BORDISM AND INVERTIBLE FIELD THEORIES

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Abstract. The purpose of these two lectures is to provide a crash course in some of the algebraic topology ideas that will appear in Søren Galatius' PCMI lectures in summer 2019. I'll define bordism and invertible topological field theories, highlighting some of the notions from homotopy theory that are used to study them.

1. Bordism

Probably you already know the basic definitions, but let's refresh them.

Definition 1.1. Let M_0 and M_1 be closed n-manifolds. A *bordism* from M_0 to M_1 is a compact $(n+1)$-manifold X, a partition $\partial X = Y_0 \amalg Y_1$, and diffeomorphisms $\theta_i \colon Y_i \xrightarrow{\cong} M_i$. If there is a bordism from M_0 to M_1, we say M_0 and M_1 are *bordant*.

Importantly, the maps θ_i are part of the data. A few elementary observations:

- Bordism is *reflexive*: there's a bordism M to M, namely $M \times [0,1]$.
- Bordism is *symmetric*: if M and N are bordant, then N and M are bordant.
- Bordism is *transitive*: if X is a bordism from M to N, and Y is a bordism from N to P, then we can glue X and Y along N to obtain a bordism from M to P. (There are details I did not say here – if this is your first introduction to bordism, it may be helpful to fill them in.)

So bordism is an equivalence relation. The set of equivalence classes is denoted Ω_n.

The major themes of these lectures are (1) that we can upgrade this set to increasingly sophisticated algebraic structures, and (2) how to study these structures using algebraic topology.

The first step is to define an abelian group structure on Ω_n. One can take the disjoint union of bordisms: if X is a bordism from M_0 to M_1, and Y is a bordism from N_0 to N_1, then $X \amalg Y$ is a bordism from $M_0 \amalg N_0$ to $M_1 \amalg N_1$. Therefore disjoint union defines an operation on equivalence classes $\amalg \colon \Omega_n \times \Omega_n \to \Omega_n$. Disjoint union is associative and commutative (up to natural isomorphism), so this is associative and commutative, period. Moreover, the empty n-manifold is the identity for this operation, and we get an abelian group. Computing these abelian groups (and variants introduced below) was a classical problem in algebraic topology, and one of the first.

Remark 1.2 (Ring structure). The graded abelian group $\Omega_* := \bigoplus_{n\geq 0} \Omega_n$ has the structure of a ring, where multiplication is Cartesian product and the point is the identity. (Again, because this is defined on equivalence classes, there are some details to check here.) This structure is interesting in homotopy theory, but for whatever reason doesn't appear much in Segal-style TQFT. ◀

Often we care about manifolds with additional topological structure: orientations, principal G-bundles, etc. (We will not address geometric structures, such as metrics or connections.)

Definition 1.3. An *n-dimensional tangential structure* is a pointed space ξ_n and a pointed fibration $\alpha_n \colon \xi_n \to B\mathrm{O}_n$. An *$n$-manifold with ξ_n-structure* is an n-manifold M together with a lift of its classifying map $c_{TM} \colon M \to B\mathrm{O}_n$ across α_n.

Example 1.4. A large class of important examples arise from the data of a Lie group G_n and a representation $\rho_n \colon G_n \to \mathrm{O}_n$: taking classifying spaces, we get $B\rho_n \colon BG_n \to B\mathrm{O}_n$, whose associated tangential structure is called a *G_n-structure*. On an n-manifold M, a G_n-structure is equivalent to a choice of a principal G_n-bundle $P \to M$ and an isomorphism $P \times_{G_n} \mathrm{O}_n \xrightarrow{\cong} \mathcal{B}_{\mathrm{O}}(M)$. This includes many examples you may have already seen, such as orientations (SO_n), spin structures (Spin_n), and almost complex structures ($\mathrm{U}_{n/2}$). ◀

It's often useful to work independently of the dimension n. Let $\mathrm{O}_\infty \coloneqq \operatorname{colim}_n \mathrm{O}_n$, which is a topological group.

Definition 1.5. A *stable tangential structure* is a pointed space ξ and a pointed fibration $\alpha\colon \xi \to B\mathrm{O}_\infty$. For any n, this defines an n-dimensional tangential structure $\xi_n \to B\mathrm{O}_n$ as the pullback of α across the map $B\mathrm{O}_n \to B\mathrm{O}_\infty$; a *$\xi$-structure* on an n-manifold M is then a ξ_n-structure.

Most examples of tangential structures are stable: we can form $B\mathrm{SO}_\infty$, $B\mathrm{Spin}_\infty$, $B\mathrm{Spin}^c_\infty$, $B\mathrm{Pin}^\pm_\infty$, etc., and so orientations, spin structures, spin^c structures, etc. can be defined in this way.

Remark 1.6. Two nonexamples:

- An almost complex structure is not the same thing as a $B\mathrm{U}_\infty$-structure (which is called a *stably almost complex structure*): a $B\mathrm{U}_\infty$-structure on M is a choice of a complex structure on $TM \oplus \mathbb{R}^k$ for some k; in particular, M may be odd-dimensional.
- The r-fold connected covering map $\mathrm{SO}_2 \to \mathrm{SO}_2$, followed by the inclusion $\mathrm{SO}_2 \hookrightarrow \mathrm{O}_2$, defines a two-dimensional structure called an *r-spin structure*; since $\pi_1 \mathrm{SO}_m = \mathbb{Z}/2$ if $m \geq 3$, this structure cannot stabilize. ◀

Lemma 1.7. *A ξ-structure on a manifold M induces a ξ-structure on ∂M.*

That is, if $\dim M = n$, we have a ξ_n-structure on M, and this induces a ξ_{n-1}-structure on ∂M.

