Topological phases and topological field theories

Arun Debray

April 1, 2020

Overview

- The math in this talk comes from the study of topological phases of matter in condensed-matter physics
- I'll first briefly discuss topological phases of matter and what we know about modeling them mathematically
- Lots that we don't know about modeling topological phases mathematically, but we can extract and solve some questions

Topological phases of matter

- Condensed-matter physicists discovered certain materials that didn't fit into the previously understood classification of phases of matter
  - e.g., take some alloy, cool it to a certain temperature, maybe apply a magnetic field…

Topological phases of matter

- Condensed-matter physicists discovered certain materials that didn't fit into the previously understood classification of phases of matter
  - e.g., take some alloy, cool it to a certain temperature, maybe apply a magnetic field...
- These materials behave in unusual ways
  - Example: particle-like excitations that are neither bosons nor fermions

Topological phases of matter

- Condensed-matter physicists discovered certain materials that didn’t fit into the previously understood classification of phases of matter
  - e.g., take some alloy, cool it to a certain temperature, maybe apply a magnetic field…
- These materials behave in unusual ways
  - Example: particle-like excitations that are neither bosons nor fermions
- So condensed-matter theorists set out to classify these phases

Modeling topological phases

- As usual in condensed-matter physics, use lattice Hamiltonian systems
- Triangulate the ambient manifold M
- Use the combinatorial data of the triangulation to write down a Hilbert space $\mathscr{H}$ and Hamiltonian $H\colon \mathscr{H} \to \mathscr{H}$
- These must be "local," built out of things which only depend on information within a specified radius

Modeling topological phases

- As in QM, $\mathscr{H}$ is the space of states; an eigenvector with eigenvalue E is a state of the system with energy E.

Modeling topological phases

- As in QM, $\mathscr{H}$ is the space of states; an eigenvector with eigenvalue E is a state of the system with energy E.
- Lowest-energy states, called *ground states*, are vacuum states, with no particles

Modeling topological phases

- As in QM, $\mathscr{H}$ is the space of states; an eigenvector with eigenvalue E is a state of the system with energy E.
- Lowest-energy states, called *ground states*, are vacuum states, with no particles
- The next lowest-energy states correspond to particles localized to specific regions of M
  - "Next-lowest energy state" requires a gapped Hamiltonian!

Modeling topological phases

- As in QM, $\mathscr{H}$ is the space of states; an eigenvector with eigenvalue E is a state of the system with energy E.
- Lowest-energy states, called *ground states*, are vacuum states, with no particles
- The next lowest-energy states correspond to particles localized to specific regions of M
  - “Next-lowest energy state” requires a gapped Hamiltonian!
- Two such systems expected to describe same physics (“in the same phase”) if one can be deformed into another without closing the gap.

Difficulties with a straightforward approach

- Sounds straightforward, right? Take a space of all Hamiltonians, remove the gapless ones, take π_0, and voilà!

Difficulties with a straightforward approach

- Sounds straightforward, right? Take a space of all Hamiltonians, remove the gapless ones, take π_0, and voilà!
- We're **nowhere near making this a reality**
  - Usual obstructions to making QFT mathematical
  - Also a few new surprises from condensed matter (e.g. fractons)

Low-energy approach

- We have to find another way to study topological phases

Low-energy approach

- We have to find another way to study topological phases
- **Key ansatz**: there should be a way to extract a topological field theory (TFT) in the mathematical sense, as due to Atiyah-Segal, from a lattice Hamiltonian system

Low-energy approach

- We have to find another way to study topological phases
- **Key ansatz**: there should be a way to extract a topological field theory (TFT) in the mathematical sense, as due to Atiyah-Segal, from a lattice Hamiltonian system
  - Should induce an equivalence between topological phases and suitable equivalence classes of TFTs
  - Borne out of physics ideas, but not yet a mathematical theorem or even precise conjecture
  - TFT is called the *low-energy limit* of the lattice system

Low-energy approach

- We have to find another way to study topological phases
- **Key ansatz**: there should be a way to extract a topological field theory (TFT) in the mathematical sense, as due to Atiyah-Segal, from a lattice Hamiltonian system
  - Should induce an equivalence between topological phases and suitable equivalence classes of TFTs
  - Borne out of physics ideas, but not yet a mathematical theorem or even precise conjecture
  - TFT is called the *low-energy limit* of the lattice system
- Concretely, state space of TFT on N^{n-1} should be the space of ground states of lattice model on N

Low-energy approach

- We have to find another way to study topological phases
- **Key ansatz**: there should be a way to extract a topological field theory (TFT) in the mathematical sense, as due to Atiyah-Segal, from a lattice Hamiltonian system
  - Should induce an equivalence between topological phases and suitable equivalence classes of TFTs
  - Borne out of physics ideas, but not yet a mathematical theorem or even precise conjecture
  - TFT is called the *low-energy limit* of the lattice system
- Concretely, state space of TFT on N^{n-1} should be the space of ground states of lattice model on N
- Though no general approach yet, we can study examples!

