MATH 53 NOTES

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These notes were taken in Stanford's Math 53 (Ordinary Differential Equations) class in Spring 2013, taught by Akshay Venkatesh. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu.

Contents

Note that I never took Math 53, but that I audited classes in order to understand the material better. Beware of typos.

1. Overview: 4/1/13

A little bit of administrative stuff: homework will be due Thursdays, and here is the course website.

Let's start with an example:

Theorem 1.1. *Suppose the population of the United States is 315 million, and grows by* 0.75% *each year. Let* $P(t)$ *be the population in year* t*, so that* $P(2012) = 315$*. Then,* $P(t+1) = P(t) + 0.0075P(t)$*. But the population grows continuously, so consider the derivative:* $P'(t) = 0.0075P(t)$*.*

This is a simple differential equation, which is an equation relating a function $f(x)$ and its derivatives $\frac{\mathrm{d}f}{\mathrm{d}x}$, $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ etc., and the goal is to find f. This course will develop techniques for solving standard classes of ODEs.

Example 1.1. Another example is a pendulum swinging on a rod. Let θ be the angle of the rod with the vertical, so that the pendulum satisfies the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\sin\theta = 0,$$

where g is the acceleration due to gravity. This equation comes from the equation $F = ma$: $F = -mg\sin\theta$ and $a = \ell\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$.

This equation is already not so simple to solve, since the sine is a tricky function. The exact solution will be given in terms of an integral rather than familiar elementary functions. However, if θ is small, then one can approximate $\sin\theta \approx \theta$, giving the approximate differential equation $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g\theta/\ell = 0$. This is much simpler to solve.

Differential equations often model how systems change with time[1]

Example 1.2. Planetary motion is also given by a differential equation: again using $F = ma$, suppose the sun is located at the origin in the plane. Then, the system becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -C\frac{x}{(x^2+y^2)^{\frac{3}{2}}} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -C\frac{y}{(x^2+y^2)^{\frac{3}{2}}},$$

where C depends on the masses of the sun and the planet and the gravitational constant. This is a system of two differential equations, which looks trickier but isn't all that much worse.

This example is somewhat complicated, because the functions themselves are, and will resurface near the end of the course.

Other examples include calculating airflow or weather. Some of these involve partial differential equations (e.g. differentiating with respect to time and space), which are beyond the scope of this course. Many of these are too complicated to solve analytically, and instead numerical approximations are sought.

Differential equations can be classified by their order: for example, a first-order differential equation involves only the first derivative of a function. Example 1.1 is first-order, but the others given above aren't.

Differential equations can also be linear: the function y and its derivatives occur only as multiplied by a function of x. For example, $y' + y = 3$ and $xy' + x^2y = \sin x$ are both linear, but $y' + \sin y = y^2$ and the equation in Example 1.1 aren't. Linear equations are easier to solve, but many important equations are nonlinear.

Thus, the simplest differential equations are first-order and linear. These look like $A(x)y' + B(x)y = C(x)$.[2] Any linear first-order equation can be rewritten by dividing by A:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \underbrace{\frac{B(x)}{A(x)}}_{b(x)}y = \underbrace{\frac{C(x)}{A(x)}}_{c(x)}.$$

Simplifying even further, suppse $c(x)$ and $b(x)$ are constants: $y' = c - by$. These are the simplest differential equations.

Solving this requires a trick (as do differential equations in general; sadly, these tricks are somewhat hard to motivate most of the time). Dividing by $c - by$, one has

$$\begin{aligned} \frac{1}{c-by}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\log(c-by)}{-b}\right) &= 1 \qquad \text{(just using the Chain Rule)} \end{aligned}$$

Let A be a constant of integration. Then, after integrating,

$$\begin{aligned} &\Longrightarrow -\frac{\log(c-by)}{b} = x + A \\ &\Longrightarrow c - by = e^{-bx}e^{-bA} \\ &\Longrightarrow y = \frac{c}{b} - e^{-bx}\left(\frac{e^{-bA}}{b}\right). \end{aligned}$$

Then, e^{-bA}/b is some arbitrary constant, so call it K. Thus, the solution to the differential equation is $y = c/b - Ke^{-bx}$. This is the general solution to $y' = c - by$; K is an arbitrary constant, but given $y(0)$ (or y at any other value of x) it is possible to determine k.

[1]. . . as well as changes in space, though that won't be discussed today.

[2]There are plenty of first-order equations that aren't linear at all, such as $ye^yy' + x^3 = 0$.

For example, if $y' = 1 - y$ and $y(0) = 2$, then $b = c = 1$, so $y = 1 - Ke^{-x}$ for some K. Then, $2 = 1 - Ke^0$, so $K = 1$, and the specific solution is $y = 1 + e^{-x}$.

2. First-Order Linear Differential Equations: 4/3/13

Last time, we considered first-order linear differential equations with constant coefficients, such as $y' = 1 + y$. When solving, a solution was obtained by finding something which had a specified function as its derivative. This can also be thought of as multiplying by $\mathrm{d}x$:

$$\begin{aligned} \frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \implies \frac{\mathrm{d}y}{1+y} &= \mathrm{d}x \\ \implies \int \frac{\mathrm{d}y}{1+y} &= \int \mathrm{d}x \\ \log(1+y) &= x + C \end{aligned}$$

and then the equation can be solved as before. This is called separation of variables, and will resurface next week.

One can produce plots of integral curves: if the solution depends on a parameter K, it is possible to plot several curves to get an idea of the family of solutions; see Figure 1.

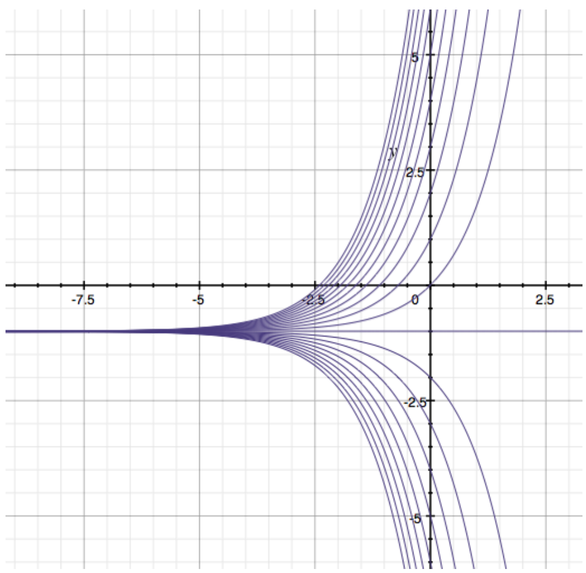


FIGURE 1. Integral curves of $y' = 1 + y$, with $y(0) \in \{-10, \ldots, 10\}$.

In fact, this can be done without a solution, plotting direction fields, and obtaining a numerical solution. See Sections 1.1 and 1.3 of the book for more details. If $y(0) = 2$ and $y' = 1 + y$, then near 0 $y' \approx -1$, so $y(0.1) \approx y(0) - 0.1 = 1.9$. This approximation is only good near zero. After successive approximations, one discovers that $y(1) \approx 1.349$. Using the exact solution, $y(1) = 1.366$, so this can be pretty useful. This method works for any first-order differential equation, linear or not.

Some applications of differential equations: population growth modeling was discussed in the previous lecture, though it can often be more nuanced: there can be many sources of population growth or loss, and they may be given by their own equations. Alternatively, one can consider the cooling of a hot object, or the concentration of (for example) a drug in someone's bloodstream. These should help with intuition for the constants given in the solutions; they provide some specific physical constant that gives the particular solution from the general formula.

Now, it is possible to generalize: consider only first-order linear differential equations, but with possibly nonconstant coefficients, such as $xy' + x^2y = \sin x$ or $e^xy' + xy = 1/(1+x)$. Any such equation can be put into the form $y' + p(x)y = q(x)$ by dividing by the y'-coefficient: the first example above becomes $y' + xy = \frac{\sin x}{x}$. Here, $p(x)$ can represent some growth or removal rate, and $q(x)$ measures the amount that is externally added or removed. Of course, there are cases in which they might represent something different, but this representation aids intuition.

To motivate the general case, $y' = 1 - y$ will be solved in a somewhat different way: rewrite it as $y' + y = 1$ and multiply it by something such that the left hand side looks like a derivative: $M(x)y' + M(x)y = M(x)$. It would be nice if we had $My' + M'y$, so we need M to be its own derivative. Take $M = e^x$:

$$\begin{aligned} e^x y' + e^x y &= e^x \\ \implies \frac{\mathrm{d}}{\mathrm{d}x}(e^x y) &= e^x \\ \implies e^x y &= e^x + C \\ y &= 1 + Ce^{-x}. \end{aligned}$$

Example 2.1. Another example, this one slightly more complicated. Suppose $y + y' = e^{-x}$. Then, multiplying by e^x, $e^x y' + e^x y = 1$, so $e^x y = x + C$, or $y = xe^{-x} + Ce^{-x}$.

Now, let's take the general case for a first-order linear differential equation $y' + p(x)y = q(x)$. Again, we will multiply by the integrating factor $M(x)$, in order to make the equation easier to integrate. Take $M = e^{\int p(x)\,\mathrm{d}x}$ (which will be motivated later; right now, it just works, since you obtain the right derivative), and take

$$\begin{aligned} M(x)q(x) &= M(x)\frac{\mathrm{d}y}{\mathrm{d}x} + M(x)p(x)y \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(M(x)y) \\ \implies M(x)y &= \int M(x)q(x)\,\mathrm{d}x \\ \implies y &= \frac{\int M(x)q(x)\,\mathrm{d}x}{M(x)}. \end{aligned}$$

A closed-form solution depends on being able to find $\int M(x)q(x)$, but it's an explicit solution in either case.

Example 2.2. In $y' + y/x = \sin x$, $p(x) = 1/x$, so $M(x) = e^{\log x} = x$. Thus, the formula is

$$y = \frac{\int x\sin x\,\mathrm{d}x}{x} = \frac{-x\cos x + \sin x}{x} + C = -\cos x + \frac{\sin x}{x} + C.$$

after integration by parts.

3. Examples of Integrating Factors: 4/5/13

Integrating factors as discussed in the last lecture seem like magic, but in solving an equation $y' + p(x)y = q(x)$, one chooses $M(x) = e^{\int p(x)\,\mathrm{d}x}$, so that by the Chain Rule $\frac{\mathrm{d}M}{\mathrm{d}x} = p(x)M(x)$. Thus, the differential equation can be multiplied by $M(x)$ to obtain $y'M + yM' = qM$, so the solution is $y = \left(\int q(x)M(x)\,\mathrm{d}x\right)/M(x)$.

Example 3.1. Consider a bank account with variable interest. let $M(t)$ be the amount of money in the account at time t, measured in years (though the specific unit isn't important conceptually), and let $I(t)$ be the rate of interest at time t: for example, 3% interest corresponds to $I(t) = 0.03$. Finally, let $Q(t)$ be the amount of money put in (or negative for removing money) in year t. Thus, $M(t)$ obeys the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = I(t)M(t) + Q(t).$$

This illustrates a common trend: the first term indicates how it would grow independent of external forces, and the second term represents external influences.

This can be solved in different ways: first, suppose $Q(t) = 0$ and $I(t) = I$ is constant. Then, $M(t) = M(0)e^{It}$, as has been done in calculus classes. If $I(t)$ can vary, then $M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u}$, where the integral represents the average value of the interest rate.

When $Q(t)$ isn't zero, it can be broken into the cases $Q(0), Q(1), \dots$, giving a solution

$$M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u} + Q(1)e^{\int_1^t I(u)\,\mathrm{d}u} + Q(2)e^{\int_2^t I(u)\,\mathrm{d}u} + \dots$$

This is an approximation assuming money is only added once at the end of each year. If it is done continuously, the sum should be replaced by an integral:

$$M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u} + \int_0^t Q(x)e^{\int_x^t I(u)\,\mathrm{d}u}\,\mathrm{d}x.$$

This could have also been derived with the method of integrating factors, or more specifically, the given general solution for first-order linear equations. The $M(0)$-term comes from the constant of integration, so a little bit of algebra is involved.

