THE COMPARISON OF TWO COHOMOLOGY OPERATIONS

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1. Introduction

The goal of this document is to make the following calculation.

Theorem 1.1. *Let M be a closed 5-manifold and $B \in H^2(M;\mathbb{Z}/2)$. Then,*

$$\langle B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B, [M]\rangle = \frac{1}{2}\langle \widetilde{w}_1 \smile \mathfrak{P}(B), [M]\rangle. \tag{1.2}$$

The right-hand side uses some unfamiliar notation, which we proceed to define.

Lemma 1.3. *If $\mathbb{Z}_{w_1}$ denotes the orientation local system, $H^1(B\mathrm{O}_1, \mathbb{Z}_{w_1}) \cong \mathbb{Z}/2$.*

Indeed, this is the group cohomology $H^1(\mathbb{Z}/2, \mathbb{Z}_\sigma)$, where $\mathbb{Z}_\sigma$ denotes $\mathbb{Z}$ with the sign action.

Definition 1.4. The pullback of the nonzero element of $H^1(B\mathrm{O}_1;\mathbb{Z}_{w_1})$ under the determinant map $B\det\colon B\mathrm{O}_n \to B\mathrm{O}_1$ is called the *twisted first Stiefel-Whitney class* $\widetilde{w}_1 \in H^1(B\mathrm{O}_n;\mathbb{Z}_{w_1})$.

Hence this defines a twisted first Stiefel-Whitney class of any real vector bundle, which lives in cohomology twisted by the orientation bundle. Its mod 2 reduction is the usual first Stiefel-Whitney class in untwisted $\mathbb{Z}/2$-cohomology. In (1.2), we consider its reduction $\widetilde{w}_1 \in H^1(M;(\mathbb{Z}/4)_{w_1})$, twisted mod 4 cohomology.

Next, $\mathfrak{P}$ denotes the *Pontrjagin square* $\mathfrak{P}\colon H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/4)$ (though it exists in greater generality). It is the realization of the idea that if you know an $x \in \mathbb{Z}$ modulo 2, you know x^2 mod 4.

On the right-hand side of (1.2), we use cup and cap products in twisted $\mathbb{Z}/4$-cohomology: if $[M]$ denotes the fundamental class in twisted $\mathbb{Z}/4$-cohomology, this is

$$H^1(M;(\mathbb{Z}/4)_{w_1}) \otimes H^4(M;\mathbb{Z}/4) \xrightarrow{\smile} H^5(M;(\mathbb{Z}/4)_{w_1}) \xrightarrow{-\frown[M]} \mathbb{Z}/4. \tag{1.5}$$

However, since $2\widetilde{w}_1 = 0$, $\langle \widetilde{w}_1 \smile \mathfrak{P}(B), [M]\rangle$ is even, and so it makes sense to divide by 2 and obtain an element of $\mathbb{Z}/2$, so we can compare with the left-hand side of (1.2).

We'll prove Theorem 1.1 in three steps:

(1) First, prove that both sides of (1.2) are cobordism invariants for a certain class of manifolds.
(2) Then, determine generating manifolds for the group of cobordism classes of those manifolds.
(3) Finally, verify (1.2) on the generators.

2. Cobordism-invariance

To capture the notion of cobordism of a manifold and a degree-2 cohomology class, we consider cobordism of manifolds with a $\mathbb{Z}/2$-gerbe, or equivalently, manifolds M together with a degree-2 $\mathbb{Z}/2$ cohomology class B, where (M, B) bounds if $M = \partial W$ for W compact and B extends over W. For the rest of this document, cobordism-invariant, cobordism groups, etc., refers to this kind of cobordism unless otherwise specified.

The classifying space for this structure is $B\mathrm{O}_n \times K(\mathbb{Z}/2, 2)$, so the cobordism groups are the homotopy groups of the Thom spectrum of the virtual bundle $(V_n - \mathbb{R}^n) \to B\mathrm{O}_n \times K(\mathbb{Z}/2, 2)$ (here $V_n \to B\mathrm{O}_n$ is the tautological bundle).

