M392C NOTES: MORSE THEORY

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These notes were taken in UT Austin's M392C (Morse Theory) class in Fall 2018, taught by Dan Freed. I live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Any mistakes in the notes are my own.

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Lecture 1.

Critical points and critical values: 8/29/18

"The victim was a topologist." (nervous laughter)

In this course, manifolds are smooth unless assumed otherwise.

Morse theory is the study of what critical points of a smooth function can tell you about the topology of its domain manifold.

Definition 1.1. Let $f\colon M \to \mathbb{R}$ be a smooth function.

- A $p \in M$ is a *critical point* if $\mathrm{d}f|_p = 0$.
- A $c \in \mathbb{R}$ is a *critical value* if there's a critical point $p \in M$ with $f(p) = c$.

The set of critical points of f is denoted $\mathrm{Crit}(f)$.

Example 1.2. Consider the standard embedding of a torus T^2 in $\mathbb{R}^3$ and let $f\colon T^2 \to \mathbb{R}$ be the x-coordinate. Then there are four critical points: the minimum and maximum, and two saddle points. These all have dfifferent images, so there are four critical values. ◀

If M is compact, so is $f(M)$, and therefore f has a maximum and a minimum: at least two critical points. (If M is noncompact, this might not be true: the identity function $\mathbb{R} \to \mathbb{R}$ has no critical points.) In the 1920s, Morse studied how the theory of critical points on M relates to its topology.

Example 1.3. On S^2, there's a function with precisely two critical points (embed $S^2 \subset \mathbb{R}^3$ in the usual way; then f is the z-coordinate). There is no function with fewer, since it must have a minimum and a maximum. ◀

What about other surfaces? Is there a function on T^2 or $\mathbb{RP}^2$ with only two critical points?

Well, that was a loaded question – we'll prove early on in the course that the answer is no.

Theorem 1.4. *Let M be a compact n-manifold and $f\colon M \to \mathbb{R}$ be a smooth function with exactly two nondegenerate critical points. Then M is homeomorphic to a sphere.*

So, it "is" a sphere. But some things depend on what your definition of "is" is — Milnor constructed *exotic 7-spheres*, which are homeomorphic but not diffeomorphic to the usual S^7, and Kervaire had already produced topological 10-manifolds with no smooth structure. Freedman later constructed topological 4-manifolds with no smooth structure. In lower dimensions there are no issues: smooth structures exist and are unique in the usual sense. In dimension 4, there are some topological manifolds with a countably infinite number of distinct smooth structures. One of the most important open problems in geometric topology is to determine whether there are multiple smooth structures on S^4, and how many there are if so.

Morse studied the critical point theory for the energy functional on the based loop space ΩM of M, which is an infinite-dimensional manifold. This produced results such as the following.

Theorem 1.5 (Morse)**.** *For any $p, q \in S^n$ and any Riemannian metric on S^n, there are infinitely many geodesics from p to q.*

And you can go backwards, using critical points to study the differential topology of ΩM. Bott and Samelson extended this to study the loop spaces of symmetric spaces, and used this to prove a very important theorem.

Theorem 1.6 (Bott periodicity)**.** *Let $\mathrm{U} \coloneqq \varinjlim_{n\to\infty} \mathrm{U}_n$, which is called the infinite unitary group.*[1] *Then*

$$\pi_q \mathrm{U} \cong \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

This theorem is at the foundation of a great deal of homotopy theory.

The traditional course in Morse theory (e.g. following Milnor) walks through these in a streamlined way. These days, one uses the critical-point data of a Morse function on M to build a CW structure (which recovers the homotopy theory of M), or better, a handlebody decomposition of M (which gives its smooth structure). We could also study Smale's approach to Morse theory, which has the flavor of dynamical systems, studying gradient flow and the stable and unstable manifolds. This leads to an infinite-dimensional version due to Floer, and its consequences in geometric topology, and to its dual perspective due to Witten, which we probably won't have time to cover. Our course could also get into applications to symplectic and complex geometry.

Milnor's Morse theory book is a classic, and we'll use it at the beginning. There's a more recent book by Nicolescu, which in addition to the standard stuff has a lot of examples and some nonstandard topics; we'll also use it. There will be additional references.

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Let M be a manifold and $(x^1, \dots, x^n)$ be a local coordinate system (or, we're working on an open subset of affine n-space $\mathbb{A}^n$). One defines the first derivative using coordinates, but then finds that it's intrinsic: if

[1] The map $\mathrm{U}_n \to \mathrm{U}_{n+1}$ sends $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$x = x(y)$ is a change of coordinates (so $x = x(y^1, \dots, y^n)$), then

$$\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i = \frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^i}{\partial y^\beta}\,\mathrm{d}y^\beta = \frac{\partial f}{\partial y^\alpha}\,\mathrm{d}y^\alpha, \tag{1.7}$$

and so this is usually just called $\mathrm{d}f$, and can even be defined intrinsically. For critical points we're also interested in second derivatives, but the second derivative isn't usually intrinsic:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right) + \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}^2 x}{\mathrm{d}y^2}. \tag{1.8}$$

The second term depends on our choice of x, so it's nonintrinsic. In general one needs more data, such as a connection, to define intrinsic higher derivatives. But at a critical point, the second term vanishes, and the second derivative is intrinsic![2]

Definition 1.9. Let $f\colon M \to \mathbb{R}$ and $p \in \mathrm{Crit}(f)$. Then the *Hessian* of f at p is the function $\mathrm{Hess}_p(f)\colon T_pM \times T_pM \to \mathbb{R}$ sending $\xi_1, \xi_2 \mapsto \xi_1(\xi_2 f)(p)$, where we extend ξ_2 to a vector field near p.

Of course, one must check this is independent of the extension. Suppose η is a vector field vanishing at p. Then

$$\xi_1 \cdot (\eta f)(p) = \eta(\xi_1 f)(p) + [\eta, \xi] \cdot f(p) = 0 + 0 = 0, \tag{1.10}$$

so everything is good.

Lemma 1.11. *The Hessian is a symmetric bilinear form.*

Proof. Extend both ξ_1 and ξ_2 to vector fields in a neighborhood of p. Then

$$\xi_1 \cdot (\xi_2 f)(p) - \xi_2(\xi_1 f)(p) = [\xi_1, \xi_2] f(p) = 0. \tag{1.12}$$

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In order to study the Hessian, let's study bilinear forms more generally. Let V be a finite-dimensional real vector space and $B\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form.

Definition 1.13. The *kernel* of B is the set K of $\xi \in V$ with $B(\xi, \eta) = 0$ for all η. If $K = 0$, we say B is *nondegenerate*.

Equivalently, B determines a map $b\colon V \to V^*$ sending $\xi \mapsto (\eta \mapsto B(\xi, \eta))$, and $K = \ker(b)$. Any symmetric bilinear form descends to a nondegeneratr form $\widetilde{B}\colon V/K \times V/K \to \mathbb{R}$.

Example 1.14.

(1) If B is *positive definite*, meaning $B(\xi, \xi) > 0$ for all $\xi \neq 0$, then B is an *inner product*.
(2) On $V = \mathbb{R}^3$, consider the nondegenerate and indefinite form

$$B((\xi^1, \xi^2, \xi^3), (\eta^1, \eta^2, \eta^3)) \coloneqq \xi^1\eta^1 - \xi^2\eta^2 - \xi^3\eta^3. \tag{1.15}$$

The *null cone*, namely the subspace of ξ with $B(\xi, \xi) = 0$, is a cone opening in the x-direction. We can restrict B to the subspace $\{(x, 0, 0)\}$, where it becomes positive definite, or to the subspace $\{(0, y, z)\}$, where it's negative definite. ◂

However, we can't canonically define anything like *the* maximal positive or negative definite subspace — the only canonical subspace is the kernel. We can fix this by adding more structure.

Lemma 1.16. *Let $N, N' \subset V$ be maximal subspaces of V on which B is negative definite. Then* $\dim N = \dim N'$.

This is called the *index* of B.

Proof. Since N and N' don't intersect K, we can pass to V/K, and therefore assume without loss of generality that B is nondegenerate. Assume $\dim N' < \dim N$; then, $V = N \oplus N^\perp$. Let $\pi\colon V \twoheadrightarrow N$ be a projection onto N, which has kernel $N^\perp$. Then $\pi(N')$ is a proper subspace of N. Let $\eta \in N$ be a nonzero vector with $B(\eta, \pi(N')) = 0$. Then $B(\eta, N') = 0$, and so $B(\xi + \eta, \xi + \eta) < 0$ for all $\xi \in N'$, and therefore N' isn't maximal. ⊠

[2] This generalizes: if the first n derivatives vanish at x, the $(n+1)$st derivative is intrinsic.

Applying the same proof to $-N$, there's a maximal dimension of a positive-definite subspace P. So B determines three numbers, $\dim K$ (the *nullity*), $\lambda \coloneqq \dim N$ (the *index*), and $\rho \coloneqq \dim P$. This doesn't have a name, but the *signature* is $\rho - \lambda$. In Morse theory we'll be particularly concerned with the index.

Proposition 1.17. *There exists a basis of V, $e_1, \dots, e_\lambda, e_{\lambda+1}, \dots, e_{\lambda+\rho}, e_{\lambda+\rho+1}, \dots, e_n$, such that*

$$B(e_i, e_j) = 0, \qquad i \neq j, B(e_i, e_i) = \begin{cases} -2, & 1 \le i \le \lambda, \\ 2, & \lambda + 1 \le i \le \lambda + \rho \\ 0, & \textit{otherwise.} \end{cases} \tag{1.18}$$

Proof. We have the kernel $K \subset V$, and can choose a complement V' for it; then $B|_{V'}$ is nondegenerate. Let $N \subset V'$ be a maximal negative definite subspace, and $N^\perp$ be its orthogonal complement with respect to $B|_{V'}$. Then $V = N \oplus N^\perp \oplus K$, and we can choose these bases in each subspace. ⊠

Remark 1.19. If we choose an inner product $\langle -, - \rangle$ on V and define $T \colon V \to V$ by

$$B(\xi_1, \xi_2) = \langle \xi_1, T\xi_2 \rangle \tag{1.20}$$

for all $\xi_1, \xi_2 \in V$, then T is symmetric and therefore diagonalizable. ◀

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With the linear algebra interlude over, let's get back to topology. The Hessian is a very useful invariant, e.g. defining the curvature of embedded hypersurfaces in $\mathbb{R}^n$.

Definition 1.21. Let $f \colon M \to \mathbb{R}$ be smooth.

(1) A $p \in \mathrm{Crit}(f)$ is *nondegenerate* if $\mathrm{Hess}_p(f)$ is nondegenerate.
(2) If every critical function is nondegenerate, f is called a *Morse function*.

Example 1.22. For example, on the torus as above, the y-coordinate is a Morse function. But the z-coordinate is not Morse: there's a whole circle of maxima, and another one of minima, and therefore the Hessians on these circles cannot be nondegenerate. ◀

Example 1.23. For another example, consider $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. This isn't Morse: it has one critical point, which is degenerate. Unlike the previous example, this is a degenerate critical point which is isolated. ◀

Example 1.24. Let V be a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$,[3] and let $T \colon V \to V$ be a symmetric linear operator with distinct eigenvalues (i.e. its eigenspaces are one-dimensional). Then $\mathbb{P}(V)$, the set of lines through the origin (i.e. one-dimensional subspaces) in V is a closed manifold. Define $f \colon \mathbb{P}(V) \to \mathbb{R}$ by

$$L \longmapsto \frac{\langle \xi, T\xi \rangle}{\langle \xi, \xi \rangle}, \qquad \xi \in L \setminus 0. \tag{1.25}$$

It's a course exercise to show the critical points of f are the eigenlines of T, and to compute their Hessians and their indices.

It may be useful to know that there's a canonical identification $T_L\mathbb{P}(V) \cong \mathrm{Hom}(L, V/L)$. This also generalizes to Grassmannians. ◀

The next thing we'll study is a canonical local coordinate system around a critical point of a Morse function (the Morse lemma). It's a bit bizarre to build coordinates out of nothing, so we'll start with an arbitrary coordinate system and deform it. We will employ a very general tool to do this, namely flows of vector fields. This may be review if you like differential geometry.

Definition 1.26. Suppose ξ is a vector field on M. A curve $\gamma \colon (a, b) \to M$ is an *integral curve* of ξ if for $t \in (a, b)$, $\dot\gamma(t) = \xi|_{\gamma(t)}$.

Theorem 1.27. *Integral curves exist: for all $p \in M$, there exists an $\varepsilon > 0$ and an integral curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ for ξ with $\gamma(0) = p$.*

[3] With a little more work, we can make this work over the quaternions.

This is a geometric reskinning of existence of solutions to ODEs, as well as smooth dependence on initial data (whose proof is trickier). If you don't know the proof, you should go read it!

We can also allow ξ to depend on t with a trick: consider the vector field $\frac{\partial}{\partial t} + \xi_t$ on $(a, b) \times M$. By the theorem, integral curves exist, and since this vector field projects onto $\frac{\partial}{\partial t}$ on (a, b), the integral curve we get projects onto the integral curve for $\frac{\partial}{\partial t}$. So what we've constructed is exactly the graph of γ. In ODE, this is known as the non-autonomous case.

We'd like to do this everywhere on a manifold at once.

Definition 1.28. A *flow* is a function $\varphi\colon (a, b) \times M \to M$ such that $\varphi(t, -)\colon M \to M$ is a diffeomorphism.

We'd like to say that vector fields give rise to flows. Certainly, we can differentiate flows, to obtain a time-dependent vector field $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \xi_t$.

Example 1.29. For a quick example of nonexistence of flow for all time, consider $\xi = \frac{\partial}{\partial t}$ on $\mathbb{R} \setminus \{0\}$. You can't flow from a negative number forever, since you'll run into a hole. Now maybe you think this is the problem, but there's not so much difference with just $\mathbb{R}$ and the vector fields $t\frac{\partial}{\partial t}$ or $t^2\frac{\partial}{\partial t}$, where you will reach infinity in finite time. ◀

One of the issues with global-time existence of flow is that the metric might not be complete. But it's not the only obstruction, as we saw above.

Theorem 1.30. *Let ξ_t be a family of vector fields for $t \in (t_-, t_+)$, where $t_- < 0$ and $t_+ > 0$.*

1. *Given a $p \in M$, there are neighborhoods of p $U' \subset U$ and an $\varepsilon > 0$ such that there's a flow $\varphi\colon (-\varepsilon, \varepsilon) \times U' \to U$ with $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \xi_t$.*
2. *If M has a complete Riemannian metric and there's a $C > 0$ in which $|\xi_t| \le C$, then the flow is global: we can replace $(-\varepsilon, \varepsilon)$ with (t_-, t_+).*

A compact manifold is complete in any Riemannian metric, so for ξ arbitrary, global flows exist.

Remark 1.31. If ξ is *static*, i.e. independent of t, then $t \mapsto \varphi_t$ is a *one-parameter group*, i.e. $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$. ◀

Example 1.32. Let M be a Riemannian manifold and $f\colon M \to \mathbb{R}$ be smooth. Define its *gradient vector field* by

$$\mathrm{d}f|_p(\eta) \coloneqq \langle \eta, \mathrm{grad}_p f\rangle \tag{1.33}$$

for all $\eta \in T_pM$. ◀

Let's (try to) flow by $-\,\mathrm{grad}\, f$.

Definition 1.34. Let $\omega \in \Omega^*(M)$ and ξ be a vector field with local flow φ generated by ξ. The *Lie derivative* is

$$\mathcal{L}_\xi \omega \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^* \omega,$$

which is also a differential form, homogeneous of degree k if ω is.

Theorem 1.35 (H. Cartan)**.** $\mathcal{L}_\xi \omega = (\mathrm{d}\iota_\xi + \iota_\xi \mathrm{d})\omega$. *Here ι_ξ denotes contracting with ξ.*

With this in our pockets, let's turn to the Morse lemma.

Lemma 1.36 (Morse lemma)**.** *Let $f\colon M \to \mathbb{R}$ be smooth and p be a nondegenerate critical point of f of index λ. Then there exist local coordinates $x^1, \dots, x^n$ near p with $x^i(p) = 0$ and*

$$f(x^1, \dots, x^n) = f(p) - ((x^1)^2 + \cdots (x^\lambda)^2) + ((x^{\lambda+1})^2 + \cdots + (x^n)^2).$$

The proof employs a technique of Moser. Moser used this to provide a nice proof of Darboux's theorem, that symplectic manifolds all look like affine space locally.

Lemma 1.37. *Let $U \subset \mathbb{R}^n$ be a star-shaped open set with respect to the origin and $g\colon U \to \mathbb{R}$ be such that $g(0) = 0$. Then there exist $g_i\colon U \to \mathbb{R}$ with $g(x) = x^i g_i(x)$.*

Proof. Well, just let

$$g_i(x) = \int_0^1 \frac{\partial g}{\partial x^i}(tx)\,\mathrm{d}t. \tag{1.38}$$

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Proof of Lemma 1.36. Choose local coordinates $x^1, \dots, x^n$ such that

$$\frac{1}{2}\operatorname{Hess}_p(f) = \left(-\left(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \dots + \mathrm{d}x^\lambda \otimes \mathrm{d}x^\lambda\right) + \left(\mathrm{d}x^{\lambda+1} \otimes \mathrm{d}x^{\lambda+1} + \cdots \mathrm{d}x^n \otimes \mathrm{d}x^n\right)\right)_p. \tag{1.39}$$

Since we're only asking for this at p, we can start with any coordinate system and then apply Lemma 1.37. Set

$$h(x) \coloneqq f(p) - ((x^1)^2 + \cdots (x^\lambda)^2) + ((x^{\lambda+1})^2 + \dots + (x^n)^2) - f(x). \tag{1.40}$$

We're hoping for this to be zero. Also set

$$\alpha_t \coloneqq (1-t)\Bigg(\underbrace{-\left(x^1\,\mathrm{d}x^1 + \dots + x^\lambda\,\mathrm{d}x^\lambda\right) + \left(x^{\lambda+1}\,\mathrm{d}x^{\lambda+1} + \cdots x^n\,\mathrm{d}x^n\right)}_{\alpha_0}\Bigg) + t\,\mathrm{d}f, \tag{1.41}$$

for $t \in [0,1]$. We claim that in a neighborhood of $x = 0$, we can find a vector field ξ_t such that $\iota_{\xi_t}\alpha_t = h$; in particular, h does not depend on t; and such that $\xi_t(p = 0) = 0$. We'll then use this to move the coordinates; at p everything looks right, so we'll use this to move the coordinates elsewhere.

Assuming the claim, let φ_t be the local flow generated by ξ_t, which exists at least in a neighborhood of U. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\alpha_t &= \varphi_t^*\mathcal{L}_{\xi_t}\alpha_t + \varphi_t^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t\right)\\ &= \varphi_t^*(\mathrm{d}\iota_{\xi_t}\alpha_t + \iota_{\xi_t}\,\mathrm{d}\alpha_t - \mathrm{d}h).\end{aligned}$$

Since α_t is exact,

$$= \varphi_t^*(\varphi_t^*\,\mathrm{d}(\iota_{\xi_t}\alpha_t - h)) = 0.$$

Therefore $\varphi_1^*(\mathrm{d}f) = \varphi_1^*\alpha_1 = \varphi_0^*\alpha_0 = \alpha_0$. In particular, φ_1 is a local diffeomorphism fixing $p = 0$, and it pulls $\mathrm{d}f$ back to d of something quadratic. Therefore $\varphi_1^* f$ is quadratic, and has the desired form.

Now we need to prove the claim. Observe $\alpha_t(0) = 0$ and $h(0) = 0$. Then write

$$\begin{aligned}\alpha_t(x) &= A_{ij}(t,x)x^j\,\mathrm{d}x^i\\ h(x) &= h_j(x)x^j\\ \xi_t &= \xi^k(t,x)\frac{\partial}{\partial x^k},\end{aligned}$$

so $\iota_{\xi_t}\alpha_t h$ is equivalent to

$$A_{ij}(t,x)x^j\xi^i(t,x) = h_j(x)x^j, \tag{1.42}$$

which is implied by

$$A_{ij}(t,x)\xi^j(t,x) = h_j(x). \tag{1.43}$$

Since $(A_{ij}(t,0))$ is invertible, we can solve this in some neighborhood of $x = 0$ uniform in t (it remains invertible in that neighborhood). ⊠

Lecture 2.

Sublevel sets: 9/5/18

Last time, we proved the Morse lemma: if $f\colon M \to \mathbb{R}$ is a smooth function and $p \in M$ is a nondegenerate critical point, then there are local coordinates $x^1, \dots, x^n$ with $x(p) = 0$ and

$$f(x) = f(p) - ((x^1)^2 + \dots + (x^\lambda))^2 + ((x^{\lambda+1})^2 + \dots + (x^n)^2). \tag{2.1}$$

In this case we can define the Hessian; λ is its index, which is the maximal dimension d such that there's a d-dimensional subspace $N \subset T_pM$ on which the Hessian is negative definite.

Corollary 2.2. *A nondegenerate critical point is isolated.*

Recall that a smooth function is called Morse if all of its critical points are nondegenerate.

Corollary 2.3. *If f is a Morse function, then* $\operatorname{Crit}(f) \subset M$ *is discrete. If M is compact, then* $\operatorname{Crit}(f)$ *is finite.*

So Morse functions are really nice. But they're nontheless generic.

Theorem 2.4. *Let M be a smooth manifold.*

(1) *M admits a Morse function; in fact, Morse functions are dense in $C^\infty(M)$.*
(2) *M admits a proper Morse function.*[4]

To make precise the notion of density of Morse functions, we need to specify a topology on $C^\infty(M)$; that can be done, but we're not going to do it here. Proofs will be given in the next section.

Definition 2.5. Let $f\colon M \to \mathbb{R}$ be smooth and $a \in \mathbb{R}$. Then define $M^a := f^{-1}((\infty, a])$, which is called a *sublevel set.*

See Figure 1 for examples of sublevel sets. Sublevel sets of M define a filtration of M indexed by $\mathbb{R}$.

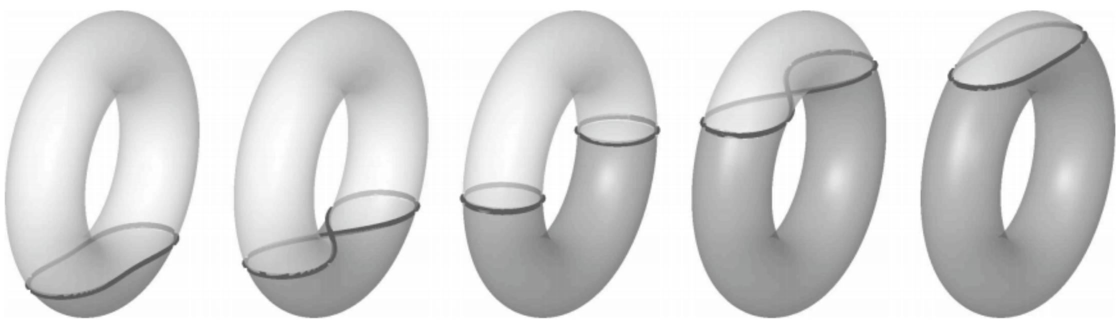


Figure 1. Sublevel sets for the standard height function on a torus. We can also get the empty 2-manifold $\varnothing^2$ for sublevel sets for a below the minimum, and T^2 for sublevel sets for a above the maximum.

The second fundamental theorem of Morse theory, which we'll do next time, is about handles and handlebodies, and that when you cross a critical point, the diffeomorphism type of the sublevel set changes precisely by adding a handle.

We probably should have already mentioned an important theorem from differential topology.

Theorem 2.6. *If a is a regular value, $f^{-1}(a) \subset M$ is a manifold, and M^a is a manifold with $\partial M^a = f^{-1}(a)$.*

Since a point is compact, and an interval is compact, choosing proper Morse functions allows us to get compact level sets for $f^{-1}(a)$. Moreover, the preimage of $[a, b]$ is a compact manifold with boundary $f^{-1}(a) \amalg f^{-1}(b)$ (here a and b should be regular values), i.e. a *bordism* from $f^{-1}(a)$ to $f^{-1}(b)$.

This perspective, involving handles and differential topology, is geometric, and is due to Smale in the 1960s or so. But there's another, homotopical approach, where one uses a Morse function to define a CW structure. This not only shows that all manifolds have CW structures, which is nice, but also is a gateway to good calculations of homology and cohomology. The idea is to think of handle attachment by collapsing the "irrelevant" dimensions, so that instead of attaching a handle, you can attach a k-cell (depending on the index), and so on.

But the simplest question you can ask is: if a and b are regular values with no critical values in $[a, b]$, how do M^a and M^b differ? The answer is, more or less, they don't.

Theorem 2.7. *Let $f\colon M \to \mathbb{R}$ be a smooth function and $a < b$ such that every $y \in [a, b]$ is regular for f. Assume $f^{-1}([a, b])$ is compact. Then,*

(1) *M^a and M^b are diffeomorphic.*
(2) *M^a is a deformation retract of M^b: in particular, inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*[5]

[4]Recall that a *proper map* is a map $f\colon X \to Y$ such that the preimage of any compact set in Y is compact.

[5]Recall that given an inclusion $i\colon A \hookrightarrow X$, a map $r\colon X \to A$ is a *deformation retraction* if theres a homotopy $h\colon [0, 1] \times X \to X$ such that $h_0 = \mathrm{id}_X$ and $h_1 = i \circ r$, and such that $r \circ i = \mathrm{id}_A$.

Again, we have a smooth manifold statement and a homotopical statement.

Proof. First, introduce a Riemannian metric on M. This additional data is necessary so that we can measure things (such as lengths and angles and so on). Riemannian metrics exist on all smooth manifolds; let's talk about why. An inner product on V is a positive definite bilinear pairing; these form a convex space in $\mathrm{Sym}^2 V^*$. In fact, it's a convex cone, because if $a > 0$ and g is an inner product, ag is also an inner product.

Now let M be a smooth manifold and $\mathfrak{U}$ be an atlas. Each open $U \in \mathfrak{U}$ is diffeomorphic to affine space, so we can introduce the standard Euclidean metric on it. We can then use a partition of unity to sum these metrics into a global one: because inner products form a convex space and the partition of unity is a locally finite convex combination, this works.

From the Riemannian metric, we obtain a vector field $\operatorname{grad} f$ with $\operatorname{grad}_p f = 0$ iff f is a critical point. This flows in the direction of increasing height; we want to push M^b down to M^a, so we'll flow along $-\operatorname{grad} f$. But we don't want to flow too much beyond that, so let's introduce a cutoff function $\rho\colon M \to \mathbb{R}^{\geq 0}$ such that

$$\rho(x) = \begin{cases} \dfrac{1}{\|\operatorname{grad} f\|^2}, & x \in f^{-1}([a,b]) \\ 0 & \text{outside } U, \end{cases} \tag{2.8}$$

where U is an open neighborhood of $\overline{f^{-1}([a,b])}$ whose closure is compact.

Set $\xi := -\rho \operatorname{grad} f$. Then ξ generates a global flow $\varphi_t\colon M \to M$. If $p \in M$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(p)) = \left\langle \operatorname{grad} f, \frac{\mathrm{d}\varphi_t(p)}{\mathrm{d}t} \right\rangle = -\rho\|\operatorname{grad} f\|^2. \tag{2.9}$$

In $f^{-1}([a,vb])$ this is just -1, and outside of U, this is the identity. In particular, $\varphi_{b-a}\colon M^b \to M^a$ is a diffeomorphism: its inverse is φ_{a-b}.

For the second part, we can define the requisite homotopy $h\colon [0,1] \times M^b \to M^b$ by

$$h(t,p) := \begin{cases} p, & p \in M^a \\ \varphi_{t(f(p)-a)}, & p \in f^{-1}([a,b]). \end{cases} \tag{2.10}$$

⊠

Exercise 2.11. Let $M = \mathbb{R}$ and $f(x) = (\log x)^2$. Make the theorem explicit in this case.

Let $M = \mathrm{GL}_n(\mathbb{R})$ (resp., $\mathrm{GL}_n(\mathbb{C})$). Show that M deformation retracts onto O_n (resp. U_n). Make the theorem explicit for $f(A) = \mathrm{tr}(\log(A^*A))$.

