# FALL 2018 HOMOTOPY THEORY SEMINAR

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These notes were taken in the homotopy theory learning seminar in Fall 2018. I live-TeXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Riccardo Pedrotti for fixing a few typos and for providing the notes for §9.

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## 1. Overview: 9/5/18

This short overview was given by Richard.

In the beginning, there were homotopy groups $\pi_n(X) := [S^n, X]$. Homotopy theory begins with the study of these groups, which are hard to calculate. Even the homotopy groups of the spheres, $\pi_k(S^n)$, are complicated. However, there are patterns.

**Theorem 1.1** (Freudenthal suspension theorem). *For $n \geq k+2$, $\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1})$.*

The first few of these stable homotopy groups are $\pi_n(S^n) = \mathbb{Z}$, $\pi_{n+1}(S^n) = \mathbb{Z}/2$, $\pi_{n+2}(S^n) = \mathbb{Z}?2$, $\pi_{n+3}(S^n) = \mathbb{Z}/24$, $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$, $\pi_{n+6}(S^n) = \mathbb{Z}/2$, and $\pi_{n+7}(S^n) = \mathbb{Z}/120$.

You can encode all of this stability data in one place using spectra. There's an object $\mathbb{S}$ called the *sphere spectrum* built in a precise way from spheres, and the homotopy groups of $\mathbb{S}$ are the stable homotopy groups of the spheres.

These stable homotopy groups are very hard to calculate. However, we can work locally (at primes), which simplifies the problem a little bit.

**Theorem 1.2** (Fracture square). *Let $X$ be a space, $X_\mathbb{Q}$ be its rationalization, and for $p$ a prime let $X_p$ denote the $p$-completion of $X$. Then the following square is a homotopy pullback:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_\mathbb{Q} \arrow[d] \\
\displaystyle\prod_{p \text{ prime}} X_p \arrow[r] & \left(\displaystyle\prod_{p \text{ prime}} X_p\right)_\mathbb{Q}.
\end{tikzcd}$$

Here $\pi_*(X_p) = \pi_*(X) \otimes \mathbb{Z}_p$ and $\pi_*(X_\mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$. The upshot of Theorem 1.2 is that these groups determine the original homotopy groups of $X$.

The rational homotopy groups of spheres are known, due to an old theorem of Serre. Over $p$, there are other techniques, such as the Adams and Adams-Novikov spectral sequences. The Adams-Novikov spectral sequences uses a filtration on $X_p$ to produce a spectral sequence with $E_2$-term

$$E_2^{*,*} = \operatorname{Ext}_{BP_*BP}(BP_*, BP_*(X)), \tag{1.3}$$

and converging to $\pi_*(X)_{(p)}$ ($p$-local, not $p$-complete!). Here $BP$ is a spectrum, but you don't actually need to know much about it (yet): $BP_*$ is some algebra, and $BP_*BP$ is a Hopf algebra, and they can be described explicitly. We'll learn more about this spectral sequence in time.

If you look at a picture of the $E_\infty$-page of the Adams-Novikov spectral sequence for any $p$ (maybe just $p$ odd for now), there are strong patterns: a pattern along the bottom, which is the $\alpha$-family (said to be $v_1$-periodic), and some periodic things along the diagonal (said to be $v_2$-periodic), containing the $\beta$-family. Both of these are families in the homotopy groups of spheres, providing structue in the complicated story — we don't know the stable homotopy groups of spheres past about 60, so producing families is very helpful for our understanding! In a similar way, one can find $v_3$-periodic elements, including something called the $\gamma$-family, and so forth.

Of course, there's a lot of work to do even from here: how to we get here from the $E_2$-page? Do the extension problems go away, giving us actual elements of the stable stem? For the $\alpha$-, $\beta$-, and $\gamma$-families, these are known, and there are even geometric interpretations for small $n$ (up to 3 or 4) and large $p$ (usually something like $p > 5$ or $p > 7$). Specifically, if $V(0)$ denotes cofiber of the multiplication-by-$p$ map $\mathbb{S} \to \mathbb{S}$, the $\alpha$-family comes from self-maps $\Sigma^k V(0) \to V(0)$, together with the maps to and from $\Sigma^k \mathbb{S}$ coming from the cofiber sequence. There are less explicit complexes $V(1)$ and $V(2)$ which give you the $\beta$- and $\gamma$-families, and there is a similar story.

## 2. Introduction to spectra: 9/12/18

The goal of studying Hopf algebroids is to study the cohomology operations of things such as $MU$ and $BP$, but the definition of a Hopf algebroid doesn't look helpful at first. So first we'll look at algebraic stacks, because there's an interpretation of the Adams and Adams-Novikov spectral sequences in this setting.

Let $\mathscr{X}$ be an algebraic stack and $U = \operatorname{Spec} A$ be an affine atlas for $\mathscr{X}$, coming with a faithfully flat map $U \to X$. Let $R\colon U \times_{\mathscr{X}} U$, which admits two maps to $U$; we say $R \rightrightarrows U$ is a *presentation* of $\mathscr{X}$, or $\mathscr{X} = [U/R]$. We would like to be able to recover $\mathscr{X}$ from its presentation, and in order to do that we need stacks to be valued in groupoids.

That is, the functor of points of a stack is a functor $\mathscr{X}\colon \mathsf{CAlg} \to \mathsf{Grpd}$.[1] Examples include sending a scheme $B$ to the groupoid of families of elliptic curves over $B$ (so $\mathscr{X}$ is the moduli of elliptic curves).

These two ways to think of a stack are closely related: $\mathscr{X}(B)$ is a groupoid, so it has a set of objects, which is precisely $U(B)$, and a set of morphisms, which is precisely $R(B)$. The two maps $R(B) \rightrightarrows U(B)$ send a morphism to its source, resp. target.

Let $E$ be a good ring spectrum (so we have a well-behaved $E$-based Adams spectral sequence). Then $\operatorname{Spec} E_*E \rightrightarrows \operatorname{Spec} E_*$ is a groupoid object in affine schemes; the quotient stack is denoted $\mathscr{X}_E$. The descent spectral sequence for this stack is exactly the $E$-based Adams spectral sequence.[2]

**Definition 2.1.** A *Hopf algebroid* is a groupoid object in the category of affine schemes, or equivalently, a cogroupoid object $(A, \Gamma)$ in $\mathsf{CAlg}$.

So given a Hopf algebroid $(A, \Gamma)$, we have a commutative algebra of objects $A$ and a commutative algebra $\Gamma$ of morphisms. Then, opposite to the source and target maps, we have *left and right unit* maps $\eta_L, \eta_R\colon A \to \Gamma$, and a *comultiplication* $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$ dual to composition. Dual to the identity morphism is an *augmentation map* $\varepsilon\colon \Gamma \to A$ and dual to the inverse is the *conjugation* map $c\colon \Gamma \to \Gamma$. These satisfy a number of relations which you would want them to satisfy.

- $\varepsilon \circ \eta_L = \varepsilon \circ \eta_R = \mathrm{id}$: the sorce and target of an identity morphism is that object itself.
- Comultiplication is coassociative, i.e. the following diagram commutes.

$$\begin{tikzcd} \Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \Gamma \times_A \Gamma \arrow[d, "\Delta\otimes\mathrm{id}"] \\ \Gamma \otimes_A \Gamma \arrow[r, "\mathrm{id}\otimes\Delta"'] & \Gamma \otimes_A \Gamma \otimes_A \Gamma. \end{tikzcd} \tag{2.2}$$

[1] Here $\mathsf{CAlg}$ is the category of commutative algebras over some ground field or ring. Rok denoted this $\mathsf{CAlg}^\heartsuit$; I don't work implicitly in the derived world.

[2] TODO: I don't know what this means.

- There are a few more, which are a bit more tedious to write down.

Now we can say something about the relationship between Hopf algebroids and Hopf algebras. A Hopf algebroid is precisely (dual to) a group object in affine schemes, and a group is just a groupoid with a single object. So we should be able to recover the usual definition of a Hopf algebra as a "Hopf algebroid with a single object." This is precisely what happens in the case $\eta_L = \eta_R$.

**Example 2.3.** Let $BG$ be the classifying stack of a group $G$, assumed to be flat and algebraic. This has a presentation $G \rightrightarrows \mathrm{pt}$. If we take rings of functions on these schemes, we get $k \to \mathscr{O}(G)$, i.e. a presentation of $\mathscr{O}(G)$ as a Hopf algebra. This is reassuring, because we said that Hopf algebras are (dual to) group objects in schemes. $\blacktriangleleft$

**Example 2.4.** The examples we care more abou are when $\mathscr{X}$ is the stack of formal group laws (so over $\mathbb{Z}$) or $p$-typical formal group laws (over $\mathbb{Z}_{(p)}$). These arise from the Adams construction, e.g.

$$\mathscr{M}_{\mathrm{FGL}} = [\operatorname{Spec} \pi_* MU / \operatorname{Spec} MU_* MU]. \tag{2.5}$$

To get the $p$-typical group laws, replace $MU$ with $BP$.

To recover the classical Adams spectral sequence, we can let $\mathscr{X}$ be the stacky quotient of $\operatorname{Spec} \mathbb{F}_p$ by $\operatorname{Spec} \mathscr{A}_p^\vee$. At least when $p = 2$, there's a deep and mysterious result: $\operatorname{Spec} \mathscr{A}_2^\vee \cong \operatorname{Aut}_{\mathbb{F}_2}(\widehat{\mathbb{G}}_a)$, the automorphisms of the additive formal group. Something analogous is true over other primes, even if the statement is slightly different. Thus the stack we get is a classifying space $B \operatorname{Aut}_{\mathbb{F}_2}(\widehat{\mathbb{G}}_a)$. $\blacktriangleleft$

Finally we need to make sense of comodules over a Hopf algebroid — but essentially by definition, these are the same thing as quasicoherent sheaves over the corresponding quotient stack. I unfortunately missed Rok's talk, but he gave the last 10 minutes as the first 10 minutes of the second week, so here it is.

Recall that a spectrum $X$ is a sequence of pointed spaces $\{X_n\}_{n\in\mathbb{Z}}$ together with weak equivalences $X_n \simeq \Omega X_{n+1}$. There's a functor $\Sigma^\infty$ from spaces to spectra which turns several topological concepts into algebraic ones that make $\mathsf{Sp}$ behave like the derived category $\mathscr{D}(R)$ of $R$-modules for $R$ a commutative ring. Here's a dictionary:

- $\Sigma^\infty \mathrm{pt}$ is the *zero spectrum*, which corresponds to the *zero complex* of $R$-modules (zero in every degree).
- $\Sigma^\infty S^0$, denoted $\mathbb{S}$, is the *sphere spectrum*, which corresponds to $R$ as an $R$-module.
- Suspension of spaces is sent to suspension of spectra, which corresponds to the shift functor $[1]$ of a derived category.
- The (based) loop space functor $\Omega$ maps to *de*suspension of spectra, which corresponds to the shift functor $[-1]$ in the derived category.
- Wedge sum of spaces turns into wedge sum of spectra, which can be thought of as a direct sum, and corresponds to the direct sum of complexes of $R$-modules.
- Smash product of spaces turns into smash product of spectra, which is their tensor product, and corresponds to the derived tensor product ${}^{\mathbf{L}}\otimes_R$ of complexes.
- Stable homotopy groups of spaces map to homotopy groups of spectra, which behave like cohomology groups in the derived category.