Lemma 2.1. *Let $E \to X$ and $F \to Y$ be virtual vector bundles. The Thom space of $E \boxplus F \to X \times Y$ is* $\mathrm{Thom}(E) \wedge \mathrm{Thom}(F)$.

We're looking at $(V_n - \mathbb{R}^n) \boxplus 0$, hence obtain $MO \wedge K(\mathbb{Z}/2, 2)_+$ (the Thom space of the zero bundle on X is $X/\varnothing = X_+$). The cobordism group we want is π_5 of this spectrum.

Proposition 2.2. *The quantity* $\langle \widetilde{w}_1 \smile \mathfrak{P}(B), [M]\rangle$ *is a cobordism invariant, and in particular a group homomorphism* $\Omega_5^{\mathrm{O}}(K(\mathbb{Z}/2,2)) \to \mathbb{Z}/2$.

Proof. This quantity is additive under disjoint union, so it suffices to show that it vanishes when M bounds. Let (M,B) bound, i.e. M is a closed 5-manifold, $B \in H^2(M;\mathbb{Z}/2)$, and there's a compact manifold W and a $\widehat{B} \in H^2(W;\mathbb{Z}/2)$ such that $M = \partial W$ and if $i\colon M \hookrightarrow W$ is inclusion, $B = i^*\widehat{B}$. Then, $TW|_M \cong TM \oplus \underline{\mathbb{R}}$ (using the outward normal vector field), so $i^*\widetilde{w}_1(W) = \widetilde{w}_1(M)$. By naturality $i^*\mathfrak{P}(\widehat{B}) = \mathfrak{P}(B)$. In the long exact sequence for (W,M),

$$H^n(W;(\mathbb{Z}/4)_{w_1}) \xrightarrow{i^*} H^n(M;(\mathbb{Z}/4)_{w_1}) \xrightarrow{\delta} H^{n+1}(W,M;(\mathbb{Z}/4)_{w_1}), \tag{2.3}$$

so $\widetilde{w}_1(M)\mathfrak{P}(B) \in \mathrm{Im}(i^*) = \ker(\delta)$.

Let $[W,M] \in H_{n+1}(W,M;(\mathbb{Z}/4)_{w_1})$ denote the fundamental class of the pair, and $[M] \in H_n(M;(\mathbb{Z}/4)_{w_1})$ denote the fundamental class. Under the connecting morphism $\partial\colon H_{n+1}(W,M;(\mathbb{Z}/4)_{w_1}) \to H_n(M;(\mathbb{Z}/4)_{w_1})$, $[W,M] \mapsto [M]$. Lefschetz duality gives us a version of Stokes' theorem: if $x \in H^n(M;(\mathbb{Z}/4)_{w_1})$, then

$$\langle x, \partial[W,M]\rangle = \langle \delta x, [W]\rangle. \tag{2.4}$$

Hence

$$\langle \widetilde{w}_1(M)\mathfrak{P}(B), [M]\rangle = \langle \widetilde{w}_1(M)\mathfrak{P}(B), \partial[W,M]\rangle = \langle \delta(\widetilde{w}_1(M)\mathfrak{P}(B)), [W,M]\rangle = 0. \tag{2.5}$$ ⊠

Proposition 2.6 (Conner-Floyd [CF64])**.** *For any* $a \in H^i(X;\mathbb{Z}/2)$ *and degree-*$(n-i)$*-polynomial* p *in the Stiefel-Whitney classes, the* Whitney number

$$\phi_{p,a}\colon (M,f) \longmapsto \langle p(M)f^*(a), [M]\rangle \tag{2.7}$$

is a cobordism invariant for the cobordism theory of unoriented n*-manifolds with a map to* X*, and moreover a group homomorphism* $\Omega_n^{\mathrm{O}}(X) \to \mathbb{Z}/2$.

