M382D NOTES: DIFFERENTIAL TOPOLOGY

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These notes were taken in UT Austin's Math 382D (Differential Topology) class in Spring 2016, taught by Lorenzo Sadun. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Adrian Clough, Parker Hund, Max Reistenberg, and Thérèse Wu for finding and correcting a few mistakes.

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Lecture 1.

The Inverse and Implicit Function Theorems: 1/20/16

"The most important lesson of the start of this class is the proper pronunciation of my name [Sadun]: it rhymes with 'balloon.' "

We're basically going to march through the textbook (Guillemin and Pollack), with a little more in the beginning and a little more in the end; however, we're going to be a bit more abstract, talking about manifolds more abstractly, rather than just embedding them in $\mathbb{R}^n$, though the theorems are mostly the same. At the beginning, we'll discuss the analytic underpinnings to differential topology in more detail, and at the end, we'll hopefully have time to discuss de Rham cohomology.

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$. Its derivative is $\mathrm{d}f$; what exactly is this? There are several possible answers.

- It's the best linear approximation to f at a given point.
- It's the matrix of partial derivatives.

What we need to do is make good, rigorous sense of this, moreso than in multivariable calculus, and relate the two notions.

Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at an $a \in \mathbb{R}^n$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|} = 0. \tag{1.2}$$

In this case, L is called the *differential* of f at a, written $\mathrm{d}f|_a$.

Since $h \in \mathbb{R}^n$, but the vector in the numerator is in $\mathbb{R}^m$, so it's quite important to have the magnitudes there, or else it would make no sense.

Another way to rewrite this is that $f(a+h) = f(a) + L(h) + o(\text{small})$, i.e. along with some small error (whatever that means). This makes sense of the first notion: L is a linear approximation to f near a. Now, let's make sense of the second notion.

Theorem 1.3. *If f is differentiable at a, then $\mathrm{d}f$ is given by the matrix $\left(\frac{\partial f^i}{\partial x^j}\right)$.*

Proof. The idea: if f is differentiable at a, then (1.2) holds for $h \to 0$ along any path!

So let's take $\mathbf{e}_j$ be a unit vector and $h = t\mathbf{e}_j$ as $t \to 0$ in $\mathbb{R}$. Then, (1.2) reduces to

$$L(t\mathbf{e}_j) = \frac{f(a_1, a_2, \dots, a_j + t, a_{j+1}, \dots, a_n) - f(a)}{t},$$

and as $t \to 0$, this shows $L(\mathbf{e}_j)^i = \frac{\partial f^i}{\partial x^j}$. ⊠

In particular, if f is differentiable, then all partial derivatives exist. The converse is *false*: there exist functions whose partial derivatives exist at a point a, but are not differentiable. In fact, one can construct a function whose directional derivatives all exist, but is not differentiable! There will be an example on the first homework. The idea is that directional derivatives record linear paths, but differentiability requires all paths, and so making things fail along, say, a quadratic, will produce these strange counterexamples.

Nonetheless, if all partial derivatives exist, then we're almost there.

Theorem 1.4. *Suppose all partial derivatives of f exist at a and are continuous on a neighborhood of a; then, f is differentiable at a.*

In calculus, one can formulate several "guiding" ideas, e.g. the whole change is the sum of the individual changes, the whole is the (possibly infinite) sum of the parts, and so forth. One particular one is: *one variable at a time*. This principle will guide the proof of this theorem.

Proof. The proof will be given for $m = 2$ and $n = 1$, but you can figure out the small details needed to generalize it; for larger n, just repeat the argument for each component.

We want to compute

$$\begin{aligned} &f(a_1+h_1, a_2+h_2) - f(a_1, a_2) \\ &\qquad = f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2) + f(a_1+h_1, a_2) - f(a_1, a_2) \end{aligned}$$

Regrouping, this is two single-variable questions. In particular, we can apply the mean value theorem: there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} &= \left.\frac{\partial f}{\partial x^2}\right|_{(a_1+h_1, a_2+c_2)} h_2 + \left.\frac{\partial f}{\partial x^1}\right|_{(a_1+c_1, a_2)} h_1 \\ &= \left(\left.\frac{\partial f}{\partial x^1}\right|_{a_1+c_1, a_2} - \left.\frac{\partial f}{\partial x^1}\right|_a \right) h_1 + \left(\left.\frac{\partial f}{\partial x^2}\right|_{a_1+h_1, a_2+c_2} - \left.\frac{\partial f}{\partial x^2}\right|_a \right) h_2 + \left(\left.\frac{\partial f}{\partial x^1}\right|_a, \left.\frac{\partial f}{\partial x^2}\right|_a \right) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \end{aligned}$$

but since the partials are continuous, the left two terms go to 0, and since the last term is linear, it goes to 0 as $h \to 0$. ⊠

We'll often talk about *smooth* functions in this class, which are functions for which all higher-order derivatives exist and are continuous. Thus, they don't have the problems that one counterexample had.

Since we're going to be making linear approximations to maps, then we should discuss what happens when you perturb linear maps a little bit. Recall that if $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then its image $\mathrm{Im}(L) \subset \mathbb{R}^m$ and its kernel $\ker(L) \subset \mathbb{R}^n$.

Suppose $n \le m$; then, L is said to have *full rank* if $\operatorname{rank} L = n$. This is an open condition: every full-rank linear function can be perturbed a little bit and stay linear. This will be very useful: if a (possibly nonlinear) function's differential has full rank, then one can say some interesting things about it.

If $n \ge m$, then full rank means rank m. This is once again stable (an open condition): one can write such a linear map as $L = (A \mid B)$, where A is an invertible $m \times m$ matrix, and invertibility is an open condition (since it's given by the determinant, which is a continuous function).

To actually figure out whether a linear map has full rank, write down its matrix and row-reduce it, using Gaussian elimination. Then, you can read off a basis for the kernel, determining the free variables and the relations determining the other variables. In general, for a k-dimensional subspace of $\mathbb{R}^n$, you can pick k variables arbitrarily and these force the remaining $n-k$ variables. The point is: *the subspace is the graph of a function*.

Now, we can apply this to more general smooth functions.

Theorem 1.5. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth, $a \in \mathbb{R}^n$, and $\mathrm{d}f|_a$ has full rank.*

(1) (Inverse function theorem) *If $n = m$, then there is a neighborhood U of a such that $f|_U$ is invertible, with a smooth inverse.*
(2) (Implicit function theorem) *If $n \ge m$, there is a neighborhood U of a such that $U \cap f^{-1}(f(a))$ is the graph of some smooth function $g : \mathbb{R}^{n-m} \to \mathbb{R}^m$ (up to permutation of indices).*
(3) (Immersion theorem) *If $n \le m$, there's a neighborhood U of a such that $f(U)$ is the graph of a smooth $g : \mathbb{R}^n \to \mathbb{R}^m$.*

This time, the results are local rather than global, but once again, full rank means (local) invertibility when $m = n$, and more generally means that we can write all the points sent to $f(a)$ (analogous to a kernel) as the graph of a smooth function.

It's possible to sharpen these theorems slightly: instead of maximal rank, you can use that if $\mathrm{d}f|_a$ has block form with the square block invertible, then similar statements hold.

The content of these theorems, the way to think of them, is that in these cases, smooth functions locally behave like linear ones. But this is not too much of a surprise: differentiability means exactly that a function can be locally well approximated by a linear function. The point of the proof is that the higher-order terms also vanish.

For example, if $m = n = 1$, then full rank means the derivative is nonzero at a. In this case, it's increasing or decreasing in a neighborhood of a, and therefore invertible. On the other hand, if the derivative is 0, then bad things happen, because it's controlled by the higher-order derivatives, so one can have a noninvertible function (e.g. a constant) or an invertible function whose inverse isn't smooth (e.g. $y = x^3$ at $x = 0$).

This is not the last time in this class that maximal rank implies nice analytic results.

We're going to prove (2); then, as linear-algebraic corollaries, we'll recover the other two.

Lecture 2.

The Contraction Mapping Theorem: 1/22/16

Today, we're going to prove the generalized inverse function theorem, Theorem 1.5. We'll start with the case where $m = n$, which is also the simplest in the linear case (full rank means invertible, almost tautologically).

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. If $\mathrm{d}f|_a$ is invertible, then*

(1) f is invertible on a neighborhood of a,
(2) f^{-1} is smooth on a neighborhood of a, and
(3) $\mathrm{d}(f^{-1})|_{f(a)} = (\mathrm{d}f|_a)^{-1}$.

Proof of part (1). Without loss of generality, we can assume that $a = f(a) = 0$ by translating. We can also assume that $\mathrm{d}f|_a = I$, by precomposing with $\mathrm{d}f|_a^{-1}$:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[u, "\mathrm{d}f|_a^{-1}"] \arrow[ur] &
\end{tikzcd}$$

If we prove the result for the diagonal arrow, then it is also true for f. Since the domain and codomain of f are different in this proof, we're going to call the former X and the latter Y, so $f : X \to Y$.

Now, since f is smooth, its derivative is continuous, so there's a neighborhood of a in X given by the x such that $\|\mathrm{d}f|_x - I\| < 1/2$.[1] And by shrinking this neighborhood, we can assume that it is a closed ball C.

On C, f is injective: if $x_1, x_2 \in C$, then since C is convex, then there's a line $\gamma(t) = x_1 + tv$ (where $v = x_2 - x_1$) joining x_1 to x_2, and $\frac{\mathrm{d}f}{\mathrm{d}t} = (\mathrm{d}f|_{\gamma(t)})v$. Therefore

$$\begin{aligned}
f(x_2) - f(x_1) &= \left(\int_0^1 \mathrm{d}f|_{\gamma(t)}\,\mathrm{d}t\right)v \\
&= \int_0^1 \big((\mathrm{d}f|_{\gamma(t)} - I) + I\big)v\,\mathrm{d}t \\
&= x_2 - x_1 + \int_0^1 (\mathrm{d}f|_{\gamma(t)} - I)v\,\mathrm{d}t.
\end{aligned}$$

We can bound the integral:

$$\left|\int_0^1 \big(\mathrm{d}f|_{\gamma(t)} - I\big)v\right| \le \int_0^1 \big|(\mathrm{d}f|_{\gamma(t)} - I)v\big|\,\mathrm{d}t \le \int_0^1 \frac{1}{2}|v|\,\mathrm{d}t = \frac{|v|}{2}.$$

Thus, since $x_2 - x_1 = v$, then $f(x_2) - f(x_1)$ has magnitude at least $v/2$, so in particular it can't be zero. Thus, f is injective on C. The point is, since $\mathrm{d}f$ is close to the identity on C, we get an error term that we can make small.

[1] There are many different norms on the space of $n \times n$ matrices, but since this is a finite-dimensional vector space, they are all equivalent. However, for this proof we're going to take the *operator norm* $\|A\| = \sup_{v \in S^{n-1}} |Av|$.

To construct an inverse, we need to make it surjective on a neighborhood of $f(a)$ in Y. The way to do this is called the contraction mapping principle, but we'll do it by hand for now and recover the general principle later.

To be precise, we'll iterate with a "poor-man's Newton's method:" if $y \in Y$, then given x_n, let $x_{n+1} = x_0 - (f(x_0) - y) = y + x_0 - f(x_0)$ (since we're using the derivative at the origin instead of at x, and this is just the identity). A fixed point of this iteration is a preimage of y. Specifically, we'll want $x_0 = a$, since we're trying to bound the distance of our fixed point from a.

Since

$$x_{n+1} - x_n = y + x_n - f(x_n) - (y + x_{n-1} - f(x_{n-1})) = (x_n - x_{n-1}) - (f(x_n) - f(x_{n-1})),$$

then $|x_{n+1} - x_n| < (1/2)|x_n - x_{n-1}|$, so in particular, this is a Cauchy sequence! Thus, it must converge, and to a value with magnitude no more than $2|y|$ (since $f(x_0) = f(a) = 0$). Thus, if C has radius R, then for any y in the ball of radius $1/2$ from the origin (in Y), y has a preimage x, so f is surjective on this neighborhood. ⊠

Now, we can discuss the contraction mapping principle more generally.

Definition 2.2. Let X be a complete metric space and $T : X \to X$ be a continuous map such that $d(T(x), T(y)) \le cd(x, y)$ for all $x, y \in X$ and some $c \in [0, 1)$. Then, T is called a *contraction mapping*.

Theorem 2.3 (Contraction mapping principle)**.** *If X is a complete metric space and T a contraction mapping on X, then there's a unique fixed point x (i.e. $T(x) = x$).*

Proof. Uniqueness is pretty simple: if T has two fixed points x and x' such that $x \ne x'$, then $d(T(x), T(x')) \le cd(x, x') = d(T(x), T(x'))$, and $c < 1$, so this is a contradiction, so $x = x'$.

Existence is basically the proof we just saw: pick an arbitrary $x_0 \in X$ and let $x_{n+1} = T(x_n)$. Then, $d(x_m, x_n) \le c^{|n-m-1|} d(x_n, x_{n-1})$, so this sequence is Cauchy, and has a limit x. Then, since T is continuous, $T(x) = x$. ⊠

Now, back to the theorem.

Proof of Theorem 2.1, part (2). Once again, we assume $f(0) = 0$. By the fundamental theorem of calculus, on our neighborhood of 0,

$$y = f(x) = \int_0^1 \mathrm{d}f|_{tx}(x)\,\mathrm{d}t.$$

Since we assumed $\mathrm{d}f|_0 = I$, and f is smooth, then $\mathrm{d}f$ is continuous, so for any $\varepsilon > 0$, there's a neighborhood U of 0 such that for all $x \in U$, $\mathrm{d}f|_x = I + A$, where $\|A\| < \varepsilon$. When we integrate this, this means $y = x + o(|x|)$: $\mathrm{d}f$ is "small in x." Hence, $|x| - \varepsilon < |y| < |x| + \varepsilon$, so since U is bounded, this puts a bound on x in terms of y, too; in other words, $x = y + o(|y|)$ (this is little-o, because we can do this for any $\varepsilon > 0$, though the neighborhood may change). This is exactly what it means for f^{-1} to be differentiable at $y = f(0)$, and its derivative is the identity! In general, if $\mathrm{d}f|_0 \ne I$, but is still invertible, then we get that $\mathrm{d}f^{-1}|_{f(0)} = (\mathrm{d}f|_0)^{-1}$.

We'd like this to extend to a neighborhood of the origin. Since $\mathrm{d}f|_0$ is invertible, and $\mathrm{d}f$ is continuous, then locally a neighborhood of 0 corresponds to a neighborhood of $\mathrm{d}f|_0$ in the space of $n \times n$ matrices, and vice versa. But the set of invertible matrices is open in the space of matrices, so there's a neighborhood V of 0 such that $\mathrm{d}f|_x$ is invertible for all $x \in V$, so for each $x \in V$, $\mathrm{d}f^{-1}|_{f(x)} = (\mathrm{d}f|_x)^{-1}$. Then, matrix inversion is a continuous function on the subspace of invertible matrices, so this means $\mathrm{d}f^{-1}$ is continuous in a neighborhood of $f(0)$.

This gives us one derivative; we wanted infinitely many. Using the chain rule,

$$\frac{\partial(\mathrm{d}f^{-1})}{\partial y} = \frac{\partial(\mathrm{d}f)^{-1}}{\partial x}\frac{\partial x}{\partial y},$$

and $\frac{\partial x}{\partial y} = (\mathrm{d}f)^{-1}$. So we want to understand derivatives of matrices. Let A be some invertible matrix-valued function, so that $AA^{-1} = I$. Thus, using the product rule, $A'A^{-1} = A(A^{-1})' = 0$, so rearranging, $(A^{-1})' = A^{-1}A'A^{-1}$. That is, the derivative inverse can be specified in terms of the inverse and the derivative of A. In particular, this means $\frac{\partial(\mathrm{d}f^{-1})}{\partial y}$ is a product of continuous functions ($\frac{\partial(\mathrm{d}f)}{\partial x}$ and $(\mathrm{d}f)^{-1}$), so it is continuous. By the same argument, so is the partial derivative in the x-direction, so by Theorem 1.4, $\mathrm{d}f^{-1}$ is differentiable. This can be repeated as an inductive argument to show that $\mathrm{d}f^{-1}$ is differentiable as many times as $\mathrm{d}f$ is, and by smoothness, this is infinitely often. ⊠

We can use this to recover the rest of Theorem 1.5 as corollaries.

Proof of Theorem 1.5, part (2). First, for the implicit function theorem, let $n > m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth with full rank, and choose a basis in which $\mathrm{d}f|_a = (A \mid B)$ in block form, where A is an invertible $m \times m$ matrix. The theorem statement is that we can write the first m coordinates as a function of the last $n-m$ coordinates: specifically, that there exists a neighborhood U of a such that $U \cap f^{-1}(f(a)) = U \cap \{g(y), y\}$ for some smooth $g : \mathbb{R}^{n-m} \to \mathbb{R}^m$.[2]

Now, the proof. Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$, and let

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ y \end{pmatrix}.$$

Hence,

$$\mathrm{d}F|_a = \left(\begin{array}{c|c} A & B \\ \hline 0 & I \end{array}\right).$$

This is invertible, since A is: $\det(\mathrm{d}F|_a) = \det(A) \neq 0$. Thus, we apply the inverse function theorem to F to conclude that a smooth F^{-1} exists, and so if π_1 denotes projection onto the first component, $x = \pi_1 \circ F^{-1}(0, y) = g(y)$. ⊠

Lecture 3.

Manifolds: 1/25/16

"Erase any notes you have of the last eight minutes! But the first 40 minutes were okay."

Recall that we've been discussion Theorem 1.5, a collection of results called the inverse function theorem, the implicit function theorem, and the immersion theorem. These are local (not global) results, and generalize similar results for linear maps: not all matrices are square, but if a matrix has full rank, it can be written in two blocks, one of which is invertible. Using this with $\mathrm{d}f|_a$ as our matrix is the idea behind proving Theorem 1.5: the first several variables determine the remaining variables.

However, we don't know which variables they are: you may have to permute $x_1, \dots, x_n$ to get the last variables as smooth functions of the first ones. For example, for a circle, the tangent line is horizontal sometimes (so we can't always parameterize in terms of x_2) and vertical at other times (so we can't only use x_1).

Before we prove the immersion theorem (part (3) of Theorem 1.5), let's recall what tools we use to talk about curves in the plane.

(1) A common technique is using a *parameterized curve*, the image of a smooth $\gamma(t) : \mathbb{R} \to \mathbb{R}^2$ whose derivative is never zero (to avoid singularities). For example, $f(t) = (t^2, t^3)$ has a zero at the origin, but the curve one obtains is $y = \pm x^{3/2}$, which has a cusp at $(0,0)$. This is the content of the immersion theorem.
(2) Another way to describe curves is as level sets: $f(x,y) = c$, most famously the circle. This is the content of the implicit function theorem: this looks like a graph-like curve locally.
(3) This brings us to the most simple method: graphs of functions, just like in calculus.

And the point of Theorem 1.5 is that these three approaches give you the same sets, *up to permutation of variables* (and that a curve is the graph of a function only locally). We have these three pictures of what higher-dimensional surfaces look like.

And that means that when we talk about manifolds, which are the analogue of higher-dimensional surfaces, we should keep these things in mind: a manifold may be defined abstractly, but we understand manifolds through these three visualizations.

Proof of Theorem 1.5, part (3). We're going to prove the equivalent statement that if the first n rows of $\mathrm{d}f|_a$ are linearly independent, then the remaining $m-n$ variables are smooth functions in the first n.

[2]For example, if $n = 2$ and $m = 1$, consider $f(x) = |x|^2 - 1$, and $a = (\cos\theta, \sin\theta)$. Then, $f^{-1}(f(a))$ is the unit circle, so the implicit function is telling us that locally, the circle is a function of x_1 in terms of x_2, or vice versa.

Since $f : \mathbb{R}^n \to \mathbb{R}^m$, then let π_1 denote projection onto the first n coordinates, so we have a commutative diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[dr, "\pi_1 \circ f"'] & \mathbb{R}^m \arrow[d, "\pi_1"] \\ & \mathbb{R}^n. \end{tikzcd}$$

In block form, $\mathrm{d}f|_a = \binom{A}{B}$, where A is invertible, and therefore $\mathrm{d}(\pi_1 \circ f)|_a = A$. This is invertible, so $(\pi_1 \circ f)^{-1}$ has an inverse in a neighborhood of a, by the inverse function theorem. Thus, if π_2 denotes projection onto the last $m-n$ coordinates, then $g = \pi_2 \circ f \circ (\pi_1 \circ f)^{-1}$ writes the last $m-n$ coordinates in terms of the first n, as desired. ⊠

Now, we're ready to talk about smooth manifolds.

Definition 3.1. A *k-manifold* X in $\mathbb{R}^n$ is a set that locally looks like one of the descriptions (1), (2), or (3) for a smooth surface. That is, it satisfies one of the following descriptions.

(1) For every $p \in X$, there's a neighborhood U of p where one can write $N-k$ variables in smooth functions of the remaining k variables, i.e. there is a neighborhood $V \subset \mathbb{R}^k$ and a smooth $g : V \to \mathbb{R}^{N-k}$ such that $X \cap U = \{(x, g(x)) : x \in V\}$ (up to permutation).
(2) X is locally the image of a smooth map, i.e. for every $p \in X$, there's a neighborhood U of p and a smooth $f : \mathbb{R}^k \to \mathbb{R}^N$ with full rank such that the image of f in U is $X \cap U$. This is the "parameterized curve" analogue.
(3) Locally, X is the level set of a smooth map $f : \mathbb{R}^N \to \mathbb{R}^{N-k}$ with full rank.

If k is understood from context (or not important), X will also be called a *manifold*.

The big theorem is that these three conditions are equivalent, and this follows directly from Theorem 1.5.

For example, suppose we have the graph of a smooth function $y = x^2$. How can we write this as the image of a smooth map? Well, $(x, y) = (t, t^2)$ has nonzero derivative, and we can do exactly the same thing (locally) for a manifold in general. And it's the level set $f(x, y) = 0$, where $f(x, y) = y - x^2$, and the same thing works (locally) for manifolds: for a general graph $\mathbf{y} = g(\mathbf{x})$, this is the level set of $f(\mathbf{y}, \mathbf{x}) = \mathbf{y} - g(\mathbf{x})$, whose derivative $\mathrm{d}f$ has block matrix form $(I \mid -\mathrm{d}g)$, which has full rank. Neat.

And perhaps most useful for now, something that's locally a graph is really easy to visualize: it's the bedrock on which one first defined curves and surfaces.

Now, that's a manifold in $\mathbb{R}^n$. As far as Guillemin and Pollack are concerned, that's the only kind of manifold there is, but we want to talk about abstract manifolds, but that means we'll need one more important property.

Suppose $X \subset \mathbb{R}^N$ is a manifold, and $p \in X$. We're going to look at a neighborhood of p as the image of a smooth $g_1 : \mathbb{R}^k \to \mathbb{R}^N$; this is the most common and most fundamental description of a manifold. However, this is not in general unique; suppose $g_2 : \mathbb{R}^k \to \mathbb{R}^N$ lands in a different neighborhood of p — though, by restricting to their intersection, we can assume we have two smooth maps (sometimes called *charts*) into the same neighborhood, and they both have inverses, so we have a well-defined function $g_2^{-1} \circ g_1 : \mathbb{R}^k \to \mathbb{R}^k$. Is it smooth?

Theorem 3.2. *$g_2^{-1} \circ g_1$ is smooth.*

The key assumption here is that $\mathrm{d}g_1$ and $\mathrm{d}g_2$ both have maximal rank.

Definition 3.3. The *tangent space* to X at p, denoted T_pX, is $\mathrm{Im}(\mathrm{d}g_1|_{g_1^{-1}(p)})$; it is a k-dimensional subspace of $\mathbb{R}^N$.

This is the set of velocity vectors of paths through p, which makes sense, because such a path must come from a path downstairs in $\mathbb{R}^k$, since g_1 is locally invertible.

Lemma 3.4. *The tangent space is independent of choice of g_1.*

The idea is that any velocity vector must come from a path in both $\mathrm{Im}(\mathrm{d}g_1|_{g_1^{-1}(p)})$ and $\mathrm{Im}(\mathrm{d}g_2|_{g_2^{-1}(p)})$, so these two images are the same.

Then, we'll punt the proof of Theorem 3.2 to next lecture.

Lecture 4.

Abstract Manifolds: 1/27/16

Last time, we were talking about change of variables, but we were missing a lemma that's important for the proof, but not really the right way to view manifolds.

Let X be a k-dimensional manifold in $\mathbb{R}^n$, so for any $p \in X$, there's a map ϕ from the neighborhood of the origin in $\mathbb{R}^k$ to a neighborhood of p in X, where $\phi(0) = p$ and $\mathrm{d}\phi|_0$ has rank k. We'd like a local inverse to ϕ, which we'll call F; it's a map from a neighborhood of $\mathbb{R}^n$ to a neighborhood of $\mathbb{R}^k$. We'd like F to be smooth, and we want $F \circ \phi = \mathrm{id}|_{\mathbb{R}^k}$.

By permuting coordinates, we can assume that the first k rows of $\mathrm{d}\phi$ are linearly independent. That is, $\mathrm{d}\phi|_0$ has block form $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$, where A is invertible. Then, define $\widetilde{\phi} : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ sending $(x, y)^\mathrm{T} \to \phi(x) + (0, y)^\mathrm{T}$,[3] so that $\widetilde{\phi}(x, 0)) = \phi(x)$. ϕ and $\widetilde{\phi}$ fit into the following diagram.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, hook, "x \mapsto (x,0)"] \arrow[rr, bend right, "\phi"'] & \mathbb{R}^n \arrow[r, "\widetilde{\phi}"] & \mathbb{R}^n \end{tikzcd}$$

Thus, by the chain rule,

$$\mathrm{d}\widetilde{\phi}|_0 = \left(\begin{array}{c|c} A & 0 \\ \hline B & I \end{array}\right),$$

so $\mathrm{d}\widetilde{\phi}|_0$ has full rank! Thus, in a neighborhood of p, it has an inverse, and certainly the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ has a left inverse π (projection onto the first k coordinates), so we can let $F = \pi \circ \widetilde{\phi}^{-1}$, because

$$F \circ \phi(x) = F \circ \widetilde{\phi}(x, 0) = \pi \circ \widetilde{\phi}^{-1} \circ \widetilde{\phi}((x, 0)) = \pi(x, 0) = x.$$

Likewise, $\phi \circ F = \mathrm{id}|_X$, since every point in our neighborhood is in the image of ϕ.

This is how we talk about smoothness on manifolds: we don't know what smoothness means on some arbitrary submanifold, so we'll use the fact that we can locally pretend we're in $\mathbb{R}^n$ to talk about smoothness.

Suppose $\phi, \psi : \mathbb{R}^k \rightrightarrows X$ are two such smooth coordinate maps; we'd like to find a smooth function g from a neighborhood in $\mathbb{R}^k$ to a neighborhood in $\mathbb{R}^k$ relating them (again, locally). But we have a local inverse to ψ called F, so since we want $\psi = \phi \circ g$, then define $g = F \circ \psi$, because $\phi \circ g = \phi \circ F \circ \psi = \psi$. And g is the composition of two smooth functions, so it's smooth (this is Theorem 3.2). This is our change-of-coordinates operation.

Theorem 4.1. *A function $g : X \to \mathbb{R}^m$ can be extended to a smooth map G on a neighborhood of p in $\mathbb{R}^n$ iff $g \circ \phi$ is smooth.*

This is another notion of smooth: the first one determines smoothness by coordinates, and the second says that smooth functions on a submanifold are restrictions of smooth functions $\mathbb{R}^n \to \mathbb{R}^m$. But the theorem says that they're totally equivalent.

Proof. Suppose such a smooth extension G exists; since $G|_X = g$ and $\mathrm{Im}(\phi) \subset X$, then $G \circ \phi = g \circ \phi$. G and ϕ are smooth, so $G \circ \phi = g \circ \phi$ is smooth.

Conversely, if $g \circ \phi$ is smooth, then let $G = g \circ \phi \circ F$, which is a smooth map (since it's a composition of two smooth functions) out of a neighborhood of p in $\mathbb{R}^n$. ⊠

This extrinsic definition is the one Guillemin and Pollack use throughout their book; the other notion doesn't depend on an embedding into $\mathbb{R}^n$, but we had to check that it was independent of change of coordinates (which by Theorem 3.2 is smooth, so we're OK). This means we can make the following definition.

Definition 4.2.

- A *chart* $\mathbb{R}^k \to X$ for a topological space X is a continuous map that's a homeomorphism onto its image.
- An *(abstract) smooth k-manifold* is a Hausdorff space X equipped with charts $\varphi_\alpha : \mathbb{R}^k \to X$ such that
  (1) every point in X is in the image of some chart, and

[3] $\widetilde{\phi}$ is pronounced "phi-twiddle."

(2) for every pair of overlapping charts φ_α and φ_β, the change-of-coordinates map $\varphi_\beta^{-1} \circ \varphi_\alpha : \mathbb{R}^k \to \mathbb{R}^k$ is smooth.

The definition is sometimes written in terms of neighborhoods in $\mathbb{R}^k$, so each chart is a map $U \to X$, where $U \subset \mathbb{R}^k$, but this is completely equivalent to the given definition, since $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism (and there are many others, e.g. $e^x/(1+e^x)$). The point is that every point has a neighborhood homeomorphic to $\mathbb{R}^k$, even if we think of neighborhoods as little balls much of the time.

There are lots of different categories of manifolds: a C^n *manifold* has the same definition, but we require the change-of-coordinates maps to merely be C^n (n times continuously differentiable); an *analytic manifold* requires the change-of-coordinates maps to be analytic; and in the same way one can define *complex-analytic manifolds* (holomorphic change-of-coordinates maps) and *algebraic manifolds*. For a *topological manifold* we just require the change-of-coordinates maps to be continuous, which is always true for a covering of charts. But in this class, the degree of regularity we care about is smoothness.

Definition 4.3. Let X be a manifold and $f : X \to \mathbb{R}^n$ be continuous. Then, f is *smooth* if for every chart $\varphi_\alpha : \mathbb{R}^k \to X$, the composition $f \circ \varphi_\alpha$ is smooth.

This is just like the definition of smoothness for manifolds living in $\mathbb{R}^n$.

Example 4.4. Let X be the set of lines in $\mathbb{R}^2$ (*not* just the set of lines through the origin). This is a manifold, but we want to show this. Using point-slope form, we can define a map $\phi_1 : \mathbb{R}^2 \to X$ sending $(a,b) \mapsto \{(x,y) : y = ax + b\}$, which covers all lines that aren't vertical. We need to handle the vertical lines with another chart, $\phi_2 : \mathbb{R}^2 \to X$ sending $(c,d) \mapsto x = cy + d$.

These charts intersect for all lines that are neither vertical nor horizontal, so the change-of-coordinates map describes $c = 1/a$ and $d = -b/a$, i.e. $g(a) = (1/a, -b/a)$. And since we're restricted to non-vertical lines, $a \neq 0$, so this is smooth, and $g^{-1}(c,d) = (1/c, -d/c)$, which is also smooth (since we're not looking at horizontal lines). Thus, we're described X as a manifold.

It turns out that X is a Möbius band. A line may be described by a direction (an angle coordinate) and an offset (intersection with the x-axis, heading in the specified direction). However, there are two descriptions, given by flipping the direction: $(\theta, D) \sim (\theta + \pi, -D)$. Thus, this is the quotient of an infinitely long cylinder by half a rotation and a twist, giving us a Möbius band.

One thing we haven't talked much about is: why do manifolds need to be Hausdorff? This makes our example much less terrible: here's just one creature we avoid with this condition.

Example 4.5 (Line with two origins)**.** Take two copies of $\mathbb{R}^2$, and identify $(x,1) \sim (x,2)$ for all $x \neq 0$. Thus, we seem to have one copy of $\mathbb{R}$, but two different copies of the origin. The charts are perfectly nice: any interval on either copy of $\mathbb{R}$ is a chart for this space, but every neighborhood of one of the origins contains the other, so it isn't Hausdorff (it is T_1, though). See Figure 1 for a (not perfectly accurate) depiction of this space. We don't want to

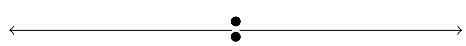


FIGURE 1. Depiction of the line with two origins. Note, however, that the two origins are technically infinitely close together.

have spaces like this one, so we require manifolds to be Hausdorff.

Tune in Friday to learn how to determine when two manifolds are equivalent. Is the same space with different charts a different manifold?

Lecture 5.

Examples of Manifolds and Tangent Vectors: 1/29/16

"How do you make the unit disc into a manifold? With pie charts."

Today, we're going to make the notion of a manifold more familiar by giving some more examples of what structures can arise: specifically, the 2-sphere S^2 and the projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$. Then, we'll move to discussing tangent vectors and how to define smooth maps between manifolds.

Example 5.1 (2-sphere)**.** The concrete 2-*sphere* is $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}|^2 = 1\}$. Why is this a manifold?

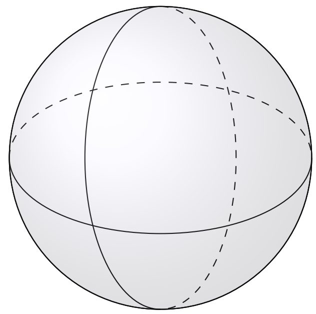


FIGURE 2. The 2-sphere, an example of a manifold.

We can put charts on this surface as follows: if $z > 0$, then we have a chart $(u, v, \sqrt{1-u^2-v^2})$, and if $z < 0$, then the chart is $(u, v, -\sqrt{1-u^2-v^2})$. Similarly, if $y > 0$, we have $(u, \sqrt{1-u^2-v^2}, v)$, and similarly for $y < 0$ and for x. However, since $\mathbf{0} \notin S^2$, then this covers all of S^3, and one can check that the transition maps are smooth and the chart maps have full rank.

Another way to realize this is that if $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x, y, z) = x^2 + y^2 + z^2$, then f is smooth and $S^2 = f^{-1}(1)$. Thus, S^2 is the level set of a smooth function whose derivative $\mathrm{d}f = (2x, 2y, 2z)$ has full rank, so by the implicit function theorem, it must be a manifold.

That is, you can see S^2 is a manifold using maps into it, or maps out of it.

Example 5.2 (Real projective space)**.** $\mathbb{RP}^n$, *real projective space*, is defined to be the set of lines through the origin in $\mathbb{R}^{n+1}$. Any nonzero point in $\mathbb{R}^{n+1}$ defines a line through the origin, and scaling a point doesn't change this line. Thus, $\mathbb{RP}^n = \{\mathbf{r} \in \mathbb{R}^{n+1} \setminus 0\}/(\mathbf{r} \sim \lambda\mathbf{r}$ for $\lambda \in \mathbb{R} \setminus 0)$. We have coordinates $(x_0, \dots, x_n)$ for $\mathbb{R}^{n+1}$, and want to make coordinates on $\mathbb{RP}^n$.

The set $U_0 = \{\mathbf{x} : x_0 \neq 0\}$ is open, and $(x_0, x_1, \dots, x_n) \sim (1, x_1/x_0, \dots, x_n/x_0)$ in $\mathbb{RP}^n$, so we get a chart on U_0. We're parameterizing non-horizontal lines by their slope (or, well, the reciprocal of it). Thus, we have a map $\psi_0 : \mathbb{R}^n \to \mathbb{RP}^n$ sending $(x_1, \dots, x_n) \mapsto [(1, x_1, \dots, x_n)]$ (where brackets denote the equivalence class in $\mathbb{RP}^n$).

We can do this with every coordinate: let $\psi_1 : \mathbb{R}^n \to \mathbb{RP}^n$ send $(x_1, \dots, x_n) \mapsto [(x_1, 1, x_2, \dots, x_n)]$, and so forth. Then, since every point in $\mathbb{RP}^n$ has a nonzero coordinate, then this covers $\mathbb{RP}^n$. Are the transition maps smooth? $\mathbb{RP}^2$ will illustrate how it works: if $[1, a, b] = [c, 1, d]$, then $c = 1/a$ and $d = b/a$, which is smooth (because in these charts, a and c are nonzero).

By the way, $\mathbb{RP}^1$ is just a circle. More generally, one can also realize $\mathbb{RP}^n$ as the unit sphere with opposite points identified (every vector can be scaled to a unit vector, but then $\mathbf{x} \sim -\mathbf{x}$). However, $\mathbb{RP}^2$, etc., are more interesting spaces.

Example 5.3 (Complex projective space)**.** We can also refer to *complex projective space*, $\mathbb{CP}^n$. The idea of "lines through the origin" is the same, but, despite what algebraic geometers call it, a one-dimensional complex subspace looks a lot more like a (real) plane than a real line. In any case, one-dimensional complex subspaces of $\mathbb{C}^{n+1}$ are given by nonzero vectors, so we define $\mathbb{CP}^n = \{\mathbf{r} \in \mathbb{C}^{n+1} \setminus 0\}/(\mathbf{r} \sim \lambda\mathbf{r}, \lambda \in \mathbb{C} \setminus 0)$. Now, the same definitions of charts give us $\psi_k : \mathbb{C}^n \to \mathbb{CP}^n$, but since we know how to map $\mathbb{R}^{2n} \to \mathbb{C}^n$, this works just fine.

In this case, the first interesting complex projective space is $\mathbb{CP}^1$. Our two charts are $[1, a]$ and $[b, 1]$, and their overlap is everything but the two points $[1, 0]$ and $[0, 1]$. In other words, every point is of the form $[z, 1]$ for some $z \in \mathbb{C}$ or $[1, 0]$: that is $[1, 0]$ is a "point at infinity" ∞, whose reciprocal is 0! So $\mathbb{CP}^1$ is the complex numbers plus one extra point. We can actually realize this as S^2 using a map called *stereographic projection*: the sphere sits inside $\mathbb{R}^3$, and the xy-plane can be identified with $\mathbb{C}$. Then, the line between the north pole $(0, 0, 1)$ and a given $(u, v, 0)$ (corresponding to $[u + vi, 0]$) intersects the sphere at a single point, which is defined to be the image of the projection $\mathbb{CP}^1 \to S^2$. However, the point at infinity isn't identified in this way, and neither is the north pole;

thus, the north pole can be made the point at infinity. This is a great exercise to work out yourself, e.g. how it relates to the change of charts if you use the south pole instead. In fact, it will be on the homework![4]

Tangent vectors. In order to discuss tangent vectors concretely, we'll work in $\mathbb{R}^n$ for now. At every point $p \in \mathbb{R}^n$, there's a tangent space $T_p\mathbb{R}^n$ of vectors based at p, which is an n-dimensional vector space. And you can take the union of all of the tangent vectors and call it the *tangent bundle*: these are pairs (p, v), where $p \in \mathbb{R}^n$ and v is a vector originating at p. This is a $2n$-dimensional vector space, and this is cool and all, but it doesn't really tell us anything. We'd like a better way to characterize tangent vectors.

One way to define a tangent vector is the velocity vector of a smooth curve through p, and another way is as a derivation (or, as we saw on the homework, the directional derivatives $v = \sum v^i \partial_i$). These are related in a natural way: if $\gamma : \mathbb{R} \to \mathbb{R}^n$ is smooth and has $\gamma(0) = p$, and $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, then one could ask how fast f changes along the path γ. This is

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\bigg|_{t=0} = \sum_{i=1}^n \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i} = v \cdot \nabla f.$$

That is, the space of possible velocities *is* the space of directional derivatives: in the way we just described, curves do act as first-order differential operators. And in coordinates, the tangent vectors are just n-tuples of numbers (like with any basis). You'll need to be used to working with all of these perspectives and switching between them.

Now, let's generalize to an n-dimensional submanifold X of $\mathbb{R}^N$. For any $p \in X$, let $\phi : \mathbb{R}^n \to X$ send $a \mapsto p$; then, we can define the *tangent space* of X at p to be $T_pX = \mathrm{Im}(\mathrm{d}\phi|_a)$, which is necessarily an n-dimensional subspace of $\mathbb{R}^N$, as $\mathrm{d}\phi|_a$ has full rank. These are "vectors living at p," and we'll be able to relate these to velocities and directional derivatives, too.

However, we need to show that this is independent of chart: if $\psi : b \mapsto p$ is another chart for X, we know that in neighborhoods of a, b, and p, the change-of-coordinates is a diffeomorphism $g : b \mapsto a$. Then, $\psi = \phi \circ g$, and these are smooth, so the chain rule says $\mathrm{d}\psi|_b = \mathrm{d}\phi|_a \circ \mathrm{d}g|_b$. But since g is a diffeomorphism, $\mathrm{d}g_b$ is invertible, so its image is all of $\mathbb{R}^n$; thus, $\mathrm{Im}\,\mathrm{d}\psi|_b = \mathrm{d}\phi|_a(\mathbb{R}^n) = \mathrm{Im}(\mathrm{d}\phi|_a)$, and this is indeed independent of coordinates.

Thus, since $T_pX \subset \mathbb{R}^N$, then we can realize the tangent bundle as $TX \subset T\mathbb{R}^N$: $TX = \{(p, v) \mid p \in X \text{ and } v \in T_pX\}$. This tangent bundle sits inside $T\mathbb{R}^N = \mathbb{R}^{2N}$, so we know what it means for it to be a manifold, and can write down charts, and so forth.

Another interesting insight is that smooth curves through p correspond to smooth curves through $a \in \mathbb{R}^n$ through ϕ, and so we can relate the other definitions of tangent vectors to this definition of T_pX. The point is: local coordinates allow us to translate the notions of tangent vectors to submanifolds of $\mathbb{R}^N$; we'll be able to turn this into talking about abstract manifolds and derivatives of maps between manifolds.

Lecture 6.

Smooth Maps Between Manifolds: 2/1/16

We're going to talk more about tangent spaces today. We've already talked about what they are in $\mathbb{R}^n$, but in order to talk about them for abstract manifolds, we'll transfer the notion from $\mathbb{R}^n$. This is very general: since manifolds are defined to locally look like Euclidean space, everything we do with manifolds will involve constructing a notion in $\mathbb{R}^n$ and showing that it still works when one passes to manifolds.

