points on S^2 as follows: $(s, h) \mapsto (h\cos s, h\sin s, h)$, where $s \in [0, 2\pi]$ and $h \in [-1, 1]$. Thus, s represents the angle and h the height.

By the definition of the exponential function, $f(t, s) = \exp_N tv(s) = (\cos s \sin t, \sin s \sin t, \cos t)$, and therefore

$$\frac{\partial f}{\partial s}(t, s) = (-\sin s \sin t, \cos s \sin t, 0).$$

Thus, $\left|\frac{\partial f}{\partial s}\right| = \sin t = t - t^3/3 + o(t^3)$, so $K_p(T_NS^2) = 1$ after expanding the series.

Definition. Let $\gamma : [0, a] \to M$ be a geodesic. Then, the point $\gamma(t_0)$ is a conjugate point to $\gamma(0)$ along γ with $t_0 \in (0, a]$ if there exists a nonzero Jacobi field J along γ such that $J(0) = J(t_0) = 0$.

The multiplicity of $\gamma(t_0)$ is the maximum number of such Jacobi fields.

Proposition 16.6. *The multiplicity is at most $n - 1$, where $n = \dim M$.*

Proof. The Jacobi field $t\gamma'(t)$ is nonzero when $t \neq 0$, so of the n possible, at least one is missing. Furthermore, Example 16.5 gives an example achieving multiplcity $n - 1$ (i.e. 1). ☒

Conjugate points are closely related to critical points of the exponential map.

Proposition 16.7. *Let γ be a geodesic and $p = \gamma(0)$. Then, the point $q = \gamma(t_0)$ is conjugate to p along γ iff $v_0 = t_0\gamma'(0)$ is a critical point if $\exp_p$, and the multiplicity of q is $\dim(\ker((D\exp_p)_{v_0}))$.*

Proof. If q is conjugate, then there exists a J such that $J(0) = J(t_0) = 0$, and since $J(0) = 0$, then $J(t) = (D\exp_p)_{t\gamma'(0)}tJ'(0)$, and $J(t_0) = (D\exp_p)_{t_0\gamma'(0)}t_0J'(0) = 0$, and therefore $tJ'(0) \in \ker((D\exp_p)_{v_0})$. Since $J'(0) = 0$, then the kernel is nontrivial, so this map is noninjective, and therefore not surjective, so v_0 is a critical point of $\exp_p$.

Multiplicity is not much harder to check: each linearly independent vector gives a Jacobi field and vice versa. ☒

17. The Cartan-Hadamard Theorem: 10/29/14

18. The Path Lifting Property and Covering Maps: 10/31/14

"So, I was thinking of going as a principal line bundle for Halloween."

Last time, we were proving the Cartan-Hadamard Theorem, which is recalled here.

Theorem 18.1 (Cartan-Hadamard). *If M is a simply connected, complete manifold with nonpositive sectional curvature everywhere, then $\exp : T_pM \to M$ is a diffeomorphism.*

The only thing left is to prove the following lemma.

Lemma 18.2. *If $f : \widetilde{M} \to M$ is a local diffeomorphsim, $\widetilde{M}$ is a complete Riemannian manifold, and M is Riemannian such that $|Df_p(v)| \geq |v|$ for all $p \in M$ and $v \in T_pM$, then f is a covering map.*

In some specific cases, f being a covering map is equivalent to a statement known as the path lifting property: that for all $c : [0,1] \to M$ and $q \in \widetilde{M}$ such that $f(q) = c(0)$, there exists a $\bar{c} : [0,1] \to \widetilde{M}$ such that $\bar{c}(0) = q$ and $f \circ \bar{c} = c$.

Proposition 18.3. *If $f : \widetilde{M} \to M$ is a local homeomorphism of Riemannian manifolds, then f is a covering map iff it satisfies the path lifting property.*

Proof. The forward direction is pretty easy, so let's worry about the reverse direction. Let V be a simply connected neighborhood of $p \in M$, which can be taken because M is a manifold, and write $f^{-1}(V) = \coprod_\alpha V_\alpha$, where V_α is path-connected and open, and the union is disjoint. Thus, we want $f|_{V_\alpha} : V_\alpha \to V$ to be a homeomorphism.

Suppose there's a $q \in V$ such that $q \notin f(V_\alpha)$. Then, connect p and q by a path α contained within V; then, the path lifting property guarantees this path lifts to an $\widetilde{\alpha}$. Since $q \notin f(V_\alpha)$, then $\widetilde{\alpha}(1) \notin V_\alpha$, but since $q \in V$, then $\widetilde{\alpha} \in f^{-1}(V)$, so it's in one of the other V_α, which is a contradiction, since the V_α are maximal path-connected components. Thus, $f|_{V_\alpha}$ is onto.

Why is it one-to-one? Suppose there's a $p \in V$ such that $p_1, p_2 \in V_2$ satisfy $f(p_1) = f(p_2) = p$, where $p_1 \neq p_2$. Then, draw a curve $\widetilde{\alpha}$ connecting p_1 and p_2, and project it to V, where it becomes a loop, and thus is homotopic to a point.

Here we use the fact that the path lifting property implies the homotopy lifting property (proven by lifting each curve in a family, and then showing that the lifted paths remain a homotopy), so $\widetilde{\alpha}$ is homotopic to the preimage of a point, and therefore $p_1 = p_2$. ⊠

This can't be generalized too much: the path lifting property isn't equivalent to f being a covering map on more general topological spaces. For example, we need path-connected components to be identical to connected components.

Proof of Lemma 18.2. To prove this lemma, we now only need to show that f satisfies the path lifting property. Thus, suppose $p \in M$ and $c : [0,1] \to M$ is such that $c(0) = p$ and $q \in \widetilde{M}$ is such that $f(q) = p$.

Since f is a local diffeomorphism, then there exists an $\varepsilon > 0$ such that $\bar{c} : [0, \varepsilon] \to \widetilde{M}$ exists (taking the preimage of a small neighborhood of c); then, keep extending this curve. The conditions that $\widetilde{M}$ is complete and the restriction on the lengths of the tangent vectors allow us to keep going. Specifically, if $A = \{t \in [0,1] \mid \bar{c} \text{ is defined on } [0,t]\}$; then, A is open, because f is a local diffeomorphism, so if $t_0 \in A$, it can be extended in some neighborhood of t_0.

Now, to show that $A = [0,1]$, we merely have to show that A is also closed. Let $\{t_n\}$ be an increasing sequence and let $t_0 = \lim t_n$. We want to show that $t_0 \in A$. First, we'll show that $\{\bar{c}(t_n)\}$ is contained within a compact $K \subset M$; if not, then it's unbounded (if it's bounded, then we can just take K to be its closure; since $\widetilde{M}$ is complete, closed and bounded implies compact). Thus, $d(\bar{c}(0), \bar{c}(t_n)) \to \infty$, so

$$\begin{aligned} \ell(c([0,t_n])) &= \int_0^{t_n} |c'(t)|\, \mathrm{d}t = \int_0^{t_n} |Df_{\bar{c}(t)}(\bar{c}'(t))|\, \mathrm{d}t \\ &\geq \int_0^{t_n} |c'(t)|\, \mathrm{d}t = \ell(\bar{c}([0,t_n])) \\ &\geq d(\bar{c}(0), \bar{c}(t_n)), \end{aligned}$$

but this is a contradiction, because the curve must have finite length.

Thus, the curve has a point r that is an accumulation point of $\{\bar{c}(t_n)\}$; let V be a neighborhood of r such that $f|_V$ is a homeomorphism (which can be done because f is a local diffeomorphism), and let $t_0 = \lim t_n$. We want $t_0 \in A$. Well, there is an interval $I \subset [0,1]$ such that $t_0 \in I$, $c(I) \subset f(V)$, but that means we can pull this back: the lift of $c(I)$ extends $\bar{c} : [0, t_n] \to M$, but since $t_0 \in [0, t_n] \cup I$, then $\bar{c}$ is defined on $[0, t_0]$, and therefore $t_0 \in A$. ⊠

Variational Formulas. In order to prove some results, we'll need to write down the first and second variational formulas for the energy of a curve.

Definition. Let $c : [0, a] \to M$ be a piecewise smooth curve; then, the energy of c is

$$E(c) = \int_0^a \langle c'(t), c'(t)\rangle \, dt.$$

Lemma 18.4. *$L(c)^2 \le aE(c)$, with equality iff c is parameterized by a parameter proportional to arc length.*

Proof. Use the Schwarz inequality:

$$\left(\int fg\right)^2 \le \int f^2 \int g^2,$$

with equality iff f is proportional to g. Then, let $f = 1$ and $g = |c'(t)|$. ⊠

That energy depends on parameterization is very interesting, and means that energy can be used to determine information about the parameterization of the curve.