Example 3.2. Consider the equation $y' = \sin(t) - y$. This approximates a common phenomenon: it models the temerature of an object that's being alternately heated and cooled (for example, a body of water heated in the day and cooled at night, or some sort of yearly temperature variation). Without the sine term, $y' = -y$ is Newton's Law of Cooling, but adding it in gives a reasonably crude model. To solve, rewrite it as $y' + y = \sin t$ and multiply by $M(t) = e^t$. Thus,

$$\begin{aligned} e^t y' + e^t y &= e^t \sin t \\ \implies y &= \frac{\int e^t \sin t\,\mathrm{d}t}{e^t}. \end{aligned}$$

This is an integration by parts:

$$\begin{aligned} \int e^t \sin t\,\mathrm{d}t &= e^t \sin t - \int e^t \cos t\,\mathrm{d}t \\ &= e^t \sin t - e^t \cos t - \int e^t \sin t\,\mathrm{d}t \\ \implies 2\int e^t \sin t\,\mathrm{d}t &= e^t \sin t - e^t \cos t \\ \implies \int e^t \sin t\,\mathrm{d}t &= \frac{e^t \sin t - e^t \cos t}{2} + C. \end{aligned}$$

Plugging this back into the differential equation,

$$y = \frac{\sin t - \cos t}{2} + Ce^{-t}.$$

The second term is not very large, particularly as t becomes large. However, $\sin t - \cos t = \sqrt{2}\sin(t - \pi/4)$, so the temperature oscillates with the heating and cooling, but it las behind by $1/8$ of a full heating-cooling cycle. Of course, this is an approximation, but it still says interesting things: a more realistic model is $y' = \sin t - ay$, where a is a constant representing how quickly the object cools or heats. Then, the lag depends on a: it becomes lesser as a increases.

Exercise 3.1. Solve the refined model $y' = \sin t - ay$, and determine how the lag depends on a.

Moving to the next topic, we will now discuss first-order nonlinear differential equations. This moves into Chapter 2 of the textbook, so Section 1.3 is skipped. This is the study of equations $y' = f(x, y)$ for any f, such as $y' = x^2y^3\sin y + e^y$.

Of course it won't be possible to solve every first-order differential equation, but many classes of them can be solved. Additionally, many more real-world system are modelled by nonlinear differential equations.

Example 3.3. Consider a falling object that encounters some air resistance. This is most relaistic for a spherical object, as otherwise the air resistance would vary in complicated ways. Let $v(t)$ be the speed at time t, so that v' is the acceleration. Let g be the acceleration due to gravity, so that the equation is $v' = g - av^2$, where the second term is a model of the air resistance. a is some constant, and the term is quadratic because as an object moves faster, it hits the air faster, but it also travels through more air. This will be solved exactly next week.

When $g = av^2$, the object achieves a constant speed. Using this terminal velocity, it is possible to determine the value of a.

Example 3.4. This example isn't necessarily as realistic, but is commonly used. The logistic model of population growth takes $p(t)$ to be the population at time t. The model takes some additional factor that illustrates that a space cna only hold a fixed carrying capacity of the population, producing the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = cP(t)\left(1 - \frac{P(t)}{A}\right). \tag{1}$$

When $P(t) = A$, the change is zero, and when it's greater, it becomes negative, indicating that A is the largest population supported by the environment.

4. Separation of Variables: 4/8/13

The basic idea behind separation of variables (section 2.1 in the book) is to split $\mathrm{d}x$ and $\mathrm{d}y$, then move all things in terms of x and $\mathrm{d}x$ to one side, all terms involving y and $\mathrm{d}y$ to the other, and integrate.

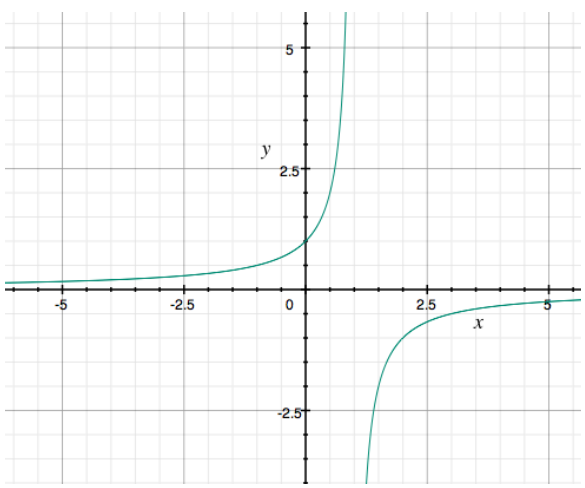
Example 4.1. Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$. This is linear, so it could be solved by any of the techniques presented last week, but let's try separation of variables: it becomes separated as

$$\begin{aligned} \frac{\mathrm{d}y}{y} &= x\,\mathrm{d}x \\ \Longrightarrow \int \frac{\mathrm{d}y}{y} &= \int x\,\mathrm{d}x \\ \Longrightarrow \log|y| &= \frac{x^2}{2} + C \\ \Longrightarrow y &= Ke^{x^2/2}, \end{aligned}$$

where $K = e^{\pm C}$ depending on how the absolute value is taken. This is the same solution as would be obtained with the integrating factor $e^{x^2/2}$.

Example 4.2. Another fairly easy example is $\frac{\mathrm{d}y}{\mathrm{d}x} = y+1$. Then, $\mathrm{d}y/(y+1) = \mathrm{d}x$, and after integrating, $\log|y+1| = x+C$. Thus, $y = Ke^x - 1$, by roughly the same logic as in the previous example.

Example 4.3. Now, for a more interesting example: $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$. This isn't linear, but can be solved by separation of variables, and more generally, any differential equation in which $y' = f(y)$ and there is no dependence on t can always be solved by separation of variables. Suppose $y(0) = 1$; then, $\mathrm{d}y/y^2 = \mathrm{d}t$, so $-1/y = t + C$. Given the initial condition, $C = -1$, so $y = 1/(1-t)$. Notice that this graph diverges at $t = 1$; thus, it can't perfectly model a real phenomenon. However, no linear differential equation will diverge like this.



Example 4.4. Consider logistic population growth, as in Example 3.4, given by (1). Here $cP(t)$ represents the natural growth without any restrictions, and $(1 - P/A)$ is the factor corresponding to the carrying capacity. This isn't the most precise model, but it has the right ideas and is quite simple. It can be solved exactly: today, call c r instead, and A will be denoted K. Then,

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{r}{K}P(K-P) \\ \Longrightarrow K\frac{\mathrm{d}P}{P(K-P)} &= r\,\mathrm{d}t \\ \Longrightarrow K\int \frac{\mathrm{d}P}{P(K-P)} &= rt + C. \end{aligned}$$

Now, integrate by partial fractions: $K/P(K-P) = 1/P + 1/(K-P)$, so

$$\int \frac{K}{P(K-P)}\,\mathrm{d}P = \int \left(\frac{1}{P} + \frac{1}{K-P}\right)\mathrm{d}P = \log P - \log(K-P) + C.$$

$$\Longrightarrow \log P - \log(K-P) = rt + C$$

$$\Longrightarrow \frac{P(t)}{K-P(t)} = De^{rt}$$

$$\Longrightarrow P(t) = (De^{rt})(K-P(t))$$

$$\Longrightarrow P(t)(1+De^{rt}) = KDe^{rt}$$

$$\Longrightarrow P(t) = K\frac{De^{rt}}{1+De^{rt}}.$$

As $t \to \infty$, the De^{rt}-term dominates the 1 in the deominator, so $P(t) \to K$, no matter whether it starts above or below it. The initial growth is exponential, however. The value of r determines the time scale: larger r implies the population stabilizes more quickly.

Example 4.5. Consider a falling body with air resistance, given by Example 3.3, so $v' = g - av^2$, where a is the air resistance. This is going to be kind of algebraically ugly, as it involves using partial fractions to integrate $\mathrm{d}v/(g-av^2) = \mathrm{d}t$:

$$\frac{1}{g-av^2} = \frac{1}{\left(\sqrt{g} - \sqrt{a}v\right)\left(\sqrt{g} + \sqrt{a}v\right)} = \frac{1}{2\sqrt{g}}\left(\frac{1}{\sqrt{g} - \sqrt{a}v} + \frac{1}{\sqrt{g} + \sqrt{a}v}\right)$$

$$\Longrightarrow \int \frac{\mathrm{d}v}{g-av^2} = \frac{1}{2\sqrt{g}\sqrt{a}}\left(\log(\sqrt{g} + \sqrt{a}v) - \log(\sqrt{g} - \sqrt{a}v)\right) + C.$$

If this is confusing, use $g = a = 1$, which illustrates the concept but isn't quite as messy. It would also be helpful to know how the general answer depends on g and a, however.

Now, the general equation can be solved:

$$\frac{1}{2\sqrt{g}\sqrt{a}}\left(\log(\sqrt{g} + \sqrt{a}v) - \log(\sqrt{g} - \sqrt{a}v)\right) = t + C$$

$$\Longrightarrow \frac{\sqrt{g} + \sqrt{a}v}{\sqrt{g} - \sqrt{a}v} = Ke^{2\sqrt{ag}t}.$$

As in the logistic case, solve for v, first multiplying by $\sqrt{g} - \sqrt{a}v$. Then,

$$v = \sqrt{\frac{g}{a}}\frac{Ke^{2\sqrt{ag}t} - 1}{Ke^{2\sqrt{ag}t} + 1}.$$

This looks like the solution in Example 4.4 in that it has exponentials in the numerator and the deominator.

if the body starts at rest (i.e. dropped rather than thrown), then $v(0) = 0$, so $K = 1$. Thus,

$$v(t) = \sqrt{\frac{g}{a}}\frac{e^{2\sqrt{ag}t} - 1}{e^{2\sqrt{ag}t} + 1} = \tanh\left(\sqrt{ag}t\right).$$

As $t \to \infty$, the exponential term dominates, and $v(t) \to \sqrt{g/a}$, called the terminal velocity. One could also calculate how long it takes to reach half of the original velocity, which can be shown to be $\log(2)/(2\sqrt{ag})$. Thus, as a gets smaller, the air resistance decreases, and the maximum speed increases and it takes longer to reach the terminal velocity, which is exactly what one might expect.

5. Autonomous Equations: 4/10/13

The logistic equation and falling body examples presented in the previous lecture (Examples 4.4 and 4.5 respectively) are examples of autonomous equations: those in which $y' = f(y)$, and there is no x-dependence.

In the logistic equation, looking at the signs of the two terms can indicate the rough behavior of the system without explicitly solving. This can be generalized: if $y' = \sin y$, then y increases when $y \in (0, \pi)$ and decreases when $y \in (\pi, 2\pi)$, and so on. This has logistic-like properties: the zeros of $f(y)$ are equilibrium points, and curves converge towards

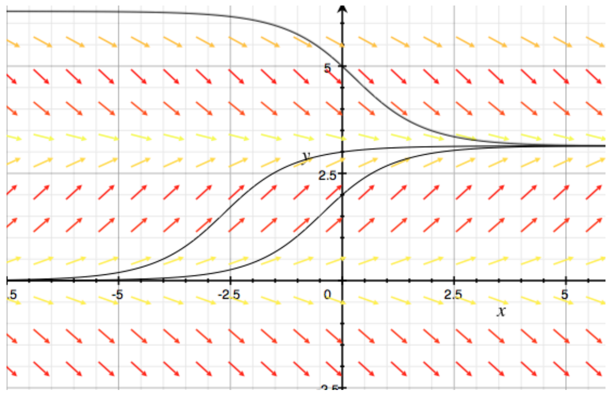


FIGURE 2. Logistic-like properties of the differential equation $y' = \sin y$.

them. here, $y = 0$ and $y = 2\pi$ are unstable equilibria, and $y = \pi$ is stable. Additionally, if $y = g(x)$ is a solution, then $y = g(x + c)$ is also a solution for any constant c, just as in the logisitc equation.

$y' = \sin y$ can actually be solved exactly, but the solution is somewhat complicated. This illustrates the more general fact that all autonomous equaations can be solved by a separation of variables: take $\mathrm{d}y/f(y) = \mathrm{d}x$. This gives $\int \mathrm{d}y/\sin y$, which can be solved by the following trick: let $u = \cos y$:

$$\int \frac{\mathrm{d}y}{\sin y} = \int \frac{\sin y\,\mathrm{d}y}{\sin^2 y} = -\int \frac{\mathrm{d}u}{1 - u^2}.$$

After this, it gets a bit messy, but it's perfectly solvable using partial fractions.