∽ · ∼

Now we'll do a short review of some Riemannian geometry. Let A be an affine space modeled on a vector space V and $\eta\colon A \to V$ be a smooth function to some vector space. We can define the directional derivative in the direction of an $\eta \in V$ by

$$D_\xi \eta := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(p + t\xi). \tag{2.12}$$

If we're on a smooth manifold M, though, we can't make sense of $p + t\xi$. Instead, we'd like to choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = \xi$, and use this to define the directional derivative. However, we then have a problem: as t varies, $\eta(\gamma(t))$ lives in different vector spaces, so we can't define their difference, which is important for taking the derivative. So we need to introduce more structure in order to define directional derivatives.

Definition 2.13. Let M be a smooth manifold. A *covariant derivative* on $TM \to M$, also called a *linear connection*, i a bilinear map $\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that

1. (linearity over functions) if $f \in C^\infty(M)$, then $\nabla_{f\xi}\eta = f\nabla_\xi\eta$.
2. (Leibniz rule) if $g \in C^\infty(M)$, then $\nabla_\xi(g\eta) = (\xi \cdot g)\eta + g\nabla_\xi\eta$.

The first condition implies $\nabla_\xi\eta|_p$ depends only on $\xi|_p$, which expresses tensoriality.

Definition 2.14. ∇ is *torsion-free* if

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.15}$$

If $\langle -,-\rangle$ is a Riemannian metric on M, then ∇ is *orthogonal* with repsect to g if

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{2.16}$$

Remarkably, these exist and are unique! This is a foundational theorem in Riemannian geometry.

Theorem 2.17. *For any Riemannian manifold (M,g), there's a unique torsion-free orthogonal connection on TM.*

This connection is called the *Levi-Civita connection.* It turns out this can be explicitly constructed with a straightedge and compass, though it would take a while.

Exercise 2.18. Prove Theorem 2.17 by explicitly writing a formula for $\langle \nabla_X Y, Z\rangle$ and using the torsion-free and orthogonal conditions to expand it out, hence defining $\nabla_X Y$.

There are lots of different ways to say the proof, but it's really a formula proof, and no synthetic proof exists. There are special classes of manifolds (e.g. Kähler manifolds) on which a synthetic proof exists.

If (M,g) is a Riemannian manifold and $N \hookrightarrow M$ is an immersed submanifold, then it inherits a Riemannian metric: a subspace of an inner product space gains an inner product by restriction, and doing this for all $T_pN \subset T_pM$ defines the metric on N. Moreover, if $X, Y \in \mathcal{X}(M)$ and $p \in N$, then $\nabla_X^M Y|_p \in T_pM$ need not be in T_pN. But $T_pM = T_pN \oplus \nu_p$, where ν_p is the normal bundle; to choose this splitting we needed to use the metric.

Using this, let $II(X,Y)$ denote the component of $\nabla_X^M Y|_p$ in ν_p, where ∇^M denotes the Levi-Civita connection on M.

Lemma 2.19. *$II(X,Y)$ is linear over functions in both of its arguments, and $II(X,Y) = II(Y,X)$; in particular, it's a symmetric bilinear form.*

The proof is a calculation. $II(X,Y)$ is called the *second fundamental form.*[6] Moreover, it expresses the difference between ∇^M and ∇^N.

Lemma 2.20. *The tangential component of $\nabla_X^M Y$ is $\nabla_X^N Y$.*

If Z is a normal vector field to N in M, we can define $II^Z(X,Y) := \langle II(X,Y), Z\rangle$. Then II^Z is a symmetric bilinear form $T_pM \times T_pM \to \mathbb{R}$, and we know what the invariants of symmetric bilinear forms are. We can also define $S\colon T_pM \to T_pM$ by $\langle S(X), Y\rangle = II(X,Y)$. This is symmetric, so we can diagonalize, and therefore recover an orthonornal basis $e_1, \dots, e_m$ of T_pM (up to units and reordering) such that $Se_j = \lambda_j e_j$ for some $\lambda_j \in \mathbb{R}$. These λ_j are expressing the amount of curvature in various directions — unless they coincide (this is called an *umbilic point*). S is called the *shape operator*, as it determines the local shape of the surface.

Lecture 3.

: 9/5/18

Lecture 4.

Handles and handlebodies: 9/12/18

Today, Riccardo and George spoke about the smooth perspective on Morse theory, where a Morse function defines a handlebody structure on the ambient manifold.

Definition 4.1. If $k, m \in \mathbb{N}$ with $0 \le k \le m$, an *n-dimensional k-handle* is a copy of $D^k \times D^{n-k}$ attached to a manifold X via an embedding $\varphi\colon \partial D^k \times D^{n-k} \hookrightarrow \partial X$.

Inside $D^k \times D^{n-k}$ we have a few distinguished subsets, which also have names in the context of a handle.

- The *attaching sphere* or *attaching region* is the submanifold $\partial D^k \times \{0\}$ of the k-handle, which corresponds to where X meets the k-handle.
- The *core* is $D^k \times \{0\}$. The handle retracts onto its core, so this contains all of the homotopical information about the handle: $X \cup_\varphi (D^k \times D^{n-k})$ is homotopy equivalent to just attaching the core to X.
- $\{0\} \times D^{n-k}$ is called the *cocore* or *belt sphere.*

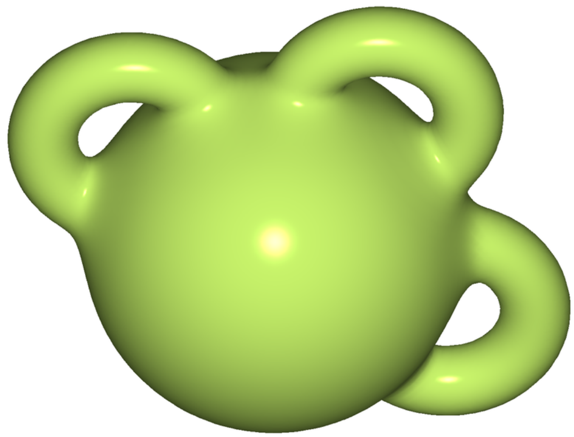


FIGURE 2. Three 2-dimensional 1-handles attached to S^2 minus three discs. Source: https://en.wikipedia.org/wiki/Handle_decomposition.

Sometimes k is also called the *index*.

Definition 4.2. Let X be a compact n-manifold with boundary $\partial X = \partial_- X \amalg \partial_+ X$. A *handle decomposition* of X (relative to $\partial_- X$) is an identification of X with a manifold obtained from $\partial_- X \times I$ by attaching handles. A manifold with a given handle decomposition is called a *relative handlebody* built on $\partial_- X$.

Recall that an isotopy between embeddings $\varphi_0, \varphi_1 \colon X \to Y$ is a homotopy such that φ_t is also a diffeomorphism.

Theorem 4.3 (Isotopy extension theorem)**.** *Let Y be a compact manifold. Then any smooth isotopy $Y \times I \to \mathrm{Int} X$ can be extended to an ambient isotopy $\phi_t \colon X \to X$.*[7]

Proposition 4.4. *An isotopy $h \colon [0,1] \times \partial D^k \times D^{n-k} \to \partial X$ for a handle H specifies a diffeomorphism $X \cup_{\varphi_0} H \cong X \cup_{\varphi_1} H$ (at least up to ambient isotopy).*

Proof. By Theorem 4.3, we can extend h to an ambient isotopy $\Phi \colon [0,1] \times \partial X \to \partial X$. ☒

Proposition 4.5. *The isotopy class of $\varphi \colon \partial D^k \times \partial D^{n-k} \to \partial X$ only depends on the following data:*

- *an embedding $\varphi_0 \colon \partial D^k \times \{0\} \to \partial X$[8] with trivial normal bundle, and*
- *a normal framing of $\varphi_0(S^{k-1})$, i.e. an identification of the normal bundle with $S^{k-1} \times \mathbb{R}^{n-k}$.*

Proof. This is basically the tubular neighborhood theorem, which says that an embedding $\varphi \colon \partial D^k \times D^{n-k} \to \partial X$ can be constructed from the restriction to $\varphi_0 \colon \partial D^k \times \{0\} \to \partial X$ and a choice of a framing. ☒

Remark 4.6. In fact, if $2(\ell+1) \le m$, then any two *homotopic* embeddings of an ℓ-manifold into an m-manifold are isotopic. This is related to the Whitney embedding theorem. ◀

Great, so what data determines a framing? Pick one framing of the normal bundle of $S^{k-1} \hookrightarrow \partial X$. Given another framing f, their "difference" is a map $S^{k-1} \to \mathrm{GL}_{n-k}(\mathbb{R})$. The Gram-Schmidt process is a retraction $\mathrm{GL}_{n-k}(\mathbb{R}) \simeq \mathrm{O}_{n-k}$, so $\pi_{n-1}\mathrm{O}_{n-k}$ acts on the set of framings modulo isotopy.

For example, $\pi_0 \mathrm{O}_1 \cong \mathbb{Z}/2$, which corresponds to the annulus and the Möbius strip. But in general, for $(n-1)$-handles for $n \ne 2$, there's a unique choice of framing, because $\pi_{n-2}\mathrm{O}_1 \cong \pi_{n-1}\mathrm{O}_0 = 1$.

Remark 4.7. A handle has corners, which need to be smoothed. This is possible, but there are details that have to be worked out, and which are mostly not discussed. However, they are worked out in Kosinski's book. ◀

[6] The "first fundamental form" is another word for the inner product on T_pN.
[7] TODO: not clear how X and Y are related. Presumably Y embeds in X?
[8] You could think of this as a knot in ∂X, though this is only literally true when $k = 2$.

In the second half, George provided some examples of handlebodies. The first observation is that, by retracting each handle to its core, a handle decomposition of M describes a CW decomposition (relative to $\partial_- I$, or just a CW decomposition if $\partial_- I = \varnothing$) of a space homotopy equivalent to M.

Theorem 4.8. *Every pair $(X, \partial_- X)$ admits a handle decomposition, where X is a compact manifold and $\partial_- X$ is a union of components of $\partial_- X$.*

We'll see the proof in Dan's lecture later today. The idea is that given a Morse function f and a critical point p with $c \coloneqq f(p)$, $f^{-1}((-\infty, c+\varepsilon]) = f^{-1}((-\infty, c-\varepsilon]) \cup H$, where there are no critical points in $[c-\varepsilon, c+\varepsilon]$ and H is attached to $f^{-1}((-\infty, c-\varepsilon])$ as a handle.

Example 4.9. Let Σ be the closed, connected, oriented surface with genus g. Start with a disc D, and add two 2-dimensional 1-handles h_1 and h_2 such that, traversing along ∂D, the boundary components of h_1 and h_2 alternate. The resulting manifold with boundary is diffeomorphic to a cylinder plus a 2-dimensional 1-handle with one boundary component attached to each component of the boundary of the cylinder.

If we stop here, attaching a 2-handle in the only way we can, we get a torus. More generally, you can attach g pairs of 1-handles as we did, with alternating boundary components. Then closing off with a 2-handle, you get Σ. ◀

Example 4.10. Take a disc and attach a 1-handle by a twist, then attach a 2-handle in the only way possible. Then you obtain $\mathbb{RP}^2$: you can count the number of 1-cells of the corresponding CW complex is 1. ◀

This process is very noncanonical: one can realize S^2 with $2k$ handles by attaching $(k-1)$ 1-handles to a disc to divide the boundary into k components, then adding k 2-handles to close off the boundary. So the manifold isn't just the handle data — you can describe the same manifold in multiple ways.

Example 4.11. Let's construct a handle decomposition for $\mathbb{CP}^n$. Let $\varphi_i \colon \mathbb{C}^n \to \mathbb{CP}^n$ send

$$(z_1, \dots, z_n) \longmapsto [z_1 : z_2 : \dots : z_i : 1 : z_{i+1} : \dots : z_n],$$

and let $B_i \coloneqq \varphi_i(D^2 \times \cdots \times D^2)$. The pairwise intersections of these B_is are subsets of their boundaries, and more generally,

$$B_k \cap \bigcup_{1 \le i < k} B_i = \varphi_k\big(\partial(D_1^2 \times \cdots \times D_k^2) \times D_{k+1}^2 \times \cdots \times D_n^2\big). \tag{4.12}$$

That is, adding $B_k$ias attaching a $2n$-dimensional $2k$-handle. So even though we haven't drawn a picture, we've still specified a handle decomposition. ◀

We've been somewhat sloppy about order, but it turns out that actually doesn't matter.

Proposition 4.13. *Any handle decomposition of a compact pair $(X, \partial_- X)$ can be modified by isotopy such that the handles are attached in increasing order of index.*

TODO: I missed the proof.

Lecture 5.

Handles and Morse theory: 9/12/18

"I'd better prepare for an annoying question, then!" (Picks up colored chalk)

Recall the first theorem of Morse theory: if we have two regular values a and b, $a < b$, and there are no critical values in $[a, b]$, then flow by $-\operatorname{grad} f$ on $f^{-1}([a, b])$ flows $f^{-1}(b)$ to $f^{-1}(a)$, and in particular $f^{-1}([a, b]) \cong [a, b] \times f^{-1}(a)$. This assumes $f^{-1}([a, b])$ is compact.

But at critical points, the topology can and does change.

Theorem 5.1. *Let p be a nondegenerate critical point of a smooth $f \colon M \to \mathbb{R}$ of index λ. Let $c \coloneqq f(p)$ and $\varepsilon > 0$ be such that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact with unique critical point c. Then $M^{c+\varepsilon}$ is diffeomorphic to $M^{c-\varepsilon} \cup_\varphi H$, where H is an index-λ handle and $\varphi \colon \partial D^\lambda \times D^{n-\lambda} \to f^{-1}(c-\varepsilon)$ is an embedding.*

If $\varepsilon' < \varepsilon$, we can replace ε by ε'.

Proof. Set $c = 0$ for convenience. By Lemma 1.36, we can find a system of coordinates $x = (x^1, \dots, x^n) \colon U \to \mathbb{R}^n$ with $x(p) = 0$, $x(U) \supset \overline{B_\varepsilon(0)}$, and

$$f = -((x^1)^2 + \cdots + (x^\lambda)^2) + ((x^{\lambda+1})^2 + \cdots + (x^n)^2) \tag{5.2}$$

on U. Let

$$H \coloneqq \{q \in M^\varepsilon \cap U \mid (x^1)^2 + \cdots + (x^\lambda)^2 \le \varepsilon/2\} \tag{5.3}$$

and $N^\varepsilon \coloneqq \overline{M^\varepsilon \setminus H}$. We'll show (1) H is a handle of index λ, (2) this identifies $\partial H \cap \partial N^\varepsilon \cong \partial D^\lambda \times D^{n-\lambda}$, and (3) $N^\varepsilon \cong M^{-\varepsilon}$. If all of these are true, then the theorem follows.

For the first claim, consider the function

$$\psi \colon D^\lambda(\sqrt{\varepsilon/2}) \times D^{n-\lambda} \longrightarrow H \tag{5.4a}$$

defined by

$$\psi((u^1, \dots, u^\lambda), (v^1, \dots, v^{n-\lambda})) \coloneqq \big(u^1, \dots, u^\lambda, cv^1, \dots, cv^\lambda\big), \tag{5.4b}$$

where

$$c = \frac{2}{3}\left(1 + \frac{(U^1)^2 + \cdots + (u^\lambda)^2}{\varepsilon}\right). \tag{5.4c}$$

It remains to check this is a diffeomorphism, but we've been given a completely explicit formula so that's not very hard.[9] The second claim is "clear," meaning that if you trace through the definition of ψ and track what happens to $\partial D^\lambda \times D^{n-\lambda}$, you'll see it.

For the last claim, let $g \coloneqq f|_{N^\varepsilon} \colon N^\varepsilon \to \mathbb{R}$. Then $g^{-1}([-\varepsilon, \varepsilon])$ is compact and contains no critical points, so by Theorem 2.7, $N^\varepsilon \cong M^{-\varepsilon}$. ⊠

Corollary 5.5. *Any manifold M admits a handle decomposition.*

Proof. Use a proper Morse function. ⊠

If M is noncompact, we may need an infinite number of handles, which is fine; it'll be countable, because M is countable and nondegenerate critical points are isolated.

You can think of these handle attachments in terms of surgery. Say $M = S^1$, so the only handles are 0- and 1-handles (which look like $\cup$ and $\cap$).

If $M = T^2$ with the standard height function, we first attach a 2-dimensional 0-handle, and then a 1-handle, then another 1-handle, and finally a 2-handle.

These surgeries come with the manifolds-with-boundary $C \coloneqq f^{-1}([c-\varepsilon, c+\varepsilon])$, which is also helpful to have around. If $B_\pm \coloneqq f^{-1}(c \pm \varepsilon)$, then C is a *bordism* between B_- and B_+: it's a compact manifold together with an identification $\partial C = B_- \amalg B_+$. Compactness is important here: otherwise ever manifold is bordant to the empty set via $M \times [0, \infty)$, and that's not very exciting. If you restrict to compact bordisms, there are manifolds which don't bound: $\mathbb{RP}^2$ is the simplest example.

Since we know the bordism is n-dimensional and corresponds to an index-λ critical point, we have very explicit descriptions of these three manifolds: if $A \coloneqq B_- \setminus S^{\lambda+1} \times D^{n-\lambda}$, then

$$C \cong B_- \cup_{S^{\lambda-1} \times D^{n-\lambda}} D^\lambda \times D^{n-\lambda} \tag{5.6a}$$

$$B_- \cong A \cup_{S^{\lambda-1} \times S^{n-\lambda-1}} S^{\lambda-1} \times D^{n-\lambda} \tag{5.6b}$$

$$B_+ \cong A \cup_{S^{\lambda-1} \times S^{n-\lambda-1}} D^\lambda \times S^{n-\lambda-1}. \tag{5.6c}$$

Now we'll switch to the homotopical story, which is broadly similar in its relationship to Morse theory but is otherwise pretty different.

Definition 5.7. Let Y be a space and $\psi \colon S^{\lambda-1} \to Y$ be a continuous map. Then, forming the space $X \coloneqq Y \cup_\psi D^\lambda$ is called attaching a cell to Y via ψ, and ψ is called the *attaching map*.

Definition 5.8. A *CW complex* or *cell complex* is a space constructed by successively attaching 0-cells, 1-cells, 2-cells, etc., in order, to $\varnothing$.[10]

[9] This way of giving a proof sketch is appealing, because the explicit formula isn't so bad, and the audience really can fill in all the details.

[10] If you want to attach infinitely many cells, use the weak topology.

Whiteead first defined CW complexes in an equivalent but different-looking way; you can see this definition in the appendix of Hatcher's book.

Theorem 5.9. *With notation as in Theorem 5.1, $M^{c+\varepsilon} \simeq M^{c-\varepsilon} \cup_\psi D^\lambda$ for some $\psi\colon S^{\lambda-1} \to M^{c-\varepsilon}$.*

Remark 5.10. In the smooth case, we glued along open sets, which was important in order to know what the smooth structure is. In this setting, where we only care about the homotopy type, we can glue along closed sets without any issues. ◀

Proof of Theorem 5.9. Again we set $c = 0$. Take

$$\psi\colon (u^1, \dots, u^\lambda) \longmapsto (u^1, \dots, u^\lambda, 0, \dots, 0) \tag{5.11}$$

composed with the diffeomorphism $\partial N^\varepsilon \cong \partial M^{-\varepsilon} = f^{-1}(-\varepsilon)$ given by the third claim in the proof of Theorem 5.1. We'll construct a deformation retraction of $N^\varepsilon \cup H = M^\varepsilon$ into $N^\varepsilon \cup_\psi D^\lambda$ which is the identity outside

$$V \coloneqq \left\{ q \in M^\varepsilon \cap U \mid (x^1)^2 + \cdots + (x^\lambda)^2 \le \frac{3\varepsilon}{4} \right\}. \tag{5.12}$$

Let $\rho\colon M^\varepsilon \to [0,1]$ be a smooth function equal to 0 outside V and equal to 1 on H, and let

$$\xi \coloneqq -\rho\left(x^{\lambda+1} \frac{\partial}{\partial x^{\lambda+1}} + \cdots + x^n \frac{\partial}{\partial x^n} \right). \tag{5.13}$$

Flow along $-x\partial_x$ flows to the origin, since the integral curves are of the form $x = Ce^{-t}$. Therefore flowing to infinity deformation retracts $\mathbb{R}$ onto the origin. Instead ξ retracts H onto $H \cap D^\lambda$, and then smoothly softens to zero outside of H. In particular, ξ generates a flow φ, and $\lim_{t\to\infty} \varphi_t$ is the desired retraction. ⊠

Corollary 5.14. *M has the homotopy type of a CW complex, with a λ-cell for each critical point of index λ.*

This is not a trivial corollary (several pages in Milnor's book). One problem is that we'd like to attach the cells in order of dimension, which can be done using a rearrangement theorem, using a *self-indexing* Morse function: the critical points of index k are on $f^{-1}(k)$. These exist. Another, easier, issue is that we'd like the attaching maps to be cellular, but this can be easily fixed using the cellular approximation theorem.

We didn't have time to get to the next theorem, but it's interesting.

Theorem 5.15 (Reeb)**.** *Let M be a compact n-manifold and $f\colon M \to \mathbb{R}$ have exactly two critical points, each nondegenerate. Then $M \approx S^n$.*

That is, M is homeomorphic to S^n. Milnor looked at some examples and discovered something surprising, that some of them aren't diffeomorphic to S^n! He looked specifically at S^7, but this is true in many other dimensions too.

Lecture 6.

Morse theory and homology: 9/26/18

"This is called the Morse inequalities, which is strange because they're equalities."

First we'll discuss the proof of Theorem 5.15, that any manifold M with a function f with exactly two critical points, both nondegenerate, is homeomorphic to a sphere.

Proof of Theorem 5.15. Let p_0 be the first critical point and p_0 be the second, and without loss of generality assume $f(p_i) = i$. Choose Morse coordinates $x^1, \dots, x^n$ on an open neighborhood U of p_0: $x^i(p_0) = 0$, $B_0(2\varepsilon) \subset x(U)$, and on U,

$$f = (x^1)^2 + \cdots + (x^n)^2. \tag{6.1}$$

Now we choose a Riemannian metric on M which on $f^{-1}((-\infty, 2\varepsilon))$ is the standard Riemannian metric on $B_{2\varepsilon}(0)$: $(\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$. Let $\xi \coloneqq (\operatorname{grad} f)/|\operatorname{grad} f|^2$ on $f^{-1}([\varepsilon, 1-\delta])$ for some $\delta > 0$, and let φ_t be the flow ξ generates. Observe $\xi \cdot f = 1$ everywhere.

Define $h\colon B \to M \setminus \{p_1\}$ by

$$x = (x^1, \dots, x^n) \longmapsto \begin{cases} \text{the corresponding point in } U \subset M, & |x| \le \varepsilon \\ \varphi_{1-\varepsilon}(\varepsilon x/r), & \varepsilon \le r = |x| < 1. \end{cases} \tag{6.2}$$

Then check that h is a diffeomorphism: smoothness follows from properties of flow, and the inverse function theorem tells you the inverse is smooth. Then one can extend h to a *homeo*morphism $\widetilde{h}\colon S^n \approx D^n/\partial D^n \to M$, which sends $[\partial D^n] \mapsto p_1$. ⊠

In general, we cannot make M diffeomorphic to S^n.

Recall that we showed in Corollary 5.14 that a Morse function f on M defines a CW complex homotopic to M. This has consequences for the homology and cohomology of M. Specifically, the homology of M is that of a chain complex

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n \longleftarrow 0 \tag{6.3}$$

where C_q is free abelian of rank c_q, the number of critical points of index q. In particular, if M is closed, so its CW complex is finite, each c_q is finite, and this is smaller than, but quasi-isomorphic to, the singular chain complex used to define homology, and computations with it may be easier.

Corollary 6.4 (Lacunary[11] principle)**.** *If for every $c_q, c_{q'}$ nonzero, we have $|q'-q| \geq 2$, then $H_*(M)$ is torsion-free.*

Proof. Well this means all maps ∂ in (6.3) are zero, and therefore the chain complex computes its own homology, and each C_q is torsion-free. ⊠

Let k be a field, and define $C_q(k) \coloneqq C_q \otimes k$. Then $H_*(M;k)$ is the homology of the induced chain complex

$$0 \longleftarrow C_0(k) \xleftarrow{\partial} C_1(k) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n(k) \longleftarrow 0 \tag{6.5}$$

Definition 6.6. The *Betti numbers* of M are $h_q(k) \coloneqq \dim_k H_q(M;k)$. If we don't specify k, it's assumed to be $\mathbb{Q}$.

Example 6.7. $M = \mathbb{RP}^n$ has a CW structure with a cell in every dimension, and its CW chain complex is

$$0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \longleftarrow \mathbb{Z} \longleftarrow 0. \tag{6.8}$$

If $k = \mathbb{F}_2$, then all of the boundary maps on $C_*(\mathbb{F}_2)$ are 0, so the homology is $\mathbb{F}_2$ in every dimension, and $h_q(\mathbb{F}_2) = 1$ for all q. But over $\mathbb{Q}$, they're nonzero:

$$h_q = \begin{cases} 1, & q = 0 \text{ or } q = n \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.9}$$

◄

Definition 6.10. The *Euler characteristic* or *Euler number* of M is

$$\chi(M) \coloneqq \sum_{q=0}^{n} (-1)^q c_q. \tag{6.11}$$

This turns out to equal $\sum (-1)^q h_q(k)$ for all fields k.

Theorem 6.12 (Morse inequalities)**.** *Define*

$$M_t \coloneqq \sum_{q=0}^{n} c_q t^q \qquad \text{and} \qquad P_t(k) = \sum_{q=0}^{n} h_q(k) t^q. \tag{6.13}$$

Then there's a polynomial R_t whose coefficients are nonnegative integers and such that

$$M_t - P_t(k) = (1-t)R_t. \tag{6.14}$$

$P_t(k)$ is called the *Poincaré polynomial* of M.

Proof. As usual, let $B_q(k)$ denote the group of *q-boundaries* (in the image of $\partial\colon C_{q+1}(k) \to C_q(k)$) and $Z_q(k)$ denote the group of *q-cycles* (in the kernel of $\partial\colon C_q(k) \to C_{q-1}(k)$). Let $b_q(k) = \dim_k B_q(k)$. From the short exact sequences

$$0 \longrightarrow Z_q(k) \longrightarrow C_q(k) \xrightarrow{\partial} B_{q-1}(k) \longrightarrow 0 \tag{6.15a}$$

$$0 \longrightarrow B_q(k) \longrightarrow Z_q(k) \longrightarrow H_q(k) \longrightarrow 0, \tag{6.15b}$$

[11]"Lacunary" means pertaining to gaps.

we see

$$c_q = h_q(k) + b_q(k) + b_{q-1}(k), \tag{6.16}$$

so we can set

$$R_t \coloneqq \sum_{q=0}^{n} b_q(k) t^q. \tag{6.17}$$

⊠

Corollary 6.18. *$c_0 \ge h_0(k)$, $c_1 - c_0 \ge h_1(k) - h_0(k)$, and so on: for any m,*

$$\sum_{q=0}^{m} (-1)^q c_q \ge \sum_{q=0}^{m} (-1)^q h_q(k). \tag{6.19}$$

For example, in the lacunary situation of Corollary 6.4, $R_t = 0$ and $M_t = P_t(k)$.

Corollary 6.20. *If $f\colon M \to \mathbb{R}$ is Morse, the Morse inequalities Corollary 6.18 hold where c_q is the number of critical points of index q.*

This provides information about critical points: there must be at least as many index-q critical points as the rank of $H_q(M; k)$, for any field k. For example, the homology of $\mathbb{CP}^n$ has one free term in each even degree, so we know Morse functions on $\mathbb{CP}^n$ have at least those critical points, though we may hope for the lacunary situation and a minimal number of critical points.

Definition 6.21. A Morse function $f\colon M \to \mathbb{R}$ is *perfect over k* if $c_q = h_q(k)$ for all q. If this holds for all k, we call f *perfect*.

The existence of a perfect Morse function implies $h_q(k) = h_q(\mathbb{Q})$ for all fields k, which means that $H_*(M)$ is torsion-free. (The converse is probably not true.) Thus, for example, $\mathbb{RP}^n$ cannot have a perfect Morse function unless $n \le 1$.