At an $x \in \mathbb{R}^n$, the tangent space $T_x\mathbb{R}^n$ can be thought of arrows based at x, or as velocities of smooth paths through x, or as derivations[5] at x (the equivalence of these was a problem on the last homework). Then, the tangent bundle is $T\mathbb{R}^n = \{(x, v) \mid v \in T_x\mathbb{R}^n\}$, which is isomorphic (as vector spaces) to $\mathbb{R}^n \times \mathbb{R}^n$; thus, we can give it the topology of $\mathbb{R}^{2n}$: two vectors are close if either their basepoints or their directions are close.

First, we generalize this slightly to a k-dimensional manifold $X \subset \mathbb{R}^N$. If $x \in X$, then x is in the image of a chart $U \subset \mathbb{R}^k$ under the chart map ϕ. Let a be the preimage of x; then, we defined $T_xX = \mathrm{Im}\,\mathrm{d}\phi|_a \subset T_x\mathbb{R}^N$, and we showed that this was independent of the chart used to construct this, because change-of-charts maps are smooth. This is also the space of velocities of paths through X, or the derivations at x on X (i.e. using $C^\infty(X)$ instead of $C^\infty(\mathbb{R}^N)$; this is the same as $C^\infty(\mathbb{R}^k)$ through ϕ). This is a little more work than we had to do for $\mathbb{R}^n$, but

[4]Stereographic projection works for the n-sphere and $\mathbb{R}^n$ for all n, so $S^n = \mathbb{R}^n \cup \{\infty\}$, in a sense; however, it won't correspond to projective space in higher dimensions.

[5]Recall that you can also think of derivations as directional derivatives.

everything is still the same, because everything (derivations, paths) is the same in $\mathbb{R}^k$ and X, at least near x. Then, the tangent bundle is $TX = \{(x,v) : x \in X, v \in T_xX\} \subset T\mathbb{R}^N$, which is a $2k$-dimensional manifold.

So from this perspective, do we even need $\mathbb{R}^N$? Not really: if you're working in an abstract manifold, pulling derivations back to a chart in $\mathbb{R}^k$ still works, so one can define tangent vectors and tangent bundles on abstract manifolds, which have the same properties (though an abstract tangent manifold doesn't naturally sit inside $T\mathbb{R}^N$).

Now, we want to talk about maps between manifolds, and what derivatives of those maps mean. If we're inside $\mathbb{R}^N$, this is easy: a smooth function on a manifold inside $\mathbb{R}^N$ is the restriction of a smooth function on a neighborhood in $\mathbb{R}^N$; courtesy of the inverse function theorem, you could construct these, but generally don't. Instead, you use charts: a map between manifolds $f : X \to Y$ (where X is k-dimensional and Y is ℓ-dimensional) can be defined in terms of neighborhoods. If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$ are neighborhoods with charts $\phi : U \to X$ and $\psi : V \to Y$ such that $\phi(a) = p$ and $\psi(b) = f(p)$, then f can be understood on $\mathbb{R}^k$ and $\mathbb{R}^\ell$; let $h = \psi^{-1} \circ f \circ \phi$, which fits into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift right, "\phi^{-1}"'] & Y \arrow[d, shift right, "\psi^{-1}"'] \\ U \arrow[r, "h"] \arrow[u, shift right, "\phi"'] & V. \arrow[u, shift right, "\psi"'] \end{tikzcd} \tag{6.1}$$

We say that f is *smooth* if h is smooth. One has to show that this is independent of the choice of charts (which it is, for the reason that the change-of-charts map is smooth, and compositions of smooth functions are smooth), and that this agrees with the definition given above (which is a homework exercise).

Next, derivatives. We can take a derivative $\mathrm{d}h|_a : T_a\mathbb{R}^k \to T_b\mathbb{R}^\ell$, and we want to turn this into a map $\mathrm{d}f|_p : T_pX \to T_{f(p)}Y$, or $\mathrm{d}f : TX \to TY$. What this means depends on your definition of tangent vector, so we'll give a few definitions. It's important to prove that they're equivalent, but this follows from the chain rule.

- First, let's suppose v is a derivation on X at p; we'd like $\mathrm{d}f|_p(v)$ to be a derivation at $f(p)$; hence, if $g \in C^\infty(Y)$, then we can pull it back to X: $g \circ f \in C^\infty(X)$, so we can define $(\mathrm{d}f|_p(v))(g) = v(g \circ f)$.
- Next, suppose v is the velocity vector of a $\gamma : \mathbb{R} \to X$. Then, $f \circ \gamma$ is a path in Y, so we can let $\mathrm{d}f_P(v)$ be the velocity of $f \circ \gamma$. Again, we compose with f, but it's a little strange that in one case, we pull back, and in the other case, we pull back. This is an example of a useful mantra: *vectors push forward; functions pull back*. This will come back when we talk about differential forms later.
- The arrow definition is stranger: suppose $v = \mathrm{d}\phi|_a(w)$ for a $w \in T_a\mathbb{R}^\ell$. We don't know anything about abstract arrows, but we can push it forward with $\mathrm{d}h|_a$: $\mathrm{d}h|_a(w) \in T_b\mathbb{R}^\ell$ corresponds through ψ to a tangent vector at $f(p)$. In other words, $\mathrm{d}f|_p(v) = \mathrm{d}\psi_b \circ \mathrm{d}h|_a \circ \mathrm{d}\phi^{-1}|_p(v)$, and you can check that this is independent of choice of charts. That is: there's a commutative diagram (6.1) of spaces, and the tangent spaces also form a commutative diagram!

Exercise 6.2. Prove that these notions of derivative are all the same (using the chain rule).

We're going to move interchangeably between these pictures, so it's important to know how to translate between them.

Now that we've translated the notion of derivative to smooth maps between manifolds, we can translate all the nice theorems about them too.

Theorem 6.3 (Inverse function theorem). *Suppose X and Y are k-dimensional manifolds. If $f : X \to Y$ is smooth and $\mathrm{d}f|_p$ is invertible, then there's a neighborhood $U \subset X$ of p such that $f|_U$ is a diffeomorphism onto its image.*

In other words, *f is locally a diffeomorphism* in a neighborhood of p.

Proof. Recall our commutative diagram (6.1). Since $\mathrm{d}\phi|_a$ and $\mathrm{d}\psi|_b$ are invertible, then $\mathrm{d}f|_p$ is invertible iff $\mathrm{d}h|_a$ is. Hence, h is locally a diffeomorphism $\mathbb{R}^k \to \mathbb{R}^k$, so since ϕ and ψ are, then f is. ⊠

We've already done the unpleasant analysis, so now we can just do definition chasing. Similarly, using this diagram, you can define the inverse of f locally, by chasing it across the commutative diagram (as h^{-1} already exists).

The next question is what happens when X and Y have different dimensions. If Y is ℓ-dimensional, with $k < \ell$, then $\mathrm{d}f|_p$ is a skinny matrix, with block from $\binom{A}{B}$, and A is invertible. Then, there are diffeomorphisms $\phi : \mathbb{R}^k \to \mathbb{R}^k$ and $\psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "f"] & \mathbb{R}^\ell \\ \mathbb{R}^k \arrow[u, leftrightarrow, "\phi"] \arrow[r, "x\mapsto(x,0)"] & \mathbb{R}^\ell \arrow[u, leftrightarrow, "\psi"] \end{tikzcd} \tag{6.4}$$

The map $h(x) = (x, 0)$ on the bottom is known as the *canonical immersion*, and is the simplest way to put $\mathbb{R}^k$ into $\mathbb{R}^\ell$.

Why is this true? We know the image of f is a graph of points $(x, g(x))$ for some smooth g. Thus, $\psi(x, y) = (x, y - g(x))$, so

$$\mathrm{d}\psi|_{f(p)} = \left(\begin{array}{c|c} I & 0 \\ \hline -\mathrm{d}g^{\mathrm{T}} & I \end{array}\right).$$

Thus, this is invertible, so we can use the inverse function theorem om $\psi \circ f$.

In other words, if π_1 denotes projection onto the first coordinate (the $\mathbb{R}^k$ one), then $\mathrm{d}\pi_1 \circ f)|_a = A$. This is invertible, so $\pi_1 \circ f$ is locally a diffeomorphism! Thus, we let it be ϕ in (6.4), and thus, the map along the bottom really is the canonical immersion. In other words, we've sketched the proof of the following theorem.

Theorem 6.5. *If $k < \ell$ and $\mathrm{d}f|_p$ has rank k, then there are coordinates such that $h(x) = (x, 0)$.*

And this translates to manifolds in exactly the same way as before. This kind of argument (working in local coordinates and using it to translate things from $\mathbb{R}^k$ to manifolds) is very common in this subject, and can be summarized as "think locally, act globally."

Lecture 7.

Immersions and Submersions: 2/3/16

"Differential topology is a language, and as a language, is best learned through immersion."

Immersions.

Definition 7.1. Let X be a k-dimensional manifold and Y be an ℓ-dimensional manifold.

- A smooth map $f : X \to Y$ is an *immersion* if $\mathrm{d}f$ has full rank k everywhere.
- f is a *local immersion* at an $a \in X$ if $\mathrm{d}f|_a$ has rank k (which means it has full rank on a neighborhood of a).

Both of these force $k \le \ell$.

If $f : X \to Y$ has rank k at a, then there are coordinate charts $\phi : U \to X$ and $\psi : V \to Y$ (with $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$) such that $h = \psi^{-1} \circ f \circ \phi$ looks like the canonical immersion $x \mapsto (x, 0)$. Thus, locally, an immersion has a pretty nice image. Moreover, since $\mathrm{d}(f \circ \phi) = \mathrm{d}f \circ \mathrm{d}\phi$, and both $\mathrm{d}f$ and $\mathrm{d}\phi$ are injective, then $\mathrm{d}(f \circ \phi)$ is also injective. So $f \circ \phi$ looks suspiciously like a coordinate chart.

The big question is, *if f is an immersion, is its image a manifold?*

Just because $\mathrm{d}f$ is injective everywhere does not imply f is. For example, you could map $S^1 \to \mathbb{R}^2$ as a figure-8; then, at the intersection point, the manifold locally looks like a pair of coordinate axes, which is not a manifold (it doesn't look like $\mathbb{R}^n$ locally). Okay, great, so if f is an injective immersion, is its image a manifold?

What we'd like to say is that a neighborhood of $f(a)$ comes from a neighborhood of a. However, we'll still need another condition.

Example 7.2. The torus T^2 can be realized as a rectangle with opposite edges identified, as in Figure 3. Thus, we can smoothly map $\mathbb{R} \hookrightarrow T^2$ as a line in this rectangle (wrapping around the identifications), but if the slope of this line is irrational, then there will be countably many disjoint intervals in each neighborhood of any point, and this means that the image isn't a manifold.

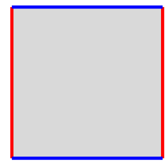


FIGURE 3. The torus can be realized as a rectangle with opposite sides identified, so glue the red sides together and the blue sides together.

One way to work around this is to restrict to immersions that are homeomorphisms onto their image. But another way to think of this: the issue with $\mathbb{R} \hookrightarrow T^2$ was that very distant points ended up nearby. There's a nice way to formalize this.

Definition 7.3. A map $f : X \to Y$ of topological spaces is *proper* if for every compact $K \subset Y$, $f^{-1}(K)$ is compact in X.

Proper maps need not be immersions: the double cover map $\theta \mapsto 2\theta : S^1 \to S^1$ is smooth and proper, but every point has two images.

But a proper injective immersion is sufficient.

Definition 7.4. A smooth map $f : X \to Y$ of manifolds is an *embedding* if it is a proper injective immersion.

Remark. A proper injective map is sometimes called a *topological embedding*. This might be enough to imply that it's an immersion (though the textbook sticks with requiring that f is an immersion).

The quality of being proper is sometimes called *properness*, but *propriety* sounds better.

Theorem 7.5. *Let $f : X \to Y$ be an embedding. Then,* $\mathrm{Im}(f)$ *is a submanifold of* Y.

Proof sketch. For any $a \in X$, consider neighborhoods of $f(a)$. Since f is proper, there's a neighborhood of $f(a)$ that is the image of only finitely many neighborhoods in X, and since f is injective, then they all must be positive distances from each other. Thus, we can shrink our neighborhood to one that only contains the neighborhood for $f(a)$, and then since f is an embedding, a chart for a makes a chart for $f(a)$, so we win. ⊠

Much of the time, we're going to be looking at compact manifolds, for which propriety is redundant: if X is compact, then any continuous map $X \to Y$ (where Y is Hausdorff) is proper (since the preimage of a closed set under a continuous map is closed, and a closed subset of a compact space is compact).

Submersions. Immersions aren't the only way full rank can happen; since full rank is such a nice condition, let's look at another case of it.

Definition 7.6. Let X be a k-dimensional manifold, Y be a manifold, and $f : X \to Y$ be smooth.

- f is a *submersion* if $\mathrm{d}f$ is surjective everywhere.
- f is a *local submersion* near an $a \in X$ if $\mathrm{d}f$ is surjective on a neighborhood of a (equivalently, at a).

This time, these imply that $k \geq \dim Y$.

Just as immersions locally look like the canonical immersion, submersions locally look like the *canonical submersion* $\pi : \mathbb{R}^k \to \mathbb{R}^\ell$ sending $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_\ell)$.

Theorem 7.7 (Local submersion theorem)**.** *Let $f : X \to Y$ be a local submersion near a. Then, there are coordinate charts $\phi : U \to X$ and $\psi : V \to Y$ such that in these coordinates, f looks like the canonical submersion, i.e. $h = \psi^{-1} \circ f \circ \phi$ sends $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_\ell)$.*

Proof. Start with any coordinate charts $\phi : U \to X$ and $\psi : V \to Y$.

Since f has full rank at a, then $\mathrm{d}f|_a = \binom{A}{L}$, where L is a fat matrix with full rank. Since it's linear, there's a smooth map $H : U \to V \times \mathbb{R}^{\ell - k}$ sending $x \mapsto (h(x), L(x))$. Thus, $\mathrm{d}H|_a = \binom{\mathrm{d}h|_a}{L}$, and each of these blocks has full rank, so $\mathrm{d}H|_a$ does too. Thus, since H is square, it's locally invertible, and $\psi^{-1} \circ f \circ \phi' = h \circ H^{-1}$, so using a new coordinate chart ϕ', $h \circ H^{-1}$ is our change-of-charts map, and it's the canonical submersion. ⊠

Theorem 7.8. *Let $f : X \to Y$ be a submersion and $y \in Y$. Then, $f^{-1}(y)$ is a submanifold of X of codimension equal to* $\dim Y$.

Proof sketch. Again, we can check in neighborhoods: let $a \in f^{-1}(y)$; thus, in a neighborhood U of a in X, f looks like the canonical submersion, by Theorem 7.7. In particular, composing with the canonical submersion in a chart for a gives a chart for $U \cap f^{-1}(a)$. ⊠

Because Theorem 7.7 provides a neighborhood in X, rather than in Y, the nuance between embeddings and immersions doesn't come up for submersions.

We can get a stronger result: consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$. Yes, something bad happens at 0, but for the preimage of 1, we don't really care. We can formalize this.

Definition 7.9. Let $f : X \to Y$ be a smooth map of manifolds and $y \in Y$.

- y is a *regular value* of f if $\mathrm{d}f|_a$ is surjective for every $a \in f^{-1}(y)$.
- Otherwise, y is called a *critical value*.

Regular values are extremely important.

Theorem 7.10. *Let y be a regular value for $f : X \to Y$; then, $f^{-1}(y)$ is a submanifold of X with codimension equal to* $\dim Y$.

The proof is exactly the same as for Theorem 7.8, since that proof only required local data.

Example 7.11. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = x_1^2 - x_2^2$, so that $\mathrm{d}f = \begin{pmatrix} 2x_1 \\ -2x_2 \end{pmatrix}$. Thus, $\mathrm{d}f|_{(x_1,x_2)}$ is surjective whenever $(x_1, x_2) \neq (0,0)$, so these are regular values, but at the origin, $\mathrm{d}f|_{(0,0)}$ isn't surjective (it's the zero matrix). Hence, 0 is the only critical value. And lo, the preimage of 0 isn't a manifold, though the preimage everywhere else is.

One interesting nuance is that there are many points in $f^{-1}(0)$ where f is locally a submersion (in fact, all but the origin); but it only takes one bad point to make a set not a manifold.

One important thing to keep in mind is that critical values live in Y, the codomain. We'll hear "points" for things in X and "values" for things in Y, as in the following definition. Be careful to keep them separate!

Definition 7.12. Let $f : X \to Y$ be a smooth map of manifolds and $x \in X$.

- If $\mathrm{d}f|_x$ isn't surjective, then x is a *critical point*.
- Otherwise, x is a *regular point*.

In Example 7.11, the origin is the only critical point.

There's a nice theorem from real analysis about this, which we will not prove.

Theorem 7.13 (Sard)**.** *If $f : X \to Y$ is smooth, then the set of critical* values *of f has measure zero.*

You might wonder: what measure are we using? Well, that's a tricky question: the standard measure on $\mathbb{R}^n$ isn't preserved by change-of-charts maps. However, the condition of having measure zero is preserved, so a set having measure zero in a manifold is well-defined.

Also, another caveat: the critical *points* in X may not have measure zero (e.g. the zero map $\mathbb{R}^m \to \mathbb{R}^n$ — points not in the image of f are regular, since the condition is vacuously satisfied). The point is: there are lots of regular values, which is the aspect of Sard's theorem that we'll use.

Lecture 8.

Transversality: 2/5/16

Note: I missed the first eight minutes of lecture today; I'll fill in any missing details later.

Recall that if $f : X \to Y$ is smooth and y is a regular value for f, then $f^{-1}(y)$ is a submanifold of X. We want to understand a generalization: if $Z \subset Y$ is a submanifold, when is $f^{-1}(Z)$ a submanifold of X? Locally, we know Z is the zero set of a smooth function $g : Y \to \mathbb{R}^{\ell - k}$ (where X is k-dimensional and Y is ℓ-dimensional). In particular, $f^{-1}(Z) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$. Thus, $f^{-1}(Z)$ is a submanifold when 0 is a regular value of $g \circ f$. In particular, this forces $\mathrm{d}(g \circ f)|_x$ to be surjective.

This motivates an extremely important definition.

Definition 8.1. Let $f : X \to Y$ be smooth and $Z \subset Y$ be a submanifold. Then, f is *transverse* to Z, written $f \pitchfork Z$, if for all $x \in f^{-1}(Z)$, $\mathrm{Im}(\mathrm{d}f|_x) + T_{f(z)}Y = T_{f(y)}Y$.

An important special case is when both X and Z are submanifolds of Y and $f : X \to Y$ is inclusion. Then, $f^{-1}(Z) = Z \cap X$, and this is a submanifold if $f \pitchfork Z$. The derivative of inclusion is also inclusion on tangent spaces, so this condition means that $T_x X + T_x Z = T_x Y$. In this case, one simply says X is transverse to Z, written $X \pitchfork Z$.

Intuitively, transversality means that the infinitesimal angle of intersection is not parallel: if it is, then they share tangent vectors, and so we don't get the entire tangent space.

Suppose $p(x)$ is a 17^{th}-order polynomial. Then, we know some conditions on how it intersects the x-axis: it must intersect at least once, and in fact an odd number of times, if the intersection is transverse (no multiple roots). However, if it's not transverse, we have a multiple real root, and it can intersect an even number of times. Strange things happen when you perturb a double root slightly: it can become two real roots, or two complex roots. However, we're going to prove that if you start with a transverse intersection of submanifolds, it's stable under slight perturbations (the number of intersections is the same).

Generally, two curves in $\mathbb{R}^3$ cannot intersect transversely… unless they never intersect at all, in which case they vacuously satisfy the definition. But this set of 0 intersection points is stable, after all. The way to gain intuition about transversality is to think of it in terms of this stability of intersections.

In summary, we've proven the following theorem.

Theorem 8.2. *The following are equivalent for a smooth map $f : X \to Y$ and a submanifold $Z \subset X$.*

- *f is transverse to Z.*
- *$\operatorname{Im}(\mathrm{d}f|_x) + T_{f(x)}Z = T_{f(x)}Y$ for all $x \in f^{-1}(Z)$.*
- *Locally, 0 is a regular value of $g \circ f$, where g is a local submersion $Y \to \mathbb{R}^{\ell-k}$ defined on a neighborhood, and on this neighborhood $Z = g^{-1}(0)$.*

Moreover, each of these implies that $f^{-1}(Z)$ is a submanifold of X.

The converse, however, is not true: the submanifolds $y = x^2$ and $y = 0$ intersect non-transversely at 0, but a point is a zero-dimensional manifold. However, there do exist non-transverse intersections where the intersection is not a manifold.

Homotopy. We want to make precise this fuzzy notion that if you mess with an intersection a little bit, transversality guarantees its stability. The way to slightly change a submanifold is a homotopy.

Definition 8.3. Let X and Y be topological spaces and $f_0, f_1 : X \rightrightarrows Y$ be two continuous functions. Then, a *homotopy* from f_0 to f_1 is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$. If there exists a homotopy between f_0 and f_1, one says that they're *homotopic*, and writes $f_0 \sim f_1$.

This is a topological notion: starting with two functions, we generate a whole family of them interpolating between F_0 and f_1: for every $t \in [0,1]$, we have the interpolator $f_t(x) = F(t,x)$.

For example, if $f_0, f_1 : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ are given by $f_0(x) = 0$ and $f_1(x) = x$, then $F(t,x) = tx$ is a homotopy between them.

We would like to introduce smoothness to this definition, but $[0,1] \times X$ is not a manifold: for any $x \in X$, $(0,x)$ doesn't have a neighborhood diffeomorphic to a Euclidean space. So we don't know what it means to be smooth on the boundary.

There are two inequivalent ways to make this precise if f_0 and f_1 are smooth.

- We could require that F is smooth on the manifold $(0,1) \times X$ and continuous on $[0,1] \times X$. Since we knew f_0 and f_1 are smooth, this seems reasonable.
- A stronger notion of smooth homotopy is that F can be extended $(-\varepsilon, 1+\varepsilon) \times X \to Y$.

For the most part, we'll only need the weaker definition of smooth homotopy. The homotopy $F(t,x) = \sqrt{t}x$ between $f_0(x) = 0$ and $f_1(x) = x$ satisfies the weaker definition, but not the stronger one.

For various properties of maps, we want to know whether they're preserved under this notion. Specifically, let X and Y be smooth manifolds, and P be a property of maps $X \to Y$ (e.g. immersion, submersion, proper, embedding, injective, rank is at most 3, it's smooth, it's analytic, …). If $f_0 : X \to Y$ and F is a homotopy, does f_t have the property P for all sufficiently small t? This is what we mean by stability; if this is the case for all homotopies, P is said to be *stable*.

The first thing we'll see is that it's very hard to preserve *any* properties if X isn't compact; for example, one could define a homotopy that changes things more and more as one goes out to infinity. So this is generally studied when X is compact, and indeed, under this assumption, a whole bunch of properties are stable, including transversality.

One example is that when X and Y are vector spaces, a *linear homotopy* (a homotopy of linear maps for which all the intermediate maps are linear) locally preserves full rank: this is a stable property. Not having full rank is not stable, however.

Now, the flipside is that certain properties are generic, i.e. if a map doesn't have the property, you can bump it a little bit and make it have that property.

Definition 8.4. A property P is *generic* if for any f_0, there's a homotopy F for f_0 and an $\varepsilon > 0$ such that f_t has P for all $t \in (0, \varepsilon)$.

This is existence: the constant homotopy might not work if f_0 doesn't have property P.

The best properties are both generic and stable: you can change a map a little bit and it has the property. And the big punchline is: *transversality is both generic and stable*. We cannot prove this yet, but it's a major stop on this highway. Next time, we'll be able to prove that a lot of properties are stable, and talk about genericity.

Lecture 9.

Properties Stable Under Homotopy: 2/8/16

"Welcome to UT! I hope I won't do anything to scare you away."

We're in the middle of talking about smooth homotopies $f_0 \sim f_1$ of manifolds, which are smooth maps $F : I \times X \to Y$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(X)$. Then, we defined $f_t(x) = F(t, x)$. There are two nuances to this.

- Guillemin and Pollack define this as a map $X \times I \to Y$. *A priori*, this makes no difference whatsoever, but when we begin to talk about oriented manifolds, it will be easier to orient this if we use the convention $I \times X$.
- What does "smooth" mean on the boundary? To Guillemin and Pollack, all manifolds live in some ambient space, so this really means it can be extended to an open neighborhood of the boundary. But we find it more useful to require the partial derivative in x to not vanish.

  As an example, let $X = Y = \mathbb{R}$, $f_0(x) = x$, and $f_1(x) = x + \sin x$. As continuous maps, these are clearly homotopic, and one example of the homotopy is

$$F(t, x) = \begin{cases} x + t \sin\left(\dfrac{x}{t^2}\right), & t \neq 0 \\ x, & t = 0. \end{cases}$$

  Is this smooth? Well, what do you want smoothness to be? We're looking for a stability condition on transversality, but this homotopy sends something transverse to the real line to something not transverse to it, no matter how short you travel along it. And indeed, $\frac{\partial F}{\partial x}$ isn't continuous in t. Hence, for the purposes of stability, we'll require that a smooth homotopy have all partial derivatives of x continuous in t.

Under this definition, we do have some nice stability (i.e. if f_0 has a property, then so does f_t for $t > 0$ sufficiently small).

Theorem 9.1. *Let X be a compact smooth manifold, Y be a smooth manifold. Then, the following properties are stable under smooth homotopies $F : I \times X \to Y$:*

(1) local diffeomorphisms,
(2) immersions,
(3) submersions,
(4) embeddings,
(5) transversality with respect to a fixed closed submanifold $Z \subset Y$, and
(6) diffeomorphisms.

Partial proof. Suppose f_0 is a local diffeomorphism, so for any $a \in X$, $\mathrm{d}f_0|_a$ is invertible. This is true in a neighborhood of a, because the derivative having full rank is an open condition. Thus, for each $a \in X$, there's a neighborhood $U_a \subset X$ of a and a $\varepsilon_a > 0$ such that on $U_a \times (0, \varepsilon)$, $\mathrm{d}f$ has full rank. However, since X is compact, we can cover it by only finitely many of these U_a, and then take ε to be the minimum of those finitely many ε_a; thus, for $t \in (0, \varepsilon)$ and all $x \in X$, $\mathrm{d}f_t|_x$ has full rank; this proves (1).

Since the conditions on immersions and submersions are that the derivative has full rank, the same proof applies, *mutatis mutandis*, to prove (2) and (3).

Now, let's look at (5). We defined transversality to mean that for all $x \in f^{-1}(Z)$, $\mathrm{Im}(\mathrm{d}f|_x) + T_{f(x)}Z = T_{f(x)}Y$. We proved there's a map $g : Y \to \mathbb{R}^{\dim Y - \dim Z}$ that sends a neighborhood of $f(x)$ in Z to 0, and such that $f \circ g$ is a submersion. Thus $f_t \circ g$ is a submersion for sufficiently small t, and so $f_t \pitchfork Z$.

The final two, (4) and (6), depend on global topological behavior, and so we'll leave them to be exercises, but the proofs are not dissimilar. ⊠

For part (5), the stipulation that Z is closed is important: an open submanifold can be infinitesimally close to another submanifold without intersecting it (e.g. the distance between $(0,1)$ and $[1,2]$ is 0). Another important thing we depend on is that the derivatives with respect to x are continuous in t, because that allowed us to prove the first three parts. We had an explicit counterexample for (6),[6] but there are also counterexamples for the other five parts if you don't have the right notion of smoothness.

Next, let's talk about Sard's theorem, Theorem 7.13, which states that if $f : X \to Y$ is smooth, then its set of critical values has measure zero.

Partial proof of Theorem 7.13. First, we can reduce this to a statement about neighborhoods in X and Y: if we know it in charts, then we can take a countable union of charts in X (which exists because X is second countable), and a countable union of sets with measure zero still has measure zero.

Hence, we may assume without loss of generality that $X = (0,1)^k$ and $Y = (0,1)^\ell$. If $k = \ell$, let C be the set of critical points, so $f(C)$ is the set of critical values. Since C is the points where $\det \mathrm{d}f|_x = 0$, let $C_\varepsilon = \{x \in X : \det \mathrm{d}f|_x < \varepsilon\}$. Then, $|f(C)| \le |f(C_\varepsilon)| < \varepsilon$ for each $\varepsilon > 0$, so $|f(C)| = 0$. This estimate comes from the fact that the determinant of the derivative measures how much f changes volume locally, so small determinants in the unit cube squish their image into a small space. The idea here is that there may be a lot of critical points, but they're squashed together.

To apply this when $k \neq \ell$, you have to do some extra linear algebra: if you have a fat matrix without full rank, what does it do to volume, and what does a small perturbation do to volume? The takeaway will be that the image will have proper codimension, and therefore automatically is measure zero. But this isn't topology, so we're not going to dwell on it. ⊠

A Five-Minute Crash Course in Morse Theory. One cool use of Sard's theorem is Morse theory. This will be a short digression.

Let X be a compact manifold (the canonical example is a torus) and $f : X \to \mathbb{R}$ be a smooth function (in the example, a height function). Consider the sets $f^{-1}((-\infty, a))$ for $a \in \mathbb{R}$. Since X is compact, there's a minimum a_0, and for values of a just a little bit greater than a_0, you get the behavior of X in a neighborhood of that minimum, but they're all the same until you get to the donut hole.

That is, at a critical value of f, there's something interesting topologically going on, and nothing topologically happens at the regular values. You need f to have a condition that makes its behavior particularly clean around critical values, but such f exists, but the result is a decomposition of X into pieces associated with its critical values.

So we need to understand how f behaves around critical values, meaning a power series expansion

$$f(x) = f(a) + \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_a (x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^n \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_a (x_i - a_i)(x_j - a_j) + o(x^3).$$

If x is a critical point, then the first derivatives vanish, so to make this nondegenerate, we just need that the second derivatives don't vanish at each critical point. Such a function is called a *Morse function*, and a critical point satisfying this is called *nondegenerate*.

The fact that Morse functions exist, and in fact can be made from a perturbation of any function, is a consequence of Sard's theorem.

[6] Technically, we didn't start with a compact X, but the noncompactness of $\mathbb{R}$ was never needed, and we could replace it with its one-point compactification S^1 without changing the essence of the argument.

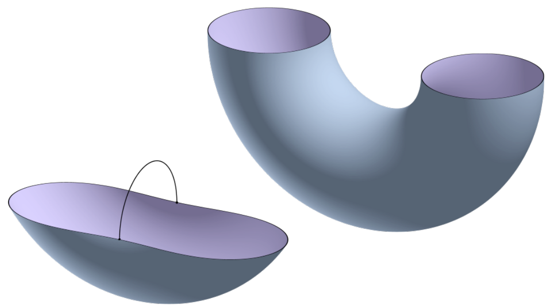


FIGURE 4. Adding a bridge at a critical point of f.

Lecture 10.

May the Morse Be With You: 2/10/16

Last time, we briefly started talking about Morse theory. Today, we'll slow down and go in more detail.

Definition 10.1. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is *Morse* if, whenever $\mathrm{d}f|_x$ is smooth, the Hessian at x is invertible.

The awesome fact is that garden-variety functions are Morse, or, in different words, Morse functions are generic.

Theorem 10.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let $f_a(x) = f(x) + a \cdot x$. Then, for almost all $a \in \mathbb{R}^n$, f_a is Morse.*

"Almost all" means that the statement is true except on a set of measure zero.

Proof. Define $g_a(x) = \nabla f_a = (\partial_1 f_a, \partial_2 f_a, \dots, \partial_n f_a) = a + \nabla f$, and $\mathrm{d}g_a$ is the Hessian of f_a. Hence, f_a is Morse iff 0 is a regular value of g_a iff $-a$ is a regular value of g. By Sard's theorem (Theorem 7.13), regular values have full measure, so almost every $-a$ is a regular value, and therefore almost every f_a is Morse. ⊠

Those were Morse functions on Euclidean space. What about on manifolds?

Definition 10.3. If X is a smooth manifold, a smooth $f : X \to \mathbb{R}$ is Morse if whenever $\mathrm{d}f|_x = 0$, the Hessian at x in local coordinates is invertible.

At every critical point x, there's a chart $\psi : \mathbb{R}^k \to X$, and this statement is equivalent to $f \circ \psi$ being Morse as a function $\mathbb{R}^k \to \mathbb{R}$.

Remark. Since we chose a chart to make this definition, we need to know that it's independent of choice of charts, so suppose $\phi : \mathbb{R}^k \to X$ is another chart for a neighborhood of x, and let g be the change-of-charts map for ϕ and ψ. The fact that it's a diffeomorphism means that the critical points of $f \circ \phi$ and $f \circ \psi \circ g$ are the same, and using the Chain rule, $H(f \circ \phi) = (\mathrm{d}g)H(f \circ \psi)\mathrm{d}g^{\mathrm{T}}$, and since $\mathrm{d}g$ is invertible, this is linear in $\mathrm{d}(f \circ \psi)$, and therefore one is invertible when the other does. This argument should be fleshed out a bit, but the point is that Morseness doesn't depend on which local coordinates you use.

Now, we can prove an analogue of Theorem 10.2 for submanifolds of $\mathbb{R}^n$. There's an analogue for abstract manifolds, but it's a little harder to state, since we can't take the dot product abstractly.

Theorem 10.4. *Let X be a k-dimensional submanifold of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ be smooth. Then, if $f_a(x)$ is defined as in Theorem 10.2, then for almost every $a \in \mathbb{R}^n$, f_a is Morse.*

Proof. We're going to work in charts: because Euclidean space is separable, X can be covered by countably many charts, or more precisely, every cover of X by charts has a countable subcover. Thus, if we prove that on each chart, the set of a which fail has measure zero, then the total set of such a is a countable union of sets of measure zero, and thus has measure zero. And every point in X has a neighborhood to which the immersion theorem applies, so we can cover X by countably many neighborhoods in which it applies.

We can write $a = (b, c)$, where b denotes the first k coordinates and c denotes the last $n-k$ coordinates. Around a given point p, we can thus write $f_a(x) = f(x) + c \cdot (x_{k+1}, \dots, x_n) + b \cdot (x_1, \dots, x_k)$. And because $f(x) + c \cdot (x_{k+1}, \dots, x_n)$ is smooth, then f_a is Morse for almost every b. By Fubini's theorem, the set of (b, c) where b doesn't work also has measure zero, so f_a is Morse for almost every a. ⊠

Morse functions are a dime a dozen, if not a dime a countably many! And there are lots of useful things you can do with a Morse function, e.g. looking at the topology of a manifold using preimages of intervals under Morse functions.

Embeddings of Manifolds. We're going to make a series of increasingly strong statements about how to embed abstract manifolds into $\mathbb{R}^N$ for sufficiently large N.

Theorem 10.5 (Whitney embedding theorem)**.** *Let X be an abstract k-dimensional manifold.*

(1) There's an embedding $X \hookrightarrow \mathbb{R}^N$ for some N.
(2) There's an injective immersion $X \hookrightarrow \mathbb{R}^{2k+1}$.
(3) There's an embedding $X \hookrightarrow \mathbb{R}^{2k+1}$.
(4) There's an immersion $X \hookrightarrow \mathbb{R}^{2k}$.
(5) There's an embedding $X \hookrightarrow \mathbb{R}^{2k}$.

One consequence is that the Guillemin-and-Pollack approach to manifolds captures all diffeomorphism classes of manifolds. Of course, this theorem is not in the textbook. Parts (2), (3), (4), and (5) are all due to Whitney.

We'll attack this as follows. First, we'll prove (1) for X compact, and then prove (2), (3), and (4) assuming (1) in generality (the details aren't that different). (5) is extremely difficult to prove.

To prove these statements, we'll rely heavily on the concept of a partition of unity. We'll discuss these more on Friday (and provide a proof of existence).

Definition 10.6. Let X be a smooth manifold.

- Let $\rho : X \to \mathbb{R}$ be a smooth function. Then, its *support* is the closed set $\operatorname{supp} \rho = \overline{\{x : \rho(x) \neq 0\}}$ (the closure of where it's nonzero).
- If $U \subset X$ is open and $K \supset U$ is compact, a *bump function* $\rho : X \to \mathbb{R}$ is a smooth function such that $\rho|_U = 1$ and $\operatorname{supp} \rho \subseteq K$.

That is, a bump function is smooth, but if K isn't much bigger than U, it has to change from 1 to 0 smoothly and quickly.

Definition 10.7. Let $X \subset \mathbb{R}^n$ be a manifold and $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X. Then, a collection of smooth functions $\rho_i : X \to \mathbb{R}$ (also indexed by I) is a *partition of unity* if it satisfies the following axioms.

- $\operatorname{supp}(\rho_i) \subset U_i$ for each i.
- For every $x \in X$, there's a neighborhood of x on which only finitely many ρ_i are nonzero.[7]
- $\sum_{i \in I} \rho_i(x) = 1$ for all $x \in X$. (This makes sense, because at each point, it's a finite sum.)

Bump functions can be used to construct these, as we will show at some point.

Proof of Theorem 10.5, part (1). Since X is compact, there's an $s \in \mathbb{N}$ and an open cover $\mathfrak{U}$ of X by s coordinate charts. Let $\{\rho_i\}$ be a partition of unity indexed by $\mathfrak{U}$.

On the chart U_1, we have coordinates $(x_1, \dots, x_k)$, and the function $\widetilde{g}_1 : U_1 \to \mathbb{R}^{k+1}$ sending $x \mapsto (\rho_1, \rho_1 x_1, \dots, \rho_1 x_k)$ is smooth and supported in U_1, so we can extend it to all of X by defining it to be 0 outside of U_1. Thus, this formula defines a smooth $g_1 : X \to \mathbb{R}^{k+1}$. The same construction defines functions $g_2, \dots, g_s : X \to \mathbb{R}^{k+1}$.

Now, our embedding will be $j : X \to \mathbb{R}^{s(k+1)}$, defined by $j(x) = (g_1(x), \dots, g_s(x))$, which is smooth and has full rank (since each point is in a chart U_i, where g_i has full rank k, so j has to have rank k as well). Then, it's injective, because if $j(x_1) = j(x_2)$, then $\rho_i(x_1) = \rho_i(x_2)$ for some i where this quantity is nonzero. Thus, they lie in the same chart, so their coordinates in that chart agree (since $g_i(x_1) = g_i(x_2)$), and therefore j is injective. And since X is compact, it's proper, so j is an embedding. ⊠

[7] This can be thought of as a form of "ρ reduction."

Lecture 11.

Partitions of Unity and the Whitney Embedding Theorem: 2/12/16

Last time, we found a way to immerse any manifold in high-dimensional Euclidean space (well, we did this for compact manifolds, but you can do this for noncompact ones as well). The key ingredient was a partition of unity, but we haven't shown that this exists.

"Little Spivak" (Spivak's *Calculus on Manifolds*) provides a great, detailed proof that partitions of unity exist for Euclidean spaces. The starting point is the existence of bump functions: for any open $U \subset \mathbb{R}$ and a compact $V \subset U$, there's a smooth $\rho : \mathbb{R}^n \to \mathbb{R}$ that's 1 on V and 0 outside of U. If V is almost all of U, this could change pretty quickly.

Then, we'll bootstrap this to a compact $K \subset \mathbb{R}^n$. For any open cover $\mathfrak{U}$ of K, we can restrict to a finite subcover, and finding a partition of unity for that subcover finds one for the open cover. Thus, if $U_1, \dots, U_m$ is this finite subcover, then let $W_i = \overline{\bigcup_{j\neq i} U_i}$. Thus, $W_i \subset U_i$ is closed, and since it's a closed subset of K, then it's compact. Thus, there's a bump function ψ_j for it, and the required partition of unity is $\rho_j = \psi_j / \sum_{i=1}^n \psi_i$: now, their sum is 1, and globally this is a finite sum, so it's locally finite, too.

The next step is to generalize this to a set X that can be *exhausted by compact sets*: there are compact $K_1, K_2, \dots$ and open sets $U_1, U_2, \dots$ such that $K_1 \subset U_1 \subset K_2 \subset U_2 \subset \cdots$, and $X = \bigcup K_i$. The trick is to work with only $K_i \cup K_{i+1}$ at each level, so now we get an infinite collection that's locally finite, and then scale to get the sum equal to 1. (This is the argument that Spivak goes into more detail about.)

We'd also like to apply this to open sets, so if $A \subset \mathbb{R}^n$ is any open set, let $K_n = \{x \in A \mid |x| \leq n \text{ and } d(x, A^c) \geq 1/n\}$ (so the points not too close to the boundary). These are compact, and their union is A (since A doesn't contain any of its boundary points), so this is an exhaustion by open sets, and therefore A has a partition of unity.

Now, *any* subset of $\mathbb{R}^n$ can be covered by open sets, so we can make this work on any set, and since we can express manifolds as unions of coordinate charts, we can generalize this to manifolds too. There's certainly an argument to be made, but the key ideas are familiar.

Now, let's return to Theorem 10.5. Last time, we proved that if X is a manifold, there's an injective immersion $X \hookrightarrow \mathbb{R}^N$ for some large N, but for part (1), we'd like it to be an embedding. X is either compact or not compact.

- If X is compact, an injective immersion $X \hookrightarrow Y$ is already an embedding: maps are always proper.
- If X isn't compact, then there's an open cover $\mathfrak{U} = \{U_i\}$ with no finite subcover. We can assume without loss of generality that $\mathfrak{U}$ is countable, by the second-countability requirement on manifolds, and likewise assume that $\overline{U_i}$ is compact (we can refine the cover if need be).

  Now, pick a partition of unity $\{\rho_i\}$ for it, and let $f(x) = \sum_{n\in\mathbb{N}} n\rho_n(x)$. In a neighborhood of every point, f is a sum of finitely many smooth functions, so it's smooth.[8] Now, if $K \subset \mathbb{R}^n$ is any compact set, then it's contained in $[-N, N]$ for some N, so if we show the preimage of $[-N, N]$ is compact, then $f^{-1}(K)$ will be a closed subset of a compact set, and therefore compact. But $f^{-1}([-N, N])$ is contained in the union of the closures of finitely many elements of $\mathfrak{U}$, and hence is compact.