Proposition 18.5. *Let $p, q \in M$ and $\gamma : [0, a] \to M$ be a length-minimizing geodesic such that $\gamma(0) = p$ and $\gamma(a) = q$. Then, for all curves c connecting p and q, we have $E(\gamma) \le E(c)$, with equality iff c is a length-minimizing geodesic parameterized by a parameter proportional to arc length.*

Proof. Since γ is a geodesic, then it's parameterized proportional to arc length, so by Lemma 18.4, $aE(\gamma) = L(\gamma)^2 \le L(c)^2 \le aE(c)$, with equality iff c is a length-minimizing curve and therefore a geodesic (for the first inequality) and that c is parameterized by arc length (second inequality). ⊠

The conclusion is, if we assume *a priori* that there exists a length-minimizing curve, then length-minimizing is equivalent to energy-minimizing. But next time, we'll be able to use the first variational formula to show that, independent of a geodesic already existing, energy minimizing curves are critical points of the length functional.

19. Theme and Variations: 11/3/14

Last time, we defined the energy of a piecewise smooth curve $c : [0, a] \to M$ as

$$E(c) = \int_0^a \langle c'(t), c'(t)\rangle \, dt.$$

The goal today is to show that the critical points of the energy function are geodesics.

Definition. A variation of c is a continuous $f : [0, a] \times (-\varepsilon, \varepsilon) \to M$ such that:

1. $f(t, 0) = c(t)$ for all $t \in [0, a]$.
2. There exists a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = a$ such that for each i, $f|_{[t_i,t_{i+1}]\times(-\varepsilon,\varepsilon)}$ is smooth.

f is called proper if $f(0, s) = 0$ and $f(a, s) = c(a)$ for any $s \in (-\varepsilon, \varepsilon)$, and the variation is called smooth if f is smooth (which necessarily implies that c is).

Condition (2) is stronger than each $f(t, s)$ being smooth for a given s; rather, it requires s to vary smoothly as well, and a single partition to work for all of the curves.

Definition. If f is a variation as above and an $s \in (-\varepsilon, \varepsilon)$ is fixed, the curve $f_s(t) = f(t, s)$ is called a curve in the variation. For example, $c(t) = f_0(t)$. Similarly, if $t \in [0, a]$ is fixed, $f_t(s) = f(t, s)$ is called a transversal curve.

Immediately, we see that each curve in the variation is piecewise smooth, and each transversal curve is smooth.

Definition. A piecewise smooth vector field $V(t)$ along a curve c is called a variational field of a variation f along c if for all $t \in [0, a]$, $V(t) = \frac{\partial f}{\partial s}(t, 0)$.

In other words, the variational field at a point is the derivative of the transveral curve through that point at $s = 0$.

Definition. Define the energy function of this variation $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$E(s) = E(f_s) = \int_0^a \left\langle \frac{\partial f}{\partial t}(t, s), \frac{\partial f}{\partial t}(t, s)\right\rangle dt.$$

Proposition 19.1 (First Variation of Energy). *Let f be a smooth, proper variation and E be its energy functional. Then,*

$$\frac{1}{2}E'(0) = -\int_0^a \left\langle V(t), \frac{D}{dt}c'(t)\right\rangle dt.$$

More generally, if f is any more general variation, then

$$\frac{1}{2}E'(0) = -\int_0^a \left\langle V(t), \frac{D}{dt}c'(t)\right\rangle dt - \sum_{i=1}^{k}\langle V(t_i), c'(t_i^+) - c'(t_i^-)\rangle - \langle V(0), c'(0)\rangle + \langle V(a), c'(a)\rangle,$$

where $c'(t_i^+) = \lim_{t\to t+i^+} c'(t)$ *and* $c'(t_i^-) = \lim_{t\to t_i^-} c'(t)$.[8]

Proof. The proof will be a calculation.

$$\begin{aligned} E(s) &= \int_0^a \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t \\ &= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t. \end{aligned}$$

Now, we can differentiate with respect to s, and use the fact that the covariant derivative is compatible with the metric.

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t = \int_{t_i}^{t_{i+1}} 2\left\langle \frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t.$$

Since ∇ is symmetric and $[\partial_s, \partial_t] = 0$, then

$$\begin{aligned} &= 2\int_{t_i}^{t_{i+1}} \left\langle \frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t \\ &= 2\int_{t_i}^{t_{i+1}} \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t - 2\int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial s}, \frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial t}\right\rangle \,\mathrm{d}t. \end{aligned}$$

Letting $s = 0$,

$$= \left[2\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\right]_{t_i}^{t_{i+1}} - 2\int_{t_i}^{t_{i+1}} \left\langle V(t), \frac{\mathrm{D}}{\mathrm{d}t}c'(t)\right\rangle \,\mathrm{d}t + 2(\langle V(t_{i+1}), c'(t_{i+1}^-)\rangle - \langle V(t_i), c'(t_i^+)\rangle),$$

and then taking the sum from 0 to k, we're done. ⊠

Proposition 19.2. *A piecewise smooth curve* $c : [0, a] \to M$ *is a geodesic iff for each proper variation* f *of* c*, we have* $E'(0) = 0$.

Proof. If c is a geodesic, then $\frac{\mathrm{D}}{\mathrm{d}t}c'(t) = 0$, and c is smooth. Since f is proper, then $V(0) = V(a) = 0$, so by Proposition 19.1,

$$E'(0) = -\int_0^a \langle V(t), 0\rangle \,\mathrm{d}t - \sum_{i=1}^{k}\langle V(t_i), 0\rangle - \langle 0, c'(0)\rangle + \langle 0, c'(0)\rangle = 0.$$

In the other direction, we know $E'(0) = 0$ for all proper variations, so let's choose some special variations.

Lemma 19.3. *Given a piecewise smooth vector field* $V(t)$ *along* c*, there exists a variation* f *such that* $V(t) = \frac{\partial f}{\partial s}(t, 0)$*, and if* $V(0) = V(a) = 0$*, then* f *can be chosen to be proper.*

We'll come back and prove this later.

Let $V(t) = g(t)\frac{\mathrm{D}}{\mathrm{d}t}(c'(t))$, where $g : [0, a] \to \mathbb{R}$ is such that $g(t) > 0$ if $t \neq t_i$ and $g(t_i) = 0$; then, by Lemma 19.3, there's a proper variation f such that $\frac{\partial f}{\partial s} = g(t)\frac{\mathrm{D}}{\mathrm{d}t}c'(t)$. Then, by Proposition 19.1,

$$\frac{1}{2}E'(0) = -\int_0^a \left\langle g(t)\frac{\mathrm{D}}{\mathrm{d}t}c'(t), \frac{\mathrm{D}}{\mathrm{d}t}c'(t)\right\rangle = -\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} g(t)\left|\frac{\mathrm{D}}{\mathrm{d}t}c'(t)\right|^2 \,\mathrm{d}t.$$

Since $g(t)$ is strictly positive on this interval, and the left-hand side is equal to 0, then on each open interval (t_i, t_{i+1}), with $i = 0, \dots, k$, $\frac{\mathrm{D}}{\mathrm{d}t}c'(t) = 0$, so c is a piecewise geodesic.[9]

To address the boundaries, let $U(t)$ be some smooth vector field, and then

$$V(t) = \begin{cases} c'(t_i^+) - c'(t_i^-), & t = t_i \\ 0, & t = 0, a \\ U(t), & t \neq t_i. \end{cases}$$

Lemma 19.3 once again produces a proper variation f that has V as its variational field. Thus,

$$\frac{1}{2}E'(0) = -\sum_{i=1}^{k}|c'(t_i^+) - c'(t_i^-)|^2 = 0,$$

so $c'(t_i^+) = c'(t_i^-)$, and therefore c is at least C^1 smooth, but since it's piecewise geodesic, this forces it to be C^∞ smooth as well (since $c(t)$ is the exponential of something, which is therefore smooth), and therefore c is a geodesic.

Now, all we have to do is prove Lemma 19.3. The most obvious thing to do to get a curve from a vector field is the exponential mapping, so let $f(t, s) = \exp_{c(t)} sV(t)$. Since $c([0, a])$ is compact in M (because of the existence of totally normal

[8] These limits are occasionally known as subderivatives, though we won't use that name here.

[9] These one-sided limits exist because c is piecewise smooth, i.e. it's a piecewise manifold-with-boundary.

neighborhoods at each point, so it's bounded), there exists a $\delta > 0$ such that $\exp_{c(t)}$ for $t \in [0, a]$ is defined for all $v \in T_{c(t)}M$ with $|v| < \delta$.

Thus, let $N = \max_{t \in [0,a]} |V(t)|$, and $\varepsilon = \delta/n$; then, $f(t, s) = \exp_{c(t)} sV(t)$ is well-defined, and piecewise smooth as needed because of the properties of the exponential map. And, of course, since the exponential map is tangent to V at $s = 0$, then we're good.

If $V(0) = V(a) = 0$, then $f(0, s) = \exp_{c(0)} 0 = c(0)$, so in this case it may be chosen to be proper. ⊠

20. The Second Variation of the Energy Functional: 11/5/14

21. Weinstein's Theorem: 11/9/14

Today's goal is to prove the following theorem.