Example 6.22. Let SU_3 denote the group of complex 3×3 matrices A such that $\det A = 1$ and $A^*A = I$, where $*$ denotes Hermitian conjugate. This is an eight-dimensional Lie group: a Hermitian matrix is determined by three complex numbers above the diagonal and three real numbers on the diagonal, so U_3 is nine-dimensional, and requiring $\det A = 1$ cuts it down one more dimension.

The Lie algebra of SU_3, denoted $\mathfrak{su}_3$, is the Lie algebra of 3×3 compelx matrices X with trace zero and $X^* + X = 0$. This contained within it the subalgebra $\mathfrak{t}$ of diagonal matrices, with entries $\lambda_1, \lambda_2, \lambda_3$ all in $i\mathbb{R}$ and summing to zero. This is a two-dimensional vector space with three distinguished lines $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, and $\lambda_1 = \lambda_3$.

There's an SU_2-action on $\mathfrak{su}_2$ by conjugation: given a $P \in \mathfrak{su}_2$, let M_P denote its orbit, called the *adjoint orbit* of P. It's a fact that every adjoint orbit intersects $\mathfrak{t}$ nontrivially, in an orbit of the symmetric group S_3 acting on $\mathfrak{t}$ by permuting the diagonal entries; this is a jazzed-up version of the fact that a skew-Hermitian matrix is diagonalizable.

There are three kinds of orbits.

(1) The generic situation (the *generic orbits*) occurs when A is diagonalizable, so we may assume A is diagonal. The space of such orbits is a 2-dimensional torus, since it's given by the diagonal matrices in SU_3, which are specified by data of three unit complex numbers whose product is 1. Therefore the orbit is a 6-manifold, a homogeneous space of the form SU_3/T^2. This is a complex manifold (in fact a Kähler manifold), called the *flag manifold* of SU_3. Call this M.

(2) Another orbit type has $\lambda_1 = \lambda_2$, where its Jordan form is block diagonal (one 2×2 block, one 1×1 block). In this case, the stabilizer is the special unitary matrices which have that form, which is denoted $S(\mathrm{U}_2 \times \mathrm{U}_1)$, and what we get is a 4-manifold. Each matrix in an orbit is determined by a complex line, so the orbit is precisely $\mathbb{CP}^2$.

(3) The zero matrix is unaffected by conjugation. This is the last kind of orbit.

The vector space $\mathfrak{su}_3$ has a metric,

$$\langle X, Y \rangle = -\operatorname{tr}(XY). \tag{6.23}$$

This is SU_3-invariant, and for $Z \in \mathfrak{su}_3$,

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0. \tag{6.24}$$

Therefore if $\mathrm{ad}_P\colon \mathfrak{su}_3 \to \mathfrak{su}_3$ sends $X \mapsto [P, X]$, $T_P M_P$ is the image of ad_P, and therefore the normal space is $\ker(\mathrm{ad}_P)$.

For an adjoint orbit M, let $f\colon M \to \mathbb{R}$ be

$$P \longmapsto \frac{1}{2}\mathrm{dist}(Q, P)^2, \tag{6.25}$$

where Q is some matrix not in this orbit.

Theorem 6.26.
1. $\mathrm{crit}(f) = M \cap \mathfrak{t}$.
2. *f is Morse iff Q isn't on the three lines $\{\lambda_i = \lambda_j\}$.*
3. *The index of a $P \in \mathrm{crit}(f)$ is twice the number of times the line from P to Q intersects the lines $\{\lambda_i = \lambda_j\}$.*

The indices are even, so the lacunary principle applies, and we can read off the Betti numbers from these intersections, and see Poincaré duality. We in particular conclude
1. $H_*(M)$ and $H_*(\mathbb{CP}^2)$ are torsion-free.
2. $\mathbb{CP}^2$ has a CW structure with one 0-cell, one 2-cell, and one 4-cell.
3. The flag manifold has a CW structure with one 0-cell, two 2-cells, two 4-cells, and one 6-cell.
4. For generic P, $H^2(M_P) \cong \mathbb{Z}^2$, which we can interpret as the group of line bundles on the orbit.

◀

This applies to general connected compact Lie groups, though requires more theory. Bott then applies this to loop spaces, which are infinite-dimensional.

Lecture 7.

Knots and total curvature: 9/26/18

Jonathan, then Sebastian, gave this part of the lecture, where they discussed integrating the curvature of a knot and the Fary-Milnor theorem.

Definition 7.1. A *knot* is a smooth embedding $K\colon S^1 \to \mathbb{E}^3$.

To do geometry with knots, we'll want to parameterize the knot, by defining a function $x\colon \mathbb{R}|to\mathbb{E}^3$ with $x(s_1) = x(s_2)$ iff $s_2 - s_1 = Ln$ for a fixed constant $L \in \mathbb{R}$ and $n \in \mathbb{Z}$. Assume $|x'(s)| = 1$.

Definition 7.2. The *absolute curvature* of K at $x_0 \in K$ is $|\kappa(s)| = |x''(s_0)|$, where $x(s_0) = x_0$. The *total curvature* is

$$T_K := \int_0^L |\kappa(s)|\,\mathrm{d}s.$$

Absolute curvature has units of $1/L$, and the total curvature is dimensionless.

Theorem 7.3 (Fáry-Milnor)**.** *If the total curvature of K is less than 4π, then K is unknotted (i.e. isotopic to a trivial embedding).*

This theorem was proven at about the same time by both Fáry and Milnor. Milnor was about 19 years old.

Example 7.4. Consider the unknot as the unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$, with parameterization $(R\cos s, R\sin s, 0)$. Then $|\kappa(s)| = 1/R$, and the total curvature is 2π. ◀

Example 7.5. The embedding

$$x(s) = (4\cos 2s + 2\cos s, 4\sin 2s - 2\sin s, \sin 3s) \tag{7.6}$$

defines a knot called a *trefoil*. In this case $T_K \approx 13.04$ (for reference, $4\pi \approx 12.57$). ◀

Pick a $v \in S^2$ and define $h_v\colon K \to \mathbb{R}$ by $h_v(x) = \langle x, v\rangle$.

Definition 7.7. Let $\mu_K(v)$ be $\#\,\mathrm{crit}(h_v)$ when h_v is Morse, and zero otherwise, which defines a function $\mu_K\colon S^2 \to \mathbb{Z}$.

This function is integrable (in the sense of Lesbegue), and we let

$$\overline{\mu}_K \coloneqq \frac{1}{4\pi}\int_{S^2} \mu_K(v)\,\mathrm{d}A. \tag{7.8}$$

This is the average number of critical points of h_v over $v \in S^2$.

Definition 7.9. Let (M_0, g_0) and (M_1, g_1) be compact Riemannian manifolds of the same dimension and $f\colon M_0 \to M_1$ be a smooth map. The *Jacobian* of f is a function $|J_f|\colon M_0 \to [0,\infty)$, defined as follows: at $x_0 \in M_0$, if $\{e_i\}$ is an orthonormal basis of $T_{x_0}M_0$, let $G_F(x_0)$ denote the matrix whose $(i,j)^{\text{th}}$ entry is $g_1(\mathrm{d}f|_{x_0}(e_i), \mathrm{d}f|_{x_0}(e_j))$. Then,

$$|J_f|(x_0) \coloneqq \sqrt{\det G_F(x_0)}. \tag{7.10}$$

There's an argument to show this doesn't depend on the orthonormal basis we chose.

Definition 7.11. Suppose $f\colon M_0 \to M_1$ is a smooth function between compact manifolds of the same dimension. Let $N_f\colon M_1 \to \mathbb{Z}$ send x_1 to the cardinality of its preimage if x_1 is a regular value, and 0 if it's a critical value.

Theorem 7.12 (Co-area formula)**.** *N_f is measurable, and*

$$\int_{M_1} N_f(x_1)\,\mathrm{d}V = \int_{M_0} |J_f|(x_0)\,\mathrm{d}V.$$

Proof idea. There's a fairly simple calculation which gets across the idea, but not the details:

$$\int_{M_0} |J_f|(x_0)\,\mathrm{d}V = \int_{M_1}\left(\int_{f^{-1}(x_1)} \mathrm{d}V_{f^{-1}(x_1)}\right)\mathrm{d}V = \int_{M_1} N_f(x_1)\,\mathrm{d}V. \tag{7.13}$$ ☒

There's another interpretation of this theorem: the Riemannian metric on M_0 defines a measure μ_0, and we can push it forward to M_1. Then, Theorem 7.12 says that $f_*\mu_0$ is absolutely continuous with respect to the Riemannian measure μ_1 of M_1, and that it's a multiple by the function N_f.

Let $S(\nu)$ denote the normal bundle of the knot, the set of pairs $(x, v) \in K \times S^2$ with $v \perp \dot{x}$, and let $\rho_K\colon S(\nu) \to S^2$ send $(x, v) \mapsto v$.

Lemma 7.14. *Given a $v \in S^2$, $\mu_k(v) = N_{\rho_K}(v)$. That is, v is a nondegenerate critical point iff v is a regular value of ρ_K, and $\#\operatorname{crit}(h_v) = \#\rho_K^{-1}(v)$.*

Proof. Fix a $v \in S^2$. Then $x(s) \in \operatorname{Crit}(h_v)$ iff $h_v'(s) = (v, x'(x)) = 0$, which is true precisely when $v \perp \dot{x}(s)$, i.e. when $(x(s), v) \in S(\nu)$, which is equivalent to $(x(s), v) \in \rho_K^{-1}(v)$.

Now suppose $x_0 \in \operatorname{crit}(h_v)$, so $\langle v, \ddot{x}(s_0))\rangle = 0$. Fix $\mathbf{e}_1(s)$ such that $(x(s), \mathbf{e}_1(s))$ is a section of $S(\nu)$ and $\mathbf{e}_1(s_0) = v$, and let $\mathbf{e}_2(s) \coloneqq \dot{x}(s) \times \mathbf{e}_1(s)$; since $\mathbf{e}_1(x) \perp \dot{x}(s)$, this gives us something nonzero. We also have that (TODO: I'm not sure what the notation meant exactly here). ☒

By the coarea formula.

$$\begin{aligned}\overline{\mu}_K &= \frac{1}{4\pi}\int_{S^2} \mu_K(v)\,\mathrm{d}A\\ &= \frac{1}{4\pi}\int_{S^2} N_{\rho_K}(v)\,\mathrm{d}A\\ &= \frac{1}{4\pi}\int_{S(\nu)} |J_{\rho_K}|\,\mathrm{d}A.\end{aligned}$$

We put the metric $g_{S(\nu)} \coloneqq \mathrm{d}s^2 + \mathrm{d}\theta^2$ on $S(\nu)$, and then compute:

$$\rho_K(s,\theta) = v(s,\theta) = \cos(\theta)\mathbf{e}_1(s) + \sin(\theta)\mathbf{e}_2(s), \tag{7.15}$$

and the Jacobian is

$$|J_K|^2 = \begin{vmatrix}\langle v_s, v_s\rangle_E & \langle v_\theta, v_s\rangle_E\\ \langle v_s, v_\theta\rangle_E & \langle v_\theta, v_\theta\rangle_E\end{vmatrix}, \tag{7.16}$$

where

$$v_s = \cos\theta \mathbf{e}_1'(s) + \sin\theta \mathbf{e}_2'(s) \tag{7.17a}$$

$$v_\theta = -\sin\theta \mathbf{e}_1(s) + \cos\theta \mathbf{e}_2(s). \tag{7.17b}$$

Let $A(s) = (a_{ij}(s))$, where $a_{ij}(s) = \langle \mathbf{e}_i(s), \mathbf{e}_j'(s)\rangle$. This is a skew-symmetric matrix:

$$A(s) = \begin{pmatrix} 0 & -\alpha(s) & -\beta(s) \\ \alpha(s) & 0 & -\gamma(s) \\ \beta(s) & \gamma(s) & 0 \end{pmatrix}. \tag{7.18}$$

Then

$$\langle v_\theta, v_\theta\rangle = 1 \tag{7.19a}$$

$$\langle v_s, v_s\rangle = (\alpha(s)\cos\theta + \beta(s)\sin\theta)^2 + \gamma(s)^2 \tag{7.19b}$$

$$\langle v_s, v_\theta\rangle = \langle \mathbf{e}_1'(s), \mathbf{e}_2(s)\rangle = \alpha_{12}(s) = \gamma(s). \tag{7.19c}$$

This means the Jacobian is

$$\begin{aligned} |J_{N_f}| &= |\alpha(s)\cos\theta + \beta(s)\sin\theta| \\ &= |(\alpha(s), \beta(s)) \cdot (\cos\theta, \sin\theta)|. \end{aligned} \tag{7.20}$$

Therefore

$$\begin{aligned} \int_0^L \left(\int_0^{2\pi} |(\alpha(s), \beta(s)) \cdot (\cos\theta, \sin\theta)|\,\mathrm{d}\theta\right)\mathrm{d}s &= \int_0^{2\pi} |\alpha(s), \beta(s)| \cdot |\cos(\theta - \varphi)|\,\mathrm{d}\theta \\ &= 4\sqrt{\alpha(s)^2 + \beta(s)^2} \\ &= 4|\mathbf{e}_0'(s)|. \end{aligned}$$

Milnor defined the *crookedness* of a knot to be $c_K := (1/2)\overline{\mu}_K$ and

$$T_K := \int_0^L |\kappa(s)|\,\mathrm{d}s = \pi \cdot \overline{\mu}_K = 2\pi c_K. \tag{7.21}$$

Corollary 7.22. *Any knot has total curvature at least 2π.*

Proof. Since any Morse function has a minimum, $c_K \geq 1$; then invoke (7.21). ☒

Corollary 7.23. *If K is planar and convex, then $T_K = 2\pi$.*

Proof. Convexity means any Morse function has a unique minimum, so $c_K = 1$, and then we use (7.21). ☒

In fact, the converse is true.

Proof sketch of Theorem 7.3. If $T_K < 4\pi$, then $c_K < 2$, which means $c_K(v) = 1$ for all v. (TODO: how does this suffice? I'm really confused — maybe I have some definitions wrong) ☒

Chern and Lashof generalized this to higher-dimensional immersions $M \hookrightarrow \mathbb{R}^N$. For example, consider a compact, oriented surface Σ with genus g embedded in $\mathbb{R}^3$, and with total curvature $(2g+2)\cdot 2\pi$ iff the surface lies on one side of the tangent plane at each point of positive Gaußian curvature.

Lecture 8.

Submanifolds of Euclidean space: 10/1/18

> *"Please ask questions, it's boring to just be up here by myself. Actually, that's not true; I love it."*

Let E be a Euclidean space modeled on a real finite-dimensional inner product space V, and M be an n-dimensional submanifold of E. In this setup there is some additional structure; the first thing we'll do today is discuss that structure.

Definition 8.1. The *first fundamental form* on M is the induced metric on M, $I_p\colon T_pM \times T_pM \to \mathbb{R}$.

In more detail, if $p \in M$, T_pE is canonically identified with V, and $T_pM \subset T_pE = V$, so given $\xi, \eta \in T_pM$, $I_p(\xi, \eta) = \langle \xi, \eta \rangle$ taken in V. The *normal bundle* $NM \to M$ is the vector bundle whose fiber at a $p \in M$ is the orthogonal complement of T_pM inside V. For all p there's a direct-sum splitting $V = N_pM \oplus T_pM$, splitting a vector ξ into its tangential and normal components $\xi^\top$ and $\xi^\perp$, respectively.

Definition 8.2. The *second fundamental form* on M, denoted $I\!I_p \colon T_pM \times T_pM \to N_pM$, sends $\xi_1, \xi_2 \mapsto (D_{\xi_1}\xi_2)^\perp$.

To make sense of this, we employ a common trick in geometry: extend ξ_1 and ξ_2 to vector fields in a neighborhood of p, then show it's independent of that extension.

Lemma 8.3. *This is independent of the extension of ξ_2, and is symmetric in ξ_1 and ξ_2.*

Proof. It suffices to show that $\varphi\colon \xi_2 \mapsto (D_{\xi_1}\xi_2)^\perp$ is *linear over functions*, i.e.

$$\varphi(f\xi_2) = f(p)\varphi(\xi_2). \tag{8.4}$$

This is a calculation:

$$(D_{\xi_1}(f\xi_2))^\perp(p) = ((\xi_1 \cdot f)(p) \cdot \xi_2(p) + f(p)D_{\xi_1}\xi_2(p))^\perp \tag{8.5}$$

$$= f(p)(D_{\xi_1}(\xi_2))(p), \tag{8.6}$$

since ξ_1 and ξ_2 are purely tangential.

We'll return to symmetry in a little bit. ⊠

If we chose the tangential component instead of the normal one, we wouldn't get something linear over functions; instead, we'd get a connection, and in fact the Levi-Civita connection.

Definition 8.7. If $\nu \in N_pM$, define $I\!I_p(\nu)\colon T_pM \times T_pM \to \mathbb{R}$ by

$$\xi_1, \xi_2 \longmapsto \langle I\!I_p(\xi_1, \xi_2), \nu \rangle = \langle D_{\xi_1}\xi_2, \nu \rangle. \tag{8.8}$$

If ν is extended to a normal vector field in a neighborhood of p, then $\langle \xi_2, \nu \rangle = 0$, so

$$0 = \xi_1 \cdot \langle \xi_2, \nu \rangle = \langle D_{\xi_1}\xi_2, \nu \rangle + \langle \xi_2, D_{\xi_1}\nu \rangle. \tag{8.9}$$

Since I_p is nondegenerate, we define the *shape operator*, a self-adjoint operator $S_p(\nu)\colon T_pM \to T_pM$ by

$$I\!I_p(\nu)(\xi_1, \xi_2) = I_p(S_p(\nu)\xi_1, \xi_2) = \langle S_p(\nu)\xi_1, \xi_2 \rangle. \tag{8.10}$$

Example 8.11. Suppose $\dim V = 2$ and $\dim M = 1$. Then the normal bundle is one-dimensional; a consistent choice of unit normal ν_p on the plane curve M is called a *co-orientation*. In this case, the shape operator for ν_p is exactly the signed curvature of M at p. ◀

For surfaces in 3-space, the shape operator is also related to curvature as it's classically studied, though the description is a little more complicated.

Suppose $q \in E \setminus M$, and define $f\colon M \to \mathbb{R}$ by

$$f(p) := \frac{1}{2}\mathrm{dist}_E(p, q)^2 = \frac{1}{2}\langle \nu_p, \nu_p \rangle, \tag{8.12}$$

where $\nu\colon M \to V$ sends $p \mapsto q - p_t$, where p_t is a vector field with $p_0 = p$ and $\dot{p}_0 = \xi$. Then

$$\mathrm{d}f_p(\xi) = \langle D_\xi \nu, \nu \rangle = -\langle \xi, \nu \rangle, \tag{8.13}$$

since $\xi \in T_pM$. That is, p is a critical point of f iff $q - p \perp T_pM$. In this case, the Hessian is

$$\begin{aligned} \mathrm{Hess}_p\, f(\xi_1, \xi_2) &= \xi_1 \cdot (\xi_2 f)(p) = \frac{1}{2}\xi_1\xi_2\langle \nu, \nu \rangle \\ &= -\xi_1\langle \xi_2, \nu \rangle \\ &= -\langle D_{\xi_1}\xi_2, \nu \rangle - \langle \xi_2, D_{\xi_1}\nu \rangle \\ &= -I\!I_p(\nu)(\xi_1, \xi_2) + \langle \xi_2, \xi_1 \rangle. \end{aligned}$$

That is,

$$\mathrm{Hess}_p(f) = I_P - I\!I_p(\nu), \tag{8.14}$$

which is a pretty formula.

The associated self-adjoint operator is $\mathrm{id}_{T_pM} - S_p(\nu)$. If $\mu_1, \ldots, \mu_n$ are the eigenvalues of $S_p(\nu)$, then

$$\dim\ker\mathrm{Hess}_p(f) = \#\{\mu_i \mid \mu_i = 1\} \tag{8.15a}$$

$$\mathrm{ind}\,\mathrm{Hess}_p(f) = \#\{\mu_i \mid \mu_i > 1\}. \tag{8.15b}$$

Lemma 8.16. *Set $q_t := p + t(q-p)$ and $f_t(p') := (1/2)|q_t - p'|$. Then*

$$\mathrm{ind}\,\mathrm{Hess}_p f = \sum_{0<t<1} \dim\ker\mathrm{Hess}_p f_t.$$

Proof. This is because

$$\mathrm{Hess}_p f_t = I_p - II_p(t\nu) = I_p - tII_p(\nu). \tag{8.17}$$ ⊠

The focal points of the manifold are exactly the points q such that the p we get is a degenerate critical point. If M is a light source, these are focal points ("bright spots") as per usual.

More precisely, let $e\colon NM \to E$ be the map $(p, v) \mapsto p + v$, the evaluation map.

Definition 8.18. A *focal point* of M is a critical value of e.

Proposition 8.19. *$q = p + \nu$ is a focal point iff $\mathrm{Hess}_p f_q$ is nondegenerate.*

Proof. Suppose (p_t, ν_t) is a curve in NM with $(p_0, \nu_0) = (P, \nu)$, $(\dot p_0, \dot\nu_0) = \lambda \in T_{(p,\nu)}NM$, such that the component in T_pM is ξ. Then

$$de_{(p,\nu)}(\lambda) = \xi + \dot\nu \in V. \tag{8.20}$$

If this vanishes, then $\dot\nu^\perp = -\xi$, so $\dot\nu^\perp = 0$. For any $\nu \in T_pM$,

$$II_p(\nu)(\xi,\eta) = -\langle D_\xi\nu, \eta\rangle = -\langle\dot\nu,\eta\rangle = \langle\xi,\eta\rangle, \tag{8.21}$$

so $S_p(\nu)\xi = \xi$ and $\xi \in \ker\mathrm{Hess}_p f_q$. ⊠

∽ · ∼

In the second half, we'll study Morse theory on adjoint orbits of SU_3 acting on $\mathfrak{su}_3$, using the technology we developed above. In this case $V = E = \mathfrak{su}_3$, an eight-dimensional real vector space with an inner product $\langle A, B\rangle = -\mathrm{tr}(AB)$. Letting $\mathfrak{t}$ denote the diagonal matrices in $\mathfrak{su}_3$, which have entries $\lambda_1, \lambda_2, \lambda_3$ whose product is 1, there's a subset Δ of three lines, in which two (or more) of the λ_i are equal. If M_P denotes the SU_3-orbit containing some $P \in \mathfrak{t}$, then SU_3-orbits in $\mathfrak{su}_3$ are in bijective correspondence to S_3-orbits in $\mathfrak{t}$ (by permuting the diagonal entries, which is also by reflection across lines $\{\lambda_i = \lambda_j\}$).

Any $A \in \mathfrak{su}_3$ defines a skew-adjoint operator $\mathrm{ad}_A\colon \mathfrak{su}_3 \to \mathfrak{su}_3$ by $B \mapsto AB - BA$.

Exercise 8.22. There are natural identifications $T_PM \cong \mathrm{Im}(\mathrm{ad}_P)$ and $N_PM \cong \ker(\mathrm{ad}_P)$.

The proof uses the SU_3-invariance of the inner product we defined.

Set $g_t := e^{tA}$ and compute $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$. If

$$P = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}, \tag{8.23}$$

then $\mathfrak{t} \subset \ker\mathrm{ad}_P$. If $\lambda_1 = \lambda_2 = \lambda$, then P commutes with block matrices (one 2×2 block, one 1×1 block); this is the normal space $\ker\mathrm{ad}_P$.

Fix an orbit M and $Q \in \mathfrak{t} \setminus (M \cap \mathfrak{t})$. Let $f\colon M \to \mathbb{R}$ send $A \mapsto (1/2)\mathrm{dist}(Q, A)^2 = (1/2)\,\mathrm{tr}(Q - A)^2$, as in (8.12).

Theorem 8.24. $\mathrm{Crit}(f) = M \cap \mathfrak{t}$. *$f$ is Morse iff $Q \notin \Delta$, and the index of $P \in \mathrm{Crit}(f)$ is twice the number of points that the open line between P and Q intersects Δ.*[12]

Corollary 8.25. *We're in the lacunary situation, so $H_*(M_P)$ is torsion-free. We also obtain a CW structure on $\mathbb{CP}^2$ with a single 0-, 2-, and 4-cell, and show that a generic M_P has Betti numbers 1, 0, 2, 0, 2, 0, 1.*

[12] The three lines of Δ intersect at the origin, so if we have to include that case, we should count it with intersection number 3.

Proof of Theorem 8.24. First, suppose that $R \in \mathfrak{t}$, $P \in M \cap \mathfrak{t}$, and $X \in \mathfrak{su}_3$. Then

$$\nu_t := e^{tX} R e^{-tX} \tag{8.26}$$

is normal to M at

$$P_t := e^{tX} P e^{-tX}. \tag{8.27}$$

Using the Leibniz rule,

$$\dot{P} = \frac{\mathrm{d}}{\mathrm{d}t} P_t \bigg|_{t=0} = XP - PX, \tag{8.28}$$

also known as $\mathrm{ad}_X P = [X, P]$. Since $D_{[X,P]}\nu = [X, R]$, then we can compute the first and second fundamental forms:

$$I_p([X_1, P], [X_2, P]) = \langle [X_1, P], [X_2, P]\rangle \tag{8.29a}$$

$$II_p(R)([X_1, P], [X_2, P]) = -\langle [X_1, R], [X_2, P]\rangle, \tag{8.29b}$$

and the shape operator is

$$S_P(R) = -\,\mathrm{ad}_R \,\mathrm{ad}_P^{-1}. \tag{8.29c}$$

That formula makes sense because on T_pM, ad_P is indeed invertible. Therefore $\ker \mathrm{Hess}_P(f)$ is the fixed points of $S_P(Q - P)$, hence the fixed points of $(\mathrm{ad}_P - \mathrm{ad}_Q)\,\mathrm{ad}_P^{-1}$, hence the fixed points of $\mathrm{id} - \mathrm{ad}_Q \,\mathrm{ad}_P^{-1}$, i.e. the kernel of ad_Q. This vanishes if $Q \notin \Delta$, i.e. it has three distinct diagonal entries.

To compute the index, we simultaneously diagonalize the action of ad_R for all $R \in \mathfrak{t}$. The commutator of the diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$ and E^i_j is $(\lambda_i - \lambda_j)E^i_j$.

Lemma 8.30. *Bott studied an infinite-dimensional version of this problem for* $\Omega\mathrm{SU}_3$. *The story is roughly similar, but the triangles are a little more complicated. The lacunary principle applies, showing that* $H_*(\Omega SU_3)$ *is torsion-free, and computing its Poincaré polynomial.*

⊠

Lecture 9.

Critical submanifolds: 10/3/18

Let $p \in M$ be a critical point for a smooth function $f\colon M \to \mathbb{R}$, and let $K_p \subset T_pM$ denote the kernel of the Hessian of f at p. We might ask whether a given $\xi \in K_p$ is *integrable* — that is, is there a curve p_t for $t \in (-\varepsilon, \varepsilon)$ with $p_0 = p$, $\dot{p}_0 = \xi$, and $p_t \in \mathrm{Crit}(f)$?

Definition 9.1. A *critical submanifold* of M is a submanifold contained inside $\mathrm{Crit}(f)$. In this case $T_pP \subset K_p$ for any $p \in P$.

Clearly the most interesting examples arise for non-Morse functions!

Definition 9.2 (Bott)**.** A critical submanifold P is *nondegenerate* if for all $p \in P$, $K_p = T_pP$.

Equivalently, the induced form

$$\mathrm{Hess}_p f\colon T_pM/T_pP \times T_pM/T_pP \longrightarrow \mathbb{R} \tag{9.3}$$

is nondegenerate. Recall that $TM|_P/TP$ is the normal bunle $N \to P$.

Example 9.4. Consider the unit sphere $S^2 \subset \mathbb{E}^3$ with coordinates x, y, z and the function $\widetilde{f}\colon S^2 \to \mathbb{R}$ given by $(x, y, z) \mapsto z^2$. There are two isolated critical points, at the extrema, and $P = \{z = 0\}$ is a nondegenerate critical submanifold. The Hessian is $2\,\mathrm{d}z \otimes \mathrm{d}z$. On P, the normal bundle is $\{(\xi^1, \xi^2, \xi^3) \mid \xi^1 = \xi^2 = 0\}$.