Proof. This quantity is additive under disjoint union, so it suffices to show that if $(M, f\colon M \to X)$ is the boundary of $(W, g\colon W \to X)$, in the sense that $M = \partial W$ and $g|_M = f$, then $\langle p(M)f^*(a), [M]\rangle = 0$. Let $i\colon M \hookrightarrow W$ be inclusion; as in the proof of Proposition 2.2, it suffices to prove $p(M)f^*(a) \in \mathrm{Im}(i^*)$. Since $f = g \circ i$, then $f^*(a) = i^*(g^*(a)) \in \mathrm{Im}(i^*)$, and since $TW|_M \cong TM \oplus \underline{\mathbb{R}}$, then $i^*w_k(W) = w_k(M)$, so $p(M)$, which is a polynomial in the Stiefel-Whitney classes of M, is also in $\mathrm{Im}(i^*)$; thus, $p(M)f^*(a) \in \mathrm{Im}(i^*)$, which suffices. ⊠

Corollary 2.8. *The quantity* $\langle B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B, [M]\rangle$ *is a cobordism invariant.*

Proof. Let B denote the tautological class in $H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$. The quantity $B\mathrm{Sq}^1B + \mathrm{Sq}^2\mathrm{Sq}^1B$ is a Whitney number for $X = K(\mathbb{Z}/2,2)$, $a = B\mathrm{Sq}^1B + \mathrm{Sq}^2\mathrm{Sq}^1B \in H^5(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, and $p = 1$, hence is a cobordism invariant. ⊠

3. Computing the cobordism group

Proposition 3.1 (Serre [Ser53])**.** $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ *is generated by* $\mathrm{Sq}^I B$*, where* $B \in H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ *is the tautological class, and we consider admissible sequences* $I = (i_1,\ldots,i_m)$ *such that* $i_j \geq 2i_{j+1}$ *and* $2i_1 - \sum_{j\geq 1} i_j < 2$.

Corollary 3.2. *The low-dimensional mod 2 homology groups of* $K(\mathbb{Z}/2,2)$ *are:*

$$H_i(K(\mathbb{Z}/2,2);\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & i = 0,2,3,4 \\ (\mathbb{Z}/2)^{\oplus 2}, & i = 5 \\ 0, & i = 1. \end{cases} \tag{3.3}$$

Proof. The low-degree generators of cohomology are 1 (degree 0), B (degree 2), Sq^1B (degree 3) B^2 (degree 4), and $\mathrm{Sq}^2\mathrm{Sq}^1B$ and $B\mathrm{Sq}^1B$ (degree 5). To get homology, use the universal coefficient theorem over the field $\mathbb{F}_2$, so all Ext terms vanish and the homology and cohomology are isomorphic. ⊠

Proposition 3.4. $\pi_5(MO \wedge K(\mathbb{Z}/2,2)_+) \cong (\mathbb{Z}/2)^{\oplus 4}$.

Proof. MO is a wedge of suspensions of Eilenberg-Mac Lane spectra $H\mathbb{F}_2$, so

$$MO \wedge K(\mathbb{Z}/2,2)_+ \cong \bigvee_k \Sigma^k H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+, \tag{3.5}$$

where k runs over the degrees of the generators of unoriented cobordism as a graded abelian group: $k = 0, 2, 4, 4, 5, \ldots$ (corresponding to pt, $\mathbb{RP}^2$, $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$, and the Wu manifold, respectively).

For the purpose of taking π_5, we can 5-truncate to obtain

$$H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+ \vee \Sigma^2 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+ \vee \Sigma^4 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+ \vee \Sigma^4 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+ \vee \Sigma^5 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+.$$