One interesting result is that it's possible to have different solutions with the same initial condition! Take $y' = y^{1/3}$, and solve by separation of variables:

$$\begin{aligned} y^{-1/3}\,\mathrm{d}y = \mathrm{d}t &\implies \frac{y^{2/3}}{2/3} = t + C \\ &\implies y^{2/3} = \frac{2t}{3} + C \\ &\implies y = \left(\frac{2t}{3} + C\right)^{3/2}. \end{aligned}$$

For another example, consider $y' = 1/y$, which by separation of variables gives $y\,\mathrm{d}y = \mathrm{d}t$, or $y = \pm\sqrt{2t + C}$. Thus, for the initial condition corresponding to $C = 0$ (i.e. $y(0) = 0$), both $y = \sqrt{2t}$ and $y = -\sqrt{2t}$ are solutions. Similarly, the first equation has the solutions $y = -(2t/3)^{3/2}$ and $y = (2t/3)^{3/2}$ at $y(0) = 0$; there's even a third, $y = 0$. For a linear first-order differential equation, this can never happen, but for general first-order differential equations, this could happen. This will be discussed again next week.

Example 5.1. Anoher example of a separable equation is $\frac{\mathrm{d}y}{\mathrm{d}x} = x/y$. This is solved by separation of variables: $y\,\mathrm{d}y - x\,\mathrm{d}x$¡ so $y^2/2 = x^2/2 + C$, or $y^2 - x^2 = C$. The direction field is given by Figure 3, and the solution can move on level sets of $y^2 - x^2$.

This illustrates the more general example. that for any separable equation $A(x)\,\mathrm{d}x = B(y)\,\mathrm{d}y$, the solution becomes $\int A(x)\,\mathrm{d}x - \int B(y)\,\mathrm{d}y = C$, so the solution moves on level sets of $\int A(x)\,\mathrm{d}x - \int B(y)\,\mathrm{d}y$.

Example 5.2. Consider a body falling towards Earth from a significant distance, with force proportional to $1/r^2$. Then, the acceleration is c/r^2, or $\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{c}{r^2}$. This can be rewritten into a first-order equation as follows: let v be the object's velocity, so that $v' = r''$. Then, $\frac{\mathrm{d}v}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}r}v$, so $\frac{\mathrm{d}v}{\mathrm{d}r}v = -\frac{c}{r^2}$. Now, this is a first-order separable equation, and can be solved by separation of variables, integrating to $v^2/2 = c/r + C$, where C is some constant.

This has a physical interpretation: $v^2/2 - c/r$ is constant, and this represents the total energy of the system: the former term is the kinetic energy, and the latter is the potential energy. Thus, the total energy is conserved.

In the general case, a separable system which moves on level sets of $\int A(t)\,\mathrm{d}t - \int B(y)\,\mathrm{d}y$ tends to represent some physically conserved quantity when the differential equation is used to model something physical. This leads to the notion of exact equations, given in Section 2.5 of the book:

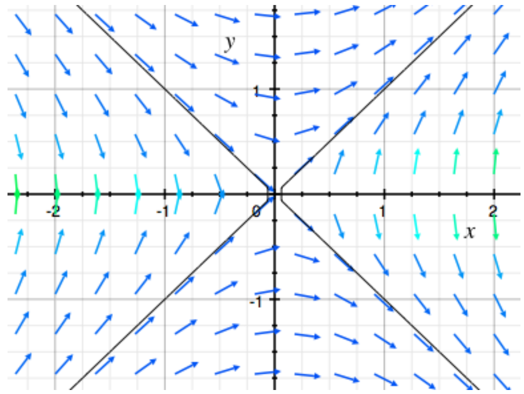


FIGURE 3. Direction field for the equation $y' = x/y$.

Definition. An exact equation is an equation for which one can find a "conserved quantity," even if it isn't separable.

A large class of inseparable equations have some relatively obvious quantity that remains constant.

Example 5.3. This is a pretty contrived example, but shoud help illustrate the idea: $y' \sin x + y \cos x + 2x = 0$. This is a linear equation, but it's not separable, so one could solve it with an integrating factor. But there'a another, simpler way: the first two terms look like the derivative of a product $y \sin x$, and the latter is the derivative of x^2. Thus, the equation becomes $\frac{\mathrm{d}}{\mathrm{d}x}(y \sin x + x^2) = 0$. Thus, $y \sin x + x^2$ is a constant, and from there it is easy to solve for y: $y = (C - x^2)/\sin x$.

Next time, we will discuss precisely how and when this sort of solution works.

6. Exact Equations: 4/12/13

With the techniques now developed, it will be possible to solve the pendulum equation given in Example 1.1. Recall that the differential equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell} \sin\theta = 0, \tag{2}$$

where g is the acceleration due to gravity. This formula is due to calculating the forces and accelerations.

This is a second-order equation, but can be broken into two first-order equations and thus solved: let $v = \frac{\mathrm{d}\theta}{\mathrm{d}t}$, so that $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}v$. Thus, the equation has simplified to

$$v\frac{\mathrm{d}v}{\mathrm{d}\theta} + \frac{g}{\ell}\sin\theta = 0,$$

which is a separable first-order equation. Integrating,

$$\begin{aligned} \frac{v^2}{2} &= \frac{g}{\ell}\cos\theta + C \\ \Longrightarrow \frac{v^2}{2} - \frac{g}{\ell}\cos\theta &= C. \end{aligned}$$

The left-hand side of the above equation is a conserved quantity, and is proportional to the energy. Then, substituting in the initial condition $\theta = \theta_0$, one can calculate that $C = -g\cos\theta_0/\ell$, which when plugged back into the equation gives

$$\begin{aligned} \frac{v^2}{2} &= \frac{g}{\ell}(\cos\theta - \cos\theta_0) \\ v &= \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_0)}. \end{aligned}$$

Thus, v is maximum when $\theta = 0$ and is 0 when $\theta = \theta_0$. However, in order to solve for θ there's another equation to solve, though once again it's separable:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_0)}$$
$$\implies \sqrt{\frac{\ell}{2g}}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \mathrm{d}t$$

There is no closed form for $\int \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - a}}$; such expressions are elliptic integrals. However, the period can still be calculated: the time it takes the pendulum to go from angle θ_0 to 0 is

$$\sqrt{\frac{\ell}{2g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

Thus, the total period is four times that. This is ugly, but works well for numerical approximations and such. However, when θ_0 is small, the elliptic integral becomes approximately $\pi/\sqrt{2}$, so the period is about $2\pi\sqrt{\ell/g}$. Of course, even the "exact" formula doesn't account for things like air resistance. The approximation can also be seen by taking $\sin\theta \approx \theta$ for small θ in the original equation (2), and then solving the resulting linear equation.

An exact equation is an equation $y' = f(y, t)$, such that the goal is to find a function $\psi(y, t)$such that if $y(t)$ is a solution of the equation, then $\psi(y(t), t)$ is constant. Exact equations, discussed in Section 2.5 of the book, are those for which it is easy to find such a ψ, and include all separable equations.[3]

Example 6.1. Take $2t + y + (x + 3y^2)y' = 0$. Then, $\psi(y, t) = y^3 + ty + t^2$, so that any solution $y(t)$ satisfies $y(t)^3 + ty(t) + t^2 = C$, giving an implcit expression for $y(t)$.

Why does this work? Take the derivative:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(y^3 + ty + t^2) = 3y^2\frac{\mathrm{d}y}{\mathrm{d}t} + t\frac{\mathrm{d}y}{\mathrm{d}t} + y + 2t = 0,$$

because that was the differential equation posed at the head of the problem.

In the general case, one has an equation $M(y, t) + N(y, t)y' = 0$. Then, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then there exists a function $\psi(y, t)$ such that $\frac{\partial \psi}{\partial t} = M$, $\frac{\partial \psi}{\partial y} = N$, and ψ is constant along solutions, so if $y(t)$ is a solution to the differential equation, then $\psi(t, y(t))$ is constant.

Though the specific method of finding these conserved quantities might not be useful, conserved quantities are extrenely important in general.

Returning to Example 6.1, $M = 2t + y$ and $N = t + 3y^2$, so $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial t} = 1$. Thus, such a ψ exists, but the result cited above isn't constructive. Fortunately, it's not too difficult to find ψ: the goal is to find the function whose partial derivatives are M and N. First, integrate $\frac{\partial \psi}{\partial t} = 2t + y$ with respect to t, treating y as a constant of integration. Thus, $\psi = t^2 + yt + C(y)$, since for each value of y, the constant of integration could be different. Similarly, integrating $\frac{\partial \psi}{\partial t} = t + 3y^2$, there's a different constant (function) of integration: $\psi = ty + y^3 + D(t)$. Then, set these expressions equal and solve: $y^3 + D(t) = C(y) + t^2$, so $\psi = ty + y^3 + t^2 + K$, where K is an actual constant of integration.

Alternatively, after integrating the first equation, one could take $\frac{\partial \psi}{\partial y} = t + \frac{\mathrm{d}C}{\mathrm{d}y}$, and solve to obtain C. Note, however, that if the partial derivatives of M and N don't agree, then this won't work, as the two expressions for ψ will be incompatible.

The next example will show that separation of variables is a special case.

Example 6.2. Suppose $A(y)\,\mathrm{d}y + B(t)\,\mathrm{d}t = 0$, or $A(y)y' + B(t) = 0$. Thus, $N(y, t) = A(y)$ and $M(y, t) = B(t)$. Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = 0$. Thus, one can integrate: $\psi = \int A(y)\,\mathrm{d}y + \int B(t)\,\mathrm{d}t = C$ for some constant C, once the above method is used, and this is exactly what separability already told us.

7. More Exact Equations: 4/15/13

Recall the defintion of an exact equation: a first order ODE $M(x.y) + N(x.y)y' = 0$ is exact when $N_x = M_y$ (partial derivatives). Thus, there is a function $\psi(x, y)$ such that $\frac{\partial \psi}{\partial y} = N$ and $\frac{\partial \psi}{\partial x} = M$ and if $y = y(x)$ is a solution to the original

[3]The professor doesn't actually think this is useful, and that it is in the syllabus of ODE courses largely by tradition. Admittedly, it is a powerful technique, but in the vast majority of cases, the equation is separable anyways, and worrying about exactness is needlessly complicated.

ODE, then $\psi(x, y(x))$ is constant. Then, one can sometimes solve for $y(x)$ in terms of x, and even if not, it can be seen implicitly.

Example 7.1. Suppose $(2y + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + (3x^2 + 2xy) = 0$. Then, $N = 2y + y^2$ and $M = 3x^2 + 2xy$. First, it is necessary to check for exactness: $\frac{\partial N}{\partial x} = 2x = \frac{\partial N}{\partial y}$, so we can proceed. The goal is to find a ψ such that $\frac{\partial \psi}{\partial y} = 2y + x^2$ and $\frac{\partial \psi}{\partial x} = 3x^2 + 2xy$.

Integrating the first equation with respect to y, $\psi = y^2 + x^2 y + C(x)$ for some function C of x, and integrating the second equation, $\psi = x^3 + x^2 y + D(y)$. Comparing these, $C(x) = x^3$ and $D(y) = y^2$, so $\psi(x, y) = x^3 + x^2 y + y^2$.[4]

Thus, if $y(x)$ is a solution, then $\psi(x, y(x))$ is constant, so suppose $y(0) = 1$, so $y(x)^2 + x^2 y(x) + x^3 = 1$. This can be solved explicitly as $y = (-x^2 + \sqrt{x^4 - 4x^3 + 4})/2$.