Exercise 1.8. Prove Lemma 1.7.

Crucial fact: there are two different ways to induce the ξ-structure on ∂M, stemming from the two trivializations of the normal bundle of $\partial M \hookrightarrow M$. The standard convention is to use the outward normal; hence, if we use the inward normal to define a ξ-structure on ∂M, we'll denote it $-\partial M$. For orientations, for example, $-\partial M$ carries the opposite orientation from ∂M.

Lemma 1.7 means we can make sense of bordisms of manifolds of ξ-structure.

Definition 1.9. Let M and N be closed n-manifolds with ξ-structure. A *bordism* X from M to N is a compact ξ-manifold X, an identification $\partial X \cong X_0 \amalg X_1$, and diffeomorphisms of ξ-manifolds $\theta_0\colon M \xrightarrow{\cong} X_0$ and $\theta_1\colon N \xrightarrow{\cong} -X_1$.

That minus sign is important!

Anyways, as before, bordism is an equivalence relation, and the equivalence classes in dimension n form an abelian group denoted $\Omega_n^{t\xi}$, under disjoint union.

Remark 1.10. We often, *but not always*, get a bordism ring $\Omega_*^{t\xi}$ as before, induced from Cartesian product. For example, this happens with orientations, spin structures, stably almost complex structures. But it's not always true that the product of ξ-manifolds admits a ξ-structure, e.g. this happens for pin$^+$ and pin$^-$ structures. ◀

Example 1.11. In this example, we'll describe a *bordism invariant*, i.e. a function $\Omega_n^{t\xi} \to A$ for some abelian group A. (Here, ξ is orientation, $n = 4$, and $A = \mathbb{Z}$.)

Given a 4-manifold M, let $p_1(M) \in H^4(M)$ denote its first Pontrjagin class. If M is oriented, we can pair that with the fundamental class to obtain $\langle p_1(M), [M]\rangle \in \mathbb{Z}$.[1] We'd like to prove this is a bordism invariant. It's not super difficult to show that it's additive under disjoint union; the key is showing that if $M = \partial W$, where W is an oriented 5-manifold, then $\langle p_1(M), [M]\rangle = 0$.

The key fact is that $TW|_M = TM \oplus \nu$, where ν is the normal bundle for $i\colon M \hookrightarrow W$, and ν is trivial. The first Pontrjagin class is *stable*, so $p_1(E \oplus \mathbb{R}) = p_1(E)$, and this means that $i^* p_1(W) = p_1(M)$. Now we can use the long exact sequence of a pair:

$$H^4(W) \xrightarrow{i^*} H^4(M) \xrightarrow{\delta} H^5(W, M), \tag{1.12}$$

so $p_1(M) \in \mathrm{Im}(i^*) = \ker(\delta)$.

[1] If you're used to thinking of $p_1(M)$ as a differential form, then $\langle p_1(M), [M]\rangle$ and $\int_M p_1(M)$ are the same number.

Let $[W, M] \in H_5(W, M; \mathbb{Z})$ denote the fundamental class of the pair: under the connecting morphism $\partial\colon H_5(W, M) \to H_4(M)$, $[W, M] \mapsto [M]$. Lefschetz duality gives us a version of Stokes' theorem: if $x \in H^4(M)$, then

$$\langle x, \partial[W, M]\rangle = \langle \delta x, [W]\rangle. \tag{1.13}$$

Hence

$$\langle p_1(M), [M]\rangle = \langle p_1(M), \partial[W, M]\rangle = \langle \delta(p_1(M)), [W, M]\rangle = 0. \tag{1.14}$$

This approach to defining bordism invariants is extremely general: for bordism of ξ-manifolds, you can do this for any cohomology classes you can define naturally with the ξ-structure, in any cohomology theory for which ξ-manifolds are oriented (or, failing that, twisted cohomology!). For example, you can use Stiefel-Whitney classes for unoriented bordism, Chern classes for (stably almost) complex bordism, or for bordism of manifolds with a principal G-bundle, you can use characteristic classes of that bundle. For spin bordism you can use KO-theory characteristic classes. ◀

As an example, Ω_2 is nontrivial: there are closed surfaces Σ with $\langle w_2(\Sigma), [\Sigma]\rangle \neq 0$, such as $\mathbb{RP}^2$. In fact, this group is $\mathbb{Z}/2$.

2. The Pontrjagin-Thom construction

Serious calculation of bordism groups proceeds through a beautiful construction of Pontrjagin and Thom, translating into a calculation in homotopy theory – and crucially, this calculation is often tractable.

Throughout, we fix a tangential structure ξ.

Definition 2.1. Let $V \to X$ be a real vector bundle. Its *Thom space*, denoted X^V, is the pointed space $D(V)/S(V)$, where $D(V)$ denotes the unit disc bundle, $S(V)$ denotes the unit sphere bundle, and the basepoint is the equivalence class of $S(V)$.

There are other, equivalent models for the Thom space, e.g. the pair $(D(V), S(V))$, or $(V, V \setminus D(V))$, which you can think of as V but with all sufficiently large vectors collapsed to a point.

Exercise 2.2. Exhibit a natural homeomorphism $X^{V\oplus\mathbb{R}} \xrightarrow{\cong} \Sigma X^V$.

This is a useful result on its own, e.g. it tells us the Thom space of a trivial bundle. But it also says that you can think of the Thom space X^V as a twisted suspension of X.

Theorem 2.3 (Whitney embedding theorem)**.** *For any $n \geq 0$, there is an $N(n)$ such that*

1. *all n-manifolds M embed into S^N, and*
2. *all embeddings $M \hookrightarrow S^N$ are isotopic.*

We could have said $\mathbb{R}^N$ instead of S^N. Whitney and Wu showed you can take $N = 2n + 2$, but we'll never need this.