The quotient of S^2 by the antipodal $(x, y, z) \sim (-x, -y, -z)$ is the *real projective plane* $\mathbb{RP}^2$, and $\widetilde{f}$ descends to a function $f\colon \mathbb{RP}^2 \to \mathbb{R}$. In this case $\mathrm{Crit}(f) = \mathbb{RP}^1 \cup \{\mathrm{pt}\}$. We'd like to write the Morse polynomial for this function, whose t^q coefficient is the number of critical points of index q, but $\mathbb{RP}^1$ contributes too many points. Instead, we use its the Poincaré polynomial, as if there were a perfect Morse function there. Thus

$$M_t(f) = t^2 + (1 - t), \tag{9.5}$$

since the isolated critical point has index 2. ◀

Remark 9.6. The idea of a nondegenerate critical submanifold was due to Bott, who found it useful for studying critical points of energy functionals on infinite-dimensional manifolds. ◀

Proposition 9.7. *Suppose $\pi\colon N \to M$ is a fiber bundle and $f\colon M \to \mathbb{R}$ is nondegenerate (i.e.* $\mathrm{Crit}(f)$ *is a nondegenerate critical submanifold). Then $\pi^* f \coloneqq f \circ \pi$ is also nondegenerate.*

Compare: pullbacks of Morse functions are generally not Morse, unless the fibers are zero-dimensional. But they are still nondegenerate in this sense.

Proof. $\mathrm{Crit}(\pi^* f) = \pi^{-1}\mathrm{Crit}(f)$, which is a fiber bundle over $\mathrm{Crit}(f)$. The index, as a locally constant function on $\mathrm{Crit}(f)$, also pulls back. ☒

Example 9.8. Consider the Hopf fibration $S^1 \to S^3 \to S^2$, and consider the standard height function $f\colon S^2 \to \mathbb{R}$, with Morse polynomial $1 + t^2$. The pullbacks of the north and south pole are circles, so the pullback Morse function is

$$M_t(f) = 1(1+t) + t^2(1+t) = 1 + t + t^2 + t^3. \tag{9.9}$$

This isn't the same as the Poincaré polynomial $1 + t^3$. The idea is that if you perturbed this function, you'd get a Morse function which splits the two critical circles into points, and then we would get $1 + t + t^2 + t^3$ as the Morse polynomial in the more restricted sense. ◀

Now suppose $f\colon M \to \mathbb{R}$ is a function and $P \subset M$ is a nondegenerate critical submanifold with normal bundle $\pi\colon N \to P$. Define $q\colon N \to \mathbb{R}$ by

$$q(\nu) \coloneqq f(\pi(\nu)) + \mathrm{Hess}_{\pi(\nu)} f(\nu, \nu). \tag{9.10}$$

Example 9.11. If $P = p$ is a point, so $N = T_pM$, the Hessian is the usual Hessian, and there's a coordinate system in which

$$\mathrm{Hess}(\nu,\nu) = -\sum_i (x^i)^2 + \sum_j (y^j)^2, \tag{9.12}$$

which is what the Morse lemma tells us. ◀

Theorem 9.13 (Parameterized Morse lemma)**.** *There exists a neighborhoof $U \subset N$ of the zero section $P \subset N$ and a tubular neighborhood $i\colon U \to M$ covering the identity map $P \to P$ such that $i^* f = q$ on U.*

The normal bundle plays the role of coordinates around P via the tubular neighborhood theorem.

Remark 9.14. $TN \to N$ has a subbundle $T(N/P) = \ker(\pi_*)$, and the annilihator of $T(N/P) \subset TN$ is $\pi^*T^*P \subset T^*N$. The spaces of sections are $\pi^*\Omega^1_P \subset \Omega^1_N$. ◀

Proof of Theorem 9.13. Fix a tubular neighborhood $j\colon N \to M$, and for $0 \le t \le 1$, set

$$h \coloneqq q - j^* f \tag{9.15}$$

$$\alpha_t = \mathrm{d}((1-t)q + t j^* f) \bmod \pi^*\Omega^1_P. \tag{9.16}$$

We claim that in a neighborhood of the zero section, there exists a time-varying vertical vector field ξ_t such that

$$\xi_t|_P = 0 \tag{9.17a}$$

$$\iota_{\xi_t}\alpha_t = h. \tag{9.17b}$$

In general we can't flow for infinite time, but for $0 \le t \le 1$, we have a flow φ_t defined on some tubular neighborhood U of P, and with codomain U' (another tubular neighborhood). Let's compute what it does to α_t. Using Cartan's formula,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\alpha_t &= \varphi_t^*\left(\mathcal{L}_{\xi_t}\alpha_t + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t\right)\\ &= \varphi_t^*(\mathrm{d}\iota_{\xi_t}\alpha_t - \mathrm{d}h)\\ &= \varphi_t^*(\mathrm{d}h - \mathrm{d}h) = 0.\end{aligned}$$

Therefore

$$\mathrm{d}((j\circ\varphi_1)^* f) = \varphi_1^*\alpha_1 = \varphi_0^*\alpha_0 = \mathrm{d}q. \tag{9.18}$$

Setting $i = j \circ \varphi_1 \colon U \to M$; then $i^* f = q$ as desired, and we're done — except we need to prove the claim. Both of the equations in (9.17) are affine conditions, so we can prove them on an open cover and patch them together using a partition of unity. That is, it suffices to produce a solution to (9.17) for the trivial bundle.

Let $p \in P$ and $x^1, \dots, x^k$ be coordinates on the trivial bundle. Write

$$h(p, x) = h_j(p, x) x^j \tag{9.19a}$$

$$\alpha_t(p, x) = A_{ij}(t, p, x) x^j \, \mathrm{d}x^i \pmod{\pi^* \Omega^1_P}. \tag{9.19b}$$

We want $h|_P = 0$ and $\alpha_t|_P 0$, so set

$$h_j(p, x) \coloneqq \int_0^1 \frac{\partial h}{\partial x^j}(p, tx) \, \mathrm{d}t, \tag{9.20}$$

and write

$$\xi_t = \xi^k(t, p, x) \frac{\partial}{\partial x^k}. \tag{9.21}$$

Then (9.17b) is the equation $A_{ij} \xi^i x^j = h_j x^j$, which is implied by $A_{ij} \xi^i = h_j$. Since $A_{ij}(t, p, 0)$ is the Hessian of f at p in N_p, it's nondegenerate, which implies $A_{ij}(t, p, x)$ is nondegenerate for x small, and we can let $\xi = A^{-1} h$. ⊠

Corollary 9.22. *Suppose M is compact and $f \colon M \to \mathbb{R}$ has its minimum on a nondegenerate critical submanifold $P \subset M$. Suppose* $\mathrm{Crit}(f) = P \cup \{p_1, \dots, p_N\}$*, where each p_i is nondegenerate of index λ_i. Then M is obtained from P by attaching n-dimensional handles of indices $\lambda_1, \dots, \lambda_N$.*

The proof is a lot like the original use of the Morse lemma to produce handle decompositions, but in this case, if c is the minimum of f, we begin at $M^{c+\varepsilon} \approx P$ using the parameterized Morse lemma and continue the argument from there.

Complex manifolds. Recall the definition of a manifold: a set M together with a cover $\mathfrak{U}$ by open subsets $U \subset A_U$ of affine spaces, and such that the change-of-charts maps are smooth. This induces a topology and a smooth structure on M.

If we replace A_U with complex affine spaces and ask for the transition maps to be holomorphic (satisfying the Cauchy-Riemann equations), we obtain a *complex manifold*.

Example 9.23. Let V be a finite-dimensional complex vector space and $\mathbb{P}(V)$ denote the set of one-dimensional subspaces of V. We'll sketch a realization of $\mathbb{P}(V)$ as a complex manifold.

Let $W \subset V$ be a codimension-1 subspace, and let $A_W = \{L \in \mathbb{P}(V) \mid L \not\subset W\}$.

Exercise 9.24. Give A_W the structure of an affine space over $\mathbb{C}$, as $\mathrm{Hom}(V/W, W)$.

Then we have a map

$$\coprod_W A_W \longrightarrow \mathbb{P}(V). \tag{9.25}$$

Exercise 9.26. Prove that the transition functions are holomorphic.

This complex manifold has an obvious line bundle $\mathscr{L}$, whose fiber over a point L is L. It's a subspace of $\mathbb{P}(V) \times V = \underline{V}$. ◀

If V is a complex vector space, we can choose a Hermitian pairing $h \colon \overline{V} \times V \to \mathbb{C}$. If $g \coloneqq \mathrm{Re}(h) \colon V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$ and $\omega \coloneqq \mathrm{Im}(h) \colon V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$, which is skew. The unitary group $\mathrm{U}(H)$ acts on V, so the data of h allows us to take the unit sphere ... TODO: I don't know what happened after that.

Lecture 10.

The Lefschetz hyperplane theorem: 10/3/18

In this part of the lecture, Ricky and Ivan spoke about the Lefschetz hyperplane theorem.

Let M be a complex manifold of (complex) dimension k, and assume it embeds biholomorphically in $\mathbb{C}^N$. At any $p \in M$, we have a chart $w \colon U \cong V \subset \mathbb{C}^N$, where $U \subset \mathbb{C}^k$. Choose a $\nu \in T_p\mathbb{C}^N \cong \mathbb{R}^{2N}$, so for all $\xi \in T_pM$, $g(\xi, \nu) = 0$ (this is the inner product in $\mathbb{R}^{2N}$). Therefore $\nu \in N_p(M)$.

The function $\phi\colon \mathbb{C}^k \to \mathbb{C}$ defined to send (here h is the Hermitian inner product)

$$z \longmapsto h(w(z), \nu) = \sum_{j=1}^{N} w_j(z)\overline{\nu}_j \tag{10.1}$$

is analytic at 0, so we can Taylor-expand it. Let $Q(z) = \sum a_{ij}^2 z^i z^j$ denote its quadratic term. The function

$$\mathrm{Re}(h(w(z), \nu)) = g(w(z), \nu) = g(w(x+iy), \nu) \tag{10.2}$$

is real analytic in x and y, and hence also has a Taylor series

$$g(w(x+iy), \nu) = \langle w(0), \nu\rangle + \text{linear} + \frac{1}{2}\sum_{i,j} g\big(\partial_{\xi^i}\partial_{\xi^j} w(0), \nu\big)\xi^i\xi^j, \tag{10.3}$$

where

$$\xi^i = \begin{cases} x^i, & \text{if } 1 \le i \le k \\ y^{i-k}, & \text{if } k+1 \le i \le 2k. \end{cases} \tag{10.4}$$

Let $Q'(x^1, \dots, x^k, y^1, \dots, y^k)$ denote the quadratic term in (10.3). We'll call the associated matrix A.

We have a basis for T_pM given by $\partial_{x^1}, \dots, \partial_{x^k}, \partial_{y^1}, \dots, \partial_{y^k}$. Let J be the automorphism sending $\partial_{x^i} \mapsto \partial_{y^i}$ and $\partial_{y^i} \mapsto -\partial_{x^i}$. Then $Q^{(()}Jv) = -Q'(v)$ and $J^{\mathrm{T}}AJ = A$.

Let v be an eigenvector for A with eigenvalue λ; then,

$$J^{-1}AJv = J^{\mathrm{T}}AJv = -Av = -\lambda v, \tag{10.5}$$

so $AJv = -\lambda(Jv)$.

Now let $L_q\colon M \to \mathbb{R}$ send

$$x \longmapsto h(q-x, q-x). \tag{10.6}$$

Saying $p \in \mathrm{Crit}(L_q)$ is equivalent to $h(q-p, \xi) = 0$ for all $\xi \in T_pM$. Let $\nu := q - p$; then, the index of the Hessian at p is the number of $\lambda \in \mathrm{Spec}(II)$ such that $0 < 1/\lambda \le \|\nu\|$. This means the index is always at most k, so using Morse theory we get a stunning result:

Corollary 10.7. *The complex manifold M has the homotopy type of a k-dimensional CW complex.*

This is cool because the real dimension of M is twice that!

Remark 10.8. The fact that M embeds in $\mathbb{C}^N$ (we say it's *affine*) is crucial for this: $\mathbb{CP}^n$ has cohomology in degree $2n$, for any n. So we also see there's no analogue of the Whitney embedding theorem. ◂

Corollary 10.9. *Let V be a complex submanifold of $\mathbb{CP}^N$ (we say it's a* projective variety*). Suppose P is a hyperplane (a $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$) in $\mathbb{CP}^n$ and P contains the singular points of V. Then for all $r \le k-1$, $H^r(V, V \cap P) = 0$. Equivalently, $H^r(V) \cong H^r(V \cap P)$ for all $r < k-1$.*

To get at this, we'll need a theorem which generalizes Poincaré duality to manifolds with boundary.

Proposition 10.10. *Let (A, X) be a pair of topological spaces (so $A \subset X$) such that X is compact Hausdorff, A is closed in X, and $X \setminus A$ is an orientable n-manifold. Then there are isomorphisms $H^r(X, A) \cong H_{n-r}(X \setminus A)$.*

Letting $X = V$ and $A = V \cap P$, $V \setminus V \cap P$ is a complex submanifold of $\mathbb{CP}^n \setminus P \cong \mathbb{C}^n$, so putting that together with Corollary 10.7, we get the vanishing result.

Now we'll discuss the Lefschetz hyperplane theorem, and a proof due to Bott, following Thom.

Theorem 10.11 (Lefschetz hyperplane theorem)**.** *Let $X \subset \mathbb{CP}^n$ be a smooth algebraic variety and H be a hyperplane in $\mathbb{CP}^n$ transverse to X. Then the induced maps $\pi_j(X \cap H) \to \pi_j(X)$ and $H_*(X \cap H) \to H_j(X)$ are isomorphisms if $j < \dim_{\mathbb{C}} X - 1$ and surjective if $j = \dim_{\mathbb{C}} X - 1$.*

This follows from another theorem.

Theorem 10.12 (Bott-Thom)**.** *In the above setting, $X \simeq (X \cap H) \cup e^{\lambda_1} \cup \dots \cup e^{\lambda_k}$ for cells e^{λ_i} which have dimension higher than $\dim_{\mathbb{C}} X$.*

This follows from another theorem.

Theorem 10.13. *If f is a (generalized) Morse function on a compact manifold X, let X_* denote the region where f attains its minimum value and λ be the minimum index of f on $X \setminus X_*$. Then $X \simeq X_* \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$ where $\lambda_j \geq \lambda$.*

This implies the previous theorem: we can find a Morse function ϕ on X such that $X_* = X \cap H$ and $\lambda \geq \dim_{\mathbb{C}} X$. This ϕ will arise as a perturbation of $h(s,s)|_X$, where h is a natural Hermitian metric and s is a global holomorphic section of a hyperplane bundlre $J^* \to \mathbb{CP}^n$, such that s vanishes on H^*. Here are a few facts about J^* (also known as $\mathscr{O}(2)$ in algebraic geometry):

(1) it has a natural Hermitian metric h.
(2) There's a natural identification of global holomorphic sections of J^* with degree-1 homogeneous polynomials in degree-1 homogeneous coordinates on $\mathbb{CP}^n$ (see Griffiths-Harris for more information).

The second point implies (TODO: I think) that there's a section s which vanishes precisely on H. Then if $\phi = h(s,s)|_X$, $H \cap X = X_*$, and it's a nondegenerate critical manifold.

Lemma 10.14. *Let $p \in H \cap X$. Then there exists a holomorphic coordinate system $(z^1, \ldots, z^m)$ of X centered at p, such that near p, $s = z^1 s^*$, where s^* is a local section of the hyperplane bundle that doesn't vanish at p.*

Proof sketch. We can assume $H = \{z_1 = 0\}$ and $p = [z^0 : 0 : \ldots : z^n]$. Since $z^0 \neq 0$, we can introduce affine coordinates $w^i := z^i/z^0$ for $i = 1, \ldots, n$. Let U_i be the usual patches of affine coordinates on $\mathbb{CP}^n$; then we've just said $p \in U_0$, so $s = Az_1$ on U_0, represented by $s_0 = A_1 z_1/z_0 = A_1 w_1$.

Pick any local holomorphic frame of J^* near p; then (TODO: ?) there's an s^* with $s = gs^*$, so $s_0 = gs_0^*$. TODO: I didn't follow the rest of the board, but w_1 is part of a holomorphic coordinate system near p with nice properties, and some transversality condition. ⊠

Ok, now $\phi = h(s,s)|_X = z^1\overline{z}^1 h(s^*, s^*)|_X$. If $z^1 = x^1 + iy^1$, then $\partial_{x^1}|_p, \partial_{y^1}|_p$ is a basis for $T_pX/T_p(H \cap X)$, and

$$\operatorname{Hess}_p h(s,s)|_X \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) = 2h(s^*, s^*)|_p \neq 0. \tag{10.15}$$

In particular, the Hessian is nondegenerate.

A perhaps bolder claim is that if $p \in X \setminus (X \cap H)$, then $\lambda(p) \geq \dim_{\mathbb{C}} X$. First we have a lemma about positivity of $J^* \to \mathbb{CP}^n$.

Lemma 10.16. *$\overline{\partial}\partial \log(h(s,s))|_p > 0$.*

That is,

$$\overline{\partial}\partial \log(h(s,s))|_p = -\left.\frac{\partial^2 \log(h(s,s))}{\partial z^\alpha \partial z^\beta}\right|_p \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta, \tag{10.17}$$

and this defines $g_{\alpha\beta}$ for a 2-form g on T_pX; positivity means this form is a positive definite Hermitian form.

For motivation, suppose $E \to X$ is a holomorphic vector bundle with a Hermitian metric h. Then there's a unique D_h whose composition with h in a holomorphic frame has $(0,1)$-component $\overline{\partial}$, and such that $D_h = \mathrm{d} + H^{-1}\partial H$, where $H = (h(z_i, z_j))$.

If E is a line bundle, then

$$D_h = \mathrm{d} + 2\log h(z.z) \tag{10.18}$$

and

$$F_{D_h} = \mathrm{d}(\partial \log h(z,z)) = \overline{\partial}\partial \log(h(z,z)). \tag{10.19}$$

TODO: I didn't follow anything after that, and some of what came before.

Lecture 11.

The h-cobordism theorem: introduction: 10/10/18

"Oh... I didn't know that was supposed to be funny."

Fix an $n \geq 1$.

Definition 11.1. Let V_0 and V_1 be closed $(n-1)$-manifolds. A *bordism* between V_0 and V_1 is a quadruple $(W, p, \theta_0, \theta_1)$ consisting of

- a compact n-manifold W with boundary,
- a smooth map $p\colon \partial W \to \{0,1\}$, and
- diffeomorphisms $\theta_i\colon V_i \to p^{-1}(i)$.

Often θ_1, θ_2, and p are implicit, and we just write $\partial W = V_0 \amalg V_1$.

Remark 11.2. It's possible to glue a bordism between V_0 and V_1 to a bordism between V_1 and V_2. For this reason it's possible to define a category whose objects are closed n-manifolds and whose morphisms are (diffeomorphism classes of) bordisms between them. ◀

Example 11.3. Let $f\colon M \to \mathbb{R}$ be a proper Morse function. If a_1 and a_2 are regular values, then $W := f^{-1}([a_1, a_2])$ is a bordism between $f^{-1}(a_1)$ and $f^{-1}(a_2)$, and if a' is a regular value between a_1 and a_2, the bordisms $f^{-1}([a_1, a'])$ and $f^{-1}([a', a_2])$ glue to $f^{-1}([a_1, a_2])$.

If $[a_1, a_2]$ consists only of regular values, then $W \cong [a_0, a_1] \times f^{-1}(a_0)$, but the converse is not true: consider the height function $f(x) = x^3 - x$ as a Morse function $f\colon \mathbb{R}^3 \to \mathbb{R}$. This has the two critical points $\{\pm 1\}$, but the bordism from -2 to 2 is diffeomorphic to $[-2, 2]$. ◀

The h-cobordism theorem involves Morse theory, but is stated in terms of bordisms.

Theorem 11.4 (h-cobordism theorem (Smale, 1956))**.** *Suppose a bordism W between V_0 and V_1 satisfies*

(1) $H_*(W, V_0) = 0$,
(2) W, V_0, *and* V_1 *are simply connected, and*
(3) $n \geq 6$.

Then W is diffeomorphic to $[0,1] \times V_0$.

Remark 11.5. For $n = 5$, this is false for smooth manifolds (work of Donaldson-Freedman), and is true for topological manifolds (work of Freedman). For $n = 4$, this is open, and it would imply the four-dimensional Poincaré conjecture (the uniqueness of the smooth structure on S^4). It's true for $n = 3$ by work of Perelman, and is true for $n < 3$ for easier reasons. ◀

To prove this just for $y = x^3$ as discussed above, you could try to "straighten out" $\mathbb{R}$. What that actually means is trying to cancel critical points by considering a path in the space of Morse functions, such as

$$f_t(x) := \frac{x^3}{3} - tx. \tag{11.6}$$

When $t > 0$, this has two roots, and hence we get two critical points at $\pm\sqrt{t}$. At $t = 0$, there's a single, degenerate critical point. For $t < 0$, there are no critical points, so we get a cylinder bordism as promised.

We can consider the space of Morse functions inside the space of all functions. This space is known to be contractible, using a subject called Cerf theory after its pioneer, J. Cerf. There are various proofs of the contractibility of this space, such as one by Eliashberg and another by Galatius. This means that, in a sense, it doesn't matter which path you take to cancel the critical points. This is one approach to the h-cobordism theorem, but not the only one.

So the main steps of the proof are:

(1) First, construct an *excellent* Morse function on W, meaning that f is constant on V_0 and V_1, and each $f(V_i)$ is a regular value. To do this, you have to think about what smoothness on a manifold-with-boundary actually means: we usually use open sets to talk about it, and we don't quite have those. So this means introducing collars to make sense of this notion.
(2) Next we want to construct a self-indexing Morse function on W. We could do some extra work to get the same critical points, which might not be needed. This means that we have critical values $0, \dots, n$, and the preimage of i contains the set of critical points of index i. For convenience, set the regular values $f(V_0) = -1/2$ and $f(V_1) = n + 1/2$.

(3) Then we cancel critical points of consecutive indices, given a sufficient condition.
(4) If n is even, we might be left with critical points in the middle dimension, which we eliminate with something called the *Whitney trick*.
(5) There's a special argument needed to eliminate the critical points of index 0 and 1. If f is a Morse function, $-f$ is too, and if f is self-indexing, $n-f$ is also a self-indexing Morse function. So this argument also cancels the critical points of indices $n-1$ and n. Therefore these kinds of arguments will often stop after the middle dimension, since then you can just turn f upside down.

The details will appear in the next few student talks.

Remark 11.7. As long as we're not looking at topological 5-manifolds, the only place the constraint on the dimension appears is in step (4). ◀

In Dan's next few lectures, he'll talk about negative gradient flow. This is a subject which has several applications: one is to actually construct the CW complex that Morse theory tells us about, using geometry; another, in the most general setup in infinite dimensions, is Floer theory. In infinite-dimensional Morse theory, say modeled on a Banach space, manifolds can't be locally compact, and so one has to produce clever arguments and ideas to work around this. Palais and Smale wrote some good papers about this, and so a nice condition replacing compactness is called the *Palais-Smale condition*. But enough of the details are present in the finite-dimensional case to be interesting, and we'll restrict ourselves to that.[13]

Our setup is a closed (or sometimes just compact) Riemannian manifold M and a Morse function $f\colon M\to\mathbb{R}$. Let $p_1,\dots,p_N$ be the critical points of f, with indices $\lambda_1,\dots,\lambda_N$. Letting $\xi \coloneqq -\operatorname{grad} f$, then for all vectors $\eta\in T_qM$,

$$-\,\mathrm{d}f|_q(\eta) = \langle \xi,\eta\rangle. \tag{11.8}$$

Let φ_t ($t\in\mathbb{R}$) denote the flow of ξ.

Lemma 11.9. *For every $q\in M$, the limits $\lim_{t\to\pm\infty}\varphi_t(q)$ exist and are critical points.*

This is *not* true for arbitrary flows/integral curves. An easy example is flow along circles parallel to the xy-plane in $S^2\subset\mathbb{R}^3$: all orbits are periodic, so no limits exist except those for the fixed points, the north and south poles.

Another obstacle is dese orbits. Consider the torus $T^2=\mathbb{R}^2/\mathbb{Z}^2$ and a constant vector field $\xi=(1,a)$, where $a\in\mathbb{R}\setminus\mathbb{Q}$. This is locally, but not globally, a gradient flow. One can show that the orbit containing the image of $(0,0)$ is dense, and therefore its limit as $t\to\infty$ cannot exist.

The existence of these limits means that the velocity decreases as time goes on. This is a very useful fact, allowing us to control its geometry, and is not true for many flows. For example, Perelman studied Ricci flow on Riemannian manifolds, and was able to obtain powerful results by interpreting it as akin to a gradient flow, discovering a similar bound on velocities. In physics, there's an analogous concept called renormalization group flow, and if it behaves like a gradient flow, it's a powerful tool to control the quantum field theory of interest (though these are not theorems, yet).

Proof of Lemma 11.9. It suffices to consider $t\to\infty$, and then replace f with $-f$ to get the result at $-\infty$.

Let

$$c \coloneqq \inf_{t\in\mathbb{R}} f(\varphi_t(q)). \tag{11.10}$$

Then

$$0 = \lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(q)) = -\lim_{t\to\infty} -|\xi_{\varphi_t(q)}|^2, \tag{11.11}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t(q)) = \mathrm{d}f|_{\varphi_t(q)}(\xi) = -\langle\xi,\xi\rangle. \tag{11.12}$$

Since the velocity decreases to zero in the limit, the limit must exist, which is left as an exercise. One doesn't need compactness for this part, only completeness.

Since $\lim_{t\to\infty}\xi_{\varphi_t(q)}=0$, then the limit is a critical point, and $f(p)=c$. ⊠

[13]Further details on the infinite-dimensional case can be found in Jürgen Jost's book, chapter 8.

So these gradient flow lines must both begin and end at critical points. We also have N special flow lines which sit at each critical point. These partition the manifold; N constant, zero-dimensional ones, and the rest *injective motions*, with everywhere nonzero velocity. This means they're embeddings of $\mathbb{R}$ into M. This flow is very simple, compared to many other flows we could write down.

From this we obtain two maps $+, -\colon M \to \mathrm{Crit}(f)$, sending $q \mapsto \lim_{t\to\infty} \varphi_t(q)$, resp. $\lim_{t\to-\infty} \varphi_t(q)$.

Definition 11.13. With M and f as above, let p be a critical point of M. Its *stable manifold* $W^s(p)$ is $\{q \in M \mid \lim_{t\to\infty} \varphi_t(q) = p\}$, and its *unstable manifold* $W^u(p)$ is $\{q \in M \mid \lim_{t\to-\infty} \varphi_t(q) = p\}$.

The key theorem is that these are actually manifolds, and in fact balls, which (if f satisfies an additional condition) give a CW decomposition of M.

Example 11.14. Consider the torus with its standard Morse function. Then gradient flow doesn't actually define a CW structure on the torus! It produces four 0-cells, but the region that flows to the middle two critical points is diffeomorphic to two intervals, not one. ◀

The issue comes down to transversality: two things intersect nontransversely, so the dimension of the intersection is larger than expected. Of course, we can tilt the function slightly to fix this problem.

Definition 11.15. A Morse function f is *Morse-Smale* if for all $p, p' \in \mathrm{Crit}(f)$, $W^s(p)$ and $W^u(p')$ intersect transversely.

Of course, this is a little funny before we proved $W^s(p)$ and $W^u(p')$ are manifolds! So on to the theorem where we do that.

Theorem 11.16. *For all $p \in \mathrm{Crit}(f)$, if λ is the index of f at p, then*

(1) *$W^u(p)$ is a submanifold of M diffeomorphic to B^λ, and*
(2) *$W^s(P)$ is a submanifold of M diffeomorphic to $B^{n-\lambda}$.*

This takes care of everything away from the critical points; then we also have a nice local model $V = T_pM$ at each critical point p, namely Morse coordinates. We will study the negative gradient flow in these coordinates.

In this setting, $q = f(p)$ plus a quadratic, so for $x \in V$,

$$q'(x) = f(p) - \frac{1}{2}\langle Lx, x\rangle \tag{11.17}$$

for some invertible self-adjoint operator $L\colon V \to V$. In particular, this is a linear vector field, and is the gradient flow.