There's a homotopical reason to believe this analogy between spectra and the derived category: the Eilenberg-Mac Lane functor $H\colon \mathsf{Ab} \to \mathsf{Sp}$ induces an equivalence between the (homotopy or $(\infty, 1)$) categories $\mathsf{Mod}_{HR}$ of $R$-module spectra and $\mathscr{D}(R)$ which sends smash product over $HR$ to the derived tensor product over $R$.

The sphere spectrum is the unit for the smash product, so we can think of spectra as the category of $\mathbb{S}$-modules, which is a very useful, and sometimes literaly, analogy.

Spectra define cohomology theories: if $E$ is a spectrum and $X$ is a space (non-pointed), then the associated cohomology theory is defined by $E^i(X) := [X, \Sigma^i E]$.

## 3. Spectral sequences: 9/17/18

Here's Ricky's talk on spectral sequences, followed (TODO) by notes from Arun's part of the talk.

Let $C = \bigoplus_{n=0}^\infty C^n$ be a graded $R$-module and assume it has a decreasing filtration by chain maps

$$C \supseteq \cdots \supseteq F^p C \supseteq F^{p+1} C \supseteq \cdots, \tag{3.1}$$

meaning that $d$ carries $F^p C^{p+q}$ into $F^p C^{p+q+1}$. (Upper indices typically correspond to decreasing filtrations.) Let's assume for now that

- $R = k$ is a field, and
- for each $n$, $F^\bullet C^n$ is finite.

Then there's a filtration on cohomology, where

$$F^pH^*(C) := \operatorname{Im}(H^*(F^pC \hookrightarrow C)) = \pi(\underbrace{F^pC^{p+q} \cap \ker(d)}_{Z_\infty^{p,q}}), \tag{3.2}$$

where $\pi\colon \ker(d) \to \ker(d)/\operatorname{Im}(d) = H^{p+q}(C)$ is the quotient map. Because

$$F^pH(C)/F^{p+1}H(C) = \pi(Z_\infty^{p,q})/\pi(Z_\infty^{p+1,q-1}) = Z_\infty^{p,q}/(Z_\infty^{p+1,q-1} + B_\infty^{p,q}), \tag{3.3}$$

where $B_\infty^{p,q} := F^pC^{p+q} \cap \operatorname{Im}(d)$.

Let $E_0^{p,q} := F^pC^{p+q}/F^{p+1}C^{p+q}$; then, the differentials induce maps $E_0^{p,q-1} \to E_0^{p,q} \to E_0^{p,q+1}$, and they satisfy $d_0^2 = 0$ because we originally had $d^2 = 0$. Then

$$\frac{\ker(d_0)}{\operatorname{Im}(d_0)} = \frac{F^pC^{p+q} \cap d^{-1}(F^{p+1}C^{p+q+1})}{\underbrace{F^pC^{p+q} \cap d(F^pC^{p+q-1})}_{B_0^{p,q}} + \underbrace{F^{p+1}C^{p+q}}_{Z_0^{p,q-1}}} = \frac{Z_1^{p,q}}{B_0^{p,q} + Z_0^{p,q-1}}. \tag{3.4}$$

Define

$$\begin{aligned} Z_r^{p,q} &:= F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}) \\ B_r^{p,q} &:= F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}) \\ E_r^{p,q} &:= Z_r^{p,q}/(Z_{r-1}^{p,q-1} + B_{r-1}^{p,q}). \end{aligned} \tag{3.5}$$

The key claim is that

$$H^*(E_r^{p,q}, d_r) = E_{r+1}^{p,q}. \tag{3.6}$$

A spectral sequence is, roughly speaking, something which behaves like this.

**Definition 3.7.** A *(cohomologically graded) spectral sequence* is a collection $\{E_r^{\bullet,\bullet}, d_r\}$ of differentially bigraded modules such that $d_r$ has bidegree $(r, 1-r)$ and such that $E_{r+1}^{p,q} = H^*(E_r^{p,q}, d_r)$. If $E_r^{p,q}$ is constant in $r$ when $p$ and $q$ are fixed after some finite number of pages $r$, then we also call it $E_\infty^{p,q}$.

The spectral sequence *converges* to $(H^*, F)$, a filtered graded $R$-module, if $E_\infty^{p,q}$ is the associated graded of $H^*$. This implies $H^r$ is a direct sum of $E_\infty^{p,q}$ over all $p + q = r$.[3]

Sometimes spectral sequences have more structure given by multiplication. In this case, we want each $E_r^{\bullet,\bullet}$ to be a *differential bigraded R-algebra*, meaning it has a multiplication map which is additive on bidegrees of homogeneous elements, and that the differential obeys a graded Leibniz rule with respect to total grading:

$$d(xy) = d(x)y + (-1)^{|x|}xd(y). \tag{3.8}$$

Suppose we took the spectral sequence of a filtered $R$-module above, but it's also an $R$-algebra. Unfortunately, the higher pages in the spectral sequence aren't $R$-algebras without some work (TODOI missed this).

**The Serre spectral sequence.** Here's Arun's example with the Serre spectral sequence.[4]

**Definition 3.9.** A *(Serre) fibration* $f : E \to X$ of topological spaces is a map such that if $\Delta^n$ denotes the $n$-simplex and one has commuting maps

$$\begin{tikzcd} \Delta^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ \Delta^n \times [0,1] \arrow[r] & X, \end{tikzcd}$$

there exists a map $G\colon \Delta^n \times [0,1] \to E$ that commutes with the maps in the diagram.

We always take $X$ to be path-connected, in which case $f^{-1}(x) \simeq f^{-1}(x')$ for all $x, x' \in X$. This preimage is called the *fiber* of $f$, and is often denoted $F$; the triple $F \to E \to X$ is called a *fiber sequence*. We will also assume $X$ is simply connected, which will allow us to obtain stronger results.

**Example 3.10.** Let $M$ be a manifold of dimension $n$. Then, $TM \to M$ is a fibration, and the fiber is $\mathbb{R}^n$. ◄

[3] If $R$ isn't a field, then it might instead be an extension that doesn't split.

[4] I learned this example from Ernie Fontes, and this presentation is adapted from his presentation of this example.

**Theorem 3.11** (Serre)**.** *Fix a coefficient ring $R$; let $f : E \to X$ be a fibration and $F$ be its fiber. Then, there exists a multiplicative spectral sequence, called the* Serre spectral sequence

$$E_2^{p,q} = H^p(X; H^q(F;R)) \Longrightarrow H^{p+q}(E;R).$$

*Proof sketch.* Let $\{X_i\}$ be the CW filtration of $X$, and let $E_i := f^{-1}(X_i)$, which induces an exhaustive filtration $\{E_i\}$ of $E$. Applying $H^q(-;R)$ defines a spectral sequence by the formalism from the first lecture. The multiplicative structure comes from the cup product on $X$. ⊠

*Remark* 3.12. Let $A$ be a multiplicative generalized cohomology theory (e.g. $K$-theory). Then, we could have applied $A$ instead of $H^q(-;R)$ and obtained a multiplicative spectral sequence

$$E_2^{p,q} = H^p(X; A^q(F)) \Longrightarrow A^{p+q}(E).$$

Letting $A = H^*(-,R)$, we recover the Serre spectral sequence, and letting $E \to X$ be the identity map $X \to X$, which is a fibration, we recover the Atiyah-Hirzebruch spectral sequence. For this reason this spectral sequence is sometimes called the *Serre-Atiyah-Hirzebruch spectral sequence*. ◀

**Example 3.13.** Let $PX := \mathsf{Top}_*(I, X)$ denote the *path space*, i.e. the maps from the unit interval to $X$. Evaluation at 0 defines a map $ev\colon PX \to X$. The path space is made of spaghetti, hence is contractible: shrink each path from time 1 to time $t$, and let $t \to 0$.

$ev\colon PX \to X$ is a fibration, and the fiber is $\Omega X$, the space of (based) loops in $X$ (i.e. based maps $S^1 \to X$). Recall that a fiber sequence induces a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \tag{3.14}$$

Since $\pi_n(PX) = 0$, this implies $\pi_n(X) \cong \pi_{n-1}(\Omega X)$.

Let's apply the Serre spectral sequence to this fibration in the case where $R = \mathbb{Q}$ and $X = S^3$. The Serre spectral sequence takes the form

$$E_2^{p,q} = H^p(S^3; H^q(\Omega S^3;\mathbb{Q})) \Longrightarrow H^{p+q}(PS^3;\mathbb{Q}).$$

We know the $E_\infty$ page already: it's 0 unless $p+q=0$, in which case it's $\mathbb{Q}$. So we're going to reverse-engineer the spectral sequence, to use the $E_\infty$ page to compute the $E_2$ page.

We also know $H^*(S^3;\mathbb{Q}) = E_{\mathbb{Q}}(X)$, where $|x| = 3$, an exterior algebra in one variable. This is also isomorphic to $\mathbb{Q}[x]/x^2$, so has a $\mathbb{Q}$ in degrees 0 and 3, and is 0 elsewhere.

We know $H^0(\Omega S^3;\mathbb{Q}) = \mathbb{Q}$, so the $E_2$ page looks like

$$\begin{array}{c|cccc}
3 & ? & & & ? \\
2 & ? & & & ? \\
1 & ? & & & ? \\
0 & 1 & & & x \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

with the missing entries equal to 0.

We know that the $(3,0)$ term has to vanish by the $E_\infty$ page, so it either *supports a differential* (has a nonzero differential mapping out of it) or *receives a differential* (has a nonzero differential mapping into it). Since this is a first-quadrant spectral sequence, all differentials out of $x$ hit 0, so it has to receive a differential. But on the $E_2$ page, this differential comes from the 0 in position $(1,1)$, so it's zero, and any differentials in page 4 or above mapping into $x$ come from the fourth quadrant, so there has to be a nonzero differential on the $E_3$ page mapping into $x$, so there's some $y \in E_2^{0,2}$, which generates a copy of $\mathbb{Q}$, such that $d_3 y = x$. There can't be more than one generator in $E_2^{0,2}$, because then either it would survive to the $E_\infty$ page (which can't happen), or it gets killed, meaning the difference of it and $y$ is not killed by $d_3$ and hence survives. Oops. Thus, $E_2^{0,2} = H^2(\Omega S^3;\mathbb{Q}) \cong \mathbb{Q}$. Hence we know $E_2^{3,2} = H^3(S^3;\mathbb{Q})$ as well, and the spectral sequence looks like

$$
\begin{tikzcd}
2 & y \arrow[rrrdd, "d_3"] & & & \mathbb{Q} \\
1 & ? & & & ? \\
0 & 1 & & & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}
$$

We can also immediately determine $E_2^{\bullet,2}$: looking at $E_2^{0,2}$, there are no differentials that hit something nonzero, or map from something nonzero to it. So anything in it survives to the $E_\infty$ page, and hence it must be zero. Thus $H^1(\Omega S^3;\mathbb{Q}) = 0$ and hence $E_2^{1,3} = 0$ too.