  The actual trick is to replace the injective immersion $g : X \hookrightarrow \mathbb{R}^N$ with $X \hookrightarrow \mathbb{R}^{N+1}$ given by $x \mapsto (f(x), g(x))$. This is still full rank and injective, and now is in fact a proper map, so we have an embedding.

Our proof of part (1) specialized to the compact case, but in the noncompact case, with an infinite cover, it's possible to reuse coordinates by preserving injectivity: if U_1 and U_2 are disjoint charts, we can map them to parts of $\mathbb{R}^k$ that don't overlap, so that the map defined by assigning them to the same tuple of slots is still injective. In the end, there will be infinitely many assigned to the same slot, which is OK, since we're also using a partition of unity. There is significant technical detail that we're skipping over, but it turns out that the condition we need is exactly paracompactness!

The next step is to reduce the dimension, to get part (2). We'd like to find a projection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ such that after composing with the embedding $f : X \hookrightarrow \mathbb{R}^N$, it's still injective. These projections are all given

[8] The local finiteness here is quite important: a countably infinite sum of smooth functions is not necessarily smooth.

by $\pi_v(x) = x - (x \cdot v)v$ for $v \in S^{N-1}$ (the unit $(N-1$-sphere in $\mathbb{R}^N$). In particular, if $\pi_v(x_1) = \pi_v(x_2)$, then $x_1 - x_2 = kv$ for some $k \in \mathbb{R}$.

Definition 11.1. If X is a manifold, the *diagonal* $\Delta \subset X \times X$ is the submanifold $\{(x,x) \mid x \in X\}$.

We can define a map $h : (X \times X) \setminus \Delta \to S^{N-1}$ given by $(x_1, x_2) \mapsto (x_1 - x_2)/|x_1 - x_2|$. This is smooth, since it's a quotient of smooth functions and the denominator is never zero (since we've left the diagonal out). And if $v \notin \mathrm{Im}(h)$, then there would be no $x_1, x_2 \in X$ such that $x_1 - x_2 = kv$, so $\pi \circ f$ would still be an embedding.

Suppose $2k < N - 1$, so that at every $x \in X$, $\mathrm{d}h|_x$ maps from a vector space of smaller dimension to one of greater dimension. Thus, it can never be surjective, so if $y \in S^{N-1}$ is a regular value, then $y \notin \mathrm{Im}(h)$. By Sard's theorem, regular values have full measure, so for almost every v, $\pi_v|_{f(X)}$ is one-to-one.

This is pretty cool, but alone it's not enough; we need the derivative to still have full rank. For any $y \in \mathbb{R}^N$, $\ker(\mathrm{d}\pi_v|_y) = \mathrm{span}\{v\}$, so $\mathrm{d}(\pi_v \circ f)|_x$ is injective iff $v \notin T_xX$ for any $x \in X$. That is, we have a map $j : TX \to \mathbb{R}^N$ defined by $(p, w) \mapsto w$, and we want a v such that $v \notin \mathrm{Im}(j)$. This means we can use exactly the same trick: $\dim(TX) = 2k$, so if $N > 2k$, then $\mathrm{d}j$ can't be surjective anywhere, so by the same line of reasoning with Sard's theorem, almost every $v \in S^{N-1}$ isn't in the image of j.

The intersection of two sets of full measure still has full measure, so as long as $N > 2k + 1$, we can find a v such that $\pi_v \circ f$ is an injective immersion, and we can do this again and again until we hit dimension $2k + 1$.

The last thing we need to check is propriety, which we could've lost. But if we don't have it, then in the same way as the proof above, we can make a proper map to $\mathbb{R}$ and then add it to our map to get an embedding $X \hookrightarrow \mathbb{R}^{2k+2}$. Then, we can project again: if we project along a v that isn't on the same coordinate that we used to stick in the proper map, then this preserves properness (and we can totally do this, since the set of permissible v has full measure).

Thus, we have an embedding $X \hookrightarrow \mathbb{R}^{2k+1}$, which is (2), and an immersion $X \hookrightarrow \mathbb{R}^{2k}$, which is (4). The final step, making the last immersion an embedding, is possible, but requires a highbrow technique called the *Whitney trick*. This means thinking like a low-dimensional topologist: failure of a projection to be injective means a crossing in your manifold (e.g. actual crossings in knot theory).

Lecture 12.

Manifolds-With-Boundary: 2/15/16

Definition 12.1. A topological space is *second countable* if it has a countable basis of open sets.

$\mathbb{R}^N$ is second countable, with a basis given by balls of rational radius centered at points in $\mathbb{Q}^N \subset \mathbb{R}^N$.

Recall that a *concrete k-manifold* is a subset of $\mathbb{R}^N$ such that each $x \in X$ has a neighborhood (in X) diffeomorphic to $\mathbb{R}^k$ (equiv. an open ball in $\mathbb{R}^k$). This X is automatically Hausdorff and second countable, because $\mathbb{R}^N$ is.

A smooth *abstract k-manifold* is a Hausdorff, second-countable topological space X with an atlas giving every point a neighborhood diffeomorphic to an open ball in $\mathbb{R}^k$, and the change-of-coordinates maps are smooth.

We waved our hands about second countability, but it's actually quite important: the Whitney embedding theorem may fail for a space which resembles a manifold but isn't second-countable.

Example 12.2 (Long line)**.** There's a standard counterexample called the *long line*, which is like the line but longer: instead of countably many copies of $[n, n+1)$, there are uncountably many! Using the axiom of choice, one can deduce the existence of an uncountable, well-ordered set Σ; then, the long line is $\Sigma \times [0,1]/\sim$, where $(n,1) \sim (n+1, 0)$ (where $+1$ denotes the successor to an $n \in \Sigma$), topologized with the dictionary ordering. This satisfies all of the requirements for a smooth manifold, except second countability.

For the most part, though, everything we've talked about has been a smooth manifold. One big exception is the parameter space for a homotopy, $[0,1] \times X$. This looks like a manifold, except for the "boundary points." We can make this precise.

Definition 12.3. A *concrete k-manifold-with-boundary*[9] is a set $X \subset \mathbb{R}^N$ such that each $x \in X$ has a neighborhood diffeomorphic either to an open ball in $\mathbb{R}^k$ or an open ball in $H^k = \{(x_1, \dots, x_k) \mid x_k \geq 0\}$.

[9] Often, one sees "manifold with boundary" instead of "manifold-with-boundary," but this is an abuse of notation: a manifold-with-boundary is not a manifold.

One can also think of H^k as $\mathbb{R}^{k-1} \times [0, \infty)$, which makes the boundary a litle easier to see. Another quick thing about the definition is that every open ball in $\mathbb{R}^k$ is diffeomorphic to one in H^k (that doesn't touch the boundary), so we can simplify to only using H^k in the definition.

Definition 12.4. The *boundary* of H^k, written ∂H^k, is the subset $\mathbb{R}^{k-1} \times \{0\}$, and the *interior* is everything else. Hence, if X is a manifold-with-boundary, its *boundary* ∂X is the subset that maps to ∂H^k in an atlas, and its *interior* X^0 is everything else.

This is *not* the same as the topological boundary and interior of a subset of $\mathbb{R}^N$! For example, $[0,1]$ embeds into $\mathbb{R}^2$ via $t \mapsto (t,0)$, and its interior in that sense is empty, but its interior as a manifold-with-boundary is $(0,1) \times \{0\}$. So to determine the boundary, work locally.

Concrete manifolds-with-boundary are automatically Hausdorff and second countable, so we can define abstract manifolds-with-boundary in pretty much the same way as before.

Definition 12.5. A smooth *(abstract) k-manifold-with-boundary* is a Hausdorff, second-countable topological space X with an atlas giving every point a neighborhood diffeomorphic to an open ball in H^k, and for which the change-of-coordinates maps are smooth.

Notice every manifold is also a manifold-with-boundary, and its boundary is empty (the empty set is a perfectly fine manifold).

The embedding theorems apply still, and therefore considering concrete manifolds-with-boundary is, up to diffeomorphism, the same as being abstract.

Another important thing to observe is that the interior and boundary of a manifold-with-boundary are disjoint, because if x maps to a point on ∂H^k, no neighborhood of it is diffeomorphic to an open ball in $\mathbb{R}^k$. Hence, the interior and boundary are disjoint, and the interior is a k-manifold (since it has no boundary).

Theorem 12.6. *If X is a manifold-with-boundary, ∂X is a $(k-1)$-manifold.*

In particular, $\partial(\partial X) = \varnothing$, and even if X is connected, ∂X might not be, e.g. $\partial([0,1]) = \{0\} \cup \{1\}$. Another way to think about this is that our manifolds-with-boundary don't have corners, so to speak.

Theorem 12.6 is nearly tautological: an open neighborhood in H^k of a point on the boundary restricts to an open neighborhood on $\mathbb{R}^{k-1}$ (since the boundary point is in $\mathbb{R}^{k-1} \times \{0\}$), and restrictions of smooth functions are smooth, etc.

Theorem 12.7. *If X is a manifold-with-boundary and Y is a manifold, $X \times Y$ is a manifold-with-boundary, and $\partial(X \times Y) = (\partial X) \times Y$.*

This follows because $H^{\ell+k} \cong \mathbb{R}^\ell \times H^k$, and charts $\phi : H^k \to X$ and $\psi : \mathbb{R}^\ell \to Y$ induce a map $(\phi, \psi) : H^k \times \mathbb{R}^\ell \to X \times Y$. However, the product of two manifolds-with-boundary may have corners (the boundary may not be a smooth manifold), so we won't consider those sorts of products.

Tangent Spaces. We'd like to apply our favorite constructions on manifolds to manifolds-with-boundary. First, the tangent space: suppose $p = \psi(a)$ in a chart for a manifold X; then, we defined $T_pX = \mathrm{Im}(\mathrm{d}\psi|_a)$. If X is instead a manifold-with-boundary and $q \in \partial X$, this definition still makes just as much sense: $T_qX = \mathrm{Im}(\mathrm{d}\psi|_{\psi^{-1}(q)})$. This means that a tangent vector is a velocity vector for a curve that can be extended in a neighborhood of q; it continues in all directions, not just those that are still in the manifold. This is nice, because it means we still have a vector space, but we can also refer to *inward-* and *outward-pointing vectors*, and in fact talk about vectors tangent to the boundary! This is the space $T_q(\partial X)$, which is a codimension-1 subspace of T_qX. Then, $T_qX \setminus T_q(\partial X)$ has two components, the inwards-pointing and outwards-pointing vectors. Thus, every tangent vector at the boundary is either inwards-pointing, outwards-pointing, or tangent to the boundary.[10]

Regular Values. Another notion we like is that of regular values. Recall that for manifolds, $f : X \to Y$ has a regular value y if $\mathrm{d}f$ is surjective on all of $f^{-1}(y)$. This still makes sense for manifolds-with-boundary: since tangent spaces are defined by smooth extensions on a neighborhood of a boundary point, we get a map of vector spaces, and life goes on.

We can also define a *boundary map* $\partial f = f|_{\partial X}$; regularity and transversality tend to require or imply things about both f and ∂f.

[10]Formally, the way to do this is to define inwards-pointing vectors on $\mathbb{R}^k$ to point in the positive direction and outwards-pointing ones to point away from it; then, one shows that for an arbitrary manifold-with-boundary X and a $q \in \partial X$, this notion is independent of chart. In other words, "you put your vector in, you put your vector out..."

Theorem 12.8 (Sard's theorem for manifolds-with-boundary). *If $f : X \to Y$ is a smooth map of manifolds-with-boundary, then the set of its regular values has full measure in Y, and the regular values of ∂f also have full measure in Y.*

We also defined regular values in order for preimages of points to be manifolds. That may not still be true, but if y is a regular value of f, then $f^{-1}(y)$ is a manifold-with-boundary with the correct codimension.

Lecture 13.

Retracts and Other Consequences of Boundaries: 2/17/16

Recall that a k-dimensional manifold-with-boundary is a second-countable, Hausdorff space for which every point has a neighborhood that is diffeomorphic to an open set in H^k (the upper half-space in $\mathbb{R}^n$): since $\mathbb{R}^k$ can be embedded in H^k, then all manifolds are manifolds-with-boundary.

We'd like to prove the following theorem, which classifies compact, connected, one-dimensional manifolds-with-boundary.

Theorem 13.1. *A nonempty, compact, connected 1-manifold-with-boundary is diffeomorphic to either $[0,1]$ or S^1.*

Lemma 13.2. *A nonempty, compact, connected 1-manifold is diffeomorphic to S^1.*

Proof. Let X be a nonempty, compact, connected 1-manifold. Each point has a neighborhood diffeomorphic to $(-1,1)$, so by compactness, we have finitely many neighborhoods $U_1, \dots, U_n$.

Let's induct. If there's only one chart, $X \cong (-1,1)$, which isn't compact, so oops. Thus, there must be at least two charts that intersect (since X is connected). The union of these two intervals has to be either an open interval (if they intersect on one side of each) or a circle (if they intersect on both sides), but if their union is an open interval, there has to be another chart, by compactness. ⊠

Proof of Theorem 13.1. Let X be such a 1-manifold. Then, either X has no boundary, in which case $X \cong S^1$ by Lemma 13.2, or it has a boundary point, and therefore a chart containing that boundary point. This chart must be diffeomorphic to $[a,b)$; this isn't compact, so there must be another chart. This chart either intersects another boundary point, giving us $[a,b]$ as desired, or doesn't; in the latter case, their union has to be a half-open interval, so there has to be another chart (until we eventually get a closed interval). ⊠

Corollary 13.3. *Let X be a compact 1-manifold-with-boundary. Then, $\#(\partial X)$ is even.*

In particular, since X is compact, this number must be finite. Later, when we talk about oriented manifolds, we'll have a way to assign orientations to the boundary; if we count points weighted with this sign, then they must sum to 0.

Rewriting this as $\#(\partial X) \equiv 0 \bmod 2$, we'll end up developing a lot of tools that count everything mod 2; once we take orientation into account, we can redo everything in $\mathbb{Z}$ instead of $\mathbb{Z}/2$ (and with the bonus of much less analysis).

Theorem 13.4. *Let X be a k-dimensional manifold-with-boundary and Y be an n-dimensional manifold. If $f : X \to Y$ is smooth and p is a regular value of both f and ∂f, then $f^{-1}(p)$ is a $(k-n)$-dimensional manifold-with-boundary, and $\partial(f^{-1}(p)) = \partial X \cap f^{-1}(p)$.*

This is a generalization of Theorem 7.10. The additional hypothesis that p is a regular value of ∂f is necessary: suppose $X = \{(u,v) \in \mathbb{R}^2 \mid u \geq -1\}$ and $Y = \mathbb{R}$. Then, $f(u,v) = u^2 + v^2$ is certainly smooth, and 1 is a regular value of f, but not for ∂f (since there, $f(v) = v^2 + 1$). And $f^{-1}(1)$ is the unit sphere, which is a manifold, sure, but its boundary is empty, and $f^{-1}(p) \cap \partial X = \{(0,-1)\}$, so the conclusion of Theorem 13.4 isn't satisfied.

Proof of Theorem 13.4. We need to show that if $f(x) = p$, then x has a neighborhood that looks like H^k, and that it's in the interior iff x is on the interior. If $x \in X^0$, then Theorem 7.10 shows there's a neighborhood of x diffeomorphic to $\mathbb{R}^{n-k}$, so instead suppose $x \in f^{-1}(p) \cap \partial X$. Then, there's a neighborhood $U \subset H^k$ and a chart $\varphi : U \to X$ sending $0 \mapsto x$, and such that $\partial X \cap \varphi(U) = \varphi(U \cap \partial H^k)$. The composition $f \circ \varphi$ is smooth, so we can extend it to a smooth $\widetilde{f}$ on an open neighborhood $V \supset U$ in $\mathbb{R}^k$. Then, Theorem 7.10 implies that $\widetilde{f}^{-1}(p)$ is a codimension-k submanifold of V containing p, so when we restrict to U, $(f \circ \varphi)^{-1}(p)$ has a neighborhood near

0 diffeomorphic to H^{k-n}, and in this neighborhood, $\partial(f\circ\varphi)^{-1}(p) = (f\circ\varphi)^{-1}(p)\cap\partial H^k$, so applying φ, we're done. ⊠

This theorem will be very important for a lot of what follows. And by Sard's theorem (Theorem 12.8), almost all points of Y are regular values for f and ∂f, so almost all of them satisfy the hypotheses of Theorem 13.4. This is useful.

Retractions.

Definition 13.5. Let $Z\subset X$ be a submanifold. Then, a *retraction* $f : X\to Z$ is a smooth map such that $f|_Z$ is the identity.

For example, we can retract the unit disc to the disc of radius 1/2, or $\mathbb{R}^n$ to a point.

Theorem 13.6. *If X is a nonempty, compact manifold-with-boundary, there is no smooth retraction $X\to\partial X$.*

Proof. Suppose such an f exists, so that there's a $p\in\partial X$ that's a regular value of both f and ∂f.[11] Thus, by Theorem 13.4, $f^{-1}(p)$ is a nonempty, compact 1-dimensional manifold-with-boundary, and $\partial f^{-1}(p) = f^{-1}(p)\cap\partial X = \{p\}$ (since ∂f is the identity, so no other points map to p). But by Corollary 13.3, every compact 1-manifold-with-boundary has an even number of points in its boundary, so this is a contradiction. ⊠

This leads us to a beautiful theorem.

Theorem 13.7 (Brouwer fixed-point theorem)**.** *Let B^n denote the closed unit ball in $\mathbb{R}^n$. Then, any smooth $f : B^n\to B^n$ has a fixed point.*

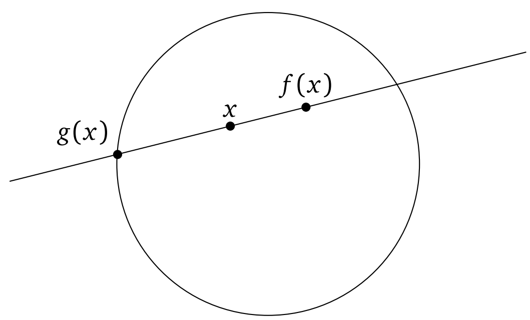


FIGURE 5. The map $g(x)$ defined in the proof of the Brouwer fixed-point theorem.

Proof. Suppose f has no fixed point; then, for every $x\in B^n$, there's a unique line through x and $f(x)$. Let $g(x)$ denote its intersection with the boundary that's closer to x, as in Figure 5. This is a smooth map $B^n\to\partial B^n$, and on the boundary it's just the identity, so it's a retraction. But we just proved that retractions don't exist. ⊠

This is also true for merely continuous maps, which can be deduced from the smooth case, as we will do on the homework.

∽ · ∼

We can also extend the notion of transversality to manifolds with boundary.

Theorem 13.8. *Let X be a manifold-with-boundary, Y be a manifold, and $Z\subset Y$ be a submanifold. Let $f : X\to Y$ be smooth and suppose $f \pitchfork Z$ and $\partial f \pitchfork Z$; then, $f^{-1}(Z)$ is a submanifold-with-boundary of X.*

This is a generalization of Theorem 8.2. The proof is the same as for that theorem, but using Theorem 13.4 instead of Theorem 7.10.

Though we're developing a lot of notions for manifolds-with-boundary, the reason we'll care about them in this class is primarily to refer to homotopies of manifolds, and in particular to understand the stability of notions such as transversality, intersection numbers, etc.

[11] In this case, $\partial f = \mathrm{id}$, so every $x\in\partial X$ is a regular value of ∂f.

Lecture 14.

The Thom Transversality Theorem: 2/19/16

Recall that if X is a compact, k-dimensional manifold (without boundary), Y is n-dimensional, Z is a closed, m-dimensional submanifold of Y, and $f : X \to Y$ is smooth and transverse to Z, then $f^{-1}(Z)$ is a compact, $(k+m-n)$-dimensional submanifold of X. As a special case, if $m = n-k$, then $f^{-1}(Z)$ is a compact, 0-dimensional manifold, so it's a finite set of points. Intersection theory starts by asking how many points are in this preimage.

We'd like to do this with general smooth maps, and so we'll need to prove the following theorems.

Theorem 14.1. *If $f : X \to Y$ is smooth and Z is a closed submanifold of Y, then f is smoothly homotopic to a $g : X \to Y$ such that $g \pitchfork Z$.*

Theorem 14.2. *Let $g_0, g_1 : X \rightrightarrows Y$ be smooth maps and Z be a closed submanifold of Y. If $g_0 \sim g_1$ and both g_0 and g_1 are transverse to Z, then there's a smooth homotopy $G : [0,1] \times X \to Y$ such that $G \pitchfork Z$.*

We'll prove Theorem 14.1 in Lecture 15, and prove Theorem 14.2 in Lecture 16.

Assuming Theorem 14.2, if $G \pitchfork Z$ and $\partial G \pitchfork Z$ (which is true for the theorem hypothesis, since ∂G is just g_1 and g_2), then $\#(g_0^{-1}(z)) \equiv \#(g_1^{-1}(Z)) \bmod 2$, because $G^{-1}(Z)$ is a compact manifold-with-boundary, and its boundary is $\{0\} \times g_0^{-1}(Z) \amalg \{1\} \times g_1^{-1}(Z)$, since this is a compact, one-dimensional manifold-with-boundary, and therefore has an even number of boundary points.

Definition 14.3. If $f : X \to Y$ is smooth and Z is a closed submanifold of Y, then define the *mod 2 intersection number* of f and Z to be $I_2(f, Z) = \#(g^{-1}(Z)) \pmod 2 \in \mathbb{Z}/2$, where $f \sim g$ and $g \pitchfork Z$.

By Theorem 14.1, such a g exists, and by Theorem 14.2, this is well-defined.

This is great, and how we'll begin doing intersection theory, but we need to prove Theorems 14.1 and 14.2. This will require some putzing around, but the conclusions are nice. We'll be able to use intersection numbers to prove that two maps aren't homotopic if they have different intersection numbers,[12] thanks to Corollary 16.4. Another interesting takeaway is that we're using manifolds-with-boundary to understand facts about plain old manifolds.

Definition 14.4. Let X be a manifold-with-boundary, Y be a manifold, and Z be a closed submanifold of Y. If S is a manifold, then a *smooth family* of maps $X \to Y$ is a smooth $F : S \times X \to Y$, where $f_s(x) = F(s, x)$ is an element of the family.

This generalizes homotopy from the interval to other parameterizations of maps; if S is path-connected, then all maps in a smooth family are homotopic. We've defined it in order to have the following theorem.

Theorem 14.5 (Thom transversality theorem)**.** *Suppose $F : S \times X \to Y$ is a smooth family of maps and $Z \subset Y$ is a closed submanifold. If $F \pitchfork Z$ and $\partial F \pitchfork Z$, then for almost all $s \in S$, $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$.*

It's hard to overstate this theorem's usefulness; certainly, it's one of the most useful theorems in the entire course. Infinite-dimensional analogues appear in functional analysis.

Example 14.6. Suppose $Y = \mathbb{R}^n$; then, $F(s, x) = f(x) + s$ is a smooth family, and $\mathrm{d}F|_{(s,x)} = (I, \mathrm{d}f|_x)$. This has full rank, and so $\mathrm{Im}(\mathrm{d}F|_{(s,x)}) = T_{F(s,x)}Y$, so $F \pitchfork Z$ for any submanifold $Z \subset Y$! The same argument works on ∂X, so the conditions of Theorem 14.5 are satisfied, so one deduces that $f(x) + s$ is transverse to Z for almost all $s \in \mathbb{R}^n$, which is pretty nice.

This is an analogue to Theorem 10.2. Other uses of the Thom transversality theorem tend to also pick a huge parameter space to give the proof more wiggle room.

Proof of Theorem 14.5. Let $W = F^{-1}(Z)$, so that W is a submanifold-with-boundary of $S \times X$, and ∂W is a submanifold of $S \times \partial X$ by Theorem 13.8; in fact, it's $W \cap (S \times \partial X)$.

Let $\pi_1 : S \times X \to S$ be projection onto the first factor, so $\pi_1(s, x) = s$. This is a smooth map, so by Sard's theorem for manifolds-with-boundary (Theorem 12.8), almost every $s \in S$ is a regular value of π and $\partial \pi$.

[12]Ultimately, this is because one can refine this story to take place in homology groups, but we won't go into detail about that right now.

We would like to prove that if s is a regular value of π_1, then $f_s \pitchfork Z$, and if it's a regular value of $\partial\pi_1$, then $\partial f_s \pitchfork Z$. This suffices to prove the theorem, because almost every s is a regular value of both. However, we've run out of nice, black-box results to use, so we'll actually have to do some calculations with tangent spaces.[13]

Let's start with the proof for the interior; the argument is the same when we pass to the boundary. $\pi^{-1}(s)$ consists of points of the form (s,x), and suppose such a point is in W. Then, $F(s,x) \in Z$, so $x \in f_s^{-1}(z)$.

Let $\alpha \in T_{f_s(x)}Y$; to show that $f_s \pitchfork Z$, we need to write it as a sum of tangent vectors in Z and in $\operatorname{Im} \mathrm{d}f_s|_x$. Since $F \pitchfork Z$, then there's a $\beta \in T_{f_s(x)}Z$ and a $(\rho,\sigma) \in T_{(s,x)}(S\times X)$ such that $\alpha = \beta + \mathrm{d}F|_{(s,x)}(\rho,\sigma)$. That is, $\rho \in T_sS$, and since π_1 is a submersion, then $\rho = \mathrm{d}\pi_1(\xi)$ for some $\xi \in T_{(s,x)}(S\times X)$. That is, there's a $\gamma \in T_{(s,x)}(S\times X)$ such that $(\rho,\sigma) = \xi + (0,\gamma)$, and therefore $\mathrm{d}F|_{(s,x)}(\rho,\sigma) = \mathrm{d}F|_{(s,x)}(\xi) + \mathrm{d}F|_{(s,x)}(0,\gamma)$. And since ξ is tangent to W, then $\mathrm{d}F|_{(s,x)}(\xi) \in T_{f_s(x)}Z$. Because $\mathrm{d}F|_{(s,x)}(0,\gamma) = \mathrm{d}f_s|_x(\gamma)$, then

$$\alpha = \underbrace{\beta + \mathrm{d}F|_{(s,x)}(\xi)}_{\in T_{f_s(x)}Z} + \underbrace{\mathrm{d}f_s|_x(\gamma)}_{\in \operatorname{Im}(\mathrm{d}f_s|_x)},$$

and since this is true at all $(s,x) \in W$, then $f_s \pitchfork Z$. ⊠

This is confusing; try working out the details out yourself, especially if you want to actually understand what's going on.

Lecture 15.

The Normal Bundle and Tubular Neighborhoods: 2/22/16

"Whenever I say $\mathbb{R}^{2n}$ I think of Star Wars."

Last time, we proved Theorem 14.5, the Thom transversality theorem: if X is a manifold-with-boundary, Y is a manifold, $Z \subset Y$ is a closed submanifold, and S is a manifold, then for a smooth family of mappings $F : S\times X \to Y$, if $F \pitchfork Z$ and $\partial F \pitchfork Z$, then for almost all $s \in S$, $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ (where $f_s(x) = F(s,x)$).

Today, we'd like to begin using this theorem to prove Theorem 14.1: that every smooth map $X \to Y$ is homotopic to a map transverse to Z. If $Y = \mathbb{R}^n$ (or an open subset of it), this is easy: let S be the the unit ball in $\mathbb{R}^n$, and define $F(s,x) = f_0(x) + s$, and therefore $\mathrm{d}F|_{(s,x)} = (I, \mathrm{d}f|_x)$ is onto, as in Example 14.

Unfortunately, this is hard to do in general, unless Y is embedded in some Euclidean space. Thanks to the Whitney embedding theorem, we will be able to do this, but this is one of the few times in this class that Guillemin and Pollack's embedded approach is necessary.

Definition 15.1. Let Y be an m-dimensional submanifold of $\mathbb{R}^n$.

- The *normal space* to a $y \in Y$ is $N_yY = (T_yY)^\perp$, which is an $(n-m)$-dimensional subspace of $T_y\mathbb{R}^n$.
- The *normal bundle* is $NY = \{(y,v) \mid y \in Y, v \in N_yY\}$.

The normal bundle is an n-dimensional manifold: near any $y \in Y$, there's a chart $\phi : \mathbb{R}^n \to Y$, and $N_yY = \ker(\mathrm{d}\phi|_a)^{\mathrm{T}}$. We proved on a previous problem set that because $\mathrm{d}\phi|_a$ has full rank, there's a basis for this kernel depending continuously on a, at least in a neighborhood of a, and this continuous choice of basis defines a chart $\mathbb{R}^m \times \mathbb{R}^{n-m} \to NY$; hence, the normal bundle is really a manifold, and if you want to, you can make this parameterization explicit. By construction, NY is a submanifold of $T\mathbb{R}^n$, i.e. $\mathbb{R}^{2n}$.[14]

There's a natural map $i : NY \to \mathbb{R}^n$ sending $(x,v) \mapsto x+v$. We'd like to know what this does locally. If we force v to be small (which can be made precise by thinking about charts), then x is the nearest point on Y to $x+v$, so the image of i is a shell around Y, thickening it just a little bit.

Definition 15.2. Let $\varepsilon > 0$; then, $Y^\varepsilon = \{x \in \mathbb{R}^n \mid d(x,Y) < \varepsilon\}$ is called a *tubular neighborhood* of Y in $\mathbb{R}^n$.

We'll also use the notation $N^\varepsilon Y = \{(y,v) \in NY \mid |v| < \varepsilon\}$.

Theorem 15.3 (Tubular neighborhood theorem)**.** *If Y is compact, there's an $\varepsilon > 0$ such that $i : N^\varepsilon Y \to Y^\varepsilon$ is a diffeomorphism.*

[13] The professor referred to this part of the proof as "analysis," which seems to me an exaggeration.

[14] Something analogous can be done with *vector bundles*, which are defined by associating a vector space smoothly to every point (to be precise, there is a basis where each basis element is smooth in Y): the total space is a manifold with the same kinds of charts as above.

This also means there's a diffeomorphism $NY \to Y^\varepsilon$.

Proof sketch. By definition, $i(N^\varepsilon Y) = Y^\varepsilon$, so why is it a diffeomorphism? We need to show both that it's injective and that it's smoothly invertible. There are two issues that can arise if ε is too large.

- The first one is curvature. Consider the unit circle in $\mathbb{R}^2$ and suppose $\varepsilon > 1$; then, there's no unique closest point on the circle to the origin, so i isn't injective. The intuition is that ε needs to be smaller than the radius of curvature locally, and by compactness, there are only finitely many radii of curvature we need to worry about.
- The other problem is "necks," where distant points on Y map to close points in $\mathbb{R}^n$. In this case, i might also not be injective unless we shrink to avoid this.

We can finesse the first issue by working in local neighborhoods. In such a neighborhood, $T_{(p,0)}NY = (T_pY \times N_pY) \cong \mathbb{R}^n$, and since $i(x,v) = x + v$, then $\mathrm{d}i|_{(p,0)} = \mathrm{id}$. Hence, i is a local diffeomorphism for some ε_j, and we need only finitely many such neighborhoods, since Y is compact, so if $\varepsilon = \min_j \varepsilon_j$, then we've resolved all issues with curvature. Then, to deal with necks, we use $\varepsilon/2$ instead, which you can check works. ⊠

The tubular neighborhood theorem as stated is false for noncompact Y: one can make a "neck" get closer and closer together off to infinity. In this case, the same proof still works for the following, weaker result.

Theorem 15.4 (Tubular neighborhood theorem for noncompact manifolds)**.** *Let Y be an arbitrary submanifold of $\mathbb{R}^n$. Then, there's a smooth $\varepsilon : Y \to (0,\infty)$ such that $i : Y^{\varepsilon(y)} \to N^{\varepsilon(y)}Y$ is a diffeomorphism.*

$Y^{\varepsilon(y)}$ and $N^{\varepsilon(y)}Y$ are defined just like Y^ε and $N^\varepsilon Y$, but where $\varepsilon(y)$ depends on $y \in Y$.

The whole point of this theorem (for us) is to construct homotopies: the proof below won't work in Y, but we can give it a little more freedom in a tubular neighborhood, and then project it back down to Y.

Proof of Theorem 14.1. We're given the data of a smooth $f : X \to Y$ and a closed submanifold $Z \subset Y$; we'll need to realize Y as a submanifold of $\mathbb{R}^N$. Let $\pi : NY \twoheadrightarrow Y$ be the usual projection $(y,v) \mapsto y$; by the tubular neighborhood theorem (specifically Theorem 15.4), there's an embedding $i : NY \xrightarrow{\sim} Y^{\varepsilon(y)} \hookrightarrow \mathbb{R}^N$ onto neighborhood of Y.

Let B be the open unit ball in $\mathbb{R}^N$ and define $F : B \times X \to Y$ by $F(s,x) = \pi \circ i^{-1}(f(x) + \varepsilon(f(x))s)$, so that F is a smooth family of maps $B \times X \to Y$, $F(0,x) = f(x)$, and $\mathrm{d}F = \mathrm{d}\pi \circ (\varepsilon(f(x))I, M)$ for some matrix M. The point is that $\mathrm{d}F$ is everywhere a composition of two surjective maps, so F is a submersion, and $F \pitchfork Z$ automatically! Thus, by Theorem 14.5, $f_s(x) = F(s,x)$ is transverse to Z for almost all $s \in B$, so we can pick such an s and a path from 0 to s in B, which defines a homotopy of maps $X \to Y$. ⊠

We're almost done with our construction of the foundations of unoriented intersection theory.

Lecture 16.

The Extension Theorem: 2/24/16

Let's recall the path that we've taken in the last few lectures. Two lectures ago, we proved the Thom transversality theorem, Theorem 14.5: if X is a manifold-with-boundary, Y is a manifold, and Z is a closed submanifold of Y, then if S is a manifold and $F : S \times X \to Y$ is a smooth family of mappings such that $F \pitchfork Z$ and $\partial F \pitchfork Z$, then for almost all $s \in S$, $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ (where $f_s(x) = F(s,x)$).

We then used this to prove Theorem 14.1, that for every smooth $f : X \to Y$, there's a smooth $g : X \to Y$ homotopic to f that's transverse to Z (and $\partial g \pitchfork Z$). We proved this using the tubular neighborhood theorem for noncompact manifolds, Theorem 15.4.

The next step is to generalize this to an extension result: if we already know f is transverse to Z on a submanifold, can we make g agree with f there?

Definition 16.1. Let X, Y, and Z be as before, and $C \subseteq X$ be a closed subset. Then, f is *transverse to Z on C*, written $f \pitchfork Z$ on C, if for all $x \in C \cap f^{-1}(Z)$, $\mathrm{Im}(\mathrm{d}f|_x) + T_{f(x)}Z = T_{f(x)}Y$.

This is different than just restricting f to C, because even if C is a submanifold, we're considering $\mathrm{d}f|_x(T_xX)$, not $\mathrm{d}f|_x(T_xC)$, which is generally smaller.

Theorem 16.2 (Extension)**.** *Let X be a manifold-with-boundary, $C \subseteq X$ be closed, and $f : X \to Y$ be a smooth map such that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $\partial X \cap C$. Then, there's a smooth $g : X \to Y$ homotopic to f such that $g \pitchfork Z$, $\partial g \pitchfork Z$, and $g|_C = f|_C$.*

Theorem 14.5 is a special case of this, where $C = \varnothing$.

The key application of Theorem 16.2 is that two homotopic maps which are both transverse to Z can be realized through a homotopy transverse to Z, which is Theorem 14.2.

Proof of Theorem 14.2. Apply Theorem 16.2 to $X = [0,1] \times M$ and $C = \partial X$. ⊠

Proof of Theorem 16.2. Let's start with an $x \in C$, so $\operatorname{Im}(\mathrm{d}f|_x) + T_{f(x)}Z = T_{f(x)}Y$, or $x \notin f^{-1}(Z)$. In either case, we can extend to a neighborhood of x: if $x \notin f^{-1}(Z)$, then since Z is closed, so is $f^{-1}(Z)$, and so there's an open neighborhood of x not in $f^{-1}(Z)$. If $x \in f^{-1}(Z)$, then we use the fact that transversality is stable: it can be expressed as the condition that the matrix $\mathrm{d}f|_x$ has full rank, which is an open condition.

Applying this to every $x \in C$, we have an open neighborhood U containing C such that $f \pitchfork Z$ on U. We have an open cover of X given by $\mathfrak{U} = \{U, C^c\}$, so there exists a partition of unity $\{\theta_1, \theta_2\}$ subordinate to $\mathfrak{U}$, where θ_1 is supported in U and θ_2 is supported in C^c.

Using the Whitney embedding theorem, we can embed $Y \hookrightarrow \mathbb{R}^N$ for some large N. Let $\varepsilon : Y \to (0, \infty)$ be such that the tubular neighborhood $Y^{\varepsilon(y)} = \{(y, v) \in NY : |v| < \varepsilon(y)\}$ is diffeomorphic to NY, and let $\pi : Y^{\varepsilon(y)} \to Y$ be projection back onto Y. Then, define $F : [0,1] \times X \to Y$ by $F(t,x) = \pi(f(x) + \theta_2(x)\varepsilon(f(x))t)$. Now, if $x \in C$, $F(s,x) = f(x)$, because $\theta_2|_C = 0$. And just as in the proof of Theorem 14.1, $F \pitchfork Z$ and $\partial F \pitchfork Z$ (since the derivatives have the same rank), so we can choose $g = f_s$ for almost any s. ⊠

Anyways, we were proving all of these technical theorems for the purpose of intersection theory, right? Let's recall the setup from a few lectures ago: we have a compact manifold X, an arbitrary manifold Y, and a closed submanifold $Z \subset Y$, where $\dim X + \dim Z = \dim Y$. Suppose $f : X \to Z$; we want to understand the intersection $\operatorname{Im}(f) \cap Z$.

In the case where $f \pitchfork Z$, we defined the mod 2 intersection number $I_2(f, Z) = \#(f^{-1}(Z)) \bmod 2$. It's not obvious why this is finite, but since $f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold of X with codimension $\operatorname{codim}_Y Z = \dim X$, so it's a 0-dimensional submanifold of a compact manifold, meaning it's a finite set of points.

If f isn't transverse to Z, then we defined the mod 2 intersection number by choosing a g homotopic to f and such that $g \pitchfork Z$, which we can do by Theorem 14.1. Then, $I_2(f, Z) = I_2(g, Z)$. However, we need to show that this is independent of the choice of g.

Proposition 16.3. *Let X be a compact manifold, Y be an arbitrary manifold, and $Z \subset Y$ be a closed submanifold such that $\dim X + \dim Z = \dim Y$. If $g_0, g_1 : X \to Y$ are two smooth functions such that $g_0 \pitchfork Z$, $g_1 \pitchfork Z$, and $g_0 \sim g_1$, then $I_2(g_0, Z) = I_2(g_1, Z)$.*

Proof. By Theorem 14.2, there's a $G : [0,1] \times X \to Y$ such that $G \pitchfork Z$ and $\partial G \pitchfork Z$, so $G^{-1}(Z)$ is a compact 1-manifold-with-boundary. By Corollary 13.3, it must have an even number of boundary points, but the boundary points are just $g_0^{-1}(Z) \amalg g_1^{-1}(Z)$, meaning that $\#(g_0^{-1}(Z)) \equiv \#(g_1^{-1}(Z)) \bmod 2$. ⊠

Corollary 16.4. *$I_2(f, Z)$ is a homotopy invariant of f: if $f \sim g$, then $I_2(f, Z) = I_2(g, Z)$.*

This is true even when f isn't transverse to Z, since for the purposes of intersection number we can replace it by something that is transverse.

Example 16.5. Let $f : S^1 \to \mathbb{R}^2 \setminus 0$, which is some loop in the plane that avoids the origin. Let Z be a ray in any particular direction; then, what is $I_2(f; Z)$? This isn't technically the number of intersections; it's the number of intersections where we adjust degenerate intersections to obtain transversality.

In this case, $I_2(f, Z)$ is the winding number mod 2, keeping track of whether the loop winds an even number of times around the origin or an odd number. Unfortunately, this isn't enough information to determine the difference between a path which winds around 0 times (and is homotopic to a constant map) and one which winds around twice (which is not null-homotopic), but it's not nothing. Later on, we will define an intersection number valued in $\mathbb{Z}$ rather than in $\mathbb{Z}/2$, which will distinguish these two cases.

Lecture 17.

Intersection Theory: 2/26/16

Throughout this lecture, X will denote a compact manifold, Y an arbitrary manifold, and Z a closed submanifold of Y such that $\dim X + \dim Z = \dim Y$. We let $f : X \to Y$ be a smooth map.

Recall that we've defined the mod 2 intersection number $I_2(f, Z)$ to be $\#(g^{-1}(Z)) \bmod 2$, where $g \sim f$ and $g \pitchfork Z$. We've proven that such a g exists for all f (Theorem 14.1), and that $I_2(f, Z)$ is independent of our choice of g (Theorem 16.3).

We can also prove the following result.[15]

Theorem 17.1. *Suppose $X = \partial W$ for a compact W and $f : X \to Y$ extends to a smooth $F : W \to Y$. Then, $I_2(f, Z) = 0$.*

Proof. Let $G : W \to Y$ be homotopic to F such that $G \pitchfork Z$ and $\partial G \pitchfork Z$; let $g = \partial G$. The homotopy $G \sim F$ induces a homotopy $g \sim f$, so by Corollary 16.4, $I_2(f, Z) = I_2(g, Z)$. But $I_2(g, Z) = \#\partial(G^{-1}(Z))$, and $G^{-1}(Z)$ is a compact one-manifold with boundary, so it has an even number of boundary points, and thus $I_2(g, Z) = 0$. ⊠

Compactness of W is necessary in this theorem; for example, choose any X, f, and Z such that $I_2(f, Z) = 1$ and let $W = X \times [0, \infty)$. For example, one could let X be the unit circle in $Y = \mathbb{R}^2 \setminus 0$, f be the inclusion map, and Z be the positive x-axis. Then, $F(x, t) = f(x)$ is a smooth extension of f on W, but it doesn't satisfy the conclusion of Theorem 17.1. Some intuition for this theorem is that if X is the boundary of a disc, then it's null-homotopic; this is just one specific instance, but might be illuminating.