Next we diagonalize, introducing coordinates $x^1, \dots, x^n$ such that

$$q = f(p) - \frac{1}{2}\big((x^1)^2 + \cdots + (x^\lambda)^2\big) + \frac{1}{2}\big((x^{\lambda+1})^2 + \cdots + (x^n)^2\big). \tag{11.18}$$

Then for $i = 1, \dots, \lambda$ and $j = \lambda+1, \dots, n$,

$$L\frac{\partial}{\partial x^i} = \alpha_i \frac{\partial}{\partial x^i} \tag{11.19a}$$

$$L\frac{\partial}{\partial x^j} = -\beta_j \frac{\partial}{\partial x^j}, \tag{11.19b}$$

for some $\alpha_i, \beta_j > 0$, and[14]

$$\pm\,\xi = \sum_i \alpha_i x^i \frac{\partial}{\partial x^i} - \sum_j \beta_j x^j \frac{\partial}{\partial x^j}. \tag{11.20}$$

So flow lines look like hyperbolas (or hyperboloids in general) avoiding the origin, plus flow lines along the coordinate axes, incoming along some directions and outgoing along others. Then one can deform the stable manifold to... TODO: I missed this part. An argument using Moser's principle seems to work but fails for $\lambda \neq 0, n$, and in general the deformation is topological, not smooth.

[14] Despite the presence of lower and upper indices, we're not using any summation convention.

Lecture 12.

The h-cobordism theorem: 10/10/18

Today, Riccardo and Cameron spoke, as the first student lecture on the h-cobordism theorem.

Definition 12.1. An *elementary bordism* is one which admits an associated Morse function with a single critical point.

In particular, if $f\colon M \to \mathbb{R}$ is a proper Morse function and $a_1, a_2 \in \mathbb{R}$ are regular values such that (a_1, a_2) contains a single critical point, then $f^{-1}([a_1, a_2])$ is an elementary bordism from $f^{-1}(a_1)$ to $f^{-1}(a_2)$.

Example 12.2. The pair-of-pants bordism from two circles to one circle is an elementary bordism. ◀

Every bordism can be written as a successive composition (by gluing) of elementary bordisms (except of course those with *no* critical points, which are products with $[a_1, a_2]$). We would like to rearrange the components of this composition. Specifically, if X is a bordism from C to C' and Y is a bordism from D to D', such that $\operatorname{ind}(C) = \operatorname{ind}(D')$ and $\operatorname{ind}(C') = \operatorname{ind}(D)$, then we can *rearrange* X into Y.

The idea is that if p and p' are two critical points of a Morse function whose stable and unstable manifolds never intersect, we can "move p past p'," and more specifically move f in the space of Morse functions so that $f(p) < f(p')$.

Definition 12.3. A vector field ξ is *gradient-like* if it's the gradient of some Morse function f, so $\mathrm{d}f(\xi) > 0$ away from the critical points.

A gradient-like vector field for f in a Morse neighborhood looks like

$$\xi = -x^1\partial_1 - \ldots - x^\lambda\partial_\lambda + x^{\lambda+1}\partial_{\lambda+1} + \cdots + x^n\partial_n. \tag{12.4}$$

We will let $K_p := W^s(p) \cup W^u(p)$. If there are only two critical points, this is compact, but this need not be true in general.

Theorem 12.5. *Let (W, V_0, V_1) be a bordism with associated Morse function $f\colon W \to [0,1]$ having two critical points p, p'. Suppose that for some choice of gradient-like ξ, the sets K_p and $K_{p'}$ are disjoint. Let $a, a' \in (0,1)$; then there exists a Morse function g such that*

1. *f is still gradient-like for g,*
2. *the critical points of g are p and p', and $g(p) = a$ and $g(p') = a'$, and*
3. *g agrees with f near V_0 and V_1, and near p and p', $g - f$ is constant.*

Proof sketch. Let $K := K_p \cup K_{p'}$, and let $\pi\colon W \setminus K \to V_0$ be a smooth projection. Let $\mu\colon V \to [0,1]$ be a function which is 0 in a neighborhood of $V_0 \cap K_p$ and 1 in a neighborhood of $V_0 \cap K_{p'}$.

We claim we can extend μ to a $\overline{\mu}\colon W \to [0,1]$ whch is 0 on K_p and 1 on $K_{p'}$, and which is constant along flow lines of ξ. More specifically, we claim there exists a $G\colon [0,1] \times [0,1] \to [0,1]$ such that

1. for all x and y, $\frac{\partial G}{\partial x} > 0$ and $G(x,y)$ increases from 0 to 1,
2. $G(f(p), 0) = a$ and $G(f(p'), 1) = a'$,
3. $G(x,y) = x$ for x near 0 or 1 and for all y,
4. $\frac{\partial G}{\partial x}(x, 0) = 1$ for x in a neighborhood of $f(p)$, and
5. $\frac{\partial G}{\partial x}(x, 1) = 1$ for x in a neighborhood of $f(p')$.

This is plausible, and isn't the most interesting part of the proof, so we'll skip it.

The next claim is that $g(q) = G(f(q), \overline{\mu}(q))$ is the required Morse function. For example, we know it must differ from f by a constant near p and p' because they have the same derivative. The other properties aren't too much harder. ⊠

We can amplify this to a broader result.

Theorem 12.6. *Let (W, V_0, V_1) be a bordism with associated Morse function $f\colon W \to [0,1]$ whose critical points are partitioned into two sets $P = \{p_1, \ldots, p_m\}$ and $P' = \{p_1', \ldots, p_n'\}$, such that $f|_P$ and $f|_{P'}$ are constant. Let $a, a' \in (0,1)$; then, there's a Morse function g such that*

1. *f is still gradient-like for g,*
2. *the critical points of g are $p_1, \ldots, p_m$ and $p_1', \ldots, p_n'$, and $g(p_i) = a$ and $g(p_j') = a'$, and*
3. *g agrees with f near V_0 and V_1, and near each p_i and p_j', $g - f$ is constant.*

The proof is analogous.

Definition 12.7. Let a be a regular value of a Morse function f and p be a critical point. Then (assuming Theorem 11.16) $W^s(p) \cap f^{-1}(a)$ and $W^u(p) \cap f^{-1}(a)$ are either empty or spheres; in the latter case they're called the *stable* (resp. *unstable*) *spheres* of p at a, and denoted $S^s(p)$ and $S^u(p)$ (as long as a is clear from context).

Theorem 12.8. *Let (W, V_0, V_1) be a bordism with associated Morse function $f\colon W \to [0,1]$ whose critical points p, resp. p' have indices λ, resp. λ', and assume $\lambda' \geq \lambda$. Without loss of generality assume $f(p) < 1/2 < f(p')$; then, it's possible to alter ξ on a prescribed neighborhood of $f^{-1}(1/2)$ in such a way that with respect to the new ξ, $\overline{S^u(p)} \cap \overline{S^s(p')} = \varnothing$.*

Here, we're taking the spheres at $a = 1/2$.

Proof. We know $\dim S^s(p) = n - \lambda - 1$ and $\dim S^u(p) = \lambda' - 1$, where $\lambda := \operatorname{ind} p$ and $\lambda' := \operatorname{ind} p'$. Then by transversality there exists an $h_t\colon I \times V \to V$ such that $h_0 = \mathrm{id}_V$ and (possibly more axioms I didn't catch, TODO). Letting $H(t, x) = (t, h_t(x))$... I didn't follow what happened next, but I think we used H to "straighten out" the flow. ⊠

Finally, we'll need one more lemma.

Lemma 12.9. *Given (W, V_0, V_1) and f as above, and a vector field ξ gradient-like for f, let $V = f^{-1}(b)$, where b is a regular value, and let $h\colon V \to V$ be a diffeomorphism isotopic to the identity. If $f^{-1}([a, b])$ doesn't contain any critical points, it's possible to construct a new gradient-like vector field $\overline{\xi}$ for f such that $\overline{\xi}$ and ξ coincide outside of $f^{-1}([a, b])$ and $\overline{\varphi} := h \circ \varphi$, where $\overline{\varphi}$ and φ are the diffeomorphisms $f; (a) \to V$ obtained by following trajectories.*

Our goal is to prove the following theorem.

Theorem 12.10 (Final rearrangement theorem)**.** *Any bordism c may be expressed as a composition of bordisms $C = C_0 \circ \cdots \circ C_n$, where $n - 1 = \dim C$, and where each bordism C_k admits a Morse function with just one critical value and all critical points are of index k.*

We cannot assume each C_i is elementary! For example, consider two circles as a bordism $\varnothing \to \varnothing$: it has two critical points of index 0 and two critical points of index 1.

Theorem 12.11. *With notation as above, the gradient-like vector field ξ may be chosen such that $S^s(p)$ intersects $S^u(p')$ transversely.*

Theorem 12.12. *In the setting as above, if $S^s(p)$ and $S^u(p)$ intersect transversely and at a single point, then $W \cong V \times [0,1]$.*

Now we can use this to simplify some cobordisms. We will always adhere to the notation that W is a bordism from V_0 to V_1, with an associated Morse function f. p will denote a critical point of f, ξ will denote gradient flow, and ξ' be a modified gradient flow, with g a function such that ξ' is gradient-like for g and $g = f$ in a neighborhood of ∂W.

Consider a function $v\colon \mathbb{R} \to \mathbb{R}$ which in a neighborhood $N(0)$ of 0 looks like $v(t) = t$ and in a neighborhood $N(1)$ of 1 looks like $v(t) = 1 - t$, and which is positive on $(0, 1)$ and negative on the complement of $[0, 1]$. We specify

$$\int_0^1 v(t)\,\mathrm{d}t = \frac{1}{2}(f(p') - f(p)). \tag{12.13}$$

Then let

$$V(x^1) = f(p) + 2\int_0^{x^1} v(t)\,\mathrm{d}t. \tag{12.14}$$

If $x^1 \in N(0)$, this is $f(p) + (x')^2$, and if $x^1 \in N(1)$, this is $f(p') - (x_1 - 1)^2$.

The multivariate version of this is to let

$$F(x^1, \dots, x^n) = f(p) + V(x^1) - (x^2)^2 - \cdots - (x^{\lambda+1})^2 + (x^{\lambda+2})^2 + \cdots + (x^n)^2. \tag{12.15}$$

In a neighborhood of 0, this looks like $f(p)+(x^1)^2-\cdots+\cdots$, and in a neighborhood of 1, we let $y^1 = x^1 - 1$, and get $F(y^1, x^2, \ldots, x^n) = f(p') - ((y^1)^2 + \cdots) + \cdots$.

Let T denote the orbit of the flow out from p.

Lemma 12.16. *For any open U containing T, there exists an open $U' \subset U$ such that no flows start in U', leave U, then come back to U'.*

This is a kind of uniqueness result.

Proof. Introduce a Riemannian metric, so we can make our manifold a metric space. If the theorem is false, then we can produce a sequence of flows φ_n and their points t_n, s_n, and r_n where the flow is in U', is in U, and then is in U' again.

If $T_{s'}$ is the segment of γ' from V_0 to s, then $d(T_{s'}, T)$ is a continuous function on s', so has a minimum on this compact set, which is realized by some $d(T_s, T)$, and this T_s will cause a contradiction (TODO: as soon as I understand the proof...) ⊠

Note: Cameron continued his lecture the next Wednesday: the notes from here until the end of this section are from 10/17. We're in the middle of proving a cancellation theorem for critical points by modifying ξ in the neighborhood of the unique trajectory between two critical points p and p'.

The key is to reduce the general story to the local model. If we're sure that a flow line leaves the neighborhood of the flow line T from p to p', then it goes from V_0 to V_1. If we can modify ξ such that every flow line that enters the neighborhood once leaves once, then we also don't have to worry about those.

Specifically, we claim that for every neighborhood U of T, there's a "safe" neighborhood $U' \subset U$ such that if γ is a flow line which is in U' at t_0 and leaves U at $t_1 > t_0$, then γ does not return to U'. The proof is to proceed by contradiction — given a point in a sufficiently small neighborhood of an $s \in T$, we can get a flow line T_s violating the claim, and $\psi(s) = d(T, T_s)$ is continuous; then one can use this to produce a sequence of points with contradictory properties.

Then (TODO) we have to shrink the neighborhood a few more times, such that (1) we're in a compact set and (2) something else. In particular, we can modify ξ inside this compact set, and leave it the same everywhere else, using a bump function (so we have this compact set inside an open set whose closure is compact and in U').

TODO: I didn't understand the modification and therefore wasn't able to follow the proof.

Now we want to show that, after making this modification, we get a product cobordism for the modified flow.

Lemma 12.17. *For $q \in M$, define $\tau_0(q)$ to be the time to flow to q from V_0 and $\tau_1(q)$ to be the time to flow from q to V_1. Then τ_0, τ_1 are smooth.*

To prove this, we need another lemma.

Lemma 12.18. *Let M be a manifold, $D \subset M$ be a codimension-1 embedded disc and ξ be a C^∞ vector field on M tranverse to D at some point $x_0 \in D$. Then there's a neighborhood U of x_0 in D and an $\varepsilon > 0$ and an embedding $\Phi\colon (-\varepsilon, \varepsilon) \times U \to M$ such that $\Phi(0,0) = x_0$, $\Phi|_{U\times\{0\}}$ is $x \mapsto x$, and $\Phi_*(\partial_t) = \xi$.*

We'll call this sort of neighborhood a *striated neighborhood*.

Proof. Since D is embedded, there are coordinates $u^1, \ldots, u^n$ for a chart U such that $V = D \cap U$ is included in $\mathbb{R}^n$ as the last $n-1$ coordinates. Transversaliy of ξ means that $\mathrm{d}u^1(\xi) > 0$. Then, the fundamental theorem of ODEs shows there's a neighborhood $\mathcal{V}$ of $\{0\} \times V$ in $\mathbb{R} \times V$ such that $(t, x) \mapsto \psi_t(x)$ is a map $\mathcal{V} \to U$. Then $\mathcal{V}$ contains a neighborhood of $(0,0)$ of the form $(-\varepsilon, \varepsilon) \times U$, so all we need to do is check that $\mathrm{d}\psi|_{(0,0)}$ is an isomorphism. For $i \geq 2$,

$$\mathrm{d}\psi\left(\frac{\partial}{\partial u^i}\right) = \frac{\partial}{\partial u^i}, \tag{12.19}$$

and if $i = 1$, we get ξ, and $\psi_0 = \mathrm{id}$, so $\mathrm{d}\psi$ is an isomorphism at $(0,0)$. ⊠

Corollary 12.20. *Let x be on a flow line of ξ which departs from x_0. Then there's a neighborhood W of x, and a smooth $\tau\colon W \to \mathbb{R}$ such that, for $x' \in W$, $\psi_{-\tau(x')}(x') \in D$.*

Proof. Take the striated neighborhood we just constructed and flow x into the neighborhood. What we end up with has an open neighborhood W' in the striated neighborhood; then we flow W' back to get W. ⊠

TODO: I wasn't able to follow what happened after that, but we define

$$g(u,q) \coloneqq \int_0^u \lambda(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}(t,q) + (1-\lambda(t))k(q)\,\mathrm{d}t, \tag{12.21}$$

where

$$k(q) = \frac{1 - \int_0^1 \lambda(t)\frac{\partial \Phi f}{\partial t}(t,q)\,\mathrm{d}t}{\int_0^1 (1-\lambda(t))\,\mathrm{d}t}. \tag{12.22}$$

Then it remains to check that $g = f$ near V_0 and V_1, which is some calculations with the formula, and that the total integral is 1. Again, these are calculations.

Lecture 13.

The stable and unstable manifolds: 10/17/18

"At the time, they were new... funny how that works."

Recall that if $f\colon M \to \mathbb{R}$ is a Morse function on a Riemannian manifold M and $p \in \mathrm{Crit}(M)$ has index λ, we can use the metric to define a gradient vector field ξ for f, and let $\varphi_t\colon M \to M$ be the flow generated by ξ.

Last time, we defined the unstable manifold (also descending manifold) $W^u(p)$, the $q \in M$ which flow to p at time infinity: formally speaking, we ask for $\lim_{t\to-\infty}\varphi_t(p) = q$. We obtain the stable manifold (ascending manifold) by replacing $-\infty$ with ∞.

The first theorem of today is: the name "manifold" is justified.

Theorem 13.1. *$W^u(p)$, resp. $W^s(p)$, is a submanifold of M diffeomorphic to B^λ, resp. $B^{n-\lambda}$.*

Last time, we discussed the local model around p for this setup, in $V = T_pM$, which picks up an inner product from the metric. In Morse coordinates,

$$f = f(p) - \big((x^1)^2 + \cdots + (x^\lambda)^2\big) + \big((x^{\lambda+1})^2 + \cdots + (x^n)^2\big), \tag{13.2}$$

and $\mathrm{Hess}_p f(v_1, v_2) = \langle v_1, Lv_2\rangle$ for some linear operator $L\colon V \to V$. There's a direct-sum decomposition $V = V' \oplus V''$, where we write $x = (x', x'')$ and $x' = (x^1, \dots, x^\lambda)$ and $x'' = (x^{\lambda+1}, \dots, x^n)$. Then there are $\alpha_i, \beta_j > 0$ such that for $1 \le i \le \lambda$ and $\lambda + 1 \le j \le n$,

$$Le_i' = -\alpha_i e_i' \tag{13.3a}$$

$$Le_j'' = \beta_j e_j''. \tag{13.3b}$$

In this case, $\xi^0 = -L$ is the gradient of $f\colon V \to V$, so on V'', flow lines go towards the origin, and on V', they go away from the origin. Explicitly,

$$\varphi_t^0(x', x'') = (e^{-tL}x', e^{-tL}x''), \tag{13.4}$$

and $W^s(\xi^0) = V''$ and $W^u(\xi^0) = V'$.

Therefore in a neighborhood of $0 \in V$, $\xi = \xi^0 + \eta = -L + \eta$, for some $\eta\colon U \to V$ with $\eta(0) = 0$. Our goal is to show that $W^s(p) \cap U$ is the graph of a function $x' = g(x'')$.

Suppose $y = y(t)$ is a flow line of ξ; then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Ly + \eta(y), \tag{13.5}$$

and therefore

$$\frac{\mathrm{d}e^{tL}y}{\mathrm{d}t} = e^{tL}\left(Ly + \frac{\mathrm{d}y}{\mathrm{d}t}\right) = e^{tL}\eta(y). \tag{13.6}$$

Since

$$e^{tL}y(t) = e^{\tau L}y(\tau) + \int_\tau^t e^{sL}\eta(y(s))\,\mathrm{d}s, \tag{13.7}$$

then

$$y(t) = e^{-(t-\tau)L} y(\tau) + \int_\tau^t e^{-(t-s)L} \eta(y(s))\, \mathrm{d}s. \tag{13.8}$$

The projections π' and π'' (onto V' and V'', respectively) commute with L, hence with e^{uL} for all u by the spectral theorem. Therefore

$$\pi' y(t) = e^{-(t-\tau)L} \pi'(y(\tau)) + \int_\tau^t e^{-(t-s)L} \pi' \eta(y(s))\, \mathrm{d}s, \tag{13.9a}$$

so in the limit $\tau \to \infty$,

$$= \int_\infty^t e^{-(t-s)L} \pi' \eta(y(s))\, \mathrm{d}s. \tag{13.9b}$$

Setting $\tau = 0$, we have

$$\pi'' y(t) = e^{-tL} \pi'' y(0) + \int_0^t e^{-(t-s)L} \pi'' \eta(y(s))\, \mathrm{d}s \tag{13.10}$$

$$y(t) = e^{-tL} x'' + \int_0^t e^{-(t-s)L} \pi'' \eta(y(s)) - \int_t^0 e^{-(t-s)L} \pi' \eta(y(s))\, \mathrm{d}s, \tag{13.11}$$

where $x'' \coloneqq \pi'' y(0)$.

We can interpret the right-hand side of (13.11) as a map on a space of such functions $y(t)$, $t \geq 0$, and then applying the contraction mapping principle. We'll skip the details, but the ideas are

- define the space of functions and show that it's a complete metric space,
- prove the right-hand side of (13.11) is a contraction, and
- if $y_{x''}$ is a fixed point, let $g(x'') = \pi' y_{x''}(0)$.

We want to use the inverse function theorem (in its general form, in Banach spaces) to show that near p, this is the graph of a submanifold. It suffices to show the graph of g is $W^s(p) \cap U$, then the implicit function theorem. The details of all of this are in Jost's book.

Finally, we want to get from a neighborhood of p to the whole thing; you can flow your chart along the rest of the manifold to get more charts. Now, why is it diffeomorphic to a ball? We can define a map $B_\varepsilon(0) \subset V''$ to $W^s(p)$ by

$$y'' \longmapsto \varphi_{-t}(0, y''), \qquad t = \frac{|y'|/\varepsilon}{1 - |y'|/\varepsilon}, \tag{13.12}$$

and one can check this is a diffeomorphism.

With this theorem out of the way, we have two partitions of M:

$$M = \coprod_{p \in \mathrm{Crit}(f)} W^s(p) = \coprod_{p \in \mathrm{Crit}(f)} W^u(p). \tag{13.13}$$

These are not descriptions as CW complexes.

If $p, p' \in \mathrm{Crit}(f)$, let $\widehat{M}(p, p') \coloneqq W^u(p) \cap W^s(p')$, or the union of the flow lines which flow to p as $t \to -\infty$ and to p' as $t \to \infty$. The group $\mathbb{R}$ acts on $\widehat{M}(p, p')$ by "time translation:" $s \in \mathbb{R}$ acts by $\gamma \mapsto (t \mapsto \gamma(t - s))$. Let $M(p, p') \coloneqq \widehat{M}(p, p')/\mathbb{R}$. And, of course, all of this depends on the metric.

Definition 13.14. The pair (f, g) of a Morse function and a Riemannian metric is *Morse-Smale* if, for all $p, p' \in \mathrm{Crit}(f)$, $W^u(p) \pitchfork W^s(p)$.

Theorem 13.15. *For fixed f there exists a dense set of metrics g such that (f, g) is Morse-Smale.*

One could say "Morse-Smale functions and metrics are generic," but what does generic mean? The answer is that it's second category, in the sense of Baire; this is exactly what Sard's theorem does for you.

If (f, g) is Morse-Smale, then $\widehat{M}(p, p')$ is indeed a manifold (we'll prove that later), hence has dimension $\lambda - \lambda'$. We would like to understand these moduli spaces; they arose as spaces of solutions to parameterized families of ODEs: specifically, the variable is $\gamma\colon \mathbb{R} \to M$, the equation is

$$\gamma'(t) + \mathrm{grad}^{(g)}(f)\Big|_{\gamma(t)} = 0, \tag{13.16}$$

and since $\mathbb{R}$ is noncompact, we must specify the boundary conditions $\lim_{t\to-\infty}\gamma(t) = p$ and $\lim_{t\to\infty}\gamma(t) = p'$. The parameter is the metric g, and the domain is the product of the space of metrics and the space of maps from $\mathbb{R}$ to M. Then the ODE is

$$F(g,\gamma) = \left(t \longmapsto \gamma'(t) + \operatorname{grad}^{(g)} f\Big|_{\gamma(t)}\right). \tag{13.17}$$

This is a map to a different vector space at each point, so it's really a section of a vector bundle. The fiber at (g,γ) is the space of C^∞ sections of $\gamma^*TM \to \mathbb{R}$, and these stitch together into an (infinite-dimensional) vector bundle E. We're computing F^{-1} of the zero section of E.

We want this to have a good space of solutions: the moduli space should be a manifold. Next time, we'll show this for generic (g,γ). Then we need to divide by the symmetry, in order to pass to $M(p,p')$; and we'll argue that it has a good compactness property; that is, it's not compact, but has a natural, geometric compactification. Once we have this, we'll use it to define topological invariants. This submanifold defines a homology class inside that large function space, and this can be used to do interesting things; in fact, it can be enhanced to a bordism class, which is stronger.

Remark 13.18. This is indicative of a general scheme in geometric analysis, whose details were worked out starting in the 1960s, but more completely in the 1970s for anti-self-dual connections on a 4-manifold. This construction was done, at least in special cases, in a special paper of Atiyah-Hitchin-Singer, following analogous earlier work of Kuranishi in complex geometry. Some of the analytic work for compactness was done by Uhlenbeck and Taubes, and then Donaldson put it together to obtain topological invariants called Donaldson invariants, which are quite powerful. (In the 1990s, another set of equations, the Seiberg-Witten equations, were studied, and they have simpler proofs of the theorems in this package but are also powerful.)

For us, this will be easier because we have an ODE, rather than a PDE. This was set up in the finite-dimensional case after Floer, who did it in the infinite-dimensional case. There are other versions of this in other parts of geometric analysis, including Gromov-Witten theory; using ODEs simplifies the theory.

What invariants do we obtain? Well, we get the homology of M, unfortunately — but the real reward is learning the story along the way. The kind of amazing thing that comes in later appears in physics, where it admits an interpretation as inside some quantum field theory, and understanding of quantum field theory can help prove mathematical statements. If we have first-order ODE, the QFT will be supersymmetric; this was first studied by Witten in the 1980s. ◀

We'll hear more of this story in the next few days; to finish today, we'll discuss a finite-dimensional model which might be helpful to keep around as a simpler example.

Consider the function $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ sending

$$g,\gamma \longmapsto \gamma^2 - g. \tag{13.19}$$

If $F(g,\gamma) = 0$ and 0 is a regular value (which is obviously is, because $\mathrm{d}F = 2\gamma\,\mathrm{d}\gamma - \mathrm{d}g$ and the second term is surjective on its own), then $F^{-1}(0)$ is a manifold. Let π be the projection onto the g-axis; then regular values of π are closely related to regular values of 0.

To prove these facts, we'll need to generalize the inverse function theorem and Sard's theorem to function spaces, which is where we'll start next time.

Remark 13.20. Cameron's lecture was a continuation of his lecture from last time, and hence was appended to the previous section. ◀

Lecture 14.

Calculus on Banach spaces: 10/24/18

Let V and W be Banach spaces and $U \subset V$ be open. If $f\colon U\to W$ is a continuous function, $p\in U$, and $\xi\in V$, we can define the directional derivative $\xi f(p)$ as usual, as

$$\lim_{t\to 0}\frac{1}{t}(f(p+t\xi) - f(p)). \tag{14.1}$$

The key theorem is that this is linear in ξ. One can define C^1 functions, and therefore higher derivatives, by asking for $\mathrm{d}f\colon U\to \operatorname{Hom}(V,W)$ to be continuous, differentiable, etc. Thus one can follow the same path as in calculus on finite-dimensional spaces, eventually leading to the inverse and implicit function theorems.

Thus one can define *Banach manifolds* by patching together open subsets U of a model Banach space using smooth functions.

Example 14.2. Let M and N be finite-dimensional smooth manifolds.

(1) The space $\mathrm{Map}^k(M,N)$ of C^k maps $M \to N$ is a Banach manifold.
(2) The space of C^k metrics on M, denoted $\mathrm{Met}^k(M)$, is a Banach manifold. Similarly, spaces of connections are Banach manifolds.
(3) If $k \geq (\dim M)/p$, the Sobolev space $L^p_k(M,N)$ is a Banach manifold. (The constraint is needed to guarantee that such maps are continuous.) ◀

Suppose X and Y are Banach manifolds and $f\colon X \to Y$ is a smooth map. If $y \in Y$, when is $f^{-1}(y)$ a finite-dimensional submanifold of X?

Example 14.3. Let M be a finite-dimensional Riemannian manifold and $h\colon M \to \mathbb{R}$ be a Morse function. Let $p_\pm \in \mathrm{Crit}(f)$, $\xi = -\operatorname{grad} h$, and Γ be the Banach manifold of differentiable maps $\gamma\colon \mathbb{R} \to M$ with $\lim_{t\to\pm\infty} \gamma(t) = p_\pm$.

The equation for an integral curve f of ξ is

$$\underbrace{\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \xi_{\gamma(t)}}_{f(\gamma)} = 0 \tag{14.4}$$

for all $t \in \mathbb{R}$. That is, $f(\gamma)$ is a section of γ^*TM. More generally, set E_γ to be the sections of γ^*TM with $\lim_{t\to\pm\infty} s(t) = 0$; then E is a fiber bundle over Γ, and f is a section.