Why does this procedure work? If $Ny' + M = 0$ and the partials agree, it's necessary to check that the ψ defined above always exists, and that $\psi(x, y(x))$ is constant for every solution of the equation. Notice that for any twice-differentiable function g, $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$, so that order in evaluating partial derivatives doesn't matter. Thus, if ψ exists, then $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}$ and $\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial M}{\partial y}$, so the partials agree and $\frac{\partial N}{\partial x} = \frac{\partial N}{\partial y}$. Then, if $\psi(x, y(x))$ is constant, then $\frac{\mathrm{d}}{\mathrm{d}x}\psi(x, y(x)) = 0$. This requires using the two-variable chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}$$

where $f(u, v)$ and u and v are functions of x. Then, for ψ,

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = M + N\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

by the given differential equation. Finally, it is necessary to establish why such a ψ should exist. See Theorem 2.5.1 in the textbook for a more detailed discussion; the gist of it is that the integration method given always works: given N and M, let $\psi = \int N\,\mathrm{d}y + A(x)$, so that

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(\int N\,\mathrm{d}y + A(x)\right) = \int \frac{\partial N}{\partial x}\,\mathrm{d}y + \frac{\mathrm{d}A(x)}{\mathrm{d}x} = \int \frac{\partial M}{\partial y}\,\mathrm{d}y + \frac{\mathrm{d}A}{\mathrm{d}x} = M(y) + B(x) + A'(x).$$

Thus, one can choose $A(x)$ such that $A'(x) + B(x) = 0$, so that $\frac{\partial \psi}{\partial x} = M(y)$ as needed.

Example 7.2. Suppose $(2y/x + x)y' + (3x - 2y) = 0$. This isn't exact: $\frac{\partial N}{\partial x} = 1 - 2y/x^2$ and $\frac{\partial M}{\partial y} = 2$. However, after multiplying by x, the equation from Example 7.1 is obtained, and the eqation can be solved.

In general, multiplying by an integrating factor can lead to solutions to more equations. The book has a more thorough discussion on this.

One can also talk about numerical solutions. This is covered in Section 2.6 of the book, and goes into quite a bit of detail. Most of it won't be necessary for this class.

Example 7.3. Let $y' = y^2$ and $y(0) = 1$. This can be solved exactly by separation of variables, where $y = 1/(1 - x)$. This will be used so that the numerical value can be compared to the actual value: specifically, $y(0.9) = 10$.

Supposing this weren't known, one could use Euler's method to approximate y by the tangent line. $y(0) = 1$, so $y(0, 1) \approx y(0) + y'(0)(0.1) = 1.1$, $y(0.2) \approx y(0.1) + y'(0.1)(0.1) = 1.21$. Continuing in this way, $y(0.9) \approx 4.3$.

This isn't very good; how might it be done better? The most obvious solution is to increase the step size. Using steps of 0.05, the approximation is $y(0.9) \approx 8.28$, and if the step size is 0.001, the approximation is $y(0.9) \approx 9.78$. This is actually close, but the tradeoff is that a lot more has to be calculated.

Thankfully, there are much better methods to approximate equations (e.g. using the tangent line in the midpoint of an interval).

8. Numerical Methods and Existence and Uniqueness: 4/17/13

Return again to the equation $y' = y^2$, with $y(0) = 1$. The exact solution is $y(x) = 1/(1 - x)$, but if this weren't known, how could it be approximated? Euler's method was discussed in the previous lecture, but as shown, since each aproximation depends on the previous one, this method becomes quite inacurate after repeated approximation. This can be made better by reducing the step size. For exams, it's helpful to know the basic idea, but in practice, there are much better methods (which won't be on the exam).

[4] Alternatively, after $\psi = y^2 + x^2 y + C(x)$ is obtained, one could differentiate with respect to x: $\frac{\partial \psi}{\partial x} = 2xy + C'(x) = 3x^2 + 2xy$, so $C'(x) = 3x^2$ and $C(x) = x^3$. The constant term will be irrelevant.

Numerical methods for solving differential equations are closely related to numerical methods for integration. For example, if a function is given as a collection of points, its integral can be approximated by summing the areas of rectangles at the points of the function. To improve this approximation, the size of the rectangle should be reduced. But with almost no extra work, one could draw trapezoids between the x-axis and two points on the graph, which provides a much better approximation. This is analogous to an improvement of Euler's method, and there are further refinements (e.g. Simpson's rule) that correspond to even better methods of solving ODEs numerically.

There are many variations on this method, but the general idea is always the same: instad of using $y'(0)$ (for example) as the slope, use the average of $y'(0)$ and $y'(1)$. Thus, the approximation as a whole is given as

$$y(0.1) \approx y(0) + (0.1)\left[\frac{y'(0) + y'(0.1)}{2}\right].$$

Using this method, $y(0.9) \approx 8.13$, which is much better than the conventional Euler method. Then, decreasing the step size to 0.01 gives an approximation of 9.96, and to 0.001 gives a step size of 9.9996. This is an enormous accuracy improvement for relatively little work. Further improvements (e.g. the Runge-Kutta methods) exist, but won't really be discussed here. These methods are usually more accurate for a given step size, but the formulas are more complicated.

Numerical approximation is a fine way to solve differential equations, but use with care: the step size must be chosen appropriately and errors tend to accumulate.

Moving to existence and uniqueness (section 2.3 of the book), suppose one has a first-order ODE $y' = f(t, y)$ with some initial data $y(t_0) = y_0$. Does a solution $y(t)$ exist for all t, and is it unique? In general, this is not the case: taking $y' = y^2$ and $y(0) = 1$, the solution was shown to be $y(x) = 1/(1-x)$, which goes to infinity in finite time, and if $y' = 2\sqrt{y}$, then $y(x) = 0$ and $y(x) = x^2$ are both solutions satisfying $y(0) = 0$. The first caseis concerning when making a model: infinity doesn't happen in the real world, so the model probably needs to be refined.

Here are some important facts about existence and uniqueness:

(1) For a linear, first-order differential equation, solutions always exist, and are always unique.
(2) More generally, if $y' = f(y, t)$ and $y(t_0) = y_0$, a solution always exists for a short time: there exists a solution $y(t)$ defined for $t_0 \le t \le t_1$ for some $t_1 > t_0$. Globally, it's not clear what might happen, but at least locally a solution exists.[5]
(3) If $y' = f(t, y)$ *and* $\frac{\partial f}{\partial y}$ *is continuous*, then the solution through $y(t_0) = y_0$ is unique within some interval $[t_0, t_1]$, where $t_1 > t_0$. In the book, this is Theorem 2.3.2.

The example for which uniqueness fails has $\frac{\partial f}{\partial y} = 1/\sqrt{y}$, which isn't continuous at 0, so the above condition doesn't apply.

While formal proofs won't be given, some intuition and reasoning for items 1 and 2 cam be given, and the latter is particularly interesting (though not all that useful in practice).

Notice that 1 has already been shown: using integrating factors, any linear first-order differential equation has already been solved, giving a unique formula. For 2, an approximation method will be used: a solution will be guessed, and then iteration will converge to the actual solution. This is called Picard iteration: given $y' = f(y, t)$, integrate from t_0 to $t = x$ to obtain $y(x) - y_0 = \int_{t_0}^{x} f(y(t), t)\,\mathrm{d}t$, giving the next approximation as

$$y_1(t) = y_0 + \int_{t_0}^{x} f(y, t)\,\mathrm{d}t.$$

Then, one can repeat, setting y_2 on the left-hand side and $y_1 = y$ on the right-hand side. This will converge to a solution.

9. Review of Weeks 1 to 3: 4/19/13

First, recall some terminology: there are lots of classes of differential equations, including linear, first-order, autonomous, etc. It would be useful to know what these are.

Here are some methods for solving equations:

- Consider a first-order linear differential equation with constant coefficients, such as $y' = y + 1$. These have explicit solutions given by a formula. Notice they are both linear and separable (see below), and can be solved in either way.
- Linear first-order differential equations, such as $\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 1$, can be solved by integrating factors.

[5]Technically, it's necessary for f to be well-behaved: specifying that it's differentiable will do. This is also true in the linear case, but in practice discontinuities wouldn't make any sense.

Example 9.1. Suppose $y' = y/x+$. Since a linear differential equation looks like $y' = p(x)y + q(x)$, this is linear, and can be solved with integrating factors. Intutively, this can be thought of as a model in which $p(x)$ is some sort of interest-like quantity, and $q(x)$ represents some outside change. This isn't essential, but having the model is useful for intuition.

This can be solved by rewriting t as $y' - p(x)y = q(x)$ and multiplying by the integrating factor $e^{-\int p(x)\,\mathrm{d}x}$. Then, this can be integrated: the integrating factor is $M(x) = e^{-\int 1/x\,\mathrm{d}x} = e^{-\log x} = 1/x$, so

$$\begin{aligned}\frac{1}{x}y' - \frac{y}{x^2} &= \frac{1}{x}\\ \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \frac{1}{x}\\ \frac{y}{x} &= \log x + C\\ \implies y &= x\log x + Cx.\end{aligned}$$

- Separable equations, such as $y' = y/x$, can be solved by separation of variables.

  Example 9.2. Suppose $y' = x/y$. Then, the idea in separation of variables is to move all x terms to one side and all y terms to the other. This implies that $y\,\mathrm{d}y = x\,\mathrm{d}x$, which can be integrated:

$$\begin{aligned}\int x\,\mathrm{d}x &= \int y\,\mathrm{d}y\\ \frac{y^2}{2} &= \frac{x^2}{2} + C\\ y^2 &= x^2 + 2C\\ y &= \pm\sqrt{D + x^2},\end{aligned}$$

  where $D = 2C$ is a constant of integration determined by the initial conditions (e.g. if $y(0) = 3$, then $D = 9$ and the sign of the square root is forced: $y = \sqrt{9 + x^2}$).

  On the midterm, you may be asked to integrate things using partial fractions, integrating by parts, etc., but nothing overly tedious or tricky will be necessary.

- Separable equations can be further generalized to exact equations: $M + Ny' = 0$, where $\frac{\partial M}{\partial X} = \frac{\partial N}{\partial y}$. This implies there is a function $\psi(x, y)$ such that $\frac{\partial\psi}{\partial x} = M$ and $\frac{\partial\psi}{\partial y} = N$. ψ can be found by integrating, and leads to the solutions: any solutiion of the original differential equation $y = y(x)$ satisfies $\psi(x, y(x))$ is constant.

  Example 9.3. Suppose $2xy^4 + \cos x + (4x^2y^3 + \cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. First, this should be checked for exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 8xy^3$, so this is in fact exact.

  Then, ψ can be found by integrating: $\frac{\partial\psi}{\partial x} = M$, so $\psi = x^2y^4 + \sin x + f(y)$. Then, this can be plugged into $\frac{\partial\psi}{\partial y} = N$: $\frac{\partial\psi}{\partial y} = 4x^2y^3 + f'(y) = N = 4x^2y^3 + \cos y$. Thus, $f(y) = \sin y + C$, so $\psi(x, y) = x^2y^4 + \sin x + \sin y$, so if $y = y(x)$ is a solution to the original differential equation, $x^2y^4 + \sin x + \sin y$ is constant, and the specific value is determined by the initial data.

  Sometimes, an explicit solution in terms of y can be given; the above example doesn't allow that, however.

Equations can be graphed as a slope field: if $y' = f(y, x)$, one can visualize solutions by following the slope field. A special case happens for autonomous equations, where f is independent of x, and a graph can be visualized by just graphing $f(y)$. This leads to the idea of equilibrium solutions being stable or unstable. Since computers are so good at graphing, there's no reason to actually do this by hand, but the important concept is that solution trajectories will follow these direction fields. For an autonomous equation, looking at $f(y)$ indicates where the solution is increasing or decreasing.

An equilibrium solution, corresponding to a zero of an autonomous equation, is a constant solution to the equation (e.g. $y(t) = k\pi$ for $k \in \mathbb{Z}$ when $y'(x) = \sin x$). Here, $y = 0$ is an unstable solution: a small perturbation leads to different long-term behavior, but $y = \pi$ is stable: nearby solution curves are pulled in.

Sometimes, a solution may be neither stable nor unstable, in which case it is called semistable. If $y'(x) = x^2$, then $y = 0$ is semistable: solution curves below it converge to it, but those above it diverge.

As for numerical solutions, there is Euler's method, which approximates a solution by its tangent line, and several better approximations, but only Euler's method will be tested.

Existence and uniqueness: for linear, first-order equations, solutions exist and are unique given some initial data. In the general case, however, all that is known is that solutions exist locally and they might not be unique. Relatedly, y can go to infinity in a finite time.