Fix an n-manifold M. If $m \geq N$, let $\nu_m \to M$ denote the normal bundle of any embedding $M \hookrightarrow S^m$; as all such embeddings are isotopic, this bundle is well-defined up to isomorphism. And the inclusion $S^m \hookrightarrow S^{m+1}$ as the equator induces an isomorphism $\nu_{m+1} \xrightarrow{\cong} \nu_m \oplus \underline{\mathbb{R}}$.

Exercise 2.4. Prove this.

So the stable equivalence class of ν_m doesn't depend on m, and is called the *stable normal bundle* of M. We can make sense of a ξ-structure on ν (called a *normal ξ-structure*): pick any m and ask for a ξ_{m-n}-structure on ν_m, and these pass between different choices of m. Bordism of manifolds with normal ξ-structure is denoted Ω_n^ξ.

Let $S_k \to BO_k$ denote the tautological bundle, and $V_k \to \xi_k$ denote its pullback across $\xi_k \to BO_k$.

Definition 2.5. Now, let M be an n-manifold with a normal ξ-structure. Given $m \geq N$, the classifying map $\varphi\colon \nu_m \to V_{n-m}$ extends to a map $S^m \to \xi_{m-n}^{V_{m-n}}$ via *Pontrjagin-Thom collapse*, as follows. Let $f\colon [0, \infty) \to (0, \infty)$ be smooth with $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Choose a metric on ν_m; then we define a map $f_M\colon S^m \to V_{m-n}$ by

$$f_M(x) = \begin{cases} \dfrac{\varphi(x)}{f(|\varphi(x)|)}, & x \in \nu_m \\ 0, & x \in S^m \setminus \nu_m. \end{cases} \tag{2.6}$$

Choose a basepoint for S^m not on ν; then we can take the quotient of V_{n-m} by the subspace of sufficiently large vectors and obtain a pointed map $S^m \to \xi_{n-m}^{V_{n-m}}$.

So, to reiterate: given a manifold M with a normal ξ-structure, we get a map $f_M\colon S^m \to \xi_{n-m}^{V_{n-m}}$.

Theorem 2.7 (Pontrjagin-Thom)**.** *If M and M' are bordant as normal ξ-manifolds, f_M and $f_{M'}$ are homotopic, and conversely. Thus Pontrjagin-Thom collapse defines a bijection $\Omega_n^\xi \xrightarrow{\cong} [S^m, \xi_{n-m}^{V_{n-m}}] = \pi_m(\xi_{n-m}^{V_{n-m}})$. The inverse is: given a map $S^m \to D(V_{n-m})$ sending the basepoint to $S(V_{n-m})$, take the preimage of a regular value.*

Both sides are abelian groups, and this is in fact an isomorphism of abelian groups!

Next, we remove the dependence on m.

Theorem 2.8 (Freudenthal suspension theorem)**.** *For any pointed, k-connected space X, the natural map $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism for $n \le 2k$.*

The suspension of a k-connected space is $(k+1)$-connected, so for any space X and $n \ge 0$, the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$ eventually stabilizes. Call the limiting value the n^{th} *stable homotopy group* of X.

Remark 2.9. This information can be very rich; for example, S^1 has only one nonzero homotopy group, but its stable homotopy groups are rich and complicated, and infinitely many are nonzero. ◂

Stable homotopy theory is the subfield which studies stable phenomena such as this one. Stable data associated to a space is encoded in the following object.

Definition 2.10. A *spectrum* E is a sequence of pointed spaces $E_0, E_1, \ldots$ together with *structure maps* $t_n\colon \Sigma E_n \to E_{n+1}$. A map between spectra $E \to F$ is data of maps $E_n \to F_n$ which intertwine the structure maps.

We can define a homotopy of maps of spectra by smashing with the unit interval, just as for spaces.

Example 2.11 (Suspension spectra)**.** Let X be a pointed space. The *suspension spectrum* of X, denoted $\Sigma^\infty X$, is the spectrum whose n^{th} space is $\Sigma^n X$, and whose structure maps $\Sigma\Sigma^n X = \Sigma^{n+1} X \to \Sigma^{n+1} X$ are all the identity.

If $X = S^0$, $\Sigma^\infty S^0$ is called the *sphere spectrum* and denoted $\mathbb{S}$. ◂

The homotopy groups of $\Sigma^\infty X$ are the stable homotopy groups of X.

Example 2.12 (Thom spectra)**.** Let ξ be a tangential structure. We define a spectrum $M\xi$, called the *Thom spectrum* of ξ, to have n^{th} space $\xi_n^{V_n}$. The pullback of V_{n+1} along $\xi_n \to \xi_{n+1}$ splits as $V_n \oplus \underline{\mathbb{R}}$, so we get

$$\Sigma \xi_n^{V_n} = \xi_n^{V_n \oplus \underline{\mathbb{R}}} \longrightarrow \xi_{n+1}^{V_{n+1}}, \tag{2.13}$$

which is the structure map. ◂

All of the stuff above establishes an isomorphism between the bordism group of n-manifolds with normal ξ-structure and $\pi_n(M\xi)$. This, followed up understanding the homotopy theory of $M\xi$, is how bordism groups are calculated.

Remark 2.14 (Tangential and normal structures). There is a map $s\colon B\mathrm{O}_\infty \to B\mathrm{O}_\infty$ which "switches the tangent and normal bundles:" if $\varphi\colon M \to B\mathrm{O}_\infty$ classifies TM, then $\varphi \circ s$ classifies the stable normal bundle of M.[2] Given a tangential structure $\xi \to B\mathrm{O}_\infty$, we can pull back by s to obtain another tangential structure $\xi^\perp \to B\mathrm{O}_\infty$. On a manifold M, tangential ξ-structures correspond to normal $\xi^\perp$-structures, and vice versa.