The multiplicative structure tells us that the generator of $E_2^{3,2}$ must be $y \cdot x$. Thus, the spectral sequence looks like

$$
\begin{tikzcd}
2 & y \arrow[rrrdd, "d_3"] & & & yx \\
1 & & & & \\
0 & 1 & & & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}
$$

But now $yx$ has to die, and the only way that can happen is if it's hit by $d_3$ of the $E_2^{0,4}$ term, which turns out to be $y^2$. This is because $d_3 y = x$, so

$$d_3(y^2) = d_3(y)y + (-1)^2 y d_3(y) = 2xy.$$

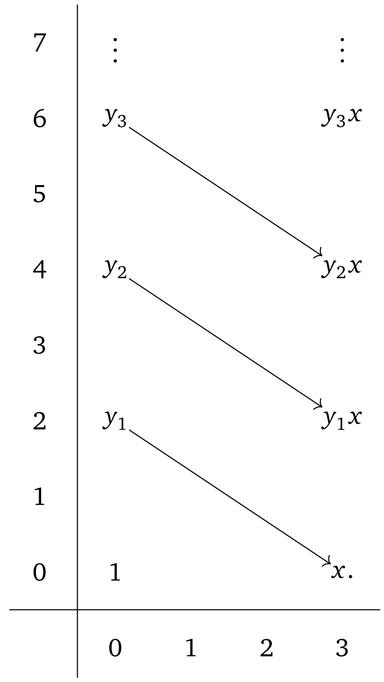
Thus $d_3$ is multiplication by 2. Hence the spectral sequence looks like

$$
\begin{tikzcd}
4 & y^2 \arrow[rrrdd, "\cdot 2"] & & & y^2x \\
3 & & & & \\
2 & y \arrow[rrrdd, "\cdot 1"] & & & yx \\
1 & & & & \\
0 & 1 & & & x. \\
 & 0 & 1 & 2 & 3
\end{tikzcd}
$$

But now we need $y^2x$ to vanish, and it's hit by $y^3 \in E_2^{0,6}$ via $d_3$, which is multiplication by 3, and so on. Inductively we can conclude that

$$H^*(\Omega S^3;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

Much of this argument, but not all of it, works with $\mathbb{Q}$ replaced by $\mathbb{Z}$. The difference is that multiplication by 2, 3, etc. is not an isomorphism, so we have a sequence of generators $y_1, y_2, \dots$:




Now we have to figure out the multiplicative structure. We know $y_1^2 = c_1y_2$ for some $c_1 \in \mathbb{Z}$, so since $d_3$ is an isomorphism, let's compute: we know $d_3(y_2) = y_1x$ by construction, and $d_3(y_1^2) = 2y_1x$ for the same reason as over $\mathbb{Q}$, so $y_1^2 = 2y_2$.

A similar calculation in general shows that $y_1^n = n!y_n$, as

$$\begin{aligned}
d_3(y_1^n) &= d_3(y_1y_1^{n-1}) = d_3(y_1)y_1^{n-1} + y_1(n-1)!d(y_{n-1}) \\
&= xy_1^{n-1} + y_1(n-1)!xy_{n-2} \\
&= x(n-1)!y_{n-1} + (n-1)y_{n-1}x(n-1)! \\
&= n!xy_{n-1},
\end{aligned}$$

but $d_3(n!y_n) = n!xy_{n-1}$. Hence the ring structure on $H^*(\Omega S^3)$ is a divided power algebra.

**Definition 3.15.** A *divided power algebra* on a single generator $x$ in degree $k$, denoted $\Gamma(x)$, is the free algebra generated by $\{x_i\}_{i\geq 1}$ where $|x_i| = ki$, subject to the relations

$$x_ix_+j = \binom{i+j}{j}x_{i+j} \qquad \text{and} \qquad x_i = \frac{x^i}{i!}.$$

Thus $H^*(\Omega S^3) \cong \Gamma(y)$ with $|y| = 2$.

## 4. First steps with the Adams spectral sequence: 9/24/18

Today's talk was given by Riccardo and Alberto.

Fix $R$ a commutative ring and $M$ an $R$-module.

**Definition 4.1.** A *left exact* functor $F\colon \mathsf{Mod}_R \to \mathsf{Ab}$ is a functor which sends a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{4.2}$$

to an exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C), \tag{4.3}$$

which may not necessarily complete to an exact sequence.

The easiest example of a left exact functor which isn't exact is $\mathrm{Hom}_R(-,M)$ for certain choices of $M$.

**Lemma 4.4.** *With $R$ and $M$ as above, $\mathrm{Hom}_R(-,M)$ is exact iff $M$ is projective.*

So if we'd like to understand what happens when we hit exact sequences with $\mathrm{Hom}_R(-,M)$ for $M$ not projective, it would be good to approximate $M$ by projectives.

**Definition 4.5.** A *projective resolution* of $M$ is an exact sequence

$$\cdots \longrightarrow P_j \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

written $P_\bullet \to M$, such that each $P_j$ is projective.

**Lemma 4.6** (Fundamental lemma of homological algebra)**.** *Any two projective resolutions of $M$ are chain homotopy equivalent.*

This makes the following definition independent of $P_\bullet \to M$.

**Definition 4.7.** Let $N$ be another $R$-module. The $i^{\text{th}}$ *Ext group* is $\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}(P_\bullet, N))$, where $P_\bullet \to M$ is a projective resolution.

**Theorem 4.8.** *Let $R$, $M$, and $N$ be as above.*

(1) $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$.
(2) *A short exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules induces a long exact sequence*

$$0 \longrightarrow \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \xrightarrow{\delta} \mathrm{Ext}^1_R(M,A) \longrightarrow \cdots \tag{4.9}$$

*with natural maps* $\mathrm{Ext}^i_R(M.C) \to \mathrm{Ext}^{i+1}_R(M,A)$.

Now we assume $R$ is a graded ring and $M$ is a graded $R$-module. We will use $\Sigma^r$ to denote shift by $R$, i.e. $\Sigma^r M$ is the graded $R$-module with $(\Sigma^r M)^t := M^{t-r}$.

**Example 4.10.** In this setting $\mathrm{Hom}_R(M,N)$ is also a graded object, with $\mathrm{Hom}^i_R(M,N) := \mathrm{Hom}_R(M,\Sigma^i N)$ (the latter are degree-preservving maps). ◂

This implies Ext is *bi*graded: $\mathrm{Ext}^{r,s}_R(M,N) :=$TODO. There's a pairing called the Yoneda product on Ext groups, which has signature

$$\mathrm{Ext}^{s,t}_R(M,N) \otimes \mathrm{Ext}^{s,t}_R(L,N) \longrightarrow \mathrm{Ext}^{s+s',t+t'}_R(L,N). \tag{4.11}$$

The Adams spectral sequence involves bigraded Ext for a specific choice of $R$, so let's turn to that choice of $R$.

**Definition 4.12.** A *cohomology operation of degree $k$*[5] is a natural transformation $\gamma\colon H^*(-,\mathbb{F}_2) \to H^{*+k}(-;\mathbb{F}_2)$. If it commutes with the suspension isomorphism, we say $\gamma$ is *stable*.

**Definition 4.13.** The *Steenrod algebra $\mathscr{A}$* is the graded, noncommutative, infinitely generated $\mathbb{F}_2$-algebra of stable cohomology operations: in degree $k$ it is the degree-$k$ stable cohomology operations.

Since $H^n(-;\mathbb{F}_2) \cong [-,K(\mathbb{F}_2,n)]$, and these Eilenberg-Mac Lane spaces are the constituents in the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$, then essentially by the Yoneda lemma, $\mathscr{A} \cong H\mathbb{F}_2^*(H\mathbb{F}_2)$. This implies no stable cohomology operations of negative degre exist (since $H\mathbb{F}_2$ is connective).

**Theorem 4.14.** *For all $k \ge 0$, there is a stable cohomology operation $\mathrm{Sq}^k$ of degree $k$ with the following properties:*

- $\mathrm{Sq}^0 = \mathrm{id}$ *and* $\mathrm{Sq}^1$ *is the* Bockstein, *the natural transformation $H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z}/2)$ coming from the connecting morphism in the long exact sequence in cohomology induced from the short exact sequence* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.
- *If $x \in H^k(X;\mathbb{Z}/2)$, then* $\mathrm{Sq}^k(x) = x^2$.
- *If $x \in H^i(X;\mathbb{Z}/2)$ and $i < k$, then* $\mathrm{Sq}^k(x) = 0$.
- *(Cartan formula)*

$$\mathrm{Sq}^k(x \smile y) = \sum_{i+j=k} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$

*The Steenrod algebra is generated by these elements, and these properties characterize them.*

[5]In general one can consider other coefficient groups than $\mathbb{F}_2$.

In fact, these generators have redundancies: $\mathscr{A}$ is generated by $\mathrm{Sq}^{2^i}$ for $i \geq 0$.

**Example 4.15.** We can use this to show the Hopf fibration $\eta\colon S^3 \to S^2$ is nontrivial. This is the quotient of $S^3$ by the $\mathrm{U}_1$-action on it as the unit sphere in $\mathbb{C}^2$; the quotient is $\mathbb{CP}^1$, also known as $S^2$. It suffices to know that the cofiber of $\eta$, which has the homology of $S^3 \wedge S^2$, isn't homotopic to $S^3 \wedge S^2$, and you can check this by showing its cohomology has a different $\mathscr{A}$-module structure. ◂

This data all enters into a spectral sequence called the *Adams spectral sequence*. Fix spaces (or spectra) $X$ and $Y$; then, the spectral sequence has $E_2$-page

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)), \tag{4.16}$$

and which converges to $[X, Y_{(2)}^\vee]_{t-s}$. This means stable homotopy classes of maps between $X$ and the 2-completion of $Y$. (There are analogues of this, and of the Steenrod algebra, over other primes.) This completion on groups gives you $\varprojlim_n G/2^n$, and does something similar for spaces.

If $X = Y$, the Yoneda product on $\mathrm{Ext}_{\mathscr{A}}^{s,t}$ induces a product on the $E_2$-page of the Adams spectral sequence.

Since $\mathscr{A}$ isn't finitely generated, the Adams spectral sequence is complicated, but there's a clever simple application using connective *ko*-theory (a version of *KO*-theory with no nonzero negative homotopy groups). One can compute that

$$H^*(ko; \mathbb{Z}/2) \cong \mathscr{A} \otimes_{\mathscr{A}(1)} \mathbb{Z}/2, \tag{4.17}$$

where $\mathscr{A}(1) = \langle \mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ inside $\mathscr{A}$. The change-of-rings formula for Hom induces a change-of-rings formula for Ext:

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{\mathscr{A}(1)} \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathscr{A}(1)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2). \tag{4.18}$$

This is much nicer: $\mathscr{A}(1)$ is 8-dimensional, making all of the algebra simpler. Moreover, there's a traditional diagrammatic way to describe $\mathscr{A}(1)$-module structures, in which $\mathrm{Sq}^1$-actions are given by straight lines and $\mathrm{Sq}^2$-actions are given by curly lines. For example, $\mathscr{A}(1)$ is drawn in Figure 1.

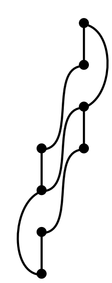


FIGURE 1. The algebra $\mathscr{A}(1)$: the vertical stratification is the degree, the straight lines are $\mathrm{Sq}^1$, and the curvy lines are $\mathrm{Sq}^2$.

For example, we can draw a projective resolution of $\mathbb{Z}/2$ as an $\mathscr{A}(1)$-module (on the board, but not really live-TEXable in time). If you work out a few terms, you'll see that there's a pattern of the kernel, so the terms in the resolution are always of the form $\Sigma^{m_1}\mathscr{A}(1) \oplus \Sigma^{m_2}\mathscr{A}(1)$. Since

$$\mathrm{Hom}^s(\Sigma^r \mathscr{A}(1), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & r = s \\ 0, & \text{otherwise,} \end{cases} \tag{4.19}$$

passing to the $E_2$-page is relatively simple once you have the resolution. Looking at a picture of the $E_2$-page, one sees infinitely many dots for $t - s = 0$ or 4 (or 8, etc.), one dot each in $t - s = 1, 2$, and $9, 10$, etc., and no places where there could be nontrivial differentials. Therefore, if you can resolve an extension problem you've proven Bott periodicity for *ko*-theory.

