Physics 40 Series Notes

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December 9, 2012

Contents

4 Kinematics in Two Dimensions

4.1 Acceleration

Most of this isn't terribly new, but they do decompose $\mathbf{a}$ into $\mathbf{a}_{\parallel}$, the component parallel to $\mathbf{v}$, and $\mathbf{a}_{\perp}$, the component normal to $\mathbf{v}$. The former is responsible for the change in speed, and the latter for the change in direction of velocity.

4.2 Two-Dimensional Kinematics

One can view two-dimensional kinematics problems as pairs of one-dimensional problems, as the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ all expand into $\mathbf{v}_x\hat{\imath} + \mathbf{v}_y\hat{\jmath}$ and so on. Not much new here.
The book also uses θ to decompose the velocity, which is standard trig: $v_x = v\cos\theta$ and $v_y = v\sin\theta$, so $v_x^2 + v_y^2 = v^2$.

4.3 Projectile Motion

A projectile moves in two dimensions under the influence of only gravity. Generally, air resistance is ignored, so projectiles follow parabolic trajectories.
The start of a projectile's motion occurs at a launch angle θ, so that $v_{ix} = v_i\cos\theta$ and $v_{iy} = v_i\sin\theta$, where v_i is the initial speed. (Remember that due to sign conventions, not all of these values are necesarily positive.)
Additionally, as in high school physics, $a_x = 0$ and $a_y = -g$.
Then, the parametric equations are two one-dimensional problems: $x_f = x_i + v_{ix}\Delta t$, $y_f = y_i + v_{iy}\Delta t - \frac{1}{2}g(\Delta t)^2$, v_x is constant, and $v_{fy} = v_{iy} - g\Delta t$. (Here, I have followed the textbook's notation, even though I don't particularly like it.)
With a little rearranging, one obtains $y = y_0 - \frac{gx^2}{2v_0^2}$, proving that projectile motion is parabolic.
One can also use some intuitive reasoning for this, thinking of a parabolic trajectory as how the object "falls" $\frac{gt^2}{2}$ from the straight-line path it would take without gravity. Additionally, knowing that the horizontal and vertical components have no effects on each other explains that a projectile launched horizontally falls in the same time as one that is dropped at rest.
Another shortcut equation is that a projectile that lands at the same elevation it was fired from travels distance $x = \frac{v_0^2\sin(2\theta)}{g}$, so the maximum distance happens when $\theta = 45°$. Also, projectiles launched at θ_1 and θ_2, where $\theta_1 + \theta_2 = 90°$, travel the same distance.

4.4 Relative Motion

This section deals with the idea that if two people use different coordinate systems, they may get different results for kinematics, even though they agree on relative positions and velocities.
A reference frame is a coordinate system that someone makes measurements in. It can have its own position or even velocity.
The laws of motion are valid in inertial reference frames, for which: 1. the orientations are all the same ($x /\!/ x'$), 2. the origins are the same at $t = 0$, 3. the z-axis is not being considered, and 4. the relative velocity of the two frames, $\mathbf{V}$, is constant.
Using the Galilean transformation of position, one can map from one reference frame to another. If $\mathbf{r}$ is measured in a reference frame S and $\mathbf{r}'$ is measured in S', let $\mathbf{R}$ be the difference in their positions (i.e. relative position) and $\mathbf{V}$ be their relatie velocity. Then, $\mathbf{r} = \mathbf{r}' + \mathbf{R} = \mathbf{r}' + t\mathbf{V}$. This can be broken down into components, for which it is most useful: $x = x' + V_x t$ and $y = y' + V_y t$. Think of this in terms of transformations.
One interesting consequence of this is that free-fall and projectile motion are the same motion, just under different reference frames. For a reference frame with $\mathbf{V} = \mathbf{v}_x$, projectile motion looks like free fall.
The Galilean tranformation of velocity is not very different: $\mathbf{v} = \mathbf{v}' + \mathbf{V}$ so $v_x = v_x' + V_x$ and $v_y = v_y' + V_y$. Once again, $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of an object measured relative to different reference frames, and $\mathbf{V}$ is the velocity of the frame itself.

4.5 Uniform Circular Motion

"The best way to handle circular motion is with circular logic."

Uniform circular motion occurs when a particle moves at a constant speed around a circle of radius r. In this case, $\mathbf{v}$ is always tangent to the circle, and it always has the same length.
The time in which the particle completes one revolution of the circle is the period, or T. It is related to v, the speed, in that $v = \frac{2\pi r}{T}$.
It can be helpful to define the position in polar-like coordinates, and so the angular position of the particle is the coordinate θ. Generally, radians are used, which make calculating arc length s awfully convenient: $s = r\theta$. (The radian is a defined dimensionless unit, which means it can appear or disappear without warning. With practice, this becomes less sketchy.)
Angular displacement is the change $\Delta\theta = \theta_t - \theta_i$. It is very analogous to the linear displacement defined earlier. In fact, one can

define average angular velocity as $\frac{\Delta\theta}{\Delta t}$ and instantaneous angular velocity as $\omega \equiv \frac{\mathrm{d}\theta}{\mathrm{d}t}$. The SI units are $\frac{\mathrm{rad}}{\mathrm{s}}$, but degrees per second, revolutions per second, and revolutions per minute (rpm) are also used.
Notice that a particle is in uniform circular motion iff ω is constant.
ω can be either positive or negative, just like the linear v, and the notions of slope correspond nicely, too. Thus, $\theta_f = \theta_i + \omega\Delta t$. (Once again, this looks familiar. I think you get the picture.)
Angular velocity is also closely related to the period: $|\omega| = \frac{2\pi}{T}$.

4.6 Velocity and Acceleration in Uniform Circular Motion

For a particle in circular motion (must it be uniform?), the velocity vector is tangent to the circle, so it only has a tangential component v_t. It is the rate of change of arc length with respect to time, or $\frac{\mathrm{d}s}{\mathrm{d}t}$; since $s = r\theta$, this gives $v_t = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega r$ (provided ω is in SI units). Thus, v_t is positive for counterclockwise motion and negative for clockwise motion (which is just a sign convention). Because the normal component of velocity is zero, the particle's speed is just $v = |v_t| = |\omega|r$. (If the sign of ω is clear the absolute value signs may be dropped.)
The acceleration of uniform circular motion is called centripetal acceleration. This is not a new force, but just a naming convention. This acceleration always points towards the center of the circle. Thus, $\mathbf{v} \perp \mathbf{a}$. Since $\Delta\mathbf{v}$ always has the same length, the magnitude of the centripetal acceleration is constant. With a little math, one can show that $a = |\mathbf{a}| = \frac{v^2}{r} = \omega^2 r$ (because $v = \omega r$). (Note: since the direction is changing, the acceleration is not constant, even though its magnitude is.)

4.7 Nonuniform Circular Motion and Angular Acceleration

Nonuniform circular motion is, unsurprisingly, circular motion in which the speed changes. In this case, the acceleration is not always in the direction of the center; the tangential acceleration a_t is in the direction of the velocity v_t, in addition to the radial acceleration (or centripetal acceleration) a_r. The acceleration vector points ahead of the center, so to speak, if the particle speeds up, and behind the center if it slows down.
The magnitude of the overall acceleration is $a = \sqrt{a_r^2 + a_t^2}$. Additionally, $a_t = \frac{\mathrm{d}v_t}{\mathrm{d}t}$.
If a_t is constant, kinematics come to the rescue: $s_f = s_i + v_{it}\Delta t + \frac{a_t(\Delta t)^2}{2}$, and $v_{ft} = v_{it}s + a_t\Delta t$.
Circular motion is also applicable to rotating solid objects because $\Delta\theta_1 = \Delta\theta_2$ even if their radii are different. As such, their angular velocities are identical, so one can refer to the angular velocity ω of the wheel.
If the rotation is nonuniform, then the tangential acceleration (a_t) is nonzero, and $a_t = r\frac{\mathrm{d}\omega}{\mathrm{d}t}$.
One can also define angular acceleration $\alpha \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t}$, with units of radians per second squared. (It's got all the implications and tricks of regular velocity, so be careful; in particular, speeding up while rotiating counterclockwise causes a negative angular acceleration).
So $a_t = r\alpha$, so two points on a wheel have the same angular acceleration but (usually) different tangential accelerations. Compare this equation with $v_t = r\omega$ for tangential and angular velocity.
(Once again, the idea of α as the slope of ω with respect to t still holds.)
The rotational kinematic equations, each of which has a linear counterpart, are $\theta_f = \theta_i + \omega_i\Delta t + \frac{1}{2}\alpha(\Delta t)^2$, $\omega_f = \omega_i + \alpha t$, and $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$.

5 Force and Motion

The first four subsections have been skipped due to complete redundancy.

5.1 Newton's Second Law

The net force of an object subject to forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ is $\mathbf{F}_{\text{net}} = \sum_{i=1}^{n} \mathbf{F}_i$. An object of mass m subject to forces $\mathbf{F}_i$ will undergo acceleration $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$; this means that an object accelerates in the direction of the net force. (The form $\mathbf{F} = m\mathbf{a}$ is also seen, and is of course equivalent.)
The units of force are Newtons, or N. $1\,\mathrm{N} = 1\frac{\mathrm{kg\,m}}{\mathrm{s}^2}$. One Newton is the amount of force necessary to accelerate a mass of 1 kg at 1 $\frac{\mathrm{m}}{\mathrm{s}^2}$.
Conceptually, a force is an interaction between two objects, which is in the spirit of Newton's Third Law.

5.2 Newton's First Law

Since the time of the Greeks, scientists have wondered what the "natural state" of an object is Aristotle concluded it was an object at rest, but eventually Newton discovered it is when $\mathbf{a} = \mathbf{0}$. Newton's first law, or the Law of Inertia, states that an object at rest will remain at rest, and an object in motion will move in a straight line with constant velocity iff the net force on the object is zero.
An object for which $\mathbf{F}_{\text{net}} = \mathbf{0}$ is in mechanical equilibrium; if it is at rest, it is called static equilibrium, and if it is in motion, dynamic equilibrium.
This law allows for a more formal definition of a force (i.e. that which changes an object's velocity). However, thanks to various

forms of friction, this is at odds with our everyday sense of intuition. (There is no such thing as a "force of motion," for example, as that's just what we think of as overcoming friction. Newton's First Law and inertia do the rest.) Another common misconception is that air pressure is a significant force in many problems. It certainly can be, but since air pressure is exerted in all directions, the net air pressure is effectively zero. Exceptions to this include vacuum suction.

Newton's first law seems to be a special case of the second law when $\mathbf{a} = \mathbf{0}$, which would imply it is unnecessary, but since the definition of a force in the second law stems from the first law, it is necessary to help develop the second law. Additionally, inertial reference frames can help disambiguate the two. As defined in section 4.4, inertial reference frames are ones in which Newton's Laws are valid. The First Law is a very good way to check if a frame is inertial (if the frame is not accelerating). This means, though, that the Earth is not an inertial reference frame, since it is undergoing uniform circular motion (and rotation). However, this acceleration is not large enough to make a significant difference in most experiments.

(The last subsection has also been skipped due to complete redundancy.)

6 Dynamics I: Motion Along a Line

6.1 Equilibrium

An object on which the net force is zero is said to be in equilibrium (which might be static or dynamic; see section 5.2). To solve equilibrium problems, set $\mathbf{F}_{\text{net}} = \mathbf{0}$ and solve the vector equations ($\sum_i (F_i)_x = 0$, etc.). These problems pop up frequently (e.g. the last problem on the second problem set, which required some vector magic but was otherwise pretty simple). An example of dynamic equilibrium would be towing a car up a frictionless hill at a constant velocity, for which the gravitational, normal, and tension forces all cancel each other out.

6.2 Using Newton's Second Law

For cases where the acceleration is nonzero, one is often asked to find velocities or positions. This either means forces are given (so that one finds acceleration and then uses kinematics to solve for the rest), or that position or velocity is given (so that one finds acceleration and then computes the required force).

6.3 Mass, Weight, and Gravity

Though these are interchangeable terms in common parlance, they have quite different meanings in science and engineering. Mass is an intrinsic property of an object that describes its inertia, and in a sense also measures how much matter is in the object. It is a scalar.

Gravity is an attractive, long-range force between any two objects. If the two objects have masses m_1 and m_2 and are separated by a distance r, then the gravitational force is $F = \frac{Gm_1m_2}{r^2}$, where $G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$. G is one of the fundamental constants of nature. Observe that the gravitational force is a vector in the direction from one object to the other, but Newton's Law of Gravitation, the above equation, only gives the magnitude. Gravitational force gets weaker as objects get farther away, and it's already pretty weak; the force between objects on a human scale is negligible.

This complicated force can be approximated in many settings. For example, if an object is at a height h above the surface of a planet with radius R, then if $h \ll R$ the magnitude can be approximated as $\frac{Gm_1m_2}{R^2}$.

Another technique is the flat-Earth approximation, which works best when close to the surface. In this case, $\mathbf{F}_G \approx -mg\hat{\jmath}$, where $-\hat{\jmath}$ points straight down and $g = \frac{Gm_1}{R^2}$ (if $m_1 \gg m_2$). g, sometimes called the gravitational field of a planet, is a property of a specific planet. And it happens to coincide with the other use for g introduced so far: the accelertion due to gravity. The mass cancels out of the equation, showing that all objects have the same free-fall acceleration regardless of mass (when not comparable to the planet's mass, so that the approximation is still valid). Newton's law of gravitation predicts that $g = 9.83 \frac{\text{m}}{\text{s}^2}$; the discrepancy is due to the Earth's rotation.

Using the reference frame of the Earth, Newton's laws are technially invalid, because of its rotation, but near the surface the difference is miniscule. (This is the same idea behind artificial gravity in space stations in science fiction, for which a new reference frame is defined for rotation).

Weight is (as Dr. Simon would say) an entirely different animal. Weight is a measurement of force (and therefore measured in Newtons). (The definition in the book is a bit fuzzy, in that it relies on a specific stationary calibrated scale...)

One interesting implication is that one cannot feel his own weight, only acceleration or deceleration. The weight changes if one's vertical acceleration is nonzero (as hundreds of elevator homework problems will soon attest). As such, any object in free fall has no weight, which produces the sensation of weightlessness found on the Space Shuttle.

6.4 Friction

Static friction, $\mathbf{f}_s$, is the force on an object that keeps it from slipping. If a force $\mathbf{F}$ is exerted on an object that is not moving, it can be shown that $f_s = F$, that their magnitudes are equal, and that they point in opposite directions. Thus, the magnitude of

static friction depends on how hard one pushes; it is a responsive force.
The static friction force for any given object has a maximum magnitude $f_{s\,\max}$, so that if $f_s < f_{s\,\max}$, the object remains at rest; if $f_{s\,\max} = f_s$, the object slips, and $f_{s\,\max} < f_s$ is not possible.
Experiments have shown that $f_{s\,\max}$ is proportional to the magnitude of the normal force. The constant of proportionality, μ_s, is a dimensionless quantity called the coefficient of static friction: $f_{s\,\max} = \mu_s n$.
Once an object has started to slide, kinetic friction, or $\mathbf{f}_k$, takes over. This force has a nearly constant magnitude, and $f_k < f_{s\,\max}$, which is why it is easier to move something than to start it moving. The direction of $\mathbf{f}_k$ is always opposite the direction of motion. The kinetic friction force is also proportional to the magnitude of the normal force: $f_k = \mu_k n$, where μ_k is the coefficient of kinetic friction. It can be shown that $\mu_k < \mu_s$.
A third type of friction is rolling friction, which occurs because the portion of a wheel that is in contact with the ground is stationary with respect to the surface rather then sliding. Since no wheel is perfectly round, the bottom of a wheel flattens slightly, and the friction between this flat part of the wheel (large for rubber wheels, much less for steel wheels — but still greater than zero) is called rolling friction. Once again, there is a constant of proportionality, the coefficient of rolling friction μ_r, with the magnitude of the normal force, and $f_r = \mu_r n$. The direction of rolling friction is opposite the direction of motion.
These equations are models, not laws; they have some minor inaccuracies, in particular not accounting for surface area or speed, which have very minor effects.
Consider an object on an inclined plane. At some angle θ, called the angle of repose, the component of gravitational force in the direction of inclination is enough to overcome static friction, and the object begins to slip.
Friction occurs because, on the microscopic scale, objects are irregularly shaped and the high and low points jam up into each other. Additionally, chemical bonds may moentarily form, which require energy to break. This is most notable for static friction, which is why f_k is smaller.

6.5 Drag

Drag ($\mathbf{D}$) is conceptually similar to friction, but is a force exerted by air. It is opposite in direction to $\mathbf{v}$ and increases in magnitude as the object's speed increases. This makes it more complex than conventional friction, especially because it depends on the shape of an object.
One model of drag is nearly accurate for objects on a human scale moving at speeds we are familiar with and in the troposphere. For these objects, the magnitude of drag is $\frac{Av^2}{4}$, where A is the cross-sectional area of the object in the direction of drag; the direction is opposite the direction of motion. (If you do the dimensional analysis closely enough, the $\frac{1}{4}$ has units of $\frac{\text{kg}}{\text{m}^3}$).
Because drag depends on an object's size, but not its mass, it has a larger effect on less massive and less dense objects.
Drag also creates a terminal speed v_{term}, at which $D = F_G$ for a falling object. This causes the net force to equal zero. A quick calculation shows that $v_{\text{term}} = \sqrt{\frac{4mg}{A}}$; thus, a more massive object has a higher terminal speed than a less massive one with same cros-sectional area. Though this is most commonly applied to free fall, it works just as well for horizontal motion (though g will have to be replaced, of course).
(The last section just consists of more examples.)

7 Newton's Third Law

7.1 Interacting Objects

An interaction is the mutual influence of two objects on each other; in particular, each object exerts an equal and opposite force on the other. This pair of forces is called an action/reaction pair. (Notation is often something like $\mathbf{F}_{A \text{ on } B}$; it is important to distinguish the agent of the force from the recipient.) However, the name "action-reaction pair" is somewhat misleading in that neither is explicitly the action or the reaction. Both exist simultaneously or not at all.
Thus far, only forces acting on a single object have been considered (as in free-body diagrams). However, for Newton's Third Law, one needs to consider systems and their environments (objects external to the system of objects one wishes to analyze). This leads to a new type of diagram, the interaction diagram. The aim is to capture internal interactions and external forces (the forces the environment exerts on the system).

7.2 Analyzing Interacting Objects

After drawing an interaction diagram, in which every force between two objects is represented as a line, draw free-body diagrams for every object in the system. it is customary to connect two action-reaction forces with a dashed line.
Friction is an example of a propulsion force; in order to move forward, a person exerts a backwards static frictional force on the ground, and the corresponding reaction force pushes forward on the person, who thus accelerates forward. Other examples of propulsion include rocket thrust (though they don't push against anything; rather, they expel gases that push the rocket forward).

7.3 Newton's Third Law

. . . is that every action has an equal and opposite reaction; same magnitude, different directions. It characterizes force as an interaction between objects, and shows that the action/reaction forces happen on different objects.
Importantly, the forces are equal, but the accelerations aren't if the masses are unequal. This is why the gravitational force of a tennis ball on the Earth can be effectively ignored. Generally, the lighter mass will do most or nearly all of the accelerating.
Another helpful piece of information for solving a problem is an acceleration constraint: two objects that remain in contact necessarily have the same acceleration. (Sometimes, as in the first problem of Homework 3, they have the same magnitude but different signs, so be careful.)

7.4 Ropes and Pulleys: a closer look at tension

The key aspect of tension is that it pulls equally in both directions; since any particular cross-section of a rope is in equilibrium, the forces acting on it in each direction of the rope must have equal magnitudes.
The massless spring approximation can be used when the mass of the rope is small relative to the objects it connects, and states that $T_{A \text{ on } B} = T_{B \text{ on } A}$, or that the tension in a massless string is constant. This means that A and B can be treated as an action/reaction pair if connected by a massless string!
Pulleys often get involved in these sorts of problems. The magnitude of the tension on a massless string that goes through a pulley is constant if the pulley is both massless and frictionless.

(The last section consists only of example problems.)

8 Dynamics II: Motion in a Plane

8.1 Dynamics in Two Dimensions

Like in kinematics, this is exactly like solving two one-dimensional dynamics problems. However, it will be trickier than the previous dynamics problems because you can't necessarily pick an axis for which one of a_x or a_y is zero. In any case, solve for accelerations, forces, and velocities as before.
This chapter also uses Newton's laws to prove that projectiles travel in parabolic paths, which is more or less redundant.