Theorem 21.1 (Weinstein). *Let M be a compact Riemannian manifold with positive sectional curvature everywhere (i.e. $K_p(\sigma) > 0$ for all $p \in M$ and two-dimensional $\sigma \subset T_pM$), and let f be an isometry that is orientation-preserving if $n = \dim M$ is even, and orientation-reversing if n is odd. Then, f has a fixed point.*

It is conjectured that this remains true if f is merely a diffeomorphism, but this is an open question. It would imply the following conjecture, too:

Conjecture 21.2 (Weinstein). *$S^2 \times S^2$ has no metric with positive sectional curvature.*

The idea is that there exists an orientation-preserving diffeomorphism $S^2 \times S^2 \to S^2 \times S^2$ with no fixed point.

Proof of Theorem 21.1. Suppose otherwise, that there's no q such that $f(q) = q$. Let $p \in M$ be such that $d(p, f(p))$ achieves the minimum, which exists because M is compact. Then, by the Hopf-Rinow Theorem, there's a geodesic $\gamma : [0, \ell] \to M$ such that $\gamma(0) = p$, $\gamma(\ell = f(p)$, and $|\gamma(t)| = 1$. The proof idea is to use the second variation of the energy functional to find a smaller curve and establish a contradiction.

Let $P : T_{f(p)}M \to T_pM$ be parallel transport along γ from $f(p)$ to p, and let $\widetilde{A} = P \circ Df_p : T_pM \to T_pM$. We want to find a fixed point v of $\widetilde{A}$ such that $v \perp \gamma'(0)$, i.e. $\langle v, \gamma'(0)\rangle = 0$. $\widetilde{A}(\gamma'(0)) = P \circ Df_p(\gamma'(0)) = P((f \circ \gamma')(0)) = P(\gamma'(\ell)) = \gamma'(0)$ assuming that $(f \circ \gamma')(0) = \gamma'(\ell)$, i.e. the curve γ and $f \circ \gamma$ connecting p to $f(p)$ to $f^2(p)$ is smooth. Why is this true? Since f is an isometry, so that it preserves distances, if p' is on the line from p to $f(p)$, then

$$d(p', f(p')) \leq d(p', f(p)) + d(f(p), f(p')) = d(p', f(p) + d(p, p') = d(p, f(p)).$$

Since $d(p, f(p))$ is a minimum, this forces $d(p', f(p')) = d(p, f(p)) + d(f(p), f(p'))$, and therefore the curve is smooth at $f(p)$ (which was the only point in doubt), and so it's a smooth length-minimizing curve and therefore a geodesic.

We know from its definition that $\widetilde{A}$ is an orthogonal linear transformation on T_pM. Let A be the restriction of $\widetilde{A}$ to the (orthogonal) complement of $\gamma'(0)$.

Lemma 21.3. *If $A : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is orthogonal, and $\det(A) = (-1)^n$, then there's a $v \in \mathbb{R}^{n-1}$ such that $A(v) = v$.*

Proof. If n is even, then the characteristic polynomial is of the form $\det(A - \lambda I) = \cdots + \det A$, and is of degree $n - 1$. The existence of a fixed point is equivalent to 1 being an eigenvalue, but $\det(A) = 1 = \prod \lambda_i$; since there are an odd number of eigenvalues and the complex eigenvalues come in pairs, then one of them must be 1.

Similarly, if A is odd, we have a polynomial of even degree, then $\prod \lambda_i = -1$, and the complex eigenvalues come in pairs, so there must be at least one pair of real eigenvalues (since the product of two complex conjugates is positive); then, if all of the real eigenvalues are -1, there's an even number, so the product is positive, which is a contradiction, so one must have value 1. ⊠

Thus, in our specific case, A has an invariant vector $v \in T_pM$, i.e. $A(v) = v$.

Since $\widetilde{A}$ restricts to 1 on $\gamma'(0)$, then $\det(A) = \det(\widetilde{A}) = \det P \det(Df_p)$. Parallel transport is locally orientation-preserving (which we might not have mentioned before, but is true, by looking at the exponential map), so $\det P = 1$, and thus $\det(Df_p) = (-1)^n$.

Now, let $V(t)$ be parallel along γ such that $V(0) = v$ (i.e. dragging v along by P), and let $\beta(s) = \exp_p sv$. Thus, $(f \circ \beta)'(0) = V(\ell)$, because $P \circ Df(V) = V$, so $(f \circ \beta)'(0) = Df_p(V) = P^{-1}(V) = V(\ell)$.

Let $h : [0, \ell] \times (-\varepsilon, \varepsilon) \to M$ be given by $h(t, s) = \exp_{\gamma(t)} sV(t)$, and write $h_s(t) = h(t, s)$. This variation connects $\beta(s)$ and $f(\beta(s))$, but it's not proper, so we have to use the generalized form of the second variational formula (on the homework). Specifically, since V is parallel, terms in the following cancel out:

$$\begin{aligned}\frac{1}{2}E''(0) &= -\int_0^\ell (\langle V(t), V''(t)\rangle + R(\gamma'(t), V(t))\gamma'(t))\,\mathrm{d}t + \left\langle \frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial h}{\partial s}, \gamma' \right\rangle\Bigg|_{(0,0)}^{(\ell,0)} + \left\langle v, \frac{\partial V}{\partial t}\right\rangle\Bigg|_0^\ell \\ &= -\int_0^\ell K(\gamma'(t), V(t))\,\mathrm{d}t < 0,\end{aligned}$$

since M has positive sectional curvature everywhere. Thus, the energy is convex, so it has a global maximum h_s such that for other curves c in the variation, $E(c) < E(h_s)$ unless $c = h_s$. But then, using Cauchy-Schwarz, $(\ell(c))^2 \leq \ell E(c) < \ell E(\gamma) = \ell(\gamma)^2$, so $\ell(c) < \ell(\gamma)$, which is a contradiction. ⊠

As a corollary, we recover a beautiful theorem of Synge.

Corollary 21.4 (Synge). *If M is a compact Riemannian manifold with positive sectional curvature everywhere, then:*

(1) *If M is orientable and even-dimensional, then it's simply connected: $\pi_1(M) = 1$.*
(2) *If n is odd, then M is orientable.*

Proof of Corollary 21.4, part 2. Suppose M is not orientable; then, there exists a double covering $\pi : \overline{M} \to M$, which is a local diffeomorphim, and we can choose this $\overline{M}$ to be orientable. Let $k : \overline{M} \to \overline{M}$ be a nontrivial deck transformation, i.e. $k \neq \mathrm{id}$ and $\pi \circ k = \pi$. Thus, k has no fixed point. Since M is non-orientable, then k must be orientation-reversing.

Since M is compact and $\overline{M}$ is a finite cover, then $\overline{M}$ also is compact, and since M has positive sectional curvature and π is a local isometry, then $\overline{M}$ also has positive sectional curvature. Then, k is an isometry, so the conditions of Theorem 21.1 hold, so k has a fixed point, which is a contradiction. ⊠

The other part of the proof will appear on the homework; it does involve a cover which isn't *a priori* compact, but ends up being so (somehow). A finite cover might not work, but the universal cover might.

22. The Chern-Weil Theorem and Connections on a Vector Bundle: 11/12/14

"Assume we talked about differential forms on Monday."

We'll spend much of the rest of the class discussing the Chern-Weil Theorem, including a bit of its history.

Everyone knows the Gauss-Bonnet Theorem, which is similar.

Theorem 22.1 (Gauss-Bonnet). *Let Σ be a compact, orientable, Riemannian closed surface and k be its Gauss curvature. Then,*

$$\int_\Sigma K\,\mathrm{d}A = 2\pi\chi(E).$$

Here, χ denotes the Euler characteristic.

One can ask a related question: for a compact, orientable, Riemannian manifold of dimension $2m$, is there a two-form ω whose integral is the Euler characteristic? It turns out the answer is yes.

Theorem 22.2 (Generalized Gauss-Bonnet). *If M is a compact, orientable, Riemannian manifold of dimension $2m$, then there exists a $2m$-form called the Pfaff form, $\mathrm{Pf}(k)$, such that $\int_M \mathrm{Pf}(k) = \chi(M)$.*

This theorem isn't incredibly useful (Szego did use it to compute areas of some fundamental domains), but it led Chern to introduce the idea of Chern classes, which we'll see again.

In order to introduce these notions, we'll need the idea of differential forms, which people in the class have seen before, and vector bundles, which we will talk about.

Definition. Let M be a smooth manifold.[10] Then, a vector bundle ζ over M consists of the following:

(1) a smooth manifold $E = E(\zeta)$, called the total space,
(2) a smooth map $\pi : E \to M$, called the projection, and
(3) for all $p \in M$, the structure of a vector space on $\pi^{-1}(p)$ (called a fiber).

These are subject to a condition called local triviality: for all $p \in M$, there exists a neighborhood $U \subset M$ of p, an integer n, and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that for all b, $x \mapsto h(b, x)$ defines an isomorphism between $\mathbb{R}^n$ and $\pi^{-1}(p)$.