Recall that we've seen two spectral sequences so far: the Serre spectral sequence, with signature

$$E_2^{p,q} = H^p(B; H^q(F)) \Longrightarrow H^{p+q}(F), \tag{5.1}$$

and the Adams spectral sequence, with signature

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_*}(H^*(Y;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_*(Y)^\wedge_{(2)}. \tag{5.2}$$

This converges when $Y$ is connective and of finite type. The algera $\mathscr{A}$, called the *Steenrod algebra*, is $[H\mathbb{F}_2, H\mathbb{F}_2] = H\mathbb{F}_2^* H\mathbb{F}_2$.

The Adams spectral sequence is pretty amazing, and it would be nice to generalize it. There are versions over other primes, using $\mathscr{A}_p := H\mathbb{F}_p^* H\mathbb{F}_p$, but these are also kind of messy. One idea is to dualize everything: let $\mathscr{A}^\vee := \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = H\mathbb{F}_{p*} H\mathbb{F}_p$, and to use the homology of $Y$ instead.

We built the Serre spectral sequence from the Postnikov tower for the total space, which is a "resolution" of the space by spaces in which we've killed off homotopy groups. Dual to that, there's an *Adams tower* which kills off cohomology: starting with $Y$, define spaces $\cdots \to Y_2 \to Y_1 \to Y$ such that

(1) the cofiber $Ci_j$ of the map $i_j \colon Y_{j+1} \to Y_j$ is a wedge of $H\mathbb{F}_p$s, and
(2) such that the induced map on cohomology $H^*(Ci_j) \to H^*(Y_j)$ is an epimorphism.

**Exercise 5.3.** In this setting, the sequence

$$0 \longleftarrow H^*(Y) \longleftarrow H^*(Ci_0) \longleftarrow H^*(\Sigma Ci_1) \longleftarrow H^*(\Sigma^2 Ci_2) \longleftarrow \cdots \tag{5.4}$$

is a resolution of $H^*(Y)$ as $\mathscr{A}$-modules.

**Proposition 5.5.** *Adams towers exist for all $Y$.*

*Proof.* Let $\overline{H\mathbb{F}_p}$ be the fiber of the unit map $\epsilon \colon \mathbb{S} \to H\mathbb{F}_p$. Then we can let

$$\begin{aligned} Y_s &:= (\overline{H\mathbb{F}_p})^{\wedge s} \wedge Y_0 \\ Ci_s &:= H\mathbb{F}_p \wedge Y_s, \end{aligned} \tag{5.6}$$

and check that these satisfy the criteria. ⊠

*Remark* 5.7. Via some cosimplicial nonsense,[6] Adams towers for $Y$ are equivalent to cosimplicial resolutions of $Y$, which is a kind of Dold-Kan correspondence. The cosimplicial resolution corresponding to the Adams tower we described above is

$$\mathrm{Tot}_n(CB^\bullet(H\mathbb{F}_p) \wedge Y), \tag{5.8}$$

where $CB^\bullet(H\mathbb{F}_p)$ is a *cobar construction*:

$$H\mathbb{F}_p \rightrightarrows H\mathbb{F}_p \wedge H\mathbb{F}_p \Rrightarrow H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge H\mathbb{F}_p \Rrightarrow \cdots \tag{5.9}$$

◀

Anyways, taking $\pi_*$ of our cobar resolution, we get a resolution for $H_*(Y)$, which is a good first step for the generalized Adams spectral sequence.

We want to produce an $E$-based Adams spectral sequence, where $E$ is a commutative ring spectrum, meaning we want resolutions, an $E_2$-term which uses Ext, and some nice convergence result.

The maps $E \rightrightarrows E \wedge E$ (unit smash identity, identity smash unit) induce on homotopy groups left and right $E_*$-actions on $E_*E = \pi_*(E \wedge E)$. This is the first step in the sequence

$$E \rightrightarrows E \wedge E \Rrightarrow E \wedge E \wedge E \Rrightarrow \cdots, \tag{5.10}$$

or smashing with a space $X$,

$$X \wedge E \rightrightarrows X \wedge E \wedge E \Rrightarrow X \wedge E \wedge E \wedge E \Rrightarrow \cdots. \tag{5.11}$$

Suppose that $E_*E$ is a flat $E_*$-module. Then the map

$$\mathrm{id} \wedge m \wedge \mathrm{id} \colon (E \wedge E) \wedge_E (E \wedge X) \longrightarrow E \wedge E \wedge X \tag{5.12}$$

[6] Not to be confused with simplicial cononsense.

induces on homotopy groups am isomorphism

$$E_*E \otimes_{E_*} E_*X \cong \pi_*(E \wedge E \wedge X), \tag{5.13}$$

and we don't have to take the derived tensor product! Therefore in this situation we can use the Künneth spectral sequence to compute the left-hand side: if $M$ and $N$ are $R$-modules,

$$E_2^{p,q} = \mathrm{Tor}^{R_*}(M_*, N_*) \Longrightarrow \pi_*(M \otimes_R N). \tag{5.14}$$

This looks closer to what we want the generalized Adams spectral sequence to look like.

*Remark* 5.15. For $E = H\mathbb{Z}$ or $ku$, $E_*E$ is not flat over $E_*$; however, this does work for $H\mathbb{F}_p$, $MU$, and $BP$. ◂

The pair $(E_*, E_*E)$ is a *Hopf algebroid*: it has maps $E_*E \to E_*$ and $E_* \to E_*E$ together with a *comultiplication map*

$$\Delta \colon E_*E \longrightarrow E_*E \otimes_{E_*} E_*E. \tag{5.16}$$

If $E$ is commutative it's even a commutative Hopf algebroid. There's a little more structure (e.g. an antipode).

**Definition 5.17.** An $(E_*, E_*E)$-comodule is a left $E_*$-module $M$ together with an $E_*$-linear map $M \to E_*E \otimes_{E_*} M$.

The category $\mathsf{Comod}_{E_*E}$ has a cotensor product, and is an abelian category, which allows us to make sense of things such as Ext. This leads eventually to the *Adams-Novikov spectral sequence(s)*, a family of spectral sequences using this idea:

$$\mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*(X)) \Longrightarrow \pi_*(X), \tag{5.18}$$

or working at a prime $p$,

$$\mathrm{Ext}_{BP_*BP_*BP}(BP_*, BP_*(X)) \Longrightarrow \pi_*(X)^\wedge_{(p)}. \tag{5.19}$$

The general $E$-based Adams spectral sequence for computing $\pi_*(L_E Y)$ has nice convergence properties when

(1) $E$ and $Y$ are both connective,
(2) $\pi_0 E \subset \mathbb{Q}$ or is $\mathbb{Z}/n$, and
(3) $E_*E$ is concentrated in even degrees.

Next week we'll discus multiplicative structures.

## 6. First computations with the Adams spectral sequence: 10/8/18

Riccardo and Alberto spoke today. Some parts of this talk are mechanical, or can be done by a computer program.

The first thing we need to do is compute the $E_2$ page for the Adams spectral sequence. Specifically, we will define a minimal $\mathscr{A}$-resolution $P_\bullet$ of $\mathbb{F}_2$. Following Rognes' notes, we will let $g_{s,i}$ denote a degree-$i$ generator in Adams filtration $s$.

The first thing we need is the augmentation $\varepsilon \colon P_0 \to \mathbb{F}_2$ (so in Adams filtration zero). There will be a generator $g_{0,0}$ in degree 0, and $P_0 = \mathscr{A}[g_{0,0}]$ as an $\mathscr{A}$-module. The kernel of $\varepsilon$ is the augmentation ideal $I(\mathscr{A})$ of $\mathscr{A}$ (sometimes also denoted $\overline{\mathscr{A}}$).

For Adams filtration $s = 1$, we need a surjective map $\partial_1 \colon P_1 \to \ker(\varepsilon)$. Using the Adams relations, we know $\mathscr{A}$ is generated by $\{\mathrm{Sq}^{2^n} \mid n \in \mathbb{N}\}$. The first thing we need to hit is $\mathrm{Sq}^1[g_{0,0}]$, which we can hit with $g_{1,0}$, but then we don't hit $\mathrm{Sq}^2 g_{0,0}$, so we define another generator $g_{1,1}$ and send

$$g_{1,1} \longmapsto \mathrm{Sq}^2 g_{0,0}. \tag{6.1}$$

Using that this map must intertwine the $\mathscr{A}$-actions, you next don't hit $\mathrm{Sq}^4 g_{0,0}$, so you add another generator, then $\mathrm{Sq}^8 g_{0,0}$, and so on: you have one generator for each power of 2. We will only be computing in low degrees, so our $P_1$ will be

$$P_1 \coloneqq \mathscr{A}[g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}]. \tag{6.2}$$

Now we move to $s = 2$.

- The first thing we need to hit is $\mathrm{Sq}^1 g_{1,0}$, so we add a generator $g_{2,0} \mapsto \mathrm{Sq}^1 g_{1,0}$.
- Since $\mathrm{Sq}^3 g_{1,0} + \mathrm{Sq}^2 g_{1,1}$ represents the Ádem relation $\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^2 = 0$. Therefore we need to hit it with $\partial_2$ of a generator $g_{2,1}$.
- Continuing in a similar way, we'll add more geneators $g_{2,2}, \ldots$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | $\vdots$ | | | | | | | | |
| 4 | $g_{4,0}$ | | | | | | | $g_{4,1}$ | |
| 3 | $g_{3,0}$ | | | $g_{3,1}$ | | | | $g_{3,2}$ | $g_{3,3}$ |
| 2 | $g_{2,0}$ | | $g_{2,1}$ | $g_{2,2}$ | | | $g_{1,2}$ | $g_{2,5}$ | $g_{2,5}$ |
| 1 | $g_{1,0}$ | $g_{1,1}$ | | $g_{1,2}$ | | | | $g_{1,3}$ | |
| 0 | $g_{0,0}$ | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

FIGURE 2. The generators for a minimal $\mathscr{A}$-resolution for $\mathbb{F}_2$ in low degrees.

We obtain the following table.

Then we need to hom into $\mathbb{F}_2$, but that doesn't change anything, and we obtain the low degrees of the $E_2$-term of the Adams spectral sequence. Some of these are well-known, e.g. the dual to $\gamma_{1,0}$, known as $h_0$, is the multiplication by 2 map $\mathbb{S} \to \mathbb{S}$; the dual to $\gamma_{1,1}$, written as $h_1$, represents the Hopf fibration, and more: $h_i$ represents the dual of $\gamma_{1,i}$.

However, there will also be nonzero differentials, and we will need to determine some of them. In general this is incredibly difficult, but in low degrees we can make some headway,

**Theorem 6.3** (Moss)**.** *Let $X$, $Y$, and $Z$ be spectra, with $Y$ and $Z$ bounded below, and with $H_*(Y)$ and $H_*(Z)$ finite type. Then there's a pairing of Adams spectral sequences*

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \longrightarrow E_r^{*,*}(X,Z), \tag{6.4}$$

*which for $r = 2$ coincides with the Yoneda product of Ext terms. This produce converges to the composition pairing*

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \longrightarrow [X, Z_2^\wedge]_*, \tag{6.5}$$

*and with respect to this pairing, differentials behave as derivations.*

TODO: I missed the thing immediately after that.