10. Systems of Differential Equation: 4/22/13

For the next three weeks, this course will look at systems of differential equations. An example of this is $\frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = y - x$. Given $x(0) = 3$ and $y(0) = 1$, what are $x(t)$ and $y(t)$? For another example, suppose $\frac{\mathrm{d}x}{\mathrm{d}t} = x + y + z$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y - z$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = x$. This is a system of three first-order ODEs, and since each influences the other, all three must be solved simultaneously. It turns out that solving systems of ODEs is not much harder than solving single equations, and several previously discussed techniques still apply. Terms such as order, linear, etc., still apply, and are generalized in the straightforward way to multiple variables.

Example 10.1. Consider the double pendulum, where one rod can swing at the bottom of another rod, which swings from a fixed point. This is a complicated system, and can exhibit some interesting behavior.

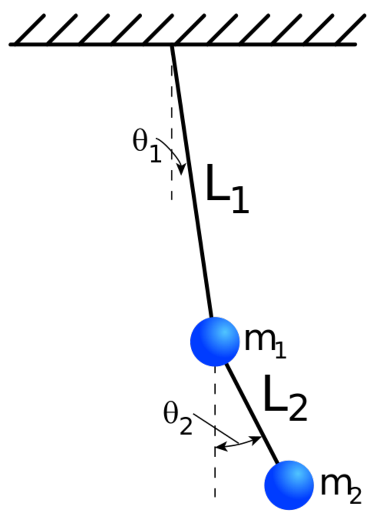


FIGURE 4. A double pendulum. Source

Recall that the single pendulum equation is $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g\sin\theta/\ell = 0$, and for small θ, $\sin\theta \approx \theta$. The precise equation is very long and complicated, but the small-angle approximation gives

$$\begin{aligned}\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \frac{g}{\ell}(2\theta_1 - \theta_2) &= 0\\ \frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} + \frac{g}{\ell}(2\theta_2 - \theta_1) &= 0.\end{aligned}$$

This is an example of systems of differential equations arising from interacting physical systems, which is a common source.

Any second-order equation can be replaced by a system of two first-order equations, and more generally, any n^{th}-order equation can be replaced by n first-order equations. The general procedure can be illustrated by an example:

Example 10.2. Suppose that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0. \tag{3}$$

Let $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, so $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -v - xy$ after substituting in from (3). Thus, the following system of first-order equations is obtained for y and v in terms of x:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -v - xy \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = v.$$

This could seem confusing, but becomes more clear after some examples.

Similarly, for higher-order equations, one makes substitutions for higher-order derivatives. If $y'' + y' + y - x = 0$, then one makes the substitution $v = y'$ and $u = y''$ to obtain a system of three first-order equations.

Just as the general form for a first-order ODE is $y' = f(x, y)$, the general form of a system of two first-order equations is $\frac{\mathrm{d}u}{\mathrm{d}x} = f_1(x, u, v)$, and $\frac{\mathrm{d}v}{\mathrm{d}x} = f_2(x, u, v)$. Similarly, a first-order linear equation looks like $y' = p(x)y + q(x)$, and in the case of two equations is generalized to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = a(x)u + b(x)v + e(x)$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = c(x)u + d(x)v + f(x).$$

Example 10.3. Consider the motion of the Earth around the Sun. From $F = ma$, one obtains

$$\frac{\mathrm{d}^2x}{\mathrm{d}t} = \frac{-Cx}{(x^2+y^2)^{3/2}} \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t} = \frac{-Cy}{(x^2+y^2)^{3/2}},$$

where the Sun is at the origin and the Earth traces out the trajectory $(x(t), y(t))$. This is a system of nonlinear, second-order differential equations, and is kind of complicated. Another example of where systems arise is when an object has more than one degree of freedom.

Example 10.4. Suppose one has a chemical reaction $A + B \to C$. Let $a(t)$ be the amount of molecule A present at time t, and similarly for $b(t)$ and B, and $c(t)$ and C. A typical rate law for the reaction is $\frac{\mathrm{d}a}{\mathrm{d}t} = -Ca(t)b(t)$, and $\frac{\mathrm{d}b}{\mathrm{d}t} = -Ca(t)b(t)$ as well,[6] giving a system of first-order, nonlinear dfferential equations. This is a nuance: products of the two dependent variables are not allowed in a system of linear equations. This equation can actually be solved by converting it to a single equation and then using separation of variables.

An important way to understand these systems is to think of them in the linear-algebraic sense: if $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y + t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y + t^2$, then this can be rewritten in matrix notation: $\frac{\mathrm{d}V}{\mathrm{d}t} = PV + Q$, where the product is matrix multiplication, and $V(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and $Q = \begin{bmatrix} t \\ t^2 \end{bmatrix}$. Using this formalism, many of the previous solution methods will apply, but with scalars replaced with matrices. This can be seen by explicitly computing $PV + Q$, which will evaluate to

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 2x + y + t \\ x + 2y + t \end{bmatrix},$$

which is the same system, but written in a way that looks much more like the previous examples.

Example 10.5. Suppose $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x = 2y$. This system can be solved from the matrix viewpoint. First, a sustitution will be made: though it looks like it comes from nowhere, it will be explained in a future lecture. It has to do with eigenvectors, but don't worry about that yet; it will make sense. Let $x = u + v$ and $y = u - v$, which gives

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}v}{\mathrm{d}t} = 2(u + v) + (u - v) \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} - \frac{\mathrm{d}v}{\mathrm{d}t} = (u + v) + 2(u - v).$$

Then, u and v can be obtained, and the reverse substitution can be made in the end. Then, the equations simplify, $\frac{\mathrm{d}u}{\mathrm{d}t} = 3u$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = v$ after solving. Now, these two equations are completely independent, and can be solved separately: $u(t) = C_1e^{3t}$ and $v(t) = C_2e^t$. Thus, $x(t) = C_1e^{3t} + C_2e^t$ and $y = C_1e^{3t} - C_2e^t$.

11. Review of Linear Algebra: 4/24/13

In order to solve systems of linear equations, eigenvalues and eigenvectors are necessary to obtain good coordinates for the system, so they will be reviewed.

Let A be a 2×2 matrix. Everything will generalize nicely to $n \times n$, but this will have easier intuition.

Definition. The determinant of a matrix is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

This indicates how the area of a region changes under action by A: if $R \subset \mathbb{R}^2$ and $A(R) = R'$, then $\operatorname{area}(R') = |\det A| \operatorname{area}(R)$. This formula will not be used much in this course, but it's important in order to understand what the determinant does. Here are some facts about the determinant:

- If $\det A \neq 0$, then A is invertible; that is, there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

[6] There are plenty of reactions where more complicated rate laws exist, because of some sort of intermediate process. However, the equations given above are correct for many reactions.

- if $\det A = 0$, then A has a null space; that is, there is a nonzero vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $A\mathbf{v} = \mathbf{0}$.

Definition. A vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is called an eigenvector of A if $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, or $A\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Then, λ is called an eigenvalue of A.

Example 11.1. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3, because

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}.$$

Additionally, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Eigenvectors are important in differential equations because every eigenvector of A gives a solution to the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$. Specifically, let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an eigenvector of A with eigenvalue λ. Then, $x = v_1 e^{\lambda t}$ and $y = v_2 e^{\lambda t}$ is a solution to the indicated differential equation. One can also think of this as $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{v}e^{\lambda t}$.

Example 11.2. When $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $x = e^{3t}$ and $y = e^{3t}$ is a solution, by Example 11.1

Why does this method work? Set $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{v}e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$, and it will be shown that it satisfies the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} \lambda v_1 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} \end{bmatrix} = \lambda e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = A\mathbf{v}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t}.$$

Thus, the equation is satisfied. These turn out to give all possible solutions, as will be seen later.

Returning to the review of linear algebra, suppose λ is an eigenvalue of A. Then, $A\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v}$, so $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Then, take the determinant: $\det(A - \lambda I) = 0$ is a quadratic polynomial in λ whose roots yield the eigenvalues of A. This is called the characteristic polynomial of A.

In the case of the matrix A from Example 11.1, $\det(A - \lambda I) = (2 - \lambda)^2 - 1$. Thus, $\lambda = 1$ and $\lambda = 3$ are the two eigenvalues.

There are some wrinkles in this solution process: the eigenvalues could be complex or repeated. This is fine, but will have to be dealt with later.

Assuming these issues don't pop up, the eigenvectors can be found: if $A\mathbf{v} = \lambda_1\mathbf{v}$, then $(A - \lambda_1 I)\mathbf{v} = 0$, so the eigenvectors are the null space of $A - \lambda_1 I$.

For the matrix A we've been working with, an eigenvector for $\lambda_1 = 3$ is in the null space of $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, so $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ works, as does any scalar multiple of it.

Now, it is possible to obtain the general solution to $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$. This is the analogue to the linear equation with constant coefficients. This will correspond to Theorems 3.3.1 and 3.3.2 in the book. For now, assume the eigenvalues of A are distinct: $\lambda_1 \neq \lambda_2$, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors. Then, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_1 e^{\lambda_2 t}\mathbf{v}_2,$$

where c_1 and c_2 are constant determined by the initial conditions. Note that λ_1 and λ_2 might be complex, and that their corresponding eigenvectors also might have complex entries.

It turns out that all solutions of the equation come from these eigenvalues. First observe that the solution can be rewritten as

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

for some constants b_1 and b_2. However, this involves taking the exponential of a matrix... which is in fact a perfectly legitimate operation.

Definition. If M is a square matrix, then its exponential can be given by two equivalent definitions:

$$e^M = \lim_{n\to\infty}\left(I + \frac{M}{n}\right)^n = 1 + M + \frac{M^2}{2} + \cdots$$

This is advantageous because it allows the solution to be rewritten so that it looks like the single-variable case. However, it requires computing the matrix exponential, which in practice boils down to the same eigenvalue and eigenvector calculations as before.

12. Systems of Linear Equations: 4/26/13

Recall that the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$ has the general solution $\begin{bmatrix} x \\ y \end{bmatrix} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$, where λ_1 and λ_2 are the distinct eigenvalues of A and $\mathbf{v}_1$ and $\mathbf{v}_2$ are their corresponding eigenvectors. However, this compact solution contains a lot of information: $\lambda_1, \lambda_2 > 0$ looks very different to when they have opposite sign, or when they are complex, etc.

The derivation can be understood by setting $U = [\mathbf{v}_1\ \mathbf{v}_2]$, and make the substitution $\begin{bmatrix} x \\ y \end{bmatrix} = U\begin{bmatrix} u \\ v \end{bmatrix}$. Then, the equation becomes $\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda_1 u$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \lambda_2 v$, which is easy to solve. This is because $AU = [A\mathbf{v}_1\ A\mathbf{v}_2] = [\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2]$.

Example 12.1. If $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x + 2y$, then the general solution is (as solved in the previous lecture) $x = c_1 e^{3t} + c_2 e^t$ and $y = c_1 e^{3t} - c - 2e^t$. See Figure 5 to see the behavior of this system.

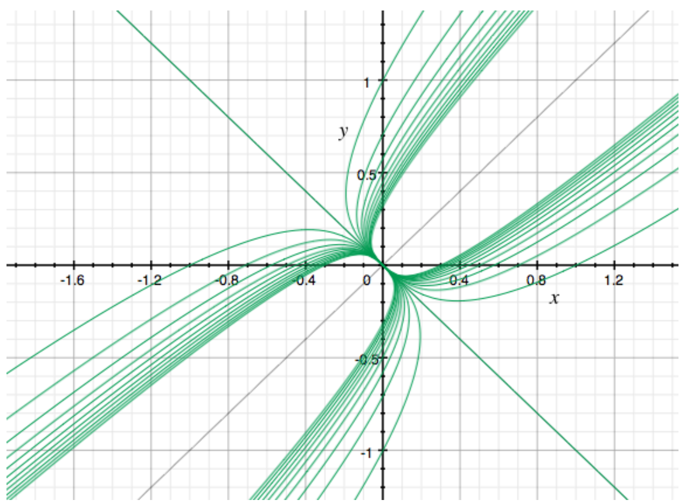


FIGURE 5. Solutions to $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x + 2y$. Equations with two positive eigenvalues all look more or less like this: solutions come from the direction of the eigenvector corresponding to the smallest eigenvalue (in this case, the direction $y = -x$) and leave in the direction of the eigenvector corresponding to the greater eigenvalue.