Sometimes, the "big guns" of fundamental group or homology class can be useful to get intuition about this: for example, consider two circles in a torus T, one around the center hole and one "perpendicular" to it, around a slice of it. Since the first circle is nontrivial in $H_1(T)$, we know it doesn't extend to a boundary. In particular, we have the following corollary.[16]

Corollary 17.2. *Suppose $I_2(f; Z) \neq 0$; then, f doesn't extend smoothly on any compact manifold that X bounds.*

Winding Number. We can use this to define the winding number of a function. Suppose $f : S^1 \to \mathbb{C} \setminus p$ is smooth, so the induced map $\widehat{f}_p : S^1 \to S^1$ defined by $\widehat{f}_p(x) = (f(x) - p)/|f(x) - p|$ is smooth. We'd like to formalize the intuition that the "number of times that f wraps around S^1" is homotopy-invariant. For example, the identity map winds around once, and $\theta \mapsto 2\theta$ wraps around twice.

Definition 17.3. Suppose X and Y are manifolds with the same dimension, Y is connected, and $y_0 \in Y$. Then, the *mod 2 degree* of a smooth $f : X \to Y$ is $\deg_2 f = I_2(f, \{y_0\})$.

Because $\{y_0\}$ is a 0-dimensional manifold, $\dim X + \dim(\{y_0\}) = \dim Y$, so we can take this mod 2 intersection number. Eventually, we'll do oriented intersection theory and everything will be over $\mathbb{Z}$ instead of $\mathbb{Z}/2$.

Theorem 17.4. *The mod 2 degree doesn't depend on one's choice of y_0.*

The book's proof uses the stack of records theorem (which was on our homework); we'll supply a different one.

Proof. Let $y_1 \in Y$ be a different point; since Y is a connected manifold and therefore path-connected, there's a path $p : [0, 1] \to Y$ such that $p(0) = y_0$ and $p(1) = y_1$. We can replace f with a $g : X \to Y$ such that $g \sim f$ and $g \pitchfork \mathrm{Im}(p)$ and $g \pitchfork \partial\, \mathrm{Im}(p)$. Then, $I_2(f, y_i) = I_2(g, y_i)$ and $g^{-1}(p)$ is a 1-dimensional manifold-with-boundary. Since $g^{-1}(p) \subset X$, then it's compact, and a compact 1-dimensional manifold-with-boundary has an even number of boundary points, by Corollary 13.3. But its boundary is just the points in $g^{-1}(y_0)$ and $g^{-1}(y_1)$, and therefore they have the same number of points. ⊠

Definition 17.5. Suppose X is an n-dimensional manifold and $f : X \to \mathbb{R}^{n+1} \setminus p$ is smooth. Then, we let $\widehat{f}_p = (f(x) - p)/|f(x) - p|$, which is a smooth map $X \to S^n$. Define the *winding number* of f and p to be $W_2(f, p) = \deg_2(\widehat{f}_p)$.

[15] If you're also thinking about the intersection form in mod 2 homology, this has a very homological interpretation, since the intersection form has to vanish on boundaries.

[16] Yes, this is just the contrapositive of Theorem 17.1. I'm sorry too.

Since this is defined as an intersection number, it's immediately homotopy-invariant. And in the case $n = 1$, this recovers the notion of actually winding around the origin.

This has a neat consequence.

Theorem 17.6. *Let $p(z) \in \mathbb{C}[z]$ be a polynomial of odd degree; then, p has a root.*

This is part of the fundamental theorem of algebra. Maybe it seems like a roundabout way to do this, but the proof immediately generalizes to polynomials plus bounded functions. Once we get to oriented intersection theory, we'll be able to distinguish the situation of 2 roots and 0 roots and prove the rest of the fundamental theorem of algebra.

Proof. The idea of the proof is that if $d = \deg f$, then for sufficiently large z, the z^d term dominates $f(z)$ and we can replace f with $g(z) = z^d$ to calculate that $W_2(f, 0) = 1$. Then, we'll show that if f has no roots, then $W_2(f, 0) = 0$.

Since the z^d term dominates all other polynomial terms, there's some large circle $C \subset \mathbb{C}$ centered at the origin such that all of the roots of f (if any exist) are strictly inside C, so the map $\widehat{f} : C \to S^1$ sending $z \mapsto f(z)/|f(z)|$ is well-defined and smooth. Let $g : C \to S^1$ send $z \mapsto z^d/|z^d|$, and let $F(t, z) = z^d + (1-t)f_*(z)$, where $f_*(z)$ is the terms of f that have degree less than d. Then, $\widehat{F}(t, z) = F(t, z)/|F(t, z)|$ is a smooth homotopy $\widehat{f} \sim g$, so $\deg_2(\widehat{f}) = \deg_2(g)$. A nonzero $p \in \mathbb{C}$ has d preimages under g, so $\deg_2(\widehat{f}) = \deg_2(g) = 1$.

Suppose f has no roots; then we can extend $\widehat{f}(z) = f(z)/|f(z)|$ to the interior of C. Thus, by Theorem 17.1, $I_2(\widehat{f}, 0) = 0$, so $\deg_2(\widehat{f}) = 0$, which is a contradiction. ⊠

This proof strategy depends only on the asymptotic behavior of f, so we have the following corollary.

Corollary 17.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be $f(z) = z^d + O(|z^{d-1}|)$, where d is odd; then, f has a root.*

Does this seem silly? Perhaps, but it's still impressive just how much mathematics can be done with this tangible, concrete differential topology.

Just as with transversality, we can recast intersection theory for two submanifolds, rather than submanifolds and maps.

Definition 17.8. Suppose $X \subset Y$ as a submanifold; then, define the *mod* 2 *intersection number* of X and Z to be $I_2(X, Z) = I_2(i_X, Z)$, where $i_X : X \hookrightarrow Y$ is the inclusion.

This allows us to define the mod 2 intersection number of X with itself, if $\dim X = (1/2)\dim Y$; this seems counterintuitive, but the point is we have to take a small homotopy of X and intersect that with X. For example, if $Y = \mathbb{RP}^2$ and $X = \mathbb{RP}^1$, then $I_2(X, X) = 1$: the way it's wrapped around inside $\mathbb{RP}^2$ (as antipodal points are identified), a small perturbation must intersect X once (mod 2): you can end up with 1 point, or 3, or … On the other hand, consider a circle in the torus; you can push it a little ways off, and then it doesn't intersect itself at all.

These self-intersection numbers are telling you something interesting about how the circles embed in tori versus in $\mathbb{RP}^2$.

If X is an arbitrary compact manifold, it embeds into $X \times X$ as the diagonal Δ; we can ask what $I_2(\Delta, \Delta)$ is. This, we will learn, is the Euler characteristic mod 2 (and when we learn oriented intersection theory, if X is an oriented manifold, we can recover the entire Euler characteristic). So there's an awful lot of topology that can be recovered with this intersection theory. Next lecture, we'll cover two such applications: the Jordan curve theorem, and the Borsuk-Ulam theorem.

Theorem 17.9 (smooth Jordan curve theorem)**.** *Let $p : S^1 \to \mathbb{R}^2$ be a smooth embedding. Then, $\mathbb{R}^2 \setminus p(S^1)$ has two components (an "inside" and an "outside").*

We'll use winding numbers to generalize this to all $(n-1)$-manifolds in $\mathbb{R}^n$. Then, we'll use this to prove the Borsuk-Ulam theorem, Theorem 18.4.

Again, it's surprising how much we can recover, though only in the smooth case and only mod 2; soon, we will redo much of this from an oriented perspective and in $\mathbb{Z}$.

Lecture 18.

The Jordan Curve Theorem and the Borsuk-Ulam Theorem: 2/29/16

We've been immersed in a story about intersection theory and winding numbers mod 2. If X is a compact manifold, Y is any manifold, and Z is a closed submanifold of Y such that $\dim X + \dim Z = \dim Y$, then we can talk about the mod 2 intersection number $I_2(f, Z)$ for a smooth $f : X \to Y$, which we defined as the number of points in $\widetilde{f}^{-1}(Z)$ mod 2, where $\widetilde{f} \sim f$ and is transverse to Z; we've proven that such an $\widetilde{f}$ necessarily exists, and that the intersection number is independent of such a choice of f.

If $\dim X = \dim Y$ and Y is connected, we can define the degree as $\deg_2 f = I_2(f, \mathrm{pt})$, which we showed is independent of the choice of point in Y (Theorem 17.4), and used this to define the winding number mod 2 $W_2(f, p) = \deg_2 \widetilde{f}_p$, where $p \notin \operatorname{Im}(f)$ and $\widetilde{f}_p : X \to S^k$ is defined by $x \mapsto (f(x) - p)/|f(x) - p|$.

We proved this with an extension theorem, Theorem 17.1: if $X = \partial W$, for W compact, and $f : X \to Y$ extends to a smooth map on W, then $I_2(f, Z) = 0$. In particular, if $Y = \mathbb{R}^{k+1}$ and X is k-dimensional, then in this situation, if $F : W \to \mathbb{R}^{k+1}$ is the extension to a compact W (with $\partial W = X$, as before), then if $W_2(f, p) \neq 0$, then $F^{-1}(p) \neq \varnothing$ (since it has 1 mod 2 elements).

We used this to prove part of the fundamental theorem of algebra, but it's considerably more general. For example, we generalized it to Corollary 17.7; here's another direction it could go in.

Definition 18.1. If X is a compact manifold, Y is a manifold of the same dimension as X, $p \in Y$, and $f : X \to Y$ is smooth, to calculate $\#(f^{-1}(p))$ *with multiplicity* means to choose a $g : X \to Y$ homotopic to f and transverse to $\{p\}$, and calculate $\#(g^{-1}(p))$.

For now, this is only defined mod 2; in oriented intersection theory, we'll count with sign. In any case, this agrees with the notion of the multiplicity of a polynomial.

Theorem 18.2. *Let X be a compact k-dimensional manifold such that $X = \partial W$, where W is compact. Suppose a smooth $f : X \to \mathbb{R}^{k+1}$ extends to a smooth $F : W \to \mathbb{R}^{k+1}$ and p is a regular value of F not in* $\operatorname{Im}(f)$*. Then,* $\#(F^{-1}(p))$ mod 2*, counted with multiplicity, is equal to $W_2(f, p)$.*

The point is that we'd like to understand the roots of F, at least mod 2, if Y has some sort of zero.

Proof. Without loss of generality, we can homotope F to make it transverse to $\{p\}$, so now we're just counting roots.

Since p is a regular value for F and $\dim(W) = \dim(\mathbb{R}^{k+1})$, then F is a local diffeomorphism at each root. Thus, we can choose a small ball B_{x_i} around each root x_i such that F is a local diffeomorphism on B_{x_i}, and such that $F(B_{x_i})$ doesn't intersect $\partial(\operatorname{Im}(F)) = \operatorname{Im}(f)$. Let $\widehat{W} = \operatorname{Im}(F) \setminus \bigcup_i B_{x_i}$, i.e. $\operatorname{Im}(F)$ with these balls taken out. Thus, $\partial \widehat{W}$ is the disjoint union of $\operatorname{Im}(f)$ and the balls around each root (since we can shrink them if necessary to not intersect $\operatorname{Im}(f)$, since $p \notin \operatorname{Im}(f)$). We can calculate the winding number around each root

Hence, we can directly calculate that around each root r_i, $W_2(f, r_i) = 1$, so adding them up, $W_2(f, p)$ is the number of roots of F, at least mod 2. ⊠

The cooler thing we'll do today is prove the Jordan curve theorem, which says that many kinds of smooth embeddings divide the plane into an inside and an outside. This is a generalization of Theorem 17.9

Theorem 18.3 (Jordan curve theorem)**.** *Let X be a compact, connected, nonempty k-dimensional manifold and $f : X \hookrightarrow \mathbb{R}^{k+1}$ be an embedding, Then, $\mathbb{R}^{k+1} \setminus X$ has precisely two path components.*

Proof. First, we'd like to find p and q that are on "opposite sides" of X. By the tubular neighborhood theorem, there exists a neighborhood U of X in $\mathbb{R}^{k+1}$ is diffeomorphic to NX, so for some $x \in X$, we can pick $p, q \in \mathbb{R}^{k+1}$ that are in opposite path components of $T_{f(x)}\mathbb{R}^{k+1} \setminus T_{f(x)} \operatorname{Im}(f)$.

Let V be a chart for X and W denote the portion of U that covers V_1 under the projection $NX \twoheadrightarrow X$ sending $(x, v) \mapsto x$.

For any $x \in \mathbb{R}^{k+1} \setminus X$, we would like to find a path from x to W that does not intersect X. Since W is the normal bundle over a chart for X and X has codimension 1, $W \setminus X$ has two path components; thus, since we'll construct a path connecting $x \in X$ to some point in $W \setminus X$, X also has at most two path components.

If $x \in U$, then regarding the tubular neighborhood as NX, $x = (y_0, v)$ for some $y_0 \in X$ and $v \in N_{y_0} X \setminus 0$. Let $y_1 \in V$; since X is a connected, it's path-connected, so there is a smooth path $\gamma[0,1] \to X$ joining y_0 and y_1. Then, the path $t \mapsto (\gamma(t), v)$ joins x to $(y_1, v) \in W$, and doesn't intersect X. If $x \notin U$, then there's a straight line from x to some $y \in X$, but since U is a neighborhood of X, then that line must pass through U before hitting y, so cut it off at some $z \in U$ and then use the previous argument to connect z to a point in W. Thus, every $x \in X \setminus U$ can be connected in $X \setminus U$ to $W \setminus U$, so $X \setminus U$ has at most two path components.

Next, we'll show there are at least two path components. Recall that U is a tubular neighborhood of X, and let γ be a smooth path in U joining p and q. Since p and q are in different path components of $U \setminus X$, then $I_2(\gamma, X) = 1$: there are two path components of $U \setminus X$, so γ must cross X an odd number of times. Now, let γ' be any other smooth path from p to q in $\mathbb{R}^n$ — then, $\gamma \sim \gamma'$ through the smooth homotopy $F : [0,1] \times [0,1] \to \mathbb{R}^{k+1}$ sending $(t,s) \mapsto t\gamma'(s) + (1-t)\gamma(s)$. Thus, $I_2(\gamma', X) = 1$, by the homotopy-invariance of winding number. As such, p and q must live in different path components: if they didn't, there would be a path γ' connecting them in $\mathbb{R}^{k+1} \setminus X$, but that would imply $I_2(\gamma', X) = 0$.[17] ⊠

The next interesting corollary of this winding number theory is the Borsuk-Ulam theorem.

Theorem 18.4 (Borsuk-Ulam). *Let $f : S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ be an* odd *smooth map (i.e. $f(-x) = -f(x)$). Then, $W_2(f, 0) = 1$.*

Proof. We'll induct on k. First, suppose $k = 1$ and let $U \subset S^1$ be the upper semicircle (as a closed submanifold-with-boundary, so it contains $(1,0)$ and $(-1,0)$); let R be any ray starting at the origin not pointing in the direction of $f(\pm 1, 0)$, so that $W_2(f, 0) = \#(\mathrm{Im}(f) \cap R)$ (assuming transversality as usual). Let $-R$ be the ray starting at 0 and going in the opposite direction, and $\ell = R \cup -R$ be the line defined by these rays.

Since $f(1,0) = -f(-1,0)$, the path $f(U)$ must cross ℓ an odd number of times (after all but one of the crossings, they would have to be in the same component of $\mathbb{R}^2 \setminus \ell$). But since f is odd, this 1/2 the number of times that all of $\mathrm{Im}(f)$ crosses ℓ, and therefore the number of times that $\mathrm{Im}(f)$ crosses R, which is $W_2(f, 0)$.

For the inductive step, we have the usual inclusion $S^{k-1} \hookrightarrow S^k$ as the equator. We'll use Theorem 18.2, and by Sard's theorem, we can find a ray that doesn't intersect $\mathrm{Im}(f|_{S^{k-1}})$ and repeat the same argument for the upper and lower hemispheres of S^k.

TODO what happened? :(⊠

Corollary 18.5. *There are two antipodal points on the Earth that currently have the same temperature and barometric pressure.*

In this case, the odd function is the difference in the temperature and pressure here and the temperature and pressure at the antipodal point. Obviously this proof doesn't come with a construction.

Lecture 19.

Getting Oriented: 3/2/16

We're going to move to a different chapter today, so let's review where we came from and where we're going. In chapter 2 of the textbook, we introduced and developed some powerful machinery:

- manifolds-with-boundary;
- the classification of 1-manifolds with or without boundary;
- homotopy;
- the Thom Transversality theorem (Theorem 14.5);
- normal bundles and the tubular neighborhood theorem; and
- the intersection number mod 2, the mod 2 degree, and the mod 2 winding number.

This is a lot of machinery, and we've proven only four results that aren't technical theorems about our machinery: the Brouwer fixed-point theorem, the Jordan curve theorem (Theorem 18.3), half of the fundamental theorem of algebra, and the Borsuk-Ulam theorem (Theorem 18.4).

It would be nice to get more out of these technical tools. In the next chapter, we're going to introduce one big piece of machinery, orientation, and use it to define intersection numbers, degrees, and winding numbers valued in $\mathbb{Z}$. This will allow us to extract more results.

- We'll define the Euler characteristic and learn some things about it;
- Lefschetz fixed-point theory;
- the full fundamental theorem of algebra; and
- some results on vector fields (e.g. there's no nonvanishing vector field on S^2).

[17] Another approach is to use the winding numbers of f around a ray starting at p and starting at q; they must differ by 1, so p and q must be in different path components.

To do this, we need to know what intersection theory means in $\mathbb{Z}$ rather than in $\mathbb{Z}/2$. Imagine a loop in $\mathbb{R}^2 \setminus \{0\}$, which is a smooth map $S^1 \to \mathbb{R}^2 \setminus 0$. One such loop winds around twice, clockwise; another winds around twice, counterclockwise; and a third is a constant map. With the intersection theory we've developed so far, we can't tell any of these apart, since their intersection numbers are all 0 mod 2. The goal is to be able to define an intersection number "with sign," tracking the direction as well as the number. We would also like to do this in a way such that the intersection number is homotopy-invariant, and this is where the "with sign" is important: with a homotopy, you can distort a curve to add two more intersections with, say, the x-axis, but one will be going "upward" and the other "downward," which the signed intersection number will cancel out.

All of this requires a notion of what direction one travels on a curve. More generally, this comes from the idea of an orientation on a manifold. Thus, for the next several lectures, we have three goals.

(1) Define orientation on manifolds.
(2) Define the sign of an intersection point, to define $I(f, Z)$.
(3) In order to do this, we'll need to induce orientations: if $F : X \to Y$ is a smooth map, where Y is a manifold, X is a manifold-with-boundary, and Z is a closed submanifold, then if $F \pitchfork Z$ and $\partial F \pitchfork Z$, there is an induced orientation on $F^{-1}(Z)$.
(4) We'll need to relate $\partial(F^{-1}(Z))$ to $(\partial F)^{-1}(Z)$. Assuming some transversality, these are the same as unoriented manifolds, but their orientations may be different.

Part (3) is often the hardest, simply because there are many opportunities for sign errors. (4) is also somewhat of a headache. But the results we can obtain using oriented intersection theory are much more powerful, and justify the more confusing introduction of orientation. So then why do mod 2 intersection theory at all? Not every manifold is orientable, so in some cases that's all we can do. And seeing the simpler case first is also illustrative.

Most of the concepts we defined for manifolds in this class were initially constructed in linear-algebraic terms on $\mathbb{R}^n$, and then transferred to manifolds using coordinate charts. Orientation will be no different.

Let V be an n-dimensional real vector space and $\mathscr{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be a basis for V.[18] If $\mathscr{B}' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ is another basis for V, then the change-of-basis map $L : \mathbf{b}_i \mapsto \mathbf{b}'_i$ is an invertible linear map whose determinant is nonzero. There are two options: it's negative or it's positive.

Let's define an equivalence relation on bases, where $\mathscr{B} \sim \mathscr{B}'$ if $\det(L) > 0$. This is indeed an equivalence relation (the change-of-basis matrix from $\mathscr{B}$ to itself is I_n, and matrices with positive determinant are closed under taking inverses and multiplication), and there are two equivalence classes.

Definition 19.1. An *orientation* on V is a choice of an equivalence class of bases. A basis in the chosen equivalence class is called *positively oriented*, and one in the other equivalence class is called *negatively oriented*.

In some sense, we consider some bases to be normal, and the others to be flipped.

On $\mathbb{R}^n$, then *standard basis* is the class of $(e_1, \dots, e_n)$. Thus, for $\mathbb{R}^1$, a positively oriented basis is a choice of a positive number, and a negatively oriented basis is a choice of a negative number.

On $\mathbb{R}^2$, we can compute what orientation class we're in through the cross product: $\mathbf{b}_1 \times \mathbf{b}_2$ is positive iff $(\mathbf{b}_1, \mathbf{b}_2)$ is positively oriented relative to the standard basis. This is the content of the right-hand rule. Similarly, on $\mathbb{R}^3$, we can use the triple product to compute whether something is positively oriented (since this is exactly the determinant of the change-of-basis matrix).

Suppose V_1 and V_2 are vector spaces and $W = V_1 \oplus V_2$. If $(\mathbf{b}_1, \dots, \mathbf{b}_m)$ is a basis for V_1 and $(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is a basis for V_2, we can define a basis for W as $(\mathbf{b}_1, \dots, \mathbf{b}_m, \mathbf{c}_1, \dots, \mathbf{c}_n)$. This respects the equivalence relation we defined, so a choice of orientations on V_1 and V_2 induces a choice of orientation on W.

As oriented vector spaces, $V_1 \oplus V_2$ is not necessarily equal to $V_2 \oplus V_1$. To switch the basis into $(\mathbf{c}_1, \dots, \mathbf{c}_n, \mathbf{b}_1, \dots, \mathbf{b}_m)$, we need to make mn transpositions (of adjacent elements), and each transposition flips the sign. Thus, $V_1 \oplus V_2 \cong V_2 \oplus V_1$ iff mn is even.

Similarly, when $W = V_1 \oplus V_2$, given an orientation of W and of V_1, there's a unique orientation of V_2 which gives W the direct-sum orientation. If you know any two out of the three orientations, the third is defined.

Finally, none of this really makes sense when V is 0-dimensional. A choice of orientation on a zero-dimensional vector space is just a formal choice of + or −: either an empty basis is oriented, or it isn't, and it's hard to define this much more geometrically. This is actually useful: for example, if $V_2 = W$ as vector spaces, but with opposite orientation, we can regard $W = 0 \oplus V_2$ as oriented vector spaces, where 0 has the negative orientation.

[18] Bases are always ordered.

Orientation on Manifolds. We'd like orientation on manifolds to mean a consistent choice of orientation on all tangent spaces. This means that in a coordinate neighborhood, the orientations all agree, in the sense that the chart is orientation-preserving.

If X is a manifold and $x \in X$, then $(\mathrm{d}\phi|_{\phi^{-1}(x)}(e_1), \dots, \mathrm{d}\phi|_{\phi^{-1}(x)}(e_n))$ is an oriented basis of T_xX, and makes the chart orientation-preserving: if two charts induce the same orientation on T_xX, then their change-of-charts map is an orientation-preserving map on vector spaces.

Definition 19.2. Let X be a manifold.

- An *oriented atlas* is an atlas (ϕ_i, U_i) for X such that all change-of-charts maps are orientation-preserving: $\det(\mathrm{d}(\phi_i^{-1} \circ \phi_j)) > 0$ everywhere.
- An *orientation* for X is an equivalence class of choices of oriented atlases (equivalently, a maximal oriented atlas), with $\mathscr{A}_1 \sim \mathscr{A}_2$ if the change-of-charts maps are all orientation-preserving.
- A manifold that admits an orientation is called *orientable*; a manifold along with a choice of orientation is called *oriented*.

The last point is subtle: the former means one could choose an orientation, and the latter means we already have.

Eventually, we'll use this to define signed intersection number: if X, Y, and Z are oriented manifolds, where X is compact, Z is a submanifold of Y, and $\dim X + \dim Y = \dim Z$, let $f : X \to Y$ be transverse to Z. We'll define the sign of a point $x \in f^{-1}(Z)$ to be the orientation of $\mathrm{d}f|_x$ that makes $T_zY = \mathrm{Im}(\mathrm{d}f|_x) \oplus T_zZ$ correct as oriented vector spaces.

Lecture 20.

Orientations on Manifolds: 3/4/16

"Replace that proof with a handwave, which I will be generous and not put on your homework."

Recall that a manifold is oriented if we can continuously orient each tangent space. An equivalent way to phrase this is that we can pick charts (U_i, ϕ_i) such that the change-of-charts maps $\phi_j \circ \phi_i^{-1}$ all have positive determinant.

Now, how many orientations can a manifold have? Sometimes, the answer is none: the Möbius strip is everyone favorite example of a manifold that has no orientation at all. Like a vector space, we could pick two orientations on a manifold, but if the manifold isn't connected, then we would have more.

Theorem 20.1. *If X is a manifold, then any two orientations agree on an open set and disagree on an open set.*

Corollary 20.2. *If X is a nonempty, connected manifold, then it has exactly two orientations.*

Proof. By Theorem 20.1, he sets where they agree and the sets where they disagree are both clopen sets, and on a connected manifold, the only clopen sets are the empty set and the whole space. ⊠

Proof of Theorem 20.1. Let $x \in X$ and let φ be a chart for a neighborhood of x. We can consider the change-of-charts map g from φ to itself, starting with the first orientation and ending with the second. If these orientations agree, then $\det(\mathrm{d}g|_{\varphi^{-1}(x)})$ is positive, and therefore positive in a neighborhood of x, and so the orientations agree in a neighborhood of x. And if these orientations degree, then $\det(\mathrm{d}g|_{\varphi^{-1}(x)})$ is negative, and therefore negative in a neighborhood of x, and so the orientations disagree in a neighborhood of x. ⊠

We can use this to learn more about 2-manifolds.

Theorem 20.3. *A 2-manifold X is nonorientable iff it contains a Möbius strip.*

We'd like to prove this without using the classification of 2-manifolds, since that's a bit too high-powered.

Proof. Suppose X contains a Möbius strip M. Then, any orientation of X restricts to an orientation of M (since they're the same dimension), but that's not possible, so X isn't orientable.

Conversely, suppose that X is nonorientable. Then, it must have a nonorientable disc inside it; consider its tubular neighborhood. Since the disc is nonorientable, then the normal bundle must be glued to itself with a twist, and therefore is a Möbius strip. ⊠

In general, the same proof for an n-manifold shows that a nonorientable loop's tubular neighborhood is glued in the same way.

Theorem 20.4. *An n-manifold is nonorientable iff it contains a $[0,1]\times D^{n-1}/\sim$, where $\sim$ glues the edges together in an orientation-reversing way.*

The key trick here is that if L is a loop inside a manifold, its normal bundle is obtained by gluing $[0,1]\times D^{n-1}$ across its boundary, which makes sense: if we remove a point of the loop, the normal bundle can be straightened out.

Awesome. What about manifolds-with-boundary? Most of this carries through, but we need to also orient the tangent space at the boundary. This is scarcely different: at each half-space in $T(\partial X)$, we're extending to a neighborhood in the full space, so the tangent space at the boundary has full dimension. Thus, if we have oriented charts as normal, we have an orientation of the tangent space at the boundary too, and it's consistent with the orientation on the interior.

The next thing we'd like to prove is an analogue of Theorem 12.6: the boundary of an oriented manifold-with-boundary is not just a manifold, but has an orientation.

Theorem 20.5. *If X is an orientable manifold-with-boundary, then an orientation of X induces an orientation of ∂X.*

Proof. Let's pick an orientation of X as a choice of orientation of T_xX for each $x\in X$. Now, at an $x\in\partial X$, there's an *outward normal* n pointing away from the interior in $N_x(\partial X)$. Hence, if $\mathscr{B}=(b_1,\dots,b_{k-1})$ is a basis for $T_x\partial X$, then we declare it to be positively oriented iff $(n,b_1,\dots,b_{k-1})$ is positively oriented in T_xX; hence (and there's a little more to check here), this defines a consistent orientation on ∂X. ⊠

See Figure 6 for what the induced orientation looks like in practice. There is a convention here: if we placed the normal vector last, we would get the opposite convention, but we chose this one to get the counterclockwise orientation of the circle from the usual orientation of the unit disc in $\mathbb{R}^2$. This convention is also useful for generalizing the fundamental theorem of calculus: the boundary is a difference between two things, and that's what makes oriented intersection theory work.

FIGURE 6. The induced orientation on the boundary of a manifold-with-boundary.

It's worth seeing what this does to homotopies, our favorite examples of manifolds-with-boundary, because this will be crucial: if $W=[0,1]\times X$, then $\partial W=X_1-X_0$, in the sense that the copy of X at 1 has positive orientation, and the copy of X at 0 has negative orientation. This is ultimately why we'll get 0 out of homotopies when we get to oriented intersection theory.

Corollary 20.6. *If X is a compact, oriented 1-manifold, then the signed sum of the boundary components of X is 0.*

Proof. By Theorem 13.1, we can reduce to the connected components $[0,1]$ and S^1; the former has one $+$ and one $-$ by Figure 6, and the latter has no boundary at all! Thus, in both cases, we get 0. ⊠

Compactness is necessary, since $[0,1)$ has one boundary point.

Next time, we'll add transversality to this recipe, and figure out how to orient a preimage, making rigorous our fuzzy notion of how oriented intersection theory should work.

Lecture 21.

Orientations and Preimages: 3/7/16

The next step in our quest towards oriented intersection theory is orienting the preimage of a nice map.

Theorem 21.1. *Let X, Y, and Z be orientable manifolds (X may be a manifold-with-boundary) and $f:X\to Y$ be a smooth map such that $f\pitchfork Z$ (and $\partial f\pitchfork Z$ if $\partial X\neq\varnothing$). Then, $f^{-1}(Z)$ is orientable.*

Proof. Let $W = f^{-1}(Z)$ and $w \in W$, so that $z = f(w) \in Z$. Thus, $T_wX = N_wW \oplus T_wW$, where the normal bundle is for $W \subset X$. In particular, if $H = N_wW$ and W is a k-dimensional submanifold of X, then H is an $(n-k)$-dimensional subspace of T_wX.

At z, $T_zY = \mathrm{Im}(\mathrm{d}f|_w) + T_zZ$, and $\mathrm{Im}(\mathrm{d}f|_w)$ splits as the things in the tangent bundle and those in the normal bundle. Then, since $\mathrm{d}f|_w(T_wW) \subset T_zZ$, this really splits as $T_zY = \mathrm{d}f|_w(H) \oplus T_zZ$. Hence, $\mathrm{d}f|_w$ is an isomorphism of H onto $\mathrm{d}f|_w(H)$. Since Y and Z are orientable, an orientation for them induces an orientation on $\mathrm{d}f|_w(H)$, and therefore on H, and since X is orientable, the orientation on H and T_wX induces an orientation on T_wW. This assignment is smooth because in a neighborhood of z and w, this is consistent (since in a sufficiently small neighborhood, it might as well be taking place in $\mathbb{R}^n$). ⊠

This proof goes through for the boundary, but we have two ways to orient ∂W: using the method above or using the induced orientation from $\partial W = f^{-1}(Z) \cap \partial X$. Surprisingly, these can be different.

Theorem 21.2. *With the same notation as in Theorem 21.1, $(\partial f)^{-1}(Z) = (-1)^{\operatorname{codim} Z}\partial(f^{-1}(Z))$;*[19] *that is, the former is oriented as in the proof of Theorem 21.1 applied to ∂f, and the latter is the induced orientation on the boundary when that construction is applied to f.*

This is annoying, but we're forced to care because of homotopy: if W is compact and $f_0, f_1 : W \to Y$ are homotopic through a homotopy $F : [0,1] \times W \to Y$, let $X = [0,1] \times W$. If $F \pitchfork Z$, $\partial F \pitchfork Z$, and $\dim W + \dim Z = \dim Y$, so that we can do intersection theory mod 2. If everything is oriented, we'd like to define the intersection number (not mod 2) $I(f_0, Z)$ to be the number of points in $f_0^{-1}(Z)$ counted with sign, but Theorem 21.2 means this is more subtle than one might like.

The fact that $f^{-1}(Z)$ is 0-dimensional means that the decomposition we had in the proof of Theorem 21.1 simplifies: $T_wX = H \oplus T_wW$, but T_wW is 0-dimensional, so $H = T_wX$, or $T_zY = \mathrm{d}f|_w(T_wX) \oplus T_zZ$. In this case, we say that the point w has positive sign if the orientations on X and Z induce the orientation on Y in this splitting, and has negative sign if it induces the opposite orientation.

After doing the same thing to f_1, we can worry about the homotopy F: specifically, $(\partial F)^{-1}(Z) = \{1\} \times f_1^{-1}(Z) - \{0\} \times f_0^{-1}(Z)$. In particular, since $F^{-1}(Z)$ is a compact 1-manifold-with-boundary, then we will prove below that $(\partial F)^{-1}(Z)$ has an equal number of positively and negatively signed points. Then, however, if we invert all the signs, nothing changes, so $\partial(F^{-1}(Z))$ also has an equal number of positively and negatively signed points. This has the important corollary that it doesn't matter which way we orient the boundary: homotopies still preserve oriented intersection number.

Proof of Theorem 21.2. Let $w \in W$ and $z = f(w) \in Z$, and choose local coordinates in a neighborhood U of z such that $Z = \{y \in Y : y_1 = y_2 = \cdots y_{n-\ell} = 0\}$. We can also pick coordinates for X such that f is the identity on the first $m - \ell$ coordinates (we don't have any control over the rest). That is, in a neighborhood of w, we have coordinates in X given by $(x_1, \dots, x_n)$ where $f(x_i) = y_i$ for $1 \le i \le m - \ell$. In particular, this means $f^{-1}(Z) = \{x : x_1 = \cdots = x_{m-\ell} = 0\}$.

We chose these coordinates because they give us very nice bases for the tangent space. If $(e_1, \dots, e_m)$ is a basis for T_zY, then the induced basis on T_zZ is $(e_{m+1-\ell}, \dots, e_m)$, and the induced basis on $\mathrm{d}f|_w(H) = T_wX$ is $(e_1, \dots, e_{m-\ell})$. Thus, $(e_{m+\ell-1}, \dots, e_n)$ is a basis for T_wW.

In particular (winding through what all this actually means), $(e_{m+1-\ell}, \dots, e_{n-1})$ is the induced basis for $(\partial f)^{-1}(Z)$, and $(e_{m+1-\ell}, \dots, e_{n-1}, e_n)$ is a basis for $f^{-1}(Z)$. We need to relate this to $(-e_n, e_{m+1-\ell}, \dots, e_{n-1})$, but got confused and may have a sign error. ⊠

Lecture 22.

The Oriented Intersection Number: 3/9/16

"I don't specifically know who [remembers this rule]... but I bet that Dan Freed is one of them."

Note: I was late today, so material from the first bit of class may be missing.

The thing about orientations is that there are a bunch of conventions floating around. Some of them are more important than others.

[19] This means they have the same orientation if $\operatorname{codim} Z$ is even, and opposite orientation if $\operatorname{codim} Z$ is odd.

Here are some aspects that are particularly important.

- The idea of orientation: how to define it, what it means, and so forth.
- Given an orientation of X, how can you obtain an orientation of ∂X?
- No matter what your convention, $\partial([0,1]\times X) = (\{1\}\times X) - (\{0\}\times X)$.
- If X is a compact, oriented 1-manifold-with-boundary, then ∂X has 0 endpoints, counted with sign.
- The definition of the oriented intersection number:[20] if X, Y, and Z are oriented manifolds, where Z is closed in Y and X is compact, $\dim X + \dim Z = \dim Y$, and $f : X \to Y$, then $I(f,Z) = \#(g^{-1}(Z))$ counted with sign, where $g \sim f$ and $g \pitchfork Z$ (positive if $\mathrm{Im}(\mathrm{d}f|_x) \oplus T_z Z = T_z Y$ as oriented spaces, and $-$ otherwise).
- If $f_0 \sim f_1$, $I(f_0, Z) = I(f_1, Z)$.

However, these things are less important. It's important to work through these once, to see that they exist and to get an idea of the construction, but after that it's not very crucial.

- Given general oriented manifolds X, Y, and Z and a smooth $f : X \to Y$, how to obtain an orientation of $f^{-1}(Z)$ (assuming transversality).
- The difference in signs between $(\partial f)^{-1}(Z)$ and $\partial(f^{-1}(Z))$, as in Theorem 21.2.

One nuance of intersection theory is that f and Z are different with respect to I: you can make homotopies of f, but not Z. We'll prove on the homework (in the mod 2 setting, though the proof is very similar in the oriented case) that if $f : X \to Y$ and $g : Z \to Y$, then $I(f, g(Z)) = I(f\times g, \Delta)$ (where $f \times g$ is the product map $X \times Z \to Y \times Y$). Hence, we can think of $I(f,Z) = I(f, i_Z) = I(f\times i_Z, \Delta)$, where $i_Z : Z \hookrightarrow Y$ is the inclusion map, and therefore if X and Y are both closed manifolds of Y, $I(X,Z) = (-1)^{(\dim X)(\dim Z)} I(Z,X)$, and both of these are $I(X\times Z, \Delta)$. One needs to say more to turn this into a proof, but the point is that the intersection number is invariant under homotopies of either argument.

Next, we would like to define the degree of a map.

Definition 22.1. Let X and Y be oriented manifolds of the same dimension, such that Y is connected and X is compact. Then, we define the *degree* of f to be $\deg f = I(f, \{y\})$ for any $y \in Y$.

The idea of the proof of Theorem 17.4 applies, though a few minor details are different.

One useful application of the oriented degree is that $\deg f$ is positive if $\mathrm{d}f|_x$ preserves orientation for all x, and is negative if $\mathrm{d}f|_x$ reserves orientation.

Example 22.2. One good way to see what the degree is ("signed number of preimages") is to draw pictures with circles. For example, suppose $h : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, and we'll defined $f : S^1 \to S^1$ by $f(e^{i\theta}) = e^{ih(\theta)}$. Now, the degree is the number of times $h(\theta) = y_0 \bmod 2\pi$ for any y_0: for example, if it rockets past y_0, and then turns around and comes back in the opposite direction, the degree would be 0. This is much clearer if you draw a picture.

Again, we can use this to define winding numbers.

Definition 22.3. If X is a k-dimensional, compact manifold, $f : X \to \mathbb{R}^{k+1}$ is smooth, and $p \notin \mathrm{Im}(f)$, the *winding number* of f is $W(f,p) = \deg(\widehat{f_p})$, where $\widehat{f}_p(x) = (f(x)-p)/|f(x)-p|$, which defines a smooth map $X \to S^k$.

Such a p must exist because X is compact; even if it weren't, by dimensionality a regular value is something not in $\mathrm{Im}(f)$, so such a p exists by Sard's theorem.

Now, we've "constructed" a bunch of oriented tools: intersection numbers, degrees, and winding numbers, which we'll use to do all sorts of cool stuff… after spring break. Enjoy SxSW, everyone.

There is one more important tool we need, though, which is analogous to a very similar theorem in unoriented intersection theory.

Theorem 22.4 (Extension)**.** *Let X be a compact oriented manifold such that $X = \partial W$, where W is also compact and oriented. If $f : X \to Y$ extends to a smooth $F : W \to Y$, then $I(f,Z) = 0$.*

Proof. The proof idea is the same too: we can homotope F such that F and f are transverse to Z, so without loss of generality, we can assume $f \pitchfork Z$ and $F \pitchfork Z$. Hence, $F^{-1}(Z)$ is a compact, 1-manifold-with-boundary, and $f^{-1}(Z) = (-1)^{\operatorname{codim} Z}\partial(F^{-1}(Z))$, and there are 0 points (counted with sign) in $\partial(F^{-1}(Z))$, so the signed number of points in $f^{-1}(Z)$ is 0. ⊠

We can also go back and prove the fundamental theorem of algebra, in all generality, as we promised.

[20] I'm not actually sure we've defined this yet, nor have we proven the theorems that guarantee this is well-defined.

Theorem 22.5 (Fundamental theorem of algebra)**.** *Let $f \in \mathbb{C}[z]$ be an n^{th}-degree polynomial; then, f has n roots, counted with multiplicity.*

Proof sketch. The first step is to consider some huge loop about the origin in $\mathbb{C}$, with radius R. The goal is to pick R large enough that on this loop, f has no roots and $f(z)$ looks like $g(z) = z^n$. In particular, $f \sim g$, so $W(f,0) = W(g,0) = n$.

Around each root, we can consider the "local winding number" around a circle small enough to only contain that root. If the root z_0 has multiplicity k, then in this neighborhood, $f(z) = (z - z_0)^k g(z)$, where g is smooth and nonvanishing; hence, on this neighborhood, f is homotopic to $(z - z_0)^k$, which has local winding number k. Then, the total winding number is the sum of the local winding numbers. ⊠

This generalizes to something known as the argument number in complex analysis.

Definition 22.6. A function $f : \mathbb{C} \to \mathbb{C}$ is *meromorphic* if it's locally analytic (but globally, it may have some poles).

The argument principle uses the same argument (isolating things to little neighborhoods) to show that the winding number is the number of poles minus the number of roots.

Things like the fundamental theorem of algebra and the argument principle aren't the real purpose of the winding number, as we'll see in two weeks.

Lecture 23.

Exam Debriefing: 3/21/16

Today, we're going to most discuss the exam and the underlying ideas lurking within it. The exam was structured like a prelim; the "chose three of four" scenario and a similar length of time (1.5 hours rather than 1 hour). The second problem was a prelim problem several years ago; the third is easier than most prelim problems, and the fourth is harder. The goal is to get at least halfway through the prelim, to stand a good chance at passing, and the class average was 17/30, which is good! However, it was graded slightly more generously than a prelim would be.