Intuitively, a vector bundle is a structure of $\mathbb{R}^n$ attached to each point in the manifold that varies smoothly.

Example 22.3. Vector bundles show up all the time in differential geometry.

- The tangent bundle TM.
- The cotangent bundle T^*M, which at each point is given by $T_p^*M = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$.
- The exterior bundle $\Lambda^n T^*M$, where for all p, the fiber at p is $\Lambda^n T_p^*M$.

Definition. A section of ζ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of sections of ζ is denoted $\Gamma(\zeta)$.

Example 22.4.

- Sections of the tangent bundle are vector fields: $\Gamma(TM) = V(M)$.
- Sections of $\Lambda^n T^*M$ are n-forms: $\Gamma(\Lambda^n T^*M) = \Omega^n(M)$.

Vector bundles have some nice operations on them: we can take restrictions, pullbacks, direct products, direct sums (called the Whitney sum), and tensors. That is:

[10] Vector bundles can be defined over more general topological spaces, but here we will only need them on smooth manifolds.

- Let $N \overset{f}{\hookrightarrow} M$ be a submanifold; then we can map back along f to obtain a vector bundle over N from one over M:

$$\begin{array}{ccc} & & E \\ & & \big\downarrow \pi \\ N & \overset{f}{\hookrightarrow} & M \end{array}$$

- More generally, if $f : N \to M$ is any smooth map between manifolds, we can pull a vector bundle E over M back to one over M. Let $E(f^*\zeta) = \{(x,p) \in E \times N \mid \pi(x) = f(p)\}$, as

$$(f^*\pi)^{-1}(p) = \{x \in E \mid \pi(x) = f(p)\} = \pi^{-1}(f(p)).$$

  That is, the following diagram commutes:

$$\begin{array}{ccc} E(f^*\zeta) & \longrightarrow & E \\ \big\downarrow f^*\pi & & \big\downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

  Restriction is a special case of the pullback in the case where f is injective.
- If $\zeta_1 : E_1 \overset{\pi_1}{\to} M_1$ and $\zeta_2 : E_2 \overset{\pi_2}{\to} M_2$ are vector bundles, their product is given by $E(\zeta_1 \times \zeta_2) = E_1 \times E_2$, with projection map $\pi_1 \times \pi_2$ onto $M_1 \times M_2$. This is because $(\pi_1 \times \pi_2)^{-1}(p_1 \times p_2) = \pi_1^{-1}(p_1) \times \pi_2^{-1}(p_2)$.
- Suppose $\zeta_1 : E_1 \overset{\pi_1}{\to} M$ and $\zeta_2 : E_2 \overset{\pi_2}{\to} M$ are two vector bundles over the same manifold. Then, one may take the Whitney sum $\zeta_1 \oplus \zeta_2$, which is just the pullback along the diagonal map $\Delta : M \to M \times M$ given by $p \mapsto (p,p)$ of their product bundle. That is, we obtain the following diagram.

$$\begin{array}{ccc} E(\zeta_1 \oplus \zeta_2) & \longrightarrow & E_1 \times E_2 \\ \big\downarrow & & \big\downarrow \pi_1 \times \pi_2 \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

  This is a bit fancy; what it means is that at each point, the fiber is the direct sum of the two fibers of the two bundles.

  Another way to think of this is with transition maps: if ζ_1 and ζ_2 are two vector bundles over the same manifold and U and V are two charts that intersect, then there's a transition map $f_{U,V}(p) : \mathbb{R}^{\to}\mathbb{R}^n$ induced on the bundles. Thus, we get a map $f_{UV} : U \cap V \to \mathrm{GL}(n,\mathbb{R})$ for each intersection and therefore in our Whitney sum situation, two collections of functions $\{f_{UV}^{\zeta_1}\}$ and $\{f_{UV}^{\zeta_2}\}$. Now, we can take the direct sum of these two matrices, i.e. the direct sum bundle is given by the maps $f_{U,V}^{\zeta_1 \oplus \zeta_2} = f_{UV}^{\zeta_1} \oplus f_{UV}^{\zeta_2} : U \cap V \to \mathrm{GL}(2n,\mathbb{R})$ given by the block matrix $\begin{pmatrix} f_{UV}^{\zeta_1} & 0 \\ 0 & f_{UV}^{\zeta_2} \end{pmatrix}$.
- We can do the same thing with the tensor product of bundles, but in this case applied to the tensor product of matrices: if ζ_1 and ζ_2 are vector bundles over the same manifold M, then we construct the transition maps in the same way and then tensor them together to get maps $U \cap V \to \mathrm{GL}(n^2,\mathbb{R})$ for $\zeta_1 \otimes \zeta_2$. The tensor product of matrices $A \otimes B$ is the block $n \times n$ matrix whose ij^{th} term is $a_{ij}B$.

These allow us to define connections in a considerably more general scope.

Definition. A connection on ζ is a linear map $\nabla : \Gamma(\zeta) \to \Gamma(T^*M \otimes \zeta)$ that follows the Leibniz rule $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s)$ for all $f \in C^\infty(M)$ and $s \in \Gamma(\zeta)$.

This is very similar to the previous case, even though it's considerably more abstract. Note that if $\zeta = TM$, then $\nabla : V(M) \to \Gamma(T^*M \otimes TM)$, i.e. for all $X \in V(M)$, we get a map $\nabla X : TM \to TM$ sending $Y \mapsto \nabla_Y X$.

Lemma 22.5. *Any vector bundle ζ has a connection.*

Proof. The proof is similar to the earlier case, by constructing a partition of unity.

Consider a U such that $\zeta|_U$ is a product, and choose a basis $s_1, \dots, s_n$ for the sections, i.e. any $s \in \Gamma(\zeta)$ can be written as $s = f_1 s_1 + \cdots + f_n s_n$ uniquely for $f_j \in C^\infty(M)$. This can always be done locally, though in some cases not globally (e.g. invoking the Hairy Ball Theorem). Then, they can be glued together, and so forth. ⊠

One can write Christoffel-like symbols, via $\nabla(s_i) = \sum_j \omega_{ij} s_i$, and take the $n \times n$ matrix $\omega = [\omega_{ij}]$.

Now we have a connection ∇ and the map $\mathrm{d} : \Gamma(\Lambda^n T^*M) \to \Gamma(\Lambda^{n+1} T^*M)$.

Lemma 22.6. *Given ∇, there exists a unique linear map $\widehat{\nabla} : \Gamma(T^*M \otimes \zeta) \to \Gamma(\Lambda^2 T^*M \otimes \zeta)$ such that $\widehat{\nabla}(\theta \otimes s) = \mathrm{d}\theta \otimes s - \theta \wedge \nabla(s)$, and $\widehat{\nabla}(f\theta \otimes s) = \mathrm{d}f \wedge \theta \otimes s + f\widehat{\nabla}(\theta \otimes s)$.*

This looks confusing, but if one writes it in local coordinates, there's only one possible answer.

Definition. The curvature tensor $K = K_\nabla$ is the map $\widehat{\nabla} \circ \nabla : \Gamma(\zeta) \to \Gamma(\Lambda^2 T^*M \otimes \zeta)$.

Lemma 22.7. *K is in fact a tensor; for any $f \in C^\infty(M)$, $K(fs) = fK(s)$.*

This is a calculation that isn't so painful.

In local coordinates U, let $s_1, \dots, s_n$ be a basis of sections. Then,

$$\begin{aligned} K(s_i) &= \widehat{\nabla} \circ \nabla(s_i) = \widehat{\nabla}\left(\sum_j \omega_{ij} \otimes s_j\right) \\ &= \sum_j (\mathrm{d}\omega_{ij} \otimes s_j - \omega_{ij} \wedge \nabla(s_j)) \\ &= \sum_j \left(\mathrm{d}\omega_{ij} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j}\right) s_j. \end{aligned}$$

Call the term in parentheses Ω_{ij}, so we get a matrix $\Omega = [\Omega_{ij}]$, which is a matrix of two-forms. If we have a transition matrix $T : U \to V$, we get $\Omega_U = T\Omega_V T^{-1}$.

We're almost ready to define Chern classes; the next ingredient will be invariant polynomials.

Definition. Let M_n be the algebra of $n \times n$ matrices. An invariant polynomial is a function $P : M_n \to \mathbb{R}$ that is a polynomial in the entries of the matrix and such that for all non-singular T, $P(TXT^{-1}) = P(X)$.

The trace and determinant are excellent examples of these.

Lemma 22.8 (Fundamental Lemma of Chern-Weil Theory). *For all invariant polynomials P, the form*

$$P(\Omega) \in \bigoplus_{n=1}^{\infty} \Omega^{2n}(M)$$

is closed, i.e. $\mathrm{d}P(\Omega) = 0$.

These define de Rham cohomology classes, and Chern classes are obtained by choosing specific polynomials. But we'll discuss this next time.

23. The Fundamental Lemma of Chern-Weil Theory: 11/14/14

"I had lots of questions when I was a student... and the professors in general hated me."