Example: the toric code

The toric code is the *Drosophila melanogaster* of this field. Let M be a closed d-manifold with a triangulation.

- Fields: the (discrete) groupoid $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$: pairs of a principal $\mathbb{Z}/2$-bundle P on the 1-skeleton of M with a trivialization ξ on the restriction to the 0-skeleton
- State space is $\mathscr{H} := \mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$. We'll denote states by φ

Example: the toric code

- Given a vertex v, let ψ_v be the involution on $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ switching the trivialization of at v. Define the operator $A_v \colon \mathscr{H} \to \mathscr{H}$ by $A_v(\varphi) := \varphi \circ \psi_v$.

Example: the toric code

- Given a vertex v, let ψ_v be the involution on $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ switching the trivialization of at v. Define the operator $A_v\colon \mathscr{H} \to \mathscr{H}$ by $A_v(\varphi) := \varphi \circ \psi_v$.
- Given a face f and a principal bundle $P \to M^1$, let $\mathrm{Hol}_P(f)$ denote the holonomy of P around f. Define the operator $B_f\colon \mathscr{H} \to \mathscr{H}$ with $B_f(\varphi)(P, \xi) := (-1)^{\mathrm{Hol}_P(f)} \varphi(P, \xi)$.

Example: the toric code

- Given a vertex v, let ψ_v be the involution on $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ switching the trivialization of at v. Define the operator $A_v\colon \mathscr{H} \to \mathscr{H}$ by $A_v(\varphi) := \varphi \circ \psi_v$.
- Given a face f and a principal bundle $P \to M^1$, let $\mathrm{Hol}_P(f)$ denote the holonomy of P around f. Define the operator $B_f\colon \mathscr{H} \to \mathscr{H}$ with $B_f(\varphi)(P,\xi) := (-1)^{\mathrm{Hol}_P(f)}\varphi(P,\xi)$.
- The Hamiltonian is

$$H := \sum_v \underbrace{\frac{1}{2}(1 - A_v)}_{H_v} + \sum_f \underbrace{\frac{1}{2}\left(1 - B_f\right)}_{H_f}.$$

The ground states of the toric code

- Our ansatz says to get at the toric code using its low-energy TFT
- We can see what such a TFT would have to look like by studying the spaces of ground states of the toric code on various manifolds

The ground states of the toric code

- The toric code Hamiltonian is very nice: the H_v and H_f terms are commuting projectors
- This means the space of ground states is the intersection of their kernels

The ground states of the toric code

- The toric code Hamiltonian is very nice: the H_v and H_f terms are commuting projectors
- This means the space of ground states is the intersection of their kernels
- $\ker(H_v)$ is functions not depending on the trivialization at v
- $\ker(H_f)$ is the functions which vanish on principal $\mathbb{Z}/2$-bundles $P \to M^1$ which don't extend across f

The ground states of the toric code

- Upshot: the space of ground states is the space of functions on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$
- This suggests that the low-energy TFT is $\mathbb{Z}/2$ *finite gauge theory* (aka *untwisted $\mathbb{Z}/2$-Dijkgraaf-Witten theory*), which assigns to a closed $(n-1)$-manifold M the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M)$

The ground states of the toric code

- Upshot: the space of ground states is the space of functions on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$
- This suggests that the low-energy TFT is $\mathbb{Z}/2$ *finite gauge theory* (aka *untwisted $\mathbb{Z}/2$-Dijkgraaf-Witten theory*), which assigns to a closed $(n-1)$-manifold M the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M)$
- What we can say mathematically: the state space of this TFT agrees with the space of ground states of the toric code
  - Can't see the values on (most) bordisms yet

The GDS model

- The *GDS model* is a closely related example
- The H_v term is modified by a sign
- This messes up the commutation relations, so the proof we just saw doesn't work

Main theorem

In this theorem, I will say what the low-energy TFT of the GDS model is (well, to the extent that can be done mathematically) using terms I haven't defined. I'll define those terms in the next part of the talk.

Main theorem

In this theorem, I will say what the low-energy TFT of the GDS model is (well, to the extent that can be done mathematically) using terms I haven't defined. I'll define those terms in the next part of the talk.