8.2 Velocity and Acceleration in Uniform Circular Motion

For this subject, the rtz-coordinate system will be more useful than the xyz system. Consider a particle in a circular trajectory and let the r-axis (the radial axis) point from the particle (which represents the origin) to the center of the circle, the t-axis be tangential to the circle, pointing counterclockwise, and the z-axis be perpendicular to the plane of motion (basically the same z-axis as before). These three axes are orthogonal[1], but the axes move along with the particle, which is new.
A vector $\mathbf{A}$ in the plane of motion which makes an angle ϕ with the r-axis can be decomposed into radial and tangential components $A_r = A\cos\phi$ and $A_t = A\sin\phi$. Since $\mathbf{A}$ is in the plane of motion, $A_z = 0$. (This relates to the idea in section 4.1 that acceleration can be decomposed into an orthogonal component and a normal component.)
For a particle in uniform circular motion, $\mathbf{v}$ only has a tangential component v_t, and the radial and perpendicular components are zero. Thus, v_t is the rate at which the particle moves around the circle, so $v_t = \frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega r$. (Note that one should be careful with signs and units here; the magnitude of velocity will be nonnegative, but ω might not be.) The speed is $|\omega|r = v$ (sometimes the absolute value sign is dropped, so b cautious.)
Similarly, the acceleration only has a radial component $a_r = \frac{v^2}{r} = \omega^2 r$. Often, a_r is referred to as centripetal acceleration.

8.3 Dynamics of Uniform Circular Motion

A particle in uniform circular motion undergoes acceleration, and thus necessarily has a net force of magnitude $\frac{mv^2}{r}$ to prevent it from moving tangent to the circle. Thus, $\mathbf{F}_{\text{net}}$ points along the radial axis, which makes its components zero except for $(F_{\text{net}})_r = ma_r = \frac{mv^2}{r} = m\omega^2 r$. This is the key identity for these sorts of problems. (Here the lecture notes are tied in, as in notions of cars going around banked curves and experiencing static or kinetic friction.)

8.4 Circular Orbits

Orbits are an extension of projectile motion. Specifically, at some point one cannot assume the world is flat and that $\mathbf{a}_g$ always points straight down. When these assumptions are removed, a projectile launched with a sufficiently large v_{x0} will find that the ground curves out from under it as it falls. The trajectory may parallel the planet's curvature, in which case it forms a closed

[1] i.e. mutually perpendicular.

circular orbit instead of falling back down to the ground or escaping orbit.
Importantly, an orbiting projectile is in free fall. Odd, yes, but it explins why astronauts feel weightless even though there is still gravity in space.
Since stars and planets are nearly spherical, the gravitational force $\mathbf{F}_G$ points towards the center, though it still has magnitude mg. Thus, the acceleration is $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$, so it has magnitude g and points towards the center of the planet or star. Since centripetal acceleration is $a = \frac{v^2}{r}$, then the orbital velocity is just $v_{\text{orbit}} = \sqrt{rg}$. (This equality is necessary for the object to remain in a uniform circular orbit.) This can be used to calculate the period of an orbit, $T = \frac{2\pi r}{v_{\text{orbit}}} = 2\pi\sqrt{\frac{r}{g}}$.

8.5 Fictitious Forces

This section looks pretty trivial. Some actions commonly thought of as forces don't exist, because they are simple manifestations of other forces.
In a decelerating car, one feels pushed into the windshield, but this isn't a real force because the car isn't a valid inertial reference frame.
Similarly, a centrifugal force which pushes a person in a turning car away from the center of the turning circle is a fictitious force because the car undergoes acceleration. The rest of the car is moving at the person, not the other way around.[2]
Another technicality is that the Earth rotates, so it isn't a true inertial reference frame. But a minor correction and adding a fictitious force allows it to be seen as one: add to F_G a force with magnitude $-m\omega^2 R$ in the same direction. The new $\mathbf{F}_G$ is called the effective gravitational force that is slightly weaker than the real gravitational force. (On Earth, the difference is about $0.033\frac{\text{m}}{\text{s}^2}$, depending on latitude.)

8.6 Why Does the Water Stay in the Bucket?

This section deals with motion in a vertical circle, which is not uniform circular motion (the constant gravitational force implies the magnitude of the acceleration cannot be constant).
Generally, these situations can be simplified to setting the centripetal and gravitational forces against each other (though of course they sometimes point in the same direction). The car speeds up at the bottom and slows down at the top, up to a point v_{top} where the normal force is zero; were either any less, then the circular motion would be interrupted. This critical speed is $v_c = \sqrt{rg}$. In a sense, this is the speed at which gravity alone can continue the circular motion at the top. If there is not enough centripetal force, gravity makes the normal force drop to zero before the object reaches the top of the circle.
Relatedly, there is a cricical angular velocity $\omega_c = \sqrt{\frac{g}{r}}$, which is the angular velocity for which gravity alone can cause circular motion at the top.

8.7 Nonumiform Circular Motion

This is slightly more complicated in that acceleration and velocity aren't strictly normal or tangential. In particular, $a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = r\alpha$ (which is analogous to $v_t = r\omega$ for velocity). Generally, though, it's simpler to phrase nonuniform circular motion in terms of the angular kinematic equations $\theta_f = \theta + i + \omega_i \Delta t + \frac{\alpha(\Delta t)^2}{2}$ and $\omega_f = \omega_i + a\Delta t$. The centripetal equation $a_r = \frac{v^2}{r} = \omega^2 r$ is still valid. This means that the system of forces is slightly more complicated in that $F_t \neq 0$ (though F_z is).

9 Impulse and Momentum

9.1 The Impulse-Momentum Theorem

An important concept for this chapter is the idea that collisions aren't instantaneous, but occur quickly. Molecular bonds can be thought of somewhat like springs in that during collisions they briefly compress and re-expand, exerting forces and causing new accelerations.
An impulsive force is a large force exerted over a small time. Notice that it changes over time, so it tends to be written $F(t)$ and that it has a definite duration. According to Newton's Second Law, $F(t) = ma \Rightarrow m\,\mathrm{d}v = F(t)\,\mathrm{d}t$. One can integrate both sides to show that force is the rate of change of momentum, or that $\Delta p_x = \int_{t_i}^{t_f} F(t)\,\mathrm{d}t$ (in a given direction).
Formally, momentum is defined to be a vector $\mathbf{p} = m\mathbf{v}$ with units of $\frac{\text{kg m}}{\text{s}}$. It can be decomposed into x- and y-components p_x and p_y (for which the integral equations hold). Newton actually formulated his Second Law in terms of momentum, that if mass is constant than $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. (If mass is changing, differentiate $m\mathbf{a}$ with the Product Rule.)
Impulse is a scalar quantity in a given direction $J_x = \int_{t_i}^{t_f} F_x(t)\,\mathrm{d}t$. The unit are N s, which is equivalent to th units of momentum given above. Thus, $J_x = \bar{F}\Delta t$ (average force), so $\Delta p_x = J_x$. This is called the Impulse-Momentum Theorem. Conceptually, this

[2] Interestingly, the rtz system is an inertial reference frame as well, so Newton's Laws apply to it as well. However, since this system moves with respect to the particle, measurements cannot be made relative to it, so that particles aren't constantly seen to be at rest.

means that an impulse delivered to a particle (as the result of an impulsive force) changes its momentum, that $p_{fx} = p_{ix} + J_x$. You don't need to know all the details about how the force varies with time, but only the integral.
Momentum bar charts can be used to track impulse and momentum; they are an intuitive way to see which direction an impulse makes an object go in (whether the component of momentum was positive or negative).
The impulse approximation says that small forces (relative to the impulsive force) can be ignored over the time period of the impulse (e.g. the gravitational force). Importantly, though, the Impulse-Momentum Theorem only applies as the impuse is appled, and afterwards regular kinematics and dynamics take over.

9.2 Conservation of Momentum

Much of this is review.
The formal statement of this is that $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{\text{net}}$ (i.e. that the rate of change of momentum is equal to the net force acting on a system). In this case, $\mathbf{P}$ represents the total momentum in a system.
If these values are zero (as happens in an isolated system), then the total momentum is constant, and any change in monentum in one object of a system is cancelled out by changes in momenta in the rest of the objects in the system. When written over a period of time, this is $\mathbf{P}_f = \mathbf{P}_i$.
The best way to solve these sorts of problems is to choose an isolated system – this can often be more difficult than expected. In practice some approximations can be made (just like the simplification of gravity on Earth).

9.3 Inelastic Collections

A perfectly inelastic collision is one where the two objects stick together after they collide (as with lumps of clay). Similarly, elastic collisions are collisions where the colliding objects separate after colliding. Most of the interesting stuff here will be developed later.

9.4 Explosions

An explosion is, in a sense, the opposite of an inelastic collision – the particles start together and then move apart. The explosive forces that move them apart are interal forces.
Once again, one must be careful to choose a system for which momentul is conserved. But this means that a rocket and its ejected exhaust make a closed system that can be analyzed in the style of an explosion. Importantly, the gases aren't pushing against anything (especially in space, where there's nothing to push), but rather forcing the rocket upward.
The last section is pretty trivial, and as such has been omitted.

10 Energy

10.1 Kinetic and Gravitational Potential Energy

The first section is pointless and has thus been omitted.
Kinetic energy of an object of mass m and velocity v is $K = \frac{mv^2}{2}$, and the gravitational potential energy of an object of mass m at height h is $U_g = mgh$. These scalar quantities are the most basic forms of energy. Their unit is the Joule; $1\,\mathrm{J} \equiv 1\frac{\mathrm{kg\,m}^2}{\mathrm{s}^2}$, as can be obtained from the definitions of K and U_g. Conservation of energy just states that as long as energy is not lost to friction, etc., the sum of kinetic and potential energy in a system is constant.
The book discusses the notion of energy bar charts, which are a visual way of representing how much kinetic and potential energy an object has. The sum of the heights of the bars should remain constant, thanks to conservation of energy.
One tricky notion about gravitational potential energy is that it is relative; the zero for this depends entirely on where you place the coordinate system, and negative values are perfectly reasonable. What matters is the change in potential energy when an object is displaced. Doing the same calculation in different coordinate systems yields the same result.

10.2 A Closer Look at Gravitational Potential Energy

With a little mucking about with equations, one can use Newton's Second Law to prove that energy is conserved on any frictionless surface, because gravitational potential energy does not depend on the path one takes from the initial to the final point. The normal force, which changes the direction of the velocity of an object on a frictionless path, doesn't enter the equation $K_i + U_{gi} = K_f + U_{gf}$, so it does not change the magnitude of any of these quantities (and thus also the velocity).
Examples of objects in these situations include roller coasters and sleds in frictionless environments, but also a ball on a string.
The conservation of energy described here is generally referred to as conservation of mechanical energy, since the sum of an object's kinetic energy and its gravitational potential energy is called its mechanical energy. This can be written as $\Delta E_{\text{mech}} = \Delta k + \Delta U = 0$. However, this is only true in some situations; for example, mechanical energy is lost during friction. Energy is still conserved (since it is lost as heat), however.

10.3 Restoring Forces and Hooke's Law

A force that restores a system to an equilibrium position is called a restoring force, and a system with a restoring force is called elastic. A spring is an excellent example of this; it has an equilibrium length L_0 and will return to this length after being either compressed or expanded. Suppose Δs is the displacement from equilibrium (in either direction); then, a good approximation is that the force needed to stretch the spring by a displacement of Δs is $F = k\Delta s$. This linear relation has a constant of proportionality k known as the spring constant, with units $\frac{\text{N}}{\text{m}}$.
Hooke's Law is identical, save for the fact that it writes $F = -kx$. This sign change comes from the fact that F is a restoring force, so it will act in the opposite direction to Δs (or x). However, when you compute the force necessary to move the spring, it will be in the direction of the displacement, so take care.
Hooke's Law is not a fundamental law, but merely a good approximation. It will fail if a spring is stretched or compressed too far.

10.4 Elastic Potential Energy

The motion of a mass at the end of a spring is very similar to the kinematics of an object due to gravity. The two differ in that the acceleration is nonconstant in the former, and in fact the obect oscillates in position, but this sort of object-spring system has an elastic potential energy $U_s = \frac{kx^2}{2}$, and the mechanical energy $E = K + U_s$ of this system is conserved. (One notable sign convention: U_s is always positive, which is due to the x^2 term.)
If the spring is oriented vertically where there is gravity, gravitational potential energy is also included in this conservation: $E = K + U_g + U_s$ is always constant.

10.5 Elastic Collisions

A perfectly elastic collision is one in which no mechanical energy is lost (which generally implies kinetic energy is conserved, and will mean so for this section). Most collisions aren't perfectly elastic.
Still, they can be useful ways to analyze motion. It's slightly harder to analyze these than to analyze inelastic collisions since kinetic energy is also conserved, so you have two sets of equations to solve.
There's a big ugly equation that can be used to solve these sorts of things without having to think too much, and it can be reduced to simpler cases when the masses or velocities can be equated or perhaps even ignored. For example, if the two objects have equal masses and one starts at rest, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$. Most of these simplifications make intuitive sense (and if they don't, you need to play more billiards).
One excellent way to accomplish these sorts of problems is to use the center-of-mass reference frame, which is the frame of reference defined for which the system has zero momentum, makong calculations easier. Then, using $v' = v - V$, one can solve for the actual velocities.

10.6 Energy Diagrams

One can make a graph of K or U (or E) over time (or position or anything else that varies), which helps visualize motion and energy conservation. This creates potential or kinetic energy curves and the total energy curve. This makes certain types of problems (such as finding the maximum height of an object in parabolic motion) much more intuitive.
An equilibrium position is a point on an energy diagram where the graph of potential energy has a horizontal tangent line. Here, the system can remain at these energy levels indefinitely. If the point is a relative maximum, this is an unstable equilibrium, because adding a small amount of energy to the system will cause it to change drastically. Conversely, a stable equilibrium is a minimum on the graph, since a disturbance to this equilibrium will return it to that minimum. (One can think of this as the system falling to the state of lowest potential energy, which is not quite correct but very helpful.)
Interestingly enough, this already got used in Chem 31X to discover the bond length of an atomic bond. The atoms' attractive and repulsive forces balance each other out with a minimum in potential energy at a distance that is the bond length.

11 Work

11.1 The Basic Energy Model

In addition to a system's mechanical energy, there is also thermal energy E_{th} that is contained in vibrations between molecular bonds. This is positively associated with the object's temperature. Then, the system energy $E_{\text{sys}} = E_{\text{mech}} + E_{\text{th}}$. Though kinetic and potential energy can be converted into each other, thermal and mechanical energy are not normally exchanged.
Energy exchanges within the system are called energy transformations and are usually denoted $K \to E_{\text{th}}$ (i.e. kinetic energy has been converted to thermal energy, as in friction). An energy exchange between the system and its environment (or surroundings) is an energy transfer. There are two types of energy transfers: heat, which will be discussed in more detail in the future, and work. This latter term is due to forces exerted on the system by its environment, and is mechanical.
Energy can both enter and leave a system, and so work can be of either sign. If $W > 0$, then the environment does work on the

system, so the system's energy increases, but is $W < 0$, then the system does work on the environment, and the system's energy decreases. This means that $\Delta E_{\text{sys}} = \Delta K + \Delta U + \Delta E_{\text{th}} = W$.
Thus, energy can be transferred between a system and its environment, and it can be transformed within a system. This is the basic energy model.

11.2 Work and Kinetic Energy

Given the definition of work in the previous section, it is possible to derive from Newton's Second Law an expression for work in terms of force: specifically, $W = \int_{s_i}^{s_f} F_s \, \mathrm{d}s$. In words, work is the integral of the force over the distance the work is done. (In the case where the force is constant, this simplifies to the definitiong given in class, that $W = \mathbf{F} \cdot \mathbf{s} = Fs\cos\theta$. This integral shows that work has units of $\mathrm{N\,m} = \mathrm{J}$, so it agrees with the concept of $W = \Delta K$ as defined earlier. Notice, though, that if a particle changes energy but does not change in displacement, no work is done.
The statement that $W = \Delta K$, called the work-kinetic energy theorem, can be thought of as analogous to the impulse-momentum theorem (since $\Delta p = J = \int_{t_i}^{t_f} F \, \mathrm{d}t$ is akin to the equations seen for work). So impulse is force over time, but work is force over distance. if both time and distance are changing over the duration of a force, then both are present. This allows one to solve for the momentum in terms of kinetic energy: $K = \frac{p^2}{2m}$; you cannot change the kinetic energy of a particle without changing its momentum.

11.3 Calculating and Using Work

If the force is constant, then $W = F(\Delta r)\cos\theta$ as defined earlier. If the force is in the same direction as the distance, this simplifies to the familier $W = Fd$. Intriguingly, this also means that a force perpendicular to the direction of motion does no motion (as in uniform circular motion, where there is no energy change).
(If you know what a dot product is, the rest of this section is redundant, as is the next; they have been omitted.)

11.5 Force, Work and Potential Energy

Since mechanical energy is conserved for an object rolling on a track (or any equivalent scenario), the work done by gravity is independent of the path followed by the object. This may seem surprising, but it is a crucial aspect of any force that has a potential energy (since U is a state function, then it is independent of path, in an intuitive sense). Forces which satisfy independence of path are called conservative forces. Significantly, a potential energy can be associated with any conservative force, generally defined to be $\Delta U = U_f - U_i = -W$.
Note that not all forces are conservative. Friction is the canonical counterexample. Unsurprisingly, these are called nonconservative forces.
The logic behind the naming is that mechanical energy is conserved when there are no nonconservative forces. This means that using the Work-Kinetic Energy Theorem $\Delta K = W$ to explicitly compute the work will yield the same result as just representing the work done by conservative forces as potential energies.

11.6 Finding Force From Potential Energy

In order to obtain the force of an object given its change in potential energy (and assuming the force is conservative) one goes in reverse to the previous method; specifically, $F_s = -\frac{\mathrm{d}U}{\mathrm{d}s}$, where s is an arbitraty measurement of position. (This agrees with, for example, the gravitational force $-mg$ given potential energy $U = mgh$. The negatative sign occurs because the force occurs opposite to the direction of the potential energy measurement.
This also agrees with the intuition of equilibrium; when the potential energy has a slope of zero, there is no force acting on it, so the object is at equilibrium.

11.7 Thermal Energy

This section is slightly more concerned with microphysics (physics on the atomic scale) than macrophysics (physics on a larger scale or considering the object as a whole).
For example, there is a microscopic kinetic and potential energy analogous to the macroscopic one. These come from the motion of atoms and the energy stored in their atomic bonds, respectively. Interestingly, the sum of the microscopic energy of an object is greater than its macroscopic energy!
Usually, this energy is hidden from view and called thermal energy, since it is proportional to temperature. (However, it is not heat, which has a different connotation in physics.)
Microscopic energy can be used to explain dissipative forces. Atomic bonds are springlike, so when objects move past each other with friction, they briefly bond with each other and cause potential (and kinetic) energy to be stored in these bonds. This energy is considered thermal energy, and since both objects gain potential energy, both of them get warmer. Notice that the opposite process doesn't happen for dissipative forces: one cannot decrease an object's temperature with a dissiaptive force.

This is the reason work based on friction isn't calculated; work is only defined for forces acting on particles or thigns that can be modelled as particles, and as such friction would have to be calculated on an atomic level, which is rather complicated.

11.8 Conservation of Energy

Yet another look at conservation of energy will divide nonconservative forces into dissipative sources an external forces, so that the work-kinetic energy theorem becomes $\Delta K = W_c + W_{\text{diss}} + W_{\text{ext}}$, the sum of the conservtive forces, the dissipative forces, and the external forces on a system. Adding in potential energy and thermal energy shows that $\Delta K + \Delta U + \Delta E_{\text{th}} = W_{\text{ext}}$. Thus, the total energy of the system is constant for an isolated system (one for which there is no work from the outside).
Using all of this stuff, one can make more comprehensive energy bar charts that incorporate, say, thermal energy and external work as well as kinetic and potential energy.
For solving physics problems using these strategies, one should choose a good system (preferably one without dissipative forces) and then apply the equations used above.
Another generalization of this idea is that work is not the only way to transfer energy; in particular, nonmechanical energy transfer situations occur. Heat is the best example, and it would require the First Law of Thermodynamics to be written $\Delta E = W + Q$. Another unusual quality of thermal energy is that it cannot easily be transformed back into mechanical energy; this will eventually be formalized as the Second Law of Tehrmodynamics.