**Lemma 6.6.** *Let $\gamma_{s,n}$ denote the dual of $g_{s,n}$. Then the product $h_i \gamma_{s,n}$ has a nonzero coefficient $\gamma_{s+1,m}$ iff*

$$\partial_{s+1}(g_{s+1,m}) = \sum_j a_j \varphi_{s,j} \tag{6.7}$$

*contains the summand $\mathrm{Sq}^{s^i} g_{s,n}$.*

Using this, we can figure out the multiplicative structure. We know $\partial(g_{2,0} = \mathrm{Sq}^1 g_{1,0}$, so

$$\begin{aligned} \gamma_{2,0} &= h_0 \gamma_{1,0} = h_0^2 \\ \gamma_{n,0} &= h_0^n. \end{aligned} \tag{6.8}$$

Since $\partial(g_{2,1}) = \mathrm{Sq}^3 g_{1,0} + \mathrm{Sq}^2 g_{1,1}$, then $\gamma_{2,1} = h_1$ and $\gamma_{1,1} = h_1^2$. Since

$$\partial(g_{3,1}) = \mathrm{Sq}^4 g_{2,0} + \mathrm{Sq}^2 g_{2,1} + \mathrm{Sq}^1 g_{2,2}, \tag{6.9}$$

we get three more relations. In particular, we see everything except $c_0 := \gamma_{3.3}$.

Now we can see there are two possible differentials: $h_1 \to h_0^3$ and $h_3 h_1 \to h_3 h_0^3$. Later on in the sequence, differentials will be hard, but these are easy.

**Theorem 6.10.** *$E_2^{s,t} = 0$ for $0 < t - s < 2s - \varepsilon$, where $\varepsilon = 1$ if $s \equiv 0, 1 \bmod 4$, $\varepsilon = 2$ if $s \equiv 2 \bmod 4$, and $\varepsilon = 3$ if $s \equiv 3 \bmod 4$.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | $\vdots$ | | | | | | | | |
| 4 | $h_0^4$ | | | | | | | $h_3h_0^3$ | |
| 3 | $h_0^3$ | | | $h_1^3$ | | | | $h_3h_0^2$ | $c_0$ |
| 2 | $h_0^2$ | | $h_1^2$ | $h_2h_0$ | | | $h_2^2$ | $h_3h_0$ | $h_3h_1$ |
| 1 | $h_0$ | $h_1$ | | $h_2$ | | | | $h_3$ | |
| 0 | 1 | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

FIGURE 3. The $E_2$-page in small degrees, plotted in degree $(t-s, s)$.

Since $d(h_1h_0) = h_0d(h_1) + d(h_0)h_1$, we know $d(h_0) = 0$ and $h_1h_0 = 0$, so we conclude $d(h_1) = 0$. Similar methods pick off the other possible differential, so this part of the spectral sequence collapses and we know the associated graded of $\pi_*\mathbb{S}_2^\wedge$ looks the same. However, we will have extension problems!

**Theorem 6.11.** *Let $f : S^n \to S$ induce the zero map on cohomology, but be such that $\mathrm{Sq}^{n+1} : H^0(C_f) \to H^0(C_f)$ is nonzero. If $n+1 = 2^i$, then $[f]$ is detected by $h_i$.*

For example, if $f$ is multiplication by 2, we get $C_f = \Sigma^{-1}\mathbb{RP}^2$ (really $\Sigma^{-1}\Sigma^\infty\mathbb{RP}^2$), and if $f$ is the Hopf fibration, we get $C_f = \Sigma^{-2}\mathbb{CP}^2$.[7]

We solve the extension problem by detecting the Ext class of

$$0 \longrightarrow E_\infty^{s,t+s} \longrightarrow F^{0,t}/F^{s+1,s+t+1} \longrightarrow F^{0,t}/F^{s,s+t} \longrightarrow 0, \tag{6.12}$$

which we know is nontrivial exactly because it's the class in $h_0^?$, hence nontrivial. Thus we see (here everything is 2-completed) $\pi_0 = \mathbb{Z}_2$, $\pi_3 = \mathbb{Z}/8$, and $\pi_8 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

## 7. The construction of $MU$ and $BP$: 10/22/18

Today, Ty spoke about the construction of the spectra $MU$ and $BP$.

Recall that $\mathbb{CP}^\infty = BU_1 = K(\mathbb{Z}, 2)$ is the classifying space for complex line bundles, and that the homotopy class of $i : S^2 = \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ generates $\pi_2\mathbb{CP}^\infty \cong \mathbb{Z}$. This stabilizes to a map

$$i : \Sigma^2 S^2 \to \Sigma^\infty\mathbb{CP}^\infty. \tag{7.1}$$

Recall that if $E$ is a spectrum and $X$ is a pointed space, (reduced) *$E$-cohomology of $X$* is

$$\widetilde{E}^k(X) := \varprojlim_n [\Sigma^n X, E_{k+n}]. \tag{7.2}$$

**Definition 7.3.**

(1) A multiplicative cohomology theory is *complex-orientable* if the pullback map induced from (7.1),

$$i^* : \widetilde{E}^2(\mathbb{CP}^\infty) \longrightarrow \widetilde{E}^2(S^2) \tag{7.4}$$

is surjective, i.e. the unit map $\eta : \mathbb{S} \to E$ is in the image of $i^*$.

(2) A *complex orientation* on a complex-orientable cohomology theory $E$ is a choice of $x^E$ such that $i^*(x^E) = 1$.

[7] TODO: I think these are $\mathbb{RP}_{-1}^\infty$ and $\mathbb{CP}_{-2}^\infty$. Are they?

Given a complex orientation $x^E$ of $E$, we can factor the unit map $\eta\colon \mathbb{S} \to E$ as

$$\begin{tikzcd} \mathbb{S} \arrow[r, "i"] \arrow[rr, bend right, "\eta"'] & \Sigma^{\infty-2}\mathbb{CP}^\infty \arrow[r, "x^E"] & E. \end{tikzcd} \tag{7.5}$$

**Example 7.6.**

(1) If $E = H\mathbb{Z}$, whose cohomology theory is ordinary integer-valued cohomology, taking $x^{H\mathbb{Z}} = c_1 \in H^2(\mathbb{CP}^\infty)$, i.e. the first Chern class, defines a complex orientation on $H\mathbb{Z}$.

(2) For $E = KU$, complex $K$-theory, $x^{KU} = [\xi'] - 1 \in KU^0(\mathbb{CP}^\infty) = KU^2(\mathbb{CP}^\infty)$, where $\xi' \to \mathbb{CP}^\infty$ is the universal complex line bundle, defines a complex orientation on $H\mathbb{Z}$.

(3) $E = KO$ is not complex-orientable: there are isomorphisms $\widetilde{KO}^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ and $\widetilde{KO}^2(S^2) \cong \mathbb{Z}$, and the associated map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2. ◄

A complex orientation on $E$ determines an isomorphism of $E^*(\mathbb{CP}^\infty)$ with a power series ring.

**Proposition 7.7.** *Let $x^E$ be a complex orientation of a multiplicative cohomology theory $E$. Then there are isomorphisms*

(1) $E^*(\mathbb{CP}^n) \cong \pi_*(E)[i_n^*(x^E)]/(i_n^*(x^E)^{n+1})$,
(2) $E^*(\mathbb{CP}^\infty) \cong \pi_*(E)[[x^E]]$, *and*
(3) $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*(E)[[x_1, x_2]]$.

*Proof of part* (1). We'll use the Atiyah-Hirzenbruch spectral sequence to compute $E^*(\mathbb{CP}^n)$. After setting it up, one gets that an $x \in H^2(\mathbb{CP}^n; \pi_0 E)$ survives to an element in $E^2(\mathbb{CP}^n)$ iff $x$ restricts to a generator of $H^2(\mathbb{CP}^1; \pi_0 E)$. Since $x$ is complex-orientable, $i_n^* \mapsto 1 \in \pi_0 E$, so for such an $x$, such as the chosen $x^E$, $d_2$ vanishes and it's a permanent cycle corresponding to $i_n^* x \in E^2(\mathbb{CP}^n)$. ⊠

The proofs of the other two parts use a similar line of reasoning, together with something called the Milnor sequence.

There's a multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$,[8] which on cohomology defines a pullback map

$$m^*\colon \pi_*(E)[[x]] \longrightarrow \pi_*(E)[[x_1, x_2]] \tag{7.8}$$

by Proposition 7.7. Define $\mu^E(x_1, x_2) := m^*(x^E)$.

**Definition 7.9.** A *formal group law* over a ring $R$ is a power series $F(x, y) \in R[[x, y]]$ such that

(1) $F(0, x) = F(x, 0) = x$,
(2) $F(x, y) = F(y, x)$, and
(3) $F(x, F(y, z)) = F(F(x, y), z)$.

*Fact.* If $E$ is a complex-oriented cohomology theory, $\mu^E$ is a formal group law over $\pi_0 E$. ◄

**Theorem 7.10.** *There's a ring $L$ and formal group law $F(x, y) = \sum a_{ij} x^i y^j$ such that for any ring $R$ and formal group law $G$ over $E$, there's a unique ring homomorphism $\theta\colon L \to R$ such that*

$$G(x, y) = \sum \theta(a_{ij}) x^i y^j.$$

Explicitlly, $L = \mathbb{Z}[t_1, t_2, \dots]$.

**Example 7.11.**

(1) If $E = H\mathbb{Z}$ with the complex orientation defined above, $\mu^{H\mathbb{Z}}(x_1, x_2) = x_1 + x_2$. This is called the *additive formal group law* and sometimes denoted $\mathbb{G}_a$.
(2) If $E = KU$ with the complex orientation defined above, $\mu^{KU}(x_1, x_2) = x_1 + x_2 + x_1 x_2$. This is called the *multiplicative formal group law* and sometimes denoted $\mathbb{G}_m$. ◄

This leads to a natural question: can all formal group laws be realized homotopically? The answer is yes, thanks to complex bordism!

Recall that if $B$ is a space, $\Sigma^r B_+ = (B \times D^r)/(B \times S^{r-1})$. You could think of this as being built out of the trivial vector bundle, in that it's the vectors in the unit disc modulo those in the unit sphere, and try to generalize this to nontrivial vector bundles.

[8] There is a model of $BU_1$ which is a topological abelian group, and this model is not how one usually defines $\mathbb{CP}^\infty$. Nonetheless, since they're homotopic, all of the things we need it to satisfy hold in this case, so there's no loss of generality.

**Definition 7.12.** Let $\xi \to B$ be a vector bundle with a Euclidean metric. Then its *sphere bundle* is $S(\xi) := \{v \in \xi \mid \|v\| = 1\}$, and its *disc bundle* is $D(\xi) := \{v \in \xi \mid \|v\| \leq 1\}$.

The *Thom space* of $\xi$ is $B^{\xi} := D(\xi)/S(\xi)$.

Taking the Thom space defines a functor from vector bundles on $B$ to pointed spaces. If $\xi \cong \underline{\mathbb{R}}^n$, then $B^{\underline{\mathbb{R}}^n} \cong \Sigma^n B_+$.

**Example 7.13.** Let $\xi_n \to B\mathrm{U}_n$ denote the universal complex vector bundle of rank $n$. Its Thom space is denoted $MU(n)$.

The inclusion map $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ as the first $n$ coordinates induces maps $\mathrm{U}_n \hookrightarrow \mathrm{U}_{n+1}$, hence $i_n \colon B\mathrm{U}_n \to B\mathrm{U}_{n+1}$. One can show that $i_n^*\xi_{n+1} = \xi \oplus \underline{\mathbb{C}}$; therefore functoriality of the Thom space construction defines a map $B\mathrm{U}_n^{\xi_n \oplus \underline{\mathbb{C}}} \to B\mathrm{U}_{n+1}^{\xi_{n+1}}$, i.e. a map $\Sigma^2 MU(n) \to MU(n+1)$. These maps define the data of a spectrum $MU$, called the *complex bordism spectrum*, whose $2n^{\text{th}}$ space is $MU(n)$, and whose $(2n+1)^{\text{st}}$ space is $\Sigma MU(n)$, with the structure maps $\Sigma^2 MU(n) \to MU(n+1)$ as above, and $\Sigma MU(n) \to \Sigma MU(n) = \mathrm{id}$.