This is a finite coproduct, hence equivalent to a finite product. π_5 commutes with products, and finite products and sums of abelian groups are the same, so we now have

$$\begin{aligned}\pi_5(MO \wedge K(\mathbb{Z}/2,2)_+) &= \pi_5(H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+) \oplus \pi_5(\Sigma^2 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+) \oplus (\pi_5(\Sigma^4 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2))_+)^2 \\ &\quad \oplus \pi_5(\Sigma^5 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2,2)_+) \qquad (3.6a)\\ &= H_5(K(\mathbb{Z}/2,2);\mathbb{Z}/2) \oplus H_3(K(\mathbb{Z}/2,2);\mathbb{Z}/2) \oplus (H_1(K(\mathbb{Z}/2,2),\mathbb{Z}/2))^2 \oplus H_0(K(\mathbb{Z}/2,2);\mathbb{Z}/2). \qquad (3.6b)\end{aligned}$$

Plug in the results from Corollary 3.2 and we're done. ⊠

4. Finding the generators

Proposition 2.6 tells us that we can use Whitney numbers to determine whether a candidate set of generators is linearly independent. Using Proposition 3.1, the relevant cohomology classes are $1 \in H^0$, $B \in H^2$, $\mathrm{Sq}^1 B \in H^3$, $B^2 \in H^4$, and $B\mathrm{Sq}^1 B$ and $\mathrm{Sq}^2\mathrm{Sq}^1 B \in H^5$. Thus we can write down the Whitney numbers, and some of them coincide.

- When $a = 1$, we get ordinary Stiefel-Whitney numbers for 5-manifolds. Using the Wu formula, one can show they're all either 0 or determined by $w_2 w_3$.
- When $a = B$, we get $w_3 B$, $w_2 w_1 B$, and $w_1^3 B$. However, $w_2 w_1 = v_3$ on any closed manifold, and $v_3 = 0$ on any 5-manifold.
- When $a = \mathrm{Sq}^1 B$, we get $w_2 \mathrm{Sq}^1 B$ and $w_1^2 \mathrm{Sq}^1 B$. However,

  $$\mathrm{Sq}^1(w_1^2 B) = \mathrm{Sq}^1(w_1^2)B + w_1^2 \mathrm{Sq}^1 B = w_1^2 \mathrm{Sq}^1 B, \tag{4.1}$$

  and since $v_1 = w_1$,

  $$\mathrm{Sq}^1(w_1^2 B) = v_1 w_1^2 B = w_1^3 B, \tag{4.2}$$

  so this class isn't anything new; similarly, because $\mathrm{Sq}^1 w_2 = w_3$ on a 5-manifold,

  $$\mathrm{Sq}^1(w_2 B) = \mathrm{Sq}^1 w_2 B + w_2 \mathrm{Sq}^1 B = w_3 B + w_2 \mathrm{Sq}^1 B, \tag{4.3}$$

  and because $w_1 w_2 = 0$ as noted above,

  $$\mathrm{Sq}^1(w_2 B) = v_1 w_2 B = w_1 w_2 B = 0. \tag{4.4}$$

  Thus $w_2 \mathrm{Sq}^1 B = w_3 B$, and this isn't anything new either.
- When $a = B^2$, we get $w_1 B^2$. Since $w_1 = v_1$,

  $$w_1 B^2 = v_1 B^2 = \mathrm{Sq}^1(B^2) = 0. \tag{4.5}$$

- For $a = \mathrm{Sq}^2\mathrm{Sq}^1 B$, we must let $p = 1$, giving the Whitney number $\mathrm{Sq}^2\mathrm{Sq}^1 B$. This is

  $$\mathrm{Sq}^2\mathrm{Sq}^1 B = v_2 \mathrm{Sq}^1 B = (w_2 + w_1^2)\mathrm{Sq}^1 B, \tag{4.6}$$

  so this is a sum of terms we've already accounted for.
- Finally, we can take $a = B\mathrm{Sq}^1 B$, again forcing $p = 1$ and the Whitney number $B\mathrm{Sq}^1 B$.

So there are four candidate Whitney numbers: $w_3 w_2$, $w_3 B$, $w_1^3 B$, and $B\mathrm{Sq}^1 B$. We now use them to determine a generating set of $\Omega_5^{\mathrm{O}}(K(\mathbb{Z}/2,2))$. The answer is in Table 1, and the calculations follow.