To pin all this down precisely, we would need to say what kinds of maps we want to make up Γ, and what exactly the Banach manifold structure on E is. ◀

Remark 14.5. You might want to use C^∞ maps instead of C^k ones, but spaces of C^∞ maps do not have the structure of a Banach space; you're trying to control infinitely many norms at once. The local model is weaker, what's known as a *Fréchet space*, but doing calculus on Fréchet spaces is a little harder. It is possible to obtain a Fréchet manifold $C^\infty(M,N)$. ◀

Definition 14.6. Let $T\colon V \to W$ be a bounded linear map of Banach spaces. Then T is *Fredholm* if

(1) $\ker(T)$ is finite-dimensional,
(2) $T(V) \subset W$ is closed, and
(3) $\operatorname{coker}(T) \coloneqq W/T(V)$ is finite-dimensional.

Suppose $T\colon V \to W$ is a Fredholm map and F is a complement to $T(V)$, so $W = T(V) \oplus F$; then F is finite-dimensional. Choose a closed subspace $V_0 \subset V$ such that $V = V_0 \oplus \ker T$. Then $T\colon V_0 \to T(V)$ is an isomorphism.

Definition 14.7. The *index* of a Fredholm operator is $\operatorname{ind} T = \dim\ker T - \dim\operatorname{coker} T$.

Any map between finite-dimensional vector spaces is Fredholm. Here's an infinite-dimensional example.

Example 14.8. Consider the map $S\colon \ell^\infty \to \ell^\infty$ sending $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. ◀

One can show that the space of Fredholm operators from V to W is open in $\mathrm{Hom}(V,W)$. You can do this by choosing a decomposition of $W = T(V) \oplus F$ and $V = V_0 \oplus \ker T$, and $T|_{V_0}$ is invertible. Since invertibility is an open condition, it's possible to show that things sufficiently close to T are still Fredholm.

Definition 14.9. A map $f\colon X \to Y$ of Banach manifolds is *Fredholm* if $\mathrm{d}f|_p\colon T_pX \to T_pY$ is Fredholm for all $p \in X$.

Given such an f, we can choose V_0 and F as before, so that $T_pX = V_0 \oplus \ker \mathrm{d}f|_p$ and $T_{f(p)}Y = \mathrm{d}f|_p(T_pX) \oplus F$.

Lemma 14.10. *There exist local coordinate systems around* p *and* $f(p)$ *such that in these coordinates,* f *is of the form* $V_0 \oplus \ker \mathrm{d}f|_p \to \mathrm{d}f|_p(T_pX) \oplus F$ *sending* $(\xi,\eta) \mapsto (\mathrm{d}f|_p\xi, g(\xi,\eta))$, *where* $g(0,0) = 0$ *and* $\mathrm{d}g|_{(0,0)} = 0$.

The proof is an exercise. It will involve using the inverse function theorem for Banach spaces. You can also read the proof in Donaldson-Kronheimer.

Therefore the local model of $f^{-1}(q)$ is $h^{-1}(0)$, where $h\colon U \to F$, where $U \subset \ker(T)$, sends $\eta \mapsto g(0,\eta)$. That is: this preimage is given by a system finitely many equations.

We would like an analogue of Sard's theorem in this setting. However, measure theory behaves badly on infinite-dimensional vector spaces: they're not locally compact. So instead we use Baire theory.

Theorem 14.11 (Sard-Smale)**.** *If $f\colon X \to Y$ is a smooth Fredholm map of Banach manifolds, then the space of regular values is a* Baire subset *of Y, i.e. a countable intersection of open dense sets, hence is also dense.*

Often we will only need that it's nonempty!

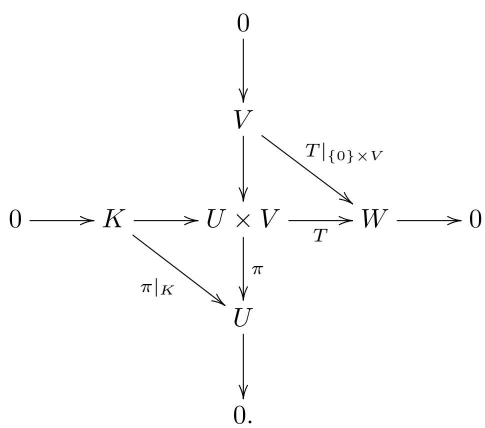
This allows us to produce regular values for the space of C^k metrics. Since C^∞ metrics are also dense in C^k metrics, we can also get to C^∞ metrics as well.

Proof. We can reduce to the local model. Observe that (α,β) is a regular value of f iff β is a regular value of $g_{\{\mathrm{d}f^{-1}\alpha\}\times\ker(\mathrm{d}f)}\colon \ker T \to F$ (or at least an open subset of $\ker T$); then use the usual Sard theorem. ⊠

Lemma 14.12. *Let U, V, and W be Banach spaces and $T\colon U\times V \to W$ be surjective with kernel K. Then $\pi|_K\colon K \to U$ is Fredholm iff $T|_{\{0\}\times V}\colon V \to W$ is Fredholm, and if so, their indices are equal.*

Proof. We have two short exact sequences fitting into a diagram

(14.13)



We claim this induces isomorphisms between their kernels, and between their cokernels. We have a map $\ker(T|_{\{0\}\times V}) \to \ker(\pi|_K)$ sending $v \mapsto (0,v)$, and using that the horizontal and vertical sequences are both short exact, this is an isomorphism. For the cokernels, we have a map $\mathrm{coker}(T|_{\{0\}\times V}) \to \mathrm{coker}(\pi|_K)$ which sends $[w] \mapsto [u]$ for $T(u,v) = w$, and you can check this defines a map, and that it's an isomorphism, which again uses short exactness. ⊠

Here's our first application.

Theorem 14.14. *Let M be a closed, finite-dimensional manifold and $k \in \mathbb{Z}^{\geq 1}$. Then the subspace of $C^k(M)$ consisting of Morse functions is Baire.*

Proof. Consider the "map" $f\colon C^k(M)\times M \to T_p^*M$ sending $(h,p) \mapsto \mathrm{d}h|_p$. Of course, this isn't really a map; it's a section f of the vector bundle $C^k(M)\times T^*M \to C^k(M)\times M$. Let Z be the zero section of this bundle; then $f^{-1}(Z) = \{(h,p) \mid p \in \mathrm{Crit}(h)\}$, which projects down to h.

We claim f is transverse to Z. Fix a metric on M and let ∇ be the Levi-Civita covariant derivative. Then for $h \in C^k(M)$ and $\eta \in T_pM$,

$$(\nabla_{(\dot h,\eta)} f)_{(h,p)} = \mathrm{d}\dot h_p + (\nabla_\eta\, \mathrm{d}h)_p. \tag{14.15}$$

To prove $f \pitchfork Z$, we would need to show that given $\alpha \in T_p^*M$, we can find $(\dot h,\eta)$ such that

$$\mathrm{d}\dot h_p + (\nabla_\eta\, \mathrm{d}h)_p = \alpha. \tag{14.16}$$

We'll set $\eta = 0$ and find $\dot{h}$. We claim that h is a regular value of $\pi\colon f^{-1}(Z) \to C^k(M)$ iff h is Morse: by Lemma 14.12, h is a regular value iff for all $p \in \mathrm{Crit}(h)$, the map

$$\eta \longmapsto (\nabla_\eta \, \mathrm{d}h)_p \tag{14.17}$$

is surjective. But at a critical point, $(\nabla \, \mathrm{d}h)_p = \mathrm{Hess}_p \, h$, so (14.17) is surjective iff the Hessian is nondegenerate at p. ⊠

Lecture 15.

The second cancellation theorem: 10/24/18

In this part of the lecture, Ricky spoke about the second cancellation theorem for critical points. As in Cameron's talk last week, we have an n-dimensional cobordism $(W; V_0, V_1)$ with $n \geq 6$, and a Morse function $f\colon W \to [0, 1]$ with critical points x, x' of indices λ and $\lambda + 1$. Let ξ be a gradient-like vector field. Without loss of generality, we assume $f(x) < 1/2$ and $f(x') > 1/2$.

Last week, we saw that if $S^s(x) \cap S^u(x')$ intersect at a single point, then $W \cong V_0 \times I$. In general, they might intersect at more than one point; then we need to count them in a slightly sophisticated way. Given a point $p \in S^s(x) \cap S^u(x')$, let $\varepsilon_p \in \{\pm 1\}$ be the local orientation. Then the *intersection number* of $S^s(x)$ and $S^u(x')$ is

$$i(S^s(x), S^u(x')) = \sum_{p \in S^s(x) \cap S^u(x')} \varepsilon_p, \tag{15.1}$$

where we isotope $S^s(x)$ until it's transverse to $S^u(x')$. This is invariant under ambient isotopies of $V_{1/2}$, as long as $S^s(x)$ and $S^u(x')$ remain transverse.

Today our goal is the following theorem.

Theorem 15.2. *With W, V, and V_0 as above (and in particular, $n \geq 6$), assume $2 \leq \lambda \leq n - 4$. If f has just two critical points and $i(S^s(x), S^u(x')) = \pm 1$, then $W \cong V_0 \times I$.*

Exercise 15.3. Is the hypothesis on the intersection number implied by the existence of only two critical points? It might follow from Poincaré duality.

Here's an overview of the proof strategy.

(1) First we'll perturb ξ such that $S^s(x) \pitchfork S^u(x')$. If $S^s(x)$ and $S^u(x')$ intersect in one point, we invoke the previous cancellation theorem; otherwise, there are $p, q \in S^s(x) \cap S^u(x')$ such that $\varepsilon_p + \varepsilon_q = 0$.
(2) In this case, we'll construct a family of functions $h_t\colon V_{1/2} \to V_{1/2}$ such that $h_0 = \mathrm{id}$, $S^s(x) \cap h_1(S^u(x')) = S^s(x) \cap S^u(x') \setminus \{p, q\}$, and $h_t = \mathrm{id}$ away from p and q.
(3) We'll build a new gradient-like flow ξ_{new} from $h-t$ such that $S^s_{\mathrm{new}}(x) \cap S^u_{\mathrm{new}}(x') = S^s(x) \cap S^u(x') \setminus \{p, q\}$. Then we can induct.

Lemma 15.4. *Suppose $h_t\colon V_{1/2} \to V_{1/2}$ is a family of functions with $h_0 = \mathrm{id}$. Then there's a gradient-like vector field ξ_{new} such that $S^u_{\mathrm{new}}(x') = h_1(S^u(x'))$.*

TODO: I missed the proof! It looked fairly explicit.

That means the bulk of what's left is in step (2). The idea is to push $S^s(x)$ out of the way of $S^u(x')$ on the region between p and q. This is where we assume $n \geq 6$, so that something called the Whitney trick works.

Theorem 15.5 (Whitney trick)**.** *Let M and M' be smooth, closed, oriented manifolds of dimensions r and s, respectively, which are embedded transversely in an $(r + s)$-dimensional manifold V. Assume $r + s \geq 5$, $s \geq 3$ and, if $r \leq 2$, then $\pi_1(V \setminus M') \to \pi_1(V)$ is an injection. Suppose there are p, q with $\varepsilon_p + \varepsilon_q = 0$ and a loop contractible in V such that an interval in the loop is in M and its complement is in M', and the intersection is $\{p, q\}$. Then there's a $h_t\colon V \to V$ such that $h_0 = \mathrm{id}$, $h_t = \mathrm{id}$, $h_t = \mathrm{id}$ near $M \cap M' \setminus \{p, q\}$, and $h_1(M) \cap M' = M \cap M' \setminus \{p, q\}$.*

From the theorem statement, we can roughly see how this theorem will be proven, but we do need to check the hypotheses. We assumed $\lambda \geq 2$ and $\lambda + 1 \leq n - 3$, so $n - \lambda - 1 \geq 3$, as required. The loop will be $V_{1/2}$; by van Kampen's theorem,

$$\pi_1(V_{1/2}) = \pi_1(D_A^{n-\lambda}(x) \cup_{S_A^{n-\lambda-1}} V_{1/2} \cup_{S_D^\lambda(x')} D_D^{\lambda+1}(x')) = \pi_1(W) = 0. \tag{15.6}$$

TODO: I tried to make notation for stable and unstable manifolds consistent between lectures, but the notation above might not be consistent.

Ricky provided a pictoral proof sketch of the Whitney trick, but I wasn't sure how to turn it into text, so unfortunately it's omitted.

(1) For the first step, our goal will be to construct a Riemannian metric $\langle\cdot,\cdot\rangle$ on V such that M and M' are *totally geodesic* that is: if ω is a geodesic with $\omega(0) \in M$ and $\dot\omega(0) \in T_{\omega(0)}M$, then $\omega \subseteq M$; and such that there are coordinate neighborhoods N_p and N_q in which $\langle\cdot,\cdot\rangle$ is Euclidean: $N_p \cap C$, $N_p \cap C'$, $N_q \cap C$, and $N_q \cap C'$ are all straight lines.

I think (TODO) there was a construction of this, but I wasn't able to follow it. Roughly, you start with metrics which locally satisfy this property, and glue them together with a partition of unity — but this isn't the final answer. You have to modify this metric in a tubular neighborhood of M, and mesh it in with the rest. Then you can check directly that M is totally geodesic.

Here begins Ricky's continuation of this talk, on October 31. Unfortunately I didn't really follow the first part, which was a recap of the proof sketch of the Whitney trick.

Last time, we found a metric $\langle\cdot,\cdot\rangle_V$ such that M and M' are totally geodesic and there are N_+p and N_q such that the metric is Euclidean in a neighborhood of them, and $M \perp M'$.

Let $\iota_C\colon C \hookrightarrow V$ be inclusion. Since C is contractible, $\iota_C^*\nu_M$ is trivial, so we cna construct a vector field perpendicular to M along C. Then, since $\varphi_1(S)$ is contractible and the Whitney trick implies there's a smooth embedding $(D^2, S^1) \hookrightarrow (V, V \setminus M)$ (this is the place where we use the dimension assumption). Since $\widetilde{\varphi}_2(D^2)$ is contractible, the normal bundle is trivial, ...

Then there was a proof by picture: TODOI couldn't write it down.

Lecture 16.

Some infinite-dimensional transversality: 10/31/18

Let M be a closed manifold, $h\colon M \to \mathbb{R}$ be a Morse function, and $p_\pm \in \mathrm{Crit}(h)$ be distinct critical points. Let $\mathrm{Met}^k(M)$ denote the space of C^k metrics on M.

We would like to exhibit a C^k metric on M such that the ascending manifold of p_+ intersects the descending manifold of p_- transversely. In fact, most metrics will satisfy that condition, in that the subspace of such metrics will be a Baire subspace of $\mathrm{Met}^k(M)$, but that comes later.

Given such a metric g, let $\xi^{(g)} := -\operatorname{grad}^{(g)} h$ be the negative gradient flow. Let P be an "appropriate" space of maps $\gamma\colon \mathbb{R} \to M$ with $\lim_{t\to\pm\infty} \gamma(t) = p_\pm$ (we'll discuss how to do this precisely near the end of lecture). Consider the *negative gradient flow equation* on P,

$$f(g,\gamma) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} - \xi^{(g)}_{\gamma(t)} = 0. \tag{16.1}$$

The function f is really a section of a bundle $E \to \mathrm{Met}^k(M) \times P$, where E is an "appropriate" (again to be clarified) bundle whose fiber at (g,γ) is the space of sections of $\gamma^*TM \to \mathbb{R}$. Once we define them precisely, E and P will be Banach manifolds, and $E \to \mathrm{Met}^k(M) \times P$ will be a vector bundle of Banach spaces. The space of solutions to (16.1) are the intersection of f with the zero section Z of this bundle. We hope for transversality.

Proposition 16.2.

(1) *f is transverse to Z, so $f^{-1}(Z) \subset \mathrm{Met}^k(M) \times P$ is a submanifold.*
(2) *The projection map $f^{-1}(Z) \to \mathrm{Met}^k(M)$ is Fredholm.*

Then we can apply Sard-Smale and conclude that the space of metrics we want is Baire (so these metrics are "generic," in that there are lots of them).

Proof. We need to differentiate f. Define $e\colon \mathrm{Met}^k(M) \times P \times \mathbb{R} \to M$ by $g,\gamma,t \mapsto \gamma(t)$. There's a natural covariant derivative on $e^*TM \to \mathrm{Met}^k(M) \times P \times \mathbb{R}$ which on the slice $\{g_0\} \times P \times \mathbb{R}$ is the pullback of $\nabla^{(g_0)}$, the covariant derivative for g_0.

Let

$$F(g,\gamma,t) \coloneqq \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - \xi^{(g)}_{\gamma(t)}, \tag{16.3}$$

which defines a section of $e^*TM \to \mathrm{Met}^k(M)\times P\times\mathbb{R}$. To compute ∇F, let η be a section of $\gamma^*TM\to\mathbb{R}$. Then ∇_η of the first term of F is

$$\begin{aligned}\nabla_\eta(\text{first term of } F) &= \frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}\\ &= \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\gamma}\\ &= \nabla_\tau\eta,\end{aligned}$$

using a homework problem and the fact that $\nabla^{(g)}$ is torsion-free, and where $\tau \coloneqq \dot\gamma = \gamma_*(\partial_t)$.

For the second term, we know that $\mathrm{d}h(\zeta) = -g(\xi^{(g)},\zeta)$ for all ζ. Differentiating with respect to g,

$$0 = -\dot g(\xi^{(g)},\zeta) - g(\nabla_{\dot g}\xi^{(g)},\zeta), \tag{16.4}$$

so

$$\nabla_{\dot g}\xi^{(g)} = -(g^{-1}\dot g)(\xi^{(g)}), \tag{16.5}$$

and, in particular, $g^{-1}\dot g$ is a linear endomorphism of $TM\to M$. And $\nabla_\eta(\text{second term}) = \nabla_\eta\xi$, so

$$\nabla f_{(g,\gamma)}(\dot g,\eta) = \nabla_\tau\eta - \nabla_\eta(\xi^{(g)}) + (g^{-1}\dot g)(\xi^{(g)}). \tag{16.6}$$

Using this, we can check the two claims in the proposition. For the first claim, set $\eta = 0$. Since $\xi^{(g)}\ne 0$ along γ, we can choose $\dot g$ to obtain an arbitrary section of $\gamma^*TM\to\mathbb{R}$.

To prove the second claim, we can invoke Lemma 14.12 for the diagram

$$\begin{array}{ccc} \mathrm{Met}^k(M)\times P & \xrightarrow{\ f\ } & E\\ \Big\downarrow{\scriptstyle\pi} & & \\ \mathrm{Met}^k(M). & & \end{array} \tag{16.7}$$

The lemma tells us to study

$$A\colon \eta \longmapsto \nabla_\tau\eta + \nabla_\eta\xi^{(g)}, \tag{16.8}$$

where $\lim_{t\to\pm\infty}\eta(t) = 0$. For (g,γ) fied, γ is a flow line for the negative gradient flow, and A is an operator on the sections of $\gamma^*TM\to\mathbb{R}$.

Since

$$\lim_{t\to\pm\infty}\nabla_\eta\xi^{(g)} = \pm\operatorname{Hess}_{p_\pm} h(\eta,\text{–}) \tag{16.9}$$

and $\frac{\mathrm{d}\eta}{\mathrm{d}t} - \lambda\eta = 0$ when λ is a nonzero real number, then $\eta(t) = Ce^{\lambda t}$ iff $t\to-\infty$ ($\lambda<0$) or $t\to\infty$ ($\lambda>0$).

Since A involves taking a derivative, it's an operator from L^2_1 sections to L^2 sections. Since $\operatorname{coker} A = \ker(A^*) = V^+_-\cap V^0_+$, A is Fredholm, and we can compute its index. ⊠

Note: Ricky's talk was a continuation of his previous talk, so the notes for that talk have been collated with the previous section.

Lecture 17.

Cancellation in the middle dimensions: 10/31/18

In this part of the lecture, Ty and Charlie spoke about cancelling in the middle dimensions.

Theorem 17.1. *Suppose $(W;V,V')$ is a cobordism of dimension $n\ge 6$ equipped with a Morse function that has no critical points of index 0, 1, $n-1$, or n. Assume W, V, and V' are simply connected and $H_*(W,V) = 0$. Then $(W;V,V')$ is a product cobordism.*

Definition 17.2. Let W be a compact, oriented smooth n-manifold with boundary M. The *induced orientation* on M is the class $[M] \in H_{n-1}(M)$ which is the image of $[W] \in H_n(W, M)$ under the map $H_n(W, M) \to H_{n-1}(M)$ in the long exact sequence for the pair (W, M).

Alternatively, if you prefer to think of an orientation as specifying which local frames are positively oriented, let v be the outward normal vector field on M. Then $(e_1, \ldots, e_{n-1})$ is a positively oriented local frame on M iff $(v, e_1, \ldots, e_{n-1})$ is positively oriented for W.[15]

Given two n-dimensional oriented cobordisms $(W; V, V')$ and $(W'; V', V'')$, we can glue them to a cobordism $(W \cup_{V'} W'; V, V'')$. Let f be a Morse function on $W \cup_{V'} W'$ with critical points $q_1, \ldots, q_\ell$ in W and $q_1', \ldots, q_m'$ in W'. We want to make sure the intersection numbers of the stable and unstable manifolds are well-defined, so we need orientations. It's enough information to define the orientation for $\eta D_d(q_i)$ to satisfy $I(D_a(q_i), D_d(q_i)) = 1$. Then $\eta S_d(q_i)$ in V' is homotopic to $\eta D_d(q_i)|_{S_d(q_i)}$.

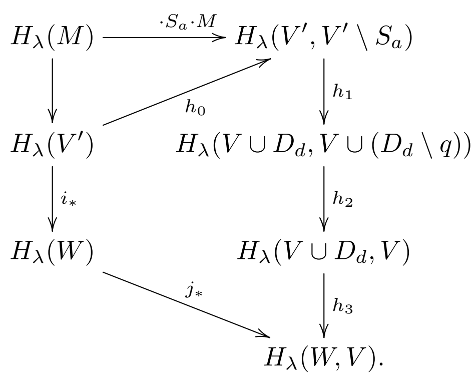
Fact. Standard tools in algebraic topology show that $H_\lambda(W, V)$ is a free abelian group on the generators $\{[D_d(q_i)]\}$, and that $H_{\lambda+1}(W \cup_{V'} W', W) \cong H_{\lambda+1}(W', V')$ is free abelian on the generators $\{[D_d'(q_i')]\}$. ◂

Lemma 17.3. *Let M be a closed, smooth, λ-dimensional manifold such that $M \hookrightarrow V'$, and let $[M] \in H_\lambda(M)$ be the fundamental class. If ℓ_* denotes the induced map in homology, then*

$$\ell_*([M]) = \sum_i S_a(q_i) \cdot M[D_d(q_i)].$$

Proof. First assume $\lambda = 1$ and set $q := q_1$, $D_d := D_d(q_1)$, and $S_a := S_a(q_1)$. We want $\ell_*([M]) = S_a \cdot M[D_d]$. Consider the diagram

(17.4)



The composition $H_\lambda(M) \to H_\lambda(W, V)$ along the left-hand side is ℓ_*. The maps along the right are defined as follows:

(1) h_1 is induced by a deformation retract $r \colon W \to V \cup D_d$ sending $V' \cdot S_a \to V \cup (D_d \setminus q)$,
(2) h_2 is given by a retraction $V \cup (D_d \setminus q) \to V$, and
(3) h_0, i_*, and j_* are induced by inclusion.

Using commutativity of the diagram (which follows because i and $r|_{V'}$ are homotopic), $h_0([M]) = S_a \cdot M\psi(\alpha)$, where $\psi \colon H_0(S_a) \to H_\lambda(V', V' \setminus S_a)$ is TODO. Therefore it suffices to show that $h_3 \circ h_2 \circ h_1(\psi(\alpha)) = [D_d]$.

Since $\psi(\alpha)$ is represented by a ball D^λ such that $S_a \pitchfork D^\lambda$ and their intersection is precisely x, and $[S_a] \cdot [D^\lambda] = 1$, which is why we picked this orientation.

We claim that $r(D^\lambda)$ is represented by $h_2^{-1} h_3^{-1}([D_d])$. Given this, TODO(I missed this part). ⊠

Corollary 17.5. *In the bases given by $\{[D_d]\}$, the boundary map*

$$\partial \colon H_{\lambda+1}(W \cup_{V'} W, W) \longrightarrow H_\lambda(W, V)$$

for the triple $(W \cup_{V'} W', W, V)$ is given by the matrix (a_{ij}), where $a_{ij} = S_a(q_i) \cdot S_d'(q_j')$.

Proof. Consider $[D_d'(q_j')] \in H_{\lambda+1}(W \cup_{V'} W', W)$. The map ∂ factors as a composition

$$H_{\lambda+1}(W \cup_{V'} W', W) \xrightarrow{\cong} H_{\lambda+1}(W', V') \xrightarrow{\delta} H_\lambda(V') \xrightarrow{i_*} H_\lambda(W) \longrightarrow H_\lambda(W, V). \tag{17.6}$$

[15]You can remember this convention, **O**uter **N**ormal **F**irst, by the mnemonic **O**ne **N**ever **F**orgets.

Recall that the cobordism $(W; V, V')$ factors as $C_0 \circ \cdots \circ C_n$, such that each C_λ has a self-indexing Morse function. If $W_\lambda := C_0 \circ \cdots \circ C_\lambda$ and $V := W_{-1}$, the W_λ filter W, so we can define a chain complex $C_\bullet$ whose λ^{th} term is

$$C_\lambda := H_\lambda(W_\lambda, W_{\lambda-1}) \cong H_*(W_\lambda, W_{\lambda-1}). \tag{17.7}$$

Its homology is $H_\lambda(W, V) \cong H_\lambda(W_{\lambda+1}, W_{\lambda-2})$ (one must check this, and also that it's a chain complex, but this is somewhat formal). Next we claim that $H_\lambda(W_{\lambda+1}, W_{\lambda+2}) \cong H_\lambda(W, V)$, which follows directly from the long exact sequence for the triple $(W, W_{\lambda+1}, W_{\lambda-2})$. ⊠

Charlie first drew a picture with an example computation of $C_\bullet$ and its homology. I don't know how to TeX that picture, unfortunately. Sorry about that.

Theorem 17.8 (Basis theorem)**.** *Let (W, V, V') be a cobordism of dimension n, f be a Morse function on W with negative gradient flow ξ, such that f has only index-λ critical points and only one critical value. Suppose W is connected and $2 \le \lambda \le n-2$. Then given a basis of $H_\lambda(W, V)$, we can construct f' and ξ' such that the new descending discs represent the basis,* $\mathrm{Crit}(f) = \mathrm{Crit}(f')$*, the critical value is the same, and $(f', \xi') = (f, \xi)$ near ∂W.*

TODO: something involving handle slides, which are ways to relate two diffeomorphic handlebody diagrams.

Lecture 18.

: 11/7/18

Lecture 19.

: 11/14/18

Lecture 20.

The h-cobordism theorem and some consequences: 11/19/18

First, leaning on the hard work of all of you, I can state the proof of the h-cobordism theorem (unless Kenny does).

Theorem 20.1 (h-cobordism (Smale))**.** *Let $(W; V, V')$ be a cobordism for which V, V', and W are simply connected, $H_*(W, V) = 0$, and $n = \dim W \ge 6$. Then W is a product cobordism: $V' \cong V$, and $W \cong [0,1] \times V$.*

Proof. Let f be a self-indexing Morse function for this cobordism. We can eliminate critical points of indices 0 and 1, and (by replacing f with $-f$) we can eliminate critical points of indices n and $n-1$. Then we eliminate all remaining critical points using cancellation in the middle dimensions. ⊠

Definition 20.2. A cobordism $(W; V, V')$ is an *h-cobordism* if V and V' are deformation retracts of W. In this case one says V is *h-cobordant* to W.

There are a few slight reformulations of the h-cobordism theorem.