Now, let's discuss the problems.

The first problem was about coordinates; do you understand coordinates? For projective space $\mathbb{CP}^n$, one often wants to think of it with $2n+2$ coordinates, realizing it as a quotient of $\mathbb{C}^{n+1}$, but it's a $2n$-dimensional manifold, and your charts should reflect that. The first part asked for two charts for $\mathbb{CP}^1$. You can write $\mathbb{CP}^1$ with coordinates $[z_0 : z_1]$; then, the two charts are $x + iy = z_0/z_1$ and $u + iv = z_1/z_0$. Thus, $u = x/(x^2+y^2)$ and $v = -y/(x^2+y^2)$ (and similarly for x and y in terms of u and v), so on the intersection of these two charts, the change-of-charts maps are smooth, since the denominators aren't zero there. Thus, these charts make $\mathbb{CP}^1$ into a smooth manifold.

The second part asked for us to calculate the derivative of the map

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

in one of our coordinate charts, at the point $[1:1]$. This is a map on $\mathbb{CP}^1$ because the matrix has full rank, so sends lines to lines. More explicitly,

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \longmapsto \begin{pmatrix} z_0 + iz_1 \\ iz_0 + z_1 \end{pmatrix} = \begin{pmatrix} z_0/z_1 + i \\ i(z_0/z_1) + 1 \end{pmatrix}.$$

In the first coordinate chart (x, y), this is $x + iy \mapsto R(x,y) = (x + (y+1)i)/((1-y) + ix)$, so in coordinates, this function is $f(x,y) = (\mathrm{Re}(R(x,y)), \mathrm{Im}(R(x,y))$. We can also calculate this:

$$R(x,y) = \frac{x + (y+1)i}{(1-y) + ix}\left(\frac{(1-y) - ix}{(1-y) - ix}\right) = \frac{(-xy + x(y+1)) + i(1 - x^2 - y^2)}{(1-y)^2 + x^2}.$$

Thus,

$$f(x,y) = \left(\frac{2x}{(1-y)^2 + x^2}, \frac{1 - x^2 - y^2}{(1-y)^2 + x^2}\right).$$

So we have a function on (part of) $\mathbb{R}^2$ to $\mathbb{R}^2$, as we should expect, and we can take $\mathrm{d}f|_{(1,1)}$ using calculus, and we'll end up with

$$\mathrm{d}f|_{(1,1)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So it just rotates the tangent space by 90°; $(1,1)$ is an eigenvector of the original matrix, so perhaps this isn't so much of a surprise. Additionally, if you did everything complex linearly, the derivative would just be multiplication by i, and you could use this to derive the answer. But the point is: $\mathbb{CP}^1$ is a two-dimensional manifold, so once you write down everything in coordinates, it's just calculus (where there is the possibility of making many numerical mistakes).

For the second problem, we have a manifold X and a smooth map $f : X \to \mathbb{R}^2$. We would like to show that there's a line $L \subset \mathbb{R}^2$ such that $f \overline{\pitchfork} L$; there's a slick way to do this and a more brute-force approach, but both are correct.

It's hard to say anything about $\mathrm{Im}(f)$; it is often not a manifold, and even if $X = S^1$ one has the Lissajous curve, which is a dense subset of the unit cube in $\mathbb{R}^n$! So transversality of various submanifolds of $\mathbb{R}^2$ is about as good as we could hope for.

One common idea that didn't work was to pick a line L', such that f might not be transverse to L', but if we bump it by a sufficiently small amount, the transverse intersection points stay transverse, and the non-transverse intersections become transverse. This almost works — the issue is that there may be infinitely many intersection points, e.g. the Lissajous curve, and so the amount we need to bump the line may need to be infinitely small.

The cleanest way to do this is to define $F : \mathbb{R}^2 \times X \to \mathbb{R}^2$ by $F(s,x) = f(x) + s$. This is a submersion, and so transverse to any line you pick, so by the Thom transversality theorem (Theorem 14.5), we can pick an $f_s(x) = F(s,x)$ that's transverse to, say, the x-axis, and therefore f is transverse to the translation of the x-axis by $-s$. This feels like overkill, like every application of the Thom transversality theorem.

The professor had thought of a slightly different proof: pick some vertical line $x = a$ in $\mathbb{R}^2$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a projection onto it. By Sard's theorem, almost all $x \in \mathbb{R}$ are regular values of $g \circ f$, and the preimage of a regular value will be a line transverse to L. This seems sneaky, but it's exactly how we motivated transversality, and so transversality follows pretty much by definition.

The philosophy of this problem was at odds to the philosophy to the first one: use abstract theorems as black boxes, rather than computing in coordinates; the reason is that in the first problem, we knew what everything was and had good control over it, but here everything is much less defined, and considerably less well-behaved.

The third problem considered a Klein bottle K and a loop $C : S^1 \hookrightarrow K$ as in Figure 7; the goal was to show that it's not null-homotopic. The first thing many people tried was to compute the self-intersection number $I_2(C,C)$,

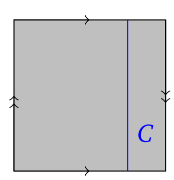


FIGURE 7. Is C null-homotopic in the Klein bottle?

but since you can push C off of itself, this is 0, which is not helpful. Instead, though, you can compute $I_2(C,Q)$, where Q is a horizontal curve; this is 1. If C were null-homotopic, then this would be 0, so C isn't null-homotopic. In fact, the fundamental group $\pi_1(K) = \mathbb{Z} * \mathbb{Z}/2$, where the $*$ denotes the free product; the class of the curve C generates the $\mathbb{Z}/2$ factor.

Nobody completely got the last problem (and the fewest people attempted it); perhaps the difficulty was that it followed not from a theorem we had in class, but a proof of it.

Suppose we have two smooth maps $f_0, f_1 : X \to Y$ such that $f_0 \sim f_1$ and there are two closed submanifolds $Z_0, Z_1 \subset Y$ such that $f_0 \overline{\pitchfork} Z_0$ and $f_1 \overline{\pitchfork} Z_1$. We would like to find a homotopy $G : [0,1] \times X \to Y$ (so $G(0,x) = f_0(x)$ and $G(1,x) = f_1(x)$) such that $G|_{(0,1)\times X}$ is transverse to both Z_0 and Z_1.

Let's start with a tubular neighborhood $\pi : Y^\varepsilon \to Y$ induced from an embedding $Y \hookrightarrow \mathbb{R}^N$. Let $S = \{s \in \mathbb{R}^{N-\dim Y} : |s| < \varepsilon\}$. Then, let $G : S \times [0,1] \times X \to Y$ be defined by $G(s,t,x) = \pi(F(t,x) + t(1-t)s)$. This is a submersion, so by the Thom transversality theorem, we can pick an s such that $G_s = G(s,\cdot,\cdot)$ is transverse to both Z_1 and Z_2 everywhere except possibly at the boundary, and so G_s is our required G.

With the test out of the way, we'll summarize some of the things we know so far about (possibly oriented) intersection theory; Wednesday, we will review what we can do. For the rest of the lecture, let X be a compact manifold, which we will usually take to be oriented; let Y be a manifold, which will usually be oriented; and let $Z \subset Y$ be a closed submanifold, which will usually be oriented, and might be compact. Suppose $\dim X + \dim Z = \dim Y$, and let $f : X \to Y$ be a smooth map. We'd like to define the intersection number $I(f,Z)$ (or $I_2(f,Z)$ if we're in the unoriented case).

Assuming everything is oriented, if $f \pitchfork Z$, we can define $I(f,Z) = \#(f^{-1}(Z))$ (this is a finite set because X is compact), where the number of points is counted with sign; if f isn't transverse, we can pick an $\widetilde{f}$ homotopic to f that's transverse to Z, and use it instead. We can always choose such an $\widetilde{f}$, by the machinery we spent so much time setting up. In particular, if $\widetilde{f}_0 \sim \widetilde{f}_1$, then if F denotes a homotopy between them, so $\widetilde{f}_1^{-1}(Z) - \widetilde{f}_0^{-1}(Z) = \partial F^{-1}(Z)$, and the boundary of an oriented 1-manifold-with-boundary has 0 boundary points, counted with sign.

Now, though, we can use all of the hard work we did at the beginning of the semester implicitly, with statements such as "without loss of generality, assume $f \pitchfork Z$," etc. But we stil need to be able to count with signs; the rule is that if $\mathrm{Im}(\mathrm{d}f|_x) \oplus T_{f(x)}Z = T_{f(x)}Y$, then count x with positive sign, and if it's equal to $-T_{f(x)}Y$, count x with negative sign. The direct sum occurs because of transversality and the dimensions of these vector spaces, and is what makes this work: there is an induced orientation on the direct sum of two oriented vector spaces.

And now that we know such a rule exists, we don't need the more general case; you should work through the general case once (maybe twice), but the important takeaway is how to use it to calculate intersection numbers. We also have Theorem 22.4, that the oriented intersection number is 0 for a map that smoothly extends from a boundary. Both the direct statement (used for calculating some intersection numbers) and its contrapositive (using intersection number to show something isn't a boundary) are useful.

Sometimes we want Z to be compact, typically when we want to interchange the roles of X and Z. For example, one would like to compare $I(X,Z)$ and $I(Z,X)$, but for this to make any sense at all, Z must also be compact. But if all you want is a target, rather than a source, it can be any closed submanifold.

Lecture 24.

(Minus) Signs of the Times: 3/23/16

Throughout this lecture, let X be a compact, oriented manifold, Y be an oriented manifold, and $Z \subset Y$ be a closed, oriented submanifold such that $\dim X + \dim Y = \dim Y$. We'll take a smooth map $f : X \to Y$, and calculate the intersection number $I(f,Z)$.

We had a homework problem that allows us to discuss the intersection number of two maps or two spaces; we proved it for the unoriented, mod 2 case, so for the rest of this paragraph, assume X, Y, and Z are *un*oriented. If Z is compact, then let $i_Z : Z \hookrightarrow Y$ denote the inclusion map, so we have a map $f \times i_Z : X \times Z \to Y \times Y$. We proved that $f \pitchfork Z$ iff $f \times i_Z \pitchfork \Delta$. Thus, we can make the following definition.

Definition 24.1. Let $g : Z \to Y$ be smooth. Then, f *is transverse to* g, written $f \pitchfork g$, if whenever $f(x) = g(z)$, then $\mathrm{Im}(\mathrm{d}f|_x) + \mathrm{Im}(\mathrm{d}g|_z) = T_{f(x)}Y$. Then, the *mod* 2 *intersection number* of f and g is $I_2(f,g) = I_2(f \times g, \Delta)$.

Thus, as we proved on the homework, $f \pitchfork g$ iff $f \times g \pitchfork \Delta$.

In the oriented case, this is not very different: however, there's a sign ambiguity, so we have to define

$$I(f,g) = (-1)^{\dim Z} I(f \times g, \Delta). \tag{24.2}$$

The classical case is where f and g are inclusions of closed submanifolds $X, Z \subset Y$. In this case, the notation is $I(X,Z) = (-1)^{\dim Z} I(X \times Z, \Delta)$. This is useful: remember Theorem 17.4? We wanted to prove that the degree, defined as the intersection number of a point, doesn't depend on the point (assuming Y is connected). But any two points in a connected manifold are connected by a path, which is a homotopy between them, and the intersection number is homotopy-invariant (again, we would have to keep track of sign numbers for the $\mathbb{Z}$-valued degree).

It would be nice to understand how $I(f \times g, \Delta)$ and $I(g \times f, \Delta)$ are related; we proved they're equal mod 2 on the homework. Suppose X is k-dimensional, so Z is $(n-k)$-dimensional. If $f \times g \pitchfork \Delta$ (which we can make true with a homotopy), then let $x \in X$ and $z \in Z$ be such that $f(x) = g(z)$; let's look at the sign of this intersection. By dimensionality, $\mathrm{Im}(\mathrm{d}f|_x) \oplus \mathrm{Im}(\mathrm{d}g|_z) = T_{f(x)}Y$, so if $(v_1, \dots, v_k)$ is a basis for T_xX and $v_{k+1}, \dots, v_n$ is a basis for T_zZ, then a basis for $\mathrm{Im}(\mathrm{d}(f \times g)|_{(x,z)})$ *with the correct orientation* is $((v_1, 0), \dots, (v_k, 0), (0, v_{k+1}), \dots, (0, v_n))$. In the same way, if we push the basis vectors for T_zZ in front, then we get a basis for $\mathrm{Im}(\mathrm{d}(g \times f)|_{(z,x)})$: $((0, v_{k+1}), \dots, (0, v_n), (v_1, 0), \dots, (v_k, 0))$.

How many permutations did we need to make? Each of the $n-k$ basis vectors for T_zZ must move across the k basis vectors of T_xX (if we do this in increasing order, then these are exactly the vectors we have to move across), so there are $k(n-k)$ moves. But then something magical happened, and I don't know what it was, because we should have proved the following proposition. TODO: where does the extra factor of n come from? Do we have to switch the signs of all the basis vectors?

Proposition 24.3. $I(f \times g, \Delta) = (-1)^{n+k(n-k)} I(g \times f, \Delta)$.

Now, we can use this to relate $I(f, g)$ and $I(g, f)$. Using (24.2),

$$\begin{aligned} I(g, f) &= (-1)^k I(g \times f, \Delta) \\ &= (-1)^{n+k(n-k)+k} I(f \times g, \Delta) \\ &= (-1)^{n+k(n-k)+n-k} I(f, g) \\ &= (-1)^{(\dim X)(\dim Z)} I(f, g). \end{aligned}$$

This is quite useful. As just one example, if $\dim Y$ is even and $\dim X = (1/2)\dim Y$, we can talk about $I(X, X)$, but we just proved that $I(X, X) = (-1)^{(\dim X)^2} I(X, X)$; in particular, if X is odd-dimensional, this is zero! So self-intersection theory is only useful inside manifolds whose dimension is divisible by 4.[21] This is at odds with the unoriented case: one can construct a 1-dimensional submanifold of the 2-dimensional Klein bottle whose self-intersection number is 1.

If X is any manifold, one can get an even-dimensional manifold by taking $X \times X$, and considering the diagonal submanifold $\Delta \subset X \times X$. Its self-intersection number is interesting.

Definition 24.4. If X is an oriented manifold, its *Euler characteristic* is $\chi(X) = I(\Delta, \Delta)$.[22]

And so by the discussion above, we have the following impressive theorem.

Theorem 24.5. *Let X be an odd-dimensional, orientable manifold; then, $\chi(X) = 0$.*

We'd like to generalize this, which will lead to Lefschetz theory. If $i : X \hookrightarrow X \times X$ is inclusion into the first factor, then $\chi(X) = I(i, \Delta)$, so for a more general smooth $f : X \to X$, we can ask whether calculating $I(i \circ f, \Delta)$ provides any information about the properties of f. It turns out this is pretty useful, and so we make the following definition.

Definition 24.6. If $f : X \to X$ is smooth, its *graph* $\mathrm{graph}(f) \subset X \times X$ is given by $\mathrm{graph}(f) = \{(x, f(x)) : x \in X\}$. Then, the *Lefschetz number* of f is $L(f) = I(\mathrm{graph}\, f, \Delta)$.

Warning: Guillemin and Pollack define $L(f) = I(\Delta, \mathrm{graph}(f))$, which is $(-1)^{\dim X}$ times the definition we (and the rest of the world) use. Be aware of this when reading the book!

We immediately know some Lefschetz numbers by homotopy invariance of the intersection number.

Corollary 24.7. *If $f \sim \mathrm{id}$, then $L(f) = \chi(X)$.*

Remark. It's possible to reframe the Euler characteristic and Lefschetz number in a more algebraic way. We won't use this, so feel free to ignore it if you haven't taken an algebraic topology class. Let X be any manifold and

[21] Here, the intersection number is often nontrivial: for example, $\mathbb{CP}^1 \subset \mathbb{CP}^2$ has nonzero intersection number.

[22] The Euler characteristic can be defined much more generally, e.g. as the alternating sum of the ranks of the homology or cohomology groups. This works much more generally, for all topological spaces! This allows an alternate proof of Theorem 24.5 using Poincaré duality. Unfortunately, though, there's no way to extend this definition to unoriented manifolds in a way that agrees with the usual definition of Euler characteristic.

$f : X \to X$ be smooth. Then, there's an induced map on homology $f_*^{(k)} : H_k(X) \otimes \mathbb{R} \to H_k(X) \otimes \mathbb{R}$. Then, the *(homological) Lefschetz number* of f is

$$L(f) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{tr}(f_*^{(k)}).$$

Therefore, we should define the Euler characteristic more generally as

$$\chi(X) = L(\mathrm{id}) = \sum_{i=0}^{\dim X} (-1)^i \dim(H_i(X) \otimes \mathbb{R}).$$

The homological Lefschetz number is about dimensions of homology; the differential topology one is about fixed points of f. We're going to use this in the next lecture.

Lecture 25.

The Lefschetz Number: 3/25/16

"Hot fudge sundaes are good!"

Throughout today's lecture, X will denote a compact, oriented manifold and $f : X \to X$ will be smooth. Last time, we defined the Lefschetz number as $L(f) = I(\Delta, \operatorname{graph} f)$ in $X \times X$. We are following Guillemin and Pollack's definition, which is **not the same** as the usual definition $I(\operatorname{graph} f, \Delta)$; the difference is a factor of $(-1)^{\dim X}$. In any case, this definition quickly implies two properties:

(1) if $f \sim g$, then $L(f) = L(g)$, and
(2) if $i_X : X \hookrightarrow X \times X$ is the inclusion sending X to the diagonal Δ, then $L(i_X) = I(\Delta, \Delta) = \chi(X)$, the Euler characteristic.

When actually calculating this, we would like to be able to make $\operatorname{graph} f \pitchfork \Delta$, and we can homotope it ot do that, but the important question is: can we realize this homotopy as the graph of some $g : X \to X$ that's homotopic to f?

Definition 25.1. f is a *Lefschetz map* if $\operatorname{graph} f \pitchfork \Delta$.

In this case, because $\dim(\operatorname{graph} f) = \dim(\Delta)$, then their intersection is a compact, 0-dimensional manifold, so a finite set of points.

Theorem 25.2. *Every f is homotopic to a Lefschetz map.*

Proof. We already know how to create a smooth family $F : S \times X \to X$ such that $\mathrm{d}F = (\mathrm{d}F_1\ \mathrm{d}F_2)$ (the S-component and the X-component), such that $\mathrm{d}F_1$ is onto. (For example, we usually construct such a map with a tubular neighborhood.) Then, if $\widehat{F}(s,x) = (x, F(s,x))$, which maps $S \times X$ to the graph of F, then $\widehat{F}$ is a submersion, which you can check directly on vectors in $T_{\widehat{F}(s,x)}(X \times X)$. Therefore it's certainly transverse to the diagonal, so by the Thom transversality theorem, for almost all s, the graph of $f_s(x) = F(s,x)$ is transverse to Δ, and $f \sim f_s$. ⊠

One useful property of the Lefschetz number is that it can provide information about fixed points. Suppose $x \in X$ is a fixed point of f, so $f(x) = x$, and let $A_x = \mathrm{d}f|_x : T_xX \to T_xX$. Suppose $(v_1, \dots, v_k)$ is an oriented basis for T_xX concordant with the orientation of X. Then, $(x,x) \in \Delta \cap \operatorname{graph}(f)$; let's compute the sign of this intersection. $T_{(x,x)}\Delta = \operatorname{span}((v_1, v_1), \dots, (v_n, v_n))$ and $\operatorname{Im}(\mathrm{d}f|_x) = \operatorname{span}((v_1, A_x v_1), \dots, (v_n, A_x v_n))$. Thus, $\operatorname{Im}(\mathrm{d}f|_x) \oplus T_{(x,x)}\Delta$ induces the existing orientation on $T_{(x,x)}(X \times X)$ iff $\operatorname{sign}(\det(A_x - I)) > 0$. In fact, we also know this can't be 0, since f is Lefschetz.

Proposition 25.3. *If f is a Lefschetz map and x is a fixed point of f, then* $\det(\mathrm{d}f|_x - I) \neq 1$.

If we count up these values for all points in the intersection, we get a formula for the Lefschetz number.

Proposition 25.4. *If f is a Lefschetz map,*

$$L(f) = \sum_{\text{fixed points } x} \operatorname{sign}(\det(\mathrm{d}f|_x - I)).$$

Corollary 25.5. *If $L(f) \neq 0$, then f has a fixed point.*

As a corollary, $\dim(A_x - I)$ doesn't depend on the orientation of X; for all orientations of X, this Lefschetz number is the same. If you work out the formula in bases, two minus signs appear, and cancel each other out. This is a reflection of the fact that the Lefschetz number is actually a homological invariant, like we mentioned last time. Homology doesn't change with respect to orientations, so the fact that we get the same result is no surprise.

Example 25.6. Let $X = S^2 \subset \mathbb{R}^3$, and let $f : S^2 \to S^2$ be rotation by $\pi/2$ about the z-axis. There are two fixed points, $(0,0,\pm 1)$ (the north and south poles). At the north pole, the matrix $A_x = f$ (since f is linear); since f is an invertible linear transformation, then $\mathrm{d}f|_{(0,0,1)}$ has full rank. The argument for $(0,0,-1)$ is exactly the same, so f is a Lefschetz map.

Now, let's compute some matrices: at both poles, $A_x = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, since we're doing the same rotation at each pole.[23] Thus, $\det(A_x - I) = 2$, so both are positive. Thus, the Lefschetz number is $1 + 1 = 2$.

One way to intuit this is that the eigenvalues of A_x are complex (since a rotation fixes no lines). Thus, they must occur in conjugate pairs, and hence their product is nonnegative (and positive as long as 0 isn't an eigenvalue).

This allows one to understand the Lefschetz number topologically from the eigenvalues around a point, as with the topological understanding of the behavior of differential equations. Suppose $\dim X = 2$; if $\mathrm{d}f|_x$ has all complex eigenvalues, then as we already remarked, the contribution of x to the Lefschetz number is $+1$. Depending on the real parts of the eigenvalues, the fixed point f can be a *source* (points flow away from it, locally), a *sink* (they flow towards it), or stable (locally, points rotate around it). So, *if a fixed point is a source, a sink, or stable, then it contributes* $+1$ *to the Lefschetz number*.

To get a negative sign, you need the eigenvalues to be real and have different signs; thus, *saddle points contribute* -1 *to the Lefschetz number*. These are very useful for understanding this construction geometrically (you can draw pictures of these!); for example, we can calculate the Lefchetz number of the rotation in Example 25.6: both the north and south poles are stable points (locally, points rotate around them), so they each contribute $+1$ to the Lefschetz number, and so $L(f) = 2$.

Since that rotation is homotopic to the identity, this also computes that $L(f) = L(\mathrm{id}) = \chi(S^2) = 2$! Cool. And if you know the Euler characteristic of a manifold, you can use it in the other direction to compute Lefschetz numbers of maps homotopic to the identity. In particular:

Corollary 25.7. *If $f : S^2 \to S^2$ is Lefschetz and homotopic to the identity, then it has a fixed point.*

This is true in more generality: f doesn't need to be Lefschetz, though this is better proven using the homological definition of the Lefschetz number.

Alternatively, you can use this to prove certain maps aren't homotopic to the identity, if their Lefschetz numbers aren't the same as the Euler characteristic. For example, consider the genus-2 surface, which has a symmetry around a "skewer" going through both holes. Reflecting about this symmetry is a smooth Lefschetz map whose Lefschetz number is positive, but the Euler characteristic is -2, so this reflection isn't homotopic to the identity!

Corollary 25.7 doesn't hold for every manifold, of course: on the torus, whose Euler characteristic is 0, a rotation by $\pi/2$ is homotopic to the identity, but has no fixed points. This suggests that if f has a nonnegative Lefschetz number, the property that it has fixed points is invariant under homotopy; if $L(f) = 0$, f may still have fixed points, but they are somehow less stable under homotopy.

This geometric intuition about behavior of the flow around fixed points is true, mostly, in more generality: the sign of $\mathrm{d}f|_x$ if x is a source is $+1$, but if x is a sink, it's $(-1)^{\dim X}$. Thus, the rotation map that we considered in Example 25.6 generalizes nicely: it's always homotopic to the identity, but its Lefschetz number changes.

Proposition 25.8.

$$\chi(S^k) = \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

If $f : S^k \to S^k$ denotes the antipodal map, then f has no fixed points, so there's a very nice result about it.

Corollary 25.9. *The antipodal map on S^k isn't homotopic to the identity if k is even.*

And on the homework, we already constructed a homotopy $f \sim \mathrm{id}$ if k is odd.

We're going to relate this to vector fields and where they vanish (corresponding to fixed points) in the next few lectures.

[23] Relevant xkcd: `https://xkcd.com/184/`.

Lecture 26.

Multiple Roots and the Lefschetz Number: 3/28/16

> *"Then, you blow it up! Well, not really, because that means something different in algebraic geometry."*

We were in the midst of discussing Lefschetz theory, so once again, throughout this lecture X will denote a compact, oriented manifold, and $f : X \to X$ will be smooth. We said that f is Lefschetz if graph $f \pitchfork \Delta$.

Recall that a *fixed point* of f is an $x \in X$ such that $f(x) = x$.

Definition 26.1. x is a *Lefschetz fixed point* if $\det(\mathrm{d}f|_x - I) \neq 0$ (so 1 is not an eigenvalue of $\mathrm{d}f|_x$).

By Propositions 25.3 and 25.4, if f is a Lefschetz map, then it has finitely many fixed points, and they're all Lefschetz.

Several times last lecture, we talked about the contribution of a fixed point to the Lefschetz number; this is easier to talk about using the following definition.

Definition 26.2. If x is a Lefschetz fixed point of f, its *local Lefschetz number* is $L_x(f) = \operatorname{sign}(\det(\mathrm{d}f|_x - I))$.

Hence, the Lefschetz number $L(f)$ is the sum of the local Lefschetz numbers (if f is Lefschetz).

$$L(f) = I(\Delta, \operatorname{graph}(f)) = \sum_{\text{fixed points } x} L_x(f).$$

Today, we are going to generalize the notions of Lefschetz number and local Lefschetz number in a way that allows points to take on more values than ± 1. This will allow us to talk about multiple roots: for example, we'd like a 7^{th}-degree polynomial to have seven roots (over $\mathbb{C}$), but $z^7 = 0$ has one root.

The solution, like many solutions in the class, is to make a small homotopy to make everything transverse: $z^7 + \varepsilon = 0$ has seven roots! Then, we will generalize the local Lefschetz number to define $L_0(z^7)$ to be the sum of the local Lefschetz numbers of the roots after that small homotopy.

This is a fine theoretical definition, but would be a nightmare to compute. Fortunately, we'll be able to provide a technique for computing this modified local Lefschetz number just in terms of the original map.

For now, let's assume that it's possible to make this homotopy in a neighborhood of such a point. This is true, and we'll prove it at the end of the lecture, but the proof technique (Sard's theorem, partitions of unity, and other "analysis") is quite different from the rest of today's story.

This means we're working in a coordinate chart, so we can just assume $X = \mathbb{R}^n$ and $a \in \mathbb{R}^n$ is a fixed point. Let $g(x) = f(x) - a$, and suppose $\mathrm{d}g|_x = \mathrm{d}f|_x - I$ is invertible. This means a is an isolated fixed point of f (isolated root of g), so there is a sphere S around a (*not* a ball) such that g doesn't vanish on S. Hence, there's a smooth map $g/|g| : S \to S^{n-1}$. What is its degree?

Since $g(x) = g(a) + \mathrm{d}g|_a(x-a) + O((x-a)^2)$ and $g(a) = 0$, then

$$\frac{g(x)}{|g(x)|} = \frac{\mathrm{d}g|_a(x-a) + O((x-a)^2)}{|\mathrm{d}g|_a(x-a) + O((x-a)^2)|}.$$

Since $O((x-a))^2$ is small, this map is homotopic to the map $h(x) = \mathrm{d}g|_a(x-a)/|\mathrm{d}g|_a(x-a)|$, which is the restriction of a linear map to the sphere, which is nice; and we can use it to calculate the degree.

For example, if $g = \mathrm{id}$, $\deg(g/|g|) = \deg(h) = 1$; if g is a reflection, then $\deg(h) = -1$, and a product of k reflections has degree $(-1)^k$. This suggests that the degree of h is a determinant.

Formalizing this intuition,

Proposition 26.3. $\mathrm{GL}(n)$ *is a smooth manifold that has two (path) components, which are the matrices where the determinant is positive, and those where the determinant is negative.*

Proof sketch. The determinant $\det : \mathrm{GL}(n) \to \mathbb{R}$ is continuous, and its image has two connected components; thus, $\mathrm{GL}(n)$ has at least two connected components.

If $A \in \mathrm{GL}(n)$, then we can without loss of generality assume A is diagonalizable (every matrix is homotopic to a diagonal matrix). Then, by rescaling along the basis vectors for a basis where A is diagonal, A is homotopic to a matrix whose eigenvalues are only 1 and -1. Then, using rotations, we can rotate $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to the identity, and so the number of positive eigenvalues mod 2 is the only invariant. ⊠

Since the degree is homotopy-invariant, this means that $\deg(g/|g|) = \operatorname{sign}(\det(\mathrm{d}g|_a))$.

Now, suppose we've perturbed f to a $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ that's Lefschetz, and let $x_1, \dots, x_m$ be the roots that we obtained from f. Let $\widetilde{g}(x) = \widetilde{f}(x) - \widetilde{f}(a)$, just like for g and f. If we restrict to a small sphere around each x_i, then

$$\sum_{i=1}^m L_{x_i}(\widetilde{f}) = \sum_{i=1}^m \deg\left(\frac{\widetilde{g}}{|\widetilde{g}|}\right).$$

But this is equal to $\deg(\widetilde{g}/|\widetilde{g}|)$ on a sphere around all of the x_i: their difference is the degree of $\widetilde{g}/|\widetilde{g}|$, which extends to the big sphere minus the small spheres; by the extension theorem, their difference must be 0. And since $g \simeq \widetilde{g}$, this implies

$$\sum_{i=1}^m L_{x_i}\widetilde{f} = \deg\left(\frac{g}{|g|}\right).$$

In other words, we have made sense of the following definition.

Definition 26.4. Let x be any fixed point of f; then, its *local Lefschetz number* is

$$L_x(f) = \deg\left(\frac{f(x) - x}{|f(x) - x|}\right)$$

on a small sphere around x.

By what we've just discussed, this is the same as the previous definition of the local Lefschetz number if x is a Lefschetz fixed point.

This is something we can compute! For example, if $f : \mathbb{C} \to \mathbb{C}$ is given by $f(z) = z^n + z$, then 0 isn't a Lefschetz fixed point, and we can quickly check that $L_0(f) = n$, as intended. We started out trying to generalize and detect multiple roots, and that's exactly what we ended up with.

However, we need to deliver on our promise.

Lemma 26.5 (Splitting)**.** *Let a be an isolated fixed point of f, so there's a neighborhood $U \subset X$ of a on which a is the only fixed point. Then, there's a smooth $f_1 : X \to X$ homotopic to f such that*

- *$f_1(x) = f(x)$ outside of U, and*
- *all the fixed points of $f_1|_U$ are Lefschetz.*

Proof. First, let's assume $X = \mathbb{R}^n$, so we can assume $a = 0$, and there are no other fixed points in a ball of radius R. We've already proven that there exists a smooth bump function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that

$$\rho(x) = \begin{cases} 1, & |x| < R/3 \\ 0, & |x| > R/2, \end{cases}$$

so if $\varepsilon > 0$ is small and S is the sphere of radius ε around 0 in $\mathbb{R}^n$, then define $F : S \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(s, x) = f(x) + \rho(x)s$.

Let s be a regular value and x be a fixed point of $f_s = F(s, \cdot)$; then, $f(x) + s = x$, so $f(x) - x = s$, and when we take derivatives, $\mathrm{d}f|_x - I$ is a surjective, hence invertible matrix, so x is a Lefschetz fixed point.

Now, use a partition of unity to generalize this to arbitrary X. ⊠

Lecture 27.

Vector Fields and the Euler Characteristic: 3/30/16

We've spent the past few days talking about Lefschetz theory: if X is a compact, oriented manifold and $f : X \to X$ is smooth, then we can define the Lefschetz number to be $L(f) = I(\Delta, \operatorname{graph}(f))$ inside $X \times X$. If f has isolated fixed points $x_1, \dots, x_n$, then $L(f) = \sum_{i=1}^n L_{x_i}(f)$, the local Lefschetz numbers, where we calculate the local Lefschetz number as the degree of $f(y) - y$ in local coordinates; if x_i is a Lefschetz fixed point, then this is the sign of the determinant of $\mathrm{d}f|_{x_i} - I$. We also stated (but didn't elaborate on) the homological viewpoint: f induces group homomorphisms $f_*^{(k)} : H_k(X) \to H_k(X)$, and the Lefschetz number can be defined by

$$L(f) = (-1)^{\dim X} \sum_k (-1)^k \operatorname{Tr} f_*^{(k)}.$$

One must show that these definitions agree; we will not do this.

If f is homotopic to the identity, then $L(f) = L(\mathrm{id}) = I(\Delta, \Delta) = \chi(X)$, the Euler characteristic. The sign ambiguity that we had in the definition of the Lefschetz number disappears here; there's one Euler characteristic, and our definition agrees with everyone else's.

Today, we're going to characterize[24] the Euler characteristic differently, as relating to the zeros of vector fields.

Definition 27.1. A *vector field* on a smooth manifold X is a *section* of the tangent bundle, i.e. a smooth map $v : X \to TX$ such that $\pi \circ v = \mathrm{id}_X$.

Here, $\pi : TX \to X$ is the canonical projection.

There's always a canonical section, called the *zero section*, sending $x \mapsto (x, 0)$. In this way, we can embed $X \hookrightarrow TX$, and since $\dim(TX) = 2\dim X$, one can refer to $I(X, X)$ inside TX, and more generally speak of intersection theory inside TX. In this way, we can interpret the story of vector fields as a parallel to Lefschetz theory, and we will maintain this perspective.

Flows. A vector field determines a *flow*: since $T_x X$ can be constructed as equivalence classes of paths $\gamma : (-1, 1) \to X$ such that $\gamma(0) = x$, and we identify the velocity vector at $t = 0$ with the tangent vector in the tangent space. What happens if we continue along this path?

For example, on $\mathbb{R}^2$, $v(x, y) = (0, x)$ is a vector field. We'd like the flow to be a map whose derivatives everywhere are this vector field: $\frac{\mathrm{d}y}{\mathrm{d}t} = x$, and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. We also need the initial condition that at time 0, we're at x, y. Thus, the flow for time t is $h_t(x, y) = (x, y + tx)$.

More generally, let's consider the vector field $v(\mathbf{x}) = A\mathbf{x}$ on $\mathbb{R}^n$. Once again we must solve a differential equation, and we'll end up with $h_t(\mathbf{x}) = e^{At}\mathbf{x}$. Here, e^{At} is the *matrix exponential*, which can be defined in two equivalent ways.

- One can define e^{At} to be the matrix that satisfies the differential equation $M'(t) = AM(t)$, such that $M(0) = I$.
- Equivalently, one can define it as the power series

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

From the first definition, we can see that $\frac{\mathrm{d}h_t}{\mathrm{d}t} = A\mathbf{x}$, so this is indeed our flow. Does the flow have any fixed points? One can imagine a rotation matrix, so for large t, $h_t(x)$ is full rotation; however, for t sufficiently small and 0 isn't an eigenvalue of A, then there are no fixed points.

More generally, given a vector field v on a manifold X, then $v(x) = Ax + O(x^2)$ for some matrix (the first-order approximation) A. The same argument about small t and eigenvalues applies, though there's a bit of effort needed to make it precise. This is a story for a differential equations class, which this class is not; in the end, we have the following result.

Theorem 27.2. *If X is a compact manifold and $h_t(x)$ is the flow for time t along the vector field v, then for sufficiently small t, the fixed points of h_t are the zeros of v.*

h_t is homotopic to the identity through the map $H(s, x) = h_s(x)$, since $h_0(x) = x$ for all x. Therefore, if v has isolated zeros, then for t sufficiently small, we can use Lefschetz theory: $L(h_t) = \chi(X)$. This is another definition of the Euler characteristic; we can also realize it as

$$\chi(X) = \sum_{x:\ v(x)=0} L_x(h_t).$$

Definition 27.3. Let v be a vector field on a smooth manifold X and x be a zero of v. Then, the *index* of x is $\mathrm{index}_x\, v = \deg(v/|v|)$ on a sufficiently small sphere around x.

Here, "sufficiently small" means that v doesn't vanish on or inside that sphere (except at x).

Lemma 27.4. *Suppose v has isolated zeros; then, $L_x(h_t) = \mathrm{index}_x\, v$.*

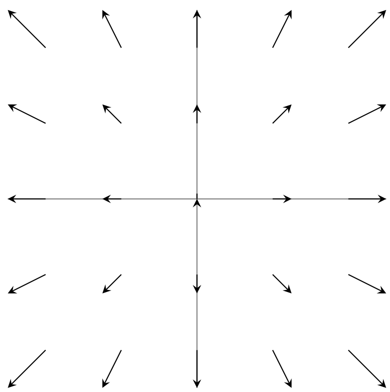
[24] No pun intended.

Proof. Let S be our sufficiently small sphere. In local coordinates, $h_t(y) = y + tv(y) + O(t^2)$; let $g_t(y) = tv(y) + O(t^2) = h_t(y) - y$; then, since $|v(x)|$ has a minimum on S which is positive (since S is compact), then we can divide by $|v(y)|$ and $|g_t(y)|$; in particular,

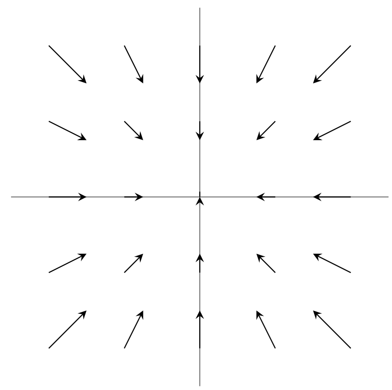
$$\frac{g_t(y)}{|g_t(y)|} = \frac{v(y) + O(t)}{|v(y) + O(t)|} \xrightarrow{t \to 0} \frac{v(y)}{|v(y)|}.$$

Thus, the left side, which calculates $L_x(h_t)$, agrees with $\deg(v/|v|)$ (since the degree is an integer, then convergence implies they must eventually be the same). ⊠

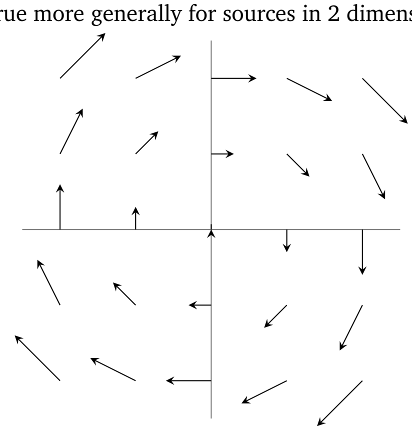
The index of a vector field at one of its zeros can be understood geometrically, especially when we can draw pictures. Consult Figure 8 for some examples.



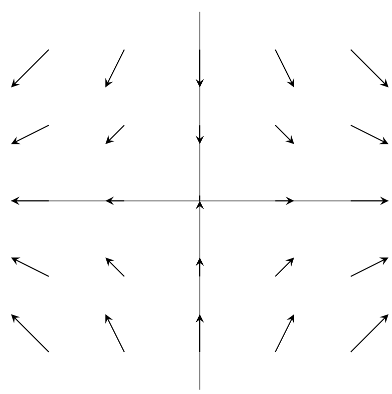
(A) This vector field, $v(\mathbf{x}) = \mathbf{x}$, has index 1 at 0. This is true more generally for sources in 2 dimensions.



(B) This vector field, $v(\mathbf{x}) = -\mathbf{x}$, has index 1 at 0. This is true more generally for sinks in 2 dimensions.



(C) A rotation vector field also has index 1 at the origin, as do rotations out of a source or into a sink.



(D) The index at a saddle point is -1.

FIGURE 8. Several vector fields on $\mathbb{R}^2$ and their indices at 0.

All the examples in Figure 8 have index 1 or -1. Can we construct something with a higher index? Regard $\mathbb{R}^2$ as $\mathbb{C}$ and consider the vector field

$$v(z) = \begin{cases} az^n, & \text{if } n > 0 \\ a(\overline{z})^{-n}, & \text{if } n < 0. \end{cases}$$

This is tricky to draw, but should be reminiscent of the magnetic field around a bar magnet; the idea is that if you draw a circle around the origin, it gets wrapped around twice.

This seems weird, and for the most part we'll end up with degrees that are ± 1. This next proposition is an example of this.

Proposition 27.5. *Let A be an invertible, $n \times n$ matrix. Then, 0 is an isolated fixed point of $v(x) = Ax$, and* $\operatorname{index}_x v = \operatorname{sign}(\det A)$.

Lecture 28.

: 4/1/16

Lecture 29.

The Hopf Degree Theorem: 4/4/16

"What does $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ *mean? Don't ask questions, just do it!"*

Today, we're going to conclude chapter 3 of the textbook with the Hopf degree theorem. We're going to provide a different proof from the one in the textbook; after this, we'll move on to the next chapter, on differential forms.

For the rest of this lecture, X will denote a compact, connected, oriented n-dimensional manifold and $f, g : X \to S^n$ will be smooth.

Theorem 29.1 (Hopf degree theorem). *For f and g as above, $\deg f = \deg g$ if and only if $f \sim g$.*

We already know one direction; we'll prove the other by showing f is homotopic to a standard form that only depends on the degree.

Proof. The reverse direction is something we already knew, so suppose $m = \deg f = \deg g$.

Without loss of generality, the south pole $s \in S^n$ is a regular value of f (since, certainly, a regular value exists, and we can rotate the sphere so it's at the south pole). Using stereographic projection, we can identify $S^n \cong \mathbb{R}^n \cup \{\infty\}$, where the point at infinity is the south pole.