11.9 Power

Power is used to measure how quickly an object does work on a system. It is defined $P = \frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t}$ and has the unit of Watts; $\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}}$. Power is also the rate of work so $P = \frac{\mathrm{d}W}{\mathrm{d}t}$ and thus also satisfies $P = \mathbf{F} \cdot \mathbf{v} = Fv\cos\theta$, where θ is the angle between the work and the velocity of the particle.

12 Rotation of a Rigid Body

12.1 Rotational Motion

A rigid body is one whose size and shape do not change as it moves (which in particular makes it much simpler to analyze in physics). Examples of these include gyroscopes, wheels, and gears. Of course, since nothing is perfect in physics, this is a model, known as the rigid-body model, and i merely a good approximation to what actually happens. A rigid body undergoes one of three types of motion: translational motion, rotational motion, and combination motion (i.e. both of the other types at once).

12.2 Rotation About the Center of Mass

An object that is not attached to some sort of axle is called unconstrained, and if there is no net force, and unconstrained object that rotates rotates around its center of mass. The center of mass is motionless, and every other point undergoes circular motion around it.
For a finite collection of point masses i at locations (x_i, y_i) and with mass m_i, the center of mass is defined as $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{M}\sum_i m_i x_i$, $\bar{y} = \frac{1}{M}\sum_i m_i y_i$, and $M = \sum_i m_i$. (Of course, in a different number of dimensions, you'd need a different number of coordinates, but expressing this in vectors would be simple enough). The center of mass is just the weighted average of the locations of the points, which makes a lot of sense.
For any realistic body, this is impractical, since it contains a distribution of mass across an infinite number of points in a region A. Then, the center of mass is $\bar{x} = \frac{1}{M}\int_A x\rho\,\mathrm{d}A$ and $\bar{y} = \frac{1}{M}\int_A y\rho\,\mathrm{d}A$, where $M = \int_A \rho\,\mathrm{d}A$ and ρ is the density function of the object at any given point. (Of course, the book doesn't go into so much detail.)

12.3 Rotational Energy

A rotating rigid body has energy due to the motion of any particular particle in it. There are more sums involved in calculating rotational kinetic energy (in particular, $K_{\text{rot}} = \sum_i \frac{m_i r_i^2 \omega^2}{2}$). The concept of a moment of inertia (oh hey, more multivariable calculus!) is defined as $I = \sum_i m_i r_i^2$ (where r_i is the distance of the particle i from the center). Then, kinetic energy can be expressed much more simply as $K_{\text{rot}} = \frac{I\omega^2}{2}$, which suggests that I is rotationally analogous to the mass in that an object with a larger moment of inertia is harder to rotate. Objects with a larger moment of inertia tend to have more mass concentrated in the rim than in the center.
Though you can calculate moments of intertia for objets, there are tables in many textbooks. It is also worth checking out the table on page 347 of the textbook.
Additionally, if the object rotates around an axle that is not located at its center of mass, then the center of mass might move up or down. This will cause a change in the potential energy of the object due to gravity (since the center of mass can be modeled as

a point mass in this case) and so conservtion of energy jumps in: $E_{\text{mech}} = K_{\text{rot}} + U_g$ is a conserved quantity plus perhaps some external work.

12.4 Calculating Moment of Intertia

Frankly, this is familiar. The book handwaves it but here's what you have to do:

$$I = \int r^2 \, \mathrm{d}m = \int (x^2 + y^2) \, \mathrm{d}m = \int_R (x^2 + y^2) \rho \, \mathrm{d}A$$

where ρ is the density function so that $\mathrm{d}m$ makes any sense. Notice a distinction with the center of mass, which can be calculate relative to any coordinate system; the moment of inertia requires the origin to be located at the pivot.
Notice that if an object can be discretely divided into objects $R = R_1 + \cdots + R_n$, then $I = \sum_{i=1}^n I_i$; that is, moment of inertia is additive.
Another very useful result is the parallel axis theorem; for an object of mass M rotating through a parallel axis that is not necessarily the center of mass, ten the moment of inertia is $I = I_{\text{cm}} + Md^2$, where I_{cm} is the moment of inertia about the center of mass and d is the distance betweenthe axis and the center of mass.

12.5 Torque

Torque is the rotational analogue of force. Suppose one applies a force $\mathbf{F}$ with magnitude F on a rigid body at a distance of r from its pivot point, and suppose that the direction of the force intersects the direction of the shortest path to the pivot point at the point where force is applied at an angle ϕ (check out the picture; it will make more sense). Then the torque is defined to be $\tau \equiv rF \sin\phi$. Colloquially speaking, torque measures the effectiveness of a force at causing an object to rotate about a pivot. The unit of torque is N m; though this can be written as J, it is not, because torque is not related to energy.
Torque is a signed quantity, just like force; the counterclockwise direction corresponds to positive torque. Notice that a force pushing straight towards or away from a pivot causes no rotation and thus has no torque.
Another nuance of torque is that it is necessarily calculated about a given point; saying that a force exerts a torque of 20 N m is meaningless; it is necessary to specify the point about which the torque has been applied.
It so happens that $F \sin\phi$ is the tangential component of the force, F_t, so the torque is just $\tau = rF_t$. This makes sense because the radial component of the force has zero torque. Alternatively one can define $d = r \sin\phi$ as the moment arm, the distance to the line of action (i.e. the line that the force is exerted on) from the pivot (specifically, the minimum distance, defined with a right angle and such as in math). Then, $|\tau| = dF$. The sign must be supplied by finding the direction in which the torque acts.
One can also define net torque defined by a set of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$. Then, $\tau_{\text{net}} = \sum_i \tau_i$. It's worth noticing that the axle exerts a force so that $\mathbf{F}_{\text{net}} = \mathbf{0}$; this force $\mathbf{F}_{\text{axle}}$ is necessarily in the radial direction and so doesn't contribute to the force.
Gravitational torque is also a possibility, exerting a downward force $m_i g$ on every particle i. Then the magnitude of the gravitational torque is $|\tau_i| = d_i m_i g$ (where d_i is the moment arm), but one must be careful with signs. Notice that in this case the moment arm for a particle i is just its $|x_i|$. A particle to the right of the axis of rotation experiences a negative torque, since gravity pushes it in a clockwise direction, and a particle left of the axis experiences a positive torque as a result. Thus $\tau_i = -m_i x_i g$, so the overall torque is $\tau_g = -Mg\bar{x}$, where $\bar{x}$ is the position of the center of mass relative to the axis of rotation. Interestingly, this is exactly what one would obtain if the center was treated as a point mass M and the gravitational force acted on it alone.[3]
One way to rotate an object without translating it is to apply two forces in opposite directions at opposite ends of an object, called a couple. Since their torques have the same sign, they exert a net torque $|\tau_{\text{net}}| = lF$, where l is the distance between the lines of action of the two forces. If an object is unconstrained, then this causes it to rotate about its center of mass.

12.6 Rotational Dynamics

Similarly to how force causes a linear acceleration, a torque causes an angular acceleration: $\tau = mr^2\alpha$ (for a single particle, at least). Generalizing to an extended object, one obtains that $\alpha = \frac{\tau_{\text{net}}}{I}$, where I is the moment of inertia as before. In the absence of a net torque, an object rotates with constant angular velocity (including the possibility that it doesn't rotate, in which case $\omega = 0$.

12.7 Rotation About a Fixed Axis

In order to solve these sorts of problems, one should identify the axis, draw a free-body diagram, and use kinematics and dynamics to identify torques and accelerations, then sove for them.
Ropes and pulleys can make this much trickier; however, is a rope passes over a rotating pulley without slipping, then its speed is necessarily the speed of the rim of the pulley, $v = |\omega|R$; if the pulley has an angluar acceleration, then the rope accelerates, and $a = |\alpha|R$. This is the analogue of linear acceleration constraints introduced when two objects are connected by a single rope.

[3] This point, the point at which gravity acts, is called the center of gravity. Whenever the gravitational force is uniform over the object, as in pretty much every case we'll consider, it is equal to the center of mass.

12.8 Static Equilibrium

The key (and in fact only) idea in this section is that for a rigid body in total equilibrium (i.e. neither a net force nor a net torque), there is no net torque in any point. Thus one can choose any point to be a pivot point, though of course some will be simpler to work with than others. Then, one can determine the net forces and torques about that point (which are, of course, zero) and then solve for the desired quantity.
This relates to ideas of balance and stability, in that an object can balance at a certain critical angle for qhich torque due to gravity will rotate the object back to equilibrium. A bit of math shows that $\theta_c = \tan^{-1}\left(\frac{t}{2h}\right)$, where h is the height of the center of mass above the bottom of the object (which is inclined in this case) and t is the object's width. In other words, at the critical angle the center of mass is directly above the base of support. This also implies a wider base of support and/or lower center of mass increases stability, since the ratio of the two is what determines the critical angle for which an objet rolls.

12.9 Rolling Motion

Rolling is a combination motion (as described in section 12.1) in which an object rotates about an axis that moves in a linear trajectory. If the object doesn't slip, then the center of mass moves forward exactly the length of the circumference as the wheel rotates once. Thus $\Delta\bar{x} = vT = 2\pi R$ (where T is the period and R the radius of the rigid body). Some rearranging results in the rolling constraint, that $v = R\omega$, the basic link between the translational and rotational motion.
For any particular particle in the rolling object, the position vector is the sum of the rotational and translational components, $\mathbf{r} = \mathbf{r}_{\text{cm}} + \mathbf{r}_{\text{rel}}$, where $\mathbf{r}_{\text{cm}}$ is the position of the center of mass of the rotating object and $\mathbf{r}_{\text{rel}}$ is the location of the particle relative to the center of mass. Specifically, this also holds true for the velocity vectors, so the particle's velocity has circular and translational components. In particular, $v_{\text{rel}} = -R\omega$ (indicating that the motion is clockwise if the particle travels in the forward x-direction), but $v_{\text{cm}} = R\omega$; thus, the point at the bottom of a rolling object is instantaneously at rest. This explains why static friction and not kinetic friction is used for rolling rotating objects, as discussed earlier. Then, calculating the velocity at the top is jsut as straightforward: $v = 2R\omega = 2v_{\text{cm}}$.
One can also calculate the kinetic energy of a rolling object. The end result will be unsurprising, but first we have to get there. Since the bottom of the rotating object, a point P, is instantaneously at rest, it can be thought of as an instantaneous axis of rotation. From this perspective, the object's motion is just pure rotation about P, so $K = \frac{I_P\omega^2}{2}$. But this can be calculated using the parallel axis theorem: $I_P = I_{\text{cm}} + MR^2$, so $K = \frac{I_{\text{cm}}\omega^2}{2} + \frac{Mv_{\text{cm}}^2}{2} = K_{\text{rot}} + K_{\text{cm}}$. This intuitively makes sense, and says that the rolling motion of a rigid body can be simply described as a translation of a center of mass plus the rotation about it.
This leads to a scenario which calculates what type of wheel rolls the fastest down a hill; it ends up being the type with the lowest moment of inertia. The conclusion is that the acceleration of a rolling object can be significantly less than that of a rolling particle.

12.10 The Vector Description of Rotational Motion

For more general rotational motion, it will be useful to think of these quantities (angular velocity, torque, etc.) as vectors instead of just scalars.
The angular velocity vector is defined to have magnitude $|\omega|$ as given before (of course), but the direction is given erpendicular to the axis of rotation, according to the right-hand rule (i.e. upwards if counterclockwise). Thus, if an object rotates in the xy-plane, ω points in the direction of the z-axis.
This book defines the vector $\mathbf{A} \times \mathbf{B}$ as the vector with length $AB\sin\theta$, where θ is the angle between the two vectors, and where the direction is given by the right-hand rule.
Torque, then, is written as $rF\sin\phi$, which seems natural to write as a cross product. Define the torque vector $\tau \equiv \mathbf{r} \times \mathbf{F}$; this gives the magnitude of the torque vector as $|\tau|$ and the direction perpendicular to the axis of rotation. Then, the scalar torque defined in section 12.5 is just the component of this vector along the axis of rotation, τ_x.
A final important concept is that of angular momentum. Suppose $\mathbf{p}$ is the momentum vector of an object and $\mathbf{r}$ is its position (not that the object is not necessarily rotating). Then, the angular momentum is $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$. Its magnitude is $L = mrv\sin\beta$, where β is the angle between the position and monentum vectors. This vector is (again) perpendicular to the plane of motion.
Angular momentum is the rotational analogue of linear momentum in the same way that torque is the analogue of force; notably, angular momentum is about the point from which $\mathbf{r}$ is measured. Its units are $\frac{\text{kg m}}{\text{s}^2}$.
If one measures $\mathbf{r}$ from the center of the circle, β becomes $90°$ and life becomes awesome; specifically, $\mathbf{L}$ lies only along the z-axis and its magnitude is $L_z = mrv_t$. Sign conventions dictate that L_z has the same sign as ω; clockwise is negative and counterclockwise is positive.
There is a rotational analogue to Newton's Second Law which states that $\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \tau_{\text{net}}$, or that a net torque causes a change in a particle's angular momentum.

12.11 Angular Momentum of a Rigid Body

The above equations are just for a sinle particle, but this is hardly problematic, because the total angular momentum for a set of particles is just the sum of the angular momenta of the particles, and in particular $\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}\mathbf{L}_i}{\mathrm{d}t} = \sum_i \tau_i = \tau_{\text{net}}$. Thus the rate

of change of the angular momentum of a system is equal to the net torque on that system; specifically, angular momentum is conserved if there is no net torque. (This implies all the nice stuff you've seen before: angular momentum is conserved in an isolated system; $\mathbf{L}_f = \mathbf{L}_i$, etc.)
Yet another counterpart of linear motion that has an analogue here is the result that $\mathbf{P} = M\mathbf{v}$; however, it is limited. For objects that are not spherical, the angular momentum and angular velocity vectors do not necessarily point in the same direction. However, you can use the relationship $\mathbf{L} = I\omega$ for the rotation of a symmetrical object about the axis of symmetry (e.g. a cylinder, or a sphere).
This implies that when angular momentum is conserved, the direction of the rotation necessarily remains unchanged. This is why gyroscopes can be used to indicate direction: once the axis of a gyroscope points north, it will not deviate from that path while a ship (or plane or spacecraft) moves.

26 Electric Charges and Forces

26.2 Charge

The first section is pointless and can be skipped. Similarly, that which is redundant in this section has been omitted.
The fundamental unit of charge, denoted as e, is the charge of one proton (if positive), and the charge of one electron (if negative). Unfortunately, to quantify e, it will be necessary to define a unit of charge, which will happen in a bit.
The charge of an object, q, is defined to be the difference in the charges of its protons and electrons: $q = e(N_p - N_e)$. This means that the charge of an object varies in discrete steps (even though on the macroscopic level these are hard to see); this is known as charge quantization.
Charge is conserved rather like mass or energy; it can be transferred between objects, but it is neither created nor destroyed (that is, the net charge is constant).
The text then discusses charge diagrams, which are relatively straightforward. Remember to distinguish between surface charges and interior charges, and to only show net charges (i.e. cancel all possible positive and negative charges in the same area).

26.3 Insulators and Conductors

The concept of metallic bonding with its negative "sea of electrons" and its positive ion cores is familiar from chemistry. In electrostatics, it has the important consequence that electrons in a metal move quickly and easily, forming a current. This meets the definition of a conductor. (Ions also are conductors, but here the charge carriers are whole ions, not electrons).
Insulators (i.e. those things which aren't conductors) can be charged by rubbing to transfer charges between objects. This tends not to work with conductors (since they can move around so easily), but it can still happen. Also note that conduction happens for these purposes instantaneously, since the electrons move very quickly.
Generally, if there are no forces or changes in charge, the system is said to be in electrostatic equilibrium. In an isolated conductor, any excess charge must be located on the surface of a conductor as a result of equilibrium; if there were an excess charge in the boundary, the interior wouldn't be neutral, and things would move around.
An object can be discharged, so that it returns to neutrality (or at least much closer to it). This is typically done by contact with some larger object. If the object in question is the Earth, the object is grounded. Objects are grounded in order to prevent charge buildup (as in circuits).
Additionally, a neutral object can be polarized, so that the object is still neutral but has slightly separated its positive and negative charges. A charged rod will polarize a metal because it attracts opposite charges within the metal. This creates a slight net force on the object called the polarization force due to the separation of charges.
Insulators don't let their electrons move around, but they still display some polarized effects. This is because each individual atom in the insulator is stretched, with the nucleus moving away from the center. This creates an electric dipole, and a polarization force upon each atom.
Induction is another way to charge things. But this was done in lecture.

27 The Electric Field

27.1 Electric Field Models

The idea behind this section is that, while electric fields can be fiendishly complicated, four simpler common models allow for a solid understanding of electrostatics. These four fields are:

- The electric field of a point charge, which was addressed in a previous section and is just $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$, where $\hat{r}$ is the unit vector pointing outwards from the charge q.
- The electric field of an infinitely charged wire.

- The electric field of a charged, infinite plane.
- The electric field of a charged sphere.

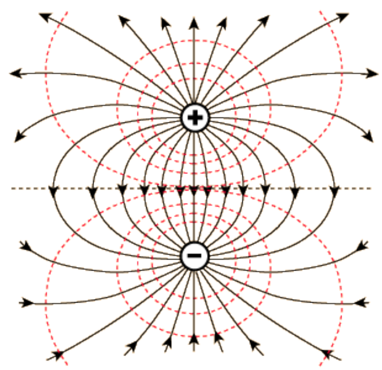
The text then discusses how electric fields obey the principle of superposition, which isn't really that much of a surprise.

27.2 The Electric Field of Multiple Point Charges

This isn't quite difficult, given that it's the most straightforward application of the principle of superposition. However, it will be important to consider which charges make up the electric field and which simply experience it.
The book then walks through some limiting cases that are already known, showing one obtains the correct setups.
A specific case that the text considers is the electric dipole, which is a set of two equal and opposite charges separated by a distance s. Dipoles are common in nature, and can be permanent (as in the polarity of a water molecule) or induced (as in a polarized magnet).
Suppose the dipole lies along the y-axis and let the charges be of magnitude $+q$ and $-q$. Then, some math shows that

$$\mathbf{E} = (E_+ + E_-)\hat{\jmath} = \frac{q}{4\pi\varepsilon_0}\left(\frac{2ys}{\left(y-\frac{s}{2}\right)^2\left(y+\frac{s}{2}\right)^2}\right)\hat{\jmath} \approx \frac{1}{4\pi\varepsilon_0}\frac{2qs}{y^3}\hat{\jmath}$$

It is also helpful to define the dipole moment $\mathbf{p}$, which has magnitude qs and is in the direction from the negative charge to the positive charge. Then, the electric field of a dipole is just $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\mathbf{p}}{r^3}$.
Similarly, for a charge in the plane perpendicular to the dipole (the x-axis or the z-axis), the field is approximately $\mathbf{E} = \frac{-1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{r^3}$.
Notice that these fields are weaker than those given by Coulomb's Law, because the dipole is electrically neutral and therefore partly cancels itself out.
Here the book intorduces the concept of field lines, which are the integral curves that are solutions to the vector field defined by the electric field (i.e. lines that follow the charge vectors). By convention they always flow from positive charges to negative ones.



Neither vector fields nor field lines are perfect approximations, but they are useful; simply, there is no precise way to illustrate an electric field.

27.3 The Electric Field of a Continuous Charge Distribution

Many objects on the human scale do not show their atomic charge, and thus it is acceptable to think of them as continuous and assign density. Charge is similar: for a rod of length L with charge Q uniformly distributed throughout its length, one can define the linear charge density $\lambda = \frac{Q}{L}$. The units are $\frac{\text{C}}{\text{m}}$, and this quantity represents the charge per meter of length. Charged surfaces have a surface charge density over an area A given by $\eta = \frac{Q}{A}$. This has (unsurprisingly) units of $\frac{\text{C}}{\text{m}^2}$.[4]
These objects will be assumed to be uniformly charged, so that charge is distributed equally over the object.
Generally, to solve stuff on this, one divides the total charge Q into many small charges ΔQ, and then adding the separate electric fields $\Delta\mathbf{E}$ of each ΔQ. Then, the sum becomes an integral once you go to infinity. Consider, for example, an infinite line of charge with constant (linear) charge density λ. Then, the electric field is described by $\mathbf{E} = \frac{2\lambda}{4\pi\varepsilon_0 r}$, where r and λ are signed quantities (in particular, the field for a positive line of charge points outwards: to the left on the left side of the line, and to the right on the right side of the line).
Though of course no real infinite line of charge exists, this can be used to make predictions about finite and long ones, since the ends of a wire don't make much of a difference in most cases.