Recall that last time, we talked about connections on a vector bundle $\eta : E \xrightarrow{\pi} M$. $\Gamma(\zeta)$ is the space of sections of the bundle, i.e. maps $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. Then, a connection is a linear map $\Gamma(\zeta) \to \Gamma(T^*M \otimes \zeta)$ such that the Leibniz condition is satisfied, i.e. $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s)$. In a neighborhood U, we can pick a basis of sections $s_1, \dots, s_n$, and write $\nabla(s_i) = \sum_j \omega_{ij} \otimes s_i$, where ω_{ij} are one-forms on U, and locally, the connection is determined by a matrix $\omega = [\omega_{ij}]_{n\times n}$ of one-forms on U.

Given a connection, there is a unique operator $\widehat{\nabla} : \Gamma(T^*M \otimes \zeta) \to \Gamma((\Lambda^2 T^*M) \otimes \zeta)$ such that $\widehat{\nabla}(\theta \otimes s) = \mathrm{d}\theta \otimes s - \theta \wedge \widehat{\nabla}(s)$ and $\widehat{\nabla}(f(\theta \otimes s)) = \mathrm{d}f \wedge \theta \otimes s + f\widehat{\nabla}(\theta \otimes s)$. Then, the curvature tensor $K = K_\nabla$ is just $\widehat{\nabla} \circ \nabla : \Gamma(\zeta) \to \Gamma(\Lambda^2 T^*M \otimes \zeta)$. That this is a tensor implies that $K(fs) = fK(s)$ for $f \in C^\infty(M)$.

Again, choose a basis $s_1, \dots, s_n$ of sections locally (i.e. in a neighborhood U, which is equivalent to local triviality of the vector bundle). Then,

$$K(s_i) = \sum_j \Omega_{ij} \otimes s_j, \qquad \text{where} \qquad \Omega_{ij} = \mathrm{d}\omega_{ij} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j}.$$

In matrix notation, we let $\Omega = [\Omega_{ij}]$, which is an $n \times n$ matrix of 2-forms. Since K is a tensor, then if V is another neighborhood, then on $U \cap V$, $\Omega_V = T_{U,V}\Omega_V T_{U,V}^{-1}$.

Exercise 4. Find this $T_{U,V} : U \cap V \to \mathrm{GL}(n, \mathbb{R})$ in terms of the transition maps $\phi_{U,V}$ of M; recall that the transition functions are also of the form $\phi_{U,V} : U \cap V \to \mathrm{GL}(n, \mathbb{R})$.

Finally, recall Lemma 22.8, which says that for any invariant polynomial $P : M_n \to \mathbb{R}$, $P(K)$ is a closed differential form, i.e. $\mathrm{d}P(K) = 0$.

This is a fine lemma, but what exactly does it mean? $P(K) \in \bigoplus_{k=0}^\infty \Omega^{2k}(M)$, so on U, $P(\Omega_U) \in \bigoplus \Omega^{2k}(U)$. Since P is invariant, then $P(\Omega_U) = P(\Omega_V)$ on $U \cap V$, so we really get a globally defined form.

Proof of Lemma 22.8. We want to calculate $\mathrm{d}P(K)$, and we may as well work within some local neighborhood U. Then, define the matrix $P'(A) = \left[\frac{\partial P}{\partial A_{ij}}\right]^{\mathrm{T}} = \left[\frac{\partial P}{\partial A_{ji}}\right]$ (i.e. differentiate with respect to the variable corresponding to that entry), so that

$$\mathrm{d}P(\Omega) = \sum_{i,j} \frac{\partial P}{\partial \Omega_{ij}} \wedge \mathrm{d}\Omega_{ij} = \mathrm{Tr}(P'(\Omega) \wedge \mathrm{d}\Omega).$$

This might seem a little opaque, but if you write out these two matrices, the trace is the only thing it could be.

Now, we want the Bianchi identity, which looks a little different in this more abstract formalism.

Proposition 23.1 (Bianchi identity). $\mathrm{d}\Omega = \omega\wedge\Omega - \Omega\wedge\omega$.

Proof. We know $\Omega = \mathrm{d}\omega - \omega\wedge\omega$, so

$$\begin{aligned}\mathrm{d}\Omega &= -\mathrm{d}\omega\wedge\omega + \omega\wedge\mathrm{d}\omega + \omega\wedge\omega\wedge\omega - \omega\wedge\omega\wedge\omega\\ &= \omega\wedge\Omega - \Omega\wedge\omega.\end{aligned}$$

⊠

Exercise 5. Determine the way in which this Bianchi identity is related to the previous one, Proposition 13.3.

Equipped with this Bianchi identity, we know that

$$\mathrm{d}P(\Omega) = \mathrm{Tr}(P'(\Omega)\wedge\omega\wedge\Omega - P'(\Omega)\wedge\Omega\wedge\omega).$$

Thus, we want to switch the order of $P'(\Omega)\wedge\Omega$; if this is true, then the trace is of the form $\mathrm{Tr}(AB-BA)$, which is of course zero, so we would win.

We can show that $P'(A)A = AP'(A)$: since P is invariant, then $P((I+tE_{ji})A) = P(A(I+tE_{ji}))$, where E_{ji} is the matrix with zeroes everywhere except for a 1 in the ij^{th} entry. This is because $P(AB) = P(BA)$ for invertible A and B (and this can even be generalized a little bit). Then, differentiate with respect to t:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P((I+tE_{ji})A) = \sum_\alpha A_{i\alpha}\frac{\partial P}{\partial A_{j\alpha}} \tag{10a}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P(A(I+tE_{ji})) = \sum_\alpha \frac{\partial P}{\partial A_{\alpha i}}A_{\alpha j}. \tag{10b}$$

Thus, these are equal, but (10a) is the ij^{th} entry of $P'(A)A$, and (10b) is that of $AP'(A)$. ⊠

Example 23.2. This is a bit out there, so let's look at an example. E_{13} has a 1 in the first row and third column, so

$$I + tE_{13} = \begin{pmatrix}1 & 0 & t\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

so when we multiply by the matrix A,

$$\begin{pmatrix}1 & 0 & t\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11}+ta_{31} & a_{12}+ta_{32} & a_{13}+ta_{33}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Thus, after differentiating we end up with

$$\frac{\partial P}{\partial a_{11}}a_{31} + \frac{\partial P}{\partial a_{12}}a_{32} + \frac{\partial P}{\partial a_{13}}a_{33}.$$

This proof comes from Milnor's *Characteristic Classes*,[11] and it's completely unclear how he thought it up. A lot of calculations are cleverly hidden within the manipulation of these differential forms.

There's a lemma that all invariant polynomials are symmetric polynomials, which leads to the results we'll eventually show about Chern classes.

Note that $P(K_\nabla)$ depends on the choice of connection. However, there's still a sort-of independence result.

Lemma 23.3. *The cohomology class*

$$[P(K)] \in \bigoplus_{k=0}^{\infty} H^{2k}(M,\mathbb{R})$$

(i.e. the even part of the de Rham cohomology) is independent of the choice of ∇.

We don't have time to prove this today, but we'll state the most important facts and use them next lecture:

(1) The space of connections is convex; that is, if ∇_0 and ∇_1 are connections, then $t\nabla_0 + (1-t)\nabla_1$ is still a connection, for $0\le t\le 1$.
(2) Secondly, connections pull back.

[11]See http://www.amazon.com/Characteristic-Classes-AM-76-John-Milnor/dp/0691081220.

Lemma 23.4. *If $f : N \to M$ is a smooth map of manifolds and ζ is a vector bundle over M, then a connection ∇ on ζ can be pulled back uniquely to an $f^*\nabla$ such that the following diagram commutes.*

$$\begin{array}{ccc} \Gamma(\zeta) & \xrightarrow{\nabla} & \Gamma(T^*M \otimes \zeta) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \Gamma(f^*\zeta) & \xrightarrow{f^*\nabla} & \Gamma(T^*N \otimes f^*\zeta) \end{array} \tag{11}$$

This is true because it's true locally: $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$, so if the diagram commutes, we would have something of the form

$$(f^*\nabla)(s_i') = \sum_j \omega_{ij}' \otimes s_j',$$

but the only possible choices are $\omega_{ij}' = f^*(\omega_{ij})$, and uniqueness ends up implying existence as usual.

24. Pontryagin Classes: 11/17/14

Last time, we saw the curvature map $K_\nabla = \widehat{\nabla} \circ \nabla : \Gamma(\zeta) \to \Gamma(\Lambda^2 T^*M \otimes \zeta)$, and that it is really a tensor: $K_\nabla(fs) = fK_\nabla(s)$. In local coordinates, this is given by a matrix of two-forms. If $P : M_n \to \mathbb{R}$ is an invariant polynomial, there's a $P(K) \in \bigoplus_{k=0}^\infty \Omega^{2k}(M)$ by $P(K)|_U = P(\Omega)$, and Lemma 22.8 showed this is a closed form.