Example 12.2. If the eigenvalues have opposite signs, the graphs look completely different. Take $y' = -x$ and $x' = -y$, so that $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, with charactersistic polynomial $\lambda^2 - 1$. Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, corresponding respectively to eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, the general solution is $x = c_1 e^t + c_2 e^{-t}$ and $y = -c_1 e^t + c_2 e^{-t}$.

This can actually be solved by separation of variables, dividing the equations to obtain $y' = c/y$, or $y^2 - x^2 = C$. This is the same solution set, but stated implicitly rather than with parameters.

Initial value problems are a little more interesting in the multivariable case. Since there will be multiple pieces of initial data and multiple constants to solve for, there will be a system of linear equations to solve. This is not difficult, but one should be aware of it.

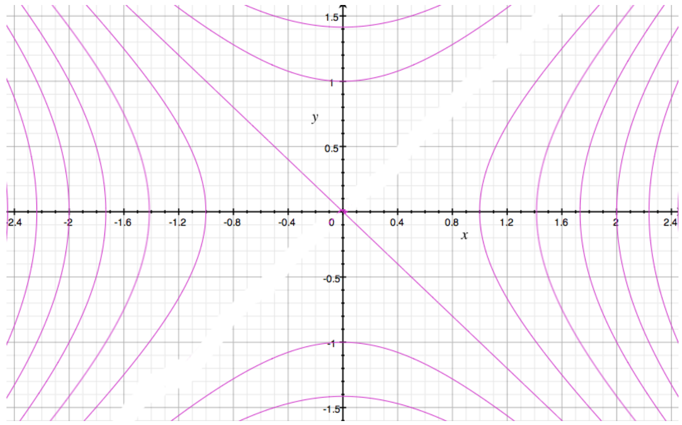


FIGURE 6. Solutions to an equation with one positive and one negative eigenvalue. They move from the eigenvector with the negative eigenvalue (here in the direction $y = x$) to the direction of the eigenvector of the positive eigenvalue ($y = -x$ here).

13. Complex Eigenvalues: 4/29/13

First, some small notation: a set of solutions as in Figure 6 are called a saddle, and those in Figure 5 are called a nodal source when both are positive (since all trajectories leave the origin) or a nodal sink when they're both negative (and all trajectories converge to the origin). What happens if 0 is an eigenvalue? Consider $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. Then, the characteristic polynomial is $(2-\lambda)^2 - 4$, so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 0$, corresponding to eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Thus, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c - 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Thus, solutions all run parallel to $\mathbf{v}_1$, heading away from the $\mathbf{v}_2$-axis, unless they are on the $\mathbf{v}_2$-axis: then, they don't move. This can be seen as the limiting behavior of the other cases.

Example 13.1. Eigenvalues can also be complex. Consider the pendulum equation, in which $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\sin\theta/\ell$. This has been solved before, but is a useful example. Suppose θ is small, so that $\sin\theta \approx \theta$. This becomes a system of first-order equations by setting $v = \frac{\mathrm{d}\theta}{\mathrm{d}t}$, so that $\frac{\mathrm{d}v}{\mathrm{d}t} = g\theta/\ell$. Then, the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & -\frac{g}{\ell} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \end{bmatrix},$$

which ends up hving complex eigenvalues: the characteristic polynomial is $\lambda^2 + g/\ell = 0$, so the eigenvalues are $\lambda = \pm i\sqrt{g/\ell}$.

For simplicity, suppose $\sqrt{g/\ell} = 1$, and switch to x and y instead of θ and v. Then, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-it}.$$

This looks a bit fishy, but if the initial conditions are real, then it will stay real for all time: for example, if $x(0) = 1$ and $y(0) = 0$, then $ic_1 - ic_2 = 1$ and $c_1 + c_2 = 0$. This can be solved to show that $c_1 = 1/2i$ and $c_2 = -1/2i$. Thus, the complex numbers cancel out in this equation: specifically, $x(t) = (e^{it} + e^{-it})/2 = \cos t$ and $y(t) = (e^{it} - e^{-it})/2i = \sin t$.

This seems like it would make no sense, but $e^{it} = \cos t + i\sin t$, so $e^{-it} = \cos t - i\sin t$, so $e^{it} + e^{-it} = 2\cos t$ and $e^{it} - e^{-it} = 2i\sin t$.[7]

[7] See also: this link.

This gives another way to write the general solution: if λ_1 is complex and corresponds to eigenvalue $\mathbf{v}_1$, then in some sense these contain all of the necessary information for the other eigenvalue: the general solution can be written $c_1 \operatorname{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + c_2 \operatorname{Im}(\mathbf{v}_1 e^{\lambda_1 t})$. Both equations are valid for any equation, but the constants will be different.

In Example 13.1 discussed above, this second form can be found as

$$\mathbf{v} - 1e^{\lambda_1 t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} = \begin{bmatrix} i(\cos t + i\sin t) \\ \cos t + i \sin t \end{bmatrix} = \begin{bmatrix} -\sin t + i\cos t \\ \cos t + i\sin t \end{bmatrix},$$

so the general solution is $c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ sint \end{bmatrix}$.

14. Complex Exponentials: 5/1/13

Recall that if the eigenvalues of A are complex (if one is, then the other is), then the general solution to $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \operatorname{Re}(\mathbf{v}_1 e^{\lambda_1 t}) + c_2 \operatorname{Im}(\mathbf{v}_1 e^{\lambda_1 t})$. These solutions tend to have trigonometric functions in the solution, and in some sense they oscillate. Also, note the lack of λ_2 or $\mathbf{v}_2$ in the solution: their information can be extracted from λ_1 and $\mathbf{v}_1$.

Since the complex exponential comes up so often, it's worth reviewing. The goal is to understand that $e^{it} = \cos t + i\sin t$, which is called Euler's formula. The picture is that e^{it} lives on the unit circle in the complex numbers. For a plausibility argument, recall that $\cos(x+y) = \cos x\cos y + \sin x \sin y$ and $\sin(x+y) = \sin x\cos y + \sin y\cos x$. Thus, if $g(x) = \cos x + i\sin x$, so

$$\begin{aligned} g(x)g(y) &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &- (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y) \\ &= g(x+y). \end{aligned}$$

There are several proofs of this formula, including a geometric one.

Proof 1. Set $g(x) = \cos x + i\sin x$ as before and $f(x) = e^{ix}$. Then, $\frac{\mathrm{d}f}{\mathrm{d}x} = ie^{ix} = if(x)$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = -\sin x + i\cos x = ig(x)$. Thus, both f and g satisfy the differential equation $y' = iy$. Additionally, they have the same initial conditions: $f(0) = g(0) = 1$. Thus, by uniqueness of differential equations, $f(x) = g(x)$. ⊠

The second proof is more historical, and is due to Euler:

Proof 2. Write out the Taylor expansions of f and g and of $\cos x$ and $\sin x$: $e^x = \sum_{n=0}^\infty x^n/n!$, so

$$e^{ix} = \sum_{n=0}^\infty \frac{i^n x^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^{2n}x^{2n}}{(2n)!} + i\sum_{n=0}^\infty \frac{(-1)^{2n+1}x^{2n+1}}{(2n+1)!} = \cos x + i\sin x.$$ ⊠

Example 14.1. Suppose $A = \begin{bmatrix} 8 & -5 \\ 2 & 1 \end{bmatrix}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, with the initial condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This will be a more representative example of the case of complex eigenvalues than Example 13.1. Then, the characteristic polynomial is $\lambda^2 - 4\lambda + 13 = 0$, so $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$. Notice that the eigenvalues are complex conjugates of each other: $\lambda_2 = \overline{\lambda_1}$ (recall that $\overline{a+bi} = a - bi$). This is because a quadratic polynomial with one (strictly) complex root has two complex roots that are conjugates of each other, and the determinant is given by a polynomial.

The eigenvectors also exhibit this behavior: after some computation, $\mathbf{v}_1 = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$ (though any multiple of $\mathbf{v}_1$ works, even complex multiples). Since $\lambda_2 = \overline{\lambda_1}$, then $\mathbf{v}_2 = \overline{\mathbf{v}_1}$, so $\mathbf{v}_2 = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$. This is why $\mathbf{v}_2$ and λ_2 don't appear in the general solution. Thus,

$$\mathbf{v}_1 e^{\lambda_1 t} = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix} e^{(2+3i)t} = e^{2t}\begin{bmatrix} (1+3i)(\cos 3t + i\sin 3t) \\ 2(\cos 3t + i\sin 3t) \end{bmatrix},$$

so the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^{2t}(\cos 3t - 3\sin 3t) \\ 2e^{2t}\cos 3t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(3\cos 3t + \sin 3t) \\ 2e^{2t}\sin 3t \end{bmatrix}.$$

Unfortunately, this sort of messiness tends to be inevitable. With the initial conditions, $c_1 = 1$ and $c_2 = 0$, so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{2t}(\cos 3t - 3\sin 3t) \\ 2e^{2t}\cos 3t \end{bmatrix}$. Notice that a real differential equation with a real initial condition has a real solution.

Geometrically, this graph oscillates, but the oscillations in the x- and y-directions are staggered, and the exponential term causes them to increase. This can be rewritten to make the lag clearer: $1 + 3i = \sqrt{10}e^{i\theta}$, where $\theta = \arctan 3 \approx 71°$. This indicates exactly what the lag is.

15. Repeated Eigenvalues: 5/6/13

Example 15.1. Recall the double pendulum equation from Example 10.1, which was given by

$$\begin{aligned} \frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} &= \frac{g}{\ell}(-2\theta_1 + \theta_2) \\ \frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} &= \frac{g}{\ell}(2\theta_1 - 2\theta_2). \end{aligned}$$

This will be solved by breaking it into a system of four first-order equations: let $u = \theta_1'$ and $v = \theta_2'$, and suppose $g/\ell = 1$ for ease of algebra. The four equations are

$$\begin{aligned} \frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= u & \frac{\mathrm{d}\theta_2}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= -2\theta_1 + \theta_2 & \frac{\mathrm{d}v}{\mathrm{d}t} &= 2\theta_1 - 2\theta_2 \end{aligned}$$

which gives the matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \theta_1 \\ \theta_2 \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ u \\ v \end{bmatrix}.$$

All of the theory still works, but there will be four eigenvalues rather than two. In particular, the general solution is given by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ u \\ v \end{bmatrix} = \sum_{i=1}^{4} c_i \mathbf{v}_i e^{\lambda_i t},$$

where $\lambda_1, \dots, \lambda_4$ are the eigenvalues and $\mathbf{v}_1, \dots, \mathbf{v}_4$ are their corresponding eigenvectors.

After some unfortunate algebra, the characteristic polynomial is $\lambda^4 + 4\lambda^2 + 2 = 0$. Thus, the solutions are given by the roots of $-2 \pm \sqrt{2}$, and are thus $\lambda = \pm i\sqrt{2+\sqrt{2}} \approx 1.848i$ and $\lambda = \pm i\sqrt{2-\sqrt{2}} \approx 0.765i$. For general g, ℓ, these are instead $\lambda = \pm i\sqrt{g/\ell}\sqrt{2 \pm \sqrt{2}}$. Thus, the solutions will be given by sines and cosines of $1.848\sqrt{g/\ell}t$ and $0.765\sqrt{g/\ell}t$.

Physically, this means that there are two modes of oscillation with different frequencies $1.848\sqrt{g/\ell}$ and $0.765\sqrt{g/\ell}$, and a typical solution will combine both modes, so it's not as obvious that it is periodic. Compare with the single pendulum, which only has frequencies $\sqrt{g/\ell}$.

In all of the cases mentioned before, there was a caveat that eigenvalues cannot be repeated. What happens if they are?

Example 15.2. Suppose $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, so the eigenvalues are $\lambda = 1, 1$ given by a repeated root. Then, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector, so $c\mathbf{v}e^{\lambda t}$ is a solution, but it's not the completely general solution.

Notice that one of the equations doesn't depend on x. This is no coincidence: in a system of repeated eigenvalues, this tends to happen. Thus, $y(t) = e^t$ if $y(0) = 1$ and $x(0) = 1$, so we also have $\frac{\mathrm{d}x}{\mathrm{d}t} = x + e^t$. This can be solved with the integrting factor e^{-t}, yielding $e^{-t}\frac{\mathrm{d}x}{\mathrm{d}t} - e^{-t}x = 1$, so $e^{-t}x = t + C$. and $x = te^t + e^t$.

