# FALL 2016 ALGEBRAIC GEOMETRY SEMINAR

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## 1. What is a scheme?: 8/31/16

Today, Tom provided an introduction and overview, with the goal of understanding what a *scheme*, the central object of study in algebraic geometry, is. We'll start with sheaves, a way of understanding locality of things in geometry, then discuss locally ringed spaces, the spectrum of a ring, and finally schemes, their properties, and a little bit about morphisms of schemes. Finally, we'll learn a little about varieties.

### 1.1. Sheaves.

**Definition 1.1.** A *presheaf of rings*[1] $\mathscr{F}$ on a topological space $X$ consists of the data

(1) for every open $U \subset X$, a ring $\mathscr{F}(U)$, and
(2) for every inclusion of open sets $V \subset U$, a ring homomorphism $\rho^U_V : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ called the *restriction map*,

such that for every nested inclusion of opens $W \subset V \subset U$, the restriction maps compose: $\rho^U_W = \rho^V_W \circ \rho^U_V$.

Elements of $\mathscr{F}(U)$ are called *sections*.

The idea is that $\mathscr{F}(U)$ is some collection of data on $U$, such as the continuous real-valued functions on $U$, which define a ring. Given such a function, we can restrict it to a $V \subset U$, and this is exactly what the restriction map does. If I want to further restrict to another subset, it doesn't matter whether I restrict to $V$ first.

Presheaves have some problems, and we define sheaves to fix these problems.

**Definition 1.2.** A *sheaf* $\mathscr{F}$ on a space $X$ is a presheaf such that

(1) sections can be computed locally: if $U \subset X$ is open, $\mathfrak{U}$ is an open cover of $U$, and $s \in \mathscr{F}(U)$, then if $\rho^U_{U_i}(s) = 0$ for all $U_i \in \mathfrak{U}$, then $s = 0$.
(2) compatible sections can be glued: with $U$ and $\mathfrak{U}$ as above, suppose we have data of $s_i \in \mathscr{F}(U_i)$ for each $U_i \in \mathfrak{U}$ such that for all $U_i, U_j \in \mathfrak{U}$, $\rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j)$, then there is a section $s \in \mathscr{F}(U)$ such that $\rho^U_{U_i}(s) = s_i$ for all $U_i \in \mathfrak{U}$.

[1] One can talk about presheaves of sets, groups, or of any other category, by replacing "rings" in this definition by "sets," "groups," or whatever you're using.

To understand this intuitively, think about continuous real-valued functions, which can be uniquely determined from local data, and can be glued together from compatible functions on an open cover.

**Example 1.3.** Let $X$ be a space. We've already been referring to the sheaf $C_X$ of continuous $\mathbb{R}$-valued functions: $C_X(U) = \{f : U \longrightarrow \mathbb{R} \text{ continuous}\}$. Restriction of functions defines a restriction map, functions are determined by local data, and compatible functions may be glued together.

This is a good example of sheaves for your intuition: sheaves in general behave a lot like a sheaf of functions, and it's convenient to think of the restriction map as actual restriction of functions.

**Example 1.4.** Let $X$ be a manifold; then, we can define the sheaf $C_X^\infty$ of smooth functions: $C_X^\infty(U)$ is the ring of smooth functions $U \longrightarrow \mathbb{R}$. This is very similar, but it's interesting that this sheaf uniquely determines the smooth structure on the manifold $X$.

That is, smooth structure is determined by what you call smooth functions. This is a rule that applies more generally in geometry: a geometric structure is determined by the sheaf of functions to some base that we allow.

*Remark.* The empty set is an open subset of a space $X$. You can prove or define (depending on your taste for empty arguments) that for any sheaf $\mathscr{F}$ on $X$, $\mathscr{F}(\varnothing) = 0$.

**Definition 1.5.** Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a space $X$. Then, a morphism of sheaves $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ is the data of for all open $U \subset X$, a ring homomorphism $\varphi(U) : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ that commutes with restriction in the following sense: for all inclusions of open sets $V \subset U$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \big\downarrow \rho_V^U & & \big\downarrow \rho_V^U \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V). \end{array}$$

That is, we want to map in a way that doesn't affect how we restrict. A sheaf is data parametrized by a topological space, and we want a morphism of sheaves to respect this parametrization.

**Definition 1.6.** Let $\mathscr{F}$ be a sheaf on $X$ and $p \in X$. Then, the *stalk* of $\mathscr{F}$ at $p$ is

$$\mathscr{F}_p = \{(s, U) \mid U \subset X \text{ is open}, s \in \mathscr{F}(U)\}/\sim,$$

where $(s, U) \sim (t, V)$ if there's an open $W \subset U \cap V$ containing $p$ such that $\rho_W^U(s) = \rho_W^V(t)$.

That is, we define two functions to be equivalent if they agree on any neighborhood of the point. These are sort of infinitesimal data of functions near the point $p$.

1.2. **Locally ringed spaces.**

**Definition 1.7.** A *local ring* is a ring $A$ with a unique maximal ideal $\mathfrak{m} \subset A$, often denoted $(A, \mathfrak{m})$.

This is the same as saying $A^\times = A \setminus \mathfrak{m}$: everything outside the maximal ideal is invertible.

**Definition 1.8.** A *locally ringed space* is a pair $(X, \mathscr{O}_X)$, where $X$ is a topological space and $\mathscr{O}_X$ is a sheaf of rings, such that all stalks $\mathscr{O}_{X,p}$ are local rings.

**Example 1.9.** Manifolds are examples of locally ringed spaces: if $X$ is a manifold, let $\mathscr{O}_X = C_X^\infty$, the smooth, real-valued functions. Let $p \in X$ and $\mathfrak{m}_{X,p}$ be the functions vanishing at $p$ inside $\mathscr{O}_{X,p}$, which is an ideal. Then, any $f \in \mathscr{O}_{X,p} \setminus \mathfrak{m}_{X,p}$ is a unit: since it doesn't vanish at $p$, there's an open neighborhood $U$ of $p$ on which $f$ doesn't vanish, so $1/f$ is smooth on $U$, and therefore defines an inverse to $f$ in $\mathscr{O}_{X,p}$.

This locally ringed formalism is surprisingly useful: the maximal ideal of a stalk will always be functions vanishing at a point, even in weirder situations.

Of course, we want to understand morphisms of locally ringed spaces.

**Definition 1.10.** A *morphism of locally ringed spaces* $(X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ is the data $(\varphi, \varphi^\#)$ of a continuous map $\varphi : X \longrightarrow Y$ and a morphism of sheaves $\varphi^\# : \varphi_*\mathscr{O}_Y \longrightarrow \mathscr{O}_X$ such that the induced map on stalks preserves the notion of vanishing at a point, i.e. for every $p \in X$, the preimage of the maximal ideal $\mathfrak{m}_{X,p}$ is contained in $\mathfrak{m}_{Y,\varphi(p)}$.

Here, $\varphi_*\mathscr{O}_X$ is the *pushforward* of $\mathscr{O}_X$, which attaches to every open $U \subset Y$ the ring $\varphi_*\mathscr{O}_X(U) = \mathscr{O}_X(\varphi^{-1}(U))$: since $\varphi$ is continuous, this is again an open set.

The pushforward is an important definition in its own right. It's necessary to check that it actually defines a sheaf, but this isn't too complicated.

As an example, a smooth map $\varphi : X \longrightarrow Y$ of manifolds defines a morphism of locally ringed spaces: $\varphi$ is continuous, and a continuous map $f : V \longrightarrow \mathbb{R}$ is sent to the map $f \circ \varphi : \varphi^{-1}(V) \longrightarrow \mathbb{R}$. This is called the *pullback* of $f$. This is curious: we could have started with a merely continuous function that sends smooth functions to smooth functions, and it's forced to be smooth. Thus, the geometry of smooth manifolds is determined entirely by their structure as locally ringed spaces! Similarly, we'll define schemes to be certain kinds of locally ringed spaces.

1.3. **The spectrum of a ring.** A scheme is a particular kind of locally ringed space, locally isomorphic to Spec $A$ for rings $A$, in the same way that a manifold is locally $\mathbb{R}^n$. Let's discuss the local model better.

**Definition 1.11.** The *spectrum* of a (commutative) ring $A$ is Spec $A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is prime}\}$.

Let's briefly recall localization of rings.

**Definition 1.12.** If $A$ is a ring and $S \subset A$ is a sub*set* such that $1 \in S$ and whenever $x, y \in S$, then $xy \in S$, we call $S$ a *multiplicative subset*. Then, we can define the *localization* $S^{-1}A$ to be the ring of fractions $\{a/s \mid a \in A, s \in S\}$, where $a/s = b/s'$ iff there exists a $t \in S$ such that $t(s'b - sa) = 0$.

This is strongly reminiscent of the field of fractions of an integral domain, for which $S = A \setminus 0$; the equivalence relation is what allows us to know that $1/2 = 2/4$. For example, if $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus 0$, then $S^{-1}A = \mathbb{Q}$. In the same sense, a more general localization is akin to formally adding inverses of $S$.

**Example 1.13.** Let $\mathfrak{p} \subset A$ be a prime ideal. Then, $S = A \setminus \mathfrak{p}$ is multiplicative, since if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. The localization $S^{-1}A$ is denoted $A_\mathfrak{p}$, the set of fractions $a/s$ where $a \in A$ and $s \notin \mathfrak{p}$, with some equivalence relation. This makes everything except $\mathfrak{p}$ for units, so the image of $\mathfrak{p}$ is maximal in $A_\mathfrak{p}$.

Similarly, if $f \in A$, we can define $S = (f)$. The localization $S^{-1}A$ is denoted $A_f$, fractions of the form $a/f^n$; this makes $f$ into a unit.

We need to define Spec $A$ as a topological space, and then place a sheaf structure on it. With this structure, Spec $A$ will be an *affine scheme*.

**Definition 1.14.** Let $I \subset A$ be an ideal. Then, let $D(I) \subset \operatorname{Spec} A$ be the set of prime ideals not containing $I$; if $I = \{f\}$, $D(I) = D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. We define the topology on Spec $A$ to have as its open sets $D(I)$ for all ideals $I$.

One has to check that these are closed under finite intersection and arbitrary union, but this is true, so Spec $A$ is indeed a topological space.

**Example 1.15.** For example, Spec $\mathbb{Z}$ as a set is the set of prime numbers and 0, since these account for all the ideals. The topology is curious: $(0) \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset \mathbb{Z}$, so the zero ideal "lives everywhere."

The open sets are $D(a)$, the set of primes not dividing $a$, unless $a = 0$, in which case we get $\emptyset$.

The open set $D(f)$ is actually isomorphic as a topological space to $\operatorname{Spec}(A_f)$; for this reason, it's called a *distinguished affine open*.

Now, we just need to define the structure sheaf $\mathscr{O}_A$: what are the functions on Spec $A$? We define $\mathscr{O}_A(U)$ to be the ring of functions $f : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ such that $f(\mathfrak{p}) \in A_\mathfrak{p}$ and for all $\mathfrak{p} \in U$, there's an $a/s \in A_\mathfrak{p}$ and an open $V \subset U$ such that for all $\mathfrak{q} \in V$, $f(\mathfrak{q}) = a/s$.

There are a bunch of equivalent definitions, but this is one of the most concrete: a section is a function to a weird space, but other definitions don't explicitly make the structure sheaf a sheaf of functions, and so it's harder to prove that the structure sheaf is, in fact, a sheaf.

