Lattice Models and TQFTs

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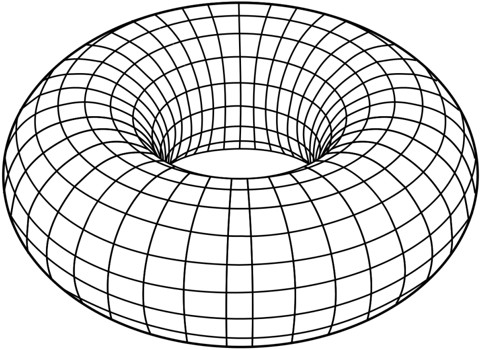
Lattice models: why

- *Lattice models* are finite approximations to quantum field theories (QFTs).
- Hope to recover original QFT as limiting case
  - This is *hard* and poorly understood: for Yang-Mills theory, would be a $1M reward
  - I'm working on a *much* simpler example
- Also used in condensed-matter systems

Lattice models: ingredients

- The system is set up on a lattice on a closed manifold
  - A collection of vertices, edges, faces, and higher-dimensional cells
  - Intuition: dimension 2, as in this picture:



Lattice models: ingredients

Let E denote the set of $(n-1)$-cells of the lattice.

- The *fields* are the set of *spin configurations*, functions $E \to \{\uparrow, \downarrow\}$.
  - That is, a choice of *spin-up* or *spin-down* for each cell
- The *state space* $\mathcal{H}$ is the vector space of complex-valued functions on the fields
  - Parameterizes the states the system can be in
  - Each function is determined by local data (the *locality principle*)

Lattice models: ingredients

- A *Hamiltonian*, a linear function $H : \mathcal{H} \to \mathcal{H}$ which determines how the system changes over time
- Named after William Rowan Hamilton (1805–65)
  - "William Rowan Hamilton. // My name is William Rowan Hamilton, // And no one uses my quaternions,// but just you wait, just you wait..."
  - (N.B.: you can find the full parody by searching "William Rowan Hamilton" on Youtube!)

Energy levels

- The Hamiltonian assigns *energy levels* to states.
  - If $H\mathbf{x} = c\mathbf{x}$ for a number c, $\mathbf{x}$ is a state with energy level c.
- The states with lowest energy are called *ground states*, and correspond to the vacuum
- Higher-energy states thought of as particles living at points on the lattice
  - (Disclaimer: might not actually be particles)

Energy levels

Examples of energy levels:

- Person before coffee: ground state
- Person after coffee: higher-energy state
- (Curiously, coffee itself is also in a ground state)

Examples of lattice models: toric code

- The *toric code* has a Hamiltonian

$$H = \sum_{\text{faces } f} H_f + \sum_{\text{vertices } v} H_v,$$

  where

$$H_f = \frac{1}{2}\left(1 - \prod_{e \in \partial f} Z_e\right), \qquad H_v = \frac{1}{2}\left(1 - \prod_{v \in \partial e} X_e\right).$$

  (Here Z_e and X_e are Pauli operators.)
- The *Ising model* is a well-known lattice model for ferromagnetism.
  - If you put it on a cylinder, is it the Ising on the cake?

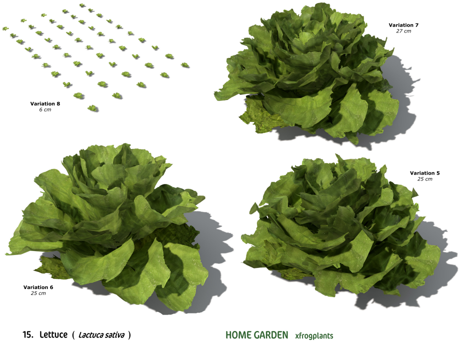
Examples of lattice models: GDS

- The *generalized double semion* (GDS) lattice model has a similar Hamiltonian, but instead

$$H_f = \frac{1}{2}\left(1 - (-1)^{\chi(\uparrow_f)+1} \prod_{v \in \partial e} X_e\right).$$

Here χ is the Euler characteristic and $\uparrow_f$ is the subset of ∂f in the spin-$\uparrow$ state.

Examples of lattice models



No wait, this is a lettuce model.