[4]There's also a volume charge density $\rho = \frac{Q}{V}$ with units of $\frac{\text{C}}{\text{m}^3}$, but that doesn't come up until later.

27.4 The Electric Fields of Rings, Disks, Planes, and Spheres

27.4.1 A Ring of Charge

Calculating the electric field in the direction of the z-axis (i.e. perpandicular to and through the center) of a thin ring of radius R and total charge Q can be done by observing that charges on opposite sides of the ring partially cancel out. Specifically, divide it up polarly into i segments, and then observe that only the z-component of the respective fields matters. After some math, the field of a segment i is $E_i = \frac{1}{4\pi\varepsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}}\Delta Q$, so the total charge is $E = \frac{1}{4\pi\varepsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}}\sum_{i=1}^{N}\Delta Q$. Interestingly, this doesn't even need an integral, so the field is $\mathbf{E} = \frac{zQ\hat{k}}{4\pi\varepsilon_0(z^2+R^2)^{\frac{3}{2}}}$.
For a positively charged ring, this means that as one goes away from the center of the ring on the z-axis (in either direction) the force is repulsive, first slightly, then strongly, then asympotically to zero. The field at the center of the ring is zero.

27.4.2 A Disk of Charge

This model consists of a circular disk that is uniformly charged (but still not a real object, since it has no thickness). Its areal charge density is $\eta = \frac{Q}{\pi R^2}$, given its radius R.
In order to calculate the electric field at a point on the z-axis of this circle (i.e. on a line perpendicular to the plane of the circle that goes through its center) one should divide the disk into N rings of radius r_i and charge ΔQ_i.
Using this notation, the electric field of ring i can be calculated as in 27.4.1. Summing this is even more straightforward, and eventually yields

$$\mathbf{E}_z = \lim_{N\to\infty}\frac{\eta z}{2\varepsilon_0}\sum_{i=1}^{n}\frac{r_i\Delta r}{(z_i^2+r_i^2)^{\frac{3}{2}}} = \frac{\eta z}{2\varepsilon_0}\int_0^R \frac{r\,\mathrm{d}r}{(z^2+r^2}^{\frac{3}{2}} = \frac{\eta\hat{k}}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)$$

Note that this is only valid for $z > 0$; if $z < 0$, the the field has the same magnitude, but the opposite direction. A simplification of this is $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}$ when $z \gg R$, which is the field of a point charge, so this limiting case makes sense.

27.4.3 A Plane of Charge

This is even more easy, now that the hard work is finished: let $R \to \infty$, and the field becomes $E = \frac{\eta}{2\varepsilon_0}$ – which is constant. Everywhere. This is pretty impressie, and is a reminder that infinity does some weird stuff. In a sense, there's no scale to determine how far away an object is, since it is never far relative to the infinite.
It is necessary to consider sign, and note that this holds true (again) for positive z: for negative values, simply flip the sign.

27.4.4 A sphere of charge

It is possible to find the charge of the electric field of a sphere in the same way, but it's much more difficult. A simpler procedure using some more advanced techniques will occur in 28.5.
For a sphere of radius R and charge Q, and a point at $r \geq R$, the electric field is $\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{r}$. This is just the same as a point charge once you are far away enough, which makes sense. The field lines point away from the origin for a positive charge and towards the origin for the negative charge.

27.5 Parallel-Plate Capacitor

The parallel-plate capacitor is a set of two disks of charge $+Q$ and $-Q$, placed face-to-face at a distance d apart. The net charge is zero, and the area of each disk is A. Some really nice simplifications happen on the inside; if the edge of the capacitor is ignored, since it doesn't contribute much to the field, then the capacitor produces no electric field outide of the two plates (since they generate equal fields in opposite directions, since they are treated as opposite charged infinite planes). Inside the capacitor plates, the electric field has magnitude $\frac{\eta}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$, and is in the direction from positive to negative.
The weak field outside of the capacitor, known as the fringe field, is relatively weak, and in most cases isn't worth worrying about. In the end, even the shape of the electrodes is mostly unimportant, as long as they are close to each other relative to their width. Here the book creates the notion of a uniform electric field (which has the same direction and magnitude at every point in space). It is interesting because it applies within capacitors and is essentially analogous to a gravitational field — so calculating trajectories is rather familiar.

27.6 Motion of a Charged Particle In An Electric Field

From familiar dynamics, one can derive that $\mathbf{a} = \frac{q}{m}\mathbf{E}$; thus, the ration $\frac{q}{m}$, called the charge-to-mass ratio, is important in determining trajetories.
In a uniform field, acceleration is constant, and particles move in parabolic trajectories. (Sometimes, a charged particle moving parallel to the electric field can end up in one-dimensional motion, rather like a projectile dropped vertically.)

The electron gun, which accelerates electrons to a high speed, uses parallel electrodes to generate uniform electric fields (or approximations of them) which accelerate electrons.
Circular motion is also possible, such as a negatively charged particle orbiting a positively charged sphere. This can happen if $|q|E = \frac{mv^2}{r}$.

27.7 Motion of a Dipole in an Electric Field

It is fairly straightforward to show that the net force on a dipole in a uniform electric field is zero; the force on one pole is equal in magnitude and opposite in direction to the force on the other pole, so the dipole overall does not move. However, depending on the orientation of the dipole, the net torque might not be zero; in particular, the torque will rotate the dipole until it is aligned with the field. The magnitude of the torque is $\tau = pe \sin\theta$, so the vector is just $\tau = \mathbf{p} \times \mathbf{E}$. Once the dipole is aligned with the field, its net torque is zero and it does not move.
In a nonuniform field, motion of dipoles is possible. Since any finite-sized object has charge related to its distance, a dipole will experience a net force towards any charged object (simply because the force farther away is weaker), and in a system of multiple charged objects, will have a net force in the direction of the strongest one.

28 Gauss' Law

28.1 Symmetry

An important property of electric fields is that the symmetry of the field matches the symmetry of the charge distribution. For example, the electric field of a cylinder cannot have any components parallel to the cylindrical axis or tangential to the cross-section, so the field mist point outwards from the cylinder at any point.
Electrostatics most often deals with three fundamental symmetries: planar, cylindrical, and spherical. Though these can be idealized, they are still useful tools.

28.2 Flux

The flux of a closed surface (in the context of electrostatics, a closed surface through which an electric field passes is a Gaussian surface) of some vector field (like the electric field) is a measure of how it "flows" through that surface. This is pretty hand-wavy; see a math textbook for a better definition. However: there is an outward (positive) flux around a net positive charge, an inward (negative) flux for a region containing a negative charge, and zero flux if there is no net charge contained by the Gaussian surface.

28.3 Calculating Electric Flux

Basically, most of this is redundant if you've done multivariable calculus. For a constant electric field, define the area vector $\mathbf{A} = A\hat{n}$ to be the vector of the area A of a surface in the direction of the normal to it. Then, the flux through this surface is just $\Phi = \mathbf{E} \cdot \mathbf{A} = EA\cos\theta$. (This also only works if the surface is planar...).
If the field is nonuniform, things get less trivial. Flux is linear, so it can be added up from smaller components, suggesting integrals. In particular, surface integrals. That the flux satisfies $\Phi = \int_S \mathbf{E} \cdot \mathrm{d}A$ is nothing new here.[5]
A number of simplifications allow this to be easier when, say, the electric field is tangent to or normal to the surface. In the former case, the flux is zero; in the latter, it is $\Phi = EA$.
In a closed surface, nothing changes: $\Phi = \oint_A \mathbf{E} \cdot \mathrm{d}\mathbf{A}$; but this time, $\mathrm{d}\mathbf{A}$ is always oriented towards the outside.

28.4 Gauss' Law

Electric flux for a point charge can easily be shown to be $\Phi_\mathrm{e} = \frac{q}{\varepsilon_0}$ by using laws of symmetry of a sphere as the Gaussian surface. Then, it can be shown that flux is independent of surface shape and radius if the charge inside the surface is constant. (This can be done by approximating the surface as a piecewise union of circles with different radii, and showing that for point charges this doesn't change anything.)
Then, it will be important to consider how the flux is affected by charges outside of the surface. Since everything cancels out (there is no net flow) due to this, the total flux through a closed surface that contains no charge is zero.
Using this all and the principle of superposition, one can obtain a powerful formula called Gauss' Law. The flux through any closed surface containing a net charge of Q is just $\Phi = \frac{Q}{\varepsilon_0}$.

[5] For those less mathematically inclined, the book evaluates surface integrals by removing everything that is not $\mathrm{d}A$ from the integral (e.g. E, $\cos\theta$) and then calculates the surface area, which is $\int_A \mathrm{d}A$.

28.5 Using Gauss' Law

One should be careful to only use Gauss' Law on closed surfaces, but there it can be quite powerful. In particular, it means that the flux through a surface surrounding a sphere of charge is equal to $\frac{Q}{\varepsilon_0}$, but is also equal to EA (since the field lines are normal to the surface). Then, you're done up to fiddling with rearranging stuff.

28.6 Conductors in Electrostatic Equilibrium

Consider a charged conductor in electrostatic equilibrium. The electric field is zero within this conductor, since it is in equilibrium, but using Gauss' Law, one can learn some other neat stuff.
Consider a Gaussian surface barely contained within the conductor's outer surface - since $\mathbf{E} = 0$, then the flux is zero as well, which implies the charge inside is zero. Thus, all the charge on a charged conductor must lie on the outside surface.
In particular, the presence of a net charge implies an electric field of some sort outside of the conductor, At the surface, this must be normal to the conductor's surface (because any tangent component would cause motion that is not allowed), and using Gauss' Law, the strength of the field at this surface is $\frac{\eta}{\varepsilon_0}$. (Generally, charge density is not constant for the surface of a conductor, but if it is known for a point, you have solved it.)
Gauss' Law can also be used to analyze regions with holes in them. Since all charges are on the exterior, the charge in and around the hole, as well as the electric field, must be zero.[6] Unfortunately, this makes the exterior field a bit more complicated.
If there is a charge within the hole (e.g. some point charge somewhere), then the flux is still zero and $Q = 0$. Thus, there must be a set of negative charges on the surface of the hole (for this case, assume a neutral conductor). If the charge inside the hole has charge q, then negative charge $-q$ moves into the surface of the hole, and positive charge q moves to the surface. (Of course, if the signs are flipped, then flip them.)

29 The Electric Potential

29.1 Electric Potential Energy

The idea behind electric energy is once again the similarity between the electric force and the gravitational force, so the various properties of work, kinetic and potential energy, and the concept of a conservative force are all essentially the same.
In particular, many calculations are made under the flat-Earth assumption that is accurate for small objects; then, the gravitational field is a uniform field. This leads to the familiar concepts of potential energy as a function of height, etc.
The parallel-plate capacitor also has a uniform electric field pointing from the positive plate to the negative one. So define an s-axis which is zero at the negative plate and increases in the direction of the positive plate (so that the electric field points downward relative to s, like the gravitational field and y). In this case, the electric field exerts a constant force $F = qE$ on a charge q, so the work done is $W = qE\Delta s$. (It will be important to keep track of signs, because of charges, and whether Δs is positive or negative.)
And the work is just the change in kinetic energy... so one can define the electric potential energy as $U = U_0 + qEs$, where U_0 is the potential energy at the negative plate (often designated to be zero). Notice that the potential energy increases with s when $q > 0$, but it decreases when s increases when $q < 0$.[7]

29.2 The Potential Energy of Point Charges

Consider two point charges q_1 and q_2 (which are for now of the same sign) in a frame of reference where q_1 does not move, but q_2 moves from x_i to x_f along the axis separating them. Then,

$$W = \int_{x_i}^{x_f} F_{1\to 2}(x)\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{Kq_1q_2}{x^2}\,\mathrm{d}x = \left[-\frac{Kq_1q_2}{x}\right]_{x_i}^{x_f} = \frac{Kq_1q_2}{x_i} - \frac{Kq_1q_2}{x_f}$$

From this, it seems sensible to define the electric potential energy of two point charges separated by a distance r to be $U = \frac{Kq_1q_2}{r}$ (with of course the alternate formulation involving ε_0). This is the energy of the system, not just of q_1 or q_2. However, it is valid for any charges (regardless of sign), and since the electric field for a sphere is identical to that of a point charge, this works for spheres of charge as well.
This already has some interesting applications: if you combine it with kinetic energy, for example, you can calculate two moving particles' closest approach (if they repel) or farthest separation (if they attract).
This calculation also relied on the notion that the electric force was conservative. This can be easily calculated, but should be no surprise given that it functions entirely similarly to the gravitational force, which is also conservative.
It also ignored the zero of potential energy; U_0 is by custom zero, but where does it happen? Taking the limit of U as $r \to \infty$, it is

[6] The process of using a metal conductor to shield something from an electric field is called screening, and a wire mesh called a Faraday Cage can efficiently accomplish this.

[7] This is really not the potential energy of the charge q, but the potential energy of the charge-capacitor system. However, at this point it isn't really incredibly important.

zero only when the particles are infinitely far apart. (Once again, this agrees with gravitational potential energy of two objects interacting). So one can think of the potential energy as the amount of interaction. However, there is also the notion of negative energy, which can be thought of as having less energy than if separated at infinity (which makes sense if the two particles have opposite charges and are mutually attractive). In this case, the particles are bound to each other and cannot escape. However, if $E > 0$, then a particle can escape to infinity (of course extremely slowly). if the system has $E = 0$, then escape is possible after some finite escape velocity.
If there are more than two point charges, then the potential energy is the sum of all pairs of charges: $U = \sum_{1 \leq i < j \leq n} \frac{K q_i q_j}{r_{ij}}$ where r_{ij} is the distance between particles i and j and q_i is the charge of particle i.

29.3 The Potential Energy of a Dipole

Consider a dipole in an electric field $\mathbf{E}$. It will rotate until it is aligned with the field, and one can calculate the work done on the dipole during this rotation. The work done in rotating it through a small angle $\mathrm{d}\phi$ is $\mathrm{d}W = -pE \sin\phi \,\mathrm{d}\phi$, so the total work is $W = -pE \int_{\phi_i}^{\phi_f} \sin\phi \,\mathrm{d}\phi = pE(\cos\phi_f - \cos\phi_i)$. Thus, the potential energy of a dipole is $U = -pE\cos\phi = -\mathbf{p} \cdot \mathbf{E}$. Notice that there is a point of stable equilibrium at $\phi = 0$ and points of unstable equilibrium at $\phi = \pm\pi$.

29.4 The Electric Potential

Electric potential is motivated by a very similar idea as the electric field: if two charged particles have an electric potential energy, where is it located? So like the electric field, in which the sources are separate from the charges, one can define an electric potential V so that the potential energy of a charge q is $U = qV$.
As it happens, this is the definition: the electric potential of a set of source charges, given a test charge q, is $V = \frac{U}{q}$, where U is the potential energy of the test charge due to the source charges. Like the electric field, this is a property of the source charges and is independent of q.
The electric potential has units of volts: $1\,\mathrm{V} = \frac{1\,\mathrm{J}}{1\,\mathrm{C}}$.
There are multiple representations of this somewhat abstract idea. It can be thought of as the ability of a set of charges to have an interaction with another charge. However, do not confuse the potential with the potential energy – the two are quite different, albeit related. Another sense of this is that once the potential is known (as a scalar field), the source charges can be thought of as "offstage;" they can be ignored as long as the potential is known for calculating interactions.
Since potential energy is often relative to some U_0, it is useful to think of the electric potential difference $\Delta V = V_f - V_i$, which is often called voltage. Unsurprisingly, $\Delta U = q\Delta V$. This leads to another expression for conservation of energy: $K_i + qV_i = K_f + qV_f$. This can be combined with other forms of conservation to analyze some interesting systems.

29.5 The Electric Potential Inside a Parallel-Plate Capacitor

From Section 27.5, the electric field of a capacitor of plates with charge density η is $\frac{-\eta\hat{s}}{\varepsilon_0}$, where s is the capacitor axis as in Section 29.1. From this one can deduce that the potential energy of a test charge q is $U = qEs$, which just means the electric potential is $V = Es$. Like the electric field, this exists at all points inside the capacitor, and does not depend on the presence of a test charge (though one is required for stuff to happen).
One can also calculate the potential difference between two plates a distance d apart as $\Delta V_C = Ed$. In circuitry, this is known as the voltage across the capacitor, or just the capacitor voltage. If this voltage is given, one can find the electric field as $E = \frac{\Delta V_C}{d}$, which is actually more convenient in practice thanks to voltmeters. Additionally, this leads to the result that $\frac{\mathrm{N}}{\mathrm{C}} = \frac{\mathrm{V}}{\mathrm{m}}$, and the latter is the more standard set of units for an electric field. This equation can also be used to calculate the potential as $V = \frac{s}{d}\Delta V_C$. This means it increase linearly from 0 at the negative plate to ΔV_C at the positive plate.
There are many graphical representations of the potential. For example, one could use a two-dimensional map of s versus V, or a three-dimensional map of s versus yz versus V. Alternatively, one could draw contour lines on a two-dimensional map or equipotential surfaces (i.e. level curves on a three-dimensional graph). From mathematics this is the relatively familiar problem of displaying a graph $f(x, y, z)$ of three variables in three dimensions. [8]
Another interesting aside is that a good way to get a capacitor with a certain voltage is to connect a battery with the same voltage. Batteries, it turns out, are excellent sources of potential.

29.6 The Electric Potential of a Point Charge

From the potential energy of two charges discussed in section 29.2, it is pretty simple to obtain the potential $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ for a point charge q. It varies inversely with r, and chooses the zero for potential energy at $r = \infty$. The contour maps are circles or spheres centered at the point charge.
A charged sphere also has this equation: if a sphere has radius R and charge Q, then $V = \frac{Q}{4\pi\varepsilon_0 r}$ when $r \geq R$. This means one can

[8] This has the very suspicious consequence that the electric field vectors are always orthogonal – I mean perpendicular – to the equipotential surfaces or contour lines, and always points in the direction of the steepest decrease in the electric field. Does this mean that $\mathbf{E} = -\nabla V$? Stay tuned.

calculate the potential at the surface as $V_0 = \frac{Q}{4\pi\varepsilon_0 R}$. This, this sphere has total charge $Q = 4\pi\varepsilon_0 R V_0$, so the electric potential for $R \geq r$ is just $V = \frac{R}{r}V_0$.

29.7 The Electric Potential of Many Charges

The electric potential obeys the principle of superposition, which makes finding the field for something like a dipole pretty straightforward. However, finding the potential for continuous distributions of charge is a bit harder, and resembles the additive and integral methods used to find the electric field.

30 Potential and Field

30.1 Connecting Potential and Field

The electric potential and the electric field are both in a sense theoretical constructs, since they don't actually do anything until one adds a test charge. However, they are also just different ways of looking at the same thing, which makes sense because no new physics was used to distinguish one from the other, only mathematics.

For example, by analogy with the work-energy theorem stating that $\Delta U = -\int_i^f \mathbf{F} \cdot \mathrm{d}\mathbf{s}$, since $\mathbf{F} = q\mathbf{E}$, then $\Delta V = -\int_i^f \mathbf{E} \cdot \mathrm{d}\mathbf{s}$.[9]

So now that the potential difference has been found, in order to find the potential, there must be some V_0 to compare to. Often, this zero point lies at infinity.

30.2 Sources of Electric Potential

Any separation of charge can create a potential difference, since it creates an electric field. For example, a Van de Graff generator mechanically transports charges around with a conveyor belt, creating a fairly powerful electric field.

However, the most common source of electric potential is a battery. Generally, a battery consists of chemicals called electrolytes that undergo electrochemical reactions that move charge around. A charge escalator model can be used for this: the charges are separated by the reaction "lifting" them from the negative terminal to the positive one. However, this increase in potential energy requires work, so when the battery's reactions complete, it dies.

Several electrical quantities can be associated with the battery. Depending on the specific chemicals it employs, the battery has a potential difference ΔV_b. In an ideal battery (no internal energy losses), then a test charge gains potential energy $\Delta U = W_b$, the work that the battery does on the charge.