Distinguished opens are particularly nice, in that $\mathscr{O}_X(D(f)) \cong A_f$. Moreover, for any $\mathfrak{p} \in \operatorname{Spec} A$, one can show $\mathscr{O}_{A,\mathfrak{p}} \cong A_\mathfrak{p}$. $A_\mathfrak{p}$ is a local ring, with (the image of) $\mathfrak{p}$ as its unique maximal ideal.

1.4. **Examples.** First, let's understand Spec $\mathbb{Z}$ as a scheme, not just a topological space. $D(6)$ is the set of all primes except 2 and 3, plus the zero ideal. The acceptable functions on it are isomorphic to $\mathbb{Z}_6 = \{a/6^n \mid n \geq 0, a \in \mathbb{Z}\} = \mathbb{Z}[1/6]$. Thinking of these as functions, the function $21/6$ has value $21/6$ — but in different rings. Over (5), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(5)}$; at (7), $21/6$ takes the value $21/6 \in \mathbb{Z}_{(7)}$. Here, $\mathbb{Z}_{(5)}$ is the ring of fractions whose denominators aren't divisible by 5. We can make sense of this for all primes except 2 or 3, and the function $21/6$ can't exist there (since dividing by zero is bad). At (0), the value is $21/6 \in \mathbb{Z}_{(0)} = \mathbb{Q}$.

Next, we'll do a more geometric example.

**Example 1.16.** Let $k$ be a field (if you like, $k = \mathbb{C}$ makes for good geometric intuition). We define *affine $n$-space* $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$. All prime ideals of $\mathbb{A}_k^1$ look like $(f)$ for some $f \in k[x]$; this prime ideal is prime iff $f$ is irreducible. If $k$ is algebraically closed, e.g. $k = \mathbb{C}$, this is only the case when $f(x) = x - a$ or $f(x) = 0$.

We associate the point $(x - a)$ to the point $a \in \mathbb{C}$, so we have a complex line of points plus the zero ideal, which is weird: it somehow lives everywhere.

$\mathbb{A}_{\mathbb{C}}^2$ is a little stranger: not only do we have a $\mathbb{C}^2$ worth of points $(a, b)$ corresponding to $(x - a, y - b)$, and $(0)$ which is once again everywhere, there are additional prime ideals: $(y - x^2)$ is a prime ideal, and it somehow lives at the entire curve $\{y = x^2\} \subset \mathbb{C}^2$. This is disorienting, but sometimes is useful.

## 2. But really, what is a scheme?: 9/7/16

These are Arun's lecture notes on the functor of points, another way to understand schemes and algebraic geometry that's particularly useful in the world of algebraic groups.

There doesn't seem to be a canonical reference for functor-of-points-style algebraic geometry. The discussion in the comments of [Kam09] provides interesting perspective, though no math. [Mil15, §1.a] has a brief introduction. [Vak15] and [Sta16] have the information, but it's scattered.

### 2.1. Functors and Yoneda's lemma.

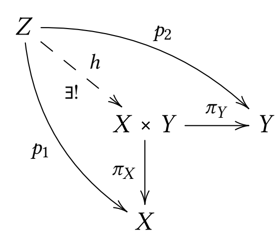
The functor of points describes maps to schemes as "universal" or "natural" families of objects; to understand these cleanly, we need a dash of category theory. This section more or less follows [Sta16, Tag 001L].

**Definition 2.1.** A *category* $\mathsf{C}$ is a collection of *objects* and for every pair of objects $X.Y \in \mathsf{C}$ a set of *morphisms* $\operatorname{Hom}_{\mathsf{C}}(X, Y)$, such that there is always an identity $\mathrm{id}_X \in \operatorname{Hom}_{\mathsf{C}}(X, X)$ and we can compose morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ into a morphism $g \circ f : X \longrightarrow Z$.

For example: $\mathsf{Set}$ of sets and functions, $\mathsf{Grp}$ of groups and group homomorphisms, $\mathsf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps, $\mathsf{Top}$ of topological spaces and continuous functions.

Categories are useful for defining *universal properties.* We know what the product of two sets is, of rings, of spaces, of manifolds, ... but if we can unify these definitions, we know how to define the product in unfamiliar situations: in particular, this is how we'll define the product of schemes.

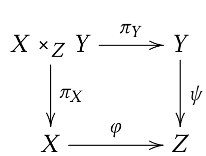
**Definition 2.2.** Let $X, Y \in \mathsf{C}$. The *product* of $X$ and $Y$, denoted $X \times Y$, is the *terminal object* with a pair of morphisms $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$. That is, for any object $Z \in \mathsf{C}$ with maps $p_1 : Z \longrightarrow X$ and $p_2 : Z \longrightarrow Y$, there exists a unique map $h : Z \longrightarrow X \times Y$ such that the following diagram commutes.




From this definition, one can show that the product doesn't always exist, but when it does, it's determined up to unique isomorphism respecting $\pi_X$ and $\pi_Y$. The product in any category you've seen is an instance of this universal definition, and $\pi_X$ and $\pi_Y$ are the projection maps.

Here's a more general example, which we'll need later.

**Definition 2.3.** Let $\varphi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ be morphisms in a category $\mathsf{C}$. Then, the *fiber product* (also *pullback* and *base change*) $X \times_Z Y$ is the terminal object with a pair of morphisms $\pi_X : X \times_Z Y \longrightarrow X$ and $\pi_Y : X \times_Z \longrightarrow Y$ such that the following diagram commutes.




Once again, this might not exist, but is uniquely determined if it does.

**Exercise 2.4.** If you haven't seen this before, show that in $\mathsf{Set}$, the fiber product of $\varphi : X \to Z$ and $\psi : Y \to Z$ is $X \times_Z Y = \{(x, y) : \varphi(x) = \psi(y)\}$.

Given any category $\mathsf{C}$, we can define the *opposite category* $\mathsf{C}^{\mathsf{op}}$ by reversing the arrows: $\mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$.

**Definition 2.5.** A *(covariant) functor* $F : \mathsf{C} \to \mathsf{D}$ is a structure-preserving map of categories: to every object $X \in \mathsf{C}$, we associate $F(X) \in \mathsf{D}$, and to every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ in $\mathsf{C}$, we associate $F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$.

A functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is sometimes called a *contravariant functor* from $\mathsf{C}$ to $\mathsf{D}$: it sends every $f : X \to Y$ in $\mathsf{C}$ to $F(f) : F(Y) \to F(X)$.

Functors abound in mathematics. For example: fundamental group, homotopy, homology, and cohomology groups; free groups, abelian groups, or algebras; pullback of functions; the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ sending a group to its underlying set.

**Example 2.6** (Functor of points)**.** Given an object $X \in \mathsf{C}$ the contravariant functor $h_X = \mathrm{Hom}_{\mathsf{C}}(-, X) : \mathsf{C} \to \mathsf{Set}$ (sending $Y$ to the set of morphisms $Y \to X$) is called the *functor of points* of $X$. This is contravariant because of precomposition: a map $\varphi : Y \to Z$ induces a map $\varphi^* : h_X(Z) \to h_X(Y)$, called *pullback*: it sends $f : Z \to X$ to $f \circ \varphi : Y \to X$.

The functor-of-points approach to algebraic geometry is to understand a geometric object $X$ through its functor $h_X$, which typically has a cleaner description. This is like understanding a function $f : S^1 \to \mathbb{R}$ through its Fourier coefficients, with Hom playing the role of an inner product.

The Yoneda lemma is the statement that this "inner product" is nondegenerate.

**Lemma 2.7** (Yoneda)**.** *Let $X, Y \in \mathsf{C}$. An isomorphism $h_X \xrightarrow{\sim} h_Y$ uniquely determines an isomorphism $X \xrightarrow{\sim} Y$.*

In particular, $X$ is completely and canonically determined by $h_X$.

Finally, we mention a few categorical facts about scheme theory.

**Definition 2.8.** A *duality of categories* is a pair of adjoint, contravariant functors inducing an equivalence of opposite categories. That is, a duality of categories $\mathsf{C}$ and $\mathsf{D}$ is the data of two functors $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}^{\mathsf{op}}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functor.

This term is nonstandard.

**Theorem 2.9** ("Fundamental theorem", [Sta16, Tag 01HX])**.** *The following pairs of functors define dualities of categories.*

*(1)* $(\Gamma, \mathrm{Spec})$ *between* $\mathsf{AffSch}$ *and* $\mathsf{Ring}$*: the ring of global sections* $\Gamma(X)$ *of an affine scheme* $X$ *and the spectrum* $\mathrm{Spec}\, A$ *of a commutative ring, respectively.*

*(2) Fixing a ring $A$,* $(\Gamma, \mathrm{Spec})$ *between* $\mathsf{AffSch}_A$ *and* $\mathsf{Alg}_A$*: the same functors, but between the category of affine schemes over $A$*[2] *and the category of $A$-algebras.*

*(3) Fixing a field $k$,* $(I, V)$ *between* $\mathsf{AffVar}_k$ *and* $\mathsf{RedAlg}_k$*: analogous functors between the category of affine varieties over a field $k$ and finitely generated reduced $k$-algebras.*

Moreover, because every scheme (resp. scheme over $A$, variety over $k$) can be constructed by gluing together affine schemes (resp. affine $A$-schemes, affine varieties over $k$), a functor of points $h_X : \mathsf{Sch}^{\mathsf{op}} \to \mathsf{Set}$ is determined by its restriction to $h_X : \mathsf{AffSch}^{\mathsf{op}} \to \mathsf{Set}$ (we'll explain this more in a bit), which by the above theorem is equivalent to a covariant functor $h_X^{\mathsf{op}} : \mathsf{Ring} \to \mathsf{Set}$. That is, functors of points on this geometric category can be understood in terms of purely algebraic data. In the same way, a functor of points on schemes over $A$ is determined by a contravariant functor $\mathsf{Alg}_A \to \mathsf{Set}$, and one from $k$-varieties is determined by a covariant functor $\mathsf{RedAlg}_k \to \mathsf{Set}$.

2.2. **Representability.** Our game plan is to write down examples of these functors and try to understand the geometry of the objects associated to them. The first obstacle is that not every functor is $h_X$ for some $X$.

**Definition 2.10.** Let $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ be a functor. If $X \in \mathsf{C}$ is such that $F \cong h_X$, then $F$ is called *representable*.

Representable functors are the ones we can do geometry with.

[2] A *scheme over a base* $A$ is the data of a scheme $X$ and a map to $\mathrm{Spec}\, A$, called the *structure map*. Morphisms of schemes over $A$ are required to commute with the structure maps.

**Example 2.11.** Algebraic geometry is built out of solution sets to systems of polynomial equations, and the functor of points is a package for solving a system of equations (the geometric constraint) over all coefficient rings simultaneously, as this example illustrates.

Let $f_1, \dots, f_n \in \mathbb{Z}[x_1, \dots, x_m]$. For any ring $A$, the unique map $\mathbb{Z} \longrightarrow A$ induces a map $\mathbb{Z}[x_1, \dots, x_m] \longrightarrow A[x_1, \dots, x_m]$. Let $F : \mathsf{Ring} \longrightarrow \mathsf{Set}$ send

$$A \longmapsto \{(a_1, \dots, a_m) \in A^n \mid f_1(a_1, \dots, a_m) = \cdots = f_n(a_1, \dots, a_m) = 0\},$$

the set of solutions to $f_1, \dots, f_n$ over $A$. Using (2.9), $F$ is representable, and is represented by the scheme $X = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_m]/(f_1, \dots, f_n)$.