Often, $\xi^\perp \cong \xi$: this is true for G-structures for $G = \mathrm{O}_\infty$, SO_∞, and Spin_∞. One counterexample is that this switches pin^+ and pin^- structures. ◂

Example 2.15 (Unoriented bordism)**.** Thom studied $\pi_* MO$ and showed that it's a polynomial $\mathbb{F}_2$-algebra with a generator in degree n for each n not equal to $2^j - 1$. In many dimensions, these generators can be represented by $\mathbb{RP}^n$, but not always; for example, this fails for $n = 5$, where for a generator we instead can take the *Wu manifold* $\mathrm{SU}_3/\mathrm{SO}_3$.

As a consequence of Thom's work, we know that Stiefel-Whitney numbers are a complete invariant for unoriented bordism. ◂

[2] Good exercise: actually define this map!

Example 2.16 (Oriented bordism)**.** Wall, building on work of Thom, Averbuch, Milnor, and Novikov, computed $\pi_* MSO$. Salient facts: all torsion is of order 2, and $\pi_* MSO \otimes \mathbb{Q}$ is a polynomial algebra with one generator in each degree $4k$, which we can take to be $\mathbb{CP}^{2k}$. The lowest-degree torsion is $\Omega_5^{\mathrm{SO}} = \mathbb{Z}/2$, also generated by the Wu manifold. Oriented bordism is determined by Stiefel-Whitney and Pontrjagin numbers. ◀

Example 2.17 (Spin bordism (low dimensions))**.** Anderson-Brown-Peterson determined the spin bordism groups, building on work of Milnor. There is a map $MSpin_* \to ko_*$ defined by Atiyah-Bott-Shapiro (roughly speaking, take the index or mod 2 index of the Dirac operator, depending on dimension), and it is an isomorphism on degrees less than 8. ◀

Example 2.18 (Framed bordism)**.** A framing is a ξ-structure for $\xi = \mathrm{pt}$.[3] Therefore the Thom spectrum $Mfr = \mathbb{S}$, which tells us something I always found profound: the stable homotopy groups of the spheres are the framed bordism groups. ◀

3. Bordism categories and topological field theories

Fix a dimension n and a tangential (n-)structure ξ. When we checked that $\Omega_{n-1}^{t\xi}$ is an abelian group, we had to check the following (and also a few more axioms).

(1) Bordism is reflexive, meaning M is bordant to M, via the cylinder $M \times [0,1]$.
(2) Bordism is transitive (meaning if M is bordant to N and N is bordant to P, then M is bordant to P). We proved this by gluing bordisms: if X is a bordism from M to N and Y is a bordism from N to P, then $X \cup_Y N$ is a bordism from M to P.
(3) Disjoint union is associative and commutative up to natural isomorphism, and the empty $(n-1)$-manifold is a unit for it.

These allow us to extract a more elaborate object out of bordism: a symmetric monoidal category. I'm not going to get into the precise definition of a symmetric monoidal category here, but it consists of data of:

- a category C,
- a functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$,
- a unit $1 \in \mathsf{C}$, and also
- choices of natural isomorphisms ensuring that $\otimes$ is associative and commutative up to natural isomorphism, and that 1 is the unit for it (e.g. a natural isomorphism $1 \otimes - \Rightarrow \mathrm{id}$).

Similarly, we can ask for a functor F between symmetric monoidal categories to be symmetric monoidal, which involves additional data of natural isomorphisms $F(1) \cong 1$ and $F(x \otimes y) \cong F(x) \otimes F(y)$.

Definition 3.1. The *bordism category* Bord_n^ξ is the symmetric monoidal category defined by the following data.

- The objects are closed $(n-1)$-manifolds with ξ-structure.
- The morphisms $M \to N$ are the diffeomorphism classes of[4] bordisms from M to N as ξ-manifolds. Composition is gluing of bordisms.
- The monoidal unit is the empty ξ-manifold.
- The monoidal product is disjoint union.

So, of course,

Definition 3.2. A *topological quantum field theory (of ξ-manifolds)* is a symmetric monoidal functor $Z\colon \mathsf{Bord}_n^\xi \to (\mathsf{Vect}_\mathbb{C}, \otimes)$.

Sometimes people use other target categories, e.g. $\mathsf{sVect}_\mathbb{C}$, $\mathbb{Z}/2$-graded vector spaces with the tensor product using the Koszul sign rule.

Definition 3.3. Let $(\mathsf{C}, \otimes, 1)$ be a symmetric monoidal category and $x \in \mathsf{C}$. We say x is *invertible* if there is an object $x^{-1} \in \mathsf{C}$ and an isomorphism $x \otimes x^{-1} \xrightarrow{\cong} 1$.

The inverse x^{-1} is unique up to unique isomorphism, so long as it exists.

[3] Well, $\mathrm{pt} \to BO_\infty$ isn't quite a fibration, but we can replace pt with a contractible space and everything's OK.

[4] We must take diffeomorphism classes so that composition is associative on the nose (rather than up to some isomorphism). Diffeomorphisms are taken rel boundary.

Definition 3.4. A *Picard groupoid* is a symmetric monoidal category which is a groupoid (so every morphism is invertible under composition) and such that every object is $\otimes$-invertible.

Given any symmetric monoidal category C, the subcategory $\mathsf{C}^\times$ of invertible objects and invertible morphisms is a Picard groupoid. For example, $\mathsf{Vect}_\mathbb{C}^\times$ is the category of complex lines and nonzero linear maps between them.