There's another way to solve the system: look instead at $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, where $a \approx 1$. This shouldn't have repeated eigenvalues, but will approach the solution we look for when $a \to 1$. Here, the eigenvalues are $\lambda_1 = 1$ and

$\lambda_2 = a$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1-a \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1-a \end{bmatrix} e^t + d \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{at}.$$

With the initial conditions $x(0) = y(0) = 1$, $d = 1 - 1/(1-a)$, which goes to infinity as $a \to 1$. However, c and d will sort of cancel each other out, so that there is a valid solution: when $a \to 1$, $y(t) \to e^t$, which is fine, and then $x(t) = e^{at} + (re^t - e^{at})/(1-a)$, so as $a \to 1$, $x(t) \to e^t + te^{at}/1$.

More generally, suppose A has only one eigenvalue λ and that A is not diagonal (which is an easy case anyways). Then, the general soluton to $\mathbf{x}' = A\mathbf{x}$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{w}te^{\lambda t} + \mathbf{v}e^{\lambda t}$, where $\mathbf{w}$ is an eigenvector of λ and $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{w}$. The apparent lack of constants is because they can be absorbed into the eigenvector.

16. Variation of Parameters I: 5/8/13

Though this has a new name, this is just the use of integrating factors in the context of systems of differential equations.

Up to now, the only systems that have been considered are homogeneous, where there are no terms just in t. An inhomogeneous system is one of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}.$$

Recall that if $\frac{\mathrm{d}x}{\mathrm{d}t} = 3x + e^t$, the integrating factor is e^{-3t}, which can be multiplied in. But one can alternately make a substitution $x = e^{3t}u$, which turns the equation into $\frac{\mathrm{d}u}{\mathrm{d}t} = e^{2u}$. For systems, the exponential in the integrating factor (or substitution) is replaced by a matrix exponential.

To solve a system $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = A\mathbf{v} + \mathbf{f}$, where $\mathbf{v} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$, then a substitution $\mathbf{v} = X(t)\mathbf{u}$, where $X(t)$ is a 2×2 matrix, and then the objective is to solve for $\mathbf{u}$. The goal is to choose X to make life simpler ($X = e^{At}$ will work). Specifically, after substitution, $X(t)\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{\mathrm{d}X}{\mathrm{d}t}\mathbf{u} = AX\mathbf{u} + \mathbf{f}$, since the product rule still holds on matrices. After simplifying, $X\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{f}$, or $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = X^{-1}\mathbf{f}$. Thus, $\mathbf{u} = \int X^{-1}\mathbf{f}\,\mathrm{d}t$, where X and $\mathbf{f}$ both depend on t. Thus, the general solution to $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = A\mathbf{v} + \mathbf{f}$ is

$$\mathbf{v}(t) = X(t) \int X(t)^{-1}\mathbf{f}(t)\,\mathrm{d}t,$$

where $X(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = AX(t)$. In order to actually obtain X, solve $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}$, obtaining two vector solutions $\mathbf{x}_1 = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$. It is necessary that these solutions be independent. Then, $X(t) = [\mathbf{x}_1(t)\ \mathbf{x}_2(t)] = \begin{bmatrix} x_1(t) & y_1(t) \\ x_2(t) & y_2(t) \end{bmatrix}$. Notice that the $\mathbf{f}$ term is ignored in this calculation. The idea here is that if it's possible to solve the equation without $\mathbf{f}$, then it is possible to solve with $\mathbf{f}$ just as easily.

Example 16.1. Suppose

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4}$$

Then, A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$, corresponding to eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. A general solution to the homogeneous equation is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$, so two independent solutions are $\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$ and $\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, so $X = \begin{bmatrix} e^{3t} & e^t \\ e^{3t} & e^{-t} \end{bmatrix}$. Thus, the general solution to (4) is $X(t) \int X(t)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{d}t$, which is just algebra. To specifically obtain $X^{-1}\mathbf{f}$, one could compute the matrix inverse, but a potentially less error-prone method is to solve $X\mathbf{y} = \mathbf{f}$ for $\mathbf{y}$, which is a conventional linear, non-differential system. The inverse of a 2×2 matrix can be given by an explicit formula:

$$B^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det B} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where the determinant is $\det B = ad - bc$. Thus, $\det X(t) = -2e^{4t}$, so

$$X^{-1} = \frac{1}{-2e^{4t}}\begin{bmatrix} -e^t & -e^t \\ -e^{3t} & e^{3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \end{bmatrix},$$

so $X^{-1}\mathbf{f} = \begin{bmatrix} e^{-3t}/2 \\ w^{-t}/2 \end{bmatrix}$. Then, the vetor can be integrated termwise, and $\int X(t)\mathbf{f}\,\mathrm{d}t = \begin{bmatrix} -e^{-3t}/6 \\ -e^{-t}/2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then, the general solution is found by multiplying this by X *again*.

17. Variation of Parameters II: 5/10/13

Example 17.1. Suppose $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, which has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$/ Then, the general solution to $\mathbf{x}' = A\mathbf{x}$ is $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t$. Then, let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$, so that $X(t) = \begin{bmatrix} e^{3t} & e^t \\ e^{3t} & e{-}t \end{bmatrix}$.

Then, to solve the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, variation of parameters can be used. $\det = -2e^{4t}$, so $X(t)^{-1} = \frac{1}{\det X}\begin{bmatrix} -e^t & -e^t \\ -e^{3t} & e^{3t} \end{bmatrix}$, so that $X^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-3t}/2 \\ e^{-t}/2 \end{bmatrix}$, so the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = X(t)\int\begin{bmatrix} e^{-3t}/2 \\ e^{-t}/2 \end{bmatrix} = X(t)\begin{bmatrix} -e^{-3t}/6 + c_1 \\ -e^{-t}/2 + c_2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}.$$

Notice that the general solution for the inhomogeneous equation is the general solution for the inhomogeneous solution plus some additional data.

Alternatively, if $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$, then $X(t)\begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-2t}/2 \\ 1/2 \end{bmatrix}$ (which can also be found by $X\mathbf{y} = \mathbf{f}$ if you don't want to compute X^{-1}), so the general solution is

$$X(t)\begin{bmatrix} -e^{-2t}/4 + c_1 \\ t/2 + c_2 \end{bmatrix} = \begin{bmatrix} -e^t/4 + te^t/2 \\ -e^t/4 - te^t/2 \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}.$$

In order to invert $X(t)$, it's necessary to compute its determinant. This happens to be simpler than in the general case, motivating the idea of a Wronskian.

Definition. If $\mathbf{v}_1(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ and $\mathbf{v}_2(t) = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$ are solutions to the homogeneous equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, then the Wronskian of $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$W(t) = \det\begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix} = ke^{(a+d)t},$$

where k is some constant and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

This is modtly helpful when the differential equation is messy.

Example 17.2. Suppose $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, for which a general solution is $c_1\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$. Then, the Wronskian of $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ and $\begin{bmatrix} \cos t \\ -\sin n \end{bmatrix}$ is $\det\begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} = -\sin^2 t - \cos^2 t = -1$ as predicted.

The reason this makes sense is that the Wronskian satifies the differential equation $W'(t) = (a+d)W(t)$, which is a simple differential equation and yields the formula given above. This is because the determinant is given by $x_1y_2 - x_2y_1$, so using the product rule,

$$\begin{aligned} W'(t) &= x_1'y_2 + x_1y_2' - x_2'y_1 - x_2y_1' \\ &= (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1) \\ &= (a+d)(x - 1y_2 - x_2y_1) = (a+d)W(t), \end{aligned}$$

after some messy algebra and using the fact that $\mathbf{x}' = A\mathbf{x}$. This can be generalized to larger systems, and is even valid if the entries in A depend on t.

Definition. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are two solutions of an equation and $X(t) = [\mathbf{v}_1\ \mathbf{v}_2]$. Then, they are said to be independent if their Wronskian is nonzero.

The following statements are equivalent:

(1) $\mathbf{v}_1$ and $\mathbf{v}_2$ are a fundamental set of solutions.
(2) $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$.
(3) $X(t)$ is invertible for all t.
(4) Any solutin to the equation is of the form $c_1\mathbf{v}_1(t) + c_2\mathbf{v}_2(t)$.

Example 17.3. If A is as in Example 17.2, then $\mathbf{v}_1 = \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$ has Wronskian -1 and is thus a fundamental set of solutions. There are many such sets; take $\mathbf{v}_2$ as before and $\mathbf{v}_1 = \begin{bmatrix}\sin t + \cos t\\ \cos t - \sin t\end{bmatrix}$, so that the Wronskian is still the same. However, $\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$ and $\begin{bmatrix}2\sin t\\ 2\cos t\end{bmatrix}$ is not such a soluton set.

18. Review of Weeks 4 to 6: 5/13/13

Suppose $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x\\ y\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix}$, for some 2×2 matrix that doesn't depend on A. Then, suppose λ_1, λ_2 are distict eigenvalues of A corresponding to eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ respectively, then the general solution is $c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$. This works whether the eigenvalues are real or complex.

If $\lambda - 1 = \lambda_2$ and A isn't diagonal (which is an easy case, because the equations are unrelated), then let $\mathbf{v}_1$ be an eigenvector and $\mathbf{v}_2$ be given by $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$. Then, the general solution is $c_1\mathbf{v}_1te^{\lambda_1 t} + (c_1\mathbf{v}_2 + c_2\mathbf{v}_1)e^{\lambda_1 t}$. Alternatively, this can be given as $\mathbf{u}te^{\lambda_1 t} + \mathbf{w}e^{\lambda_1 t}$, for every eigenvector $\mathbf{u}$ and every solution $\mathbf{w}$ to $A\mathbf{w} - \lambda_1\mathbf{w} = \mathbf{u}$.

If λ_1 and λ_2 are complex, then they are conjugates of each other, as are $\mathbf{v}_1$ and $\mathbf{v}_2$, soa general solution is $c_1\operatorname{Re}(\mathbf{v}_1e^{\lambda_1 t}) + c_2\operatorname{Im}(\mathbf{v}_1e^{\lambda_1 t})$, which can be made more real by taking $e^{i\theta} = \cos\theta + i\sin\theta$.

If $\lambda_1 > \lambda_2 > 0$, then the phase portrait (what happens geometrically), solutions proceed from the direction of the eienvector corresponding to the smaller eigenvalue and proceed to the eigenvector corresponding to the larger eigenvalue. This is known as a nodal source. If both eigenvalues are negative, the trajectories are reversed, and converge at zero. This is called a nodal sink.

If the eigenvalues have opposite signs, the solutions form hyperbolas with the eigenvectors as the asympotes. This is called a saddle. If $\lambda_2 = 0$, then the trajectories are just lines perpendicular to $\mathbf{v}_2$.

In the case of repeated eigenvalues, there are spirals that start pointing in one direction of $\mathbf{v}_1$ and end up pointing in the other. This is called an improper nodal source, and can be thought of as the limit of the nodal source as the eigenvectors appproach each other.

In the case of complex eigenvalues, $\lambda_1, \lambda_2 = \alpha \pm i\beta$. If $\alpha > 0$, then trajectories spiral away from the origin, and the graph is called a spiral source. This corresponds to oscillations of increasing magnitude. if $\alpha < 0$, then one has a spiral sink: the trajectories converge to zero, corresponding to oscilaltions of decreasing magnitude. In order to determine whether the spirals travel clockwise or counterclockwise, it is necessaery to compute a sample direction in the direction field; then, all trajectories will travel in that direction.

For variation of parameters, consider an inhomogeneous equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x\\ y\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}f(t)\\ g(t)\end{bmatrix}$, where A is a 2×2 real matrix. Then, one finds two solutions $\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix}$ and $\begin{bmatrix}x_2(t)\\ y_2(t)\end{bmatrix}$ of the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and take the matrix $X(t) = \begin{bmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{bmatrix}$. The Wronskian of these two vectors is $W(t) = \det(X(t)) = Ce^{(\operatorname{Tr} A)t}$. If the Wronskian is nonzero, then these two solutions are called a fundamental set (i.e. they are independent of each other). Then, the general solution is $X(t) = \int X(t)^{-1}\begin{bmatrix}f(t)\\ g(t)\end{bmatrix}\,\mathrm{d}t$ when the Wronskian is nonzero (so that the inverse makes sense). Once again, there could be complex eigenvalues. The general solution to the homogeneous equation will be part of the answer. As it happens, this method works just as well in the case of repeated eigenvalues.