In this sense, a solution to $(f_1, \dots, f_n)$ over $A$ can be thought of as an "$A$-valued point" of the representing scheme $X$; the $A$-valued points are in natural bijection with the maps $\operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} A, X)$. More generally, if $X$ is any scheme, an *$A$-valued point* is (the image of) a map $\operatorname{Spec} A \longrightarrow X$. This is the etymology of the functor of points: to a scheme/functor $X$ we have its $\mathbb{C}$-valued points $X(\mathbb{C})$, its $\mathbb{F}_p$-valued points $X(\mathbb{F}_p)$, etc.

The following discussion is adapted from [Vak15, §9.1.6].

**Example 2.12** (Fiber products)**.** Let $X$, $Y$, and $Z$ be schemes, and $\varphi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ be specified. Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be the functor sending a scheme $W \mapsto h_X(W) \times_{h_Z(W)} h_Y(W)$. Then, $F$ is representable, and is represented by the fiber product $X \times_Z Y$.

In particular, fiber products of schemes always exist. They include restricting to the preimage of an open subset or changing the base ring (e.g. base change with $\operatorname{Spec} \mathbb{C}$ turns schemes over $\mathbb{R}$ to schemes over $\mathbb{C}$).

When we refer to the fiber product of functors $\mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$, we take them pointwise, i.e. $(F \times_H G)(X) = F(X) \times_{H(X)} G(X)$.[3]

This allows us to discuss a criterion for representability. The key is that a morphism of schemes can be glued together from compatible local data. They define a sheaf $U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, X)$, with restriction given by actual restriction of functions.

**Definition 2.13.** Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a functor such that for every scheme $X$, the map $U \mapsto F(U)$ for $U \subset X$ open defines a sheaf of sets on $X$, and for every morphism $\varphi : X \longrightarrow Y$, the induced map $F(\varphi)$ is a morphism of these sheaves; then, $F$ is called a *Zariski sheaf.*

In other words, morphisms should sheafify in a universal way. This is necessary, but not sufficient.

**Definition 2.14.** Let $h : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a functor.

- A functor $h' : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ is called an *open subfunctor* of $h$ if for all representable functors $h_X$ and maps $h_X \longrightarrow h$,[4] the pullback $h' \times_h h_X$ is representable and represents an open subscheme of $X$.
- A collection $\mathfrak{U}$ of open subfunctors of $h$ is said to *cover* $h$ if for all representable functors $h_X$ and maps $h_X \longrightarrow h$, the schemes representing $h_X \times_h F_i$ for all $F_i \in \mathfrak{U}$ are an open cover of $X$.

Once again, these are the ordinary notions universalized.

**Theorem 2.15.** *Let $F : \mathsf{Sch}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be a Zariski sheaf that has an open cover by representable functors. Then, $F$ is representable.*

**Exercise 2.16.** Let $A$ be a ring and $F : \mathsf{Sch}_A^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be the functor sending an $A$-scheme $X$ to the set of data $\{(\mathscr{L}, s_0, s_1)\}$ up to isomorphism, where $\mathscr{L}$ is a line bundle (invertible sheaf) on $X$ and $s_0$ and $s_1$ are sections with no common zero. Use Theorem 2.15 to show $F$ is representable; the representing scheme is called *projective* 1*-space* over $A$, denoted $\mathbb{P}^1_A$.

*Remark.* You may be more used to the definition of $\mathbb{P}^1_A$ as two copies of $\mathbb{A}^1_A$ glued together so one's 0 is the other's point at infinity, as for the construction of the Riemann sphere $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$.

**Example 2.17.** Another use for the functor of points is in *moduli problems*: we want to classify geometric objects as elements of some space. In general, the "moduli space of $X$-stuff" is the scheme representing the functor sending $Y$ to the set of flat families of $X$-stuff over $Y$. These functors aren't always representable, however.

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[3] This is in fact the fiber product of these functors in the *functor category* $\mathsf{Fun}(\mathsf{Sch}^{\mathsf{op}}, \mathsf{Set})$.

[4] A morphism of schemes is an *open embedding* if it factors as an isomorphism of schemes followed by an inclusion $(U, \mathscr{O}_X|_U) \hookrightarrow (X, \mathscr{O}_X)$ of an open subset. See [Vak15, §7.1].

2.3. **Group Schemes.** This section follows [Vak15, §6.6].

Informally, just as a topological group is a topological space with continuous multiplication and inversion, and a Lie group is a manifold with smooth multiplication and inversion, a group scheme is a scheme with multiplication and inversion maps that are morphisms of schemes. We will make this precise in two different ways.

**Definition 2.18.** Let $\mathsf{C}$ be a category in which finite products exist. A *group object* in $\mathsf{C}$ is an object $G \in \mathsf{C}$ together with a *multiplication map* $\mu : G \times G \longrightarrow G$, an *identity map* $e : 1 \longrightarrow G$,[5] and an *inversion map* $i : G \longrightarrow G$, all of which are morphisms in $\mathsf{C}$, such that $\mu$, $e$, and $i$ satisfy the usual axioms of a group. For example, associativity of $\mu$ means that the following diagram commutes:

$$G \times G \times G \underset{(\mathrm{id},\mu)}{\overset{(\mu,\mathrm{id})}{\rightrightarrows}} G \times G \xrightarrow{\mu} G.$$

You know what the axioms are; the trick is writing them as commutative diagrams rather than element-wise.

**Example 2.19.** Group objects encode the usual notion of "groups with additional structure:"

- A group object in $\mathsf{Set}$ is just a group.
- A group object in $\mathsf{Top}$ is a topological group.
- A group object in $\mathsf{Man}$ is a Lie group.

**Exercise 2.20.** Why are the group objects in $\mathsf{Grp}$ the abelian groups?

Group objects in $\mathsf{C}$ form a category whose morphisms are group homomorphisms that are also $\mathsf{C}$-morphisms.

**Definition 2.21.** A *group scheme* is a group object in $\mathsf{Sch}$. A *group variety* is a group in $\mathsf{Var}$.

Algebraic groups are well-behaved group varieties.

Let's functor-of-pointsify Definition 2.18. If $G \in \mathsf{C}$ is a group object and $Y \in \mathsf{C}$, we can define multiplication pointwise on $\mathrm{Hom}_{\mathsf{C}}(Y, G)$: $f \cdot g$ is the composition $\mu \circ (f, g)$ (so, if we have elements, $(f \cdot g)(x) = f(x)g(x)$), and can define the identity and inverse maps similarly. The upshot is that $h_G(Y)$ is a group, and $h_G$ sends morphisms to group homomorphisms, so we may regard it as a functor $\mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Grp}$. The converse is also true: if a representable functor factors through $\mathsf{Grp}$, it's represented by a group object. Maybe the following definition is actually a theorem.

**Definition 2.22.** A *group object* in a category $\mathsf{C}$ is an object $X \in \mathsf{C}$ whose functor of points $h_X : \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ factors through the structure-forgetting inclusion $\mathsf{Grp} \longrightarrow \mathsf{Set}$, and hence may be regarded as group-valued.

This generalizes to abelian group objects, ring objects, etc.

There's no shortage of group-valued functors around, making it easy to define group schemes.

**Example 2.23.**

(1) Fix a ring $A$. The forgetful functor $\mathrm{For} : \mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sends an $A$-algebra to its underlying abelian group, and is covariant. It is representable, and is represented by the *additive group* $\mathbb{G}_a = \mathbb{A}^1_A$, which is hence an abelian group scheme.

(2) Similarly, the group of units is a covariant functor $\mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sending $B \mapsto B^\times$; its representing $A$-scheme is called the *multiplicative group* $\mathbb{G}_m = \mathrm{Spec}\, A[x, x^{-1}]$, which is an abelian group scheme.

(3) The functor $\mathrm{GL}_n : \mathsf{Ring} \longrightarrow \mathsf{Grp}$ sending $A \mapsto \mathrm{GL}_n(A)$, the group of $n \times n$ matrices with coefficients in $A$, is represented by the *general linear group* $\mathrm{GL}_n$, an algebraic group. In the same way, one may define the *special linear group* $\mathrm{SL}_n : A \mapsto \mathrm{SL}_n(A)$, the *orthogonal group* $\mathrm{O}_n : A \mapsto O_n(A)$, and the *special orthogonal group* $\mathrm{SO}_n : A \mapsto \mathrm{SO}_n(A)$. For $n > 2$, these are all nonabelian group schemes.

(4) There is a covariant functor $\mu_n : \mathsf{Alg}_A \longrightarrow \mathsf{Ab}$ sending $B$ to its group of $n^{\text{th}}$ roots of unity, i.e. solutions $x \in B$ to $x^n = 1$. This is an instance of Example 2.11; this functor is represented by $\mathrm{Spec}\, A[x]/(x^n - 1)$.

**Exercise 2.24.** Let $A$ be a ring and $G$ be a group scheme. Show that the identity, multiplication, and inversion maps define a group structure on the set of $A$-valued points of $G$. (It will be easier to use Definition 2.22 than Definition 2.18.)

For example, the usual notion of the group $\mathrm{GL}_n(k)$ agrees with that induced on the $k$-valued points of $\mathrm{GL}_n$.

[5]Here, 1 is the *terminal object*, which is also the empty product.

Tom Gannon spoke today. Throughout today's lecture, $k$ will denote a field, and $* = \operatorname{Spec} k$ denotes the one-point variety.

The definition of an algebraic group might be a little surprising at first, but it turns out that it's just a restatement of the usual definition of a group, trying to replace the reference to elements with commutative diagrams.

**Definition 3.1.** A (blah) group (we'll be nonspecific about what (blah) means for now) consists of a (something) $G$ and maps $m : G \times G \longrightarrow G$, $e : * \longrightarrow G$, and $i : G \longrightarrow G$ such that the following diagrams commute.

**Associativity:** We have two possible ways to apply $m$ to three copies of $G$ (starting with the first factors, or starting with the last factors), and we want them to be the same:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(\mathrm{id},m)} & G \times G \\ {\scriptstyle (m,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G. \end{array}$$

**Identity:** We want to encode that multiplication with $e$ as one of the factors doesn't change anything. Since groups in general are noncommutative, we need this to be true both on the left and the right:

$$\begin{array}{ccccc} * \times G & \xrightarrow{(e,\mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id},e)} & G \times * \\ & {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \pi_1} & \\ & & G. & & \end{array}$$

Here $\pi_i$ is projection onto the $i^{\text{th}}$ component ($i \in \{1, 2\}$).

**Inverse:** In the same way, multiplying with the inverse should give you the identity, both on the left and on the right.

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{id},i)} & G \times G & \xleftarrow{(i,\mathrm{id})} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \longrightarrow & G & \longleftarrow & * \end{array}$$

A (blah) should really be some sort of category, but we don't need the full generality.

If (something) is just a set, then this definition is equivalent to that of an ordinary group.

**Definition 3.2.** If (something) is

- a scheme, then this group is called a *group scheme*.
- an affine scheme, then this group is caled an *affine group scheme*.
- a variety, this group is called an *algebraic group* or *group variety*.

We also like commutativity, but again need to specify this without using elements.