Let $\rho : [0, \infty) \to [0, \infty)$ be a smooth map such that in a neighborhood of 0, $\rho(x) = 0$, and for all $x \ge 1$, $\rho(x) = 1$. The map $x \mapsto x/(1 - \rho(x))$ is only defined in a bounded neighborhood of 0, but pushes everything where $\rho(x) > 0$ out to infinity. In particular, if $F : [0, 1] \times [0, 1) \to [0\infty)$ is defined by

$$F(t, x) = \frac{x}{1 - \rho(tx)},$$

then F is a homotopy between the "unit disc" $[0, 1)$ and the entire nonnegative real axis.

Let's apply this. Let $x_1, \dots, x_{m+k}, y_1, \dots, y_k$ be the preimages under f of the south pole s, where the x_i are positively oriented and the y_i are negatively signed. Things happened and I didn't understand them. TODO

Now, for reasons that I didn't understand, the following transformations do not affect the homotopy type.

- Adding two points to $f^{-1}(s)$ that have opposite sign.
- Moving a preimage point along a path.
- Maybe something else? : (

Now we do something with the *configuration space* $\widetilde{X}^m$ of m signed points in X. This can be obtained from X^m by adding signs and removing the subset where two or more points collide. Since X is connected, $\widetilde{X}^m$ is connected (which is a pretty quick exercise), and therefore… something. ⊠

If you too were confused by the proof, check out the one in the textbook, though it's pretty different.

An Introduction to Differential Forms. On to differential forms. The machinery that builds these up can be kind of complicated, not conceptually *per se*, but in that it's very easy to get bogged down in the details. Keep asking yourself,

- *what are differential forms?* (Both the formal definition and what they're supposed to represent.)
- *How can I calculate with them?*
- And of course, *what are they useful for?*

Let V be a finite-dimensional real vector space, and $(e_1, \dots, e_n)$ be a basis for V.[25]

Definition 29.2. An *algebra* is a real vector space V with an associative, bilinear product $\wedge : V \times V \to V$.[26] That is,for all $a, b, c \in V$ and $\lambda \in \mathbb{R}$,

[25] Though we will eventually place an orientation on V, we can do exterior algebra without an orientation, so for now V is unoriented.

[26] This definition can be made significantly more general if you work over other base fields (or rings) than $\mathbb{R}$.

- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
- $a \wedge (b + c) = a \wedge b + a \wedge c$,
- $(a + b) \wedge c = a \wedge c + b \wedge c$, and
- $\lambda(a \wedge b) = (\lambda a) \wedge b = a \wedge (\lambda b)$.

You can think of this as the statement "all the usual notions of multiplication apply," though, importantly, *commutativity is not required.* Two good examples of algebras: the algebra of smooth functions on a manifold M, called $C^\infty(M)$, with addition and multiplication taken pointwise; and the quaternion algebra $\mathbb{H}$ (this one isn't commutative).

Definition 29.3. The *exterior algebra* on the vector space V is the space generated by[27] V, and modulo the relations $v_1 \wedge v_2 = -v_2 \wedge v_1$.

This is an algebra, apparently. One of its key properties is that the relation $v \wedge v = -v \wedge v$, so any vector wedge itself is 0.

Example 29.4. Suppose $V = \mathbb{R}^2$. Then, the only wedge products of basis vectors we can take are vacuous combinations 1, e_1, and e_2, and the non-vacuous one $e_1 \wedge e_2$ (since $e_2 \wedge e_1$ is on the same line as $e_1 \wedge e_2$): all higher products are 0 since some basis vector repeats. Thus, $\Lambda(\mathbb{R}^2) = \operatorname{span}(1, e_1, e_2, e_1 \wedge e_2)$. $\Lambda^i(\mathbb{R}^2)$ denotes the terms that are wedges of i vectors, and so $\Lambda(\mathbb{R}^2) = \Lambda^0(\mathbb{R}^2) \oplus \Lambda^1(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2)$.

More generally, the same argument shows that $\Lambda^k(\mathbb{R}^n)$ is generated by all sets of k basis vectors that don't contain 2 or more copies of the same vector, so $\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k}$, and $\dim \Lambda(\mathbb{R}^n) = 2^n$.

Now, we're going to do something else that seems unmotivated; we'll show that this isn't as arbitrary as it sounds.

Definition 29.5. **TODO**: this is not right.

Fix $n \geq 0$ and let $T_0^*\mathbb{R}^n$ be the vector space spanned by n formal symbols $(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$. (Yes, they look like they should be derivatives. We'll get there.) Then, the space of *differential forms* on $\mathbb{R}^n$, denoted $\Omega^*(\mathbb{R}^n)$, is $\Lambda(T_0^*\mathbb{R}^n)$. The *differential k-forms* are $\Omega^k(\mathbb{R}^n) = \Lambda^k(T_0^*\mathbb{R}^n)$.

What this means is that

TODO there's a bunch of stuff missing here.

Motivated by the formula for the derivative, define a derivative operator $\mathrm{d} : \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

This is well-defined, because 0-forms are just functions.

Lecture 30.

The Exterior Derivative: 4/6/16

Since we're not following Guillemin and Pollack as closely for the next few weeks, the professor has posted lecture notes at http://www.ma.utexas.edu/users/sadun/S16/M382D/forms1.pdf.

We're going to define differential forms as a bunch of symbols with certain properties, and after we're familiar with how to manipulate them, we'll talk about how to use them in differential topology.

Definition 30.1. A *multi-index* $I = (i_1, \dots, i_k)$ is an ordered subset of $\{1, \dots, n\}$. k is called the *length* of the multi-index.

Formally, the symbol $\mathrm{d}x^I$ denotes $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$. More generally, a *differential k-form* on $\mathbb{R}^n$ is a symbol of the form

$$\omega = \sum_{j=1}^{\ell} \alpha_{I_j}(x)\, \mathrm{d}x^{I_j},$$

[27] Unfortunately, this wasn't defined very clearly or precisely in class; see an algebra textbook such as Dummit and Foote for an actual definition.

where $I_1, \dots, I_\ell$ are multi-indices of size k and the $\alpha_{I_j} : \mathbb{R}^n \to \mathbb{R}$ are smooth functions.

The $\wedge$ symbol obeys the relation that $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$. This has the powerful consequence that $\mathrm{d}x^i \wedge \mathrm{d}x^i = -\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$, which means that if there are any repeated terms in a multi-index I, then $\mathrm{d}x^I = 0$. Another consequence is that if $k > n$, any k-form on $\mathbb{R}^n$ is equal to 0.

We can also define $\wedge$ of two differential forms: if $\alpha = \sum \alpha_I \,\mathrm{d}x^I$ is a k-form and $\beta = \sum \beta_J \,\mathrm{d}x^J$ is an ℓ-form, then we define

$$\alpha \wedge \beta = \sum_{I,J} \alpha_I(x)\beta_J(x)\,\mathrm{d}x^I \wedge \mathrm{d}x^J.$$

In particular, if $k + \ell > n$, then this is always 0.

Is this commutative? Sometimes. We'd like to transform $\mathrm{d}x^I \wedge \mathrm{d}x^J$ to $\mathrm{d}x^J \wedge \mathrm{d}x^I$. This means we have to move each of the ℓ symbols in $\mathrm{d}x^J$ across each of the k symbols in $\mathrm{d}x^I$, so there are $k\ell$ sign flips. Thus, $\mathrm{d}x^I \wedge \mathrm{d}x^J = (-1)^{k\ell}\mathrm{d}x^J \wedge \mathrm{d}x^I$, and therefore

$$\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha.$$

This is important to remember: even-degree forms commute with everything, but for two odd-degree forms, you need to flip the sign. This is kind of odd.

We can also take "derivatives" of forms (remember, we're still technically using abstract symbols, but this will correspond to actual derivatives soon enough).

Definition 30.2. If $\alpha = \sum_I \alpha_I \,\mathrm{d}x^I$ is a k-form, then its *exterior derivative* is the $(k+1)$-form

$$\mathrm{d}\alpha = \sum_{j=1}^{n} \frac{\partial \alpha_I}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^I.$$

Intuitively, we're applying the "differential operator," also called the *exterior derivative*,

$$\mathrm{d} = \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\,\mathrm{d}x^j,$$

which might help make this easier to remember.

Example 30.3. A 0-form is just a function, so let's consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^2ye^z$. In this case, $\mathrm{d}f = 2xye^z\,\mathrm{d}x + x^2e^z\,\mathrm{d}y + x^2ye^z\,\mathrm{d}z$. This looks a lot like the gradient ∇f, which is not a coincidence.

Since mixed partials commute, but $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, then $\mathrm{d}(\mathrm{d}f) = 0$ (you can work this out for Example 30.3 if you really feel like it). This is true for all forms, not just functions, and is one of the most important properties of the exterior derivative.

The exterior derivative satisfies some properties that make it feel a lot like an actual derivative.

Theorem 30.4. *Let α be a k-form and β be an ℓ-form.*

(1) d *is linear.*
(2) d *obeys the* Leibniz rule $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$.
(3) $\mathrm{d}(\mathrm{d}\alpha) = 0$.

The Leibniz rule looks like the product rule, but since forms aren't quite commutative, we need to keep track of the sign in $\alpha \wedge \mathrm{d}\beta$.

Proof. (1) is evident from the definition. For (2), we need to do a calculation. Suppose $\alpha = \sum_I \alpha_I(x)\,\mathrm{d}x^I$ and $\beta = \sum_J \beta_J(x)\,\mathrm{d}x^J$. Then,

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= \mathrm{d}\left(\sum_{I,J} \alpha_I \beta_J \,\mathrm{d}x^I \wedge \mathrm{d}x^J\right) \\
&= \sum_{j=1}^{n} \sum_{I,J} \frac{\partial \alpha_I \beta_J}{\partial x^J}\,\mathrm{d}x^j \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{j=1}^{n} \sum_{I,J} \left(\frac{\partial \alpha_I}{\partial x^j}\beta_J + \alpha_I \frac{\partial \beta_J}{\partial x^j}\right)\mathrm{d}x^j \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J.
\end{aligned}$$

Rearranging all these terms,

$$
\begin{aligned}
&= \sum_{j=1}^{n}\left(\frac{\partial \alpha_I}{\partial x^j}\, \mathrm{d}x^j \wedge \mathrm{d}x^I \wedge \beta_J\, \mathrm{d}x^J + (-1)^k \alpha_I\, \mathrm{d}x^I \wedge \frac{\partial \beta_J}{\partial x^j}\, \mathrm{d}x^j \wedge \mathrm{d}x^J\right) \\
&= \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta.
\end{aligned}
$$

Proving part (3) is also a calculation, though thankfully a shorter one. We'll assume $\mathrm{d}(\mathrm{d}f) = 0$ for all functions (0-forms) f, which is a quick exercise. Using this,

$$
\begin{aligned}
\mathrm{d}(\mathrm{d}\alpha) &= \mathrm{d}\left(\sum_{j=1}^{n}\sum_{I} \frac{\partial \alpha_I}{\partial x^j}\, \mathrm{d}x^j \wedge \mathrm{d}x^I\right) \\
&= \sum_{i,j=1}^{n}\sum_{I} \frac{\partial \alpha_I}{\partial x^j}\frac{\partial \alpha_I}{\partial x^i}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I.
\end{aligned}
$$

Why is this zero? If $i = j$, we get a $\mathrm{d}x^i \wedge \mathrm{d}x^i$ term, which is zero. If $i \neq j$, then for each multi-index I appearing in α, we have (i, j) and (j, i),

$$
\frac{\partial \alpha_I}{\partial x^j}\frac{\partial \alpha_I}{\partial x^i}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I + \frac{\partial \alpha_I}{\partial x^j}\frac{\partial \alpha_I}{\partial x^i}\, \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,
$$

and since $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$, this is zero. ⊠

Example 30.5. For another reason you should expect d to actually be a derivative, let's look at $\mathbb{R}^3$. To a function $f : \mathbb{R}^3 \to \mathbb{R}$ there's a canonical 0-form $\omega^0_f = f$, and a canonical 3-form $\omega^3_f = f\, \mathrm{d}f \wedge \mathrm{d}y \wedge \mathrm{d}z$. To a vector field $\mathbf{v} = (v_1, v_2, v_3)$, there's a canonical 1-form $\omega^1_{\mathbf{v}} = v_1\, \mathrm{d}x + v_2\, \mathrm{d}y + v_2\, \mathrm{d}z$, and a canonical 2-form $\omega^2_{\mathbf{v}} = v_1\, \mathrm{d}y \wedge \mathrm{d}z + v_2\, \mathrm{d}z \wedge \mathrm{d}x + v_3\, \mathrm{d}x \wedge \mathrm{d}y$ (be careful with signs).

These assignments encompass a lot of basic theorems in multivariable calculus.

Exercise 30.6.

(1) Show that $\mathrm{d}\omega^0_f = \omega^1_{\nabla f}$, so the exterior derivative of a 0-form is *gradient*.
(2) Show that $\mathrm{d}\omega^1_{\mathbf{v}} = \omega^2_{\nabla\times\mathbf{v}}$, so the exterior derivative of a 1-form is *curl*.
(3) Show that $\mathrm{d}\omega^2_{\mathbf{v}} = \omega^3_{\nabla\cdot\mathbf{v}}$, so the exterior derivative of a 2-form is *divergence*.

Exercise 30.7.

(1) Show that $\omega^1_{\mathbf{v}} \wedge \omega^1_{\mathbf{w}} = \omega^2_{\mathbf{v}\times\mathbf{w}}$. That is, the wedge product of two 1-forms is *cross product*.
(2) Show that $\omega^1_{\mathbf{v}} \wedge \omega^2_{\mathbf{w}} = \omega^3_{\mathbf{v}\cdot\mathbf{w}}$. That is, the wedge of a 1-form and a 2-form is *dot product*.

Some of these things, notably gradient and divergence, generalize to arbitrary n, but the cross product and curl only make sense in $\mathbb{R}^3$ (more generally, they operate on $(n-1)$ vectors).

Now, since we know $\mathrm{d}^2 = 0$, this automatically tells us that $\nabla \times (\nabla f) = 0$ for all smooth f, and $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ for all vector fields, as well as plenty of other nice statements (e.g. we'll see how this provides a generalized framework for Stokes' theorem, the divergence theorem, etc.). In this way, d can be thought of as generalizing all the differential operators we care about in multivariable calculus.

Finally, we're briefly going to talk about the pullback of differential forms. Suppose $g : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map.[28] Pullback is a contravariant operator g^* which sends forms on $\mathbb{R}^n$ to forms on $\mathbb{R}^m$: it goes the opposite way to g.

Here's the things we'd like pullback to satisfy.

Theorem 30.8. *For a smooth $g : \mathbb{R}^m \to \mathbb{R}^n$, there's a unique operator g^* from the space of forms on $\mathbb{R}^n$ to the space of forms on $\mathbb{R}^m$ such that*

- *if f is a function on $\mathbb{R}^n$, then $g^* f = f \circ g$;*
- *if α and β are forms on $\mathbb{R}^n$, then $g^*(\alpha \wedge \beta) = (g^*\alpha) \wedge (g^*\beta)$; and*
- *$g^*(\mathrm{d}\alpha) = \mathrm{d}(g^*\alpha)$.*

[28] Once we define differential forms on manifolds, we'll be able to define pullback for any smooth map between manifolds.

This theorem will also hold when we generalize to differential forms on manifolds.

We're not going to prove this today, but to get the operator we need, the relations force us to perceive the abstract symbol $\mathrm{d}y^i$ as the derivative of the i^{th} coordinate function y^i, which agrees with how we related vector fields and functions to forms in Example 30.6. Thus, we're forced to set

$$g^*(\mathrm{d}y^i) = \mathrm{d}(g^*(y^i)) = \mathrm{d}(y^i \circ g) = \sum_{j=1}^n \frac{\partial g_i}{\partial x^j}\,\mathrm{d}x^j$$

and $g^*\alpha_I = \alpha_I \circ g$ (since this is just a function). Thus, we can define the formula for the pullback: if $\alpha = \sum_I \alpha_I\,\mathrm{d}x^I$ as usual, then

$$(g^*\alpha) = \sum_I (\alpha_I \circ g)\,\mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k}.$$

This definition was forced on us, so the uniqueness in Theorem 30.8 follows. Existence will be an exercise.

Lecture 31.

Pullback of Differential Forms: 4/8/16

We spent the first few minutes reviewing what differential forms are, albeit still from the perspective of forms as abstract symbols. Since I was late, I didn't get this all written down, but it was a review of the definitions that we've made over the past two lectures.

One particular point to note is that if $f : \mathbb{R}^n \to \mathbb{R}$, so f can be regarded as a 0-form, then we defined

$$\mathrm{d}f = \sum_{j=1}^n \frac{\partial f}{\partial x^j}\,\mathrm{d}x^j.$$

Thus, if $x^i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function, $\mathrm{d}(x^i) = \mathrm{d}x^i$ as 1-forms, which is an inkling that these forms have actual geometric meaning.

We also began talking about the pullback of differential forms. The idea is that if $g : \mathbb{R}^n \to \mathbb{R}^m$ is smooth, we'd like to send forms on $\mathbb{R}^m$ to forms on $\mathbb{R}^n$, akin to change of coordinates for functions. More generally, once we define differential forms on manifolds, we'll let $g : X \to Y$ be a smooth map of manifolds, and obtain a pullback g^* from forms on Y to forms on X. We defined the pullback not explicitly, but in order to satisfy the conditions in Theorem 30.8: that for a 0-form f, $g^*(f) = f \circ g$, and that pullback commutes with $\wedge$ and d. (Of course, we also want it to be linear.)

Writing $g(x) = (g^1(x), \dots, g^m(x))$, and $(y^1, \dots, y^m)$ for coordinates on $\mathbb{R}^m$, the properties Theorem 30.8 imposes on us lead us to conclude

$$g^*(\mathrm{d}y^j) = g^*(\mathrm{d}(y^j)) = \mathrm{d}(y^j \circ g) = \mathrm{d}g^j,$$

which means that we should define the formula for the pullback in coordinates to be

$$g^*\left(\sum_I \alpha_I(x)\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) = \sum_I \alpha_I(g(x))\,\mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k}. \tag{31.1}$$

By definition, this is linear, and plugging in a function f makes the rule $g^*(f) = f \circ g$ drop out. The definition also tells us that pullback commutes with $\wedge$, but what about d? Time for another calculation. Since g^* and d are linear, it suffices to check this for a simple k-form (i.e. for only one multi-index). In particular, suppose $\alpha = \alpha_I\,\mathrm{d}y^I$. Thus,

$$\begin{aligned} g^*\alpha &= (\alpha_I \circ g)\,\mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k} \\ \mathrm{d}(g^*\alpha) &= \mathrm{d}(\alpha_I \circ g) \wedge \mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k} + (\alpha_I \circ g)\,\mathrm{d}(\mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k}) \\ &= \mathrm{d}(\alpha_I \circ g) \wedge \mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k} \end{aligned} \tag{31.2}$$

because $\mathrm{d}^2 = 0$. The other side is

$$\begin{aligned} \mathrm{d}\alpha &= (\mathrm{d}\alpha_I) \wedge \mathrm{d}y^I \\ g^*(\mathrm{d}\alpha) &= g^*\mathrm{d}\alpha_I) \wedge g^*(\mathrm{d}y^I) \\ &= (g^*(\mathrm{d}\alpha_I)) \wedge \mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k}. \end{aligned} \tag{31.3}$$

That is, equations (31.2) and (31.3) are the same if we can show that g^* and d commute for functions. Here we use the chain rule; since $\mathrm{d}f = \sum \frac{\partial f}{\partial y^j}\,\mathrm{d}y^j$, then

$$\begin{aligned} g^*(\mathrm{d}f) &= \sum_{j=1}^{m} \frac{\partial f}{\partial y^j}(g(x))\,\mathrm{d}g^j \\ &= \sum_{i,j} \frac{\partial f}{\partial y^j}(g(x))\frac{\partial g^j}{\partial x^i}\,\mathrm{d}x^i \\ &= \sum_{i=1}^{n} \frac{\partial (f\circ g)}{\partial x^i}\,\mathrm{d}x^i \\ &= \mathrm{d}(f\circ g) = \mathrm{d}(g^*(f)). \end{aligned}$$

Hence, Theorem 30.8 is true, with (31.1) the explicit formula you may have been hoping for.

Example 31.4. Let $U\subset\mathbb{R}^2$ be the open rectangle $(0,\pi)\times(0,2\pi)$, and let (θ,ϕ) be coordinates on U. One can define *spherical coordinates* on (most of) $S^2\subset\mathbb{R}^3$ through the function $g(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Let (x,y,z) be the usual coordinates on $\mathbb{R}^3$. Let's calculate $g^*(x\,\mathrm{d}y\wedge\mathrm{d}z y\,\mathrm{d}z\wedge\mathrm{d}x + z\,\mathrm{d}x\wedge\mathrm{d}y) = \omega^2_{(x,y,z)}$.

Though you could just smack this with the formula in (31.1), let's do it in pieces.

- We know what g^* does to coordinate functions: $g^*(x) = g^1 = \sin\theta\cos\phi$.
- Thus, $g^*(\mathrm{d}x) = \mathrm{d}(g^*x) = \cos\theta\cos\phi\,\mathrm{d}\theta - \sin\theta\sin\phi\,\mathrm{d}\phi$.
- In the same way, $g^*(\mathrm{d}y) = \cos\theta\sin\phi\,\mathrm{d}\theta + \sin\theta\cos\phi\,\mathrm{d}\phi$.
- Therefore, $g^*(\mathrm{d}x\wedge\mathrm{d}y) = \cos\theta\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$.

If you keep going, the final answer you get is $\sin^2\phi\,\mathrm{d}\theta\wedge\mathrm{d}\phi$. This is familiar: it's the thing we had to add to an integral to transform it to spherical coordinates! We'll soon see this is not a coincidence, since the thing we're pulling back is another volume element.

This example speaks volumes about how to relate forms to geometry: it seems like we need to integrate them. And indeed, this is one of the things differential forms are best at: a k-form on $\mathbb{R}^n$ is made for integrating on a k-submanifold of $\mathbb{R}^n$.

Let's start with an n-form α, which must have the form $\alpha = \alpha_I(x)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$ (here, $I=(1,2,\dots,n)$). We define

$$\int_{\mathbb{R}^n}\alpha = \int_{\mathbb{R}^n}\alpha_I(x)\,\mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^n.$$

Warning: there are at least two caveats to this definition.

- By Fubini's theorem, it shouldn't make a difference to switch $\mathrm{d}x_1$ and $\mathrm{d}x_2$, but for a differential form, this will switch the sign of the integral. The key is that this integration is defined by the orientation of $\mathbb{R}^n$ where $(e_1,\dots,e_n)$ is positively oriented. If we integrated $\alpha_I(x)\,\mathrm{d}x^2\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3\wedge\cdots\wedge x^n$, then we would choose the other orientation in which $(e_2,e_1,e_3,\dots,e_n)$ is positively oriented. So this definition isn't nonsense, but you need to be careful with orientation.
- One cannot integrate an arbitrary smooth function on $\mathbb{R}^n$; you have to add some sort of integrability hypothesis, such as compact support, or more generally absolutely integrable. This is not a measure-theory class, so technical hypotheses on forms will not greatly distract us, but it's important to at least notice that we're doing this.

Since we're doing geometry, there's nothing we love more than coordinate transformations: how does the integral change under coordinate transformations? If $\alpha = \alpha_I(y)\,\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n$ and $g:\mathbb{R}^n\to\mathbb{R}^n$ is smooth, then $g^*\alpha = \alpha + I(g(x))\,\mathrm{d}g^1\wedge\cdots\wedge\mathrm{d}g^n$. Because α is an n-form, there's a simple expression for this.

In the case $n=2$, it's a bit simpler to write down in coordinates:

$$\begin{aligned} \mathrm{d}g^1 &= \frac{\partial y^1}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial y^1}{\partial x^2}\,\mathrm{d}x^2 \\ \mathrm{d}g^2 &= \frac{\partial y^2}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial y^2}{\partial x^2}\,\mathrm{d}x^2. \end{aligned}$$

Therefore

$$\mathrm{d}g^1 \wedge \mathrm{d}g^2 = \left(\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

This looks like a determinant, or Jacobian, and this is true (but less fun to prove) in general: $g^*\alpha = \alpha_I(g(x))\det(\mathrm{d}g|_x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Putting this into integrals,

$$\int_{\mathbb{R}^n} g^*\alpha = \int_{\mathbb{R}^n} \alpha_I(g(x))\det(\mathrm{d}g|_x)\,\mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^n.$$

As usual in multivariable calculus, we're just multiplying by the Jacobian to change coordinates, which is what pullback really is. Keep in mind that this will flip the sign if g reverses orientation. Another way to think of this is as a version of change-of-coordinates from multivariable calculus that preserves sign.

$$\mathrm{d}y^1\,\mathrm{d}y^2\cdots\mathrm{d}y^n = \left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|\mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^n.$$

Next time, we'll generalize this to manifolds. This will be a little trickier: we need to show that the integral, which we define with coordinates, is independent of coordinates, as long as the two coordinates define the same orientation. This will allow us to define integration and prove Stokes' theorem!

Lecture 32.

Differential Forms on Manifolds: 4/11/16

Recall that we're talking about differential forms on $\mathbb{R}^n$. We started with some purely formal definitions of forms as sums $\sum_I \alpha_I\,\mathrm{d}x^I$, where I is a multi-index and the α_I are smooth functions $\mathbb{R}^n \to \mathbb{R}$. Then, we defined

$$\mathrm{d}\alpha = \sum_{j=1}^n \sum_I \frac{\partial \alpha_I}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^I,$$

and showed that, as purely formal consequences of the definition, $\mathrm{d}(\mathrm{d}\alpha) = 0$ $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$ (if β is a k-form).

A little more geometrically, we can talk about the pullback of forms: if $g : \mathbb{R}^n \to \mathbb{R}^m$ is smooth, then we can define

$$g^*\left(\sum \alpha_I(y)\,\mathrm{d}y^I\right) = \sum \alpha_I(g(x))\,\mathrm{d}g^{i_1} \wedge \cdots \wedge \mathrm{d}g^{i_k}.$$

Then, we showed that $g^*(\mathrm{d}\alpha) = \mathrm{d}(g^*\alpha)$ and $g^*(\alpha \wedge \beta) = g^*(\alpha) \wedge g^*(\beta)$.

Finally, we made the geometry more explicit: it's possible to integrate an n-form on $\mathbb{R}^n$, which suggests that, geometrically, n-forms are things that want to be integrated. If $U, V \subset \mathbb{R}^n$ are open and $g : U \to V$ is an orientation-preserving diffeomorphism, then we showed that $\int_U g^*\alpha = \int_V \alpha$.

We want to transfer this discussion to more general manifolds, including integrating n-forms on n-manifolds. Our model example will be the 2-sphere sitting inside $\mathbb{R}^3$, for concreteness, but we need to remember that we care about manifolds that aren't embedded in Euclidean spaces. Thus, we can't just define forms on S^2 as restrictions of forms on $\mathbb{R}^3$ (well, we can, but this is not the best approach, and so we won't).

Remember how we defined smooth functions? If X is a manifold, a function $f : X \to \mathbb{R}$ is smooth if for every chart $\psi : \mathbb{R}^k \to X$ on X, $\psi^* f = f \circ \psi$ is a smooth map. That is, f is smooth in all coordinate systems. We're going to use something similar to characterize differential forms on a manifold.

For example, on S^2, we have spherical coordinates $(\theta, \phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, which work everywhere except for one longitude (the international date line, so to speak). Alternatively, one can use the usual rectangular coordinates $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$, which works everywhere in the northern hemisphere.

Thinking more generally, let X be a manifold and $V \subset X$ be open. We'll use the notation $\mathscr{O}(V)$ to denote the set of smooth functions on V. If U_1 and U_2 are two different coordinate charts for V, then we can identify $\mathscr{O}(V)$ with $\mathscr{O}(U_1)$ or $\mathscr{O}(U_2)$. In fact, if $\mathscr{U} = \{U_i : i \in I\}$ is the set of all coordinate charts for V, then we can realize

$$\mathscr{O}(V) = \left(\coprod_{i \in I} \mathscr{O}(U_i)\right)\Big/ \sim, \tag{32.1}$$

where we identify two functions if they're the same under the change-of-coordinates maps $g_{ij} : U_i \to U_j$. That is, we say $h_i \sim h_j$ if $h_i = h_j \circ g_{ij}$, or, more suggestively, $h_i = g_{ij}^* h_j$.

We'll do exactly the same thing to define smooth forms on V. We'll use the notation $\Omega_\ell^*(V)$ to denote the set of smooth (local) forms on an open set V; in the same manner as (32.1), we define

$$\Omega_\ell^*(V) = \left(\coprod_{i \in I} \Omega^*(U_i) \right) / \sim,$$

where, again, $\alpha_i \in \Omega^*(U_i)$ is equivalent to $\alpha_j \in \Omega^*(U_j)$ if $\alpha_i = g_{ij}^* \alpha_j$. From this point of view, *a form on a manifold is an equivalence class of forms on its coordinate charts*.

To define $\Omega^*(V)$, we need to patch forms on $\Omega_\ell(W_i)$ for various $W_i \subset V$. In particular, a form $\alpha \in \Omega^*(V)$ is an equivalence class of forms $\alpha_j \in \Omega_\ell^*(W_j)$, for $j \in J$, where $\{W_j : j \in J\}$ is an open cover of V. The equivalence relation is that two forms are the same if they agree as forms on Ω_ℓ^* on some open cover.

If $\psi_i : U_i \to V$ is the chart map, then its pullback defines a map $\psi_i^* : \Omega^*(V) \to \Omega^*(U_i)$, called *realization in ψ_i-coordinates*. In particular, the *realization* of α is $\alpha_i = \psi_i^* \alpha$. Since α is an equivalence class, this seems fishy, but if you pick a different representative for it, you get the same α_i, so the realization is well-defined.

Now, we have a set of differential forms $\Omega^*(X)$, but it would be nice to have algebraic operations like we did on $\mathbb{R}^n$: forms were a real vector space with wedge product, d, and pullback. Some of these are conceptually easy.

- The vector space structure can be defined in coordinates, and turns out to be independent of coordinate chart used. Thus, we simply define addition and scalar multiplication in local neighborhoods, which works.
- In the same way, $\wedge$ and d can be defined locally, which ultimately follows because they commute with pullbacks.
- Pullback is slightly more complicated. We also define it in coordinates: let $f : X \to Y$ be a smooth map, where X and Y are both n-dimensional manifolds. Then, if ψ is a chart for an open subset of X and ϕ is one for an open in Y, and if α is an n-form on Y, we can define $f^*\alpha$ in coordinates: $\psi^*(f^*\alpha) = h^*(\phi^*\alpha)$, where $h = \phi^{-1} \circ f \circ \psi$ is the change-of-coordinates map. This turns out to be independent of coordinates (which is something that has to be checked), and we can assemble this local definition into a global one.

We've done a lot of stuff, but what is a form, really? Don't worry about this right now; it should unsettle you, but we'll talk about what they actually, geometrically mean once we've proven Stokes' theorem. So hold tight for now.

Integration of Forms on Manifolds. We have a little intuition of forms as "things that should be integrated," so let's integrate them. We needed an orientation for this, so we'll only consider oriented manifolds. As such, let X be an oriented, n-dimensional manifold and α be an n-form on X.

First, suppose $V \subset X$ is open and $\psi : U \to V$ is a coordinate chart for V, so $U \subset \mathbb{R}^n$. If α is supported on V, then we can define

$$\int_V \alpha = \int_U \psi^* \alpha.$$

For example, suppose X is the unit circle inside $\mathbb{R}^2$, which is a 1-manifold. Let $\alpha = p(x,y)\,\mathrm{d}x + q(x,y)\,\mathrm{d}y$, which is a 1-form on S^1 and $\mathbb{R}^2$. The integral just becomes a line integral:

$$\int_{S^1} \alpha = \oint_{S^1} p(x,y)\,\mathrm{d}x + q(x,y)\,\mathrm{d}y.$$

We evaluate this by parameterizing it:

$$= \int_{t_0}^{t_1} \left(p(\gamma(t)) \frac{\mathrm{d}x}{\mathrm{d}t} + q(\gamma(t)) \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t.$$

Here, $\gamma^*(\mathrm{d}x) = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$, and similarly with $\gamma^*(\mathrm{d}y)$. Thus, this parameterization is really a pullback to a 1-form on $\mathbb{R}$, and then integrating as usual. Surface integrals are pullbacks to $\mathbb{R}^2$. So this definition of integrals, perhaps really abstract on first glance, is what we've really been doing all along.

Finally, what if you want to integrate a general differential form, not just supported in one chart? In this case, we choose a partition of unity to define it formally (though of course this isn't how it's done in practice): a form is a locally finite sum of forms that are supported on coordinate charts, so you can make the definition. However, this is awful for actually computing things, so there are other techniques for evaluating once you know this definition works.

Lecture 33.

Stokes' Theorem: 4/13/16

Last time, we defined differential forms on manifolds and how to integrate them: if X is an n-dimensional manifold and $\alpha \in \Omega^n(X)$ is supported in a single coordinate chart $\psi : \mathbb{R}^n \to X$, it's simple to integrate α, because we defined

$$\int_X \alpha = \int_{\mathbb{R}^n} \psi^* \alpha.$$

For this to be defined unambiguously, ψ must be consistent with the choice of orientation on X.

For a more general differential form α, we can use a partition of unity subordinate to a cover of X by charts to write α as a sum of differential forms supported in charts. But if you try to use this to calculate anything, it's very difficult. There's a better way: deform your charts so that they barely overlap, and as the amount of overlap goes to zero, the difference between $\int_X \alpha$ and the sum of the integrals of $\psi^*\alpha$ on each chart becomes zero as well, because the two only differ on the overlaps of charts.

For example, on the circle, there are two charts $(0, 2\pi)$ and $(-\pi, \pi)$ which overlap everywhere except at 0 and π. However, we can fix them to barely overlap: for any $\varepsilon > 0$, $(\pi/2 - \varepsilon, 3\pi/2 + \varepsilon)$ and $(-\pi/2 - \varepsilon, \pi/2 + \varepsilon)$ also is a cover of S^1 by charts, and so any partition of unity is 1 everywhere except on an arbitrarily small region. This provides one method to compute integrals in practice.

The thing we want to do today is Stokes' theorem, a very nice statement relating integration, the exterior derivative, and boundaries of manifolds.

Recall that if X is an oriented manifold-with-boundary, then ∂X is oriented with the convention that the outward normal is first.[29]

Theorem 33.1 (Stokes)**.** *Let X be an oriented, n-dimensional manifold-with-boundary and α be a compactly supported $(n-1)$-form on X. Then,*

$$\int_X \mathrm{d}\alpha = \int_{\partial X} \alpha.$$

In particular, this is true for all forms on a compact manifold. Compact support is necessary, however: if $X = [0, \infty)$, consider the 0-form $f(x) = e^{-x} - 1$, which doesn't vanish on X. Then,

$$\int_X \mathrm{d}f = \int_0^\infty f'(x)\,\mathrm{d}x = \int_0^\infty -e^{-x}\,\mathrm{d}x = 1.$$

However, $\int_{\partial X} f = -f(0) = 0$ (since $0 \in \partial X$ is oriented negatively). So compact support is important.

Recall that we defined $H^n = \{x \in \mathbb{R}^n \mid x^n \geq 0\}$, the vectors whose last component is nonnegative. We know $\partial H^n = \mathbb{R}^{n-1} \times \{0\}$ as unoriented manifolds, but what happens when we throw in orientation? The standard positively oriented basis on H^n is $(e_1, \dots, e_n)$, and the outward normal for ∂H^n is $-e_n$. So we choose the orientation such that $(-e_n, e_1, \dots, e_{n-1})$ is positively oriented, so we need to make $n-1$ transpositions, plus one more to change $-e_n \mapsto e_n$. Hence, as oriented manifolds, $\partial H^n = (-1)^n \mathbb{R}^{n-1} \times \{0\}$.

Example 33.2. Let's see what this looks like on $\mathbb{R}^3$. Let α be a compactly supported 2-form on H^3, so $\alpha = \alpha_{12}\,\mathrm{d}x \wedge \mathrm{d}y + \alpha_{13}\,\mathrm{d}x \wedge \mathrm{d}z + \alpha_{23}\,\mathrm{d}y \wedge \mathrm{d}z$. After a calculation,

$$\mathrm{d}\alpha = \left(\frac{\partial \alpha_{12}}{\partial x^3} - \frac{\partial \alpha_{13}}{\partial x^2} + \frac{\partial \alpha_{23}}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{33.3}$$

The integral of α is

$$\int_{H^3} \alpha = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\frac{\partial \alpha_{12}}{\partial x^3} - \frac{\partial \alpha_{13}}{\partial x^2} + \frac{\partial \alpha_{23}}{\partial x^1}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

[29] Dan Freed likes to use the mnemonic "one never forgets" for this convention.

Thus, the integral breaks up into three parts. Using the fundamental theorem of calculus,

$$\int_{-\infty}^{\infty} \frac{\partial \alpha_{23}(x^1, x^2, x^3)}{\partial x^1} = \lim_{t\to\infty} \big(\alpha_{23}(t, x^2, x^3) - \alpha_{23}(-t, x^2, x^3)\big) = 0,$$

because α is compactly supported. Thus, the third part of the integral in (33.3) is zero. In precisely the same way, the integral of $\frac{\partial \alpha_{13}}{\partial x^2}$ is also zero. However, since we're on the half-plane, the first term is different: we have the boundary at $x^3 = 0$.

$$\begin{aligned}\int_{H^3} \frac{\partial \alpha_{12}}{\partial x^3}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 &= \iint_{\mathbb{R}^2} \left(\int_0^\infty \frac{\partial \alpha_{12}}{\partial x^3}\,\mathrm{d}x^3\right)\mathrm{d}x^1\,\mathrm{d}x^2\\ &= \iint_{\mathbb{R}^2} -\alpha_{12}(x^1, x^2, 0)\,\mathrm{d}x^1\,\mathrm{d}x^2\\ &= \iint_{\partial H} \alpha.\end{aligned}$$

Thus, we've proven Stokes' theorem for H^3 and ∂H^3.

It's clear that nothing changes for H^n and ∂H^n for $n \neq 3$, though spelling out what happens with the orientations is a good thing to do. In any case, we're ready to prove Stokes' theorem in full generality.

Proof of Theorem 33.1. Let α be a compactly supported $(n-1)$-form on X. Using a partition of unity subordinate to a coordinate cover of X, we can assume that α is compactly supported in a coordinate chart $\psi : U \to X$ for some open $U \subset H^n$. (More generally, an $(n-1)$-form will be a finite sum of these forms, and since integration is linear, this will imply Stokes' theorem for such forms.) Since we've already proven Stokes' theorem for H^n and ∂H^n, and we know $\psi(U \cap \partial H^n) = \psi(U) \cap \partial X$, then

$$\begin{aligned}\int_X \mathrm{d}\alpha = \int_U \psi^*(\mathrm{d}\alpha) &= \int_{H^n} \mathrm{d}(\psi^*\alpha)\\ &= \int_{\partial H^n} \psi^*\alpha = \int_{U\cap\partial H^n} \psi^*\alpha\\ &= \int_{\partial X} \alpha.\end{aligned}$$ ⊠

For such an austere-looking theorem, the proof was surprisingly trivial. The hard work is not in the proof; rather, it's understanding what integration and forms mean, and why they're well-defined on manifolds. Once you've defined these things properly, Stokes' theorem is effectively an important triviality, a corollary of a tricky formalism.

Definition 33.4. Let β be a form on a manifold X.

- β is *closed* if $\mathrm{d}\beta = 0$.
- β is *exact* if $\beta = \mathrm{d}\alpha$ for some form α.

So exact forms are derivatives of something else. Since $\mathrm{d}^2 = 0$, then every exact form is closed. A very fruitful question is to ask, given a manifold X, to what degree is the converse true?

One somewhat obvious-looking corollary of Stokes' theorem is also pretty useful.

Corollary 33.5. *If X is a manifold without boundary, and α is any exact n-form on X, then $\int_X \alpha = 0$.*

This is because $\alpha = \mathrm{d}\beta$ for some β and $\int_X \mathrm{d}\beta = \int_\varnothing \beta = 0$.

Let's think about closed and exact forms on $X = \mathbb{R}$. A closed 0-form is a function f such that $\mathrm{d}f = 0$, meaning the space of constants. But no form is a derivative of anything else (what would a -1-form be?), though by convention 0 is exact. Thus, the space of closed forms modulo the space of exact forms is isomorphic to $\mathbb{R}$.

What about 1-forms? Every 2-form on $\mathbb{R}$ is equal to 0, so if α is a 1-form, then $\mathrm{d}\alpha = 0$. However, every 1-form is also exact, because

$$\alpha(x)\,\mathrm{d}x = \int_0^x \alpha(t)\,\mathrm{d}t,$$

by the fundamental theorem of calculus.

In general, if X is a manifold, the closed k-forms on X form a real vector space, and the exact k-forms are a subspace of the closed forms. Thus, it makes sense to take the quotient.

Definition 33.6. The k^{th} *de Rham cohomology* of X, denoted $H^k_{\text{dR}}(X)$, is the space of closed k-forms on X modulo the subspace of exact k-forms on X.

We've just shown that $H^0_{\text{dR}}(\mathbb{R}) \cong \mathbb{R}$ and $H^1_{\text{dR}}(\mathbb{R}) = 0$. More generally, we have the Poincaré lemma.

Lemma 33.7 (Poincaré)**.**

$$H^k_{\text{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n > 0. \end{cases}$$

In other words, on $\mathbb{R}^n$, every closed k-form for $k > 0$ is exact. This is not true for general manifolds; even on S^1 one can write down a counterexample. This is what makes de Rham cohomology interesting.

Lecture 34.