Example 4.7. The *Wu manifold* $W := \mathrm{SU}_3/\mathrm{SO}_3$ has cohomology ring $H^*(W;\mathbb{Z}/2) \cong \mathbb{F}_2[z_2, z_3]/(z_2^2, z_3^2)$ with $w = 1 + z_2 + z_3$, $\mathrm{Sq}(z_2) = z_2 + z_3 + z_2^2$, and $\mathrm{Sq}(z_3) = z_3 + z_2 z_3$. Hence $(W, 0)$ and (W, z_2) are linearly independent in $\Omega_5^{\mathrm{O}}(K(\mathbb{Z}/2,2))$, giving us two of the four needed generators. ◀

	$(W,0)$	(W,z_2)	$(S^1\times\mathbb{RP}^4, xy)$,	$(S^1\times\mathbb{RP}^2\times\mathbb{RP}^2, ux)$
w_3w_2	1	1	0	0
w_1^3B	0	0	1	1
w_3B	0	1	0	1
$B\mathrm{Sq}^1B$	0	1	0	0

TABLE 1. Whitney numbers for some nonbounding 5-manifolds, explained below.

Example 4.8. Consider $Y = S^1\times\mathbb{RP}^4$: if x generates $H^1(S^1;\mathbb{Z}/2)$ and y generates $H^1(\mathbb{RP}^4;\mathbb{Z}/2)$, then $H^*(Y;\mathbb{Z}/2)\cong\mathbb{F}_2[x,y]/(x^2,y^5)$ and $w(Y) = 1+y+y^4$; the $\mathcal{A}$-module structure is determined by $\mathrm{Sq}(y) = y+y^2$ and $\mathrm{Sq}(x) = x$. If $B = xy$, $w_1^3B = xy^4$ is nonzero, so $(S^1\times\mathbb{RP}^4, xy)$ is a third linearly independent element. ◂

Example 4.9. Consider $X = S^1\times\mathbb{RP}^2\times\mathbb{RP}^2$, whose cohomology is $H^*(X;\mathbb{Z}/2)\cong\mathbb{F}_2[x,u,v]/(x^2,u^3,v^3)$, where x generates $H^1(S^2;\mathbb{Z}/2)$, u generates H^1 of the first $\mathbb{RP}^2$, and v generates H^1 of the second copy. Then,

$$w(X) = (1+u+u^2)(1+v+v^2) = 1+u+v+u^2+uv+v^2+u^2v+uv^2+u^2v^2. \tag{4.10}$$

The Steenrod action is determined by $\mathrm{Sq}(u) = u+u^2$, $\mathrm{Sq}(v) = v+v^2$, and $\mathrm{Sq}(x) = x$.

When $B = ux$, $w_3B = u^2v^2x \neq 0$, but $B\mathrm{Sq}^1B = u^3x^2 = 0$, so (X, ux) is linearly independent from the previous three examples, and hence is the last generator. ◂

5. Checking on the generators

Proposition 5.1 (Massey [Mas69])**.** *Let $m,n\in\mathbb{Z}$ be such that $m\equiv n \bmod 2$ and X be a topological space. If $a\in H^m(X;\mathbb{Z}/2)$ and $y\in H^n(X;\mathbb{Z}/2)$, then*

$$\mathfrak{P}(ab) = \mathfrak{P}(a)\mathfrak{P}(b) + \theta((\mathrm{Sq}^{m-1}u)v\mathrm{Sq}^1v + u\mathrm{Sq}^1u(\mathrm{Sq}^{n-1}v)),$$

where $\theta\colon H^*(X;\mathbb{Z}/2)\to H^*(X;\mathbb{Z}/4)$ is induced by the multiplication by 2 map $(\cdot 2)\colon\mathbb{Z}/2\to\mathbb{Z}/4$.

Proof of Theorem 1.1. We've reduced the problem to verifying (1.2) on the four generators.