The maps $B\mathrm{U}_n \times B\mathrm{U}_m \to B\mathrm{U}_{m+n}$ and $B\mathrm{U}_0 \to B\mathrm{U}_n$ define multiplication and unit maps $MU \wedge MU \to MU$ and $\mathbb{S} \to MU$ making $MU$ into an $E_\infty$-ring spectrum. ◀

*Remark* 7.14. By the Pontrjagin-Thom theorem, the homotopy groups of $MU$ classify stably almost complex manifolds up to cobordism. ◀

**Theorem 7.15** (Quillen). *$MU$ is the universal complex-oriented cohomology theory, in that if $(E, x^E)$ is a complex-oriented cohomology theory, there is a unique map of ring spectra $f : MU \to E$ such that $f_*(x^{MU}) = x^E$ and $f_*(\mu^{MU}) = \mu^E$. Moreover, $\theta_{MU} \colon L \to MU_*$ is an isomorphism.*

One might want to work over a prime $p$; in this case one works with *$p$-typical group laws*, and the analogue of $MU$ is called $BP$.

**Theorem 7.16** (Brown-Peterson). *Fix a prime $p$. Then there's a retraction of $MU_{(p)}$ onto a spectrum $BP$, such that*

(1) *$BP_*$ is universal for $p$-typical formal group laws over $\mathbb{Z}_{(p)}$,*
(2) *$\pi_*(BP) \otimes \mathbb{Q} \cong \mathbb{Q}[g_*(m_{p^k-1}) \mid k > 1]$, where $g \colon MU_{(p)} \to BP$ is the rertraction, and the $m_i$ are defined by $L \otimes \mathbb{Q} \cong \mathbb{Q}[m_1, m_2, \dots]$, and*
(3) *$\pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $|v_i| = 2(p^i - 1)$.*

NOTE: Ty spoke at the beginning of the next lecture; notes for that part of his talk have been added here.

Today we'll say a little more about the genesis of $BP$.

**Theorem 7.17** (Quillen). *Fix a prime $p$. Then there is a unique map of ring spectra $\varepsilon \colon MU_{(p)} \to MU_{(p)}$ such that*

(1) *$\varepsilon$ is idempotent, and*
(2) *$\varepsilon_*[\mathbb{CP}^n] = [\mathbb{CP}^n]$ if $n = p^t - 1$ and is $0$ otherwise.*

**Theorem 7.18** (Quillen). *There exists a ring spectrum $BP$ and maps $\pi \colon MU_{(p)}$ and $\iota \colon BP \to MU_{(p)}$ such that*

(1) *$\iota \circ \pi = \varepsilon$;*
(2) *$\pi \circ \iota = \mathrm{id}_{BP}$;*
(3) *for all $X$, if $\iota_*$ and $\varepsilon_*$ denote the induced maps on $X$-homology, $\iota_*(BP_*(X)) = \mathrm{Im}(\varepsilon_*)$;*
(4) *for all $X$, if $\iota^*$ and $\varepsilon^*$ denote the induced maps on $X$-cohomology, $\iota^*(BP^*(X)) = \mathrm{Im}(\varepsilon^*)$.*

**Theorem 7.19** (Quillen). *The homology of $BP$ with respect to various coefficients is*

(1) *$H_*(BP; \mathbb{Z}/q) = 0$ if $p$ and $q$ are coprime,*
(2) *$H_*(BP) \cong \mathbb{Z}_{(p)}[\mu_{p-1}, \dots, \mu_{p^t-1}, \dots]$, and*
(3) *$H_*(BP; \mathbb{Z}/p) = \mathbb{Z}/p[\xi_1, \dots, ]$ if $p$ is odd, and if $p = 2$, $H_*(BP; \mathbb{Z}/p) = \mathbb{Z}/2[\xi_1^2, \dots]$.*

*Moreover, there are wlements $v_t \in BP_{2p^t-2}$ for $t \geq 1$ such that*

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, ],$$

*and $MU_{(p)}$ is homotopy equivalent to a wedge of suspensions of copies of $BP$.*

These together are quite difficult to prove, but we will be able to provide a sketch.

*Proof sketch.* We can assume $\varepsilon_q \colon MU_{(p)} \to MU_{(p)}$ restricts to the identity on the $(2q-4)$-skeleton of $MU_{(p)}$, which implies $\varepsilon$ as described in Theorem 7.17 is a well-defined map on $MU_{(p)}$.

**Proposition 7.20.** *Let $R \subseteq \mathbb{Q}$ be a subring and $MU_R$ denote $MU$ localized at $R$. If $f, g\colon MU_R \to MU_R$ are equal on $\pi_*$, then they are homotopic.*

*Proof sketch.* Use the diagram

$$\begin{tikzcd}
(MU_R)_* \otimes R \arrow[r, "f\otimes 1, g\otimes 1"] \arrow[d, "\eta_R"] & (MU_R)_* \otimes \mathbb{Q} \arrow[d, "\eta_R"] \\
(MU_R)_*(MU_R \otimes \mathbb{Q}) \arrow[r, "(f\otimes 1, g\otimes 1)"] & (MU_R)_*(MU_R \otimes \mathbb{Q}).
\end{tikzcd} \tag{7.21}$$

The upshot is that there can be no phantom maps present. ⊠

Then, $\varepsilon_*$ and $\varepsilon_* \circ \varepsilon_*$ both agree on the coefficient ring, so they must be homotopic.

Now for any space $X$, let $BP^*(X) \coloneqq \mathrm{Im}(\varepsilon^*(X))$. This is a direct summand of $MU^*_{(p)}(X)$, and using standard algebrai facts, you recover most of the facts in Theorem 7.18. To see that $BP$ is a spectrum, one has to argue for why it satisfies Brown representability, but most of this is straightforward. Ty did say something about what wasn't so straightforward, but (TODO) I didn't write it down. ⊠

## 8. Hopf algebroids: 10/29/18

The goal of studying Hopf algebroids is to study the cohomology operations of things such as $MU$ and $BP$, but the definition of a Hopf algebroid doesn't look helpful at first. So first we'll look at algebraic stacks, because there's an interpretation of the Adams and Adams-Novikov spectral sequences in this setting.

Let $\mathscr{X}$ be an algebraic stack and $U = \operatorname{Spec} A$ be an affine atlas for $\mathscr{X}$, coming with a faithfully flat map $U \to X$. Let $R\colon U \times_{\mathscr{X}} U$, which admits two maps to $U$; we say $R \rightrightarrows U$ is a *presentation* of $\mathscr{X}$, or $\mathscr{X} = [U/R]$. We would like to be able to recover $\mathscr{X}$ from its presentation, and in order to do that we need stacks to be valued in groupoids.

That is, the functor of points of a stack is a functor $\mathscr{X}\colon \mathsf{CAlg} \to \mathsf{Grpd}$.[9] Examples include sending a scheme $B$ to the groupoid of families of elliptic curves over $B$ (so $\mathscr{X}$ is the moduli of elliptic curves).

These two ways to think of a stack are closely related: $\mathscr{X}(B)$ is a groupoid, so it has a set of objects, which is precisely $U(B)$, and a set of morphisms, which is precisely $R(B)$. The two maps $R(B) \rightrightarrows U(B)$ send a morphism to its source, resp. target.

Let $E$ be a good ring spectrum (so we have a well-behaved $E$-based Adams spectral sequence). Then $\operatorname{Spec} E_*E \rightrightarrows \operatorname{Spec} E_*$ is a groupoid object in affine schemes; the quotient stack is denoted $\mathscr{X}_E$. The descent spectral sequence for this stack is exactly the $E$-based Adams spectral sequence.[10]

**Definition 8.1.** A *Hopf algebroid* is a groupoid object in the category of affine schemes, or equivalently, a cogroupoid object $(A, \Gamma)$ in $\mathsf{CAlg}$.

So given a Hopf algebroid $(A, \Gamma)$, we have a commutative algebra of objects $A$ and a commutative algebra $\Gamma$ of morphisms. Then, opposite to the source and target maps, we have *left and right unit* maps $\eta_L, \eta_R\colon A \to \Gamma$, and a *comultiplication* $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$ dual to composition. Dual to the identity morphism is an *augmentation map* $\varepsilon\colon \Gamma \to A$ and dual to the inverse is the *conjugation* map $c\colon \Gamma \to \Gamma$. These satisfy a number of relations which you would want them to satisfy.

- $\varepsilon \circ \eta_L = \varepsilon \circ \eta_R = \mathrm{id}$: the sorce and target of an identity morphism is that object itself.
- Comultiplication is coassociative, i.e. the following diagram commutes.

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \Gamma \times_A \Gamma \arrow[d, "\Delta\otimes\mathrm{id}"] \\
\Gamma \otimes_A \Gamma \arrow[r, "\mathrm{id}\otimes\Delta"'] & \Gamma \otimes_A \Gamma \otimes_A \Gamma.
\end{tikzcd} \tag{8.2}$$

- There are a few more, which are a bit more tedious to write down.

[9] Here $\mathsf{CAlg}$ is the category of commutative algebras over some ground field or ring. Rok denoted this $\mathsf{CAlg}^\heartsuit$; I don't work implicitly in the derived world.

[10] TODO: I don't know what this means.

Now we can say something about the relationship between Hopf algebroids and Hopf algebras. A Hopf algebroid is precisely (dual to) a group object in affine schemes, and a group is just a groupoid with a single object. So we should be able to recover the usual definition of a Hopf algebra as a “Hopf algebroid with a single object.” This is precisely what happens in the case $\eta_L = \eta_R$.

**Example 8.3.** Let $BG$ be the classifying stack of a group $G$, assumed to be flat and algebraic. This has a presentation $G \rightrightarrows \mathrm{pt}$. If we take rings of functions on these schemes, we get $k \to \mathscr{O}(G)$, i.e. a presentation of $\mathscr{O}(G)$ as a Hopf algebra. This is reassuring, because we said that Hopf algebras are (dual to) group objects in schemes. ◂

**Example 8.4.** The examples we care more abou are when $\mathscr{X}$ is the stack of formal group laws (so over $\mathbb{Z}$) or $p$-typical formal group laws (over $\mathbb{Z}_{(p)}$). These arise from the Adams construction, e.g.

$$\mathscr{M}_{\mathrm{FGL}} = [\operatorname{Spec} \pi_* MU / \operatorname{Spec} MU_* MU]. \tag{8.5}$$

To get the $p$-typical group laws, replace $MU$ with $BP$.

To recover the classical Adams spectral sequence, we can let $\mathscr{X}$ be the stacky quotient of $\operatorname{Spec} \mathbb{F}_p$ by $\operatorname{Spec} \mathscr{A}_p^\vee$. At least when $p = 2$, there’s a deep and mysterious result: $\operatorname{Spec} \mathscr{A}_2^\vee \cong \operatorname{Aut}_{\mathbb{F}_2}(\widehat{\mathbb{G}}_a)$, the automorphisms of the additive formal group. Something analogous is true over other primes, even if the statement is slightly different. Thus the stack we get is a classifying space $B \operatorname{Aut}_{\mathbb{F}_2}(\widehat{\mathbb{G}}_a)$. ◂

Finally we need to make sense of comodules over a Hopf algebroid — but essentially by definition, these are the same thing as quasicoherent sheaves over the corresponding quotient stack.