Topological quantum field theories

- A *topological quantum field theory* (TQFT) is a kind of QFT insensitive to small changes
- Because of this, can be extracted from physics and formulated purely mathematically
- Formally, a TQFT is a symmetric monoidal functor

$$Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{C}}.$$

  - (I'd explain this, but you'd get bord.)
- That means it's a certain way of assigning the following data:
  - To every n-dimensional manifold, a vector space of states dependent only on topology
  - As a space changes with time, a description of how states evolve

Why TQFTs?

- Used to study *topological phases of matter*
  - Condensed-matter systems with scale-independent behavior
  - Closely related to work of 2016 Nobel Prize in Physics winners
- Applications to algebraic topology, knot theory, and representation theory

Low-energy effective field theories

- In a lattice model, the ground states often don't depend on lattice structure: they're purely topological, and in fact form a TQFT
- Called the *low-energy effective field theory* (LE EFT) of the system
- Since it's topological, it can be studied with pure math (which is why I care).

LE EFTs of lattice models

- Simple answer for the toric code, called $\mathbb{Z}/2$-Dijkgraaf-Witten theory.
  - Gauge group $\mathbb{Z}/2$, simplest possible Lagrangian
  - (Sidenote: I used to think gauge symmetry meant one in each ear)
- For GDS, Freedman-Hastings prove that, in general, it's something new, but leave open what it is.

Finding the Lagrangian

I've found the Lagrangian description of the generalized double semion LE EFT.

- This is a formula from which the rest of the TQFT can be written down in a standard way
- Description as an integral of *cohomology classes*, certain functions on subspaces of the ambient space
- Proven via an intermediate description of the theory, though I would also like to get it directly from the Hamiltonian

Finding the Lagrangian

Theorem (D., 2016)

Let M be an n-dimensional manifold and $y \in H^1(M; \mathbb{Z}/2)$ be a cohomology class. The Lagrangian for the generalized double semion LE EFT on M is

$$\int_M w(M) \frac{y}{1 - y^2},$$

where $w(M)$ is the total Stiefel-Whitney class of M.

Applications to physics

- Working out the continuum limit would provide a good example for finding continuum limits in more complicated models
- Understanding the GDS lattice model may improve understanding of bosons and fermions' statistics

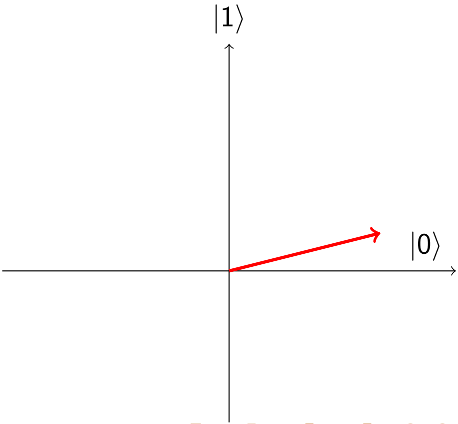
Further questions

- Investigate higher-energy states of GDS
- What is the continuum limit of the GDS lattice model?
- Realize the LE EFT as an *extended TQFT*
  - Uses *higher category theory* to compute on lower-dimensional submanifolds
  - Ties to exciting developments in algebra and topology: the cobordism hypothesis, the geometric Langlands program, ...
  - Intuition from physics (the locality principle) tells us this should be possible

Quantum computation and ECCs

It turns out lattice models have applications to quantum computation! I'll finish by briefly describing this.

- In quantum computing, rather than just taking on values in $\{0,1\}$, a bit's states are vectors in a 2-dimensional space.
- Quantum algorithms take advantage of lack of discreteness, which enables certain "shortcuts"



Error-correcting codes

- To build a quantum computer, one must overcome real-world noise.
- Sometimes there's too much noise, so the algorithm fails.
- A *quantum error-correcting code* (ECC) is a way of reducing the error.
  - Important to getting a real-world quantum computer working
  - Non-quantum error-correcting codes used in the real world (e.g. DVD encoding)
  - Because of the no-cloning theorem, naïve approaches to quantum ECCs don't work

Topological quantum computation

- Microsoft's Station Q aims to build a real-world quantum computer using *topological quantum computation*
- This is a specific physical model for a quantum bit using topological states of matter
- Topological behavior has been observed experimentally, making this plausible

The toric code is error-correcting

- The toric code lattice model is a quantum error-correcting code for topological quantum computation
- Errors raise the energy above the ground state, interpreted as (quasi)particle creation
- Topological information about particle trajectories signals whether computation is correct
- So... what about double semions?

Thanks for listening!

Do you have any questions?