**Definition 3.3.** An algebraic group $G$ is *commutative* or *abelian* if $m = m \circ \tau$, where $\tau : G \times G \longrightarrow G \times G$ is the transposition map switching the two copies of $G$ in $G \times G$.

It's also important to know what a homomorphism of algebraic groups is. Recall that for usual groups (defined over sets), a map $\varphi : (G, \cdot_G) \longrightarrow (H, \cdot_H)$ is a homomorphism if for all $g_1, g_2 \in G$, $\varphi(g_1 \cdot_G g_2) = \varphi(g_1) \cdot_H \varphi(g_2)$.

**Definition 3.4.** Let $(G, m_G)$ and $(H, m_H)$ be algebraic groups, and let $\varphi : G \longrightarrow H$ be a morphism of schemes. Then, $\varphi$ is a *homomorphism of algebraic groups* if $\varphi \circ m_g = m_h \circ (\varphi, \varphi)$.

Once again, this just means it sends multiplication to multiplication.

**Definition 3.5.** Let $(G, m_G)$ and $(H, m_H)$ be algebraic groups. Then, $H$ is a *algebraic subgroup* of $G$ if it's a subscheme of $G$ and the inclusion map $i : H \hookrightarrow G$ satisfies $i \circ m_H = m_G$.

**Example 3.6.** The *special linear group* is

$$\mathrm{SL}_n = \operatorname{Spec}(k[x_{11}, x_{12}, \dots, x_{nn}]/(\det(X) - 1)),$$

which represents the $n \times n$ matrices over $k$ with determinant 1.

Schemes have a lot of structure, e.g. they are topological spaces. Let's see what the structure of an algebraic group buys us in this context. Given a scheme $X$, we'll let $|X|$ denote the topological space of closed points of $X$.

**Definition 3.7.** At any closed point $p \in |X|$, the stalk $\mathscr{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$. The quotient $\mathscr{O}_{X,x}/\mathfrak{m}_x$ is called the *residue field*; Milne denotes this $\kappa$, and we will denote it $\mathring{k}(x)$.

The multiplication map on an algebraic group is regular. This more or less means it looks like a polynomial function locally. This means that if we fix a field $\kappa$ and let $T = \{x : \mathring{k}(x) = \kappa\}$ be the space of points with that residue field, then the multiplication map restricted to $T \times T$ maps into $T$.

If $\kappa = \overline{k}$, then $T = |G|$, and $m$ induces a map on the underlying topological space. However, it does *not* define the structure of a topological group on $|G|$ in general! However, in this case, the left multiplication map $\ell_a : G \longrightarrow G$ defined by $x \longmapsto ax$ (for an $a \in G$), is a homeomorphism of underlying topological spaces, and therefore $|G|$ is a homogeneous topological space: its automorphism group acts transitively on it (there's only one orbit).

We can also develop a notion of density. The definition is a little scary, but means intuitively that, just like a dense subset of a topological space, a function on a schematically dense set extends uniquely to one on the whole scheme.

**Definition 3.8.** Let $X$ be a scheme over an algebraically closed field $k$. Then, a subset $S \subseteq |X|$ is *schematically dense* if the restriction map $f \longmapsto \{(s, f(s)) \mid s \in S\}$ is injective.[6]

If $X$ is reduced, which is often but not always true, this is equivalent to the usual notion of density.

**Another view.** Yoneda's lemma tells us that an algebraic group $G$ determines a functor $h_G : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ (which is the functor of points):[7] specifying what this group does to all algebras allows us to determine a lot about the group in question. The functor $G \longmapsto h_G$ is fully faithful.

This functor is group-valued because we can precompose pairs of morphisms by $m : G \times G \longrightarrow G$: if $\varphi, \psi \in h_G(R)$, then their product is $(\varphi, \psi) \circ m$, and this obeys the usual axioms for a group (of sets).

This allows us to provide a better definition for $\mathrm{SL}_n$: it's the algebraic group that represents the functor $R \longmapsto \mathrm{SL}_n(R)$, where $R$ is a $k$-algebra. And we can also use it to make clean definitions about algebraic groups.

**Definition 3.9.** Let $H$ be a subscheme of the algebraic group $G$. Then,

- $H$ is an *algebraic subgroup* of $G$ if for all $k$-algebras $R$, $h_H(R)$ is a subgroup of $h_G(R)$.
- $H$ is a *normal algebraic subgroup* if for all $k$-algebras $R$, $h_H(R) \triangleleft h_G(R)$ (it's a normal subgroup).

The general pattern isn't too different: a notion on algebraic groups often comes from that notion applied to all groups that its functor of points defines.

**Proposition 3.10.** *The identity and inverse maps are uniquely determined for an algebraic group $G$. Moreover, if $\varphi : G \longrightarrow H$ is a homomorphism, then $\varphi \circ e_G = e_H$ and $i_H \circ \varphi = \varphi \circ i_G$.*

This is something we already know for groups; then, we invoke Yoneda's lemma.

**Proposition 3.11.** *The identity subscheme is a subgroup of any algebraic group.*

The proof is the same: we know this for groups (of sets), and using Yoneda's lemma, we can recover it for algebraic groups.

## 4. Nice Properties of Algebraic Groups: 9/21/16

Today, Gill spoke on the rest of chapter 1.

### 4.1. When is an algebraic group a variety?

**Definition 4.1.** Let $X$ be a scheme over a field $k$. Then, $X$ is *geometrically reduced* if $X_{\overline{k}} = X \times_k \operatorname{Spec} \overline{k}$ is a reduced scheme, i.e. for all $x \in X_{\overline{k}}$, the stalk $\mathscr{O}_{X_{\overline{k}},x}$ has no nonzero nilpotents. (Here $\overline{k}$ is the algebraic closure of $k$.)

This is stronger than being reduced, and is a good thing to have.

**Definition 4.2.** A scheme $X$ is *separated* if the diagonal map $\Delta_X : X \longrightarrow X \times X$ is a closed embedding.

[6] Question we weren't able to resolve during lecture: where should $f$ live?

[7] Milne in [Mil15] writes this as a functor $h_G : \mathsf{Alg}_k^0 \longrightarrow \mathsf{Grp}$. Here, $\mathsf{Alg}_k^0$ means the "small" (i.e. finitely generated) $k$-algebras, which suffices because he only considers schemes of finite type over $k$. This notation is confusing (since it's reminiscent of the opposite category).

These are used to define varieties, which are the kind of schemes that are used in classical algebraic geometry. Recall that we're already assuming all schemes are finite type over a field $k$.

**Definition 4.3.** A $k$-scheme $X$ is a *variety* if it is separated and geometrically reduced.

For algebraic groups, we only have to worry about the second condition.

**Proposition 4.4.** *All algebraic groups are separated.*

*Proof.* Consider the map $m \circ (\mathrm{id}, \mathrm{inv}) : G \times G \longrightarrow G$. The (image of the) diagonal is the preimage of the identity, which is a closed point; thus, the diagonal is a closed subscheme. ⊠

Recall that a set $S \subset X(k)$ is schematically dense if the assignment $f \mapsto \{(s, f(s)\}$ is injective.

*Fact.* If $G$ is a reduced algebraic group and $S \subset G(k)$ is dense in the usual (topological) sense, then $S$ is schematically dense.

This notion behaves well under field extension.

*Fact.* Let $X$ be a geometrically reduced scheme over a field $k$, $S \subset X(k)$ be schematically dense, and $k \hookrightarrow k'$ be a field extension. Then, $X_{k'}$ is reduced and $S \subset X(k')$ remains schematically dense. In particular, $S$ is schematically dense iff it's dense in $|G|$ and in $G(k)$.

Let $G$ be an algebraic group over a field $k$, so we may regard its functor of points as a covariant functor from $k$-algebras to groups. If $R$ is any $k$-algebra, the base change $G_R = G \times_k \operatorname{Spec} R$ is a covariant functor from $R$-algebras to groups, and if $A$ is a $k$-algebra, $G(A) = G_R(A \otimes R)$.

**Proposition 4.5.** *Let $G$ and $H$ be algebraic groups over $k$, $k \hookrightarrow k'$ be a field extension, and suppose $G(k')$ is dense in $G$. Then, a morphism of algebraic groups $\varphi : G \longrightarrow H$ is determined by its action on $G(k')$.*

*Proof.* Let $\varphi, \varphi' : G \rightrightarrows H$ be two morphisms agreeing on $G(k')$. Since $H$ is separated, then the equalizer $\mathrm{Eq}(\varphi, \varphi')$ is closed. It contains the subscheme where $\varphi$ and $\varphi'$ agree, but this is at least $G(k')$, which is dense, so the equalizer must be $G$ itself. ⊠

The condition that the equalizer is closed for all maps is equivalent to separability.

**Definition 4.6.** If $G$ is an algebraic group, $G^\circ$ denotes the connected component containing the identity element.

**Proposition 4.7.** *$G^\circ$ is an algebraic subgroup of $G$.*

Here are some nice facts about this subgroup.

*Fact.*

- This construction commutes with base change: for any extension $k \hookrightarrow k'$, $(G^\circ)_{k'} = (G_{k'})^\circ$.
- $G^\circ$ is geometrically connected.
- If $G$ is connected, then it is geometrically connected. In particular, $G$ is connected iff $G_{k'}$ is.

This is really nice: we don't have to worry about the distinction between connectivity and geometric connectivity.

**Proposition 4.8.** *The following are equivalent for an algebraic group $G$:*

*(1) $G$ is irreducible.*
*(2) $G$ is connected.*
*(3) $G$ is geometrically connected.*

*If $G$ is affine, these are also equivalent to $\mathscr{O}_G(G)/\mathfrak{N}$ being an integral domain. Here, $\mathfrak{N}$ is the nilradical, the ideal of nilpotent elements.*

This equivalence is not true for general schemes, e.g. the zero set $\{xy = 0\} \subset \mathbb{A}^2$, which is connected but not irreducible.

**Corollary 4.9.** *If $X$ is a connected but reducible scheme, it isn't an algebraic group.*

Smoothness is another nice property of schemes we might want to have. Once again, it's equivalent to a lot of other nice properties.

**Proposition 4.10.** *The following are equivalent for an algebraic group $G$:*

*(1) $G$ is smooth.*
*(2) $G^\circ$ is smooth.*
*(3) The stalk at the identity $\mathscr{O}_{G,e}$ is regular.*
*(4)* $\dim T_e(G) = \dim G$.[8]
*(5) $G$ is geometrically reduced.*
*(6) For all $k$-algebras $R$ and ideals $I \subset R$ such that $I^2 = 0$, the map $G(R) \longrightarrow G(R/I)$ is surjective.*

4.2. **Subgroups.** Let $S \subset X$ be a closed subset of $X$. It's a general fact from algebraic geometry that there's a unique reduced closed subscheme $S_{\mathrm{red}}$ of $S$ whose underlying topological space is $S$.

**Proposition 4.11.** *If $G_{\mathrm{red}}$ is geometrically reduced, then it's an algebraic subgroup of $G$.*

Following this, we saw a bunch of facts about algebraic subgroups. These went by too fast for me.