## 9. The Ext functor and the Steenrod algebra: 11/5/18

Riccardo generously provided his notes from his talk today.

### 9.1. An $\mathscr{A}$-resolution for $\mathbb{F}_2$.

Recall that $\mathscr{A}$ is the mod-2 Steenrod algebra. To compute the Adams $E_2$-term for the sphere spectrum, we need to compute

$$\operatorname{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = H^*(\operatorname{Hom}^t_{\mathscr{A}}(P_*, \mathbb{F}_2) \tag{9.1}$$

for any free resolution $P_*$ of $\mathbb{F}_2$. We now sketch the construction of such resolution in low degrees.

9.1.1. *Filtration $s = 0$.* We need a surjection $\epsilon\colon P_0 \to \mathbb{F}_2$, so we let $P_0 = \mathscr{A}\{g_{0,0}\}$ be the free $\mathscr{A}$-module on a single generator $g_{0,0}$ in internal degree 0. We will also use the notation $g_{s,i}$ denote the $i^{\text{th}}$ generator in filtration degree $s$.

9.1.2. *Filtration $s = 1$.* We need a surjection $\partial_1\colon P_1 \to \ker(\epsilon)$, where $\ker(\epsilon) \cong I(\mathscr{A})$. An additive basis for $\ker(\epsilon)$ is given by the admissible monomials $\mathrm{Sq}^I g_{0,0}$ for $I$ of length $\geq 1$. Using the Adem relations $\mathscr{A}$ is generated as an algebra by the family $\{\mathrm{Sq}^{2^n}\}_{n\in\mathbb{N}}$, hence it’s clear that, in order to get a surjection, we have to hit those generators.

Therefore, let us take a generator $g_{1,0}$ that maps to $\mathrm{Sq}^1 g_{0,0}$. Observe that the class $\mathrm{Sq}^2 g_{0,0}$ won’t be in the image of $\partial_1\colon \mathscr{A}\{g_{1,0}\} \to \ker(\epsilon)$ (since $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$ are independent), hence we need another generator $g_{1,1}$ mapping to $\mathrm{Sq}^2 g_{0,0}$.

Using the Adem relations one then checks the image of $\partial_1\colon \mathscr{A}\{g_{1,0}, g_{1,1}\} \to \ker(\epsilon)$. As before we notice that $\mathrm{Sq}^4$ and $\mathrm{Sq}^8$ are not hit, therefore we need to add two generators $g_{1,2}$ and $g_{1,3}$ mapping to $\mathrm{Sq}^4$ and $\mathrm{Sq}^8$ respectively. An inspection reveals that $\partial_1\colon \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}\} \to \ker(\epsilon)$ is surjective in degrees $t \leq 11$. In general it’s easy to convince yourself that we need enough $\mathscr{A}$-module generators $\{g_{1,i}\}_i$ for $P_1$ to map surjectively to the indecomposable $\mathbb{F}_2\{\mathrm{Sq}^{2^i} \mid i \geq 0\}$.

9.1.3. *Filtration $s = 2$.* We need a surjection $\partial_2\colon P_2 \to \ker(\partial_1)$. We observe that, for example, the class in lowest degree in the kernel is $\mathrm{Sq}^1 g_{1,0}$. So as we did in the previous filtration, we put a generator $g_{2,0}$ in internal degree 2 and we will map it into $\mathrm{Sq}^1 g_{1,0}$ via $\partial_2\colon P_2 \to P_1$.

Observe that this is actually a general pattern: in the next filtration we will have to map a generator into $\mathrm{Sq}^1 g_{2,0}$ and the same process will repeat at every $s$. This gives rise to the tower in degree $t - s = 0$. The first class in $\ker(\partial_1)$ that is not in the image of $\partial_2$ on $\mathscr{A}\{g_{2,0}\}$ is $\mathrm{Sq}^3 g_{1,0} + \mathrm{Sq}^2 g_{1,1}$ which corresponds to the Adem relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$. We add a second generator $g_{2,1}$ to $P_2$, in degree 4, with $\partial_2(g_{2,1}) = \mathrm{Sq}^3 g_{1,0} + \mathrm{Sq}^2 g_{1,1}$.

In general one proceeds in this way in the range of interest.

9.1.4. *Final results in the range of interest.* For each $s \geq 0$ we have $\mathrm{Hom}_{\mathscr{A}}(P_2;\mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}\{g_{s,i}\}_i, \mathbb{F}_2) \cong \prod_i \mathbb{F}_2\{\gamma_{s,i}\}$, where $\gamma_{s,i}$ is the dual of $g_{s,i}$. It will be clear later that there are at most finitely many $g_{s,i}$ in a given bidegree, so this product is finite in each degree. Then $\gamma_{s,i} \circ \partial_{s+1} = 0$, so the cocomplex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ has trivial coboundary. Hence

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Hom}^t_{\mathscr{A}}(P_s,\mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i \tag{9.2}$$

as claimed.

## 9.2. **The multiplicative structure - an introduction.**

In order to compute differentials, one of the easiest way is to make use of the multiplicative structure of the ASS.

**Theorem 9.3** (Moss 1968)**.** *Let $X, Y$ and $Z$ be spectra, with $Y$ and $Z$ bounded below and $H_*(Y)$ and $H_*(Z)$ of finite type. There is a pairing of spectral sequences*

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \to E_r^{*,*}(X,Z) \tag{9.4}$$

*which agrees for $r = 2$ with the Yoneda pairing*

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(Z),H^*(Y)) \otimes \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(Y),H^*(X)) \to \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(Z),H^*(X)) \tag{9.5}$$

*and which converges to the composition pairing*

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \to [X, Z_2^\wedge]_* \tag{9.6}$$

*the pairing is associative and unital. The differentials in the spectral sequences behave like a derivation with respect this product:*

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y). \tag{9.7}$$

It's clear that having a multiplicative structure makes computations a lot easier, so the aim of the rest of the talk will be to give tools to figure out multiplications in the second page of our ASS. To this end it's very convenient to identify the Yoneda product with the following operation:

$$\mathrm{Ext}^{u,v}_{\mathscr{A}}(H^*(Z),H^*(Y)) \otimes \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y),H^*(X)) \to \mathrm{Ext}^{s+u,v+t}_{\mathscr{A}}(H^*(Z),H^*(X)). \tag{9.8}$$

For convenience let us take minimal resolutions $P_\bullet \to H^*(Y)$ and $Q_\bullet \to H^*(Z)$ even though one can work with any resolution and the result will be the same once we consider the cohomology classes. An element $f \in \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y),H^*(X))$ can be identified with a morphism of graded $\mathscr{A}$-modules $f : P_s \to \Sigma^t H^*(X)$, similarly $g \in \mathrm{Ext}^{u,v}_{\mathscr{A}}(H^*(Z),H^*(Y))$ can be seen as a morphism of graded $\mathscr{A}$-modules $g : Q_u \to \Sigma^v H^*(Y)$.

We can lift $g$ to a chain map $g_* = \{g_n : Q_{u+n} \to P_n\}_n$ where each $g_n$ has degree $v$, making the diagram

$$\begin{tikzcd}
 & H^*(X) & & & & & \\
\cdots \arrow[r] & P_s \arrow[u, "f"] \arrow[r, "\partial_s"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \\
\cdots \arrow[r] & Q_{u+s} \arrow[u, "g_s"] \arrow[r, "\partial_{u+s}"] & \cdots \arrow[r] & Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"] & Q_u \arrow[u, "g_0"] \arrow[ur, "g"] &
\end{tikzcd} \tag{9.9}$$

The composition $f g_s : Q_{u+s} \to H^*(X)$ is then a $\mathscr{A}$-module homomorphism of degree $(v+t)$ and it's clearly a cocycle. It's easy to see that this new definition of the Yoneda product coincides with our old one when we view the ext-groups as equivalence classes of extensions.

We can finally compute the multiplication table for the second page of our ASS. Recall the minimal resolution $\epsilon : P_* \to \mathbb{F}_2$ and the notation we used for the generators.

**Lemma 9.10.** *The product $h_i \cdot \gamma_{s,n}$ contains the summand $\gamma_{s+1,m}$ if and only if $\partial_{s+1}(g_{s+1,m}) = \sum_j a_j g_{s,j}$ contains the summand $\mathrm{Sq}^{2^i} g_{s,n}$*

*Proof.* Let $\gamma_{s,n} : P_s \to \mathbb{F}_2$ be the dual to the generator $g_{s,n} \in P_s$, and let $h_i = \gamma_{1,i} : P_1 \to \mathbb{F}_2$ be the dual to $g_{1,i}$ (which maps to $\mathrm{Sq}^{2^i}$ under $\partial_1$).

$$
\begin{tikzcd}
P_{s+1} \arrow[r, "\gamma_1"] \arrow[d, "\partial_{s+1}"] & P_1 \arrow[d, "\partial_1"] \arrow[dr, "h_i"] & \\
P_s \arrow[r, "\gamma_0"] \arrow[dr, "\gamma_{s,n}"] & P_0 \arrow[d, "\epsilon"] & \mathbb{F}_2 \\
& \mathbb{F}_2 &
\end{tikzcd}
\tag{9.11}
$$

We lift $\gamma_{s,n}$ to $\gamma_0 \colon P_s \to P_0$ making the triangle commute. This means that $\gamma_0(g_{s,n}) = g_{0,0}$ and $\gamma_0(g_{s,j}) = 0$ for $j \neq n$. Then $\gamma_0 \circ \partial_{s+1}$ sends $g_{s+1,m}$ to $a_n g_{0,0}$. To lift $\gamma_0$ to $\gamma_1 \colon P_{s+1} \to P_1$ we write $a_n = \sum_k b_k \mathrm{Sq}^{2^k}$ with each $b_k \in \mathscr{A}$. Then we may take $\gamma_1(g_{s+1}, m) = \sum_k b_k g_{1,k}$, since $\partial_1(g_{1,k}) = \mathrm{Sq}^{2^k} g_{0,0}$. The coefficient of $g_{s+1,m}$ in the Yoneda product $h_i \cdot \gamma_{s,n}$ is then given by the value of $h_i \circ \gamma_1$ on $g_{s+1,m}$, which equals $h_i(\sum_k b_k g_{1,k}) = \epsilon(b_i)$. Hence $\gamma_{s+1,m}$ occurs as a summand in $h_i \cdot \gamma_{s,n}$ if and only if $\mathrm{Sq}^{2^i}$ occurs as a summand in $a_n = \sum_k b_k \mathrm{Sq}^{2^k}$. This is equivalent to the condition that $\mathrm{Sq}^{2^i}$ occurs as a summand when $a_n$ is written as a sum of admissible monomials. ⊠

## 10. Some preliminaries on the Adams-Novikov spectral sequence: 11/12/18

Throughout $p > 2$.

### 10.1. **Some facts about Hopf algebroids.**

Recall that a Hopf algebroid over a commutative ring $A$ is a cogroupoid objct in $A$-algebras, meaning a pair $(A, \Gamma)$ of commutative $A$-algebras with source and target maps $\eta_{L,R} \colon A \rightrightarrows \Gamma$, a coproduct $\Delta \colon \Gamma \to \Gamma \otimes_A \Gamma$, a counit $\varepsilon : \Gamma \to A$, and an inverse $c \colon \Gamma \to \Gamma$, satisfying some axioms.