(1) Poincare-Lefschetz duality provides an isomorphism $H_*(W, V) \cong H^{n-*}(W, V')$, so we could instead assume $H^*(W, V') = 0$.
(2) The first two assumptions (simply connected and relative homology vanishing) already imply $(W; V, V')$ is an h-cobordism, by the relative Hurewicz theorem: (the abelianization of) the first nonzero relative homotopy group is isomorphic to the first nonzero relative homology group, but we have no such group, so all relative homotopy groups vanish. Therefore using the Puppe sequence $\pi_n(X) \to \pi_n(A) \to \pi_n(X, A) \to \pi_{n-1}(X) \to \dots$, inclusion $V \hookrightarrow W$ induces an isomorphism on homotopy groups. By Whitehead's theorem, it's a homotopy equivalence, and this can be upgraded to the existence of a deformation retraction (using the fact that there exists a relative triangulation of (W, V)).

Now we discuss some corollaries of Theorem 20.1.

Corollary 20.3. *If $n \ge 5$ and two simply connected n-manifolds are h-cobordant, then they're diffeomorphic.*

Theorem 20.4 (Characterizations of D^n)**.** *Let W be a compact, simply-connected n-manifold, possibly with boundary, and assume $n \geq 6$. Then the following are equivalent.*

(1) *W is diffeomorphic to D^n.*
(2) *W is homeomorphic to D^n.*
(3) *W is contractible.*
(4) *$H_*(W) \cong H_*(\mathrm{pt})$.*

Proof. It suffices to show (4) $\implies$ (1). Let $D \hookrightarrow W$ be a smoothly embedded disc; then, $(W \setminus \operatorname{Int} D; \partial D, \partial W)$ meets the conditions of Theorem 20.1. The hypothesis on homology follows by excision:

$$H_*(W \setminus \operatorname{Int} D, \partial D) \cong H_*(W, D) = 0, \tag{20.5}$$

with the latter following from the fact that W has trivial homology and the long exact sequence of a pair.

Therefore W, regarded as a cobordism from $\varnothing$ to ∂W, is a composition of the cobordism $(D; \varnothing, \partial D)$ and $(W \setminus \operatorname{Int} D; \partial D, \partial W)$, which is a product cobordism. Therefore $W \cong D$. ⊠

Now for a much bigger fish.

Theorem 20.6 (Generalized Poincare conjecture, $n \geq 6$ (Smale))**.** *Let M be a closed, simply-connected smooth n-manifold with the integral homology of S^n. If $n \geq 5$, M is homeomorphic to S^n.*

Corollary 20.7. *In dimensions $n \geq 6$, a homotopy n-sphere is homeomorphic to an n-sphere.*

Definition 20.8. A *twisted n-sphere* is an n-manifold of the form $D^n \cup_\phi D^n$, where $\phi \colon S^{n-1} \to S^{n-1}$ is a diffeomorphism.

Hence twisted spheres admit Morse functions with just two critical points, and if M is closed and has a Morse function with two critical points, M is a twisted sphere.

Proposition 20.9. *If M is a twisted sphere, M is homeomorphic to S^n.*

Proof sketch. Let $M = D_1^n \cup_\phi D_2^n$ be a twisted n-sphere. Let $g_1 \colon D_1^n \hookrightarrow S^n$ be an embedding into the southern hemisphere. Define $g \colon M \to S^n$ by

(1) $g = g_1$ on D_1, and
(2) if $x \in D_2$, $x = tv$ for some $v \in \partial D_2$ and $t \in [0, 1]$. Then let

$$g(x) = \sin\left(\frac{\pi t}{2}\right) g_1(h^{-1}(v)) + \cos\left(\frac{\pi t}{2}\right) e_{n+1}, \tag{20.10}$$

where e_{n+1} is the last basis vector in $\mathbb{R}^{n+1}$.

One can then check this is a homeomorphism. ⊠

Proof of Theorem 20.6. Let $D \hookrightarrow M$ be a smoothly embedded disc. Then $M \setminus \operatorname{Int} D$ satisfies the conditions of Theorem 20.4, hence is also diffeomorphic to D^n. The boundaries of these two discs are identified via some diffeomorphism of $\partial D^n = S^n$, so M is a twisted sphere. ⊠

Famously, not all manifolds homeomorphic to S^n are diffeomorphic to it! These *exotic spheres* were first constructed by Milnor when $n = 7$. For general n, there are often lots of exotic n-spheres. However, we can upgrade the statement for $n = 5, 6$.

Theorem 20.11 (Kervaire-Milnor)**.** *If $n = 5$ or 6, then S^n has a unique smooth structure. Hence for these n, any smooth integral homology n-sphere is diffeomorphic to S^n.*

The general story is still being told: for example, Zhouli Xu and Guozhen Wang proved in 2016 that the only odd-dimensional spheres with a unique smooth structure are S^1, S^3, S^5, and S^{61}.

Proposition 20.12 (Characterization of the 5-disc)**.** *Let W be a compact, simply connected smooth 5-manifold with $H_*(W) \cong H_*(\mathrm{pt})$.*

(1) *If ∂W is homeomorphic to S^4, then W is homeomorphic to D^5.*
(2) *If ∂W is diffeomorphic to S^4, then W is diffeomorphic to D^5.*

We will prove this as a corollary of the generalized Poincare conjecture and the following theorems.

Theorem 20.13 (Cerf, Palais)**.** *Any two orientation-preserving embeddings of D^n into a connected oriented n-manifold are isotopic.*

For the proof, check out Palais' paper "Extending diffeomorphisms," which is only 4 pages!

Theorem 20.14 (Topological generalized Schoenflies theorem (Brown))**.** *If $\Sigma \hookrightarrow S^n$ is a topologically embedded $(n-1)$-sphere with a collar neighborhood, then $S^n \setminus \Sigma$ has two components, each homeomorphic to D^n.*

This one is only three pages.

Proof of Proposition 20.12. For the first part, let $D(W) := W \cup_{\partial W} W$ denote the *double* of W. By Theorem 20.11, $D(W)$ is diffeomorphic to S^5. The embedding $\partial W \hookrightarrow S^5$ is a topologically embedded 4-sphere with collar neighborhood, so by Brown's theorem, W is homeomorphic to D^n.

For the smooth part, let $M := W \cup_h D^5$, where h is some diffeomorphism $\partial W \to S^4$. Then M is a simply connected homology 5-sphere, hence is diffeomorphic to S^5. Now using Theorem 20.13, we can isotope the embedding $D^5 \hookrightarrow M$ to the northern hemisphere; $W = M \setminus \operatorname{Int} D^5$, so is the southern hemisphere, thus diffeomorphic to S^5. ⊠

The last application we'll cover today is the differentiable Schoenflies theorem for dimensions $n \geq 5$. This is the solution to a classic problem (dating back to the early 1900s): given an embedding $S^{n-1} \hookrightarrow S^n$, is its complement two discs? This can be formulated in the topological or smooth category. For the former, it was proved by Brown (Theorem 20.14), who does not need the h-cobordism theorem.

Example 20.15. The Alexander horned sphere shows why we need the collar neighborhood assumption in Theorem 20.14: it's an embedding $S^2 \hookrightarrow S^3$ whose complement isn't simply connected.

To construct this pathological object, start with the standard torus in $\mathbb{R}^3 \subset S^3$ and repeat the following steps *ad infinitum*:

(1) Remove a radial slice of the torus.
(2) Attach a once-punctured torus to each boundary component, such that the two tori are interlinked.
(3) Repeat for the two new tori we attached.

This defines an embedding of the sphere minus a Cantor set; it in fact extends to an embedding of all of S^2. The link of any torus is not contractible in its complement. ◀

Theorem 20.16 (Differentiable Schoenflies theorem, $n \geq 5$)**.** *Suppose $n \geq 5$ and $\Sigma \hookrightarrow S^n$ is a smoothly embedded $(n-1)$-sphere. Then there is an isotopy of S^n carrying Σ to the equator.*

Proof. By Alexander duality, $S^n \setminus \Sigma$ has two components, so Σ has a collar neighborhood. If D denotes the closure of a component of $S^n \setminus \Sigma$ in S^n, then D is a smooth, simply connected n-manifold with trivial homology, hence by **??** and Proposition 20.12 diffeomorphic to D^n. Then Theorem 20.13 allows us to isotopy D to the southern hemisphere of S^5, and Σ to the equator. ⊠

We did *not* prove that $\Sigma \hookrightarrow S^5$ is isotopic to the standard embedding; instead, it's isotopic to a map onto $S^4 \subset S^5$. To produce counterexamples of the stronger statement, begin with a diffeomorphism $g\colon S^{n-1} \to S^{n-1}$ that doesn't extend to D^n (equivalent to a twisted sphere not diffeomorphic to the standard sphere), and then compose g with the standard inclusion.

Remark 20.17. When $n = 5$, the h-cobordism theorem is true topologically (Freedman), and fails smoothly (Donaldson). When $n = 4$, the h-cobordism theorem is open, and equivalent to the (difficult) question of whether there's an exotic S^4. When $n = 3$, it's true, following from the Thurston-Perelman geometrization theorem, and when $n \leq 2$, it's vacuous. ◀

Lecture 21.

Geodesics: 11/19/18

"So if you change from hours to whatever they use in Europe..."

Today, Dan spoke about geodesics. The goal is to learn what Morse theory can say about geodesics, with the eventual goal of applying Morse theory to the infinite-dimensional space of paths on a manifold.

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, so that it has the *Levi-Civita connection*, which is characterized as the unique connection satisfying the conditions

(1) $[X,Y] = \nabla_X Y - \nabla_Y X$ (i.e. it's *torsion-free*), and
(2) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

for all $X,Y,Z \in \Omega^0_M(TM)$. One proves this by just writing down what it has to be, expanding out $2\langle\nabla_X Y, Z\rangle$ using the two properties:

$$2\langle\nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle. \tag{21.1}$$

This defines $\nabla_X Y$ for all X and Y, and then one checks this has the required properties.

In local coordinates $x^1,\dots,x^n$, let $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ be a local framing. The metric is determined by the numbers

$$g_{ij} \coloneqq \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle. \tag{21.2}$$

The matrix $g = (g_{ij})$ is invertible, because the metric is nondegenerate. We then introduce *Christoffel symbols* to capture the Levi-Civita connection in coordinates: define

$$\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}. \tag{21.3}$$

(Here we use Einstein's summation convention.) Then from the second property of the Levi-Civita connection,

$$2\Gamma^i_{jk} g_{i\ell} = \frac{\partial g_{k\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell}, \tag{21.4}$$

i.e.

$$\Gamma^i_{jk} = \frac{1}{2}\left(\frac{\partial g_{k\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell} \right). \tag{21.5}$$

Now on to geodesics.

Definition 21.6. Let $I \subset \mathbb{R}$ be an interval. A smooth curve $\gamma\colon I \to M$ is a *geodesic* if for all $t \in I$, $\nabla_{\dot\gamma}\dot\gamma(t) = 0$.

This covariant derivative is the derivative of the velocity of γ; therefore being a geodesic is a condition about constant acceleration.

In local coordinates, let $x^i = x^i(t)$, so

$$\dot\gamma(t) = \dot x^j \frac{\partial}{\partial x^j}. \tag{21.7}$$

The *geodesic equation* asks that

$$\begin{aligned} 0 &= \nabla_{\dot x^j \partial/\partial x^j}\left(\dot x^k \frac{\partial}{\partial x^k} \right) \\ &= \ddot x^k \frac{\partial}{\partial x^k} + \dot x^j \dot x^k \Gamma^i_{jk} \frac{\partial}{\partial x^i} \\ &= \left(\ddot x^i + \Gamma^i_{jk} \dot x^j \dot x^k \right) \frac{\partial}{\partial x^i}. \end{aligned} \tag{21.8}$$

This is a system of nonlinear second-order ODEs. So we appeal to the fundamental theorem of ODE to show there's a unique solution: let $y^i \coloneqq \dot x^i$, so we have a system of first-order equations

$$\begin{aligned} \dot x^i &= y^i \\ \dot y^i &= -\Gamma^i_{jk} y^j y^k. \end{aligned} \tag{21.9}$$

Therefore we have local existence of a geodesic $\gamma\colon (-\varepsilon,\varepsilon) \to M$ given specified $\gamma(0)$ and $\dot\gamma(0)$. We don't know what will happen globally: we can paste local solutions, but consider a geodesic on $\mathbb{E}^n \setminus 0$ which goes from $(1,1)$ to $(1/2,1/2)$ — it will fall off at the missing origin.

There's a good geometric interpretation of this approach. On a Riemannian manifold we have a principal O_n-bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(M) \to M$, whose fiber at an $x \in M$ is the O_n-torsor of orthonormal bases of $T_x X$. The Levi-Civita theorem tells us that there's a unique connection (namely, the Levi-Civita connection) on this principal bundle, producing horizontal subspaces on $\mathcal{B}_{\mathrm{O}}(M)$. Therefore we obtain global vector fields $\partial_1,\dots,\partial_n$ on $\mathcal{B}_{\mathrm{O}}(M)$: given a point x and a local framing $(e^1,\dots,e^n)$ near x, ∂_k is the horizontal lift to $T_{(e_1,\dots,e_k)}$ of e^k.

Theorem 21.10. *Integral curves of ∂_k map via π to geodesics of M.*

Figuring the details out is a nice exercise. In any case, you can use this to deduce theorems about geodesics whose proofs might otherwise be longer. For example, if $t \mapsto \gamma(t)$ is a geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = \xi \in T_xM$, then there's a reparameterization $\gamma_c(t) := \gamma(ct)$, for $c \in \mathbb{R}$, with $\gamma_c(0) = x$ and $\dot\gamma_c(0) = c\xi$.

Definition 21.11. Let $x \in M$. If $\xi \in T_xM$ is such that the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = \xi$ is defined on $[0, 1]$, set $\exp_x \xi = \gamma(1)$. This defines a function $\exp_x \colon U \to M$ for some open neighborhood U of the origin in T_xM.

Milnor shows this behaves nicely near 0.

Proposition 21.12. *For each $x_0 \in M$ there's a neighborhood $U \subset T_{x_0}M$ and an $\varepsilon > 0$ such that for all $x_0 \in M$, the map $\exp_x \colon B_\varepsilon(0) \to M$ is a smooth function on $B_\varepsilon(TU)$.*

Proof. The fundamental theorem of ODE gives us a neighborhood U of x_0 and $\varepsilon_1, \varepsilon_2 > 0$ such that if $x \in U$, $\xi \in T_xM$, and $|\xi| < \varepsilon_1$, then there's a unique geodesic $\gamma\colon (-2\varepsilon_2, 2\varepsilon_2) \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. Set $\varepsilon = \varepsilon_1\varepsilon_2$; if $x \in U$ and $\xi \in T_xM$ has $|\xi| < \varepsilon$, then $|\xi/\varepsilon_2| < \varepsilon_1$ so there's a unique $\gamma\colon (-2\varepsilon_2, 2\varepsilon_2) \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = \xi/\varepsilon_2$. Therefore we can reparameterize: $\gamma_{\varepsilon_2}(0) = x$ and $\dot\gamma_{\varepsilon_2}(0) = \xi$, and the domain of γ_{ε_2} is $(-2, 2)$. Therefore $\gamma_{\varepsilon_2}(1) \in M$ is defined; smoothness follows from the smooth dependence on initial conditions of the solutions of an ODE. ⊠

Consider $F\colon B_\varepsilon(U) \to M \times M$ sending $(x, \xi) \mapsto (x, \exp_x \xi)$. Then $\mathrm{d}F_{(x,0)}$ is a map $T_{(x,0)}TM \to T_xM \oplus T_xM$, and $T_{(x,0)}TM$ splits canonically as horizontal and vertical subspaces: $T_{(x,0}TM = T_xM \oplus T_xM$. Under this identification,

$$\mathrm{d}F_{(x,0)}(\dot x, \dot\xi) = (\dot x, \dot x + \dot\xi). \tag{21.13}$$

This is invertible in a neighborhood of x, so F is a local diffeomorphism at $(x_0, 0)$, and therefore there exist $U' \subset U$ and $0 < \delta < \varepsilon$ such that $F|_{B_\delta(TU')}$ is a diffeomorphism onto its image. Choose a neighborhood W of x_0 such thst the image of $F|_{B_\delta(TU')}$ contains $W \times W$. Therefore we find:

Theorem 21.14. *For all $x \in M$ there exists a neighborhood W of x and a $\delta > 0$ such that*

(1) *any two $p, q \in W$ are joined by a unique geodesic, and*
(2) *for all $p \in W$, $\exp_p \colon B_\delta(0) \subset T_pM \to M$ is a diffeomorphism onto its image in M.*

Next time we'll discuss geodesics as length-minimizers, as you might be used to thinking of them. This is true locally, but globally it might not be true (consider the long great circle arc on S^2 between two close points). We'll discuss what length is on a Riemannian manifold, and some more global concerns: the Hopf-Rinow theorem shows that on a complete Riemannian manifold, geodesics exist for all time!

Lecture 22.

Geodesics, length, and metrics: 11/28/18

"It's kind of fun, but we're not here to have fun."

Let $\gamma\colon [0, L] \to M$ be a smooth curve in a Riemannian manifold M, and let $T := \dot\gamma$, which is a vector field along γ (i.e. a section of $\gamma^*TM \to [0, L]$). We said that if $\nabla_T T = 0$, then γ is a geodesic. G

Lemma 22.1. *If γ is a geodesic, then $\|T\|$ is constant.*

Proof. Well, $T \cdot \langle T, T\rangle = 2\langle \nabla_T T, T\rangle = 0$. ⊠

Last time, we proved Theorem 21.14, which says there's a neighborhood W of any point p such that any two points of W are joined by a unique geodesic $\gamma\colon [0, 1] \to M$, and that for all $p \in W$, $\exp_p|_{B_\delta(0) \subset T_pM}$ is a diffeomorphism onto its image, for some $\delta > 0$.

Today we'll talk about how geodesics relate to length.

Definition 22.2. First, if $\omega\colon [0, 1] \to M$ is smooth, let

$$L[\omega] := \int_0^1 \mathrm{d}t\, \|\dot\omega(t)\|. \tag{22.3}$$

Let M be a connected Riemannian manifold and $p, q \in M$; then define

$$d(p,q) \coloneqq \inf\{L[\omega] \mid \omega\colon [0,1] \to M, \omega(0) = p, \omega(1) = q\}. \tag{22.4}$$

Theorem 22.5. *(M, d) is a metric space, and the metric space topology equals the topology on M we started with.*

So given a Riemannian metric on a connected manifold, we obtain a metric space structure: we know what distances are.

Remark 22.6. Let $p \in M$ and $\xi \in T_pM$. Then the geodesic $\gamma\colon [0,1] \to M$ sending $t \mapsto \exp_p(t\xi)$ has length $\|\xi\|$. This geodesic isn't necessarily length-minimizing: consider on S^1 two points p and q which are close, and consider the geodesic going "the long way" around S^1 from p to q. But within the neighborhood W from Theorem 21.14, geodesics are length-minimizing. ◀

Theorem 22.7. *Let W be as in Theorem 21.14 and $p, q \in W$. Let $\gamma\colon [0,1] \to M$ be the unique geodesic with $\gamma(0) = p$ and $\gamma(1) = q$, and $\omega\colon [0,1] \to M$ be a piecewise smooth path with $\omega(0) = p$ and $\omega(1) = q$. Then $L[\gamma] \le L[\omega]$ with equality iff $\mathrm{Im}(\gamma) = \mathrm{Im}(\omega)$ and ω is injective.*

For $0 < R < \delta$, set

$$H_R \coloneqq \{\exp_p(\xi) \colon \xi \in S_R(0) \subset T_pM\}. \tag{22.8}$$

So we've foliated $B_\delta(0)$ by spheres. We'd like to show that geodesics are perpendicular to these spheres. For surfaces, this is due to Gauss, so the general result is called the Gauss lemma.

Lemma 22.9 (Gauss)**.** *$\mathrm{d}\exp_\xi(\xi) \perp H_R$ for all $R \in (0,\delta)$ and $\xi \in S_R(0) \subset T_pM$.*

Proof. Let $c\colon (-\varepsilon, \varepsilon) \to S_R(0)$ be a curve with $c(0) = \xi$, and set $\alpha(s,t) \coloneqq \exp_p(tc(S))$. Let $T \coloneqq \frac{\partial x}{\partial t}$ and $X \coloneqq \alpha s$; then $[T, X] = 0$, which implies (by a problem on the homework from a few weeks ago) $\nabla_T X = \nabla_X T$. Then we compute:

$$\begin{aligned} T \cdot \langle T, X\rangle &= \underbrace{\langle \nabla_T T, X\rangle}_{=0} + \langle T, \nabla_T X\rangle \\ &= \langle T, \nabla_X T\rangle \\ &= \frac{1}{2} X \cdot \langle T, T\rangle = 0. \end{aligned}$$

Therefore $\langle T, X\rangle|_{s=0,t=1} = \langle T, X\rangle|_{s=0,t=0} = 0$, since $X|_{s=0,t=0} = 0$. ⊠

Let $\omega\colon [a,b] \to U_p \setminus \{p\}$. Using spherical coordinates, we can find $r\colon [a,b] \to \mathbb{R}^{>0}$ and $\xi\colon [a,b] \to S_1(0) \subset T_pM$ such that

$$\omega(t) = \exp_p(r(t)\xi(t)). \tag{22.10}$$

Lemma 22.11. *$L[\omega] \ge r(b) - r(a)$ with equality iff r is monotone and ξ is constant.*

Proof. Let $f(s,r) \coloneqq \exp_p(r\xi(s))$, so that $\omega(t) = f(t, r(t))$ and $\dot\omega(t) = X + \dot r T$, where $\frac{\partial f}{\partial s} = X$ and $\frac{\partial f}{\partial r} = T$. Therefore by the Gauss lemma, $|\dot\omega|^2 = |x|^2 + \dot r^2 \ge \dot r^2$, and therefore

$$L[\omega] = \int_a^b \mathrm{d}t\, |\dot\omega(t)| \ge \int_a^b \mathrm{d}t\, |\dot r| = |r(b) - r(a)|. \tag{22.12}$$ ⊠

Proof of Theorem 22.7. Write $q = \exp_p(\xi)$, where $|\xi| < \delta$. If $0 < R < |\xi|$, the path ω has a segment joining H_R to $H_{|\xi|}$, so $L[\omega|_{[R/|\xi|,1]}] \ge L[\gamma|_{[R/|\xi|,1]}]$. Taking $R \to 0$, we see that equality holds iff ω is radial. ⊠

Corollary 22.13. *If γ is a length-minimizing curve from $\gamma(0)$ to $\gamma(1)$, then γ is a geodesic, up to reparameterization.*

Proof. It suffices to check locally (in time), and γ is also length-minimizing on any $[a,b] \subset [0,1]$. ⊠

The Hopf-Rinow theorem is a sort of converse.

Theorem 22.14 (Hopf-Rinow)**.** *Let M be a connected Riemannian manifold. Then the following are equivalent:*

(1) (M, d) *is a complete metric space.*
(2) $\exp_p\colon T_pM \to M$ *is defined for some* $p \in M$.
(3) $\exp_p\colon T_pM \to M$ *is defined for* all $p \in M$.

Any of these three then implies that for all $p, q \in M$*, there exists a minimal geodesic* γ *from* p *to* q*, i.e.* $L[\gamma] = d(p, q)$.

Completeness is a geometric property, not a topological one: consider the open and the closed intervals, which are homeomorphic.

Proof sketch. Let's first show that (2) implies the existence of minimal geodesics. Let $D = d(p, q)$ and $p_0 \in H_{\delta/2}$ be such that $d(p_0, q)$ is minimal over $H_{\delta/2}$. Then $p_0 = \exp_p(\delta\xi/2)$ for $\xi \in T_pM$ and $|\xi| = 1$. The curve $\gamma\colon [0, D] \to M$ sending $t \mapsto \exp_p(t\xi)$ is a geodesic; we claim $\gamma(D) = q$. This would follow from $d(\gamma(t), q) = D - t$ for all $t \in [0, D]$, which is true at $t = \delta/2$.

Let t_0 be the supremum of the times at which $d(\gamma(t), q) = D - t$, so $t_0 > \delta/2$. Then $d(\gamma(t_0), q) = D - t_0$, so choose $p_0' \in H_R$ such that $d(p_0', q)$ is minimal (this is a continuous function on a compact region, so there must be such a minimum). Then

$$\begin{aligned} D - t_0 &= d(\gamma(t_0), q) \\ &= \min_{r' \in H'} \{d(\gamma(t_0), r') + d(r', q)\} \\ &= R + d(p_0', q). \end{aligned}$$

We claim that $\gamma(t_0 + R) = p_0'$, since $d(p, p_0') \geq d(p, q) - d(p_0', q) = D - d(p_0', q) = t_0 + R$. Therefore the supremum has to be D. ⊠

From now on, we assume M is complete and connected; eventually we will assume M is compact, which implies completeness. Therefore we have geodesics. We're going to study the topology of the space of paths between two points on M and relate critical points to geodesics.

Let M be a connected smooth manifold (though for now, we won't use the manifold structure) and $p \in M$.

Definition 22.15. The *path space* P_pM is the space of continuous paths $\omega\colon [0, 1] \to M$ with $\omega(0) = p$.

There's an obvious surjective map $\pi\colon P_pM \to M$ sending $\omega \mapsto \omega(1)$.

Proposition 22.16. π *is a fibration.*

In particular, the fibers are homotopy equivalent. The fiber at a $q \in M$ is the space of paths with $\omega(0) = p$ and $\omega(1) = q$. This space is denoted $\Omega_{p,q}(M)$, and its homotopy type is independent of p and q. In particular, letting $p = q$, these have the homotopy type of the based loop space ΩM.

Definition 22.17. A map $\pi\colon X \to Y$ of spaces is a *fibration* if, for every space S, map $\{0\} \times S \to P_pM$ and a homotopy of the map composed with π, namely a commutative diagram

$$\begin{array}{ccc} \{0\} \times S & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \pi \\ [0,1] \times S & \longrightarrow & Y, \end{array} \tag{22.18}$$

then the homotopy lifts to a homotopy $[0, 1] \times S \to X$.

There is a weaker notion of a *Serre fibration* where we restrict to S finite-dimensional (i.e. finite CW complexes). We can use path lifting to produce homotopy equivalences between the fibers.

Definition 22.19. A curve $\omega\colon [0, 1] \to M$ is *piecewise smooth* if it's continuous and there's a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ suc that $\omega|_{[t_{i-1}, t_i]}$ is smooth for each i.

There is an analogous definition for a homotopy $\alpha\colon S \times [0, 1] \to M$ to be piecewise smooth, where S and M are smooth manifolds: we partition $[0, 1]$ and ask for $\alpha|_{S \times [t_{i-1}, t_i]}$ to be smooth. Let $\Omega_{p,q}^{\mathrm{PS}}$ be the subset of piecewise smooth paths in $\Omega_{p,q}$; this is a subset but it will be useful to think of it as akin to a submanifold.

For example, if $\omega \in \Omega_{p,q}^{\mathrm{PS}}$, we'd think of its "tangent space" as equivalence classes of (piecewise smooth) curves $\alpha\colon (-\varepsilon, \varepsilon) \to M$. The idea is that a variation is a curve which bends in the places where p isn't smooth.