Theorem (D., 2018)

Let $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{C}}$ *be the* $\mathbb{Z}/2$*-gauge-gravity theory with Lagrangian* β *equal to the degree* n *part of* $w\alpha/(1+\alpha)$*, where* w *is the total Stiefel-Whitney class and* $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ *is the nontrivial element (thought of as a characteristic class of principal* $\mathbb{Z}/2$*-bundles).*

Then, Z *"is" the low-energy TFT of the GDS model, in that on any closed* $(n-1)$*-manifold* M*,* $Z(M)$ *is isomorphic to the space of ground states of the GDS model on* M*, and this isomorphism intertwines the natural* $MCG(M)$*-actions on each space.*

Invertible field theories

- The cobordism hypothesis says: classifying all TFTs is hard!
- Focus on an easier, but still interesting, subclass

Definition (Freed-Moore)

A topological field theory $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}$ is *invertible* if there is another TFT $Z'\colon \mathrm{Bord}_n \to \mathrm{Vect}$ such that $Z \otimes Z' \simeq \mathbf{1}$.

- Isomorphism classes of invertible TFTs (IFTs) form an abelian group under tensor product.

Invertible topological phases

- Want to define invertible topological phases (aka *symmetry-protected topological* (SPT) phases) similarly: a phase is invertible if there is another phase such that when you tensor them together, you get the trivial phase
- "Tensor" is *stacking*, placing both phases on the same material, but with no interactions between them
  - Explicitly: tensor Hilbert spaces of states together; Hamiltonian is $H := H_1 \otimes 1 + 1 \otimes H_2$
- Makes sense from physics POV, but not yet a mathematical definition
- The ansatz specializes: taking the low-energy TFT should produce an equivalence between the classifications of SPT phases and of IFTs

Classification of invertible TFTs

- If A, B are commutative monoids, $f : A \to B$ an *invertible* homomorphism (i.e. $\mathrm{Im}(f) \subset B^\times$), we can extend f to $K_0(A)$, the abelian group obtained by formally inverting all elements of A
  - Recipe: $f(x^{-1}) := f(x)^{-1}$, which exists because $f(x) \in B^\times$

Classification of invertible TFTs

- If A, B are commutative monoids, $f\colon A \to B$ an *invertible* homomorphism (i.e. $\mathrm{Im}(f) \subset B^\times$), we can extend f to $K_0(A)$, the abelian group obtained by formally inverting all elements of A
  - Recipe: $f(x^{-1}) := f(x)^{-1}$, which exists because $f(x) \in B^\times$
- Maps of abelian groups $K_0(A) \to B^\times$ are in natural bijection with invertible maps $A \to B$

Classification of invertible TFTs

- This argument categorifies: replace commutative monoids with symmetric monoidal categories, and replace abelian groups with spectra

Classification of invertible TFTs

- This argument categorifies: replace commutative monoids with symmetric monoidal categories, and replace abelian groups with spectra

Theorem (Freed-Hopkins, following Galatius-Madsen-Tillmann-Weiss, Schommer-Pries)

There's a natural isomorphism between the abelian group of n-dimensional invertible TFTs with G-structure valued in $\mathrm{sVect}_{\mathbb{C}}$ *and* $[\Sigma^n MTG_n, \Sigma^n I\mathbb{C}^\times]$.

Classification of invertible TFTs

- MTG_n is a *Madsen-Tillmann spectrum* (pull back the negative of the tautological bundle along $BG_n \to B\mathrm{O}_n$, take Thom spectrum)
- $I\mathbb{C}^\times$ is the Pontrjagin dual of the sphere, characterized by $[E, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}_{\mathrm{Ab}}(\pi_n E, \mathbb{C}^\times)$.

Classification of invertible TFTs

- *MTG_n is a Madsen-Tillmann spectrum* (pull back the negative of the tautological bundle along $BG_n \to B\mathrm{O}_n$, take Thom spectrum)
- $I\mathbb{C}^\times$ is the Pontrjagin dual of the sphere, characterized by $[E, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}_{\mathrm{Ab}}(\pi_n E, \mathbb{C}^\times)$.
- Note: Freed-Hopkins also prove a version for extended TFTs

- $\pi_n(MTG_n)$ is a bordism group, but under a stricter equivalence relation than ordinary G-bordism
- Thus ordinary bordism invariants define these kinds of bordism invariants, giving IFTs
- Freed-Hopkins prove that for *reflection-positive* IFTs, you get precisely ordinary bordism invariants
  - (Modulo a very believable conjecture)