**Definition 10.1.** A *comodule* $M$ over a Hopf algebroid $(A, \Gamma)$ is a left $A$-module together with an $A$-linear map $\psi \colon M \to \Gamma \otimes_A M$ which is counitary and coassociative.

**Theorem 10.2.** *If $\Gamma$ is a flat $A$-module, then* $\mathsf{Comod}_\Gamma$ *is abelian.*

**Lemma 10.3** (A1.2.2). *In this setting,* $\mathsf{Comod}_\Gamma$ *has enough injectives.*

Therefore we can define $\mathrm{Ext}_\Gamma^*(M, N)$ to be the right derived functor of $\mathrm{Hom}_\Gamma(M, N)$ of comodules.

**Definition 10.4.** Let $M$ be a right $\Gamma$-comodule and $N$ be a left $\Gamma$-comodule. Their *cotensor product* $M \,\square_\Gamma\, N$ is the $k$-module defined to be the kernel of

$$
\psi \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \psi \colon M \otimes_A N \longrightarrow M \otimes_A \Gamma \otimes_A N. \tag{10.5}
$$

Cotensor is commutative up to natural isomorphism. Its derived functors are called Cotor. If $M$ and $N$ have algebra structures, so does $\mathrm{Cotor}_\Gamma(M, N)$.

**Lemma 10.6.** *Assume $M$ is projective.* $\mathrm{Hom}_\Gamma(M, N) \cong \mathrm{Hom}_A(M, A) \,\square\, N$*, so* $\mathrm{Hom}_\Gamma(A, N) \cong A \,\square_\Gamma\, N$*. Therefore* $\mathrm{Ext}_\Gamma(A, M) = \mathrm{Cotor}_\Gamma(A, M)$*.*

**Theorem 10.7** (Change-of-rings (A1.3.12))**.** *Let $f : (A, \Gamma) \to (B, \Sigma)$ be a map of graded Hopf algebroids such that (some stuff you can read in A1.1.19), $M$ be a graded right $\Gamma$-comodule, and $N$ be a $B$-flat left $\Sigma$-comodule. Then*

$$
\mathrm{Cotor}_\Gamma(M, (\Gamma \otimes_A B) \,\square_\Sigma\, N) \cong \mathrm{Cotor}_\Sigma(M \otimes_A B, N). \tag{10.8}
$$

*In particular,*

$$
\mathrm{Ext}_\Gamma(A, (\Gamma \otimes_A B) \,\square_\Sigma\, N) = \mathrm{Ext}_\Sigma(B, N). \tag{10.9}
$$

### 10.2. **The Cartan-Eilenberg spectral sequence.**

**Definition 10.10.** An *extension* of Hopf algebroids is maps $i \colon (D, \Phi) \to (A, \Gamma)$ and $f : (A, \Gamma) \to (A, \Sigma)$ such that

(1) $f$ is *normal*, i.e. $\Gamma \twoheadrightarrow \Sigma$, $A \to A$ is the identity, and $\Gamma \,\square_\Sigma\, A = A \,\square_\Sigma\, \Gamma$; and
(2) $D = A \,\square_\Sigma\, A$ and $\Phi = A \,\square_\Sigma\, \Gamma_\Sigma A$ (in particular, $(D, \Phi)$ is a sub-Hopf algebroid of $(A, \Gamma)$).

**Theorem 10.11.** *Given an extension of Hopf algebroids as above, there is a* Cartan-Eilenberg spectral sequence *with $E_2 = \mathrm{Ext}_\Phi(D, \mathrm{Ext}_\Sigma(A, N))$, converging to $\mathrm{Ext}_\Gamma(A, N)$.*[11] *Here $N$ is a left $\Gamma$-comodule. TODO: differentials.*

[11]We have to assume these are connected Hopf algebroids, meaning $\Gamma \cdot A = A \cdot \Gamma = A$.

**Example 10.12.** Recall that the dual Steenrod algebra $\mathscr{A}_* \cong E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots)$ with $|\tau_i| = 2p^i - 1$ and $|\xi_i| = 2p^i - 2$. The coproduct is

If $P_*$ is the polynomial algebra on the $\xi_i$ and $E_*$ is the exterior algebra on the $\tau_j$, then $P_* \to \mathscr{A}_* \to E_*$ is an extension of Hopf algebras. Therefore we have a trigraded spectral sequence

$$E_2^{s_1,s_2,t} = \mathrm{Ext}_{P_*}^{s_1}(\mathbb{Z}/p, \mathrm{Ext}_{E_*}^{s_2,t}(\mathbb{Z}/p, \mathbb{Z}/p)) \tag{10.13}$$

converging to $\mathrm{Ext}_{\mathscr{A}_*}(\mathbb{Z}/p, \mathbb{Z}/p)$, with differentials

$$d_r \colon E_r^{s_1,s_2,t} \longrightarrow E_r^{s_1+r,s_2-r+1,t}. \tag{10.14}$$

$\blacktriangleleft$

**Theorem 10.15** (4.4.3)**.** *This spectral sequence collapses on the $E_2$ page, and there are no nontrivial extensions.*

*Proof.* We can put a different grading on $\mathscr{A}_*$, $\boxtimes$

TODO: prereqs: what is $a$?

10.3. **The algebraic Novikov spectral sequence.**

**Theorem 10.16.** *Let $(A, \Gamma)$ be an increasingly filtered Hopf algebroid with a left module $L$ and a right module $M$. Then there is a spectral sequence*

$$E_1^{s,t} = \mathrm{Cotor}_{E^0\Gamma}^{s,t}(E^0L, E^0M) \Longrightarrow \mathrm{Cotor}_\Gamma(L, M). \tag{10.17}$$

*Proof.* The filtrations on $\Gamma$ and $M$ induce one on the cobar complex $C_\Gamma(M)$, and in it, $E^0C_\Gamma(L,M) = C_{E^0\Gamma}(E^0L, E^0M)$. The filtration induces the spectral sequence in question (sort of a Grothendieck spectral sequence). $\boxtimes$

Recall from Riccardo's talk that $I = (p, v_1, v_2, \dots) \subset BP_*$.

**Theorem 10.18.** *There is a spectral sequence*

$$E_1^{s,m,t} = \mathrm{Ext}_{P_*}^{s,t}(\mathbb{Z}/p, I^m, I^{m+1}) \Longrightarrow \mathrm{Ext}_{BP_*BP}(BP_*, BP_*). \tag{10.19}$$

*Moreover, the $E_1$-page of this spectral sequence coincides with the $E_2$-page of the Cartan-Eilenberg spectral sequence.*

*Proof.* Here we're using the filtration of $BP_*(BP)$ by powers of $I$. Therefore the $E_1$-term of the spectral sequence is

$$\mathrm{Ext}_{E_0BP_*BP}(E_0BP_*, E_0BP_*). \tag{10.20}$$

Using the fact that

$$BP_*BP/I = E_0BP_*(BP) \otimes_{E_0BP_*} \mathbb{Z}/p = P_*, \tag{10.21}$$

we can apply a change-of-rings theorem...

$$\mathrm{Ext}_{P_*}(\mathbb{Z}/p, E_0BP_*) \cong \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, (E_0BP_*BP \otimes_{E_0BP_*} \mathbb{Z}/p) \,\square_{P_*}\, E_0BP_*).$$

Using the fact that $BP_*BP/I = E_0BP_*BP \otimes_{E_0BP_*} \mathbb{Z}/p = P_*$,

$$\begin{aligned} &\cong \mathrm{Ext}_{E_0BP_*BP}(E_0BP_*, P_* \,\square_{P_*}\, E_0BP_*) \\ &\cong \mathrm{Ext}_{E_0BP_*BP}(E_0BP_*, E_0BP_*). \end{aligned}$$

The two spectral sequences agree because $E_0BP_* = P(a_0, a_1, \dots) = \mathrm{Ext}_{E_*}(\mathbb{Z}/p, \mathbb{Z}/p)$. (TODO probably should justify) $\boxtimes$

## 11. The $E_2$-page of the Adams-Novikov spectral sequence: 11/19/18

Today Richard told us about some partial calculations of $\mathrm{Ext}_{BP_*BP}(BP_*BP/I_3)$ at $p = 3$, corresponding to §4.4.8 of Ravenel's book. Our goal is to compute $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, and we'll do this by understanding $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ as $n$ *de*creases.

The tool we will use to do this begins with a short exact sequence of $BP_*BP$-comodules

$$0 \longrightarrow \Sigma^{2(p^n-1)}BP_*/I^n \xrightarrow{\cdot v_n} BP_*/I_n \longrightarrow BP_*/I_{n-1} \longrightarrow 0. \tag{11.1}$$

From this, we can deduce a long exact sequence in Ext whose differential "lowers $n$":

$$\delta \colon \mathrm{Ext}_{BP_*BP}^s(BP_*, BP_*/I_{n+1}) \longrightarrow \mathrm{Ext}_{BP_*BP}^{s+1}(BP_*, \Sigma^{2(p^n-1)}BP/I_n). \tag{11.2}$$

We will also use a Bockstein spectral sequence, along with some of the tools introduced last time.

**Theorem 11.3.** *For $t < 2(p^n - 1)$ and $p > 2$,* $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*/I_n) \cong \mathrm{Ext}^{s,t}_{P_*}(\mathbb{Z}/p, \mathbb{Z}/p)$.

For $p = 3$, $P_* = P(\xi_1, \xi_2)$, as discussed last time. Consider the extension of coalgebras

$$P(\xi_1) \longrightarrow P(\xi_1, \xi_2) \longrightarrow P(\xi_2). \tag{11.4}$$

This is a cocentral extension, so we have a Cartan-Eilenberg spectral sequence like last time:

$$E_2^{s,t} = \mathrm{Ext}^s_{P(\xi_1)}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathrm{Ext}^t_{P(\xi_1)}(\mathbb{Z}/p, \mathbb{Z}/p), \tag{11.5}$$

with differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$.

Using this, it's possible to compute the $E_2$-page of the Cartan-Eilenberg spectral sequence:

$$E(h_{10}, h_{11}, h_{12}, h_{20}, h_{21}) \otimes P(b_{10}, b_{11}, b_{20}), \tag{11.6}$$

where $h_{ij} \in \mathrm{Ext}^{1,2p^j([^i-1)}_{P(\xi_i)}(\mathbb{Z}/p, \mathbb{Z}/p)$ and $b_{ij} \in \mathrm{Ext}^{2,2p^{j+1}(p^i-1)}_{P(\xi_i)}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Here Richard drew a picture, which I didn't get down. There's a multiplicative structure, and the key point is that $h_{11}h_{10} = 0$. Using sparseness, there can only be a few differentials, which are:

- $d_2(h_{20}) = h_{10}h_{11}$,
- $d_2(h_{21}) = h_{11}h_{12}$, and
- $d_3(b_{20}) = h_{12}b_{10} - h_{11}b_{11}$.

So we can deduce the $E_\infty$ page.

**Theorem 11.7.** *The $E_\infty$ page is free on $P(b_{10})$ with generators $1$, $h_{10}$, $h_{12}$, $h_{10}h_{12}$, $b_{11}$, and $h_{10}b_{11}$. There are also Massey products $g_0 = \langle h_{11}, h_{10}, h_{10}\rangle$ in bidegree $(2, 20)$ and $k_{10}\langle h_{11}, h_{11}, h_{10}\rangle$ in bidegree $(2, 28)$, which come from the fact that $h_{11}h_{10} = 0$, $h_{10}^2 = 0$, and $h_{11}^2 = 0$. In particular, $h_{10}k_0 = \pm h_{11}g_0$.*