$P(K_\nabla)$ represents a de Rham cohomology class of the manifold, and Lemma 23.3 asserts that this class is independent of ∇.

Proof of Lemma 23.3. First, we need to recall two facts from last time. Let ∇_0 and ∇_1 be connections.

(1) For all $t \in \mathbb{R}$ (not just $0 \le t \le 1$), $t\nabla_0 + (1-t)\nabla_1$ is also a connection.

(2) The connection can be pulled back from a smooth map $f : M \to N$ of manifolds; there exists a unique $\nabla' = f^*\nabla$ such that the diagram (11) commutes. Furthermore, in local coordinates, this connection ω_{ij}' is just the pullback $f^*(\omega_{ij})$.

Now, the proof uses standard differential topology techniques. Consider the projection $p : M \times \mathbb{R} \to M$ sending $(x,t) \mapsto x$, and let $i_\varepsilon : M \to M \times \mathbb{R}$ be given by $x \mapsto (x, \varepsilon)$. Finally, let $\zeta : E \to M$ be a vector bundle over M, and ∇_0 and ∇_1 be two connections on it.

Consider $\zeta' = p^*\zeta$, $\nabla_0' = p^*\nabla_0$, and $\nabla_1^* = p^*\nabla_1$, and let $\nabla = t\nabla_0' + (1-t)\nabla_1'$ be a connection on ζ'.[12] That is, restricting to the images of i_0 and i_1 are just these pullbacks. Additionally, $i_\varepsilon^*\zeta' = i_\varepsilon^* p^*\zeta = (p \circ i_\varepsilon)^*\zeta = \zeta$.

Let $i_\varepsilon^*(\nabla) = \nabla_\varepsilon$, so that locally, $i_\varepsilon^*(\omega_{ij}) = \omega_{ij}^\varepsilon$, and $[\omega_{ij}^\varepsilon]$ is the matrix of ∇_ε. That is, ω_{ij} are one-forms on $M \times \mathbb{R}$, and ω_{ij}^ε are on M. Also write $i_\varepsilon^*(\Omega_{ij}) = \Omega_{ij}^\varepsilon$, which is the matrix for K_∇^ε. This is the curvature tensor on ζ, because

$$\Omega_{ij} = \mathrm{d}\omega_{ij} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j},$$

and pulling back commutes with d and the wedge product.

Finally, $i_\varepsilon^*(P(\Omega)) = P(\Omega_\varepsilon)$. These both represent cohomology classes, so we're done once we show that i_0 is homotopic to i_1, since this implies that their pullbacks are homologous to each other (which is a standard, albeit nontrivial, proof in differential topology). Why is $i_0 \sim i_1$? Well, we have the homotopy $H(s,x) = i_s(x)$, after all. Thus, $[i_0^*(P(K_\nabla))] = [i_1^*(P(K_\nabla))]$. ⊠

Definition. Let M_n once again denote the algebra of $n \times n$ matrices. The k^{th} elementary symmetric function $\sigma_k : M_n \to \mathbb{R}$ is given by

$$\det(I + tA) = 1 + t\sigma_1(A) + \cdots + t^n\sigma_n(A).$$

For example, $\sigma_1(A) = \operatorname{Tr}(A)$ and $\sigma_n(A) = \det(A)$.

Lemma 24.1. *Every invariant polynomial $P : M_n \to \mathbb{R}$ is a polynomial in the elementary symmetric functions.*

Proof. Since P is invariant, then one can show that it's a function of the eigenvalues $\lambda_1, \dots, \lambda_n$; then, it's reasonably clearly a polynomial in $\sigma_1(x_1, \dots, x_n) = \sum x_i$, $\sigma_2(x_n) = \sum x_i x_j$, and so on, So why is it a function of the eigenvalues?

Let B be an invertible matrix such that BAB^{-1} is in Jordan canonical form;[13] then, it's certainly true for diagonal matrices, and every matrix is arbitrarily close to a diagonalizable matrix. But P is a polynomial and therefore a continuous function, and so this approximation implies it in the end. ⊠

It turns out that

$$P_k(\zeta) = (2\pi)^{-2k}\sigma_{2k}(\zeta) \in H^{2k}_{\mathrm{dR}}(M, \mathbb{R}).$$

[12] It's important to check that these are in fact connections, but this is a quick definition check.

[13] This means we're dealing with complex numbers, but that's OK, because we can just consider P as a function from the *complex* eigenvalues. Milnor used complex matrices to avoid dealing with tricky questions of invariance. Specifically, how different are the sets of $\mathrm{GL}(n,\mathbb{R})$-invariant polynomials and $\mathrm{GL}(n,\mathbb{C})$-invariant polynomials? One can make a density argument, since P is continuous and so on.

Definition. This cohomology is called the k^{th} Pontrjagin (sometimes Pontryagin) class of ζ, and their infinite sum

$$1 + P_1(\zeta) + P_2(\zeta) + \cdots \in \bigoplus_{k=0}^{\infty} H^{4k}(M, \mathbb{R})$$

is called the total Pontryagin class.

Why only the even classes? It turns out that the odd ones vanish: $\sigma_{2k+1}(\zeta) = 0$. The easiest way to see this is to choose a Riemannian metric on the bundle and pick the connection compatible with this metric; then, the matrix Ω associated with the connection will be anti-symmetric.

Definition. A Riemannian metric on a vector bundle ζ is a smooth assignment of an inner product to $\pi^{-1}(p)$ for all $p \in M$.

Definition. A connection ∇ on ζ is compatible with the Riemannian metric $\langle \cdot, \cdot \rangle_p$ if for all $s, s' \in \Gamma(\zeta)$, $\mathrm{d}\langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle$. Since $\nabla s \in \Gamma(T^*M \otimes \zeta)$ rather than just $\Gamma(\zeta)$, it isn't immediately obvious how to take the inner product of these. But since this is given by a one-form θ, then define $\langle \theta \otimes s, s' \rangle = \langle s, \theta \otimes s' \rangle = \langle s, s' \rangle \theta$.

Exercise 6. How does this definition relate to the less general definition in the case of Riemannian manifolds and tangent bundles given earlier in class? They should be the same.

Recall that by a partition of unity, a Riemannian connection always exists for the tangent bundle.

Lemma 24.2. *Given a Riemannian metric $\langle \cdot, \cdot \rangle_p$ on a vector bundle ζ, there exists a connection ∇ compatible with that metric.*

Lemma 24.3. *If ζ is a vector bundle, then if $s_1, \dots, s_n$ is a local basis of sections on a neighborhood U, then the matrix $\omega = [\omega_{ij}]$ is skew-symmetric, i.e. $\omega_{ij} = -\omega_{ji}$.*

Once these are proven, we can show that if ω is skew-symmetric, then so is Ω, and therefore plugging it into a symmetric polynomial is zero: $\sigma_{2k+1}(\Omega) = \sigma_{2k+1}(\Omega^{\mathrm{T}}) = (-1)^{2k+1}\sigma_{2k+1}(\Omega)$, forcing it to be zero.

25. The Pfaffian Form: 11/19/14

"Good mathematics shows up everywhere."

Last time, we defined the k^{th} Pontryagin class as

$$P_k(\zeta) = (2\pi)^{-2k}\sigma_{2k}(K),$$

where K is the curvature tensor and σ_{2k} is the $2k^{\text{th}}$ elementary symmetric polynomial. The reason for only considering the even numbers is that if these values are considered for $2k+1$, then they vanish.

Recall that a connection ∇ is compatible with the metric if $\mathrm{d}\langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle$ for all $s, s' \in \Gamma(\zeta)$, where $\langle \theta \otimes s, s' \rangle = \langle s, \theta \otimes s' \rangle = \langle s, s' \rangle \theta$.

Lemma 25.1. *In a neighborhood U, suppose $\zeta|_U$ is trivial, i.e. we have an orthonormal basis $s_1, \dots, s_n$ of sections, i.e. $\langle s_i, s_j \rangle_p = \delta_{ij}$. Then, the matrix $\omega = [\omega_{ij}]$ is skew-symmetric: $\omega^{\mathrm{T}} = -\omega$, or $\omega_{ij} = -\omega_{ji}$.*

Proof.

$$\begin{aligned} 0 = \mathrm{d}\langle s_i, s_j \rangle &= \langle \nabla s_i, s_j \rangle + \langle s_i, \nabla s_j \rangle \\ &= \left\langle \sum_\alpha \omega_{i\alpha} \otimes s_\alpha, s_j \right\rangle + \left\langle s_i, \sum_\alpha \omega_{j\alpha} \otimes s_j \right\rangle \\ &= \sum_\alpha \left(\langle s_\alpha, s_j \rangle \omega_{i\alpha} + \langle s_j, s_\alpha \rangle \omega_{j\alpha} \right) \\ &= \omega_{ij} + \omega_{ji}. \end{aligned}$$ ⊠

Corollary 25.2. *The matrix of the curvature tensor is also skew-symmetric, i.e. $\Omega^{\mathrm{T}} = -\Omega$ and $\Omega_{ij} = -\Omega_{ji}$.*

Proof.

$$\Omega_{ij} = \mathrm{d}\omega_{ij} = -\sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} = -\mathrm{d}\omega_{ji} + \sum_\alpha \omega_{j\alpha} \wedge \omega_{\alpha i} = -\Omega_{ji}$$ ⊠

Like in the case of manifolds, compatible connections always exist at least locally, and partitions of unity can be used to construct them globally.