One convenient formula for X^{-1} is $X^{-1} = (\operatorname{Tr} X)(I - X)/\det X$.

Example 18.1. This example will demonstrate the calculation of the Wronskian of a matrix with repeated eigenvalues: suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which has $\lambda = 1$ as its sole eigenvalue, for which an eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Solving $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$, one obtains $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, the general form of solutions to the homogeneous equation is $c-1\begin{bmatrix} 1 \\ t \end{bmatrix} e^t + \left(c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^t$, so take $c_1 = 1$ and $c_2 = 0$ to obtain $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$ and for the second $c_1 = 0$ and $c_2 = 1$ to obtain $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Thus, the Wronskian is $\begin{vmatrix} te^t & e^t \\ e^t & 0 \end{vmatrix} = -e^{2t}$.

19. Second Order Linear Systems: 5/15/13

Today's lecture will consider second-order, linear equations

$$a(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = d(x).$$

Generall,y the equations will be taken to have a, b, c constant, such as $2y'' + y' + 3y = 0$. More generally, it will be possible to obtain a general solution to $ay'' + by' + cy = 0$, which arises often in oscillating systems.

First, reduce the equation to two fist-order equations: let $v = y'$, so $v' = (-b/a)v - (c/a)y$, leading to a system

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix},$$

which has characteristic polynomial $a\lambda^2 + b\lambda + c = 0$. Notice that this looks like the original equation.[8] The general solution to this system is $y = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{21} e^{\lambda_2 t}$, where λ_1 and λ_2 are the eigenvalues and v_{11} and v_{21} are the first entries of the corresponding eigenvectors as before. The key here is that we can just obtain v by differentiating if we care. Thus, we can just define $d_1 = c_1 v_{11}$ and d_2 in the same way, leading to a simpler solution $y = d_1 e^{\lambda - 1t} + d_2 e^{\lambda_2 t}$. All of the things said about system still apply: if the eigenvalues are complex, the realification still works. But this is a lot nicer because the eigenvectors aren't necessary.

In higher orders, the same thing happens: if $2y''' + 3y' + 4y = 0$, the eigenvalues are the solutions to $2\lambda^3 + 3\lambda + 4 = 0$.

Example 19.1. Recall the pendulum equation, which has approximately the equation $\theta'' + g\theta/\ell = 0$. The general solution is thus $\theta = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, where $\lambda = \pm i\sqrt{g/\ell}$ by solving the polynomial. Thus, the general solution is

$$\theta = Ae^{i\sqrt{g/\ell}t} + Be^{-ie\sqrt{g/\ell}t} = A\cos\left(t\sqrt{\frac{g}{\ell}}\right) + B\sin\left(t\sqrt{\frac{g}{\ell}}\right).$$

Thus, the system oscillates with period $2\pi\sqrt{\ell/g}$.

One can throw in some friction (air resistance), which adds an $a\theta$ term (i.e. there is greater friction when the system is moving faster). Then, the eigenvalues are $\lambda_1, \lambda_2 = (-a \pm \sqrt{a^2 - 4g/\ell})/2$. Thus, there are two cases:

- If $a < 4g^2/\ell$, the system is underdamped (there isn't much friction). Then, the system oscillates with decreasing amplitude, so the origin is a spiral point. The period of this is $2\pi/\sqrt{g/\ell - a^2/4}$.
- If the friction is larger, then the system is overdamped: $a^2 > 4g/\ell$, so the eigenvalues are real. The origin is a nodal sink, and there is no oscilation.
- On the boundary case, the system is critically damped ($a^2 = 4g/\ell$). Then, the origin is an improper nodal sink.

There are lots of other examples: a mass m might move on a spring, which has velocity $v = x'$, so that $mv'' + kx = 0$. One could also model oscillations in time and space. For example, a string vibrates with magnitude $u(x, t)$ at a position x and time t, which satisfies the wave equation (which is a PDE) $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$. Generally, these are unpleasant to handle, but this can be solved by converting it to a system of ODEs. Here, c is a constant representing the stiffness of the string.

This PDE can actually be solved by separation of variables: assume u is the product of functions of x and t: $u(x, t) = f(x)g(t)$. Then, $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f''(x)g(t)$, and similarly for $\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$,, so one can reduce this to a system in f and g.

[8] All of this can be generalized in the obvious way to higher-order equations, leading to larger matrices but the same ideas.

20. Inhomogeneous Second-Order Equations: 5/17/13

Suppose $ay'' + by' + cy = f(t)$, and in particular, we will consider the case where the system oscillates, so $f(t) = Ae^{i\omega t}$, or $f(t) = A\cos\omega t$ when the real part is taken. A shortcut called the method of undetermined coefficients will make this easier than conventional variation of parameters.

Example 20.1. Consider a mass m that moves on a spring, and let $x(t)$ be the displacement of the mass. If the friction is given by a, the stiffness of the spring by k, and a force $f(t)$ is applied to the mass, then the system is described by the system $mx'' + ax' + kx = f(t)$.

The first step of the method of undetermined coefficients first begins by reducing to a system.:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ Ae^{i\omega t}/a \end{bmatrix},$$

so variation of parameters can be used roughly in order to understand what the solution looks like. Specifically, the coefficients on the exponential, terms will be completely ignored, so $X(t) \approx \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{bmatrix}$, so $X^{-1}(t)\mathbf{f} = \begin{bmatrix} e^{-(\lambda_1 + i\omega)t} \\ e^{-(\lambda_2 + i\omega)t} \end{bmatrix}$, and after integrating (which only does stuff with constants anyways), the solutions look loke $\begin{bmatrix} k_1 e^{i\omega t} \\ k_2 e^{i\omega t} \end{bmatrix}$, plus the solution to the homogeneous equation.

Thus, there exists a solutioon of the form $y(t) = Ce^{i\omega t}$.[9] To find C, one can just substitute into the original equation: $y' = i\omega Ce^{i\omega t}$ and $y'' = -\omega^2 Ce^{i\omega t}$, so $Ae^{i\omega t} = (-a\omega^2 + ib\omega + c)Ce^{i\omega t}$, and therefore $C = A/(-a\omega^2 + b\omega + c)$.

This method also requires a reasonable guess at the solution, and the book contain some examples of how one might reasonably guess. For example, if $y'' + 3y + 2 = t^2 - t + 1$, try $y = a_1 t^2 + a_t 2 + a_3$.

Returning to the mass on a spring, if there is no friction and no force, the equation is $x = A\cos(t\sqrt{k/m}) + B\sin(t\sqrt{k/m})$. Set $\omega_0 = \sqrt{k/m}$, the "natural frequency" of the system without friction or forcing. Then, if a and $f(t)$ are added back in, the general solution is $x(t) = Ae^{i\omega t}/(-m\omega^2 + i\omega a + k) + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. If there is friction, the homogeneous part goes to zero as $t \to \infty$, since λ_1 and λ_2 have negative real part. Thus, consider only the first term as an approximation, so the solution looks like

$$\frac{Ae^{i\omega t}}{-m\omega^2 + i\omega a + k} = \frac{A}{m} \cdot \frac{e^{i\omega t}}{-\omega^2 + i\omega a/m + k/m} = \frac{A}{m}\frac{e^{i\omega t}}{(\omega_0^2 - \omega^2) + i\omega a/m}.$$

Notice that if $\omega \approx \omega_0$, then the oscillations of $x(t)$ become very large. This is known as resonance. More explicitly, write things in polar coordinates: $\omega_0^2 - \omega_2 + i\omega a/m = r(\omega)e^{i\theta(\omega)}$. Then, the solution becomes

$$x(t) = \frac{A}{m}\frac{1}{r(\omega)}e^{i(\omega t - \theta(\omega))},$$

where $1/r(\omega)$ is the gain, or the amplitude of $x(t)$ relative to the force supplied, and $\theta(\omega)$ is the phase lag between the applied force and the movement of the mass. The gain as a function of ω looks like a normal distribution centered at $\omega = \omega_0$, and in the case where the friction is zero, $r(\omega) \to \infty$ when $\omega = \omega_0$.

Example 20.2. Consider an RLC circuit, with a capacitor and an inductor, and a voitage $V(t)$ applied to them. If the current is $i(t)$, then it satisfies the differential equation $L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + i/C = \frac{\mathrm{d}V}{\mathrm{d}t}$ (though there may also be a resistance Ri' on the wire). The gain can be used to pik out a radio signal at a particular frequency.

21. The Laplace Transform: 5/20/13

The Laplace transform is a technique that is helpful for solving inhomgeneous ODEs with constant coefficients, such as $y'' + 2y' + 3y = e^x$. It solves a similar range of problems to the techniques already discussed, but is convenient and widely used, so it is helpful to be familiar with it.

Definition. The Laplace transform of a function $f(t)$ is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

Example 21.1. If $f(t) = 1$, then

$$F(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}.$$

[9] The general solution is $Ce^{i\omega t} + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

If $f(t) = e^{at}$, then

$$F(s) = \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}.$$

The Laplace transform of a function f is also denoted $\mathcal{L}\{f\}$, such as $\mathcal{L}\{e^{2t}\} = 1/(s-2)$. This doesn't have to be real-values: $\mathcal{L}\{\cos t\} = \mathcal{L}\{(e^{it} + e^{-it})/2\} = (1/(s+i) + 1/(s-i))/2 = s/(s^2+1)$.

Here are some of its useful properties:

- $\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$, since the integral can be split up.
- $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$. Specifically, differentiation becomes something simpler.

Now, $\mathcal{L}\{-\sin t\} = s^2/(s^2+1) - 1$, so $\mathcal{L}\{\sin t\} = 1/(s^2+1)$.

The derivation for the second point is as follows:

$$\begin{aligned}\mathcal{L}\{f'\} &= \int_0^\infty f'(t)e^{-st}\,\mathrm{d}t \\ &= \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,\mathrm{d}t \text{ by integration by parts} \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}t.\end{aligned}$$

This can be applied again: $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf'(0) - f(0)$.

Example 21.2. Suppose $y'' + 4y = e^{ix}$, and $y(0) = y'(0) = 0$. One could use the method of undetermined coefficients, trying $y = Ae^{ix}$ and $y'' = -Ae^{ix}$, leading (eventually) to a solution $e^{ix}/3 + c_1e^{2ix} + c_2e^{-2ix}$.

Alaternatively, one could take the Laplace transform of both sides: $\mathcal{L}\{y''+4y\} = \mathcal{L}\{e^{ix}\}$, which becomes $(s^2+4)\mathcal{L}\{y\} = 1/(s-i)$. In general, the inverse Laplace transform will be used to proceed, but this can be solved by hand: first, use partial fractions to write

$$\frac{1}{(s^2+4)(s-i)} = \frac{1}{(s+2i)(s-2i)(s-i)} = \frac{A}{s+2i} + \frac{B}{s-2i} + \frac{C}{s-i}.$$

Then, $y(t) = Ae^{-2it} + Be^{2it} + Ce^{it}$ will work. Then, A, B, and C can be computed: $1 = A(s-2i)(s-i) + B(s+2i)(s+i) + C(s+2i)(s-2i)$, so $A = 1/3$, $B = -1/4$, and $C = -1/12$.

Often, if the differential equation is hard to solve, the Laplace transform won't provide that much help.

Unfortunately, the Laplace transform doesn't have much physical meaning, though it's closely related to the more meaningful Fourier transform

$$\mathcal{F}\{f\} = \int_{-\infty}^\infty f(t)e^{-ist}\,\mathrm{d}t,$$

which represents the amount of frequency s in $f(t)$, sort of: it will be large where the function has a period of oscillation. This lies behind how the ear hears frequencies of music.

The Fourier transform can be inverted as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathcal{F}\{f\}(s)e^{ist}\,\mathrm{d}s,$$

and a similar formula exists for the Laplace transform. Notice how similar the two formulas are.

Fourier transforms can be used to solve differential equations as well.

Example 21.3. If $y'' + 4y = f(x)$, then the solution in which $f(x) = e^{i\omega x}$ is known, so any function can be given by decomposing f into a combination of $e^{i\omega x}$, which are already solved.