*Fact.* Let $G$ be an algebraic group over a field $k$.
- Every algebraic subgroup of $G$ is a closed subscheme.
- If $G$ is affine, so are all of its algebraic subgroups.
- Let $H$ and $H'$ be algebraic subgroups of $G$ that are varieties, and $k \hookrightarrow k'$ be a field extension containing a separable closure of $k$. If $H(k') = H'(k')$, then $H = H'$.
- Let $S$ be a closed subgroup of $G(k)$. Then, there is a unique reduced subgroup $H$ of $G$ such that $H(k) = S$.

In particular, given any subgroup $S \subset G(k)$, there's a unique reduced algebraic subgroup $H$ of $G$ such that $H(k)$ is the Zariski closure of $S$ in $G(k)$. This $H$ is called the *Zariski closure* of $S$.

We're now going to extend a bunch of constructions and definitions from the land of finite groups to the world of algebraic groups. Generally, these will use the functor of points.

**Definition 4.12.** Let $G$ be an algebraic group over $k$, and $H$ be an algebraic subgroup of $G$.
- $H$ is a *normal subgroup* of $G$, written $H \trianglelefteq G$, if for all $k$-algebras $R$, $H(R) \trianglelefteq G(R)$ (that is, it's a normal subgroup in the usual sense).
- $H$ is a *characteristic subgroup* of $G$ if for all $k$-algebras $R$ and $\alpha \in \mathrm{Aut}(G)$, $\alpha(H_R) = H_R$.

**Proposition 4.13.** *$G^\circ$ is a characteristic subgroup.*

This is reassuring.

**Definition 4.14.** Let $\varphi : G \longrightarrow H$ be a morphism of algebraic groups. Its *kernel*, denoted $\ker\varphi$, is the algebraic subgroup of $G$ obtained by base changing with $e : * \longrightarrow H$, i.e.

$$\begin{array}{ccc} \ker(\varphi) & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow e \\ G & \xrightarrow{\varphi} & h. \end{array}$$

In fact, the kernel is a normal subgroup of $G$.

**Proposition 4.15.** *If $\varphi$ is a smooth, surjective map, then* $\ker(\varphi)$ *is smooth.*

We can also talk about group actions (which is a really good thing): as per usual, we'll need to rephrase it in terms of commutative diagrams.

**Definition 4.16.** A *group action* of an algebraic group $G$ on a scheme $X$ is a natural transformation $\mu : G \times X \longrightarrow X$ that is compatible with multiplication and the identity, in that the following diagrams commute.

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \mu} & G \times X \\ {\scriptstyle m \times \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle \mu} \\ G \times X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} * \times X & \xrightarrow{e \times \mathrm{id}_X} & G \times X \\ & \searrow & \downarrow {\scriptstyle \mu} \\ & & X. \end{array}$$

We can also define normalizers, centralizers, and centers, though it's then a theorem that they exist (as algebraic subgroups). They also play well with base change.

[8] It's a general fact in algebraic geometry that $\dim T_e(G) \geq \dim G$.

Today, Souparna spoke on some examples of algebraic groups. Today, we'll be thinking in the functor-of-points perspective a lot, where we consider a functor $F : \mathsf{Alg}_k \to \mathsf{Grp}$ representable if it's isomorphic to $h_A$ for some $k$-algebra $A$, where the isomorphism is in the functor category $\mathsf{Fun}(\mathsf{Alg}_k, \mathsf{Set})$.

Today, given a $k$-algebra $A$, we'll use $h_A$ to denote the covariant functor $\mathsf{Alg}_k \to \mathsf{Set}$, and $h_{\operatorname{Spec} A}$ to denote the contravariant functor $\mathsf{AffSch}_k^{\mathsf{op}} \to \mathsf{Set}$.

Suppose $F : \mathsf{Alg}_k \to \mathsf{Grp}$ is a functor and there exists a $k$-algebra $A$, an $a \in F(A)$, and a natural isomorphism $\rho : h_A \xrightarrow{\sim} F$ such that for any $k$-algebra $R$, $\rho(R) : h_A(R) \to F(R)$ sends a function $f \mapsto F(f)(a)$. In this case, we say that $(A, a)$ *represents* $F$. This is equivalent to stipulating that for any $k$-algebra $R$ and $r \in F(R)$, there's a unique map $f : A \to R$ such that $F(f) : F(A) \to GF(R)$ is defined by $a \mapsto F(f)(a) = r$.

The point is, group-valued functors of points define algebraic groups, so we'll define some algebraic groups using them,

**Example 5.1.** The *additive group* $\mathbb{G}_a$ is the functor sending a $k$-algebra to its underlying abelian group under addition: $\mathbb{G}_a : R \mapsto (R, +)$. This is represented by $(k[T], T)$.

If $G$ is an algebraic group, then pulling global sections back across the multiplication map $m : G \times G \to G$ defines a $k$-algebra map $\Delta : \mathscr{O}_G(G) \to \mathscr{O}_G(G) \otimes \mathscr{O}_G(G)$, which is called *comultiplication*. Let's describe this map for $\mathbb{G}_a$, whose global sections are $k[T]$; in the absence of concrete information on what $\mathbb{G}_a$ looks like, we can still recover the multiplication and comultiplication maps by unwinding Yoneda's lemma.

If $F$ is a group-valued functor, pointwise multiplication defines a natural multiplication $\mu : F \times F \to F$ (a natural transformation); if $F$ is represented by $h_G = h_{\mathscr{O}_G(G)}$, then we have isomorphisms $h_G \cong h_{\mathscr{O}_G(G)} \cong F$, which therefore pull back the multiplication map to a diagram

$$\begin{array}{ccc} h_{G\times G} \cong h_G \times h_G & \longrightarrow & h_G \\ \downarrow & & \downarrow \\ h_{\mathscr{O}_G(G)} \times h_{\mathscr{O}_G(G)} & \longrightarrow & h_{\mathscr{O}_G(G)} \\ \downarrow & & \downarrow \\ F \times F & \xrightarrow{\mu} & F. \end{array} \tag{5.2}$$

If $G$ is affine (which is the case for $\mathbb{G}_a$), then we can plug $G \times G$ and $\mathscr{O}_G(G) \otimes_k \mathscr{O}_G(G)$ into (5.2):

$$\begin{array}{ccc} h_{G\times G}(G \times G) & \longrightarrow & h_G(G \times G) \\ \downarrow & & \downarrow \\ h_{\mathscr{O}_G(G)}(\mathscr{O}(G) \otimes \mathscr{O}(G)) \times h_{\mathscr{O}_G(G)}(\mathscr{O}(G) \otimes \mathscr{O}(G)) & \longrightarrow & h_{\mathscr{O}_G(G)}(\mathscr{O}(G) \otimes \mathscr{O}(G)) \\ \downarrow & & \downarrow \\ F(\mathscr{O}(G) \otimes \mathscr{O}(G)) \times F(\mathscr{O}(G) \otimes \mathscr{O}(G)) & \xrightarrow{\mu} & F(\mathscr{O}(G) \otimes \mathscr{O}(G)). \end{array}$$

Starting in the upper left, we always have an identity map $\mathrm{id} \in h_{G\times G}(G \times G)$, and we can trace it around the diagram:

- Passing to $h_G(G \times G)$, we've multiplied, so $\mathrm{id}_{G\times G}$ passes to the multiplication map $m$.
- Along the left, we get the projection maps onto the two factors in $h_G \times h_G$, which is pulled back into maps in the opposite direction for $h_{\mathscr{O}(G)} \times h_{\mathscr{O}(G)}$. Thus, when we multiply across the middle arrow, we get the comultiplication map $\Delta$.
- Along the bottom, evaluating $F$ does the same thing, but with the universal element $a$, so on the bottom right, we obtain $F(\Delta)(a)$.

Now, we specialize to $\mathbb{G}_a$. When we work this out, the universal element is $a = T$, and the pullbacks of the projection maps on global sections are $\pi_1^* : T \mapsto T \otimes 1$ and $\pi_2^* : T \mapsto 1 \otimes T$. Then, we "multiply" these together in $\mathbb{G}_a(k[T])$, meaning adding them in the underlying abelian group, and obtain the comultiplication map, which is the unique map $\Delta : k[T] \to k[T] \otimes k[T]$ extending

$$T \longmapsto T \otimes 1 + 1 \otimes T.$$

**Example 5.3.** The *multiplicative group* is defined to be the functor $\mathbb{G}_m : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ sending a $k$-algebra $R$ to its group of units $R^\times$. Almost the same argument applies; $\mathbb{G}_m$ is represented by $(k[T, T^{-1}], T)$, and the comultiplication map multiplies the two projections together: $\Delta : T \mapsto T \otimes T$.

**Example 5.4.** Suppose char $k = p > 0$. Then, there's a functor $\alpha_{p^m} : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ sending $R \mapsto \{r \in R \mid r^{p^m} = 0\}$. Characteristic $p$ guarantees this is an additive subgroup.

$\alpha_{p^m}$ is represented by $(k[T]/(T^{p^m}), T)$ and the comultiplication is once again the unique map extending

$$\Delta : T \longmapsto T \otimes 1 + 1 \otimes T.$$

**Example 5.5.** Let $n \in \mathbb{N}$. Then, the $n^{\text{th}}$ *roots of unity* are the algebraic group $\mu_n : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ sending $R \mapsto \{r \in R \mid r^n = 1\}$. This is represented by $(k[T]/(T^n - 1), T)$, and has the comultiplication map $T \mapsto T \otimes T$.

*Remark.* Suppose char $k = p$ is positive. Then, there are isomorphisms

$$k[T]/(T^{p^m}) \cong k[T]/((T+1)^{p^m} - 1) \cong k[u]/(u^{p^m} - 1).$$

Thus, $\alpha_{p^m}$ and $\mu_{p^m}$ have isomorphic underlying schemes, but their group structures are different, ultimately arising from addition and multiplication.

**Example 5.6.** Let $m, n \geq 1$. Then, the functor $M_{m,n} : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ sends a $k$-algebra $R$ to the additive group of $m \times n$ matrices with values on $R$. This is represented by $k[T_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ with the universal element $(T_{ij})$: given any matrix $M$ over any $k$-algebra $R$ and a map from $(T_{ij})$ to $M$, we know where the $T_{ij}$ have to go, and therefore have defined a map from $k[T_{ij}] \longrightarrow R$.

The coordinate-free way to do this is to start with a (finite-dimensional) $k$-vector space $V$, and defining the functor $\mathrm{End}_V : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ sending $R \mapsto \mathrm{End}_R(R \otimes V)$. Choosing a basis defines a natural isomorphism $\mathrm{End}_V \cong M_{n,n}$ when $\dim V = n$.

**Example 5.7.** A related example is $\mathrm{GL}_n : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$, sending $R \mapsto \mathrm{GL}_n(R)$, the invertible $n \times n$ matrices over $R$. This is interesting because it's nonabelian. It's represented by the Zariski-open subset where matrices are invertible; specifically, we have to localize $k[T_{ij} \mid 1 \leq i, j \leq n]$ at $\det(T_{ij})$. This forces our universal element to be invertible as desired, so $\mathrm{GL}_n$ is represented by $(k[T_{ij} \mid 1 \leq i, j \leq n]_{\det(T_{ij})}, (T_{ij}))$.

The comultiplication is the unique map extending

$$\Delta : T_{ij} \longmapsto \sum_{1 \leq \ell \leq n} T_{i\ell} \otimes T_{\ell j}.$$

$\mathrm{GL}_n$ has some nice subgroups, e.g. $\mathbb{D}_n$, the functor sending a $k$-algebra $R$ to the multiplicative group of diagonal matrices in $\mathrm{GL}_n(R)$.