Some examples

- Fix a finite group G and $\beta \in H^n(BG; \mathbb{R}/\mathbb{Z})$. This defines a $\mathbb{C}^\times$-valued bordism invariant of oriented manifolds with a principal G-bundle: use the classifying map $f \colon M \to BG$ to pull back β; evaluate on the fundamental class, then exponentiate: $\exp(2\pi i \langle f^*\beta, [M]\rangle) \in \mathbb{C}^\times$. These give invertible TFTs called *classical Dijkgraaf-Witten theories*
  - First constructed by Freed-Quinn by other means
  - “Classical” here means that the cohomology class plays the role of a Lagrangian in a classical gauge theory

Some examples

- Slight variant: take $\beta \in H^n(BG; \mathbb{Z}/2)$, multiply it with Stiefel-Whitney classes of M, then evaluate and exponentiate as before, defining invertible TFTs called *classical gauge-gravity theories*
  - "gauge-gravity" indicates the Lagrangian has terms corresponding to the principal bundle ("gauge") and a characteristic class of the underlying manifold ("gravity")
- Other interesting IFTs from bordism invariants: Arf theory, Arf-Brown-Kervaire theory,…

Some upshots

- Compare this classification of IFTs to preexisting classifications of SPTs by other (physics) methods.
    - Freed-Hopkins, J. Campbell
    - The classifications match, a good sign for the ansatz

Some upshots

- Compare this classification of IFTs to preexisting classifications of SPTs by other (physics) methods.
    - Freed-Hopkins, J. Campbell
    - The classifications match, a good sign for the ansatz
- Use this classification to construct invertible TFTs
    - Then use those to construct more TFTs
    - Convenient way to define the TFT I used to get the spaces of ground states of the GDS model

Producing the quantum theory

- To obtain the gauge-gravity theory in the theorem statement, one "quantizes" the classical theory $Z_\beta^{c\ell}$
- Specifically, a finite form of path integral quantization: sum over principal $\mathbb{Z}/2$-bundles

Producing the quantum theory

- To obtain the gauge-gravity theory in the theorem statement, one "quantizes" the classical theory $Z_\beta^{c\ell}$
- Specifically, a finite form of path integral quantization: sum over principal $\mathbb{Z}/2$-bundles
  - M closed, codimension 0, this is a weighted sum of $Z_\beta^{c\ell}(M,P)$ for $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$
  - N closed, codimension 1, this is the sections of a vector bundle over the groupoid $\mathrm{Bun}_{\mathbb{Z}/2}(N)$; the fiber at P is $Z_\beta^{c\ell}(N,P)$

Producing the quantum theory

- To obtain the gauge-gravity theory in the theorem statement, one "quantizes" the classical theory $Z_\beta^{c\ell}$
- Specifically, a finite form of path integral quantization: sum over principal $\mathbb{Z}/2$-bundles
  - M closed, codimension 0, this is a weighted sum of $Z_\beta^{c\ell}(M,P)$ for $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$
  - N closed, codimension 1, this is the sections of a vector bundle over the groupoid $\mathrm{Bun}_{\mathbb{Z}/2}(N)$; the fiber at P is $Z_\beta^{c\ell}(N,P)$
- $\mathbb{Z}/2$ is finite, so these are finite sums, hence can be (and are) defined as mathematical operations on TFTs

Main theorem, redux

Theorem (D., 2018)

Let $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{C}}$ *be the* $\mathbb{Z}/2$*-gauge-gravity theory with Lagrangian* β *equal to the degree* n *part of* $w\alpha/(1+\alpha)$*, where* w *is the total Stiefel-Whitney class and* $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ *is the nontrivial element (thought of as a characteristic class of principal* $\mathbb{Z}/2$*-bundles).*

Then, Z *"is" the low-energy TFT of the GDS model, in that on any closed* $(n-1)$*-manifold* M*,* $Z(M)$ *is isomorphic to the space of ground states of the GDS model on* M*, and this isomorphism intertwines the natural* $MCG(M)$*-actions on each space.*

A few things we don't know

1. Extracting more of the TFT from the lattice model

A few things we don't know

1. Extracting more of the TFT from the lattice model
2. Comparing with conclusions of Fidkowski, Haah, Hastings, and Tantivasadakarn

A few things we don't know

1. Extracting more of the TFT from the lattice model
2. Comparing with conclusions of Fidkowski, Haah, Hastings, and Tantivasadakarn
3. Studying less-understood variants of topological phases
   - Fractons, higher-order SPTs
   - *Crystalline phases*: the symmetry group can act on space
   - Work in progress comparing a proposal by Freed-Hopkins to physicists' calculations by other methods