The work per unit charge done by the battery is called the emf, and is denoted by $\mathcal{E}$. It has units identical to potential (i.e. volts), and measures the battery's rating (so a 1.5 V battery has an emf of 1.5).[10] The emf is given by $\Delta V_b = \frac{W_b}{q} = \mathcal{E}$. Thus, a 1.5 V battery dies 1.5 J of work to move 1 C of charge. In practice, thanks to the Second Law of Thermodynamics, the potential difference between the ends of a real battery, called the terminal voltage, is slightly less than the emf.

It is useful to connect batteries in serial (attach the positive end of one to the negative end of another…and if you really want to, you can attach like ends, but this causes the total battery to lose charge). And if you place batteries in series, the potential difference is just $\Delta V = \sum \Delta V_i$ (paying attention to the orientation of the charges).

30.3 Finding the Electric Field from the Potential

So I will present this in a different way than the book, though that shouldn't change anything.

Since $\mathbf{E}$ is perpendicular to the equipotential lines and points in the direction of the greatest decrease of potential, then it is just the negative gradient of potential: $\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$.

Cool. So that makes the intuition behind this next bit somewhat easier: if you're going in some specific direction s, then the magnitude of the electric field in that direction satisfies $E_s = -\frac{\mathrm{d}V}{\mathrm{d}s}$. Geometrically, this means the electric field is the negative slope of a two-dimensional graph of potential (s versus V axes).

Since the electric field is conservative, another interesting result called Kirchhoff's loop law holds. If a particle travels in a closed loop, then $\Delta V = 0$.

There are at least three ways to explain this, so I'll start with my favorite. If a particle moves along a smooth curve C, then the potential difference is just $\Delta V = \oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r}$. And since the electric field is conservative, then this is trivially zero.

Alternatively, think of this gravitationally again: electric potential, just like electric potential energy, is a state function. So rather like how hiking on a loop trail means that all elevation differences cancel each other out, the electric potential must not change in total over the loop.

Alternatively, one can arrive here from conservation of energy: since energy isn't lost or gained, then $\Delta U = 0$, so $q\Delta V = 0$.

[9] Formally, you could state this as a line integral, but since the electric field is a conservative force it doesn't matter.

[10] Notice that emf doesn't actually stand for anything. It used to stand for electromotive force, but this was incorrect, since force and work are quite different things.

30.4 A Conductor In Electrostatic Equilibrium

As shown above, any excess charge on a conductor is located at its surface in equilibrium; thus, using Gauss' Law, one can show that the electric field inside a conductor is zero. This means that any two points within a conductor have the same potential, and in order for the electric field to be nonzero outside the conductor, the electric field on the surface must be perpendicular to it. And since the equipotential lines are perpendicular to the field, then they must roughly have the same shape as the electrode, with the approximation getting better as the distance to the surface decreases.

30.5 Capacitance and Capacitors

A capacitor can be charged by connecting it to a battery: the battery moves around charges in a current until a positive charge builds up on one plate and a negative charge builds up on the other place, but when $\Delta V_c = \Delta V_b$, and the voltage is the same as the battery's (note that this process typically takes nanoseconds), the charging stops, and the battery-capacitor system lies in equilibrium. This means that the positive plate of the capacitor has the same potential as the positive terminal of the battery, and similarly with the negative ones.

So it turns out that once a capacitor is charged, you can actually remove it from the battery and it will maintain its charge until some sort of current equalizes the potential. Given previously derived equations for a capacitor that $E = \frac{Q}{\varepsilon_0 A}$ (where A is the surface area) and $\Delta V_C = Ed$, then one can deduce that the charge on the capacitor plates is directly proportional to the potential difference between the plates. In particular, the constant of proportionality is called the capacitance, and is defined as $C = \frac{Q}{\Delta V_C} = \frac{\varepsilon_0 A}{D}$ (for a parallel-plate capacitor). Since this depends only on surface area and spacing, capacitance is a geometric property. It has units of farads, defined as $1\,\mathrm{F} = 1\frac{\mathrm{C}}{\mathrm{V}}$... which is a gigantic amount of capacitance. One is more likely to see microfarads or picofarads in practice.

From this definition, the charge of a capacitor can be written as $Q = C\Delta V_C$.

A capacitor can be more than just a parallel-plate system; any two electrodes form a capacitor, and can have a calculated capacitance. In order to arrive at this, one should calculate the charge on the electrodes and their potential difference.

Capacitors can be placed in parallel and series, and then can be represented by equivalent capacitors (that is, the several capacitors are represented by one with the same net capacitance). This in fact can be demonstrated from the definition of capacitance.

Parallel capacitors necessarily have the same potential difference, so the equivalent capacitance is $C_{\text{eq}} = \sum_i C_i$. No part of a circuit outside of the capaicitors in parallel does anything differently if these parallel capacitors are replaced with one equivalent capacitor.

Series capacitors do affect each others' potentials, but they all have the same charge. So this means that the equivalent capacitance for capacitors in series is $C_{\text{eq}} = \left(\sum_i \frac{1}{C_i}\right)^{-1}$. Thus a greater capacitance can be obtained by placing capacitors in parallel than in series; in particular, the capacitance for series is smaller than any particular capacitor in the group.

30.6 The Energy Stored in a Capacitora

Capacitors are important in circuits because they can store energy. One can calculate the energy stored in a capacitor by integrating over every charge added to it, then, one obtains $U_C = \frac{Q^2}{2C} = \frac{C(\Delta V_C)^2}{2}$. It depends on the square of the potential difference, somewhat like the spring potential energy depends on the square of the distance the spring is compressed or extended. The capacitor's potential energy is converted to kinetic energy in the current, which moves charges around.

Capacitors can be charged slowly and then discharged quickly, as in defibrillators, lasers, and flashbulbs.

The energy stored in a capacitor seems to lie within the electric field. Since the electric field is only nonzero within the capacitor, the potential energy is $U_C = \frac{\varepsilon_0}{2} V E^2$, where V is the volume of the field generated by the capacitor.

Thus, one can define the energy density of this field to be $u_E = \frac{U_C}{V} = \frac{\varepsilon_0}{2} E^2$, with units of $\frac{\mathrm{J}}{\mathrm{m}^3}$.

30.7 Dielectrics

The vacuum is given as the perfect insulator, but one can fill the space of a capacitor with another insulator instead. An insulator in an electric field is called a dielectric, so this capcitor is called a dielectric-filled capacitor. This necessarily increases the capacitance from the vacuum value C_0, since the potential difference decreases slightly, but the charge is the same.

Since an insulator is polarized in an electric field, all of its dipoles align in the direction of that field. Thus, the polarized insulator can just be represented as two plates of charge with surface charge densities η; even though the whole insulator is still neutral.

This creates an induced electric field inside the insulator, which is $E = \frac{\eta}{e_0}$ inside the capacitor (pointing from the positive to the negative direction), and zero everywhere outside. It is called a dielectric because of the two electric sheets.

When an insulator is put into a capacitor, the induced electric field $\mathbf{E}$ points opposite to the capacitor's electric field $\mathbf{E}_0$. Thus it weakens it. Define the dielectric constant as $\kappa = \frac{E_0}{E}$, so that the field strength inside a dielectric in an external field is $E = \frac{E_0}{\kappa}$. This means that $\kappa \geq 1$. κ is unitless, from its definition, and is a property of a material, like density or specific heat.

Since the electric field is still uniform, the potential difference is $\Delta V_C = \frac{\Delta V_0}{\kappa}$, so the capacitance is just $C = \kappa C_0$. This is a useful result.

Additionally, one can show that the induced surface charge density satisfies $\eta = \eta_0\left(1 - \frac{1}{\kappa}\right)$, so the induced density can go from 0 to

nearly η_0 when $]k \gg 1$.
This lost energy seems suspicious, but it was used to pull the dielectric into the capacitor's electric field. Some potential energy can also be transformed into thermal energy.
All materials have a maximum electric field they can handle before producing a spark, called the dielectric strength (for example, air has that of $3 \cdot 10^6 \, \frac{\mathrm{V}}{\mathrm{m}}$.

31 Current and Resistance

31.1 The Electron Current

Though the text has previously considered merely electrostatics, it is also possible to discuss the movement of charge. For example, if two capacitor plates are connected with a metal wire, the capacitor quickly becomes discharged, as charge moves between the plates and makes them neutral.
In general, the charges that move through a conductor are called charge carriers, and are in most cases (particularly in metals) electrons. (Think the sea of electrons in metallic bonding.) The electrons in a conductor undergo random thermal motions, but an electric field induces a net direction to these collisions, and a velocity called the drift speed v_d.
The electron current is the number of electrons passing per second through a cross-section of a conductor. Then, the current $i_e = \frac{N_e}{\Delta t}$, where N_e is the number of electrons that pass through in the time Δt. This is also of course related to drift speed; if A is the cross-sectional area of the conductor, then the electron current is $i_e = n_e A v_d$.
It is also worth asking how long it takes for current to move charges around. It ends up happening very quickly, and in most cases can be thought of as instantaneous.

31.2 Creating A Current

Since electrons lose small amounts of energy in each collision they make, a source of energy is necessary to maintain a current. Specifically, an electric field can cause the motion of charge necessary.
When one connects two parts of a wire with different charges, the surface charge density varies linearly from one end to the other. This means that current, which travels "downhill," will be constant throughout the wire, since the electric field within the wire will be uniform. Observe, however, that the surface charges don't move.
Continuing the analogy to electrons as a gas in terms of collisions, it is evident that they take linear paths in the absence of an external field. However, when a unform field is present, particles take parabolic paths. The collisions do generate heat in a frictionlike manner.
The average speed of electrons in this current is v_d, and designate the mean time between collisions as τ. Then, $v_d = \frac{e\tau E}{m}$. Plugging this into the previous equation, an electric field of strength E in a wire of cross-section A creates a current $i_e = \frac{n_e e\tau AE}{m}$, where the electron density n_e and τ are properties of the metal. Most importantly, the electron current is directly proportional to field strength.

31.3 Current and Current Density

The macroscopic definition of current is different and somewhat simpler. If Q is the amount of charge that has passed through a point of wire in a time t, then the current is $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$; for a steady current, one has $Q = I\Delta t$. The units are amperes: $1\,\mathrm{A} = 1\frac{C}{S}$.
The relation between microscopic and macroscopic current is straightforward: the electron current satisfies $I = ei_e$ since $Q = eN_e$.
It is important to know that since electrons are the charge movers in a metal, current is in the direction of flow of positive charges and thus actually opposite the flow of electrons.
Another important quantity will be the current density $J = \frac{I}{A} = n_e e v_d$, with units $\frac{\mathrm{A}}{\mathrm{m}^2}$. In a way similar to how mass density and volume give mass and charge density yields charge, one has $I = JA$ for a specific piece of metal.
Interestingly, current is conserved. It can be changed by changing the drift speed or the charge, which requires changing the electric field (or the shape of the wire), so the rate of electrons leaving any part of a circuit is the same as the rate entering the circuit.
This creates an interesting consequence at junctions, though; according to Kirchoff's Junction Law, the sum of currents leading into a junction is equal to the sum of those leading out.

31.4 Conductivity and Resistivity

Define the conductivity of a material as $\sigma = \frac{n_e e^2 \tau}{m}$. Notice that this depends only on the properties of a material (and some other factors like temperature). Then, current density is just $J = \sigma E$. This implies that current is directly related to (and caused by) an electric field on the charge carriers. Sometimes it is more convenient to use a particle's resistivity, which is just $\rho = \sigma^{-1}$. Since the units for these are somewhat awkward, it is useful to define the ohm as $1\,\Omega = \frac{1\,\mathrm{V}}{1\,\mathrm{A}}$. Then, the conductivity has units of $\Omega^{-1}\,\mathrm{m}^{-1}$ and the resistivity has units of $\Omega\,\mathrm{m}$.
At low temperatures, some materials do very strange things with regards to conductivity. At low temperatures, superconductors

actually have zero resistivity, so charges can move around in the absence of an electric field. This is of course a quantum effect, but it has some interesting applications in creating terrific currents or magnetic fields at low temperatures. There are, however, "high-temperature" superconductors (meaning 125 K instead of near absolute zero).

31.5 Resistance and Ohm's Law

Given that the field strength is constant inside a wire of constant diameter, the electric field can be easily calculated from the voltage: $E = \frac{\Delta V}{L}$. Also, using the results above, the current satisfies $I = \frac{A}{\rho}$. These equations can be combined to give that $I = \frac{\Delta V}{R}$, where $R = \frac{\rho L}{A}$ is the resistance of a conductor (Ohm's Law). Notice that this is a property of a specific conductor, depending on both geometry and substance. Resistance has the units of ohms, Ω.
So far, current has been driven by a capacitor, but it is more typically driven by a battery. This is advantageous because a battery takes much, much longer to discharge, since it is propelled by an electrochemical reaction, as opposed to a static buildup of charge.(An analogy is that a battery is an escalator, so then the current can roll down a hill and get pushed back up again.)
Some calculations: since the battery is the source of the potential difference between the ends of a wire, the voltage of the wire is equivalent to the voltage of the battery, creating an electric field $E = \frac{\Delta V}{L}$, which establishes a current $I = JA = \sigma AE = \frac{\Delta V}{R}$.
Many times, Ohm's Law is formulated in ways that make it seem confusing. For example, $V = IR$ overlooks the fact that it is a potential difference that is happening, and also suggests the current causes the voltage, rather than the opposite case (what is actually happening). Additionally, it doesn't hold true for all objects, as some materials' resistances vary during use. These are called non-ohmic devices, and include batteries and capacitors.
Some ohmic devices are useful for limiting the current in a circuit so that it doesn't deplete a battery too quickly. These resistors are made with poorly conducting materials or very thin plates of metal. There are three classes of ohmic materials:

1. Wires ideally have a very small resistivity (ideally $0\,\Omega$, so that there's no potential difference even if there is a current in it). Generally, this is not the case, but it is a useful model.

2. Resistors have resistances in the range of 10^1 to $10^6\,\Omega$, which is generally fairly specific. They are used to control the current in a circuit.

3. Insulators have high resistance (ideally infinite), so that there is no current even if there is a potential difference. All real insulators can be treated as ideal, since they ave resistance greater than $10^6\,\Omega$.

Take care to apply Ohm's Law to *only* these things, since other components of a circuit may be nonohmic.
One can show that if the wires are ideal, then since the current is conserved, then the voltage drop (and indeed the only potential difference) must occur at the resistor or resistors in a simple system.

32 Fundamentals of Circuits

32.2 Kirchhoff's Laws and the Basic Circuit

Circuit analysis is based on Kirchhoff's laws, identified in Sections 31.3 and 30.5. In particular, the loop law requires at least one potential difference to be negative, so it is crucial to know the signs of the voltages.
Using this law requires assigning a direction to the current, but this is handwaved really badly here. Then, follow the circuit loop and add up all the voltage differences. For a battery going from the negative terminal to the positive terminal, $\Delta V = \mathcal{E}$, and for a resistor traveling in the same direction, $\Delta V = IR$. If the direction is reversed, the potential difference is opposite in sign, but of the same magnitude. In particular, notice that Ohm's Law doesn't give the magnitude of the potential difference (once an orientation has been chosen).
The most fundamental circuit is a battery connected to exactly one resistor. Since there is a continuous path connecting the terminals, this circuit is complete. (The resistor might also be a resistive device with some other function, such as a lightbulb.) In this case, the battery is called the source, and the resistor the load. Notice that there are no junctions, so the current is the same in all parts of the wire. (The ideal-wire assumption will be made for basically this entire chapter.)
The potential increases by $\mathcal{E}$ at the battery, and the potential at the resistor decreases in the direction of the current. Thus, using Kirchhoff's Loop Law, $\mathcal{E} - IR = 0$, so $I = \frac{\mathcal{E}}{R}$, and the resistor's potential difference is $\Delta V = -\mathcal{E}$.

32.3 Energy and Power

When one adds, for example, light bulbs to a circuit, it becomes important to distinguish the energy from the current; thus, two light bulbs in series emit the same amount of light because they do not deplete the current. Energy is supplied to a circuit in the form of a potential difference. A short calculation shows that $P = I\mathcal{E}$, where the units are (as always for power) in watts. This represents energy tranferred per unit time (overall, it is transferred from chemical energy to potential energy to kinetic energy to thermal energy through resistors).
It's also possible to calculate the work in this system. If the average distance between collisions is d, then $W = qEd$, so the change

in thermal energy is $\Delta E_{\text{th}} = q\Delta V_R$, proportional for each charge to the voltage across the resistor, which implies that the resistor's power is exactly equal to that of the battery's! With a bit more calculation, the power dissipated by a resistor is $P_R = \frac{(\Delta V_R)^2}{R}$. If resistors are set up in series, this means that most of the power will be dissipated by the one with the largest resistance. And since the wires have resistance nearly equal to zero, they don't heat up or dissipate a meaningful amount of power.
The energy dissipated by a resistor in time t is $E = P_R \Delta t$, measured in joules. However, sometimes the units of kilowatt hours (kWh) are used.

32.4 Series Resistors

A set of resistors aligned end to end with no junctions between them are in series (or series resistors, etc.). This implies the current is conserved throughout the series. Thus, the potential differences stack, and the overall voltage is $\Delta V = I \sum_{i=1}^{n} R_i$, and the overall resistance is $R = \sum R_i$. The circuit can replace its resistors in series with this overall resistor without affecting any other component. In a sense, it's a question as to whether one steps down all at once or in several sections.
Sometimes, one can get lost in the notion that a battery only provides a constant emf (i.e. potential difference), not a fixed current. Current is a property of the battery-resistor system, so adding more resistors changes the current.
An ammeter is a device that measures current. As such, it is only useful if placed in series with a circuit. Ideally, an ammeter has no resistance (and in many cases, this can be assumed, but it is worth knowing that sometimes it will be taken into account). Often, the reading on an ammeter provides useful information about a circuit.

32.5 Real Batteries

Unlike ideal batteries, actual batteries have an internal resistance r. This could easily be modelled as an ideal battery in series with a resistor, except that sometimes the relevant information is unknown, and all that is known is the voltage of the battery (the terminal voltage). However, it can be found if a resistor with known resistance R is connected, since $I = \frac{\mathcal{E}}{R+r}$, so $\Delta V = \frac{R\mathcal{E}}{R+r}$. The battery's potential is equal to the emf only when the resistance is zero, and it decreases as the resistance increases.
A short circuit is a circuit with no resistors (just battery terminals connected). In an ideal battery, this would cause division by zero; in actual batteries, the short-circuit current is $I = \frac{\mathcal{E}}{r}$. This is the maximum posible current a battery can produce, since adding resistors lowers the current, and it is a property of the internal resistance.
Most of the time, the ideal battery assumption is valid, since $r \ll R$ in most situations.

32.6 Parallel Resistors

Resistors connected at both ends are called in parallel or parallel. Importantly, the potential differences are identical, so by Ohm's Law, $I = \sum I_i$, but $R = \left(\sum \frac{1}{R_i}\right)^{-1}$. These equivalent resistances and currents could be replaced as in the previous cases. However, it is important to be careful here and not confuse the different things going on in series and parallel.
In particular, the resistance result seems counterintuitive. The resistance went down. Though a single resistor is an obstacle for the charge, multiple resistors provide more pathways for the charge to flow through.

32.7 Getting Grounded

So far, all of these circuits have dealt only with potential differences, and the actual zero of potential has not mattered. This can be a problem, however, if one wishes to connect two different circuits, since they have no common reference point for potential. However, if one connects one point on a circuit to the ground, then all such grounded circuits have the same reference point. The wire that connects this circuit to the Earth is the ground wire.
The actual calculations for the circuit haven't changed; rather, a new quantity has been added, a total potential rather than just a potential difference. (The potential of the Earth is given as $V = 0$). Being grounded doesn't affect a circuit's behavior under normal conditions, though circuits with high voltage if they malfunction will stay at 0 V, keeping them safe. (This is the only case in which the ground wire has a current.)[11]

32.8 RC Circuits

An RC circuit is an example of a circuit that varies over time. In particular, it contains a capacitor, a switch, and a resistor (but no battery). Once the switch is blown, the capacitor discharges quickly, in a manner that can be calculated.
Kirchhoff's loop law can be used to determine that, since $I = -\frac{\mathrm{d}Q}{\mathrm{d}t}$, then $\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} = 0$. Solving this differential equation implies that $Q(t) = Q_0 e^{\frac{-t}{\tau}}$, where Q_0 is the original charge of the capacitor and $\tau = RC$ is the time constant. Notice that it decays logarithmically, so there's a nice half-life and such.
Similarly, the current satisfies $I(t) = I_0 e^{\frac{-t}{\tau}}$.
Notice that there is no set time after which a capacitor is discharged (or charged; see below), but at $t = 5\tau$, the charge has dropped

[11] Since voltage is still based on a different zero point, it can still be negative, since it's still just as relative.

to below 1%, and this is usually taken as an acceptable answer.
Charging a capacitor with a battery and a switch (and perhaps a resistor) asymptotically reaches the maximum charge according to the equation $Q(t) = Q_{\max}(1 - e^{\frac{-t}{\tau}})$.