Definition 3.5. A topological field theory $Z\colon \mathsf{Bord}_n^\xi \to \mathsf{Vect}_\mathbb{C}$ is *invertible* if it factors through $\mathsf{C}^\times \hookrightarrow \mathsf{C}$.

Equivalently:

- for any closed $(n-1)$-dimensional ξ-manifold M, $Z(M)$ is $\otimes$-invertible (so in $\mathsf{Vect}_\mathbb{C}$, a one-dimensional vector space) and for any bordism X, $Z(X)$ is invertible under composition.
- Alternatively, the symmetric monoidal category TQFT_n^ξ of TQFTs has objects symmetric monoidal functors $\mathsf{Bord}_n^\xi \to \mathsf{Vect}_\mathbb{C}$ and morphisms the symmetric monoidal natural transformations between them. The tensor product is "pointwise": $(Z_1 \otimes Z_2)(M) \coloneqq Z_1(M) \otimes Z_2(M)$, which physically corresponds to formulating two systems in the same material with no interactions. (In condensed-matter physics, this is called *stacking*). Anyways, the invertible objects of TQFT_n^ξ are precisely the invertible TQFTs.

Example 3.6 (Euler theories)**.** Let $\lambda \in \mathbb{C}^\times$. The *Euler theory* $Z_\lambda\colon \mathsf{Bord}_n^{\mathrm{O}} \to \mathsf{Vect}_\mathbb{C}^\times$ is an invertible TQFT which to every object assigns $\mathbb{C}$, and to every morphism X assigns multiplication by $\lambda^{\chi(X)}$. These compose properly because the Euler characteristic satisfies a gluing formula.

When $\lambda = 1$, this is the trivial TQFT (all maps are identity maps). For general λ, these are *deformation-equivalent* to the trivial theory, as we can move λ along a path in $\mathbb{C}^\times$ to 1. $\blacktriangleleft$

These are not the only examples – in fact, they're probably the least interesting, because they're deformation-trivial. We'll use homotopy theory to produce some more examples.

4. From invertible topological field theories to Madsen-Tillmann spectra

The supposed goal of this section is to provide a (sketch) proof of the following not-quite-theorem.

"Theorem" 4.1. *Given some sort of bordism invariant φ of n-dimensional ξ-manifolds, there is an invertible TQFT $Z_\varphi\colon \mathsf{Bord}_n^\xi \to \mathsf{sVect}_\mathbb{C}$ such that for any closed n-dimensional ξ-manifold M, $Z_\varphi(M) = \varphi(M)$, and Z_φ is unique up to some sort of isomorphism.*

To make this into a theorem, we'll have to specify what kind of bordism invariants are allowed, and what sort of isomorphism to take. There's actually more than one way to make these choices (and so more than one theorem).

There are three very different extant proof strategies for (rigorous versions of) "Theorem" 4.1.

(1) Rovi-Schoenbauer [RS18] and Kreck-Stolz-Teichner (unpublished) use the topology of bordisms to explicitly determine what kinds of invariants can arise. Their proofs are the most hands-on, but are very difficult to generalize to extended TQFT.
(2) Yonekura [Yon19] does everything by hand: beginning with a bordism invariant, he constructs an invertible TQFT realizing it. Uniqueness is also hands-on. Again, everything is elementary, and would be difficult to generalize to extended TQFT.
(3) Freed-Hopkins [FH16], building on work of Galatius-Madsen-Tillmann-Weiss [GMTW09] and Schommer-Pries [SP17], reduce to a question in stable homotopy theory whose answer, by the Pontrjagin-Thom theorem, turns into the appropriate kind of bordism invariants. This proof is by far the most involved, but is the only one that generalizes to extended invertible TQFTs, which are the ones we actually get from physics.

We will follow Freed-Hopkins, as this is close to what Galatius plans to talk about. I also expect some of the techniques and ingredients that come up to pop again in Galatius' PCMI lectures.

The proof produces from Bord_n^ξ a spectrum in the sense of stable homotopy theory, and this construction uses simplicial methods. So first I'll introduce simplicial sets. These provide a combinatorial way to do homotopy theory, which is useful in settings where geometry is more elusive.

Definition 4.2. The *simplex category* Δ is the category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving functions.

Definition 4.3. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. With natural transformations as morphisms, these form the category sSet. More generally, for any category C, a *simplicial object* in C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$.

That is, a simplicial set X is a set X_n for each $[n]$ (called the set of *n-simplices*) with compatible actions by the morphisms in Δ. A morphism of simplicial sets $X \to Y$ is a collection of maps $X_n \to Y_n$ for each n that commutes with those actions.

This doesn't seem very topological or geometric; here's another definition.

Definition 4.4. A *simplicial set* X is a collection of sets X_n for each $n \geq 0$, along with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, called the *face maps* and *degeneracy maps*, respectively, satisfying the relations

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, \quad i < j \\ s_i \circ s_j &= s_{j+1} \circ s_i, \quad i \leq j \\ d_i \circ s_j &= \begin{cases} 1, & i = j \text{ or } i = j+1 \\ s_{j-1} \circ d_i, & i < j \\ s_j \circ d_{i-1} & i > j+1. \end{cases} \end{aligned} \tag{4.5}$$

A morphism of simplicial sets $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that commute with the face and degeneracy maps.

There's geometry hiding here: one can think of a simplicial set as a generalization of a simplicial complex, where the n-simplices are, well, the n-dimensional simplices; the face maps describe which $(n-1)$-simplices live at the boundaries of which n-simplices; and the degeneracy maps encode when n-simplices are "degenerate" (if their vertices repeat, they look lower-dimensional).