Lemma 25.3. *For every vector bundle ζ, there exists a compatible connection ∇.*

Proof. Locally, within a neighborhood U, we have an orthonormal basis $s_1, \ldots, s_n$, and can choose an ω such that $\omega^{\mathrm{T}} = -\omega$ and recover the associated connection ∇. Then, choose a partition of unity, and let $\nabla = \sum_\alpha f_\alpha \nabla_\alpha$, so that

$$\begin{aligned}\langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle &= \left\langle \sum_\alpha f_\alpha \nabla_\alpha s, s' \right\rangle + \left\langle s, \sum_\alpha f_\alpha \nabla_\alpha s' \right\rangle \\ &= \sum_\alpha f_\alpha \left(\langle \nabla_\alpha s, s' \rangle + \langle s, \nabla_\alpha s' \rangle \right) \\ &= \sum_\alpha f_\alpha (\mathrm{d}\langle s, s' \rangle) = \mathrm{d}\langle s, s' \rangle. \qquad \boxtimes\end{aligned}$$

Now that we know that Ω is skew-symmetric, so $\sigma_{2k+1}(\Omega) = \sigma_{2k+1}(\Omega^{\mathrm{T}}) = \sigma_{2k+1}(-\Omega) = (-1)^{2k+1}\sigma_{2k+1}(\Omega)$, so the odd coefficients vanish. Since the cohomology class of this is independent of the choice of connection, this means for any other connection ∇', the cohomology vanishes, i.e. it is exact.

Lemma 25.4. *There exists an integer-valued polynomial* Pf*, called the Pfaffian, of the entries of each* $2n \times 2n$ *skew-symmetric matrix* A *such that* $(\mathrm{Pf}(A))^2 = \det A$ *and* $\mathrm{Pf}(BAB^{\mathrm{T}}) = \mathrm{Pf}(A)\det(B)$.

To address the sign convention, we'll pick one of the sign choices. By convention, let

$$\mathrm{Pf}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} = 1, \qquad \text{where} \qquad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can calculate the Pfaffian for small skew-symmetric matrices fairly easily; all 2×2 skew-symmetric matrices are of the form $A = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}$, for which $\det(A) = a_{12}^2$. Thus, $\mathrm{Pf}(A) = a_{12}$.

For a 4×4 matrix, it's a little more harder.

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Proof of Lemma 25.4. The idea of the proof is fairly reasonable, but it will have to be be formalized with some abstract nonsense.

For most $2n \times 2n$ skew-symmetric matrices A, there exists an $X \in M_{2n}$ such that

$$XAX^{\mathrm{T}} = \begin{pmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{pmatrix}, \tag{12}$$

and this can be found systematically, where the coefficients of X lie in the quotient field of the ring $\Lambda[a_{12}, a_{13}, \ldots, a_{2n-1}, a_{2n}]$ generated by the coefficients of the matrix.

Since (12) is true, then $\det(A)(\det(X))^2 = 1$, so $\det A = (\det(X)^{-1})^2$, so $\mathrm{Pf}(A) = \det(X)^{-1}$. This relies on the fact that this ring of polynomials is a UFD, or the fact that a rational function that is the square root of a polynomial is itself a polynomial.

For the second part of the lemma, notice that $\det(BAB^{\mathrm{T}}) = \det A(\det B)^2$, so the Pfaffian is $\mathrm{Pf}(BAB^{\mathrm{T}}) = \pm\mathrm{Pf}(A)\det(B)$; letting B be the identity shows that we must take the positive choice, so $\mathrm{Pf}(BAB^{\mathrm{T}}) = \mathrm{Pf}(A)\det(B)$.

This is great, but we have to describe how to find X. Given the matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix},$$

let $B = \begin{pmatrix} \begin{smallmatrix} 1/a_{12} & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{pmatrix}$, so that

$$BAB^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & a_{13}^* & a_{14}^* \\ -1 & 0 & a_{23}^* & a_{24}^* \\ -a_{13}^* & -a_{23}^* & 0 & a_{34} \\ -a_{14}^* & -a_{24}^* & -a_{34} & 0 \end{pmatrix},$$

where $a_{ij}^* = a_{ij}/a_{12}$. Then, let

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{23} & -a_{13}^* & 1 & 0 \\ a_{24} & -a_{14}^* & 0 & 1 \end{pmatrix},$$

so that

$$C(BAB^{\mathrm{T}})C^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & a_{34}^{**} \\ & & -a_{34}^{**} & 0 \end{pmatrix},$$

for some a_{34}^{**} that are rational in a_{ij}; then, this can be normalized by conjugating by the matrix with diagonal entries $(1, 1, 1, 1/a_{34}^{**})$, and keep going (unless it's 4×4, where we're done). ☒

Definition. A vector bundle ζ is orientable if there is a choice of transition functions such that for each $p \in U \cap V$, the image $f(p)$ of the transition function $f : U \cap V \to \mathrm{GL}(n, \mathbb{R})$ has positive determinant; this allows one to assign a global orientation on ζ.

Let ζ be an orientable $2n$-dimensional vector bundle with a Riemannian metric and a compatible connection. In a neighborhood U, choose an orthonormal basis $u_1, \ldots, u_n$ of sections, and let Ω be a skew-symmetric matrix. Then, in a different basis (with the same orientation) $s_1', \ldots, s_n'$, $\Omega' = X\Omega X^{-1}$, where $X^{\mathrm{T}} = X^{-1}$ and $\det(X) = 1$, so that $\Omega' = X\Omega X^{\mathrm{T}}$ and $\mathrm{Pf}(\Omega') = \mathrm{Pf}(\Omega)\det(X) = \mathrm{Pf}(\Omega)$. The upshot is that the Pfaffian is a globally defined form depending on the connection, so we get a $\mathrm{Pf}(K_\nabla) \in \Omega^{2n}(M)$; it does depend on the metric, though.

Even before Chern, the Euler classes were known, so the goal is to find a differential form in the curvature tensor that represents the Euler class. This is possible only if the connection is compatible with the Riemannian metric.

This means that $\mathrm{d}\,\mathrm{Pf}(K) = 0$, which is a quick calculation: since the determinant is closed, then $0 = \mathrm{d}\det(\Omega) = \mathrm{Pf}(\Omega)\,\mathrm{d}\,\mathrm{Pf}(\Omega)$.

This relates back to Theorem 22.2, which is a vast generalization of the Gauss lemma. But it can be generalized to Euler classes of vector bundles.

Theorem 25.5. *Let ζ be an orientable $2n \times 2n$-dimensional vector bundle with a Riemannian metric and a compatible connection; then, if $e(\zeta)$ denotes the Euler class of ζ, then $\mathrm{Pf}(K/2\pi) = e(\zeta)$.*

26. The Pfaffian, Euler Classes, and Chern Classes: 11/21/14

Recall that last time, we proved that the Pfaffian is closed, and the proof strategy was quite similar to that of Lemma 22.8. Specifically, we let

$$\mathrm{Pf}'(A) = \left[\frac{\partial\,\mathrm{Pf}}{\partial a_{ij}}\right]^{\mathrm{T}} = \left[\frac{\partial\,\mathrm{Pf}}{\partial a_{ji}}\right].$$

Then,

$$\begin{aligned} \mathrm{d}\,\mathrm{Pf}(\Omega) &= \sum_{i,j} \frac{\partial\,\mathrm{Pf}}{\partial \Omega_{ij}}\,\mathrm{d}\Omega_{ij} = \mathrm{Tr}(\mathrm{Pf}'(\Omega)\,\mathrm{d}\Omega) \\ &= \mathrm{Tr}(\mathrm{Pf}'(\Omega) \wedge \omega \wedge \Omega - \mathrm{Pf}'(\Omega) \wedge \Omega \wedge \omega) \end{aligned}$$

thanks to the Bianchi identity.

Claim. $\mathrm{Pf}'(\Omega) \wedge \Omega = \Omega \wedge \mathrm{Pf}'(\Omega)$.

Proof. Their difference is equal to

$$\mathrm{Tr}\left(\frac{\mathrm{Pf}'(\Omega) \wedge \omega \wedge \Omega}{X} - \frac{\Omega \wedge \mathrm{Pf}'(\Omega) \wedge \omega}{X}\right) = 0.$$

If $J_{ij}(t)$ is the matrix with t in position (i, j) and is otherwise equal to the identity, then $J_{ji}(t)^{\mathrm{T}} = J_{ij}(t)$, and since $\mathrm{Pf}(BAB^{\mathrm{T}}) = \mathrm{Pf}(A)\det(B)$, then $\mathrm{Pf}(J_{ij}(t)\Omega J_{ji}(t)) = \mathrm{Pf}(\Omega)$.