This allows us to make sense of variations of piecewise-smooth curves: let $\alpha\colon (-\varepsilon,\varepsilon)_s \times [0,1]_t \to M$ be a variation: then we have two vector fields $X \coloneqq \frac{\partial\alpha}{\partial s}$ and $T \coloneqq \frac{\partial\alpha}{\partial t}$. We define the length and energy functionals to be

$$L[\omega] \coloneqq \int_0^1 \mathrm{d}t\, \|T\| \qquad \text{and} \qquad E[\omega] \coloneqq \int_0^1 \mathrm{d}t\, \|T\|^2. \tag{22.20}$$

Exercise 22.21. Using the Cauchy-Schwarz inequality, show that

$$L_a^b[\omega]^2 \le (b-a)E_a^b[\omega].$$

Proposition 22.22. *Let M be a complete, connected Riemannian manifold and $p,q \in M$. A minimal geodesic from p to q realizes the global minimum of the energy functional $E\colon \Omega_{p,q}^{\mathrm{PS}} \to \mathbb{R}$.*

Proof. Let γ be a minimal geodesic and $\omega \in \Omega_{p,q}^{\mathrm{PS}}$. Then

$$E[\gamma] \le L[\gamma]^2 \le L[\omega]^2 \le E[\omega], \tag{22.23}$$

with equality iff ω is a reparameterized geodesic (first inequality) and has constant speed (second inequality). ⊠

Now we'll discuss the first variation formula, for the derivative in the s-direction given a variation as above. First, we compute

$$\begin{aligned} \frac{1}{2}X\langle T,T\rangle &= \langle \nabla_X T, T\rangle = \langle \nabla_T X, T\rangle \\ &= T\langle X,T\rangle - \langle X, \nabla_T T\rangle. \end{aligned} \tag{22.24}$$

Therefore

$$\mathcal{L}_\omega(x) \coloneqq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}E[\alpha_s] = \sum_i \int_{t_{i-1}}^{t_i} \mathrm{d}t\,(T\langle X,T\rangle - \langle X,\nabla_T T\rangle) \tag{22.25}$$

$$= \sum_i \left(\langle X,T\rangle\big|_{t_{i-1}}^{t_i} - \int_{t_{i-1}}^{t_i} \langle X, \nabla_T T\rangle \right). \tag{22.26}$$

Theorem 22.27. *$\mathcal{L}_\omega = 0$ iff ω is a geodesic.*

Proof. If ω is smooth,

$$\mathcal{L}_\omega(x) = -\int_0^1 \langle X, \nabla_T T\rangle = 0. \tag{22.28}$$

Conversely, we'll show that $\nabla_T T = 0$ on $[t_{i-1}, t_i]$. Take

$$X(t) \coloneqq \begin{cases} 0, & t \notin [t_{i-1}, t_i] \\ f(t)\nabla_T T, & t \in [t_{i-1}, t_i], \end{cases} \tag{22.29}$$

where f is a smooth function whose support is $[t_{i-1}, t_i]$. Then, if $T_+ \ne T_-$ at some t_i, let $X(t_i) \coloneqq T_+ - T_-$. (TODO: does this suffice?) ⊠

As Morse theorists, we're interested in the second derivative, which will mean the second variational formula. This will relate to the curvature of the Riemannian manifold, so we give a quick review of that.

Definition 22.30. Let M be a Riemannian manifold and X, Y, and Z be vector fields on M (sometimes this is written $X,Y,Z \in \mathcal{X}(M)$). The *Riemann curvature tensor* is

$$R(X,Y)Z \coloneqq \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \in \mathcal{X}(M).$$

One can write this slightly more compactly as

$$R(X,Y) \coloneqq [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \operatorname{End}(\mathcal{X}(M)). \tag{22.31}$$

Lemma 22.32. *R is linear over smooth functions in all variables, hence is a section of the bundle* $\operatorname{Hom}(TM \otimes TM \otimes TM \otimes TM, TM) \to M$.

However, it's a special kind of section, because it has a lot of symmetry.

Theorem 22.33.

(1) $\langle R(X,Y)Z,W\rangle + \langle R(Y,X)Z,W\rangle = 0$.
(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.
(3) $\langle R(X,Y)Z,W\rangle + \langle R(X,Y)W,Z\rangle = 0$.
(4) $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$.

Okay, back to the second variation. We consider a three-dimensional variation $\alpha\colon (-\varepsilon,\varepsilon)_s \times [0,1]_t \times (-\varepsilon,\varepsilon)_u \to M$; restricting to $u = 0$ and $s = 0$, we have first variations and vector fields X and Y, respectively. We want to calculate

$$\frac{\partial^2}{\partial s\partial u} E[\alpha_{s,u}]\bigg|_{s=u=0} = B_\gamma(X,Y), \tag{22.34}$$

some bilinear function of X and Y which will be the analogue of the Hessian. Since

$$\begin{aligned}\frac{1}{2}XY\langle T,T\rangle &= X\langle \nabla_Y T, T\rangle\\ &= X\langle \nabla_T Y, T\rangle = \langle \nabla_X \nabla_Y T\rangle + \langle \nabla_T Y, \nabla_X T\rangle\\ &= \langle R(X,T)Y,T\rangle + \langle \nabla_T\nabla_X Y, T\rangle + \langle \nabla_T X, \nabla_T Y\rangle\\ &= \langle -\nabla_T\nabla_T X + R(T,X)T, Y\rangle + T(\langle \nabla_X Y, T\rangle + \langle \nabla_T X, Y\rangle).\end{aligned}$$

To get the bilinear form, we integrate this.

Theorem 22.35.

$$B_\gamma(X,Y) = \sum_i \left(\langle \nabla_X Y, T\rangle + \langle \nabla_T X, Y\rangle\big|_{t_{i-1}}^{t_i} + \int_{t_{i-1}}^{t_i} \mathrm{d}t\, \langle -\nabla_T\nabla_T X + R(T,X)T, U\rangle \right).$$

Definition 22.36. Let $\gamma\colon [0,1] \to M$ be a geodesic. A vector field X along γ is *Jacobi* if $\nabla_T\nabla_T X = R(T,X)T$ (the *Jacobi equation*), i.e. the second variation vanishes. Two points p and q are *conjugate* if there exists a Jacobi vector field X such that $X(0)$ and $X(1)$ both vanish.

Theorem 22.37. $\ker(B_\gamma)$ *is the space of Jacobi fields along* γ.

Suppose $\alpha\colon (-\varepsilon,\varepsilon)_s \times [0,1]_t \to M$ is such that α_s is a geodesic for every s, and let $T := \frac{\partial\alpha}{\partial t}$ and $X := \frac{\partial\alpha}{\partial s}$. Then $\nabla_T T = 0$ for all t and s. Then

$$\begin{aligned}\nabla_T\nabla_T X &= \nabla_T\nabla_X T\\ &= R(T,X)T + \nabla_X\nabla_T T\\ &= R(T,X)T.\end{aligned}$$

So a variation of geodesics induces a Jacobi field. The converse is also true: all Jacobi fields arise from a first variation of geodesics.

Lecture 23.

Some applications of geodesics: 12/5/18

Today we'll say more about geodesics on manifolds, and give three applications: a characterization of geodesics on a sphere, due to Morse; the Freudenthal suspension theorem on homotopy groups of spheres; and Bott periodicity. Throughout this lecture, M is a connected, complete Riemannian manifold.

Recall that for $p,q \in M$, $\Omega_{p,q}$ denotes the space of piecewise smooth paths $\omega\colon [0,1] \to M$ with $\omega(0) = p$ and $\omega(1) = q$. We defined the energy functional in (22.20) and studied the first variation formula (22.24), given $T = \dot\omega$ and a "variational vector field," i.e. a vector field X along ω. If $X(0) = X(1) = 0$, this formula says

$$\mathcal{L}_\omega(X) = \sum_{\text{jumps in } T} \langle X, \Delta T\rangle - \int_0^1 \mathrm{d}t\, \langle X, \nabla_T T\rangle. \tag{23.1}$$

If $X(0) \neq 0$ or $X(1) \neq 0$, there's an additional term comparing it to $T(0)$ or $T(1)$. We also discussed the second variation formula. Without additional structure (such as a covariant derivative), one can only compute the second derivative on a manifold at a critical point; thus, assume $\omega = \gamma$ is a geodesic. Then the second variational formula is Theorem 22.35. The Jacobi equation $\nabla_T\nabla_T X = R(T,X)T$, coming from a term in this formula, can also arise by assuming X is a variation of γ entirely through geodesics: let $\Gamma\colon [0,1]^2 \to M$, $\frac{\partial}{\partial t}\big|_{s=0}\Gamma = T$, and $\frac{\partial}{\partial s}\big|_{s=0}\Gamma = X$. If $\Gamma(s_0,-)$ is a geodesic for all s_0, then

$$\begin{aligned} 0 &= \nabla_X\nabla_T T \\ &= \nabla_T\nabla_X T + \nabla_{[X,T]}T + R(X,T)T \\ &= \nabla_T\nabla_T X + R(X,T)T. \end{aligned}$$

Let $e_1,\dots,e_n$ be a basis of T_pM with $e_1 = T$, and extend e_i to vector fields along γ such that $\nabla_T e_j = 0$. We can write a vector field X along γ as

$$X(t) = f^i(t)e_i(t), \tag{23.2}$$

where $f^i\colon [0,1]\to\mathbb{R}$ is smooth. This allows us to write the Jacobi equation in coordinates. First,

$$\nabla_T\nabla_T X = \nabla_T(\dot f^i e_i) = \ddot f^i e_i. \tag{23.3}$$

Let $R(e_k,e_\ell)e_j = R^i_{jk\ell}(t)\cdot e_i$; then

$$R(T, f^\ell e_\ell)T = R^i_{11\ell}(t)f^\ell(t)e_i(t). \tag{23.4}$$

Therefore in coordinates, the Jacobi equation is

$$\ddot f^i = R^i_{11\ell}f^\ell. \tag{23.5}$$

This is a second-order linear ODE; because it's linear, solutions exist for all time. Given $f^i(0)$ and $\dot f^i(0)$, i.e. given $X(0)$ and $\nabla_T X(0)$, there is a unique solution.

Proposition 23.6. *$\ker(B_\gamma)$ is the space of Jacobi fields which vanish at $t=0$ and $t=1$. Moreover, $\dim\ker B_\gamma \ne n-1$.*

TODO: I think I missed a proof of the first part.

For the second part, the equation for f^1 is

$$\ddot f^1 = R^1_{11\ell}f^\ell = 0. \tag{23.7}$$

So $f^1(t) = a_0 + a_1(t)$. If $f^1(0) = f^1(1) = 1$, then $a_0 = a_1 = 0$.

Example 23.8. Suppose $M = \mathbb{E}^n$, i.e. $\mathbb{R}^n$ with the standard Euclidean metric. In this case, the curvature tensor vanishes, so Jacobi vector fields are linear, and if they vanish through endpoints, they must be zero. In other words, there are unique geodesics, so there can be no variations through geodesics. ◂

Example 23.9. Let $M = S^n$ with its usual metric. Then there are lots of geodesics from p to q, albeit a unique minimal one. If $q \ne -p$, there are no variations of geodesics in which p and q are fixed, so again there are no nonzero solutions to the Jacobi equations.

But if p and q are antipodal points, there are lots of variations, parameterized by the equatorial S^{n-1}, and the tangent space to this, $\ker(B_\gamma)$, is $(n-1)$-dimensional, realizing the upper bound of Proposition 23.6. ◂

Proposition 23.10. *Consider the linear map $\psi\colon T_pM\to T_qM$ defined as follows: giveb $\eta\in T_pM$, let X be the solution to the Jacobi equation with $X(0)=0$ and $\nabla_T X(0) = \eta$. Then let $\psi(\eta) \coloneqq X(1)$. This linear map is $\mathrm{d}(\exp_p)_T\colon T_pM\to T_qM$.*

The exponential map is defined on the entire tangent space because M is complete. Recall that this involves moving along the geodesic in the direction of T.

Proof. We have

$$\mathrm{d}(\exp_p)_T(\eta) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\exp_p(T+s\eta) = \mathrm{ev}_{t=1}\circ\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\exp_p(t(T+s\eta)), \tag{23.11}$$

and this is exactly what ψ is doing: $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\exp_p(t(T+s\eta))$ is exactly the Jacobi field with initial conditions $X(0)=0$ and $\nabla_T X(0) = \eta$. ⊠

Definition 23.12. We say p and q are *conjugate along* γ if there exists a nonzero Jacobi field X along γ with $X(0)=0$ and $X(1)=0$. Their *multiplicity* is the dimension of the space of such Jacobi fields.

If $q = \exp_p(T)$, then p and q are conjugate along γ iff T is a critical point of $\exp_p$; p and q are conjugate along some geodesic iff q is a critical value of $\exp_p$. Therefore for each p, the set of no-conjugate q is dense in M, by Sard's theorem.

Theorem 23.13 (Morse)**.** *For each $\tau \in [0,1]$, let $\gamma_\tau := \gamma|_{[0,\tau]}$, $B_\tau := B_{\gamma_\tau}$, and $\nu(\tau) := \dim\ker(\tau)$, the dimension of the space of Jacobi fields X along γ_τ with $X(0) = X(\tau) = 0$. Then, $\nu(\tau) = 0$ except at a finite set of τ, and*

$$\lambda = \operatorname{ind} B_\gamma = \sum_{\tau\in(0,1)} \nu(\tau). \tag{23.14}$$

TODO: then there was an example I missed.

Proof sketch. Let $\underline{t} := \{0 = t_0 < t_1 < \cdots < t_k = 1\}$ be a partition such that $\gamma([t_{i-1}, t_i]) \subset U_i$, such that any two points of U_i are joined by a unique minimal geodesic. We proved that $[0,1]$ has an open cover by such U_i, so we can pick such a $\underline{t}$.

In particular, $\gamma|_{[t_{i-1},t_i]}$ is minimal. Write $T_\gamma = V_0 \oplus V_1$, where

$$V_0 = \{x \in T_\gamma : x|_{[t_{i-1},t_i]} \text{ is Jacobi}\}. \tag{23.15}$$

So this is finite-dimensional, and T_1 is infinite-dimensional. More precisely,

$$V_0 = \bigoplus_{j=1}^{k-1} T_{\gamma(t_j)}M. \tag{23.16}$$

We can (and will) take $V_1 := \{Y \in T_\gamma \mid Y(t_j) = 0 \text{ for all } j\}$.

We claim that this direct-sum decomposition is orthogonal with respect to $B_\gamma = B_1$.[16] The next claim is that $B_1|_{V_1}$ is positive definite. First,

$$B_1(Y,Y) = \sum_i \frac{1}{2}\frac{\mathrm{d}^2 E_{t_{i-1}}^{t_i}}{\mathrm{d}s^2} \ge 0. \tag{23.17}$$

If $B_1(Y,Y) = 0$, you can check that $Y \in \ker(B_1)$, so $Y = 0$.

Next, let $\lambda(\tau) := \operatorname{ind} B_\tau$. Then,

(1) $\lambda\colon [0,1] \to \mathbb{Z}$ is monotonic nondecreasing,
(2) there's an $\varepsilon > 0$ such that $\lambda|_{[0,\varepsilon)} = 0$,
(3) λ is left continuous: for all $\tau \in [0,1]$, there's an $\varepsilon > 0$ such that $\lambda(\tau - \varepsilon) = \lambda(\tau)$, and
(4) for all $\tau \in [0,1]$, there's an ε such that $\lambda(\tau + \varepsilon) = \lambda(\tau) + \nu(\tau)$.

The upshot is that for all $k \in \operatorname{Im}(\lambda)$, $\lambda^{-1}(k) = (a,b]$ for some $a, b \in [0,1]$. The proofs of these claims lie in finite-dimensional linear algebra, as they are facts about families of bilinear forms. The second and third statements, for example, follow from the first and the fact that being negative definite is an open condition, so look at the subspace on which B is negative definite. For the fourth, you'd also use that being positive definite is an open condition; this argument is a little finickier, but the details are in Milnor. ⊠

We want to use Morse theory and geodesics to study $\Omega = \Omega_{p,q}$, as well as Ω^*, the space of C^0 paths from p to q. To do this we should put topologies on them, which we'll do with metrics.

For Ω^*, the function

$$d(\omega_1,\omega_2) := \max_{t\in[0,1]} d_M(\omega_1(t), \omega_2(t)), \tag{23.18}$$

where d_M denotes the geodesic distance on M. One can check this is a metric; its underlying topology is the compact-open topology on Ω^*.

For Ω, Milnor has a more complicated term, but we can instead use the difference of the energies of the curves:

$$d^*(\omega_1,\omega_2) = d(\omega_1,\omega_2) + |E(\omega_1), E(\omega_2)|. \tag{23.19}$$

The inclusion $i\colon \Omega \hookrightarrow \Omega^*$ is continuous, and even Lipschitz with constant 1.

Theorem 23.20. *In fact, i is a homotopy equivalence.*

[16] TODO: Dan said this aloud and I was catching up on the stuff directly above, so I missed this. Sorry!

This is an instance of a very general fact, as formulated by Palais: if you have a space of functions and "thicken it up" with functions with worse regularity, you often get a homotopy-equivalent space. Nonetheless, these spaces are infinite-dimensional, so we would like to approximate them with finite-dimensional CW complexes.

The energy functional $E\colon \Omega \to \mathbb{R}^{\geq 0}$ is continuous, and induces an increasing filtration on Ω: if $c \in \mathbb{R}$, let $\Omega^c := E^{-1}((-\infty, c])$. These spaces still aren't usually finite-dimensional, though we'll see that their homotopy types are small.

Let $\underline{t} := \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition of $[0,1]$ as before, and define

$$\Omega(\underline{t}) := \{\omega \in \Omega \mid \omega|_{t_{i-1},t_i} \text{ is geodesic}\}. \tag{23.21}$$

Then let $\Omega^c(\underline{t}) := \Omega(\underline{t}) \cap \Omega^c$. The *mesh* of $\underline{t}$ is $\max_i|t_i - t_{i-1}|$.

Proposition 23.22. *There is an $\varepsilon > 0$ such that if the mesh of $\underline{t}$ is less than ε, we can give the structure of a smooth manifold to $B := \operatorname{Int}\Omega^c(\underline{t})$ which is a deformation retract of Ω^c. Moreover,*

1. *in this setting, $E|_B$ is smooth,*
2. *for $a < c$, $B^a := \{\omega \in B \mid E(\omega) \leq a\}$ is compact and is a deformation retract of Ω^a, and*
3. $\operatorname{Crit}(E|_B)$ *is the space of unbroken geodesics γ of energy less than c, and the index and nullity agree with those of B_γ.*

Thus we can invoke Morse theory on finite-dimensional manifolds and apply it to study the topology of the infinite-dimensional path spaces. This is the approach Milnor chooses; alternatively, one could place a Banach manifold structure on the path spaces and do Morse theory there directly.

Invoking the finite-dimensional Morse theory we developed in the beginning of the class, we get:

Corollary 23.23. *If p and q are not conjugate by a geodesic of length at most $\sqrt{a}$, then Ω^a is homotopy equivalent to a CW complex with a cell of dimension λ for each geodesic of index λ with length at most $\sqrt{a}$ from p to q.*

Proof. Let C be the closed ball of radius $\sqrt{c}$ around p in M, in the geodesic distance. This is compact, and every $\omega \in \Omega^c$ has $\omega([0,1]) \subset C$: we know the length squared is less than the energy by Exercise 22.21, and the energy is less than c.

There's a $\delta > 0$ such that for all $x, y \in C$ with $d_M(x,y) < \delta$, there's a unique minimal geodesic from x to y of length less than δ. Let $\varepsilon := \delta^2/c$. If the mesh of $\underline{t}$ is less than ε and $\omega \in \Omega^c(\underline{t})$, then

$$\left(L_{t_{i-1}}^{t_i}\omega\right)^2 \leq (t_i - t_{i-1})E_{t_{i-1}}^{t_i}(\omega) \leq (t_i - t_{i-1})E(\omega) \leq (t_i - t_{i-1})c < \delta^2, \tag{23.24}$$

so $\omega|_{[t_{i-1},t_i]}$ is minimal.

The function $\operatorname{Int}\Omega^c(\underline{t}) \to M^{x(k-1)}$ sending

$$\omega \longmapsto (\omega(t_1), \ldots, \omega(t_{k-1})) \tag{23.25}$$

is a homeomorphism onto an open subset of $M^{x(k-1)}$; there's a picture for why it's a homeomorphism (TODO: which I wasn't able to get down), and the details are in Milnor. Therefore we can port the smooth structure over to B.

That E is smooth on B follows from the fact that

$$E|_B(\omega) = \sum_i \frac{d_M(\omega(t_{i-1}), \omega(t_i))^2}{t_i - t_{i-1}}. \tag{23.26}$$

Then part (2) is true because TODO. The last part follows because if γ is an unbroken geodesic of length less than $\sqrt{a}$, then $T_\gamma B$ is the space of broken Jacobi fields, which is V_1 from berore. ⊠

For our first application, we'll study geodesics on S^n, which we give the round metric with radius 1. If p and q are nonconjugate, they lie on a circle, and the minimal geodesic is also contained in this circle; it has index 0. The next geodesic, going around in the "other direction," has index $n-1$; the next has index $2(n-1)$; and so on, which we saw in an exercise. Putting this together, we learn things about the loop space of S^n.

Theorem 23.27. *There is a homotopy equivalence from ΩS^n to a CW complex with cells in dimension $k(n-1)$ for $k = 0, 1, 2, \ldots$.*

Corollary 23.28 (Morse)**.** *If $n > 2$, then for any Riemannian metric on S^n and $p, q \in S^n$ nonconjugate, there exist infinitely many geodesics from p to q.*

Well why is this? For $n > 2$, we know that the homology of the loop space is infinite. For $n = 2$, we might have differentials, since we have cells in all degrees (it turns out that the differentials don't kill the generators, but we haven't shown that).

The remaining applications are due to Bott, in the 1950s. We will assume p and q are conjugate; since we can have variations of geodesics, there can be nullity, so not quite Morse theory; this is what led Bott to develop Morse-Bott theory, where critical points can be degenerate.

Consider again S^n with the radius-1 round metric, and let p and q be antipodal. Then $\Omega^{\pi^2} \approx S^{n-1}$, because this is the minimal possible length of a curve, so we just get longitudes, and a longitude is determined by where it intersects the equatorial S^{n-1}.

Theorem 23.29. *Let M be a complete, connected Riemannian manifold, and $p, q \in M$ be of geodesic distance $\sqrt{d}$ from each other. Assume that*

(1) *Ω^d is a topological manifold, and*
(2) *every non-minimal geodesic has index at least λ_0.*

Then $\pi_i(\Omega, \Omega^d) = 0$ for $i \le \lambda_0 - 1$. In particular, the map $\pi_i\Omega^d \to \pi_i\Omega$ is an isomorphism if $i \le \lambda_0 - 2$.

Moreover, a basic theorem in algebraic topology is that $\pi_i\Omega \cong \pi_{i+1}M$, coming from the loop space and path space fibration.

Corollary 23.30 (Freudenthal suspension theorem)**.** *If $i \le 2n - 4$, $\pi_i S^{n-1} \to \pi_{i+1}S^n$ is an isomorphism.*

This is because the next critical point is in degree $2(n-1)$, and then we subtract 2. You have to check that this is the same map induced by suspension.

Next we turn to Bott periodicity. In the 1950s, there were lots of explicit computations of homotopy groups of Lie groups: spectral sequences were new and exciting,[17] and, for example, Toda made lots of computations of homotopy groups of the unitary group.

Consider the infinite unitary group U_∞, defined to be the colimit of the embeddings $\mathrm{U}_1 \hookrightarrow \mathrm{U}_2 \hookrightarrow \cdots$, topologized with the colimit topology. For every n, there is a fiber bundle

$$\begin{array}{ccc} \mathrm{U}_n & \longrightarrow & \mathrm{U}_{n+1} \\ & & \downarrow \\ & & \mathrm{U}_{n+1}/\mathrm{U}_n, \end{array} \tag{23.31}$$

in fact a principual U_n-bundle, and the base is diffeomorphic to S^{2n+1}. This is not a trivial bundle; it's trivial rationally (so the rational homotopy and homology groups are products of those of spheres, as with any compact Lie group), but the torsion story is very interesting.

Using the long exact sequence of homotopy groups associated to the fibration (23.31), we have:

Proposition 23.32. *The map $\pi_i\mathrm{U}_n \to \pi_i\mathrm{U}_{n+1}$ is an isomorphism if $i \le 2n - 1$, and is surjective if $i = 2n$. Moreover, if $i \ne 1$, this is also true for the corresponding map $\pi_i\mathrm{SU}_n \to \pi_i\mathrm{SU}_{n+1}$.*

Therefore these pass to the homotopy groups of U_∞, so are the "stable part" of the homotopy groups of U_n. Some of these are easy: $\pi_0\mathrm{U}_\infty = 0$, since each U_n is connected. $\pi_1\mathrm{U}_n = \mathbb{Z}$, and $\pi_2\mathrm{U}_n = 0$. There's a fiber bundle $\mathrm{SU}_n \to \mathrm{U}_n \to S^1$, so $\pi_3\mathrm{SU}_2 = \pi_3\mathrm{U}_2 = \mathbb{Z}$.

So we have a pattern 0, $\mathbb{Z}$, 0, $\mathbb{Z}$, and so on via a very nice pattern — which stopped at $\pi_{10}\mathrm{U}_\infty$, which Toda computed to not vanish. But Bott and Shapiro checked the calculation and showed it was incorrect, leading to the conjecture that this pattern continued forever. This it does, and this is the statement of Bott periodicity.

Theorem 23.33 (Bott periodicity)**.** *$\pi_i\mathrm{U}_\infty$ is 0 if i is even and $\mathbb{Z}$ if i is odd.*

We will approach the proof by studying Morse theory on the space of paths on $M = \mathrm{SU}_{n=2m}$ from I to $-I$. The tangent space at any point of M can be identified with the Lie algebra of SU_n, denoted $\mathfrak{su}_n$, is the algebra of skew-Hermitian matrices with trace zero.

[17] They're still exciting!

We will define a nice inner product on $\mathfrak{su}_2$ (meaning: both left and right invariant under the action of SU_2 on $\mathfrak{su}_2$), hence a Riemannian metric on SU_2. Specifically, for $T_1, T_2 \in \mathfrak{su}_n$,

$$\langle T_1, T_2 \rangle := -\operatorname{Re}(\operatorname{tr}(T_1 T_2)) = \operatorname{Re} \sum_{k,\ell} (T_1)^k_\ell \overline{(T_2)^k_\ell}. \tag{23.34}$$

The exponential map $\exp\colon T_I \mathrm{SU}_n \to \mathrm{SU}_n$ is literally the matrix exponential, as with all Lie groups.

If the matrix $T \in \mathfrak{su}_n$ is diagonalizable, with diagonal entries $ia_1, \dots, ia_n$ (imaginary because it's skew-Hermitian), then each $a_j = k_j \pi$, where k_j is an odd integer. The energy of the associated geodesic is

$$E(T_{\underline{k}}) = \pi^2 \sum_{i=1}^n k_i^2, \tag{23.35}$$

and $\Omega^{2m\pi^2}$ is diffeomorphic to the Grassmannian $\mathrm{Gr}_m(\mathbb{C}^{2m})$, because this is the homogeneous space $\mathrm{SU}_{2m}/\mathrm{S}(\mathrm{U}_m \times \mathrm{U}_m)$. This space is also diffeomorphic to $\mathrm{U}_{2m}/(\mathrm{U}_m \times \mathrm{U}_m)$ or $(\mathrm{U}_{2m}/\mathrm{U}_m)/\mathrm{U}_m$. Therefore we can write it as a quotient of the *Stiefel manifold* $\mathrm{St}_m(\mathbb{C}^{2m}) = \mathrm{U}_{2m}/\mathrm{U}_m$, which is the space of isometries $\mathbb{C}^m \hookrightarrow \mathbb{C}^{2m}$, by U_m.

Lemma 23.36. *For $i \le 2m$, $\pi_i \mathrm{St}_m(\mathbb{C}^{2m}) = 0$, and therefore $\pi_i \mathrm{Gr}_m(\mathbb{C}^{2m}) \to \pi_{i-1}\mathrm{U}_m$ is an isomorphism if $i \le 2m$.*

The key statement of Bott periodicity reduces to geodesics.

Proposition 23.37. *A nonminimal geodesic in SU_{2m} from I to $-I$ has index at least $2m+2$.*

Corollary 23.38. *If $i \le 2m$, there are isomorphisms $\pi_i \mathrm{Gr}_m(\mathbb{C}^{2m}) \cong \pi_i \Omega \mathrm{SU}_{2m} \cong \pi_{i+1}\mathrm{SU}_{2m}$ and $\pi_{i-1}\mathrm{U}_m$, establishing Bott periodicity.*

One can check that if X, Y, Z, and W are left-invariant vector fields on a Lie group G, then

$$\nabla_X Y = \frac{1}{2}[X, Y] \tag{23.39a}$$

$$\langle R(X,Y)Z, W \rangle = \frac{1}{4}\langle [X,Y], [Z,W] \rangle. \tag{23.39b}$$

Next (and harder to check), conjugate points along $\gamma(t) - = e^{t\Gamma}$ occur at $t = \pi/\sqrt{\mu}k$, where k is a nonzero integer and μ is an eigenvalue of the map $\phi\colon \mathfrak{g} \to \mathfrak{g}$ sending $X \mapsto R(T, X)T$. The multiplicity of the conjugate points is the number of such μ.

TODO: after that I didn't follow, but there was only a little more.