33 The Magnetic Field

33.1 Magnetism

The magnetic force is noticeably different than the electric force. Some key observations:

- Magnetism is like electricity in that it is a long-range force.
- Magnets, however, have two poles (north and south). They are analogous to positive and negative charges, but not identifiable with them. Like poles repel, and opposite poles attract.
- Some materials, called magnetic materials, are attracted to both poles of a magnet, as neutral objects are attracted to both positive and negative electric charges.
- Magnetic poles always appear in pairs (magnetic dipoles); no magnetic monopoles have been observed.

The Earth also has a magnetic field; since magnetic north poles point towards the north, then that pole is actually a south pole (since like poles repel).

33.2 The Discovery of the Magnetic Field

The crucial connection between electricity and magnetism is that magnetism can be caused by an electric current. A magnetic force in the direction given by the right-hand rule from the direction of the current will be induced (forming a circle around the current cross-section if field lines are drawn).
In particular, one can define a magnetic field $\mathbf{B}$ in much the same way as an electric field. Unsurprisingly, this is a vector field, and exerts a force on magnetic poles. The force on a north pole is parallel to $\mathbf{B}$, but that on a south pole is opposite. Notice that there's no established correlation yet between the magnetic field of a current and the magnetic field of a permament magnet. It is important to demonstrate that these are in fact the same force.

33.3 The Source of the Magnetic Field: Moving Charges

The analogue to Coulomb's Law for the magnetic field is the Biot-Savart law for the magnetic field of a moving point charge.
Suppose a point change q moves with velocity $\mathbf{v}$. For some point P, if the distance from the point charge to P is $\mathbf{r}$ (in the direction from q to P) and the angle between $\mathbf{r}$ and $\mathbf{v}$ is θ, then the magnitude of the magnetic field is $B = \frac{\mu_0 q v \sin\theta}{4\pi r^2}$. The units of the magnetic field strength is the tesla: $1\,\mathrm{T} = 1\frac{\mathrm{N}}{\mathrm{A\,m}}$.
The constant μ_0, called the permeability constant, is analogous to ε_0, and has the value of $\mu_0 = 4\pi \cdot 10^{-7} = 1.257 \cdot 10^{-6}\frac{\mathrm{T\,m}}{\mathrm{A}}$.
The direction of $\mathbf{B}$ is given by the right-hand rule: $\mathbf{v}, \mathbf{r}, \mathbf{B}$ defines the given orientation for $\mathbb{R}^3$. In other words, if the particle travels into the page, then the field lines of $\mathbf{B}$ are circles centered at the particle's trajectory, pointing clockwise.[12]
Since the right-hand rule is being used, the Biot-Savart Law can be formulated in terms of the cross product: $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\hat{r}}{r^2}$. Notice that this invilves a unit vector $\hat{r}$.

33.4 The Magnetic Field of a Current

Since magnetic fields obey the principle of superposition, then individual magnetic fields can be added to obtain a the field of a current. This is often an unpleasant integral, and in particular requires knowing the direction of the current at every point, but there are some common results:
For a long, straight wire, the magnitude of the magnetic field is $B = \frac{\mu_0 I}{2\pi d}$, and the direction is tangent to the circle in the right-hand direction (i.e. same direction as for the moving charge).
For a current loop with N turns (the wire in a coil N times), the field near the center is $B = \frac{\mu_0 N I}{2R}$, and its direction is out of the loop in the direction that the current entered.

[12]This means that a charge creates an electric field, but a moving charge creates a magnetic field. This might have some impressive implications...

33.5 Magnetic Dipoles

Since it is so much more difficult to obtain the off-axis formula for the field around a current loop, it is just given as flowing through the center of the loop, or in circles around any point on the ring.
But a current loop is just a magnetic dipole, then, since their magnetic fields are identical! It functions equivalently to a permanant magnet, and is called an electromagnet. However, the magnetic field inside the dipole is different in each case.
It is possible to define a magnetic dipole moment μ with magnitude equal to AI, the current times the area enclosed in the loop. The units of μ are $\mathrm{A\,m^2}$. This means that the on-axis field of the dipole is just $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{2\mu}{z^3}$. This once again looks remarkably similar to the electric field's equations.

33.6 Ampère's Law and Solenoids

Similarly to Gauss' Law, it is possible to use line integrals to make the magnetic field easier to calculate. Some simplifications can make line integrals easier to evaluate: if $\mathbf{B}$ is everywhere perpendicular to γ, then $\int_\gamma \mathbf{B} \cdot \mathrm{d}\mathbf{s} = 0$; if instead $\mathbf{B}$ is tangent to γ everywhere, then $\int_\gamma \mathbf{B} \cdot \mathrm{d}\mathbf{s} = Bl$, where l is the length of γ (however, this requires the magnitude of $\mathbf{B}$ to be constant).
Ampère investigated current in a closed loop: since $B = \frac{\mu_0 I}{2\pi d}$ but $\oint_{x^2+y^2=d^2} \mathbf{B} \cdot \mathrm{d}s = 2\pi B d$, then the field strength of a current-carrying wire is $\oint \mathbf{B} \cdot \mathrm{d}s = \mu_0 I$.
A more general result shows that if γ is any closed curve, then $\int_\gamma \mathbf{B} \cdot \mathrm{d}s = \mu_0 I$, where I is the current that passes through the interior of γ.
In this it is important to remember that the curve doesn't have to correspond to a surface or line in space, and that the signs of the current are important in calculating things.
A solenoid can generate a uniform magnetic field. A solenoid is a helical coil of wire with constant current. There may be hundreds or thousands of these turns. Thinking of a solenoid as a stack of current loops shows that the field in the center is strong and parallel to the solenoid axis, but the outside field is very weak. In particular, the ideal solenoid has a uniform field inside the loop, and zero outside of it. Of course, real solenoids are various approximations of this.
The strength of the magnetic field inside a solenoid is $B = \mu_0 n I$, where n is the number of turns and I is the current through the wire.
A finite solenoid is an electromagnet, and so its field looks like that of a magnetic dipole.

33.7 The Magnetic Force On a Moving Charge

So all of this discussion doesn't actually indicate what the field does. The magnetic field exerts a force on a current (or equivalently a moving charge). The formula is fairly straightforward: $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ for the force exerted on a charge q moving through a magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$. Notice there is no force on a stationary charge, nor on one moving parallel (or antiparallel) to the field. When there is a force, it is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. This is slightly counterintuitive, but playing with it should help fix this. In general, magnetism appears to be an interaction between moving charges on top of their electric interactions.
Since a magnetic field has no effect on charges moving in the same direction as the field (or opposite to it), only perpendicular motion needs to be considered. For example, one can consider a particle moving perpendicular to a uniform magnetic field. The particle will necessarily undergo uniform circular motion, which can be easily worked out. This motion is called cyclotron motion, and some calculation shows that the radius is $r = \frac{mv}{qB}$, where m is the mass of the particle and v is its velocity. Then, the cyclotron frequency is $f = \frac{qB}{2\pi m}$. In the more general case where the velocity isn't necessarily perpendicular, the perpendicular component causes spirals while the parallel component isn't affected, so charged particles move in somwewhat helical trajectories, spiraling around the field lines. This complicated motion is responsible for the solar wind's interaction with the Van Allen belts, which causes aurorae.
It is also used to power the cyclotron, an instrument that uses magnetic fields to accelerate particles and thus makes for some amazing particle collisions.
Another interesting result is the Hall effect — that this holds true within a conductor as well, and can be used to learn about the charge carriers inside that conductor. Positive charges within a conductor are pushed towards the bottom of the conductor (note that this depends on the charge of the carriers, not just the direction of the current, and also employs the right-hand rule). However, this deflection and polarization of the conductor can't last forever, since it builds up an electric field inside the surface. When the upward electric force balances the downward magnetic force, equilibrium is achieved; this happens when $F_m = e\frac{\Delta V}{w}$, where w is the width of the conductor. Thus, the Hall voltage is the steady-state potential difference between the two surfaces of the conductor. It is given by $\Delta V_H = w v_d B = \frac{IB}{tne}$, where t is the height (thickness) of the conductor. This is not a large number for most experiments, but it is useful for determining n or for poor conductors.

33.8 Magnetic Forces on Current-Carrying Wires

A current parallel to a magnetic field obviously has no force exerted on it, by applying the above. However, if a wire is not perpendicular, the force on the wire is given by $\mathbf{F} = I\mathbf{l} \times \mathbf{B}$, where $\mathbf{l}$ has length equal to that of the wire and goes in the direction

of the current.
This can be used to calculate some more interesting results, such as the force between two parallel wires: using the derivation in Section 33.4, if the currents are in the same direction, the force is attractive, and if they point in oposite directions, the force is repulsive. It is not a uniform field, but it is constant along the wire because it depends on the distance. The magnitude of the force is given by $F = I_1 l B_2 = \frac{\mu_0 l I_1 I_2}{2\pi d}$, where d is the distance between the two wires and I_1 and I_2 their respective currents. By Newton's 3rd Law, the two forces are equal and opposite.

33.9 Forces and Torques on Current Loops

This has some interesting consequences on current loops. In particlar, parallel current loops attract each other, and antiparallel ones repel each other. It makes sense to reformulate this in terms of north and south poles, which has the nice result that magnetic poles attract or repel because the moving charges in one current exert attractive or repulsive forces on the other current.
A current loop in a uniform magnetic field os analogous to an electric dipole in a uniform electric field, and torques and forces can be similarly computed. In particular, $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$, and so when the dipole is aligned with the field, the torque is zero.
This is the basis for how a motor works; it contains a coil of wire that rotates when current passes through it. However, if the current were constant, the magnetic field would eventually return to equilibrium, so a commutator reverses the current's direction in the coils every 180°. This keeps the motor spinning.

33.10 Magnetic Properties of Matter

Now that the physics of magnetic properties of currents is well established, it is still necessary to explain permanent magnets that exist in the absence of current. Historically, it was suggested that magnetism was some atomic property, given that hydrogen atoms are very small electric dipoles, but on the large scale these cancel out.
The key to understanding permanent magnets is to realize that electrons have an inherent magnetic moment. This is in fact the electron spin, so the magnetic moment of a filled electron shell is zero. However, atoms with odd numbers of electrons can be magnetic — though not necessarily, since if the atomic dipoles are randomly oriented, nothing interesting will happen.
In some substances, all the moments line up in the same direction. These are called ferromagnetic (and thus include iron). Often, a given piece of iron is broken into several magnetic domains, which are each oriented uniformly but aren't oriented with the other domains.
Thus, if a ferromagnet is subjected to an external magnetic field, a torque is exerted that causes everything to line up. Sometimes, the domain boundaries make this alignment less than optimal, but atomic forces can change the boundaries between the domains. Sometimes, there may even be a net magnetic dipole once the external field is removed. The object is then called a permanent magnet.

34 Electromagnetic Induction

> A woman in liquor production
> Owns a still of exquisite construction.
> The alcohol boils
> Through magnetic coils.
> She says that it's "proof by induction."

34.1 Induced Currents

Given that a current can create a magnetic field, it is natural to wonder whether a magnet can generate a current. It is in fact possible, which was discovered by Faraday. He had wrapped two coils of wire around an iron ring, and noticed that opening or closing a switch in one of the coils of wire created a current in the other wire – but only brifely. Thus, it was conjectured that a magnet can generate an electric current only if the magnetic field is changing. In this case, the current is called an induced current.

34.2 Motional emf

There are two ways of inducing a current: one can change the size or orientation of a circuit in a stationary magnetic field, or one can change the magnetic field through a stationary circuit. The effects are the same, albeit with different causes.
For example, suppose a conductor with length l moves uniformly with velocity $\mathbf{v}$ through a uniform magnetic field $\mathbf{B}$. The charge carriers in the conductor also move with velocity $\mathbf{v}$, so they experience a force $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$. The Hall effect guarantees the separation of the charge carriers as seen in Section 33.7. When the electric and magnetic fields are in equilibrium, $F_B = qvB$ and $E = vB$. Thus, the magnetic force on the charge carriers in a conductor creates an electric field inside the conductor.
With this electric field is an associated electric potential difference $\Delta V = vlB$. The motion of the wire through a magnetic field induces a potential difference between the ends of a conductor. This is similar to the potential difference between the ends of a

battery in that it can be used to generate a current, and thus it is reasonable to define $\Delta V = \mathcal{E}$ and distinguish the motional emf of an induced potential difference from the chemical emf of a battey. Then, the motional emf is just $\mathcal{E} = vlB$.
A current is only induced if there is a closed conducting loop for a current to exist in. However, if this is the case, the moving wire acts just like the battery in a circuit, so the induced current is given by $I = \frac{\mathcal{E}}{R} = \frac{vlB}{R}$, where R is the total resistance of the circuit. The current is due to magnetic forces on moving charges.
In order for the wire to move at a constant speed, though, there must be a continuous pulling force counteracting the magnetic force that creates "magnetic drag" on the wire. The pulling force has the same magnitude as the magnetic force, or $F = \frac{vl^2B^2}{R}$. This can also be thought of in terms of energy conservation: the energy dissipated by the resistance R is equal to the energy added to the circuit by the pulling force. In particular, this allows for the creation of a generator, which converts mechanical energy to electric energy.
If one pulls a sheet of metal through a magnetic field (that is not totally parallel to it), a current is induced in a whirlpool-like manner. These currents are called eddy currents. In particular, this implies that external force is necessary to pull a metal through a magnetic field.

34.3 Magnetic Flux

Suppose a surface S has area A oriented with normal vector $\mathbf{n}$ (so that $n = a$) at a given point and that $\mathbf{B}$ is a magnetic field through A. Then, the magnetic flux through A is $\Phi = \int_A \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A$. Often this is written as $\int_A \mathbf{B} \cdot \mathrm{d}\mathbf{A}$, and if $\mathbf{B}$ is uniform, this simplifies to $\Phi = \mathbf{B} \cdot \mathbf{n}$.

34.4 Lenz's Law

There is an induced current in a closed, conducting loop iff the magnetic flux through the loop is changing. Then, the direction of the current is such that the induced magnetic field opposes the change in the flux.
In order for the flux to change the magnetic field through the loop must be changing, or the loop is moving or changing size. This is because a current induces its own magnetic field, which opposes the change in flux.
There are three basic situations, in which $\mathbf{B}$ can be either steady (in which case there is no change in flux and no induced current or magnetic field), increasing (in which case the induced field goes the opposite direction), or decreasing (in which case the induced field points in the same direction).

34.5 Faraday's Law

If the magnetic flux through a loop changes, then the motional emf induced has magnitude $\mathcal{E} = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|$, with direction given by Lenz's Law. As a corollary, a coil of N turns of wire acts as N batteries in series: the emf of each individual coil adds. This formulation of emf gives $\mathcal{E} = vlB$, which is in agreement with what was derived before.
A slightly deeper insight into Faraday's Law indicates that the magnetic flux can be changed only if the loop expands or rotate, or the field itself can change. Faraday's Law takes care of these possibilities; if it is written as $\mathcal{E} = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| = \left|\mathbf{B} \cdot \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \mathbf{A} \cdot \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right|$, then the first term represents the first scenario and the second term represents the second scenario. However, they are bottled up as the rate of change, in that the emf does not depend on what caused the change.[13]

34.6 Induced Fields

Since a current is the result of doing work on charge carriers, a magnetic field is not sufficient to explain it in some cases. Specifically, there must also be an induced electric field. Interestingly, this doesn't even require induction — there is an induced electric field wherever the magnetic field is changing. Since it is not created by charges, it is called a non-Coulomb electric field (while to no surprise the electric field due to electric charges is a Coulomb electric field). Notably, the electric field is no longer just a thought experiment that makes it easier to understand the electric force; rather, it is a very real thing that explains currents.[14]
This induced field is even more peculiar in that it is nonconservative, and thus it has no potential or potential energy. Since a particle going around a loop is always pushed in the same direction by the induced field, Kirchhoff's Loop Law does not hold. However, emf can be calculated from $\mathcal{E} = \frac{W}{q}$. After some mathematics, if the field is changing but the loop is fixed, Faraday's Law can be reformulated as $\mathcal{E} = \oint_\gamma \mathbf{E} \cdot \mathrm{d}s = A\left|\frac{\mathrm{d}B}{\mathrm{d}t}\right|$. This proves that the induced electric field is independent of the conducting loop. Geometrically, this means the induced electric field circulates around the magnetic field, another divergence from Coulomb electric fields. One can show that for a solenoid the strength of the electric field inside is $E = \frac{r}{2}\left|\frac{\mathrm{d}B}{\mathrm{d}t}\right|$. In particular, if the magnetic field is unchanging, there is no electric field.
If the sign of the emf is needed, then it is worth computing as $\mathcal{E} = \oint_\gamma \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$. Generally, it's easier to determine the sign

[13]There are still some unanswered questions, though. If a wire is looped around a solenoid, the loop is completely outside of the magnetic field. But changing the current through the solenoid changes the current in the loop. This action-at-a-distance seems magical, and means at least one important detail of flux will be answered below.
[14]The book has a pretty diagram of magnetic and electric field lines perpendicular to each other and I suspect there's a pretty mathematical relationship between them. Any ideas?

from Lenz's Law.
Maxwell suggested that a changing electric field also induces a magnetic field in a very similar way (except that the induced magnetic field points in the opposite direction from the way an induced electric field would). This was a hunch without experimental evidence, but was soon verified.
In particular, it would be theoretically possible to establish a self-sustaining electromagnetic field independent of charge or current, in which a changing electric field creates a changing magnetic field, which creates a changing electric field and so on. This would create a transverse electromagnetic wave in which **E** and **B** are perpendicular to each other and to the direction of travel, and the speed of the wave is $v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$, where ε_0 and μ_0 are as given in Coulomb's and Biot-Savart's Laws. But then, $v = 3.00 \cdot 10^8 \,\frac{\text{m}}{\text{s}}$, so the electromagnetic waves in question... are just light.[15]

34.7 Induced Currents: Three Applications

34.7.1 Generators

Though a wire pulled through a magnetic field forms a simple generator, most practical generators have a slightly different structure. An outside force (e.g. the wind collected by a windmill) causes a coil to rotate in a magnetic field created by a permanent magnet. This generates a current that is removed to a circuit.
The flux varies with time as $\Phi = AB\cos\omega t$, so the emf is $\mathcal{E} = \omega ABN \sin\omega t$, where ω is the angular frequency and N is the number of turns on the coil. This means that the emf alternates in sign, producing AC voltage.

34.7.2 Transformers

Consider two coils wrapped around an iron core. One coil, called the primary coil, has N_1 turns and is driven by an alternating current $V_1 \cos\omega t$. This generates a magnetic field in the iron core that produces a current in the other coil, called the secondary coil, which causes an oscillating flux and therefore current in this coil with voltage $V_2 \cos\omega t$. The magnetic field inside the iron core is inversely proportional to N_1, but the emf in the secondary coil is directly proportional to N_2 (by Faraday's Law). Thus, the secondary voltage of an idea transformer is $V_2 = \frac{N_2}{N_1} V_1$. In a sense, the voltage is transformed across the transformer. Transformers can step-up (in which case $N_2 \gg N_1$) or step-down ($N_2 \ll N_1$), which has various applications in delivering power efficiently where high voltages are possible and then transforming into lower voltages near urban areas.

34.7.3 Metal Detectors

A metal detector consists of two coils of current: a transmitter coil with a high-frequency alternating current creates an alternating magnetic field along its axis, and a recieving coil which has an alternating induced current due to the transmitting coil. However, if a piece of metal is placed between the two coils (or strictly speaking any conductor), the induced magnetic field creates eddy currents, which decrease the induced current at the receiving coil. A circuit can detect this decrease in current and alert the user of the metal detector in some way.