Let's take $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$; acting by $J_{ij}(t)$ on the left adds t times the j^{th} row to the i^{th} row, and acting by $J_{ji}(t)$ on the right adds t times the j^{th} column to the i^{th} column. Thus, we get that

$$\underbrace{\sum_\alpha \frac{\partial\,\mathrm{Pf}}{\partial \Omega_{j\alpha}}}_{I} + \underbrace{\sum_\alpha \frac{\partial\,\mathrm{Pf}}{\partial \Omega_{\alpha i}}\Omega_{\alpha j}}_{II} = 0.$$

Then, let's split this into components.

$$\begin{aligned} I &= \sum_\alpha \Omega_{j\alpha}\frac{\partial\,\mathrm{Pf}}{\partial \Omega_{i\alpha}} = \sum_\alpha \Omega_{j\alpha}(\mathrm{Pf}'(\Omega))_{\alpha i} = (\Omega\,\mathrm{Pf}'(\Omega))_{ji}. \\ II &= \sum_\alpha (\mathrm{Pf}'(\alpha))_{i\alpha}\Omega_{\alpha j} = (\mathrm{Pf}'(\Omega)\Omega)_{ij}. \end{aligned}$$

If we can show this matrix is skew-symmetric, then we're done. We already know Ω to be skew-symmetric, and

$$\frac{\partial\,\mathrm{Pf}}{\partial\Omega_{ij}} = \frac{\partial\,\mathrm{Pf}}{\partial\Omega_{ji}}\cdot\frac{\partial\Omega_{ji}}{\partial\Omega_{ij}} = -\frac{\partial\,\mathrm{Pf}}{\partial\Omega_{ij}}.$$

Thus, Pf' is skew-symmetric, so $\Omega\cdot\mathrm{Pf}'(\Omega)$ and $\mathrm{Pf}'(\Omega)\cdot\Omega$ are too. Thus, the whole thing is skew-symmetric. ⊠

This is great, but the last thing to address is how much this depends on the connection.

Lemma 26.1. *The cohomology class $[\mathrm{Pf}(K_\nabla)]$ of the Pfaffian is independent of the choice of compatible connection ∇.*

Proof. Let ∇_0 and ∇_1 be metric-compatible connections, so that $\nabla = t\nabla_0 + (1-t)\nabla_1$ is also a compatible connection. Then, $\mathrm{Pf}(\Omega_0)$ and $\mathrm{Pf}(\Omega_1)$ can be shown to be homotopic, in a manner similar to the proof of Lemma 23.3. ⊠

Thus, we have a cohomology class $[\mathrm{Pf}(K/2\pi)] \in H^{2n}_{\mathrm{dR}}(M)$. If ζ is an orientable $2n\times 2n$ vector bundle with a Riemannian metric, then the Generalized Gauss-Bonnet Theorem asserts that it's the same cohomology class as the Euler class of ζ. Today, we'll tell the story, and then we'll fill in the details next week.

For complex vector bundles, one has Chern classes $C_k(\zeta)$, and we'll use those to get to Pontrjagin classes $P_k(\zeta)$. Since the square of the Pfaffian is the determinant and the determinant is the top Pontrjagin class (since $\det(A) = \sigma_{2n}(A)$), then we can relate Pontrjagin classes to the Pfaffian.

But first, maybe we should say what the Euler class is.

Proposition 26.2. *Let ζ be an oriented, real vector bundle with fiber dimension $\dim\pi^{-1}(b) = n$. Then, there is an Euler class $e(\zeta)\in H^n(M)$ which satisfies the following properties:*

1. *Initiality: if ζ is a bundle over a manifold M and ζ' is a bundle over a manifold N, then if $f : N \to M$ is covered by a bundle map $\zeta' \to \zeta$, then $e(f^*(\zeta) = f^*(e(\zeta))$, i.e. pullback and Euler class commute.*
2. *If $\overline{\zeta}$ is ζ with the opposite orientation, then $e(\overline{\zeta}) = -e(\zeta)$.*

Corollary 26.3. *As a consequence of (2), if n is odd, then $e(\zeta) = 0$, because in this case, there's an orientation-reversing isomorphism $f : \zeta \to \zeta$ given by $(p, v) \mapsto (p, -v)$, so $-e(\zeta) = e(\overline{\zeta}) = e(\zeta)$. (If n is even, this is orientation-preserving.)*

Corollary 26.4. *If ζ is an n-bundle and ζ' is an m-bundle over the same manifold M, then $e(\zeta\oplus\zeta') = e(\zeta)\smile e(\zeta')$ (i.e. Whitney sum becomes cup product). In particular, if $\zeta = TM$, then $\langle e(TM), [M]\rangle = \chi(M)$.*

Combining with Corollary 26.3, we have another proof that the Euler characteristic $\chi(M)$ of an odd-dimensional manifold is trivial. Additionally, we can use this to establish that $\int_M \mathrm{Pf}(K2\pi) = \chi(M)$.

That's the Euler class. How about the Chern classes? Let ζ be a complex vector bundle of $\dim_{\mathbb{C}}(\pi^{-1}(p)) = n$ with the transition maps $f_{UV} : U\cap V \to \mathrm{GL}(n,\mathbb{C})$. Then, there's a Chern class $C_k(\zeta)\in H^{2k}(M)$, for $k = 0, \dots, n$, such that $C_0(\zeta) = 1$, and a total Chern class $C(\zeta) = 1 + C_1(\zeta) + \cdots + C_n(\zeta) \in H^{2*}(M)$ such that:

1. The Whitney sum is once again the cup product: $C(\zeta\oplus\zeta') = C(\zeta)\smile C(\zeta')$.
2. Let $\overline{\zeta}$ denote the conjugate bundle (everything is replaced with its complex conjugate); then, $C_k(\overline{\zeta}) = (-1)^k C_k(\zeta)$.
3. Every complex manifold is orientable, and therefore every complex vector bundle is orientable. Every complex vector bundle gives rise to an orientable real vector bundle of dimension $2n$. Then, $c_n(\zeta) = e(\zeta)$; the top Chern class is the same as the real Euler class. This is the particular relation we want as a step from the Euler class to the Pfaffian.

If ζ is merely a real bundle, then it doesn't *a priori* have Chern classes, but we can certainly complexify it, and consider the Chern classes $C(\zeta\otimes\mathbb{C}) = 1 + C_1(\zeta\otimes\mathbb{C}) + \cdots + C_n(\zeta\otimes\mathbb{C})$. However, we know that $\zeta\otimes\mathbb{C}\cong\overline{\zeta\otimes\mathbb{C}}$ (the complexification of a real vector bundle is the same as its conjugate), given by $f : x+iy \mapsto x-iy$. This is because $f(i(x+iy)) = -i(f(x+iy)$, so the complex structure is preserved by f.

On the one hand, $C_k(\overline{\zeta\otimes\mathbb{C}}) = (-1)^k C_k(\zeta\otimes\mathbb{C}) = C_k(\zeta\otimes\mathbb{C})$. Thus, the odd Chern classes of this complexified bundle vanish: $C_{2k+1}(\zeta\otimes\mathbb{C}) = 0$. So we only need to worry about the even Chern classes, and in fact, these are just the Pontrjagin classes again! $P_k(\zeta) = (-1)^k C_{2k}(\zeta\otimes\mathbb{C})\in H^{4k}(M)$, when ζ is an orientable, $2n$-dimensional real manifold.

As real vector bundles, $\zeta\otimes\mathbb{C}\cong\zeta\oplus\zeta$ (which is, once again, the Whitney sum), in the same way that $\mathbb{C}\cong\mathbb{R}\oplus\mathbb{R}$; if $\dim(\zeta) = n$, then this isomorphism is orientation-preserving if $n(n-1)/2$ is even, and is orientation-reversing if it's odd. Intuitively, the isomorphism makes $\binom{n}{2}$ permutations amongst the basis elements (separating them out into two parts, in some sense). Thus, the Euler clas is $(-1)^n(-1)^{2n(2n-1)/2}e(\zeta\oplus\zeta) = e(\zeta\oplus\zeta) = e(\zeta)^2$.

Now we've gotten all the way to Pontrjagin classes, and therefore the determinant; but the Pfaffian is the square root of the determinant, which is just the square root of the top Pontryagin class, and therefore we have the relation with these and the Euler and Chern classes. This is the topological side of the story.

For any complex vector bundle $\zeta : E\to M$, there always exists a bundle map to the universal bundle, which is the infinite Grassmanian $\mathrm{Gr}(n,\infty)$, and a bundle $E(n,\infty)$ over it. In other words, every bundle is a pullback of this one, and any two such pullback maps are homotopic. The cohomology ring of this is a polynomial ring in the Chern classes: $H^*(\mathrm{Gr}(n,\infty)) = \Lambda[c_1(\gamma^n), c_2(\gamma^n), \dots]$, so if one wants the Chern classes of a particular bundle, they can be pulled back from these in the universal bundle.