(1) When $(M,B) = (W,0)$, both sides are 0 because $B = 0$.
(2) For $(M,B) = (W,z_2)$, we have

$$B\mathrm{Sq}^1B + \mathrm{Sq}^2\mathrm{Sq}^1B = z_2z_3 + z_2z_3 = 0, \tag{5.2}$$

and since W is orientable, $\widetilde{w}_1(W) = 0$ and the right-hand side is also 0.
(3) If $(M,B) = (S^1\times\mathbb{RP}^4, xy)$,

$$\mathrm{Sq}^1(xy) = x\mathrm{Sq}^1y + x\mathrm{Sq}^1y = xy^2 \tag{5.3a}$$

$$\mathrm{Sq}^2\mathrm{Sq}^1(xy) = (\mathrm{Sq}^2x)y^2 + \mathrm{Sq}^1x\mathrm{Sq}^1y + x\mathrm{Sq}^2(y^2) = xy^4 \tag{5.3b}$$

$$xy\mathrm{Sq}^1(xy) = x^2y^3 = 0. \tag{5.3c}$$

Hence

$$\langle B\mathrm{Sq}^1B + \mathrm{Sq}^2\mathrm{Sq}^1B, [S^1\times\mathbb{RP}^4]\rangle = \langle xy^4, [S^1\times\mathbb{RP}^4]\rangle = 1. \tag{5.4}$$

To compute the right-hand side of (1.2), apply Proposition 5.1 with $m = n = 1$:

$$\mathfrak{P}(xy) = \mathfrak{P}(x)\mathfrak{P}(y) + \theta(xy^3 + x^3y) = \theta(xy^3) \tag{5.5}$$

by degree considerations.

One can check on the generators of $H^3(S^1\times\mathbb{RP}^4;\mathbb{Z}/2)$ to show that xy^3 is not in the image of the Bockstein, hence $\theta(xy^3)\neq 0$. It lands in the piece of $H^4(S^1\times\mathbb{RP}^4;\mathbb{Z}/4)$ that is $H^1(S^1;\mathbb{Z}/4)\otimes H^3(\mathbb{RP}^4;\mathbb{Z}/4)\cong\mathbb{Z}/4\otimes\mathbb{Z}/2\cong\mathbb{Z}/2$, hence is the generator.

Now, $\widetilde{w}_1(S^1\times\mathbb{RP}^4) = \widetilde{w}_1(S^1) + \widetilde{w}_1(\mathbb{RP}^4)$. $\widetilde{w}_1(S^1) = 0$ because S^1 is orientable, and $\widetilde{w}_1(\mathbb{RP}^4) = \widetilde{y}$, the generator of $H^1(\mathbb{RP}^4;\mathbb{Z}_{w_1})$, using that the inclusion $\mathbb{RP}^4\hookrightarrow B\mathrm{O}_1$ is cellular.

Hence, in $H^5(S^1\times\mathbb{RP}^4;(\mathbb{Z}/4)_{w_1})$, $\widetilde{w}_1\mathfrak{P}(xy)$ is nonzero, so must be twice the generator. Thus

$$\langle\widetilde{w}_1\smile\mathfrak{P}(xy), [S^1\times\mathbb{RP}^4]\rangle = 2, \tag{5.6}$$

so (1.2) is valid.

(4) Finally, let $(M, B) = (S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, xy)$. We have

$$B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B = 0 + uvx + uvx = 0. \tag{5.7}$$

Using Proposition 5.1,

$$\mathfrak{P}(B) = \mathfrak{P}(ux) = \mathfrak{P}(u)\mathfrak{P}(x) + \theta(ux\mathrm{Sq}^1 x + xu\mathrm{Sq}^1 u), \tag{5.8}$$

which vanishes by degree considerations: $\mathfrak{P}(x) \in H^2(S^1; \mathbb{Z}/4) = 0$, $\mathrm{Sq}^1 x \in H^2(S^1; \mathbb{Z}/2) = 0$, and $u\mathrm{Sq}^1 u \in H^3(\mathbb{RP}^2; \mathbb{Z}/2) = 0$. ⊠

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