34.8 Inductors

Similarly to how a capacitor stores a uniform electric field and has various applications in circuits, a solenoid can do interesting things as well. First, define the inductance of a solenoid as $L = \frac{\Phi}{I}$ analogously to the capacitance[16] The units of inductance are $\frac{\text{Wb}}{\text{A}}$, and a new unit called the Henry is used for this, notated H. A coil of wire used for the purpose of providing inductance is called an inductor, and an ideal inductor is one for which there is no resistance.
Using the magnetic field of a solenoid, one can compute its inductance as $L = \frac{\mu_0 N^2 A}{l}$, where N is the number of turns and l is the length of the solenoid. Interestingly, this means inductance is a geometric property, and isn't related to current.
An inductor is at its most interesting when the current varies, since when it is stable the potential difference is zero. However, when the current changes, the inductor has an induced emf and an induced current opposite to the solenoid current and isn't related to current.
An inductor is at its most interesting when the current varies, since when it is stable the potential difference is zero. However, when the current changes, the inductor has an induced emf and an induced current opposite to the solenoid current (until a potential difference is established across the solenoid). Using Faraday's Law, the potential difference is $\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$, in a similar convention to Ohm's Law. If the current is increasing, the voltage is negative, and if it is decreasing, the potential increases in the direction of the current. If the current changes rapidly (as when a switch is closed), the inductor has a very large voltage drop, which can create a spark. This is generally how spark plugs work in cars and similar objects.
Like a capacitor, an inductor stores energy for later use. Since the electric power is $P = -LI\frac{\mathrm{d}I}{\mathrm{d}t}$, then the energy can be obtained by integrating to get $U_L = \frac{LI^2}{2}$. Notice that this resembles the equation for a capacitor, as the analogy has held constant throughout

[15] Again, I suspect there's some awesome math hiding behind this, but I don't see it.
[16] Strictly, this is the self-inductance, since in this case the solenoid creates flux on itself.

thr section. And the magnetic field energy density inside the solenoid, with units $\frac{\mathrm{J}}{\mathrm{m}^3}$ is $u_B = \frac{B^2}{2\mu_0}$, since $U_L = \frac{AlB^2}{2\mu_0}$ due to the derivation of the magnetic field in the solenoid.

34.9 LC Circuits

An LC circuit is a circuit consisting of an inductor and capacitor in parallel. This causes the circuit to oscillate at a precise, well-defined frequency, which is important in applications like telecommunications.
Consider a capacitor with charge Q_0 in this loop with $I = 0$. The capacitor instantly begins to discharge, which charges up the inductor. Then, once the capacitor's charge reaches zero, the current keeps going – the inductor resists changes in current, so it continues until the charge on the capacitor is $-Q_0$ (in the sense that everything has reversed, and I briefly traveled in the opposite direction). Then, the exact opposite process occurs, and the system returns to its starting point. This can be thought of as a spring on a block or really any system in simple harmonic motion.
Kirchhoff's law says that $\Delta V_c + \Delta V_L = 0$, so $\frac{Q}{C} - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$. Since $I = -\frac{\mathrm{d}Q}{\mathrm{d}t}$, then $\frac{Q}{C} + L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = 0$, which is a second-order differential equation for simple harmonic motion: $\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -\frac{Q}{LC}$. Thus, the solution is $Q(t) = Q_0 \cos\omega t$, where $\omega = \sqrt{\frac{1}{LC}}$ is the angular frequency. Thus, charge and inductor current oscillate shifted from each other as sine and cosine waves: $I = I\max \sin\omega t$, where $I\max = \omega Q_0$. Then, the frequency is just $f = \frac{\omega}{2\pi}$.
An LC circuit resonates in response to an electromagnetic signal at its specific frequency, so while radio is a superposition of all broadcasts at all frequencies, the LC circuit only detects the one of interest. A variable capacitor allows the frequency to change so that the frequency can be selected.

34.10 LR circuits

An LR circuit is a circuit with an inductor, a resistor, and (sometimes) a battery. Suppose the circuit has been flowing for a long time, so the current is unchanging and $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$, so the current is determined solely by the battery: $I_0 = \frac{\Delta V_{\mathrm{bat}}}{R}$. If a switch isolates the battery from the circuit, the inductor keeps the circuit going for some period of time as its magnetic field goes to zero, so the current decays instead of instantly falling to zero.
The differential equations for this circuit are very similar to those for an RC circuit, and the end result is that the current is $I(t) = I_0 e^{-\frac{Rt}{L}}$. One can define $\tau = \frac{L}{R}$ so that $I = I_0 e^{-\frac{t}{\tau}}$, then some calculations are simplified. The current undergoes exponential decay.

36 AC Circuits

36.1 AC Sources and Phasors

Many useful circuits have an emf that varies sinusoidally. Thus, they are called alternating current circuits (AC circuits). The steady-current circuits discussed in previous sections are direct current circuits, called DC circuits.
Generally, one can write the emf of an AC circuit as $\mathcal{E}(t) = \mathcal{E}_0 \cos\omega t$, where $\mathcal{E}_0$ is the maximum (or amplitude) and $\omega = 2\pi f$ the angular frequency. One can represent this as a conventional sinusoidal graph, but it is also possible to generate a phasor diagram, in which the sinusoid is graphed parametrically to yield a circle. This makes ω the angular frequency in a much more literal way, and then the instantaneous emf is just the projection of the phasor onto the x-axis.
Generally, in a circuit where current and voltage are changing, i and v are used to represent the instantaneous current and voltage, respectively, so (for example) Ohm's Law becomes $v_{\mathrm{R}} = i_{\mathrm{R}}R$ in AC circuits.[17] In a circuit consisting only of an AC source and a resistor, $v_{\mathrm{R}} = \mathcal{E}_0 \cos\omega t$; in general it is useful to write $v_{\mathrm{R}} = V_{\mathrm{R}} \cos\omega t$, since it varies sinusoidally. The current has a similar relationship, as $i_{\mathrm{R}} = I_{\mathrm{R}} \cos\omega t$. Most noticeably, the current and voltage oscillate in phase with one another.

36.2 Capacitor Circuits

In an AC circuit, a capacitor's charge obeys another oscillatory relationship $v_{\mathrm{C}} = V_{\mathrm{C}} \cos\omega t$ where V_{C} is the peak voltage across the capacitor. (In a simple circuit with only a capacitor and a source of AC emf, this is just $\mathcal{E}_0$, as before.) Solving a differential equation indicates that the current through a capacitor satisfies $i_{\mathrm{C}} = \omega C V_{\mathrm{C}} \cos\left(\omega t + \frac{\pi}{2}\right)$. The capacitor's voltage and current aren't in phase, and the current peaks before the voltage by $\frac{\pi}{2}$ radians. In some sense, this can be thought of as an object in simple harmonic motion, for which a similar relationship holds for velocity and position.
A useful notion of capacitive resistance can be introduced here: $X_{\mathrm{C}} = \frac{1}{\omega C}$. Then, $V_{\mathrm{C}} = I_{\mathrm{C}} X_{\mathrm{C}}$, and the units are ohms again. Notice that this relationship only holds for peak voltage and current, not instantaneous voltage and current. Also, since resistance depends inversely on frequency, it becimes very large at low frequencies (for whih the capacitor is a large obstacle to current). In the limiting case $\omega = 0$, which is just DC current, the current through a capacitor is zero, so the resistance idea makes sense.

[17] This use of i is why electrical engineers doing complex analysis use j for the square root of unity, causing much frustration among mathematicians and physicists trying to talk to each other!

36.3 RC Filter Circuits

RC circuits, in which a source of emf is in series with a capacitor and a resistor, generalize nicely to AC circuits. If the emf oscillates, the frequency affects the current: the lower it is, the lower the peak current. Thus, V_{R} will increase steadily as ω increases, and V_{C} will correspondingly decrease, in accordance with Kirchhoff's loop law. In particular, $\mathcal{E}_0^2 = V_{\mathrm{R}}^2 + V_{\mathrm{C}}^2$. On a phasor diagram, this becomes the Pythagorean Theorem.
For sufficiently low frequencies, the circuit is essentially capacitive, and for sufficiently high ones, the circuit is essentially resistive. However, there is a frequency at which the two voltages are equal. The crossover frequency can be easily found as $\omega_c = \frac{1}{RC}$.[18]
Sometimes, $f_c = \frac{\omega_c}{2\pi}$ is also called the crossover frequency.
Interestingly, at this crossover frequency, $V_{\mathbb{R}} = V_{\mathbb{C}} = \frac{\mathcal{E}_0}{\sqrt{2}}$. They don't add up to the peak emf, which seems odd until you realize it's a quadratic relationship.
It is worth considering these circuits as parts of a larger system. For example, consider an RC circuit with leads on either side of the capacitor. This could lead to voltage being measured or used by something else. In this case, a low-frequency input signal (well below the crossover frequency) means the output signal has no loss in magnitude. But for a sufficiently high-frequency input signal, the output is close to zero. This type of circuit is called a low-pass filter.
Analogously. a high-pass filter is one where the resistor voltage drop is used; it returns an output voltage nearly zero if the frequency is low relative to the crossover frequency, and a voltage $V \approx \mathcal{E}_0$ if the frequency is sufficiently high.
These can be useful in many applications where one wishes to separate signal from noise. However, more complicated filters are more common, since the crossover region is relatively broad.

36.4 Inductor Circuits

Much of the instantaneous inductor business is very similar to the equivalent quantities given for capacitors. The instantaneous inductor voltage is $v_{\mathrm{L}} = L\frac{\mathrm{d}i_{\mathrm{L}}}{\mathrm{d}t}$, and it satisfies $v_{\mathrm{L}} = V_{\mathrm{L}} \cos \omega t$. One can solve a differential equation to obtain $i_{\mathrm{L}} - I_{\mathrm{L}} \cos\left(\omega t - \frac{\pi}{2}\right)$, so the AC current through an inductor lags behind the voltage by a quarter period.
Inductive reactance can be defined in the same way as before, setting $X_{\mathrm{L}} = \omega L$, so that $V_{\mathrm{L}} = I_{\mathrm{L}} X_{\mathrm{L}}$. This increases as the frequency increases, which makes sense because the magnetic field changes more quickly if the frequency changes, and this increases the induced voltage across the coil.

36.5 The Series RLC Circuit

A circuit with a resistor, an inductor, and a capacitor in series is called a series RLC circuit. It is notable for its resonance behavior. Much of the circuit analysis should be familiar. For example, $i = i_{\mathrm{R}} = i + \mathrm{C} = i_{\mathrm{L}}$ and $\mathcal{E} = v_{\mathrm{R}} + v_{\mathrm{L}} + v_{\mathrm{C}}$. Supposing $V_{\mathrm{L}} > V_{\mathrm{C}}$, then the instantaneous current lags behind the emf by some angle ϕ (and if it's the other way 'round, then $\phi < 0$, which is fine; in fact, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Then, $\mathcal{E}_0^2 = V_{\mathrm{R}}^2 + (V_{\mathrm{L}}^2 - V_{\mathrm{C}}^2$, so the peak current in the circuit is

$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (X_{\mathrm{L}} - X_{\mathrm{C}})^2}} = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

The three peak voltages are $V_{\mathrm{R}} = IR$, $V_{\mathrm{L}} = IX_{\mathrm{L}}$, and $V_{\mathrm{C}} = IX_{\mathrm{C}}$.
The denominator in this equation is called the impedance of the circuit: $Z = \sqrt{R^2 + (X_{\mathrm{L}} - \mathrm{C}^2)}$. Like resistance and reactance, this is measured in ohms. This makes the previous equation simpler as just $\mathcal{E}_0 = IZ$. The phase angle can in fact be explicitly calculated (well actually it shouldn't be a surprise), and is $\phi = \tan^{-1}\left(\frac{X_{\mathrm{L}} - \mathrm{C}}{R}\right)$. In the degenerate cases of only two circuit elements, this definition agrees with rpeviously obtained results. Plenty of other relationships can be written down, such as $V_{\mathrm{R}} = \mathcal{E}_0 \cos \phi$, and so the resistor voltage only oscillates in phase if there is no phase shift.
Thanks to the combination of a capacitor and an inductor, there is very little current when the frequency is very small or very large (one of them cancels out each extreme). Thus, there is some frequency for which the current is a maximum. This is called the resonance frequency, and can be found as $\omega_0 = \frac{1}{\sqrt{LC}}$. At this frequency, the current is $I \max = \frac{\mathcal{E}_0}{R}$. This is the same as that of a purely resistive circuit, since $Z = R$ at the resonance frequency.
This is the same oscillatory frequency as for an LC circuit in Section 34.9, which makes sense. This LC circuit is analogous to a pendulum in simple harmonic motion without friction. Adding in the resistor is like dampening the oscillator;[19] just as a mechanical oscillator exhibits resonance with a large amplitude at a certain frequency, the RLC circuit exhibits resonance at a natural frequency. As R decreases, the dampening decreases and the circuit becomes more and more like the ideal one. Additionally, since the circuit is purely resistive, the phase angle is 0. Generally, it is negative when the frequency is below resonance and the capacitance overwhelms the inductance, and it is positive when the frequency is greater than ω_0, and the inductance is stronger than the capacitance.

[18] Observe that this makes dimensional sense, since RC has units of time.
[19] Given the language this book uses, I'm expecting differential equations to jump out from behind a corner and attack me. But seriously, I feel like there's something related to them here.

Because of their ability to respond to one particular frequency, resonance circuits are used in many radio and communications devices. The circuit gets better as the resistance decreases, but it will never be perfectly zero.

36.6 Power in AC Circuits

Power is also variable, and so one defines the instantaneous power to be $p = i\mathcal{E}$ (specifically, this is the power supplied to the circuit by the emf).
The power dissipated by a resistor is $p_{\mathrm{R}} = i_{\mathrm{R}} v_{\mathrm{R}} = i_{\mathrm{R}}^2 R = I_{\mathrm{R}}^2 \cos^2 \omega t$. This oscillates twice per cycle of emf, and the energy dissipation peaks when $i_{\mathrm{R}} = \pm I_{\mathrm{R}}$. Generally, the average power dissipated by the resistor is more important, and can be found by $P_{\mathrm{R}} = \frac{I_{\mathrm{R}}^2 R}{2}$. It is useful to write this as $P_{\mathrm{R}} = (I_{\mathrm{rms}}^2)R$, where $I_{\mathrm{rms}} = \frac{I_{\mathrm{R}}}{\sqrt{2}}$ is the root-mean-squared current. But comparing this to the DC equation, it's clear that the rms current is equivalent to the DC current in terms of power generation. Analogously, there exist rms voltage and emf $V_{\mathrm{rms}} = \frac{V_{\mathrm{R}}}{\sqrt{2}}$ and $\mathcal{E}_{\mathrm{rms}} = \frac{\mathcal{E}_0}{\sqrt{2}}$, so that $P_{\mathrm{R}} = \frac{V_{\mathrm{rms}}^2}{R} = I_{\mathrm{S}} V_{\mathrm{S}}$ and the power of the source is $P = I_{\mathrm{rms}} \mathcal{E}_{\mathrm{rms}}$. Voltmeters, ammeters, and other such devices in AC circuits give the rms value.
Similarly, one can define the power of a capacitor as $p_{\mathrm{C}} = v_{\mathrm{C}} i_{\mathrm{C}} = -\frac{1}{2}\omega C V_{\mathrm{C}}^2 \sin 2\omega t$. The capacitor alternately charges and discharges (so negative power just means power is being put back into the circuit). Interestingly, this means that the average power is zero. The same is true for inductors: $P_{\mathrm{C}} = P_{\mathrm{L}} = 0$.[20]
So on average, power is added by the source and dissipated by the resistor in an RLC circuit. But they are out of phase, so the average power of the source is $P = \frac{1}{2} I \mathcal{E}_0 \cos\phi = I_{\mathrm{rms}} \mathcal{E}_{\mathrm{rms}} \cos\phi$. The term $\cos\phi$ is sometimes called the power factor, and arises because the current and emf aren't in phase.
The current in an RLC circuit can be written $I = I\max\cos\phi$, and so the average power of the source is just $P = P\max\cos^2\phi$, where $P\max = \frac{I\max \mathcal{E}_0}{2}$ is the maximum power that the source can deliver to the circuit. This can only happen when $\cos\phi = 1$, which corresponds to the phase angle being zero and everything being in phase (i.e. in resonance). If the load is purely capacitive or inductive (so $\phi = \pm\frac{\pi}{2}$), the average power loss is zero.

16 Ideal Gases

16.5 The Ideal Gas Law

Sections 1 to 4 are review of things I've learned at least thrice before and as such are omitted.

It should be no surprise that for an ideal gas, $PV = nRT$, where P is pressure, V is volume, n is amount (typically measured in moles), R is the gas constant, and T is the absolute temperature. $R = 8.31 \frac{\mathrm{J}}{\mathrm{mol\,K}}$.

It is also possible to state this law in terms of molecules: if N is the number of molecules in a gas and k_B is Boltzmann's constant (given by $k_B = R/N_A$, where N_A is Avogadro's number), then $PV = Nk_BT$. $k_B = 1.38 \cdot 10^{-23} \frac{\mathrm{J}}{\mathrm{K}}$.

16.6 Ideal Gas Processes

The three quantities pressure, temperature, and volume all affect each other when the state of the gas changes. These can be graphed on a PV Diagram, which just puts volume on the x-axis and pressure on the y-axis. Since temperature can be uniquely determined from these two quantities, these represent all possible states of the gas.

Strictly speaking, the ideal gas law only applies when the gas is in thermal equilibrium, but these processes change states in such a way that does not allow this. However, if they happen very slowly, then they can be approximated as being in thermal equilibrium. These sorts of processes are called quasi-static processes, and are often good approximations to real ones. An important characteristic of quasi-static processes is that they are reversible.

An isochoric process is one which preserves volume (and thus represents a vertical line on a PV diagram). An example of this is warming a gas in a rigid container.

An isobaric process is one which preserves pressure (and thus is graphed as a horizontal line). An example of this is heating a gas in a container capped by a piston, so that the volume an temperature change so that the pressure remains constant.

An isothermal process is one which keeps temperature constant. These processes appear as hyperbolas on the PV diagram, with equations $y = k/x$. An example of uch a process is pushing a piston into a container of gases while it is submerged in a large sink of constant temperature. Individual curves that represent isothermal processes are called isotherms, and each one corresponds to a unique temperature.

[20] Of course, this assumes ideal capacitors and inductors. Real ones have a nonzero resistance and so this is not completely true. However their energy dissipation is small compared to that of resistors in these sorts of circuots.

17 Work, Heat, and the First Law of Thermodynamics

17.2 Work in Ideal-Gas Processes

Much of this involves concepts from Section 11, so it may be helpful to review it.

Importantly, unlike mechanical or thermal energy, work is not a state variable; the path one takes between states does affect the work done along that path. Thus, one will never see the quantity ΔW.

The work done *on* a gas *by* its environment in a process where the pressure and/or the volume change is $W = -\int_{V_i}^{V_f} p\,\mathrm{d}V$. Conceptually, using PV-diagrams can help visualize how work occurs in various processes:

- In an isochoric process, $\mathrm{d}V = 0$, and so $W = 0$ as well. The converse is also true; $W = 0$ implies the process is isochoric.
- In an isobaric process, p is constant and can be brought out of the integral, so $W = -p\Delta V$. The sign of ΔV clearly matters; it is positive if the gas expands, so that the gas does work on the environment, and is negative if the gas contracts, in which case the system does work on the gas.
- An isothermal process actually requires knowing a nontrivial amount of integration. There are several variations of th work equation, given by playing with $PV = nRT$:

$$W = -nRT\ln\left(\frac{V_\mathrm{f}}{V_\mathrm{i}}\right) = -p_\mathrm{i}V_\mathrm{i}\ln\left(\frac{V_\mathrm{f}}{V_\mathrm{i}}\right) = -p_\mathrm{f}V_\mathrm{f}\ln\left(\frac{V_\mathrm{f}}{V_\mathrm{i}}\right).$$

Of course, this is all only true if one uses SI units and degrees Kelvin.

17.3 Heat

Heat is a more elusive concept than work; intuitively, it is not very much like our macroscopic conception of heat, and represents instead energy transferred between a system and its environment as a result of a temperature difference between them. This requires no macroscopic motion.

Heat, like work, is not a state function, and again, it only appears as Q, never ΔQ. Heat is positive when energy enters the system (which can be done by doing work on the system), and is negative when energy leaves the system.

Note that heat is not temperature, and neither is it thermal energy; temperature is a state variable that quantifies which direction heat transfer would be in between two objects, and thermal energy is energy of a system, not between a system and its environment.