Suppose char $k = p$ is positive. Then, the *Frobenius map* $f : k \longrightarrow k$ sending $\alpha \mapsto \alpha^p$ is a ring homomorphism, essentially by the freshman's dream in characteristic $p$.

If $s : k \longrightarrow R$ is any $k$-algebra, we can define a twisted version of it, ${}_fR$, which has the underlying ring of $R$, but with the $k$-algebra structure induced by the map $s \circ f : k \longrightarrow k \longrightarrow R$.

Let $G$ be an algebraic group. We can twist it as well, defining $G^{(p)} : \mathsf{Alg}_k \longrightarrow \mathsf{Grp}$ to send $R \mapsto G({}_fR) = \mathrm{Hom}_{\mathsf{Sch}_k}(\mathrm{Spec}({}_fR), G)$ (also $\mathrm{Hom}_{\mathsf{Alg}_k}(\mathscr{O}_G(G), {}_fR)$ if $G$ is affine).[9]

If $G$ is affine, $G^{(p)}$ is represented by $(\mathscr{O}_G(G) \otimes_{k\,f} k, \varphi)$, where $\varphi : \mathscr{O}_G(G) \longrightarrow \mathscr{O}_G(G) \otimes_{k\,f} k$ is the map extending $x \mapsto x \otimes 1$.[10] In particular, $G^{(p)}$ is an affine algebraic group. In general, for nonaffine $G$, $G^{(p)}$ is an algebraic group, but not necessarily affine.

There's a natural $k$-algebra homomorphism $f_R : R \longrightarrow {}_fR$, which induces a natural $k$-algebra homomorphism $\mathrm{Spec}({}_fR) \longrightarrow \mathrm{Spec}\, R$; applying the contravariant functor $h_G = \mathrm{Hom}_{\mathsf{Sch}_k}(-, G)$ means we've defined a natural homomorphism $G(R) \longrightarrow G^{(p)}(R)$, i.e. a natural transformation of the functors $G \longrightarrow G^{(p)}$. Since the Yoneda embedding is fully faithful, this means there's a map of schemes $G \longrightarrow G^{(p)}$; the corresponding map on algebras is $\mathscr{O}(G^{(p)}) \longrightarrow \mathscr{O}(G)$ sending $c \otimes a \mapsto c \otimes a^p$.

This construction iterates, allowing us to define maps

$$G \longrightarrow G^{(p)} \longrightarrow G^{(p)2} \longrightarrow \cdots \longrightarrow G^{(p)n}.$$

We call the composition of these maps $f_n$.

[9]For a more general $k$-scheme $X$ we can define $X^{(p)}$ in the same manner, but it will not be an algebraic group in general.

[10]The twisted tensor product $R \otimes_{k\,f} k$ is sometimes also written $R \otimes_{k,f} k$.

**Proposition 5.8.** $\ker(f_n)$ *is a characteristic subgroup of* $G$.

**Definition 5.9.** We say that $G$ has *height* $\leq n$ if $f^n = 0$.

**Definition 5.10.** A homomorphism $\alpha : G \longrightarrow H$ between connected group varieties is an *isogeny* if it's surjective and has finite kernel. It's called *separable* if the kernel is étale (defined in [Mil15, p. 42]), and *central* if $\ker(\alpha) \subset Z(G)$.

Finally, we can talk about products: if $G_1, \dots, G_n$ are algebraic groups over a field $k$, the product $G = G_1 \times \cdots \times G_n$ is a scheme. We also know $h_G = h_{G_1} \times \cdots \times h_{G_n}$ is a group object in the functor category (essentially because the product of groups is a group in a natural way), and $G$ represents it, so $G$ is a group. If all of the $G_i$ are affine, then we know from algebraic geometry that $\mathscr{O}(G) \cong \mathscr{O}(G_1) \otimes \cdots \otimes \mathscr{O}(G_n)$.

The fiber products of algebraic groups over a common algebraic group are also groups, represented by the fiber product of functors $h_{G_1} \times_{h_H} h_{G_2}$, which turns out to be a group object. We know that if $G_1$, $G_2$, and $H$ are affine, then $\mathscr{O}(G_1 \times_H G_2) = \mathscr{O}(G_1) \otimes_{\mathscr{O}(H)} \mathscr{O}(G_2)$. Fiber product commutes with base change, which is a really nice tool to have.

## 6. More Examples: 10/5/16

## 7. Hopf Algebras and Affine Algebraic Groups

Today, Richard spoke; I was about 20 minutes late, so I missed some things. As usual, a field $k$ is fixed; all algebras are finitely generated, and all schemes are finite type.

Recall that an algebra over $k$ is a $k$-vector space $A$ together with a multiplication map $\mu : A \otimes A \longrightarrow A$ and a unit map $\mu : k \longrightarrow A$ satisfying a list of axioms (multiplication must be associative, and $\mu$ is a unit for multiplication) encoded as commutative diagrams.

Dually, a *coalgebra* over $k$ is a $k$-vector space $A$ together with a *comultiplication* map $\Delta : A \longrightarrow A \otimes A$ and a *counit* map $\varepsilon : A \longrightarrow k$ satisfying a list of axioms obtained by reversing the arrows in the diagrams defining algebras (and replacing multiplication by comultiplication and unit by counit).

**Definition 7.1.** A *bialgebra* is a $k$-vector space with compatible algebra and coalgebra structures, i.e. an algebra structure $(m, \mu)$ and a coalgebra structure $(\Delta, \varepsilon)$ such that either

- $m$ and $\mu$ are coalgebra homomorphisms, or
- $\Delta$ and $\varepsilon$ are algebra homomorphisms.

(The two are equivalent.)

**Definition 7.2.** A *Hopf algebra* $A$ over $k$ is a bialgebra with an *antipode* map $S : A \longrightarrow A$ such that the following diagram commutes.

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \\ \uparrow{\scriptstyle \mathrm{id}\otimes S} & & \uparrow{\scriptstyle \varepsilon\circ\mu} & & \uparrow{\scriptstyle S\otimes \mathrm{id}} \\ A \otimes A & \xleftarrow[\Delta]{} & A & \xrightarrow[\Delta]{} & A \otimes A. \end{array} \tag{7.3}$$

**Example 7.4.** Let $G$ be any group; then, the group algebra $k[G]$ of finite formal sums of elements of $G$ weighted by elements of $k$ is a Hopf algebra:

- the multiplication map is the unique linear map such that $m(1 \cdot g, 1 \cdot h) = 1 \cdot (gh)$, and
- the unit is the unique linear map such that $1_k \mapsto 1 \cdot e$.
- The comultiplication is the map induced from the diagonal map $\Delta : G \longrightarrow G \times G$, which induces a map $k[G] \longrightarrow k[G \times G] = k[G] \otimes k[G]$, and
- the counit is the trace.
- Finally, the antipode is the unique linear map extending $S(1 \cdot g) = 1 \cdot g^{-1}$.

More importantly for us, the diagrams for a Hopf algebra look suspiciously like the ones for algebraic groups. Compare associativity of multiplication and coassociativity and comultiplication:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(m,1)} & G \times G \\ \downarrow{\scriptstyle (1,m)} & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta\otimes\mathrm{id}} & A \otimes A \\ \uparrow{\scriptstyle \mathrm{id}\otimes\Delta} & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A. \end{array}$$

Compare the identity and counit diagrams:

$$\begin{array}{ccccc} \bullet \times G & \xrightarrow{(e,\mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id},e)} & G \times \bullet \\ & \searrow & \Big\downarrow m & \swarrow & \\ & & G & & \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{\mu\otimes\mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id}\otimes\mu} & A \otimes k \\ & \nwarrow & \Big\uparrow \Delta & \nearrow & \\ & & A & & \end{array}$$

Finally, compare inversion and the antipode diagram (7.3).

What we've discovered is a powerful equivalence:

**Proposition 7.5.** $(A, \Delta)$ *is a Hopf algebra iff* $(\mathrm{Spec}(A), \mathrm{Spec}\,\Delta)$ *is an affine algebraic group. In particular,* Spec *defines an equivalence of categories between (finitely generated) Hopf algebras over $k$ and affine algebraic groups over $k$.*

Maybe you didn't care about Hopf algebras before, but you should now. Let's study them some more.

**Definition 7.6.** Let $A$ be a Hopf algebra and $B$ be a $k$-subalgebra of $A$ such that

(1) $\Delta(B) \subset B \otimes B$, and
(2) $S(B) \subseteq B$.

Then, restricting $\varepsilon$ and $S$ to $B$ defines a Hopf algebra structure on $B$; $B$ is called a *sub-Hopf algebra* of $A$.

**Definition 7.7.** Let $A$ be a Hopf algebra. A *Hopf ideal* $\mathfrak{a}$ of $A$ is an ideal of $A$ such that

(1) $\Delta(\mathfrak{a}) \subset A \otimes \mathfrak{a} \oplus \mathfrak{a} \otimes A$,
(2) $\varepsilon(\mathfrak{a}) = 0$, and
(3) $S(\mathfrak{a}) \subseteq \mathfrak{a}$.

We care about Hopf ideals because, like ordinary ideals of rings, they are the kernels of homomorphisms.

**Proposition 7.8.** *Let $A$ and $B$ be Hopf algebras and $f : A \longrightarrow B$ be a $k$-algebra homomorphims. Then, $f$ is a Hopf algebra homomorphism iff* $\ker(f)$ *is a Hopf ideal.*

*Proof.* In the forward direction, the first axiom follows from the general fact that if $f : V \longrightarrow V'$ is a $k$-linear map of vector spaces, then $\ker(f \otimes f) = V \otimes \ker(f) + \ker(f) \otimes V$.

Since $f$ is a Hopf algebra homomorphism, the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow \varepsilon_A & & \Big\downarrow \varepsilon_B \\ k & = & k. \end{array}$$

If $a \in \ker(f)$, then $f(a) = 0$, so $\varepsilon_B \circ f(a) = 0$, so $\varepsilon_A(a) = 0$ as desired.

For the antipode, the argument is basically the same: since $f$ is a Hopf algebra homomorphism, the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow S_A & & \Big\downarrow S_B \\ A & \xrightarrow{f} & B. \end{array}$$

Since this diagram commutes, $S_A$ must map $\ker(f)$ to $\ker(f)$.

Conversely, suppose we know $\ker(f)$ is a Hopf ideal. Then, we will be able to put a Hopf algebra structure on $A/\mathfrak{a}$. We'll skip this for reasons of time. ⊠

**Proposition 7.9.** *There is a bijective correspondence between the algebraic subgroups of an affine algebraic group $G$ and the Hopf ideals of $\mathscr{O}_G(G)$.*

Finally, we'll sketch the proof of Cartier's theorem.

**Theorem 7.10** (Cartier)**.** *If $G$ is an affine algebraic group over $k$, where* $\mathrm{char}(k) = 0$*, then $G$ is smooth.*

The following lemma will be our criterion for smoothness.