Tensors: 4/15/16

"I like 3. *A lot of people like* 3. 47 *is good too."*

We defined the space of differential forms as a bunch of formal symbols, sums of $\mathrm{d}x^1, \dots, \mathrm{d}x^n$, subject to the anticommutativity relation $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$. One way to make this precise is to define the space of forms as the universal algebra generated by $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ subject to the anticommutativity relation. Then we did a bunch of geometry that happened to use these, which seems a little phony — differential forms can be constructed as actual, geometric objects, a representation of the abstract algebra of forms. Some people prefer the algebra, and others prefer the geometry.

Start with a real, n-dimensional vector space V and a basis $\mathscr{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. We'll let V^k denote a product of k copies of V.

Definition 34.1. A *k-tensor* is a function $T : V^k \to \mathbb{R}$ that is *multilinear*, i.e. linear in each argument if the other arguments are fixed. The space of all k-tensors on V is denoted $\mathscr{T}^k(V^*)$.[30]

Since V^0 is identified with $\mathbb{R}$, then $\mathscr{T}^0(V^*) \cong \mathbb{R}$, in the sense of constant functions, and $\mathscr{T}^1(V^*)$ is the space of linear maps $V \to \mathbb{R}$. This is the *dual space* of V, which is denoted V^*. Just as elements of V are called vectors, elements of V^* are called *covectors*.

Since we've chosen a basis for V, then if $\alpha \in V^*$, then since it's linear, then for any $\mathbf{v} = (v^1, \dots, v^n) \in V$,

$$\alpha(\mathbf{v}) = \sum_{i=1}^n v^i \alpha(\mathbf{b}_i).$$

That is, α is determined by what it does on $\mathscr{B}$. This enables us to write down a basis for V^*: if we define the coordinate functions $\phi^i(\mathbf{v}) = v^i$, then

$$\alpha(\mathbf{v}) = \sum_{i=1}^n v^i \alpha(\mathbf{b}_i) = \sum_{i=1}^n \alpha(\mathbf{b}_i)\phi^i(\mathbf{v}),$$

or $\alpha = \sum \alpha(\mathbf{b}_i)\phi^i$. Thus, the ϕ^i span V^*, and they're linearly independent: if $\alpha = \sum \alpha + i\phi^i = 0$, then $0 = \alpha(\mathbf{b}_i) = \alpha_i$, so all of the coefficients are 0. Thus, $\{\phi^i\}$ is indeed a basis for V^*, and is called the *dual basis* to $\mathscr{B}$; if you start with a different basis of V, you end up with a different basis for V^*.

[30] There's a more general story of *covariant* and *contravariant tensors*, or even tensors that are covariant on some indices and contravariant on others. All the tensors we'll use in this class are contravariant, so we won't discuss the difference.

The tensor product. Suppose $\alpha \in \mathscr{T}^k(V^*)$ and $\beta \in \mathscr{T}^\ell(V^*)$; we'd like to "multiply" them and define their tensor $\alpha \otimes \beta \in \mathscr{T}^{k+\ell}(V^*)$.[31]

Definition 34.2. If $\alpha \in \mathscr{T}^k(V^*)$ and $\beta \in \mathscr{T}^\ell(V^*)$, their *tensor product* is the function $\alpha \otimes \beta \in \mathscr{T}^{k+\ell}(V^*)$ defined by

$$\alpha \otimes \beta)(v_1, \dots, v_{k+\ell}) = \alpha(v_1, \dots, v_k)\beta(v_{k+1}, \dots, v_\ell).$$

Exercise 34.3. Prove that the tensor product satisfies the following properties for $\alpha \in \mathscr{T}^{k_1}(V^*)$, $\beta \in \mathscr{T}^{k_2}(V^*)$, $\gamma \in \mathscr{T}^{k_3}(V^*)$, and $c \in \mathbb{R}$.

(1) $(\alpha + \beta) \otimes \gamma = \alpha \otimes \gamma + \beta \otimes \gamma$.
(2) $\alpha \otimes (\beta + \gamma) = \alpha \otimes \beta + \alpha \otimes \gamma$.
(3) $(c\alpha) \otimes \beta = \alpha \otimes (c\beta) = c(\alpha \otimes \beta)$.

This allows us to define a basis for $\mathscr{T}^2(V^*)$, given by $\{\phi^i \otimes \phi^j : 1 \le i, j \le n\}$. We do need to prove this, but the argument is very similar to the argument for V^*. Let $\alpha \in \mathscr{T}^2(V^*)$ and $\alpha_{ij} = \alpha(\mathbf{b}_i, \mathbf{b}_j)$. Then, for any $\mathbf{v}, \mathbf{w} \in V$,

$$\begin{aligned}\alpha(\mathbf{v}, \mathbf{w}) &= \alpha\left(\sum_{i=1}^n v^i \mathbf{b}_i, \sum_{j=1}^n w^j \mathbf{b}_j\right)\\ &= \sum_{i,j=1}^n v^i w^j \alpha(\mathbf{b}_i, \mathbf{b}_j)\\ &= \sum_{i,j=1}^n \alpha_{ij} \phi^i \otimes \phi^j(\mathbf{v}, \mathbf{w}).\end{aligned}$$

Thus, the $\phi^i \otimes \phi^j$ span $\mathscr{T}^2(V^*)$, and they're linearly independent, because if $\alpha = \sum \alpha_{ij} \phi^i \otimes \phi^j = 0$, then $\alpha(\mathbf{b}_k, \mathbf{b}_\ell) = \alpha_{k\ell} = 0$ for all k and ℓ. Tensors of three dual basis elements define a basis of $\mathscr{T}^3(V^*)$, and those of four dual basis members for $\mathscr{T}^4(V^*)$, and so on; the arguments are precisely the same, if notationally hairier. One way to make the notation nicer (which is helpful because formalizing this argument is on the homework) is to let $I = (i_1, \dots, i_k)$ be a multi-index,[32] and define $\Phi^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$. Then, the set of Φ^I for all multi-indices I of length k will be a basis for $\mathscr{T}^k(V^*)$, meaning $\dim(\mathscr{T}^k(V^*)) = n^k$. Unlike differential forms, these are interesting for arbitrarily large k: nontrivial 17-tensors exist on $\mathbb{R}^3$, which is not true for forms.

Alternating tensors.

Definition 34.4. A k-tensor α is *alternating* if for any $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ and any permutation $\sigma \in S_k$,[33] $\alpha(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}) = (\operatorname{sign}\sigma)\alpha(\mathbf{v}_1, \dots, \mathbf{v}_k)$. The space of alternating k-tensors on V is denoted $\Lambda^k(V^*)$.

We constructed alternating forms as a subspace of $\mathscr{T}^k(V^*)$; there's another construction which realizes them as a quotient, forcing the alternating relation on the tensors. Some people prefer the second definition, but we're now following Guillemin and Pollack again, and they give the subspace definition.

A 0-tensor is trivially alternating (are you a scholar of the empty set?), as is a 1-tensor. Not all 2-tensors or higher tensors are alternating, but every 2-tensor has a symmetric piece and an alternating piece: if $\alpha \in \mathscr{T}^2(V^*)$, then let

$$\begin{aligned}\alpha_+(\mathbf{v}, \mathbf{w}) &= \frac{1}{2}(\alpha(\mathbf{v}, \mathbf{w}) + \alpha(\mathbf{w}, \mathbf{v}))\\ \alpha_-(\mathbf{v}, \mathbf{w}) &= \frac{1}{2}(\alpha(\mathbf{v}, \mathbf{w}) - \alpha(\mathbf{w}, \mathbf{v})).\end{aligned}$$

It's quick to check that $\alpha = \alpha_+ + \alpha_-$, α_+ is symmetric,[34] and α_- is alternating. We'll also call $\alpha_- = \operatorname{Alt}\alpha$. This has something to do with the representation theory of $S_2 \cong \mathbb{Z}/2$.

[31] There is also a category-theoretic definition of the tensor product that uses a universal property. We will not define it here, but the idea is that multilinear maps factor through the tensor product.

[32] Since tensor products' indices are allowed to repeat, we're going to use multi-indices whose terms can repeat. This is slightly different than what we did for differential forms.

[33] Here, S_k is the *symmetric group* on k elements, meaning the set of bijections $\{1, \dots, k\} \to \{1, \dots, k\}$. The *sign* of such a permutation is $(-1)^m$, where m is the number of transpositions needed to make that permutation.

[34] A k-tensor is *symmetric* if for any $\sigma \in S^n$, $\alpha(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)} = \alpha(\mathbf{v}_1, \dots, \mathbf{v}_k)$; in other words, you can rearrange the indices and nothing changes.

This is a nice thing, but can we do it for higher-order tensors? For a 3-tensor β, we try adding up all the permutations: $\beta(\mathbf{u},\mathbf{v},\mathbf{w})+\beta(\mathbf{v},\mathbf{w},\mathbf{u})+\beta(\mathbf{w},\mathbf{u},\mathbf{v})-\beta(\mathbf{v},\mathbf{u},\mathbf{w})-\beta(\mathbf{u},\mathbf{w},\mathbf{v})-\beta(\mathbf{w},\mathbf{v},\mathbf{u})$ is an alternating tensor, yay! However, it would be nice to have a projection $\mathscr{T}^k(V^*) \twoheadrightarrow \Lambda^k(V^*)$, meaning that we want something that's the identity on alternating tensors. If β were alternating, we'd have ended up with 6β, which is no good, so we need to average, rather than sum.

Definition 34.5. The *alternating part* of a k-tensor α is

$$\operatorname{Alt}(\alpha)(\mathbf{v}_1,\dots,\mathbf{v}_k)=\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sign}(\sigma)\alpha(\mathbf{v}_{\sigma(1)},\dots,\mathbf{v}_{\sigma(k)}).$$

This defines a linear projection $\operatorname{Alt}:\mathscr{T}^k(V^*)\twoheadrightarrow\Lambda^k(V^*)$.

The point of doing this is that it allows us to define a product on $\Lambda^k(V^*)$. The idea is that for an $\alpha\in\Lambda^k(V^*)$ and a $\beta\in\Lambda^\ell(V^*)$, we'd like to define $\alpha\wedge\beta=C_{k,\ell}\operatorname{Alt}(\alpha\otimes\beta)$ for some constants $C_{k,\ell}$. There are only two choices for $C_{k,\ell}$ that make the wedge product associative.

- Guillemin and Pollack define $C_{k,\ell}=1$. This means that
$$\phi^i\wedge\phi^j=\frac{\phi^i\otimes\phi^j-\phi^j\otimes\phi^i}{2},$$
which calculates the area of the triangle spanned by two vectors. We wanted a parallelepiped, so we're instead going to use Spivak's convention.
- The choice that we will use is $C_{k,\ell}=(k+\ell)!/k!\ell!$. This actually gives us areas of parallelepipeds and determinants, which is good.

Unlike the other times we've had to make sign choices, we are *not* going with Guillemin and Pollack's convention.

Lecture 35.

Exterior Algebra: 4/18/16

Recall that last time, we defined $\mathscr{T}^k(V^*)$ to be the space of (contravariant) k-tensors, which are multilinear maps $V\times\cdots\times V^k\to\mathbb{R}$. Such a tensor α is alternating if for every permutation $\sigma\in S_n$, $\alpha(\mathbf{v}_{\sigma(1)},\dots,\mathbf{v}_{\sigma(k)}=(\operatorname{sign}\sigma)\alpha(\mathbf{v}_1,\dots,\mathbf{k})$. We introduced a function

$$(\operatorname{Alt}\alpha)(\mathbf{v}_1,\dots,\mathbf{v}_k)=\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sign}\sigma\,\alpha(\mathbf{v}_{\sigma(1)},\dots,\mathbf{v}_{\sigma(k)}.$$

For any $\alpha\in\mathscr{T}^k(V^*)$, this is alternating, and $\operatorname{Alt}(\operatorname{Alt}(\alpha))=\operatorname{Alt}(\alpha)$, so this can be thought of as a projection $\operatorname{Alt}:\mathscr{T}^k(V^*)\to\Lambda^k(V^*)$ (the space of alternating k-tensors). Then, we *provisionally* defined the wedge product as $\alpha\wedge\beta=\operatorname{Alt}(\alpha\otimes\beta)$. This is associative: the proof idea is that if α, β, and γ are tensors, then because Alt is a projection, $\operatorname{Alt}((\alpha\otimes\beta-\operatorname{Alt}(\alpha\otimes\beta))\otimes\gamma)=0$, and therefore $\operatorname{Alt}(\alpha\otimes\beta\otimes\gamma)=(\alpha\wedge\beta)\wedge\gamma$ and $\alpha\wedge(\beta\wedge\gamma)$ (since the tensor product is associative).

This isn't quite what we wanted, though; let $\phi^1,\dots,\phi^n$ be a basis for V^*. We would like their n-fold wedge product to be $\phi^1\wedge\cdots\wedge\phi^n=\det(\phi^1\ \cdots\ \phi^n)$, but this isn't true; there's an extra factor of $n!$. Hence, we actually have to define the wedge product as

$$\alpha\wedge\beta=\frac{(k+\ell)!}{k!\ell!}\operatorname{Alt}(\alpha\otimes\beta).$$

This means that $(\alpha\wedge\beta)\wedge\gamma\neq\operatorname{Alt}(\alpha\otimes\beta\otimes\gamma)$ (there's a constant factor that you must take into account), but this wedge product is associative and sends a basis to the determinant of its matrix, as we would like. It's not commutative: if α is a k-form and β is an ℓ-form, then $\alpha\wedge\beta=(-1)^{k\ell}\beta\wedge\alpha$.

If you look at these for all k, formally

$$\Lambda^\bullet(V^*)=\bigoplus_{k=0}^\infty\Lambda^k(V^*),$$

you get a ring structure, where the multiplication is wedge product, which is associative but not commutative. This is an algebra over $\mathbb{R}$, called the *exterior algebra* of V^*.

Pullbacks. Since we're trying to recover differential forms, we should next talk about pullbacks. Let $L : V \to W$ be a linear map between vector spaces and $\alpha \in \mathscr{T}^k(W^*)$; we can define $L^*\alpha \in \mathscr{T}^k(V^*)$ by $L^*\alpha(\mathbf{v}_1, \dots, \mathbf{v}_k) = \alpha(L(\mathbf{v}_1), \dots, L(\mathbf{v}_k))$. This defines a linear map $L^* : \mathscr{T}^k(W^*) \to \mathscr{T}^k(V^*)$, the pullback map induced from L.

If $k = 1$, we recover something familiar, $L^* : W^* \to V^*$, called the *transpose*.[35] In a basis, this is really the transpose of the matrix for L: suppose $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. One thinks of V and W as spaces of column vectors, and V^* and W^* as spaces of row vectors. Hence, if $\alpha \in W^*$, we regard $\alpha = (a_1, \dots, a_m)$ as a $1 \times m$ matrix. Thus, $L^*\alpha(\mathbf{v})$ is just the multiplication $\alpha(L\mathbf{v})$: since L is $m \times n$ and $\mathbf{v}$ is $1 \times n$, then this is legit. It seems a little odd that L^* acts on α from the right, but this is why it's actually a transpose: if the entries of L are labeled L_{ij}, then the i^{th} entry of $L^*\alpha$ is

$$(L^*\alpha)_i = (\alpha L)_i = \sum_{j=1}^n \alpha_j L_{ji} = \sum_{j=1}^n L_{ij}^{\mathrm{T}} \alpha_j.$$

In other words, in a basis, this really is L^{T}.

Back to manifolds. This is not the first time we've introduced a concept for manifolds (smooth maps, orientations, etc.) by defining them on vector space and stitching them together for manifolds.

Definition 35.1.

- Let X be a smooth manifold and $x \in X$. Then, the *cotangent space* of X at x is $T_x^*X = (T_xX)^*$, i.e. the dual vector space to the tangent space.
- The cotangent spaces vary smoothly, and therefore form the *cotangent bundle* $T^*X = \{(x, \phi) \mid x \in X, \phi \in T_x^*X\}$.

That this is really a bundle isn't obvious: we'd like there to be smooth sections locally. We know such sections exist for the tangent bundle: if $x \in X$, let $\psi : U \to X$ be a coordinate neighborhood and $e_1, \dots, e_n$ be local coordinates for this chart in U. Then, for each $p \in U$, we obtain a basis for T_pX by $b_{p,i} = \mathrm{d}\psi_{\psi^{-1}(p)}(e_i)$. Since the basis $(e_1, \dots, e_n)$ induces a dual basis for $(\mathbb{R}^n)^*$, then we can do the same thing to locally obtain a smoothly varying basis of T^*X.

The same construction works for $\Lambda^k(T^*X) = \{(x, \omega) \mid x \in X, \omega \in \Lambda^k(T^*X)\}$: we have a basis in each chart, and can push it onto X.

Definition 35.2. A *k-form* on a manifold X is a section of $\Lambda^k(T^*X)$, i.e. a smooth map $\omega : X \to \Lambda^k(T^*X)$ such that $\omega(x) \in \Lambda^k(T_x^*X)$ for all $x \in X$. The space of k-forms on X is denoted $\Omega^k(X)$.

In other words, a k-form is something that looks locally like

$$\omega = \sum_I \omega_I(x) \phi^{i_1} \wedge \cdots \wedge \phi^{i_k}, \tag{35.3}$$

where $(\phi^1, \dots, \phi^n)$ is the dual basis on a coordinate chart for X.

This also helps us understand the exterior derivative. If $f : X \to \mathbb{R}$, we've already defined $\mathrm{d}f : TX \to T\mathbb{R} = \mathbb{R}$, and for every $p \in X$, $\mathrm{d}f|_p$ is a linear map $T_pX \to \mathbb{R}$; that is, $\mathrm{d}f|_p \in T_p^*X$, so $\mathrm{d}f$ is a section $X \to T^*X$, i.e. a 1-form!

If x^i is the i^{th} coordinate function (in local coordinates) and $b_{p,j}$ denotes the dual basis vector for T_p^*X that we introduced earlier, then $\mathrm{d}x^i(b_{p,j}) = 1$ if $i = j$ and is 0 otherwise — but this is the same as $\phi^i(b_{p,j})$, so they must be the same 1-tensor. So the symbols $\mathrm{d}x^i$ aren't really vague, mysterious symbols; rather, they're just the derivatives of coordinate functions, reinterpreted as 1-forms. In particular, (35.3) can be rewritten: a k-form ω is something that locally looks like

$$\omega = \sum_I \omega_I(x) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

In other words, *the definition of differential forms through tensors recovers exactly what we had before*.

We're still missing two important things that we defined before: pullbacks and the exterior derivative. Today, we're defining the exterior derivative axiomatically, rather than through a nebulous formula.

Proposition 35.4. *For each $k \geq 0$, there's a unique linear operator $\mathrm{d} : \Omega^k(X) \to \Omega^{k+1}(X)$, called the* exterior derivative*, such that*

(1) $\mathrm{d} : \Omega^0(X) \to \Omega^1(X)$ *is the usual derivative $f \mapsto \mathrm{d}f$.*

[35]This was called the adjoint map in Applied I, but that terminology is less common unless V and W are inner product spaces.

(2) $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^k\alpha\wedge\mathrm{d}\beta$.
(3) *If* $\alpha\in\Omega^{k-1}(X)$, *then* $\mathrm{d}(\mathrm{d}\alpha) = 0$.

The proof is to show that our usual formula in local coordinates satisfies these axioms: if $\alpha = \sum_I \alpha_I(x)\,\mathrm{d}x^I$, then we let

$$\mathrm{d}\alpha = \sum_I \mathrm{d}\alpha_I \mathrm{d}x^I,$$

since $\alpha_I \wedge \mathrm{d}(\mathrm{d}x^I) = 0$. This has to be the right formula: the second axiom requires us to break $\mathrm{d}\alpha$ into $\mathrm{d}(\alpha_I\wedge\mathrm{d}x^I)$ and use the rule for wedge products. Then, one has to check that the axioms are satisfied, but this follows by induction. The point is, d is not some arcane formula, but rather is a natural extension of the derivative of functions. We'll talk about pullback next time.

Lecture 36.

The Intrinsic Definition of Pullback: 4/20/16

Recall that if X is a manifold, we defined differential k-forms to be the sections of $\Lambda^k(T^*X)$, the k^{th} exterior power of the cotangent bundle, i.e. the vector bundle which at every point $x\in X$ is $\Lambda^k(T_x^*X)$. If α is a k-form, it's a way to evaluate k tangent vectors and return a number, in a way that varies smoothly over the manifold, and it's alternating.

In a coordinate neighborhood $\psi : U\to X$ of a point $x\in X$ (so $U\subset\mathbb{R}^n$), a basis $e_1,\dots,e_n$ for $\mathbb{R}^n$ and its corresponding dual basis $\mathrm{d}x^1,\dots,\mathrm{d}x^n$ induce a basis of k-forms in the neighborhood $\psi(U)$; specifically, the basis is $\{\mathrm{d}x^I\}$ where I ranges over all length-k multi-indices valued in $\{1,\dots,n\}$ whose terms are strictly increasing (this is induced from the basis of $\mathscr{T}^k(V^*)$, but some of those are sent to 0 by Alt).

Now, suppose $f : X\to Y$ is a smooth map of manifolds. We'd like to define the pullback $f^* : \Omega^k(Y)\to\Omega^k(X)$ in an intrinsic, coordinate-free way. The heuristic is that if $\alpha\in\Omega^k(Y)$, we want $f^*\alpha$ to be something that consumes k tangent vectors on X, in terms of things on Y through f. We can send tangent vectors to tangent vectors using $\mathrm{d}f : TX\to TY$, which suggests defining the pullback as

$$f^*\alpha|_x(v_1,\dots,v_k) = \alpha(\mathrm{d}f|_x(v_1),\dots,\mathrm{d}f|_x(v_k)).$$

From this definition, it's a quick computation that $f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$. More interestingly, if $g : Y\to Z$ is another smooth map, $(g\circ f)^* = f^*\circ g^*$: pullback reverses compositions. This follows from the chain rule, that $\mathrm{d}(g\circ f) = \mathrm{d}g\circ\mathrm{d}f$, and makes sense: $g^*\circ f^*$ doesn't make sense, as the domain and codomain don't match up. More explicitly, if $x\in X$ and $v_1,\dots,v_k\in T_{f(x)}Y$, then

$$\begin{aligned}(g\circ f)^*\alpha|_x(v_1,\dots,v_k) &= \alpha(\mathrm{d}(g\circ f)|_x(v_1),\dots,\mathrm{d}(g\circ f)|_x(v_k))\\ &= \alpha|_x(\mathrm{d}g|_{f(x)}(\mathrm{d}f|_x(v_1)),\dots,\mathrm{d}g|_{f(x)}(\mathrm{d}f|_x(v_k)))\\ &= g^*\alpha|_x(\mathrm{d}f|_x(v_1),\dots,\mathrm{d}f|_x(v_k))\\ &= (f^*(g^*\alpha))|_x(v_1,\dots,v_k).\end{aligned}$$

We showed that the pullback can be uniquely characterized by a few properties, so if we show those properties hold for this definition of pullback, we'll automatically know they're the same, which means we can avoid an argument with the big formula.

First, if h is a 0-form, then $f^*h = h\circ f$, because there are no tangent vectors to push forward. Then, we need to show that if $y^1,\dots,y^m$ are local coordinates for a neighborhood of $f(x)\in Y$, then

$$f^*(\mathrm{d}y^i) = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}\,\mathrm{d}x^j.$$

This is a little messier than one might expect, since you have to be explicit about what your charts are, but it's a rote computation that ultimately comes from the change-of-basis matrix.

These calculations illustrate an important principle in the topology and geometry of manifolds, the environmentalists' slogan[36] "think globally, act locally:" when you want to define things, it's important that they be intrinsic and geometric, but to compute anything you'll probably need coordinates.

[36] I was expecting "leave no trace," but that's more applicable to the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$.

As an example of this, we're going to define a form ω intrinsically. Suppose we have a continuous choice of inner product on an orientable manifold X,[37] meaning that we can compute volumes of parallelepipeds in tangent spaces and that this volume varies smoothly. The orientation means that we can compute signed volumes in a well-defined way; there's a consistent orientation on each tangent space, so the sign of the volume is unambiguous.

Then, let $\omega(v_1, \dots, v_k)$ be the signed k-dimensional volume of the parallelepiped spanned by $v_1, \dots, v_k$. This indeed defines a differential form $\omega \in \Omega^k(X)$, though you have to check it's linear. This ω is called a *volume form*; we defined it globally, but in local coordinates it helps us define integration.

At this point in the class, the professor digressed to a review of single-variable calculus, including the definition of the definite integral on $[a, b]$ as the limit of Riemann sums. If you've been able to understand anything in this course, you already know this well, so I'm not going to belabor it.

The point is that a definite integral, or a line integral in $\mathbb{R}$, is really integrating a one-form on a 1-manifold, and a surface integral is an integral of a 2-form over a 2-manifold. The idea of approximating such an integral by a Riemann sum goes through to define integrals of differential forms, though it's still just as much of a nightmare to actually compute. One takeaway is that to calculate, say, line or surface integrals, you pull the form back to Euclidean space and integrate there, where it's easier.

Lecture 37.

de Rham Cohomology: 4/22/16

We've now finished defining the basic operations (wedge product, pullback, exterior derivative, and integral) for two different constructions of differential forms, and more or less showed these are equivalent formulations.

The first was that forms on $\mathbb{R}^n$ are formal symbols $\alpha = \sum_I \alpha_I \, \mathrm{d}x^I$, where I is an ordered multi-index with distinct entries and $\alpha_I : \mathbb{R}^n \to \mathbb{R}$ is a function. We then defined the wedge product and d of forms by formulas and proved that they satisfy useful formulas such as the Leibniz rule, or $\mathrm{d}^2 = 0$. These proofs used nothing deep: they involved computations in a basis and chasing symbols around. We then defined pullback in a similar way, with a complicated-looking formula, and showed that it commutes with $\wedge$ and d. Pullback is a way of changing coordinates, and therefore shouldn't affect the integral (as long as you keep track of how the volume changes): indeed, if $g : \mathbb{R}^m \to \mathbb{R}^n$ is smooth, then $g^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (\det \mathrm{d}g)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Next, we integrated these differential forms on $\mathbb{R}^n$: the integral of an n-form $\alpha_I \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is the Riemann integral of α_I in the orientation prescribed by $(x_1, \dots, x_n)$. One important consequence is that pullbacks really do preserve this: an orientation-preserving diffeomorphism does not change the integral of a differential form.

Next, we generalized this to manifolds: a differential form on an oriented manifold is an equivalence class of a collection of differential forms on positively oriented coordinate charts. Then, all the nice formulas we had about, e.g. pullback commuting with the exterior derivative, still hold, and we can integrate forms on manifolds (in practice by pulling them back to $\mathbb{R}^n$ via coordinate charts).

After this, we defined forms in a less *ad hoc* manner: on a manifold X, a differential k-form is a section of the bundle $\Lambda^k(T^*X)$. This generalizes the usual derivative of a function $f : X \to \mathbb{R}$; since $\mathrm{d}f|_x$ is a linear operator on T_xX, $\mathrm{d}f$ is a 1-form! Another aspect of this approach is that given a basis $(e_1, \dots, e_n)$ for the tangent space in a coordinate neighborhood of X, we obtain a dual basis $(\phi^1, \dots, \phi^n)$ for the cotangent space, and therefore coordinates for the space of differential forms, $\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}$. Moreover, the symbol $\mathrm{d}x^i$ is really the derivative of the i^{th} coordinate function on a chart. In other words, forms are an intrinsic argument, but to actually compute anything you'll have to use a basis, and then you can use the formulaic approach that we presented first to get your answer. These approaches look very different, but are actually the same thing. For example, pullback is exactly the same formula for the purpose of calculation, but it's a little simpler to write down: $g^*\alpha(v_1, \dots, v_k) = \alpha(\mathrm{d}g(v_1), \dots, \mathrm{d}g(v_k))$.

Finally, we defined the integral of an n-form on an oriented n-dimensional manifold X in the same way we defined integrals in multivariable calculus: the limit of a Riemann sum over α applied to a basis of n vectors at every point. There's a lot of things to check here, e.g. it doesn't depend on which point we choose to evaluate the

[37] This is called a *Riemannian metric*; we're not going to define this precisely, but there's a theorem that these exist on any manifold.

Riemann sum once we take the limit, but this is very similar to the proof used to define the usual Riemann integral anyways. Finally, one can check this satisfies Stokes' theorem; since integrals on manifolds are defined by their pullbacks, it suffices to prove Stokes' theorem on $\mathbb{R}^n$, where it's much easier.

Again, there's the duality that to understand the concept of an integral, you use the abstract definition of a Riemann sum, and to compute anything, you have to pull it back to a subset of $\mathbb{R}^n$.

With that review out of the way, let's talk about cohomology. Let X be an n-dimensional manifold (we don't need it to be compact or even connected) and $\Omega^k(X)$ denote the space of k-forms on X. Thus, the exterior derivative is a linear operator $\mathrm{d} : \Omega^k(X) \to \Omega^{k+1}(X)$ such that $\mathrm{d}^2 = 0$. Recall that a form $\omega \in \Omega^k(X)$ is called closed if $\mathrm{d}\omega = 0$, and is exact if $\omega = \mathrm{d}\alpha$ for a $(k-1)$-form α. Since $\mathrm{d}^2 = 0$, all exact forms are closed, but the converse need not be true.

We defined the de Rham cohomology to be $H^k_{\mathrm{dR}}(X)$ (also written $H^k(X)$) to be the space of closed k-forms quotiented by the subspace of exact k-forms. *A priori*, this is a scary definition; both of these are typically infinite-dimensional vector spaces, so how can we get a handle on it? Let's start with a few computations.

Lemma 37.1. *If X is a connected manifold, $H^0_{\mathrm{dR}}(X) \cong \mathbb{R}$.*

Proof. A closed 0-form f is one whose derivative is 0, so f must be locally constant. Since X is connected, this means f is globally constant. 0 is the only exact form, so $H^0_{\mathrm{dR}}(X)$ is the space of constant functions, isomorphic to $\mathbb{R}$. ⊠

We also computed that $H^1(\mathbb{R}) = 0$, because for any 1-form α, $\alpha = \mathrm{d}f$, where

$$f(x) = \int_0^x \alpha_I(s)\,\mathrm{d}s.$$

However, $H^1(S^1) \neq 0$: we saw that the form $\alpha = x\,\mathrm{d}y - y\,\mathrm{d}x$ is closed, but not exact: a quick computation shows $\mathrm{d}\alpha = 0$, and $\int_{S^1} \alpha = 2\pi$, so by Stokes' theorem, α can't be exact (if it were, then its integral would be equal to the integral of a form over $\partial S^1 = \varnothing$, which has to be 0). One can show that α generates $H^1(S^1)$, and therefore $H^1(S^1) \cong \mathbb{R}$.

The wedge product defines a product $\wedge : H^k(X) \times H^\ell(X) \to H^{k+\ell}(X)$, where $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$. One should check that this is well-defined, in that if α and β are closed, then so is $\alpha \wedge \beta$, and that adding an exact form to either α or β doesn't change the cohomology class of their wedge product. This is a quick calculation: if α and β are closed, $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta = 0$, because both $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are 0, and adding exact forms is similar. As a consequence, $H^*(X) = \bigoplus_{k \geq 0} H^k(X)$ is a graded ring under addition of forms and wedge product.

Cohomology tells you something about spaces, but it also tells you something about maps between them: if $f : X \to Y$ is a smooth map of manifolds, we can pull back k-forms by f, obtaining a linear map $f^* : \Omega^k(Y) \to \Omega^k(X)$. Since pullback commutes with d, then the pullback of a closed k-form is closed and the pullback of an exact k-form is exact, so the pullback of a class in de Rham cohomology is well-defined. In particular, we obtain a linear map $f^\sharp : H^k(Y) \to H^k(X)$ sending $[\alpha] \mapsto [f^*\alpha]$.[38]

This map has yet more structure, because pullback commutes with wedge product, and therefore defines a ring homomorphism $f^\sharp : H^*(Y) \to H^*(X)$: $f^\sharp([\alpha] \wedge [\beta]) = f^\sharp[\alpha] \wedge f^\sharp[\beta]$. So we've extracted rings and ring homomorphisms from manifolds and smooth maps. Next time, we'll talk about how this behaves under homotopy.

Lecture 38.

Homotopy Invariance of Cohomology: 4/25/16

One of the useful properties of cohomology is its homotopy invariance: we're going to spend some time working towards the following theorem.

Theorem 38.1. *Let $f, g : X \to Y$ be smooth maps of manifolds that are homotopic. Then, $f^\sharp = g^\sharp$ as maps $H^k(Y) \to H^k(X)$.*

[38] We'll call this $f^\sharp$ because Guillemin and Pollack do; almost everyone else in the world calls this map f^* (and some even use $f^\sharp$ for the pullback on Ω^k).

We'll have to develop some machinery to prove this, and this machinery will be important in its own right.

Let X be an n-dimensional manifold and $\pi : \mathbb{R}\times X \to X$ sending $(t,x)\mapsto x$; this map has a section (the *zero section*) $s_0 : X \to \mathbb{R}\times X$ sending $x \mapsto (0,x)$. These induce maps on cohomology, $\pi^\sharp : H^k(X) \to H^k(\mathbb{R}\times X)$ and $s_0^\sharp : H^k(\mathbb{R}\times X) \to H^k(X)$.

Theorem 38.2. *These $\pi^\sharp$ and $s_0^\sharp$ are inverses of each other.*

This is notable because the maps on differential forms are *not* inverses: something special happens at the cohomological map.

Using this, Theorem 38.1 drops out as a corollary: the proof for Theorem 38.2 works just as well for the section at 1, so $s_0^\sharp$ and $s_1^\sharp$ are both inverses of $\pi^\sharp$, and therefore are the same map. That is, we've reduced the problem to proving Theorem 38.2.

Proof of Theorem 38.2. Since $\pi\circ s_0 = \mathrm{id}_X$, then one direction is easy: $s_0^*\circ\pi^* = \mathrm{id}^*$ as maps $\Omega^k(X)\to\Omega^k(X)$, and hence $s_0^\sharp\circ\pi^\sharp = \mathrm{id}_{H^k(X)}$. The other direction will be harder: $s_0\circ\pi\neq\mathrm{id}$.

Let $\alpha\in\Omega^k(\mathbb{R}\times X)$, and let $(t,x_1,\dots,x_n)$ denote local coordinates on $\mathbb{R}\times X$. Then, we can split α into indices containing $\mathrm{d}t$ and indices not containing it: in the following sum, I denotes multi-indices of length k and J denotes multi-indices of length $k-1$, so we can write α in local coordinates as

$$\alpha = \sum_I \alpha + I(t,x)\,\mathrm{d}x^I + \sum_J \beta_J(t,x)\,\mathrm{d}t\wedge\mathrm{d}x^J. \tag{38.3}$$

Since $\mathrm{d}x^I$ doesn't include $\mathrm{d}t$, then $s_0^*(\mathrm{d}x^I) = \mathrm{d}x^I$, and $s_0^*(\mathrm{d}t) = 0$. Thus,

$$s_0^*(\alpha)(x) = \sum_I \alpha_I(0,x)\,\mathrm{d}x^I,$$

and therefore

$$\pi^*s_0^*(\alpha)(t,x) = \sum_I \alpha(0,x)\,\mathrm{d}x^I.$$

That is, $\pi^*s_0^*(\alpha)$ doesn't depend on t at all!

We're going to construct linear operators $P_j : \Omega^j(\mathbb{R}\times X)\to\Omega^{j-1}(\mathbb{R}\times X)$ for $j = k, k+1$ such that $\alpha - \pi^*s_0^*\alpha = P_{k+1}(\mathrm{d}\alpha) + \mathrm{d}(P_k(\alpha))$; such an operator is called a *homotopy operator*, and this specific p is called *integration over a fiber*. That is, $1-\pi^*s_0^* = P_{k+1}\mathrm{d} + \mathrm{d}P_k$. If α is closed, then both $P_{k+1}(\mathrm{d}\alpha)$ and $\mathrm{d}(P_k\alpha)$ vanish, and so in cohomology, $[\alpha] = [\pi^*\circ s_0^*(\alpha)] = \pi^\sharp\circ s_0^\sharp([\alpha])$. In other words, to prove the theorem, it suffices to find these P_j; we'll use P to refer to the whole family of P_j.

We'll define P to be integration along a fiber of π; specifically, for a form α as in (38.3), let

$$(P\alpha)(t,x) = \sum_J\left(\int_0^t \beta_J(s,x)\,\mathrm{d}s\right)\mathrm{d}x^J.$$

Therefore

$$\mathrm{d}(P\alpha) = \sum_J \beta_J(t,x)\,\mathrm{d}t\wedge\mathrm{d}x^J + \sum_{J,j}\left(\int_0^t \frac{\partial\beta_j}{\partial x^j}(s,x)\,\mathrm{d}s\right)\mathrm{d}x^j\wedge\mathrm{d}x^J.$$

Since

$$\mathrm{d}\alpha = \sum_{j,I}\frac{\partial\alpha_I}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^I + \sum_I\frac{\partial\alpha_I}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}x^I - \sum_{j,J}\frac{\partial\beta_J}{\partial x^j}\,\mathrm{d}t\wedge\mathrm{d}x^j\wedge\mathrm{d}x^J,$$

then

$$P(\mathrm{d}\alpha) = \sum_I(\alpha_I(t,x)-\alpha_I(0,x)) - \sum_{J,j}\left(\int_0^t\frac{\partial\beta_J}{\partial x^j}(s,x)\,\mathrm{d}s\right)\mathrm{d}x^j\wedge\mathrm{d}x^J.$$

Taking the sum of these, $P(\mathrm{d}\alpha)+\mathrm{d}(P\alpha)$ is indeed $\alpha - \pi^*s_0^*\alpha$. ⊠

This tells us a lot already.

Corollary 38.4 (Poincaré lemma)**.**

$$H^k(\mathbb{R}^n) = \begin{cases}\mathbb{R}, & k=0\\ 0, & k\neq 0.\end{cases}$$

So now we know the cohomology of $\mathbb{R}^n$. Since we related d in $\mathbb{R}^3$ to classical vector calculus constructions, this also tells us the following.

- Every function on $\mathbb{R}^3$ is the divergence of some vector field.
- If v is a vector field in $\mathbb{R}^3$, $\operatorname{div}\operatorname{curl}(v) = 0$.
- If f is a function on $\mathbb{R}^3$, $\operatorname{curl}(\nabla f) = 0$.

Now, we can return to Theorem 38.1.

Proof of Theorem 38.1. First, observe that the proof of Theorem 38.2 works *mutatis mutandis* with 1 in place of 0 (so s_1 instead of the zero section, integrating from 1, etc.). Thus, $\pi^\sharp = (s_1^\sharp)^{-1}$ as well, and so $s_0^\sharp = s_1^\sharp$.

Let $f_0, f_1 : X \to Y$ be homotopic smooth maps of manifolds X and Y. Let $\widetilde{F} : [0,1] \times X \to Y$ be a homotopy realizing this, so $\widetilde{F}(0,x) = f_0(x)$ and $\widetilde{F}(1,x) = f_1(x)$. We can extend $\widetilde{F}$ to an ε-neighborhood of $[0,1]$, and therefore to all of $\mathbb{R}$ (since $(-\varepsilon, 1+\varepsilon)$ is diffeomorphic to $\mathbb{R}$). In particular, we can define $F : \mathbb{R} \times X \to Y$ such that $f_0 = F \circ s_0$ and $f_1 = F \circ s_1$. Thus,

$$f_1^\sharp = s_1^\sharp \circ F^\sharp = s_0^\sharp \circ F^\sharp = (F \circ s_0)^\sharp = f_0^\sharp. \qquad \boxtimes$$

So we know the cohomology of $\mathbb{R}^n$ is pretty trivial: it's the same as that of a point. But we also know that there are manifolds with nontrivial cohomology: we found forms that are closed but not exact on both S^1 and S^2. We'd like to calculate cohomology on a general manifold; the way to do this uses the defining property that a manifold looks like $\mathbb{R}^n$ on an open cover, and we know the cohomology of $\mathbb{R}^n$. This leads to something called the Mayer-Vietoris sequence.

Definition 38.5. Let

$$V_1 \xrightarrow{L_1} V_2 \xrightarrow{L_2} V_3 \xrightarrow{L_3} \cdots$$

be a sequence of vector spaces V_k and linear maps L_k.

- This sequence is *exact* at V_k if $\operatorname{Im}(L_k) = \ker(L_{k+1})$.
- This sequence is *exact* if it's exact at every vector space in the sequence.[39]
- An exact sequence of the form

  $$0V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \longrightarrow 0$$

  is called a *short exact sequence*.

In some sense, the de Rham cohomology measures the amount that the sequence

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \Omega^2(X) \xrightarrow{\mathrm{d}} \cdots$$

isn't exact: if it is exact, then every closed form is an exact form and the de Rham cohomology vanishes (except 0-forms, as in the cohomology of $\mathbb{R}^n$).

This formalism is what allows us to understand cohomology of manifolds in terms of that of $\mathbb{R}^n$.

Theorem 38.6 (Mayer-Vietoris)**.** *Let X be a manifold and $U, V \subset X$ be open submanifolds. Then, there is an exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(U\cup V) \arrow[r, "i^\sharp"] & H^k(U)\oplus H^k(V) \arrow[r, "j^\sharp"] & H^k(U\cap V) \arrow[dll, "\mathrm{d}^\sharp"] \\
& H^{k+1}(U\cap V) \arrow[r] & H^{k+1}(U)\oplus H^{k+1}(V) \arrow[r] & H^{k+1}(U\cup V) \arrow[dll, "\mathrm{d}^\sharp"] \\
& H^{k+2}(U\cap V) \arrow[r] & \cdots &
\end{tikzcd}$$

[39] Exact sequences can be defined in considerably more generality, e.g. for abelian groups or modules over a ring, but we only need the definition for vector spaces, where some properties of exact sequences are simpler.