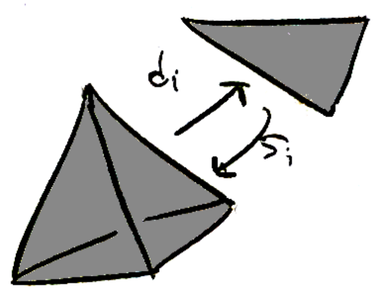

Figure 1. The standard 3-simplex, with example face and degeneracy maps.

Now, two important general constructions: from a category, we can build a simplicial set, and from a simplicial set, we can build a topological space.

Definition 4.6. Let C be a (small) category. The *nerve* of C, denoted $N\mathsf{C}$, is the simplicial set specified by the following data.

- $N\mathsf{C}_0$ is the set of objects in C.
- $N\mathsf{C}_1$ is the morphisms of C.
- $N\mathsf{C}_2$ is the set of pairs of composable morphisms $X \to Y \to Z$.
- If $n \geq 2$, $N\mathsf{C}_n$ is the set of n-tuples of composable morphisms $X_0 \to X_1 \to \cdots \to X_n$.

The degeneracy map $s_i : N\mathsf{C}_n \to N\mathsf{C}_{n+1}$ takes a string of arrows and inserts the identity at the i^{th} position. The face map $d_i : N\mathsf{C}_n \to N\mathsf{C}_{n-1}$ replaces the i^{th} and $(i+1)^{\text{th}}$ arrows with their composition, unless $i = 0$ or $i = n$, in which case it just drops the first or last arrow, respectively.

Example 4.7. Let $X_\bullet$ be a simplicial set. Its *geometric realization* $|X_\bullet|$ is the topological space formed by realizing each n-simplex of X by an actual copy of $\Delta^n \subset \mathbb{R}^n$, then using the face and degeneracy maps to glue them together. An explicit formula is

$$|X_\bullet| = \varinjlim_{\Delta^n \to X} |\Delta^n|. \tag{4.8}$$

◀

Geometric realization defines a functor $\mathsf{sSet} \to \mathsf{Top}$. This functor has a lot of nice structure – it induces an equivalence of homotopy theories (which can be made precise in a few different ways). Therefore if you have a homotopy-theoretic question about topological spaces, and it lends itself to combinatorial rather than point-set methods, you can try simplicial sets.

Example 4.9 (Classifying spaces)**.** For example, this formalism makes for a slick, if abstract, definition of the classifying space of a discrete group G. Let pt/G denote the category with a single object $*$ and with $\mathrm{Hom}(*, *) = G$; then, the geometric realization of the nerve of G is a model for BG! Obviously there's more to say here, e.g. producing EGG from a different simplicial set and showing that it's contractible.

You all care about topological groups too, and this construction works for them too. We consider pt/G as a *topological category* (meaning: the homomorphisms between two objects form a topological space, and composition is a continuous map); then, the same definition of the nerve produces a *simplicial space* (i.e. a simplicial object in Top). One can also geometrically realize simplicial spaces, via roughly the same idea: we still glue simplices together, but we begin with a space of them, rather than the disjoint union of a bunch of them.

Sure, you'll never be able to do Chern-Weil theory with this model of BG, but it has advantages: if A is an abelian group, this model for BA is a topological abelian group, and this is very manifest: you can define the group operations on $N(\mathrm{pt}/A)$ and check associativity, commutativity, and identity there, where they're pretty hands-on; then, functoriality of geometric realization automatically guarantees the image of the multiplication map also satisfies those axioms. This is much nicer than trying to reason about group models of $\mathbb{RP}^\infty$ or $\mathbb{CP}^\infty$, in my experience! It also provides a model for BG that is functorial in G and such that $B(G \times H) = BG \times BH$ (they're always homotopic, but now this is true on the nose).

So for abelian groups, we can iterate this construction, defining B^2A, B^3A, etc.; for A discrete these are the *Eilenberg-Mac Lane spaces* $K(A, n)$, defined to have $\pi_n(K(A, n)) = A$ and all other homotopy groups vanish. ◀

Now let's apply this technology to invertible field theories.

Definition 4.10. Let C be a symmetric monoidal category. Its *groupoid completion*, denoted $\overline{\mathsf{C}}$, is the Picard groupoid formed by adjoining inverses to all objects and morphisms in C.

Suppose $F\colon \mathsf{C} \to \mathsf{D}$ is a symmetric monoidal functor, where C and D are symmetric monoidal categories. If F is invertible in the sense that it factors through $\mathsf{D}^\times$, then it also factors through $\overline{\mathsf{C}}$: for any $x \in \mathsf{C}$, we must have that $F(x^{-1}) = F(x)^{-1}$, and similarly for morphisms. Hence invertible symmetric monoidal functors $\mathsf{C} \to \mathsf{D}$ are equivalent to symmetric monoidal functors $\overline{\mathsf{C}} \to \mathsf{D}^\times$.

If we pass to geometric realizations, it's easier to calculate: we can attack the problem with the tools of homotopy theory. *A priori* we could lose information by doing so, but it turns out that we don't.

Theorem 4.11 (Homotopy hypothesis, dimension 1)**.** *If* C *and* D *are groupoids, the functor* $|N{-}|$ *induces a bijection of pointed sets* $\mathrm{Hom}(\mathsf{C}, \mathsf{D}) \xrightarrow{\cong} [|N\mathsf{C}|, |N\mathsf{D}|]$.

We have Picard groupoids and symmetric monoidal functors between them, so we need a different result. Some of the steps here will be sketchy.