**Lemma 7.11.** *Let $k$ be an algebraically closed field, $G$ be an algebraic group over $k$, and $\mathfrak{m}_e \subset \mathscr{O}_{G,e}$ be the maximal ideal of the stalk of $G$ at its identity $e$. Then, $G$ is smooth if every nilpotent element of $\mathscr{O}_{G,e}$ is contained in $\mathfrak{m}_e^2$.*

*Proof sketch.* Recall that $G$ is smooth iff $\dim G = \dim T_eG$, and $T_eG \cong \mathrm{Hom}(\mathfrak{m}_e/\mathfrak{m}_e^2, k)$. This kills all the nilpotents by hypothesis, and therefore $\dim T_e(G) = \dim T_e(G_{\mathrm{red}})$. ⊠

We also need a fact from linear algebra.

**Lemma 7.12.** *Let $V$ and $V'$ be vector spaces over $k$, $W \subset V$ be a subspace, and $y \in V' \setminus 0$. If $x \in V$, then $x \in W$ iff $x \otimes y \in W \otimes V'$.*

Finally, we need an algebraic fact about Hopf algebras. The proof is a diagram chase and a little linear algebra.

**Lemma 7.13.** *Let $A$ be a Hopf algebra and $I = \ker(\varepsilon)$.*

*(1) As $k$-vector spaces, $A = k \oplus I$.*
*(2) For all $a \in I$, $\Delta(a) = a \otimes 1 + 1 \otimes a \bmod I \otimes I$.*

*Proof sketch of Theorem 7.10.* Let $A = \mathscr{O}_G(G)$, so that $\mathfrak{m}_e = \ker(\varepsilon)$. By Lemma 7.11, it suffices to show that if $a \in \mathscr{O}_{G,e} = A_{\mathfrak{m}_e}$ is nilpotent, then $a \in \mathfrak{m}_e^2$. The idea is if $a^n = 0$ but $a^{n-1} \neq 0$ in $A_{\mathfrak{m}_e}$, then we can calculate $\Delta(A^n)$ and obtain a contradiction.

## 8. Some Representation Theory: 10/19/16

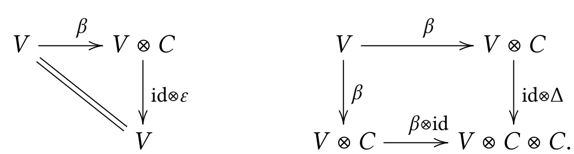
These are Arun's prepared notes for today's talk.

Throughout we fix a field $k$. Recall that if $G$ is a finite group, then a *representation* $V$ of $G$ is a $k$-vector space with a $G$-action, meaning the data of a homomorphism $\rho : G \to \mathrm{Aut}\, V$. In a basis, this means the elements of $G$ act by matrices on $V$, and matrix multiplication agrees with group multiplication.

**Definition 8.1.** Let $G$ be an algebraic group. A *representation* of $G$ is the data of a vector space $V$ and a morphism of algebraic groups $\rho : G \to \mathrm{GL}_V$. $\rho$ is *faithful* if for all $k$-algebras $A$, $\rho(A) : G(A) \to \mathrm{GL}_V(A) = \mathrm{GL}(V \otimes A)$ is injective.

### 8.1. Representations are comodules.
Let $C$ be a coalgebra, meaning it has a coassociative comultiplication map $\Delta : C \to C \otimes_k C$ and a counit map $\varepsilon : C \to k$.

**Definition 8.2.** A *$C$-comodule* is a $k$-vector space $V$ together with a $k$-linear map $\beta : V \to V \otimes_k C$ that is compatible with $\Delta$ and $\varepsilon$ in the sense that the following diagrams commute.



A *$C$-co-submodule* is a vector subspace $W \subset V$ such that $\beta(W) \subset W \otimes_k C$ (so that $\beta$ defines a comodule structure on $W$).

For $G$ a finite group, representations of $G$ over $k$ are the same data as $k[G]$-modules. Analogously, if $G$ is an algebraic group, a $G$-representation $V$ is the same data as an $\mathscr{O}(G)$-comodule structure on $V$.

- Given a representation $\rho : G \to \mathrm{GL}_V$, let $a \in G(\mathscr{O}(G))$ denote the universal element, so $\rho(a)$ is an $\mathscr{O}(G)$-linear element of $\mathrm{GL}(V \otimes \mathscr{O}(G))$, hence an $\mathscr{O}(G)$-linear map $V \otimes \mathscr{O}(G) \to V \otimes \mathscr{O}(G)$. Thus, its restriction to $V \subset V \otimes \mathscr{O}(G)$ is a $k$-linear map $\beta = \rho(a)|_V : V \to V \otimes \mathscr{O}(G)$. Associativity of multiplication on $G$ guarantees coassociativity of $\beta$, and that $1_G$ acts as the unit guarantees $\beta$ satisfies the counit axiom, so $\beta$ defines an $\mathscr{O}(G)$-comodule structure on $V$.
- On the other hand, let $\beta : V \to V \otimes \mathscr{O}(G)$ be an $\mathscr{O}(G)$-comodule, and let $\{e_i\}_{i\in I}$ be a basis for $V$. Let $r_{ij} \in \mathscr{O}(G)$ be such that
$$\beta(e_j) = \sum_{i\in I} e_i \otimes r_{ij}$$
(so there are only finitely many $j$ such that $r_{ij} \neq 0$). The comodule structure tells us that for all $i$ and $j$,
$$\Delta(r_{ij}) = \sum_{\ell\in I} r_{i\ell} \otimes r_{\ell j}$$
$$\varepsilon(r_{ij}) = \delta_{ij}.$$
Now, for any $k$-algebra $A$, $\{e_i \otimes 1\}_{i\in I}$ defines an $A$-basis for $V \otimes A$, so we can explicitly define the action of a $g \in G(A)$ to be the endomorphism with the "matrix" $(r_{ij}(g))_{i,j}$ (disclaimer: not literally a matrix unless $V$ is finite-dimensional). This allows one to define a map $\mathscr{O}(\mathrm{GL}_V) \to \mathscr{O}(G)$ which sends $T_{ij}$ (the symbol representing the $(i,j)^{\text{th}}$ matrix entry) to $r_{ij}$.

As an example application, we characterize representations of $\mathbb{G}_m$.[11]

**Proposition 8.3.** *Representations of $\mathbb{G}_m$ are $\mathbb{Z}$-graded vector spaces. That is, there is an equivalence of categories between $\mathsf{Rep}_{\mathbb{G}_m}$ and the category of $\mathbb{Z}$-graded vector spaces over $k$ which respects tensor products.*

*Proof sketch.* The key is that since $\mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$, a $\mathbb{G}_m$-representation is the same thing as a $k[x, x^{-1}]$-comodule, where $\Delta : x \mapsto x \otimes x$.

The data of a $\mathbb{G}_m$-representation on a vector space $V$ defines a map $\beta : V \longrightarrow V \otimes k[x, x^{-1}]$, which sends

$$v \longmapsto \sum_{n \in \mathbb{Z}} v_n x^n$$

(where only finitely many $v_n$ are nonzero). By linearity, $\beta(v_n) = v_n x^n$. Let $V_n \subset V$ be the subspace that's the preimage of $\operatorname{span}\{x_n\}$; then, $V = \bigoplus_{n\in\mathbb{Z}} V_n$, so $V$ is $\mathbb{Z}$-graded.

Conversely, if $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded vector space, we can define a $k[x, x^{-1}]$-comodule structure on $V$ as the unique linear map $\beta : V \longrightarrow V \otimes k[x, x^{-1}]$ such that $\beta(v_n) = v_n x^n$ for all $v_n \in V_n$.

Under this identification, $k[x, x^{-1}]$-equivariant morphisms are those which send homogeneous elements of degree $n$ to homogeneous elements of degree $n$, so this is an equivalence of categories. ⊠

## 8.2. **A few other useful results.**

**Proposition 8.4.** *Let $\rho : G \longrightarrow \mathrm{GL}_V$ be a finite-dimensional representation and $W \subset V$ be a subspace. Then, the functor $\mathrm{Stab}_G(W)$ sending a $k$-algebra $A$ to the group $\{\alpha \in G(A) \mid \alpha(W \otimes A) = W \otimes A\}$ is representable, and is represented by an algebraic subgroup of $G$.*

*Proof.* Let $\beta : \mathscr{O}(G) \longrightarrow V \otimes \mathscr{O}(G)$ be the coaction corresponding to $\rho$. Choose a complement $W^\perp$ to $W$, and let $\{e_i\}_{i\in I}$ be a basis for $V$ such that $I = J_1 \neq J_2$, $\{e_i\}_{i\in J_1}$ is a basis for $W$, and $\{e_i\}_{i\in J_2}$ is a basis for $W^\perp$. Then, there exist $a_{ij} \in \mathscr{O}(G)$ such that

$$\beta(e_j) = \sum_{i\in I} e_i \otimes a_{ij}.$$

For any $g \in G(A) = \mathrm{Hom}_{\mathsf{Alg}_k}(\mathscr{O}(G), A)$,

$$g(e_j) = \sum_{i\in I} e_i \otimes g(a_{ij}),$$

meaning that $g(W \otimes A) \subset W \otimes A$ iff $g(a_{ij}) = 0$ for all $j \in I_1$ and $i \in I_2$. Thus, $G_W$ is represented by $\mathscr{O}(G)/\langle a_{ij} \mid j \in I_1, i \in I_2\rangle$; since this is a quotient of $\mathscr{O}(G)$ and $G$ is affine, $G_W$ is represented by a closed subscheme, and since $G_W(A)$ is naturally a subgroup of $G(A)$ for all $k$-algebras $A$, then $G_W$ is an algebraic subgroup. ⊠

We're used to only thinking about finite-dimensional representations. This isn't the whole story, but fortunately it's almost the whole story: every representation can be built up from finite-dimensional representations.

**Proposition 8.5** ([Mil15], Prop. 4.6, 4.7)**.** *If $C$ is a $k$-coalgebra, every $C$-comodule is a filtered colimit of finite-dimensional sub-comodules. In particular, if $G$ is an algebraic group, every $G$-representation is a filtered colimit of finite-dimensional subrepresentations.*

*Proof.* Let $C$ be a $k$-coalgebra and $\beta : V \longrightarrow V \otimes C$ be a comodule for it; it suffices to show every $v \in V$ is contained in a finite-dimensional sub-comodule of $V$. Let $\{e_i\}_{i\in I}$ be a basis for $C$ as a $k$-vector space; then, there exist $v_i \in V$ such that

$$\beta(v) = \sum_{i\in I} v_i \otimes e_i.$$

In particular, only finitely many $v_i$ are nonzero. There are $r_{ij\ell} \in k$ such that

$$\Delta(e_i) = \sum_{j,\ell\in I} r_{ij\ell}(e_j \otimes e_\ell).$$

One of the comodule axioms was that $(\mathrm{id}_V \otimes \Delta)\circ\beta = (\beta \otimes \mathrm{id}_C)\circ\beta$. We apply each of these to $v$ to obtain

$$\sum_{i,j,k\in I} r_{ijk}(v_i \otimes e_j \otimes e_k) = \sum_k \beta(v_k) \otimes e_k$$

[11] Recall that a $\mathbb{Z}$-*graded vector space* is a vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where each $V_n$ is a subspace, and that morphisms of $\mathbb{Z}$-graded vector spaces must preserve the grading. The tensor product of $\mathbb{Z}$-graded vector spaces $V$ and $W$ is

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

inside $V \otimes \mathscr{O}(G) \otimes \mathscr{O}(G)$. When we equate the coefficients for $1 \otimes 1 \otimes e_k$, we have to factor out

$$\sum_{i,j} r_{ijk}(v_i \otimes e_j) = \beta(v_k).$$

If $W$ denotes the space spanned by $v$ and the $v_k$, then this says $\beta(W) \subset W \otimes C$, so $W$ is a $C$-sub-comodule of $V$. Since only finitely many $v_k$ were nonzero, then $W$ is finite-dimensional. ⊠

8.3. **Affine Algebraic Groups are Linear Algebraic Groups.** Matrix groups are an important source of groups; since $\mathrm{GL}_V$ is algebraic, we can ask specifically about its algebraic subgroups.