The origin of these maps isn't obvious, so let's talk about it. There are inclusions $r_u : U \cap V \hookrightarrow U$, $r_v : U \cap V \hookrightarrow V$, $i_U : U \hookrightarrow U \cup V$, and $i_V : V \hookrightarrow U \cup V$. These induce maps on differential forms, and the following diagram commutes, because $i_U \circ r_U = i_V \circ r_V$.

$$\begin{tikzcd} \Omega^k(U\cup V) \arrow[r, "i_U^*"] \arrow[d, "i_V^*"] & \Omega^k(U) \arrow[d, "r_u^*"] \\ \Omega^k(V) \arrow[r, "r_V^*"] & \Omega^k(U\cap V). \end{tikzcd}$$

Using this, we can define a sequence

$$0 \longrightarrow \Omega^k(U\cup V) \xrightarrow{i} \Omega^k(U)\oplus\Omega^k(V) \xrightarrow{j} \Omega^k(U\cap V) \longrightarrow 0, \tag{38.7}$$

where $i(\alpha) = (i_U^*\alpha, i_V^*\alpha)$ and $j(\beta,\gamma) = r_U^*\beta - r_U^*\gamma$.

Lemma 38.8. *The sequence* (38.7) *is short exact.*

This is the engine that gets the Mayer-Vietoris sequence going.

Before we prove this, let's digress a little more about small exact sequences. Suppose we have a "very short exact sequence"

$$0 \longrightarrow V \xrightarrow{L} W \longrightarrow 0.$$

Then, by exactness at V, L is injective (its kernel is the image of 0, which is 0), and by exactness at W, it's surjective (its image is the kernel of 0, which is everything). Hence, L is an isomorphism. A short exact sequence

$$0 \longrightarrow V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \longrightarrow 0$$

similarly tells us that f is injective, g is surjective, and $\mathrm{Im}(f) = \ker(g)$. It turns out that for any short exact sequence of vector spaces, there's an isomorphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r, "f"] \arrow[d, equal] & V_2 \arrow[r, "g"] \arrow[d, "\wr"] & V_3 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V_1 \arrow[r] & V_1\oplus V_3 \arrow[r] & V_3 \arrow[r] & 0. \end{tikzcd}$$

That is, the vertical maps $V_1 \to V_1$ and $V_3 \to V_3$ are the identity, and the maps on the bottom are inclusion as the first factor and projection onto the second factor, respectively.

Proof idea of Lemma 38.8. i is pretty clearly injective: if $i_U^*\alpha = 0$ and $i_V^*\alpha = 0$, then α is 0 on both U and V, and therefore has to be 0 on their union. Similarly, $\mathrm{Im}(i) = \ker(j)$ is fairly clear: $r_U^*\alpha - r_V^*\beta = 0$ means exactly that α and β agree on $U \cap V$, and are therefore in the image of i.

Exactness at $\Omega^k(U\cap V)$ is trickier: choose a partition of unity $\{\rho_U, \rho_V\}$ for $U \cup V$ subordinate to the cover $\{U, V\}$, and use it to extend any form ω on $U \cap V$ as $\rho_U\omega - \rho_V\omega$, which is a preimage under j. ⊠

Lecture 39.

Chain Complexes and the Snake Lemma: 4/27/16

We were in the middle of discussing the Mayer-Vietoris sequence. If U and V are open subsets of a manifold, there is a commutative diagram of inclusions

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U\cap V \arrow[ur, "r_U"] \arrow[dr, "r_V"'] & & U\cup V, \\ & V \arrow[ur, "i_V"'] & \end{tikzcd}$$

and therefore a corresponding commutative diagram of spaces of differential forms:

$$\begin{tikzcd}
& \Omega^k(U) \arrow[dl, "r_U^*"'] & \\
\Omega^k(U\cap V) & & \Omega^k(U\cup V). \arrow[ul, "i_U^*"'] \arrow[dl, "i_V^*"] \\
& \Omega^k(V) \arrow[ul, "r_V^*"] &
\end{tikzcd}$$

Each of these arrows is given by restricting a differential form to a subset of the domain. Using this, we defined a sequence (38.7), and proved that it's exact. The subtlety is that the map $j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ which sends $(\beta, \gamma) \mapsto r_U^*\beta - r_V^*\gamma$ is surjective.

The example to keep in mind is where U and V are coordinate charts for S^1, each covering just slightly more than half of the circle, as in Figure 9.

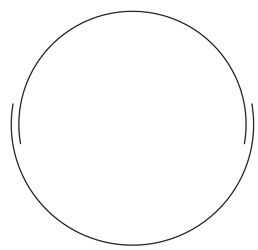


FIGURE 9. A cover of S^1 by two charts, which is a good example for the Mayer-Vietoris sequence.

We proved the surjectivity of j_k using a partition of unity $\{\rho_U, \rho_V\}$ for $U \cup V$ subordinate to its open cover $\{U, V\}$. Hence, for any $\alpha \in \Omega^k(U \cup V)$, we can write $\alpha = \rho_U\alpha + \rho_V\alpha$. Since $\rho_U\alpha$ is zero outside of U, it's the image of $(\beta, 0)$ for some $\beta \in \Omega^k(U)$, and similarly for $-\rho_V\alpha$, so we've just lifted... TODO: I think I'm missing something.

One interesting fact about the sequence (38.7) is that i_k and j_k commute with d, in the sense that $i_k \circ \mathrm{d} = \mathrm{d} \circ i_{k-1}$, and similarly for j_k. This has some useful formal consequences.

Chain complexes. We can abstract away a lot of the definitions we've made for cohomology into purely algebraic ones.

Definition 39.1. A sequence $A^\bullet$ of vector spaces and linear maps

$$A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

is a *complex* if $d^{i+1} \circ d^i = 0$ for all i (sometimes abbreviated $d^2 = 0$). In this case,

- an element in $\ker(d^k)$ is called *closed*,
- an element in $\operatorname{Im}(d^{k-1})$ is called *exact*, and
- the k^{th} *cohomology* of $A^\bullet$ is $H^k(A^\bullet) = \ker(d^k)/\operatorname{Im}(d^{k-1})$.

If $A^\bullet$ and $B^\bullet$ are two complexes, a *chain map*[40] $i^\bullet$ is a collection of maps $i^k : A^k \to B^k$ for each k that commute with the differentials, i.e. for every k the following diagram commutes.

$$\begin{tikzcd}
A^k \arrow[r, "i^k"] \arrow[d, "d^k"] & B^k \arrow[d, "d^k"] \\
A^{k+1} \arrow[r, "i^{k+1}"] & B^{k+1}.
\end{tikzcd}$$

We can also talk about, e.g. the direct sum of two complexes, which is the levelwise direct sum.

[40] Sometimes people call these *cochain maps*, and call the complexes cochain complexes, and so forth.

Definition 39.2. If $A^\bullet$, $B^\bullet$, and $C^\bullet$ are complexes and $i^\bullet : A^\bullet \to B^\bullet$ and $j^\bullet : B^\bullet \to C^\bullet$ are chain maps, then

$$0 \longrightarrow A^\bullet \xrightarrow{i^\bullet} B^\bullet \xrightarrow{j^\bullet} C^\bullet \longrightarrow 0 \tag{39.3}$$

is a *short exact sequence of chain maps* if for all k, both rows are exact in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^k \arrow[r, "i^k"] \arrow[d, "d^k"] & B^k \arrow[r, "j^k"] \arrow[d, "d^k"] & C^k \arrow[r] \arrow[d, "d^k"] & 0 \\ 0 \arrow[r] & A^{k+1} \arrow[r, "i^{k+1}"] & B^{k+1} \arrow[r, "j^{k+1}"] & C^{k+1} \arrow[r] & 0. \end{tikzcd} \tag{39.4}$$

In other words, the exact sequences in (38.7) form an exact sequence of chain maps!

If $i^\bullet : A^\bullet \to B^\bullet$ is a chain map, then it induces a map $i^\sharp : H^k(A^\bullet) \to H^k(B^\bullet)$ in cohomology: if $a \in A^k$ is closed, then $a = d^{k-1}\alpha$ for some α, so $i^k(a) = d^{k-1}(i^k(\alpha))$, so i^k sends closed forms to closed forms; in the same way it sends exact forms to exact forms, so it's well-defined on cohomology classes.

A short exact sequence of chain maps as in (39.3) induces maps $H^k(A^\bullet) \to H^k(B^\bullet) \to H^k(C^\bullet)$, but this does *not* extend to a short exact sequence of vector spaces! Something rather more interesting is true.

Lemma 39.5 (Snake)**.** *Given a short exact sequence of chain maps* (39.3)*, there exist linear maps* $d^\sharp : H^k(C^\bullet) \to H^{k+1}(A)^\bullet$ *such that the sequence*

$$\cdots \longrightarrow H^k(A^\bullet) \xrightarrow{i^\sharp} H^k(B^\bullet) \xrightarrow{j^\sharp} H^k(C^\bullet) \xrightarrow{d^\sharp} H^{k+1}(A^\bullet) \xrightarrow{i^\sharp} H^{k+1}(B^\bullet) \xrightarrow{j^\sharp} H^{k+1}(C^\bullet) \xrightarrow{d^\sharp} \cdots \tag{39.6}$$

is exact.

Proof. The trick in this proof is to come up with $d^\sharp$ by doing a diagram chase in (39.4). Let $\gamma \in C^k$ be a closed form; since the top row is exact, there's some $\beta \in B^k$ such that $j^k(\beta) = \gamma$. Since γ is closed, $0 = d^k\gamma = j^{k+1}d^k\beta$ (because the diagram commutes). Thus, since the bottom row is exact and $d^k\beta$ maps to 0, then it's in the image of i^{k+1}, so there's an $\alpha \in A^{k+1}$ (unique, since i^{k+1} is injective) such that $\alpha \mapsto d^k\beta$. We'll let $d^\sharp[\gamma] = [\alpha]$.

There's a lot to check here: why is this independent of choice of γ in its cohomology class? Why is it independent of the choice of β? Is α closed? For the last part, $d^{k+1}i^{k+1}\alpha = i^{k+2}d^{k+1}\alpha$ by commutativity, and $i^{k+1}\alpha$ is exact, so $d^{k+1}i^{k+1}\alpha = 0$. Since i^{k+2} is injective, this means $d^{k+1}\alpha = 0$, so α is closed.

Next, why doesn't this depend on β? Let β and β' be two preimages of γ. Then, $j^k(\beta - \beta') = 0$, so by exactness, there's a $\mu \in A^k$ such that $i^k(\mu) = \beta - \beta'$, so if α is the image induced from β and α' is the one induced from β', then $\mathrm{d}^k\mu = \alpha - \alpha'$: their difference is exact, so they define the same cohomology class.

If we pick a different representative for γ, the two differ by some exact form $\mathrm{d}\nu$, so the choices of $\beta \in B^k$ differ by an exact form. Therefore, when we take d^k of it, the difference vanishes! This means $d^\sharp$ is well-defined.

It remains to check exactness everywhere, which is six similar calculations that feel like more of the same diagram chases, using the fact that the original sequence was short exact. For example, let's prove that $\mathrm{Im}(j^\sharp) \subset \ker(d^\sharp)$. If $[\gamma] \in \mathrm{Im}(j^\sharp)$, then there's some closed $\beta \in B^k$ such that $[\gamma] = j^\sharp[\beta] = [j(\beta)]$. Thus, we can use this β to define $d^\sharp\alpha$ — but since β is closed, then $d^k\beta = 0$, and so when we pull back we still get 0. Thus, $\mathrm{Im}(j^\sharp) \subset \ker(d^\sharp)$. The only way to understand this is really to work through the proof, which is why it's on the homework next week. ⊠

Lecture 40.

The Snake Lemma: 4/29/16

We were in the middle of discussing the snake lemma: that if $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence of chain complexes, there's an induced long exact sequence (39.6). The most interesting part is the connecting morphism $d^\sharp : H^k(C^\bullet) \to H^{k+1}(A^\bullet)$, which is defined by a small diagram chase.

Proving exactness of the sequence at $H^k(B^\bullet)$, which we spent the first several minutes doing (and which I missed), is a diagram chase: a large number of steps that aren't individually difficult or insightful *per se*. We proved exactness at $H^k(C^\bullet)$ last lecture.

The final thing to check is exactness at $H^{k+1}(A^\bullet)$. First, suppose $[\gamma] \in H^k(C^\bullet)$; we want to prove $i^\sharp d^\sharp[\gamma] = 0$. We constructed $[\alpha] = d^\sharp[\gamma]$ by choosing some $\beta \in B^k$ that maps to γ, and then pulling $\mathrm{d}\beta$ back to something in

A^{k+1}. But this means that $i(\alpha) = d^k\beta$, so in cohomology, $i^\sharp[\alpha] = [d^k\beta] = 0$. Hence $\mathrm{Im}(d^\sharp) \subset \ker(i^\sharp)$. In the other direction, suppose $i^\sharp[\alpha] = 0$, so $[i^{k+1}\alpha] = 0$. Thus, $i^{k+1}(\alpha) = d^k\beta$ for some β, and if $\gamma = j^k\beta$, then $d^\sharp[\gamma] = [\alpha]$, so $\ker(i^\sharp) \subset \mathrm{Im}(d^\sharp)$. So the sequence is in fact exact everywhere, though the proof was no thing of beauty.

Back to forms. We introduced this formalism to better understand the Mayer-Vietoris sequence defined in (38.7) and Theorem 38.6 for a cover of a manifold X by two open subsets $\{U, V\}$. By Lemma 38.8, (38.7) is a short exact sequence for each k, and the maps commute with the exterior derivative. That is, we actually have a chain map

$$0 \longrightarrow \Omega^\bullet(X) \xrightarrow{i} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{j} \Omega^\bullet(U \cap V) \longrightarrow 0.$$

Here, $\Omega^\bullet(X)$ is the sequence $\Omega^0(X) \to \Omega^1(X) \to \Omega^2(X) \to \cdots$ and so forth, where the maps are the exterior derivative. Now, by applying the snake lemma, we get the long exact sequence in Theorem 38.6 for free.

Example 40.1. Let's use this to calculate something. Recall that we can cover S^1 by two charts as in Figure 9, one of which, V, is slightly more than the upper semicircle and the other, U, is slightly more than the lower hemisphere.

Since U and V are diffeomorphic to $\mathbb{R}$, then $H^1(U) = 0$ and $H^1(V) = 0$, and $H^0(U) = H^0(V) = \mathbb{R}$. Since $U \cap V$ is the disjoint union of two open intervals, $H^0(U \cap V) = \mathbb{R}^2$. Thus, the first part of the Mayer-Vietoris long exact sequence is

$$0 \longrightarrow H^0(S^1) \xrightarrow{i^\sharp} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^\sharp} \mathbb{R}^2 \xrightarrow{\mathrm{d}^\sharp} H^1(S^1) \longrightarrow 0.$$

You can recover $H^0(S^1)$ and $H^1(S^1)$ formally, but it helps to see what the maps are. $i^\sharp$ sends $\alpha \mapsto (\alpha, \alpha)$ and $j^\sharp$ sends $(a, b) \mapsto (a - b, a - b)$. Since this sequence is exact, $H^0(S^1) \cong \mathrm{Im}(i^\sharp) = \ker(j^\sharp) = \langle (a, a), a \in \mathbb{R} \rangle$, which is one-dimensional. Hence $H^0(S^1) \cong \mathbb{R}$, the constant functions. Similarly, since $\mathrm{d}^\sharp$ is surjective, we can determine its image to be generated by a single *bump form* $f\,\mathrm{d}\theta$, where f is supported only in $U \cap V$. Thus, $H^1(S^1) \cong \mathbb{R}$ as well. Since S^1 is one-dimensional, all higher groups vanish.

Example 40.2. A very similar argument works for S^2; let U be just a little more than the southern hemisphere, and V be just a little more than the northern hemisphere. Then, $U \cup V = S^2$ and $U \cap V$ is a belt around the equator, which is homotopic to S^1, so we know its cohomology. Thus, we have a long exact sequence

$$0 \longrightarrow H^0(S^2) \longrightarrow \mathbb{R}^2 \xrightarrow{j^\sharp} \mathbb{R} \xrightarrow{\mathrm{d}^\sharp} H^1(S^2) \longrightarrow 0 \longrightarrow H^1(S^1) \longrightarrow H^2(S^2) \longrightarrow 0.$$

Looking at the first half of the sequence, $j^\sharp$ sends $(a, b) \mapsto a - b$, which is surjective. Hence, by exactness, $\mathrm{d}^\sharp$ must be the zero map, so $H^1(S^2) = 0$. Then, since the kernel of $(a, b) \mapsto a - b$ is one-dimensional, $H^0(S^2) \cong \mathbb{R}$, which is once again the constant functions. Finally, the last part of the sequence is an isomorphism $H^2(S^2) \cong H^1(S^1) \cong \mathbb{R}$. Thus,

$$H^k(S^2) = \begin{cases} \mathbb{R}, & k = 0 \text{ or } k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

This generalizes nicely, as you might have guessed.

Theorem 40.3. *If $n > 0$,*

$$H^k(S^n) = \begin{cases} \mathbb{R}, & k = 0 \textit{ or } k = n \\ 0, & \textit{otherwise.} \end{cases}$$

The proof is similar: decompose S^n into the upper and lower hemispheres, whose cohomology groups (except for H^0) vanish, so the Mayer-Vietoris sequence implies an isomorphism $H^{k+1}(S^n) \cong H^k(S^{n-1})$ as long as $k > 0$. For $k = 0$, one has to check it explicitly, but the cohomology is just the space of constant functions as usual.

Example 40.4. The torus T provides an example with more interesting cohomology. If we bisect the torus with a plane intersecting it in two circles (sharing a bagel, not slicing it to add lox), we can let U be a little more than one half and V be a little more than the other. Then, U and V are cylinders, so they're homotopic to S^1, and $U \cap V$ is a disjoint union of two cylinders, so is homotopic to $S^1 \amalg S^1$. Thus, $H^0(U) = H^0(V) = \mathbb{R}$ and $H^0(U \cap V) = \mathbb{R}^2$, and the same is true for H^1. Hence, the Mayer-Vietoris sequence specializes to

$$0 \longrightarrow H^0(T)^{-}i^\sharp \longrightarrow \mathbb{R}^2 \xrightarrow{j^\sharp} \mathbb{R}^2 \longrightarrow H^1(T) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \xrightarrow{\mathrm{d}^\sharp} H^2(T) \longrightarrow 0. \tag{40.5}$$

Unfortunately, the exactness alone doesn't tell us what the cohomology is: the ranks of the indicated $i^\sharp$, $j^\sharp$, and $\mathrm{d}^\sharp$ could be all 1, or could be 0, 2, and 0, respectively. You have to actually figure out what the maps are doing to calculate the cohomology; the Mayer-Vietoris sequence is not just dimension-counting. In general, if you understand all of the $j^\sharp$ maps, you understand almost everything.

The issue here is that the Klein bottle admits a decomposition into the same U and V, but fit together in a different way. Hence, (40.5) holds for the Klein bottle as well, even though the cohomology in the end is different (one has one choice of ranks, and one has the other choice).

All the examples we've seen so far had finite-dimensional cohomology. We defined cohomology as a quotient of two usually infinite-dimensional spaces, so this was surprising. In general, though, cohomology may be infinite-dimensional. For example, consider the surface that's an infinite connected sum of tori, or a "genus-infinity" surface. This has infinite-dimensional H^1. However, there doesn't seem to be a way to make this work for a compact manifold.

Definition 40.6. If X is a manifold and $\mathfrak{U}$ is an open cover of X, then $\mathfrak{U}$ is a *good cover* if for all subsets $\mathfrak{U}' \subset \mathfrak{U}$, $\bigcap_{U\in\mathfrak{U}'} U$ is either empty or diffeomorphic to $\mathbb{R}^n$.

These covers are "good" because their intersections are well-behaved.

Theorem 40.7. *Every compact manifold admits a finite good cover.*

The next theorem is nice, but the method of proof is arguably more important than the theorem itself.

Theorem 40.8. *If X is a compact manifold, $H^k(X)$ is finite-dimensional for all k.*

Proof. Since X is compact, it has a finite good cover by Theorem 40.7; let's induct on the cardinality of this cover.

First, if it has a single set in its good cover, then $X \cong \mathbb{R}^n$, and we know the cohomology of $\mathbb{R}^n$ is finite-dimensional.

In the general case, suppose we've proven it up to m, and suppose X has a good cover $U_1, \ldots, U_{m+1}$. Let $U = U_1 \cup \cdots \cup U_m$ and $V = U_{m+1}$. Then, $U \cap V = (U_1 \cap U_{m+1}) \cup \cdots \cup (U_m \cap U_{m+1})$. Thus, each of U, V, and $U \cap V$ has a good cover of at most m sets, so each has finite-dimensional cohomology. Then, the Mayer-Vietoris sequence restricts to a sequence

$$H^{k-1}(U\cap V) \xrightarrow{\mathrm{d}^\sharp} H^k(X) \xrightarrow{i^\sharp} H^k(U)\oplus H^k(V).$$

Thus, since the first and last terms are finite-dimensional, the middle one must be finite-dimensional too. ⊠

There are many statements in topology which can be proven in this way, which is sometimes called the *Mayer-Vietoris argument*. However, it relied on Theorem 40.7, which we would like to prove or at least intuit.

The standard cover we provided for S^2 is not a good cover. However, you can cover S^2 with small, convex sets, and thereby obtain a good cover: a convex set is diffeomorphic to $\mathbb{R}^n$, and the intersection of two convex sets are convex. In general, one needs a Riemannian metric to understand convex sets on a general manifold, and this involves wandering into differential geometry, but you can always do this.

Lecture 41.

Good Covers and Compactly Supported Cohomology: 5/2/16

Recall that we wanted to prove Theorem 40.8, that every compact manifold has finite-dimensional de Rham cohomology. In order to do this, we invoked Theorem 40.7, that every compact manifold has a finite good cover.[41] In fact, this follows from a more general property of covers of manifolds.

Theorem 41.1. *Every manifold admits a good cover.*

Since not all manifolds are compact, the good cover doesn't need to be finite. For example, an infinite set of points with the discrete topology is a zero-dimensional manifold, and has no finite good cover (since it has infinitely many connected components).

The proof of Theorem 40.8 relies on a technique called the Mayer-Vietoris argument, which crucially uses the fact that compact manifolds have finite good covers. The technique is as follows for a property P:

(1) First, show that P holds for $\mathbb{R}^n$.

[41]Recall that an open cover is "good" if all opens in the cover are coordinate charts, and all finite intersections of opens in the cover are diffeomorphic to $\mathbb{R}^n$. It's equivalent to take the weaker notion that all intersections must be *contractible* (homotopic to a point).

(2) Then, show that if $M = U \cup V$, and P holds for U, V, and $U \cap V$, then it holds for M. Often, this uses the Mayer-Vietoris sequence.
(3) Finally, induct on the cardinality of a good cover of a manifold.

This proves that every manifold with a finite good cover has property P, and by Theorem 40.7 this includes all compact manifolds. This is exactly what we did to prove Theorem 40.8: in an exact sequence $A \to B \to C$, if A and B are finite-dimensional, then C must be finite-dimensional as well.

We want to prove Theorem 40.7, and to do so we'll need to wander into differential geometry.

Definition 41.2. A *(Riemannian) metric* on a smooth manifold M is a continuous choice of an inner product on each tangent space. That is, it's a smooth section $g : M \to \mathrm{Sym}^2(TM)$ that is positive definite at each point in M. A choice of a manifold and a Riemannian metric on it is called a *Riemannian metric*.

A Riemannian metric allows us to define the lengths of paths on a manifold: in a neighborhood, the manifold looks like $\mathbb{R}^n$ with an inner product, so we can measure lengths in coordinate charts, and therefore on the whole manifold.

Theorem 41.3. *Every smooth manifold M admits a Riemannian metric.*

Proof. Let $\{\rho_U\}$ be a partition of unity subordinate to a coordinate cover $\mathfrak{U}$ of M. On each U_i, we have a Riemannian metric g_U (the standard inner product on $\mathbb{R}^n$). A linear combination of inner products is still an inner product *as long as the coefficients are nonnegative*, since this preserves bilinearity, symmetry, and positive definiteness. Thus, $\sum_{U \in \mathfrak{U}} \rho_U g_U$ is a Riemannian metric on M. ⊠

Definition 41.4. A *geodesic* is a path in a Riemannian manifold M that is locally length-minimizing.

This is slightly vague, but we don't need to make it any more precise for this class.

Definition 41.5. A subset $N \subset M$ of a Riemannian manifold is *convex* if for any $x, y \in N$, there is a unique length-minimizing geodesic from x to y that is contained entirely in N.

Since geodesics on $\mathbb{R}^n$ are straight lines, this agrees with the usual notion of convexity on $\mathbb{R}^n$. In particular, every point on a Riemannian manifold has a convex coordinate neighborhood!

Lemma 41.6. *An intersection of convex sets is convex, and every convex set is contractible.*

This will be on the homework.

Proof of Theorem 41.1. Let $\mathfrak{U}$ be a cover of M by convex coordinate neighborhoods. Then, by Lemma 41.6, all intersections in $\mathfrak{U}$ are all contractible, so $\mathfrak{U}$ is a good cover. ⊠

Now, Theorem 40.7 drops out as a corollary by invoking compactness.

We return to topology, discussing how orientability relates to cohomology.

Theorem 41.7. *Let X be a connected, compact, n-dimensional manifold.*

(1) If X is orientable, $\dim H^n(X) = 1$.
(2) If X isn't orientable, $H^n(X) = 0$.

In this sense, cohomology detects orientability.

Partial proof. If X is compact and orientable, then it makes sense to integrate over X. Integration of differential forms is linear, and by Stokes' theorem, the integral of an exact form is 0: for any $\omega \in \Omega^{n-1}(X)$,

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega = \int_{\varnothing} \omega = 0.$$

Thus, integration over X descends to cohomology as a linear map $\int_X : H^n(X) \to \mathbb{R}$. We'd like to show that this is an isomorphism. It's not too complicated to show this is surjective: let $\mathfrak{U}$ be a finite coordinate cover of M, and pick some $V \in \mathfrak{U}$. Let f_V be a bump function on V with total integral $\int_V f_V \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = 1$ (since this is just constructing these on $\mathbb{R}^n$, which we've already done). Let $\omega_V = f_V \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, which is a differential form on V, and let $\omega_U = 0$ for every other $U \in \mathfrak{U}$. Now, if $\{\rho_U\}$ is a partition of unity subordinate to $\mathfrak{U}$, then $\sum_{U \in \mathfrak{U}} \rho_U \omega_U$ is a bump form on M with total integral 1.[42] Thus, integration hits $1 \in \mathbb{R}$, and so by linearity, is surjective. Injectivity will be harder, but it's worth noting that we've proven that if X is orientable and has a finite good cover (not just compact), then its n^{th} cohomology has dimension at least 1. ⊠

To make more progress, we need a new idea.

Definition 41.8. Let X be a manifold.

- A k-form on X is *compactly supported* if it vanishes outside of some compact subset K of X (e.g. in coordinate neighborhoods outside of K, it restricts to 0). The space of compactly supported k-forms is denoted $\Omega_c^k(X)$.
- Since d of a compactly supported form is still compactly supported (since one can check in coordinates), it makes sense to define the k^{th} *compactly supported cohomology* $H_c^k(X)$ to be the space of closed compactly supported k-forms modulo the image of $\mathrm{d} : \Omega_c^{k-1}(X) \to \Omega_c^k(X)$.

A compactly supported form may be exact, but not the derivative of a compactly supported form. This is an important distinction. Notice also that if X is compact, every form is compactly supported, so the compactly supported cohomology is the same as the ordinary cohomology.

Example 41.9. Let's look at $\mathbb{R}$ for a simple example. In this case,

$$H_c^k(\mathbb{R}) = \begin{cases} 0, & k = 0 \\ \mathbb{R}, & k = 1. \end{cases}$$

This is different from the ordinary cohomology, as $\mathbb{R}$ is noncompact.

For $H_c^0(\mathbb{R})$, every closed form on $\mathbb{R}$, compactly supported or not, is constant. However, only the zero function is a compactly supported, closed 0-form, so $H_c^0(\mathbb{R}) = 0$. For $H_c^1(\mathbb{R})$, suppose that a 1-form $f(t)\,\mathrm{d}t$ is compactly supported and exact. Then, there's an $R \in \mathbb{R}$ such that $f(t) = 0$ if $t > R$. Then, if $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$, then if $x > R$, then $F(x) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$ for $x > R$, so the total integral of f has to be 0. Since we already know integration is linear on cohomology (and precisely the same argument works for cohomology with compact supports), then this means that the integration map is injective: if the integral of a function is 0, then it has a compactly supported antiderivative. Therefore $\int_{\mathbb{R}} : H_c^1(\mathbb{R}) \to \mathbb{R}$ is an isomorphism.

Integration along the fiber allows us to generalize this, albeit with a degree shift.

Theorem 41.10. *Let X be a manifold and $k \geq 0$; then, $H_c^k(X \times \mathbb{R}) \cong H_c^{k-1}(X)$.*

This is the analogue of Theorem 38.2, but we don't quite have an isomorphism, and compactly supported cohomology is *not* homotopy-invariant.

Applying Theorem 41.10 n times, we know the compactly supported cohomology of $\mathbb{R}^n$.

Corollary 41.11.

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0, & \textit{otherwise}. \end{cases}$$

Proof of Theorem 41.10. Last time we discussed integration along the fiber, we were lucky enough to be able to use pullbacks to define the isomorphism; this time, we have to define the maps $i : \Omega_c^k(X) \to \Omega_c^{k+1}(X)$ and $j : \Omega_c^{k+1}(X) \to \Omega_c^k(X)$ ourselves.

Let $\pi : X \times \mathbb{R} \to X$ be projection and $s_0 : X \to X \times \mathbb{R}$ be the zero section; we'll let $x^1, \dots, x^n$ be the coordinates with respect to X and t be the coordinate of $\mathbb{R}$. If $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and compactly supported, let $i(\alpha) = \pi^*(\alpha) \wedge \phi\,\mathrm{d}t$, and for a general $\widetilde{\alpha} \in \Omega_c^{k+1}(X \times \mathbb{R})$ which has the form

$$\widetilde{\alpha}(x,t) = \sum_I \beta_I\,\mathrm{d}x^I + \sum_J \gamma_J\,\mathrm{d}x^J \wedge \mathrm{d}t,$$

let

$$j(\widetilde{\alpha}) = \sum_J \left(\int_{-\infty}^{\infty} \gamma_J(x,s)\,\mathrm{d}x \right) \mathrm{d}x^J.$$

This is integration along the fiber, which is OK because $\widetilde{\alpha}$ is compactly supported.

[42] If you already know that volume forms exist, you can use that to shorten this argument, but we haven't discussed them yet.

Both i and j are chain maps, meaning they commute with d: since $\mathrm{d}(\phi\,\mathrm{d}t) = 0$, because ϕ is compactly supported, then $i(\mathrm{d}\alpha) = \mathrm{d}(i\alpha)$. Showing j is a chain map is a little more elaborate, but still merely a computation: we know

$$\mathrm{d}\widetilde{\alpha} = \sum_{j=1}^{n}\sum_{I}\frac{\partial \beta_I}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^I + \sum_{I}\frac{\partial \beta_I}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}x^I + \sum_{j=1}^{n}\sum_{J}\frac{\partial \gamma_J}{\partial x^j}\,\mathrm{d}x^j\wedge\mathrm{d}x^J\wedge\mathrm{d}t,$$

and since the total integral over a fiber must vanish,

$$j(\mathrm{d}\widetilde{\alpha}) = \sum_{j=1}^{n}\sum_{J}\left(\int_{-\infty}^{\infty}\frac{\partial \gamma_J}{\partial x^j}\,\mathrm{d}t\right)\mathrm{d}x^j\wedge\mathrm{d}x^J = \mathrm{d}(j(\widetilde{\alpha})).$$

Now, we need to show that these induce isomorphisms on cohomology. $j\circ i = \mathrm{id}$ on the chain level, because integrating across ϕ gives you zero (since it's compactly supported), but in the other direction, we need to construct a homotopy operator $P : \Omega_c^{k+1}(X\times\mathbb{R}) \to \Omega_c^k(X\times\mathbb{R})$. We don't have time today to explain why, but the operator is $P(\alpha_I\,\mathrm{d}x^I) = 0$ and

$$P(\alpha_J\,\mathrm{d}x^J\wedge\mathrm{d}t) = \left(\int_{-\infty}^{t}\alpha_J(x,t)\,\mathrm{d}t - f(t)\int_{-\infty}^{\infty}\alpha_J(x,t)\,\mathrm{d}t\right)\mathrm{d}x^J,$$

where f is an antiderivative for ϕ. The rest of the proof is a calculation. ⊠

Lecture 42.

The Degree in Cohomology: 5/4/16

Recall that last time, we proved Corollary 41.11, which computes the compactly supported cohomology of $\mathbb{R}^n$: $H_c^n(\mathbb{R}^n)\cong\mathbb{R}$, and if $k\neq n$, $H_c^k(\mathbb{R}^n) = 0$. Let's write down a generator for $H_c^n(\mathbb{R}^n)$.

Lemma 42.1. *Let $\alpha\in\Omega^n(\mathbb{R}^n)$ and suppose $\alpha = \mathrm{d}\beta$ for a compactly supported $(n-1)$-form β. Then,*

$$\int_{\mathbb{R}^n}\alpha = 0.$$

Proof. We can choose a closed disc D containing the support of β, and then integrate over D: since β vanishes on the boundary of D and D is compact,

$$\int_D\alpha = \int_D\mathrm{d}\beta = \int_{\partial D}\beta = 0$$

by Stokes' theorem, and α must be 0 outside of D. ⊠

Thus, any bump form with total integral 1 generates $H^n(\mathbb{R}^n)$: every compactly supported form has some value as its integral, and for any two forms with the same integral, their difference has integral zero, and so they must be cohomologous.

We're also only partway through the proof of Theorem 41.7, that for any connected, compact manifold X, $\dim H^n(X)\le 1$, with equality iff X is orientable.

Continuation of the proof of Theorem 41.7. We've already shown that $\dim H^n(X)\ge 1$ when X is orientable, but now we can choose a generator of $H^n(X)$ to be a bump form ω supported in a single coordinate chart U, and with total integral 1.[43]

If α is any other n-form supported in U, then α and ω pull back to compactly supported forms on $\mathbb{R}^n$, and therefore α is cohomologous to a multiple of ω, since $H_c^n(\mathbb{R}^n)$ is one-dimensional.

Let $\mathfrak{U}$ be a finite cover of X by coordinate charts, and $\{\rho_U : U\in\mathfrak{U}\}$ be a partition of unity subordinate to $\mathfrak{U}$. Thus, for an arbitrary n-form α, we can write $\alpha = \sum_{U\in\mathfrak{U}}\rho_U\alpha$, and in particular α is a sum of bump forms, so without loss of generality assume α is a bump form. We've already dealt with the case where α and ω are in the same coordinate chart, but if they're not, there's a finite path of coordinate charts $U_1,\dots,U_m$ connecting the one supporting α and the one supporting ω such that two neighboring charts intersect. Thus, α is cohomologous to a bump form

[43]Since there's generally no canonical choice of volume form, by "total integral 1" we mean that, when pulled back to $\mathbb{R}^n$ in the chosen coordinate chart U, it has total integral 1.

in $U_1 \cap U_2$, which is cohomologous to something in $U_2 \cap U_3$, and so forth, and therefore α is cohomologous to something supported in $U_{m-1} \cap U_m$, which is cohomologous to ω. Thus, $H^n(X)$ is at most one-dimensional.

If X isn't orientable, we can show that $[\omega] = 0$ in $H^n(X)$: since X isn't orientable, there's a list $U_1, \dots, U_m$ of charts forming a cycle such that an odd number reverse orientation. Thus, ω is cohomologous to something with integral 1 in $U_1 \cap U_2$, and therefore something in $U_2 \cap U_3$, and so forth, but since we've reversed orientation an odd number of times, ω is cohomologous to something in $U_m \cap U_1$ with total integral -1 measured in U_1! Thus, $[\omega] = -[\omega]$, so it has to be 0. ⊠

So we know that the top degree detects orientability (or the lack thereof), and if X is orientable, integration is an isomorphism $H^n(X) \to \mathbb{R}$.

Degree of a map. Let X and Y be compact, connected, oriented manifolds of the same dimension n and $f : X \to Y$ be smooth. Then, there's a map $f^\sharp : H^n(Y) \to H^n(X)$, so it's a linear map $\mathbb{R} \to \mathbb{R}$, which therefore must be scalar multiplication by some $D \in \mathbb{R}$. That is, we've identified $H^n(X)$ with $\mathbb{R}$ through integration, and $f^\sharp[\alpha] = [f^*\alpha]$, so by "multiplication by D" we mean that for every $\alpha \in H^n(Y)$,

$$\int_X f^*\alpha = D \int_Y \alpha.$$

So we have this number D, a homotopy invariant of f. What does it mean?

Let p be a regular value of f; since $\dim X = \dim Y$, the stack of records theorem applies, producing a neighborhood $U \subset Y$ of p such that $f^{-1}(U)$ is a disjoint union of m copies of U, counted with sign, that map diffeomorphically onto U. Let α be a bump form supported in U with total integral 1, so that $f^*\alpha$ is a sum of bump forms on neighborhoods around the preimages $x_1, \dots, x_k \in f^{-1}(p)$. Since each maps diffeomorphically onto U, then we

$$\begin{aligned}\int_X f^*\alpha &= \sum_{i=1}^k \operatorname{sign}(\det(\mathrm{d}f|_{x_i})) \int_U \alpha \\ &= \deg(f) \int_Y \alpha.\end{aligned}$$

In other words, $D = \deg(f)$, so in cohomology, $f^\sharp$ characterizes and is characterized by the degree of the map. It's also interesting that the cohomology must be an integer.

The Gauss-Bonnet theorem. Here's a cute application of this machinery. Let $X \subset \mathbb{R}^3$ be an oriented surface, so the unit normal vectors $\mathbf{n}(p)$ for $p \in X$ define a smooth vector field of unit length, and therefore a smooth map $\mathbf{n} : X \to S^2$. For any $x \in S^2$, the normal vector to x (where we use the standard embedding $S^2 \hookrightarrow \mathbb{R}^3$ as the unit sphere) points in the same direction as x does from the origin, and points in the same direction $\mathbf{n}(p)$ does whenever $p \mapsto x$. Thus, T_pX and T_xS^2 are both orthogonal complements to $\mathbf{n}$, so must be the same. Thus, we obtain a map $S = \mathrm{d}\mathbf{n}|_p : T_pX \to T_{\mathbf{n}(p)}S^2 = T_pX$.

Definition 42.2. The map $S : T_pX \to T_pX$ is called the *shape operator*; its eigenvalues are the *principal curvatures* of X, and its determinant $K = \det(S)$ is called the *Gauss curvature*.

Theorem 42.3 (Gauss-Bonnet)**.**

$$\int_X K \, \mathrm{d}A = 2\pi\chi(X).$$

Proof sketch. Let ω be the surface area form on S^2. Then, $\mathbf{n}^*(\omega) = K\,\mathrm{d}A$, and hence

$$\int_X K\,\mathrm{d}A = \int_X \mathbf{n}^*(\omega) = (\deg \mathbf{n}) \int_{S^2} \omega = 4\pi(\deg \mathbf{n}).$$

There are a couple things we can do from here. The first is to homotope the immersion $X \hookrightarrow \mathbb{R}^3$ to put it in "normal form:" X is a surface, hence a connected sum of tori, and we place one at the top and hang the rest down from it in a line. In this case, the preimage of $(0,0,1)$ is all vectors with unit normal in the positive z-direction, which includes the top of X as well as the bottom of each "hole"; the former is positively signed and the latter is negatively signed (this needs to be checked; the key is that all but the top are saddle points, and the top is a local maximum). Then, if g denotes the genus of X, then $\deg(\mathbf{n}) = 1 - g = (2-2g)/2 = \chi(X)/2$.

Another possibility is to generalize to even-dimensional spheres, where we replace 2π with $\gamma_{2n}/2$, where γ_{2n} is the volume of S^{2n}. The following proof still works for this generalization. Choose any regular value $a \in S^{2n}$ of $\mathbf{n}$ such that $-a$ is also a regular value. Then, projecting the constant vector field in the direction of a down to X, we obtain a vector field on X. We can use this to determine the Euler characteristic: one has to show that the local degree is sign(det(S)), summed over the preimages of a (giving the degree) and $-a$ (also giving the degree), so $\chi(X) = 2\deg \mathbf{n}$. ⊠

We also have one last fact, whose proof we do not have time to give.

Theorem 42.4 (Poincaré duality)**.** *Let X be an orientable, n-dimensional manifold with a finite good cover. Then, there is an isomorphism $H^k(X) \cong (H_c^{n-k}(X))^*$ arising from the pairing $H^k(X) \times H_c^{n-k}(X) \to \mathbb{R}$ given by*

$$[\alpha], [\beta] \longmapsto \int_X \alpha \wedge \beta, \tag{42.5}$$

which is nondegenerate.

Since β is compact, we can take the integral of $\alpha \wedge \beta$ in (42.5).

The proof arises from another Mayer-Vietoris argument; there's a Mayer-Vietoris sequence for compactly supported cohomology in the opposite direction that looks like

$$\cdots \longrightarrow H_c^k(U \cap V) \longrightarrow H_c^k(U) \oplus H_c^k(V) \longrightarrow H_c^k(U \cup V) \longrightarrow H_c^{k+1}(U \cap V) \longrightarrow \cdots.$$

The idea is that compactly supported cohomology is dual to usual cohomology, so after taking duals (which is contravariant), the sequence should go in the other order, and therefore the sequences for H^k and $(H_c^k)^*$ are compatible, allowing a Mayer-Vietoris argument. This also resolves the issue that one increases in degree and the other decreases; if we pair k and $n-k$, we end up with the pairing for $k+1$ and $n-k-1$, which is what we should expect. So the argument is to show that the pairing is an isomorphism for $U \cup V$ assuming it for U, V, and $U \cap V$. This uses an algebraic trick known as the five lemma. There are more details we're eliding; check out Bott and Tu's book for the full story.

Corollary 42.6. *If X is a compact, orientable manifold, $H^k(X) \cong (H^{n-k}(X))^*$.*

Things can and do go wrong for manifolds without finite good covers.