(1) When C is a Picard groupoid, we'll witness some additional structure on $|N\mathsf{C}|$ called a *grouplike* E_∞*-algebra* structure.
(2) Given a grouplike E_∞-space, we can "deloop" it into an infinite loop space.
(3) Finally, an infinite loop space yields a spectrum, which we denote $|\mathsf{C}|$ and call the *classifying spectrum* of C.

We'll say more in just a second – first the theorem.

Theorem 4.12 (Stable homotopy hypothesis, dimension 1)**.** *If* C *and* D *are Picard groupoids, taking classifying spectra induces a bijection of abelian groups* $\mathrm{Hom}^\otimes(\mathsf{C}, \mathsf{D}) \cong [|\mathsf{C}|, |\mathsf{D}|]$.

This was a folklore theorem for a while: proofs or sketches can be found in [BCC93, HS05, Dri06, Pat12, JO12, GK14].

In particular, knowing $|\overline{\mathsf{Bord}_n^\xi}|$ and $|\mathsf{Vect}_\mathbb{C}^\times|$ (or whatever our target is) would allow for a proof of "Theorem" 4.1.

Let's now go into more detail about how to get a classifying spectrum, rather than a space. (Note: this part is a bit complicated, so I should skip it if I don't have time.)

First (1). Let C be a symmetric monoidal category. Functoriality of $|N-|$ means we obtain maps $|N\otimes|\colon |N\mathsf{C}|\times|N\mathsf{C}| \to |N\mathsf{C}|$ and $|N1|\colon \mathrm{pt} \to |N\mathsf{C}|$ which almost satisfy the axioms of a topological commutative monoid – but these aren't quite associative/commutative/the identity on the nose; rather, the axioms hold up to homotopy, but coherently with respect to any homotopy you might write down. Once made precise, this is the notion of an *E_∞-algebra* in spaces, or simply an *E_∞-space.* (The formal definition involves multiplication tracked by the configuration space of D^n, for all n.) An E_∞-space X is *grouplike* if its multiplication induces an abelian group structure on $\pi_0 X$ (rather than just a commutative monoid structure). This is a homotopical analogue of a topological abelian group: we have topology and a commutative multiplication, but everything holds only up to homotopy in a nice way.

Lemma 4.13. *If C is a Picard groupoid, $|N\mathsf{C}|$ is a grouplike E_∞-space.*

After all, $\pi_0|N\mathsf{C}|$ is the set of isomorphism classes of objects of C, and we're asking for each one to be $\otimes$-invertible.

Now infinite loop spaces. This story (and its effective version relating k-fold loop spaces and E_k-algebras) is part of a classic story in homotopy theory that has shaped the field a great deal. Given a group G, there's a homotopy equivalence $G \simeq \Omega BG$ (the loops on the classifying space of G), so BG is a "delooping" of G: on groups, we can invert Ω.[5]

In general BG isn't a group, so we can't do this again. But if A is a topological abelian group, recall that we saw that BA is a topological abelian group, so we can deloop again: $BA \simeq \Omega B^2 A$, and again: $B^2 A \simeq \Omega B^3 A$, and so on.

Definition 4.14. An *infinite loop space structure* on a pointed space X_0 is a sequence of pointed spaces $X_1, X_2, \dots$ and homotopy equivalences $f_n\colon X_n \xrightarrow{\simeq} \Omega X_{n+1}$ for all $n \geq 0$. A morphism of infinite loop spaces $(X_m f_n)_{n\geq 0} \to (X_n', f_n')_{n\geq 0}$ is data of continuous pointed maps $X_n \to X_n'$ for each n which intertwine f_n and f_n'.

We've just seen how to obtain an infinite loop space structure on a topological abelian group. But infinite loop spaces are homotopical notions, so we can get away with a weaker structure: if X is a grouplike E_∞-space, it has a classifying space BX, which is also a grouplike E_∞-space, and $X \simeq \Omega BX$. This in fact defines an equivalence between grouplike E_∞-spaces and infinite loop spaces.

Finally, to spectra. The functors $\Sigma, \Omega\colon \mathsf{Top}_* \to \mathsf{Top}_*$ are *adjoint*, meaning for all pairs of pointed spaces X and Y, there's a natural isomorphism $[\Sigma X, Y] = [X, \Omega Y]$.[6] Therefore if $(X_n, f_n)_{n\geq 0}$ is an infinite loop space, we can replace f_n with its image $g_n\colon \Sigma X_n \to X_{n+1}$ under the adjunction, obtaining structure maps for a spectrum.

Conversely, given a spectrum $E = \{E_n, g_n\}$, we can obtain an infinite loop space $\Omega^\infty E := \varinjlim_n \Omega^n \Sigma^n E_n$. These two notions are *almost* equivalent.

Theorem 4.15. *The two functors described above induce an equivalence of homotopy theories between infinite loop spaces and* connective *spectra, i.e. those whose negative-degree homotopy groups vanish.*

I'm making a point of carrying around both of these notions because different authors in this area use one or the other – for example, Freed-Hopkins work with spectra, but many of the papers of Galatius and collaborators use the language of infinite loop spaces.

Remark 4.16. Given an abelian group A, we saw that it defines an infinite loop space, hence a spectrum. Explicitly, this spectrum is called the *Eilenberg-Mac Lane spectrum* for A, denoted HA. These are fundamental examples in stable homotopy theory. There is a natural isomorphism $[X, \Sigma^n HA] \cong H^n(X; A)$. ◀

Great, now let's identify what these spectra actually are.

Definition 4.17. Given a stable tangential structure ξ and an $n \geq 0$, we define a spectrum $MT\xi_n$, called a *Madsen-Tillmann spectrum*, as a variant of a Thom spectrum.[7]

[5] Here ΩX is the *based loop space* of the pointed space X: we require all loops to begin and end at the basepoint.