**Definition 8.6.** A *linear algebraic group* is an algebraic subgroup of $\mathrm{GL}_V$, where $V$ is a finite-dimensional $k$-vector space.

Equivalently, it's an algebraic group $G$ with a faithful, finite-dimensional representation.

In the theory of finite groups, Cayley's theorem states that every finite group is a subgroup of a finite symmetric group. For Lie groups, the Peter-Weyl theorem states that every compact Lie group is isomorphic to a matrix group (a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$). Here's the analogous statement for algebraic groups.

**Theorem 8.7** ([Mil15], Cor. 1.29, Thm. 4.8)**.** *Let $G$ be an algebraic group. Then, $G$ is affine iff it is linear.*

*Proof.* Let $G$ be an affine algebraic group. Comultiplication $\Delta : \mathscr{O}(G) \longrightarrow \mathscr{O}(G) \otimes \mathscr{O}(G)$ makes $\mathscr{O}(G)$ into an $\mathscr{O}(G)$-comodule, and therefore defines a representation of $G$ on $\mathscr{O}(G)$, which is called the *regular representation* of $G$. Since $G$ is finite type over $k$, we may choose finitely many generators $f_1, \dots, f_n$ for $\mathscr{O}(G)$ (as a $k$-algebra), and by Proposition 8.5, there's a finite-dimensional sub-comodule $V \subset \mathscr{O}(G)$ containing these generators.

Let $\{e_1, \dots, e_m\}$ be a basis for $V$ and $a_{ij} \in \mathscr{O}(G)$ be defined by $\Delta(e_j) = \sum_i e_i \otimes a_{ij}$. Writing the coaction axioms in coordinates implies that the induced map $\rho^* : \mathscr{O}(\mathrm{GL}_V) \longrightarrow \mathscr{O}(G)$ hits each $a_{ij}$. Since $\varepsilon$ is the counit for $\Delta$, then

$$e_j = (\varepsilon \otimes \mathrm{id}_{\mathscr{O}(G)})\Delta(e_j) = \sum_i \varepsilon(e_i) a_{ij}.$$

Thus, $\mathrm{Im}(\rho^*)$ contains a basis for $V$, so contains $V$; since $\rho^*$ is an algebra homomorphism and $V$ contains the generators of $\mathscr{O}(G)$, then $\mathrm{Im}(\rho^*) = \mathscr{O}(G)$. Thus, $\rho^*$ is surjective, so $G \longrightarrow \mathrm{GL}_V$ is a closed immersion, since $G$ is affine.

Conversely, let $G$ be a linear algebraic subgroup, so that it's a closed subscheme of $\mathrm{GL}_V$ for some finite-dimensional vector space $V$. Since $\mathrm{GL}_V$ is an affine algebraic group, and closed subschemes of affine schemes are affine, then $G$ is affine. ⊠

The regular representation is quite useful, e.g. in the following result.

**Theorem 8.8.** *Every finite-dimensional representation of an algebraic group $G$ is isomorphic to a subrepresentation of a direct sum of copies of the regular representation.*

8.4. **Semisimplicity.** Semisimplicity is an important concept throughout representation theory. If $A$ is a ring, a *simple* $A$-module $M$ is a module such that if $N$ is an $A$-submodule of $M$, then $N = M$ or $N = \{0\}$. For $A = k[G]$, these are the irreducible representations of $G$. More generally, a module that is a finite direct sum of simple modules is called *semisimple*. This terminology extends to algebraic groups: we define simple representations (and comodules) the same way, and semisimple representations (comodules) in the same way.

**Theorem 8.9** (Maschke)**.** *If $\mathrm{char}(k) = 0$ and $G$ is a finite group,*[12] *then all representations of $G$ over $k$ are semisimple.*

There are several equivalent criteria for a representation (of finite groups or algebraic groups) to be semisimple:

**Proposition 8.10.** *The following are equivalent for a $G$-representation $V$:*

*(1) $V$ is semisimple, i.e. it's a direct sum of simple representations.*
*(2) $V$ is a sum of simple representations.*
*(3) If $W \subset V$ is a subrepresentation, then it has a complement $W'$ such that $V = W \oplus W'$.*

[12] More generally, we may choose any field $k$ whose characteristic doesn't divide $\#G$. If $\mathrm{char}(k) \mid \#G$, counterexamples to this theorem exist.

Today, Tom spoke about more representation theory and Chevalley's theorem.

Recall that a linear representation of an affine algebraic group $G$ on a vector space $V$ is given by any of the following equivalent data.

(1) For all $k$-schemes $X$, an action of $G(X)$ on $V \otimes \Gamma(X, \mathscr{O}_X)$.
(2) If $V_\mathfrak{a}$ denotes the functor $X \mapsto V \otimes \Gamma(X, \mathscr{O}_X)$, the data of an action of the functor $G = h_G$ on $V_\mathfrak{a}$.
(3) An $\mathscr{O}(G)$-comodule structure on $V$, i.e. a linear map $\rho : V \to V \otimes \mathscr{O}(G)$ satisfying two axioms. This is particularly useful because if $\{e_i\}$ is a basis for $V$, then we can write

$$\rho(e_i) = \sum_j e_j \otimes \alpha_{ij},$$

and thus describe the action in terms of matrices.
(4) Finally, a homomorphism of algebraic groups $G \to \mathrm{GL}_V$. Pullback on the rings of regular functions describes the group elements in terms of matrices: $k[T_{ij}]_{\det(T)} \to \mathscr{O}_G(G)$ sends $T_{ij} \mapsto a_{ij}$.

Recall also that every representation is a filtered colimit of its finite-dimensional representations (Proposition 8.5), and that affine algebraic groups are linear (Theorem 8.7). In particular, this arises from the regular representation, arising from the $\mathscr{O}(G)$-comodule structure on $\mathscr{O}(G)$ coming from the comultiplication map $\Delta : \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$. The key to proving that affine algebraic groups are linear is the following theorem.

**Theorem 9.1.** *The regular representation is faithful, and in particular, has a finite-dimensional subrepresentation on which $G$ acts faithfully.*

*Proof.* If $A = \mathrm{SO}(G)$, $A$ is a finitely-generated $k$-algebra, so let's pick a finite generating set $S$. Since $A$ as a $G$-representation is a filtered colimit of subrepresentations, we can find a finite-dimensional subrepresentation $V \subset A$ containing $S$. Let $\{e_1, \dots, e_n\}$ be a basis of $V$, and write

$$\Delta(e_j) = \sum_j e_j \otimes a_{ij},$$

so that $\mathscr{O}(\mathrm{GL}_V) \to \mathscr{O}(G)$ must have image containing these $a_{ij}$. The counit axiom for a comodule says that $(\varepsilon \otimes 1_A) \circ \Delta = 1_A$, so

$$e_i = \sum_{j=1}^n \varepsilon(e_i) a_{ji},$$

so $e_i \in \langle a_{ij} \rangle$, and therefore $V \subseteq \langle a_{ij} \rangle$, determining $A \subseteq k[a_{ij}]$. This means that the map $\mathscr{O}(\mathrm{GL}_V) \to \mathscr{O}(G)$ is surjective, so $G \to \mathrm{GL}_V$ is a closed immersion (as $G$ is affine).[13] ⊠

An algebraic group is called *linear* if it admits a faithful finite-dimensional representation, i.e. is isomorphic to an algebraic subgroup of $\mathrm{GL}_V$ for some finite-dimensional vector space $V$. The above theorem shows that every affine algebraic group is linear; the converse is also true. Thus, for an algebraic group, being affine is the same as being linear. This is not in general true for non-affine algebraic groups, which suggests that non-affine algebraic groups behave weirdly with respect to representation theory: in the non-affine case, the comodule structure doesn't uniquely determine the representation, so this whole proof and the regular representation go out the window. In fact, elliptic curves are explicit examples of algebraic groups that are not linear.

Given a representation $V$ of $G$ $\rho : V \to V \otimes \mathscr{O}(G)$, we can define another $\mathscr{O}(G)$-comodule through the map

$$1_V \otimes \Delta : V \otimes \mathscr{O}(G) \longrightarrow (V \otimes \mathscr{O}(G)) \otimes \mathscr{O}(G).$$

**Definition 9.2.** With the structure map $1_V \otimes \Delta$, $V \otimes \mathscr{O}(G)$ is called the *free $\mathscr{O}(G)$-comodule on $V$*.

A choice of isomorphism $V \cong k^n$ defines an isomorphism $V \otimes \mathscr{O}(G) \cong \mathscr{O}(G)^n$, which is more concrete, but not natural.

[13] This wasn't technically the definition of a faithful representation; we required that $G(X) \hookrightarrow \mathrm{GL}_V(X)$ for all $k$-schemes $X$, but this turns out to be equivalent to requiring $G \hookrightarrow \mathrm{GL}_V$ to be a closed immersion.

**Proposition 9.3.** *Let $\rho : V \to V \otimes \mathscr{O}(G)$ be an $\mathscr{O}(G)$-comodule. Then, the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes \mathscr{O}(G) \\ \downarrow \rho & & \downarrow 1_V \otimes \Delta \\ V \otimes \mathscr{O}(G) & \xrightarrow{\rho \otimes 1_{\mathscr{O}(G)}} & V \otimes \mathscr{O}(G) \otimes \mathscr{O}(G), \end{array}$$

*and in particular, it defines an injective homomorphism of $\mathscr{O}(G)$-comodules.*

This is a little silly, but its corollaries are not silly.

**Corollary 9.4.** *Any finite-dimensional $\mathscr{O}(G)$-comodule arises as a sub-comodule of $\mathscr{O}(G)^n$ for some $n$.*

That is, it arises as a subcomodule of a finitely generated free comodule (on the regular representation). This is quite different from the general theory of modules: it's not true that every module is a submodule of a free module.

**Theorem 9.5.** *Let $V$ be a faithful finite-dimensional representation of $G$. Then, any finite-dimensional $G$-representation $W$ can be written as a sub-comodule of a quotient of a direct sum of representations of the form*

$$\bigotimes_{i=1}^{n} (V \oplus V^\vee).$$

That is, start with $V$, direct-sum with its dual, and take some tensor power. Then, take a free representation on that, and $W$ arises as a subrepresentation of that. This is kind of cool, because any single representation knows all representations. This is not often used to construct representations in practice, but it's still good to know.

For the next theorem, we'll have to do a little more.

**Definition 9.6.** A representation $V$ is *simple* if it's nonzero and has no nonzero proper subrepresentations.

That is, its only subrepresentations are 0 and itself. This is sort of like a prime number.

**Definition 9.7.** A representation $V$ is *semisimple* if it's a direct sum of simple representations.

For finite-dimensional representations, semisimplicity is the same as complete reducibility.

**Proposition 9.8.** *A representation $V$ can be written as a sum of simple representations iff it can be written as a direct sum of a subset of those representations.*

## References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