[6] This is fairly hands-on, and is a good exercise for working with Σ in algebraic topology. But it's not at all important for grasping the key ideas in these lectures.

[7] The definition we give generalizes to define the Thom spectrum of any virtual vector bundle over any space; this one is the case $-V_n \to \xi_n$.

Recall that BO_n has a model as the Grassmannian of n-dimensional subspaces of $\mathbb{R}^\infty$. Let $\xi_{n,n+q}$ be the pullback of $\xi_n \to BO_n$ across the inclusion $\mathrm{Gr}_n(\mathbb{R}^{n+q}) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty) = BO_n$. Let $W_q \to \xi_{n,n+q}$ be the pullback of the *universal quotient bundle* over $\mathrm{Gr}_n(\mathbb{R}^{n+q})$, whose fiber at a subspace $V \subset \mathbb{R}^{n+q}$ is $\mathbb{R}^{n+q}/V$. Then $V_n \oplus W_q = \underline{\mathbb{R}}^{n+q}$. The $(n+q)^{\text{th}}$ space of $MT\xi_n$ is the Thom space $\xi_{n,n+q}^{W_q}$.

The structure maps are similar to before: the pullback of $W_{q+1} \to \xi_{n,n+q+1}$ along the map $\xi_{n,n+q} \to \xi_{n,n+q+1}$ is isomorphic to $W_q \oplus \underline{\mathbb{R}}$, and therefore we get a map of Thom spaces $\Sigma\xi_{n,n+q}^{W_q} \to \xi_{n,n+q+1}^{W_{q+1}}$, which is the structure map.

Theorem 4.18 (Galatius-Madsen-Tillmann-Weiss, Nguyen)**.** *There is a homotopy equivalence* $|\overline{\mathsf{Bord}_n^\xi}| \simeq \tau_{0:1}\Sigma^n MT\xi_n$.

Here $\tau_{0:1}$ means we *truncate*, killing all of the homotopy groups except π_0 and π_1. This is a major theorem. The codomain, fortunately, is easier: if I told you a little more it would be an exercise.

Proposition 4.19. *There are homotopy equivalences* $|\mathsf{Vect}_\mathbb{C}^\times| \simeq \Sigma H\mathbb{C}^\times$ *and* $|\mathsf{sVect}_\mathbb{C}^\times| \simeq \tau_{0:1}\Sigma I\mathbb{C}^\times$.[8]

What's $I\mathbb{C}^\times$? It's called the *Pontrjagin dual to the sphere spectrum*, a variant of a construction introduced by Brown-Comenets [BC76]. All you need to know about it is that it satisfies the universal property

$$[E, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}(\pi_n E, \mathbb{C}^\times). \tag{4.20}$$

So for our goal of classifying invertible field theories, we only need to know the cohomology (for $\mathsf{Vect}_\mathbb{C}$) or homotopy (for $\mathsf{sVect}_\mathbb{C}$) of Madsen-Tillmann spectra. Homotopy groups behave better with respect to truncation, so let's use that. As Madsen-Tillmann spectra are Thom spectra, we can use the Pontrjagin-Thom construction to explicate their homotopy groups as a certain kind of bordism groups; the answer has already been classically studied.

Definition 4.21 (Reinhardt [Rei63])**.** The *Madsen-Tillmann bordism group* $\Omega_n^{MT\xi_n}$, or the *Reinhardt bordism group*, or the *vector field bordism group*, or the *SKK bordism group*, is the group completion of the commutative monoid of ξ_n-manifolds under disjoint union, modulo the bordism relation where M bounds if it bounds a compact ξ_{n+1}-manifld W and there is a nonvanishing vector field on W which is the outward normal on M.

Unlike ordinary ξ-bordism, we have to take the group completion – it isn't already an abelian group.

Exercise 4.22. Trace through the Pontrjagin-Thom construction to see why we get Reinhardt bordism from $\pi_0 MT\xi_n$.

Because this is a finer equivalence relation than ξ-bordism in the usual sense, any ξ-bordism invariant defines a Madsen-Tillmann bordism invariant. These are the most important examples: for example, we saw that integrating characteristic classes defines cobordism invariants, so these promote to invertible TQFTs. These invertible field theories can appear as anomaly theories for anomalous global symmetries in QFTs.

Exercise 4.23. That Madsen-Tillmann bordism is finer than ordinary bordism defines a map $\Omega_n^{MT\xi_n} \to \Omega_n^{t\xi} = \Omega_n^{\xi^\perp}$. This map arises as π_n of a map $MT\xi_n \to \Sigma MT\xi_{n+1} \to \cdots \to M\xi^\perp$. Construct this stabilization map.

Freed-Hopkins apply the same line of reasoning to classify extended TQFTs, with the caveat that (1) not all of the intermediate theorems have been proven in this generality yet and (2) the proofs are harder!

- One begins with a symmetric monoidal n-category of bordisms, and some target symmetric monoidal n-category. We have some example target categories, but it's believed that the "right one" is still out there.
- Instead of Picard groupoids, we obtain *Picard n-groupoids*. There's the notion of the nerve of an n-category (hence of Picard n-groupoids), and then we geometrically realize.
- The n-dimensional stable homotopy hypothesis says this process doesn't lose any information, but this is not yet a theorem for $n > 2$.
- Schommer-Pries shows that for fully extended TQFTs, we get $|\overline{\mathsf{Bord}_n^\xi}| \simeq \tau_{0:n}\Sigma^n MT\xi_n$: we see more homotopy groups.

[8]We don't need to truncate $\Sigma H\mathbb{C}^\times$ because it only has one nonzero homotopy group: $\pi_1(\Sigma H\mathbb{C}^\times) \cong \mathbb{C}^\times$.

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