17.4 The First Law of Thermodynamics

Given the total energy of a system is the sum of its mechanical and thermal energies, if a system is at rest, then all energy is thermal. Then, the First Law of Thermodynamics states that $\Delta E_\mathrm{th} = W + Q$. Interestingly, the sum of two non-state quantities is a state function.

Note that while this provides information about the change in thermal energy, it doesn't say anything about the thermal energy as a whole. And the thermal energy is often not the only thing that changes; one may also find the pressure, volume, temperature, etc. to be affected as well.

Various special cases illustrate this law. For example, in an isochoric process (such as heating or cooling a gas in a rigid container), $W = 0$ and so $\Delta E_\mathrm{th} = Q$.

In an isothermal process, $\Delta E_\mathrm{th} = 0$, so $W = -Q$. An example of this is to reduce the pressure on a gas as its volume increases while it is heated to keep it at a constant temperature. The heat energy added to the gas is used to do work.

In an adiabatic process (i.e. one where $Q = 0$), $\Delta E_\mathrm{th} = W$, as in when a gas insulated from its surroundings is compressed. However, it is very possible for the temperature to change, even though $Q = 0$; an adiabatic compression lowers the temperature, and an adiabatic expansion increases it (since it does work on the gas).

17.5 Thermal Properties of Matter

Heat and temperature are not the same; in fact, different substances require different amounts of heat to create the same change in temperature (per unit mass, volume, amount, etc.). The specific heat of a substance is the amount of energy that raises the temperature of 1 kg of a substance by 1 K. For example, $c_{\mathrm{H_2O}} = 4190\frac{\mathrm{J}}{\mathrm{kg\,K}}$, but most metals have specific heats four to 40 times lower. In this sense, water has a high "thermal inertia" relative to, say, copper; this difference has a number of useful biological properties.

This relates heat and thermal energy: $\Delta E_\mathrm{th} = mc\Delta T$. But if $W = 0$, then this leads to the simpler formula $Q = mc\Delta T$.

The molar specific heat is the amount of energy that raises the temperature of 1 mol of substance by 1 K, and is denoted C. This satisfies the equation $Q = nC\Delta T$, where n moles of substance are being heated (or cooled). The molar specific heat for elemental solids tends to be very close to $25\frac{\mathrm{J}}{\mathrm{mol\,K}}$, which is an interesting peek into microscopic-level thermal energy.

Latent heats have been covered in innumberably many chemistry classes already and thus have been omitted.

17.6 Calorimetry

Calorimetry is a process in which several systems interact and exchange only heat energy (doing no work and losing no energy to the outside environment). Remembering that $Q_{\text{net}} = 0$ will solve most of these types of problems.

17.7 The Specific Heats of Gases

Gases are a little more interesting because heat is not a state variable with respect to pressure and volume. Thus, one can talk about specific heats for different processes:

- In an isochoric process, $Q = nC_V\Delta T$, where C_V is the molar specific heat at constant volume.
- In an isobaric process, $Q = nC_P\Delta T$, where C_P is the molar specific heat at constant pressure.

Both of these quantities have units of $\frac{\text{J}}{\text{mol K}}$.

These specific heats have some curious properties: first, they all seem identical for monatomic gases, and second, $C_P - C_V = R$. This is because ΔE_{th} is the same for any processes for which ΔT is also the same, and vice versa. Given this, one can substitute in $PV = nRT$ and discover that for an ideal gas, $C_P = C_V + R$. It is not just an observational coincidence. This also means that for any ideal-gas process, even non-isochoric ones, $\Delta E_{\text{th}} = nC_V\Delta T$ holds. What changes is the heat.

In an adiabatic process, $Q = 0$, so $\Delta E_{\text{th}} = W = nC_V\Delta T$. Since compressing a gas does work on it, an adiabatic compression raises the temperature of a gas (so, of course, an adiabatic expansion lowers the temperature of a gas).

Similarly to the other types of ideal-gas processes, adiabatic processes can be described by paths in the PV-space. Defining the specific heat ratio as $\gamma = C_P/C_V$ (which is 1.67 for a monatonic gas and 1.40 for a diatomic gas), one can show that in an adiabatic process, PV^γ is constant. These curve of constant PV^γ are called adiabats. Similarly, one can substitute to find $TV^{\gamma-1}$ is also constant in adiabatic processes.

18 The Micro/Macro Connection

18.1 Molecular Speeds and Collisions

Much of this discussion is based on statistical properties of the velocities of molecules (e.g. why the temperature represents the average kinetic energy of the molecules in a system); as such, prepare for some very magical or handwaved statistics. In general, though, the micro/macro connection is built on the idea that macroscopic properties of a system are related to the average behavior of its molecules.

A molecule moving around moves in straight-line paths between collisions (assuming an ideal gas). But the random distribution of the molecules means that the lengths of these paths vary, and so one instead finds their average. This quantity, called the mean free path, is denoted l and represents the number of collisions per unit distance.

Modelling ideal-gas molecules as billiard balls of radius r, the mean free path can be calculated to be $l = \frac{1}{4\sqrt{2}\pi(N/V)r^2}$.

18.2 Pressure in a Gas

Macroscopically, the pressure of a gas is due to an innumerable number of collisions of a gas with the walls of its container. Thus, in order to calculate the pressure, one should calculate the impulse of a single collision and then find the total number of collisions.

Suppose there are N collisions during an interval of time Δt with the same velocity v_x. Then, the net impulse is $J = 2Nmv_x$, where m is the mass of a single molecule. Thus, the average force is $F = \frac{2N}{\Delta t}mv_x$. $N/\Delta t$ is the rate of collision, which when Δt is very small (smaller than the time interval between collisions) can be calculated as $\frac{N_m}{2V}A_x$, where N_m is the number of molecules, A is the area of the wall, V presumably the volume (though the book fails to define it), and the other quantities are as before.

Thus, the average force is $F = \frac{N_m}{V}mv_x^2A$, from which pressure can be obtained.

This calculation depends on the speed of the individual molecules, which satisfy a probability distribution. Thus, this is not quite straightforward, and so one refers to the root-mean-squared-speed v_{rms}, which is exactly what it says: the square root of the average velocity squared. This is an average that makes sense, and is easier to calculate and more natural than the actual average speed.

Using this, the pressure becomes $p = \frac{N_m}{3V}mv_{\text{rms}}^2$.

18.3 Temperature

A molecule with mass m and velocity v has translational kinetic energy $\varepsilon = \frac{mv^2}{2}$ (where ε is used to denote the molecular energy and distinguish it from the system's energy E).

Using the ideal gas law and the pressure from Section 18.2, the average translational kinetic energy is $\varepsilon_{\text{avg}} = \frac{3}{2}k_BT$, where T is the temperature and k_B is Boltzmann's constant. Thus, temperature really measures the average translational kinetic energy; higher temperatures correspond to higher molecular speeds on average.

Using $\varepsilon_{\text{avg}} = \frac{1}{2}mv_{\text{rms}}^2$, one also has $v_{\text{rms}} = \sqrt{\frac{3k_BT}{m}}$, which helps conceptualize the root-mean-squared speed.

18.4 Thermal Energy and Specific Heat

In a monatomic ideal gas, the thermal energy is composed entirely of translational kinetic energy, so $E_{\text{th}} = N\varepsilon_{\text{avg}} = \frac{3}{2}Nk_BT = \frac{3}{2}nRT$. This is significant: temperature and thermal energy are related, and for a monatomic gas they are directly proportional.

Given this and that $\Delta E_{\text{th}} = nC_V\Delta T$, one has that $C_V = 3R/2 = 12.5\frac{\text{J}}{\text{mol K}}$ for a monatomic ideal gas.

In some sense, one can consider the degrees of freedome for a system as the number of different methods of storing energy (since this leads to systems of equations). In a monatomic gas, there are three degrees of freedom, since there are three directions (x, y, and z) of translational kinetic energy.

An important result of statistical physics shows that energy is equally distributed between these modes of energy storage, called the equipartition theorem. For a system of n moles at temperature T, the energy stored in each degree of freedom is $\frac{nRT}{2}$. From this, the equation for the energy of a monatomic gas follows.

Solids have six degrees of freedom: in addition to kinetic energy in any particular direction, the atomic bonds can vibrate or be compressed in any particular direction. Thus, the thermal energy of a solid is $E_{\text{th}} = 3Nk_BT = 3nRT$.

This can be used to predict the molar specific heat of a solid: if $\Delta E_{\text{th}} = nC\Delta T$, then $C = 3R = 25.0\frac{\text{J}}{\text{mol K}}$. This is not completely accurate, because the models aren't completely accurate, but it does an impressive job as a predictor.

Diatomic molecules are somewhat more difficult: they have more degrees of freedom because they may rotate and vibrate, though not along the axis of the bond. Thus, there are eight degrees of freedom, and so one would expect $E_{\text{th}} = 4k_BT$. This is not the case, and it would seem that there "should be" five degrees of freedom according to the model. This conundrum was among the major issues that statistical physics grappled with in the late 19^{th} Century.

Like everything else that didn't work at that time period, modern physics is the confounding factor, and quantum effects prevent three of the modes from being active at room temperature. Thus, the equations are $E_{\text{th}} = \frac{5}{2}Nk_BT = \frac{5}{2}nRT$ and $C_V = \frac{5R}{2}$.

18.5 Thermal Interactions and Heat

Using the above, it is possible to more precisely understand heat exchange. Consider two systems with N_1 and N_2 molecules, respectively, and that are at temperatures T_1 and T_2, respectively. They are separated by a semipermeable membrane, such that atoms can collide across the boundary but not pass through it. Then, the systems will exchange energy: if a higher-energy atom (one moving quickly) collides at the boundary with a lower-energy, slower one, the first atom will lose energy while the second gains it. Thus, heat energy is transferred by collisions across the boundary.

Since temperature measures average kinetic energy, the temperatures will tend towards an equilibrium, where the energy transfer on both sides is the same (implying the systems have the same average kinetic energies). It should be no surprise that the total energy is constant and the heat lost by one is gained by the other.

18.6 Irreversible Processes and the Second Law of Thermodynamics

A big point in statistical mechanics is that, while it is possible for collisions to transfer energy from a lower-energy system to a higher-energy one, this doesn't happen. Heat flow is an irreversible process, since it only happens in one direction. Newtonian mechanics contains no irreversible processes, as, for example, a collision between two particles is just as probable in the other direction. However, everyday life abounds with irreversible processes (such as breaking a glass beaker) in which the arrow of time points very obviously forward.

This paradox is resolved by considering the number of possibilities in each state. Equilibrium is the most likely option, which can be justified with some statistics. Tending away from equilibrium is possible, but not probable, and on macroscopic scales is so unlikely that it can be safely ignored.

A state variable called entropy can be used to quantify the probability of a macroscopic state happening spontaneously. Low entropy is associated with high-energy, ordered states, and high entropy is associated with low-energy, disordered states, such as the equilibrium. Thus, entropy is said to measure the disorder of a system.

The Second Law of Thermodynamics simply says that entropy never decreases; it is either constant (if the system is at equilibrium) or increases until the system reaches equilibrium. This is qualified by requiring that the system be isolated; adding energy from some outside source makes this less viable, but including the outsie source in the system causes the system to tend towards an equal distribution of energy overall. Unlike the First Law, this is a probabilistic statement; but it is no less true. It also establishes the direction of time.

19 Heat Engines and Refrigerators

19.1 Turning Heat Into Work

Unlike mechanics, where work is conventionally considered to be done on a system, thermodynamics sometimes speaks of work as done by the system (e.g. a heat engine) on the environment. This is a careful juggling of signs and on/by, but the two aren't mutually exclusive.

An energy reservoir is a part of the environment that is sufficiently large that its temperature doesn't change when heat is transferred into or out of it. An example is the ocean; putting a hot coal into the ocean cools the coal, but warms the vast ocean such a little bit so as not to be meaningful. A reservoir at a higher temperature than the system is called a hot reservoir, and one colder than the system is called a cold reservoir. Often, their temperatures will be written T_{H} and T_{C}, respectively. For another object, one can define the heat transferred to or from that object due to interacting with a reservoir (these heats are conventionally written Q_{H} and Q_{C}.

These sorts of processes can be formalized in an energy-transfer diagram, which represents a hot reservoir above a depiction of the system or object interacting with it, and a cold reservoir below it. Then, it is easy to depict heat moving around the system. Work is usually represented horizontally: work on the system comes in from the left as a "pipe," and work leaving the system exits at the right.

Turning work into heat is not difficult; try rubbing two sticks together until they are warmer (or if you're dedicated, catch fire). This conversion is 100% efficient; all the energy supplied by the work is turned into heat. However, the reverse is a lot harder. One can use heat to expand a gas and do work, but this is not a repeatable process, and thus doesn't work for engines. However, by the Second Law of Thermodynamics, no such engine will be perfectly efficient.

19.2 Heat Engines and Refrigerators

A heat engine is a closed-cycle device that extracts heat from a hot reservoir, does work on the environment, and puts exhaust heat into a cold reservoir. (A closed-cycle device is one that periodically returns to its initial conditions for all state variables.) Since the engine is closed-cycle, there is no net change in E_{th}, so $Q - W = 0$, and $W = Q_{\mathrm{H}} - Q_{\mathrm{C}}$ per cycle.

Practically, one is also insterested in the efficiency, which is defined as $\eta = W/Q_{\mathrm{H}}$. This is always less than 1, since a heat engine must exhaust some waste heat to a cold reservoir, and in fact practical heat engines have low efficiency: $\eta \approx 0.5$.

An example of a heat engine is a gas piston that takes a load, expands, removes the load, and then conracts back to its original position. Some heat source has to power its expansion, and a heat sink must allow its contraction.

A refrigerator is the opposite of a heat engine: it is a closed-cycle system that uses outside work to remove heat from a cold reservoir and exhaust (hopefully more) heat to a hot reservoir. In some sense, this involves pumping the heat "uphill," against its natural flow into the cold system. Efficiency of a refrigerator is $K = Q_{\mathrm{C}}/W$, which by the Second Law must be finite. In particular, a refrigerator must exhaust more heat than it removes from the inside of its system.

Still, it remains to be shown why the limits on these systems' efficiencies are so unexpectedly low.

19.3 Ideal-Gas Heat Engines

A gas heat engine can be represented as a closed cycle on a PV diagram, so that the work done is the area of the interior of the trajectory of the process. In order for this work to be positive, the curve has to have clockwise orientation.

One can solve heat-engine problems by drawing the PV diagram and then computing work, heat, etc. for each process.

A real-world example of a heat engine is the Brayton cycle, which is used in gas turbine engines. The Otto cycle and Diesel cycle operate in much the same way. At the beginning of the cycle, air undergoes an adiabatic compression ($Q = 0$ because there is no time for heat to enter or leave). Hot gas enters the combustion chamber and is combusted (which is an isobaric process that can be thought of as getting heat from the hot reservoir). Then, the air-gas mixture undergoes adiabatic expansion, spinning a turbine that does work of some sort, and the gas is cooled by flowing through a heat exchanger that transfers energy to a coolant (the cold source). Cooling is an isobaric compression.

For an adiabatic process, the pressure ratio is $r_p = p_{\max}/p_{\min}$, and $T_1 = r_p^{\frac{1-\gamma}{\gamma}} T_2$. This means that the efficiency of a Brayton cycle is $\eta_B = 1 - r_b^{\frac{1-\gamma}{\gamma}}$, so a higher pressure ratio implies better efficiency. However, the gains are increasingly marginal, and can be considered as a tradeoff.

19.4 Ideal-Gas Refrigerators

Similarly to the previous section, one can run a cycle backwards to get a refrigerator. This operates in ways that aren't necessarily surprises; however, it is important to recall that heat is always transferred from a hotter object to a colder one, so in order to extract heat from the cold reservoir, the refrigerator must have a lower temperature; in order to push heat onto the hot reservoir, the refrigerator must be warmer than it. This is slightly different than for a heat engine. So a refrigerator is not just a heat engine running backwards, though the two share many similar processes.

19.5 The Limits of Efficiency

One of the important questions in thermodynamics is: is there some limit $\eta_{\max} < 1$ on efficiency, or can one make heat engines or refrigerators that have efficiency arbitrarily close to 1?

Consider a heat engine and its corresponding refrigerator such that each has the opposite work and heat transfer as the other (same magnitude). This system is called a perfectly reversible engine, and (like all perfect things) doesn't actually exist. But it serves as a nice model — its efficiency in either direction is $\frac{Q_H - Q_C}{Q_H}$. But no heat engine can have a greater efficiency: if it did, reversing it would lead to an impossible process; one could link it to its refrigerator, and would then be able to transfer heat from a cold reservoir to a hot one.

Thus, the efficiency of the perfectly reversible heat engine is $\eta_{\max}$! This is also true, by the same argument, for refrigerators.

An engine that is perfectly reversible is also called a Carnot engine. In order for such an engine to be maximally efficient, it must only use frictionless and isothermal processes (which of course is nothing more than a nice model).

19.6 The Carnot Cycle

The definition of a Carnot engine doesn't specify whether the working substance is a gas or a liquid, but it makes no difference: the Carnot engine is maximally efficient, so nothing else really matters. In particular, any two Carnot engines operating between hot and cold reservoirs of the same temperatures (i.e. $T_{H_1} = T_{H_2}$ and $T_{C_1} = T_{C_2}$) have the same efficiency.

The Carnot cycle is an ideal-gas cycle that uses two adibatic processes and two isothermal processes to create an idealized perfectly reversible gas engine. It operates in the following steps:

1. The gas is isothermically compressed while in contact with the cold reservoir. In order to keep the temperature constant, heat energy is removed. Since there cannot be a significant temperature difference between the gas and the reservoir, this step happens very slowly.

2. The gas is adiabatically compressed while thermally isolated from the environment. This compression increases the temperature until it is equal to the temperature of the hot reservoir.

3. The gas expands isothermally while constantly at the temperature of the hot reservoir, transferring heat from the hot reservoir into the gas.

4. The gas expands adiabatically, returning to its starting conditions, as the temperature decreases to that of the cold reservoir.

One can show that $Q_C/Q_H = T_C/T_H$, so the efficiency of a Carnot engine is $\eta = 1 - T_C/T_H$.[21] This is the maximum efficiency (and similarly, for a refrigerator, the maximum efficiency is $K = \frac{T_C}{T_H - T_C}$. Since the temperatures of hot and cold reservoirs are often dictated by the condions of the application, this puts a significant upper bound on efficiency that is stricter than just the rough one given by the Second Law of Thermodynamics.

20 Traveling Waves

20.1 The Wave Model

Formally, a traveling wave is an organized disturbance traveling at a well-defined wave speed. These fall into the categories of transverse (displacement perpendicular to direction of motion) and longitudinal (displacement parallel to the direction of motion).

There are several types: mechanical waves travel over some sort of medium, electromagnetic waves travel through an electromagnetic field, and matter waves' underlie quantum physics.

20.2 One-Dimensional Waves

There are several ways to view one-dimensional waves: you can graph the displacement of a particle over a time t, or the location of a set of particles on an xy-plane at a given time t¡ or a "history graph" in three dimensions. Longitudinal waves can be represented similarly, though one may plot the x-axis against the x-axis or such.

20.3 Sinusoidal Waves

A wave source that oscillates with simple harmonic motion generates a sinusoidal wave with wave equation $D(x, t) = A \sin(kx - \omega t + \phi_0)$ (with amplitude A, wave number $k = 2\pi/\lambda$, and phase ϕ_0).

A wave on a stretched string has speed $v = \sqrt{T_s/\mu}$, where T_s is the tension on the string and μ its linear density.

[21] These temperatures must be measured in Kelvin, of course.

20.4 Waves in Two and Three Dimensions

In two dimensions, waves crests form circles, which are called wave fronts, and are spaced exactly one wavelength apart. Once the waves are sufficiently far from the source, they can be thought of as linear.

Three-dimensional waves are spherical, though when large they can be thought of as plane waves.

One can rewrite the wave equation in terms of the distance r from the source, so that $D(r,t) = A(r)\sin(kr - \omega t + \phi_0)$, where A is no longer constant (to allow for conservation of energy and such).

The phase of a wave is $\phi = kx - \omega t + \phi_0$.

The phase difference between two points on a sinusoidal wave is $\Delta\phi = 2\pi\Delta x/\lambda$. This measures the "angle" between them, in the same sense of angle as the angular frequency is measured. This depends only on the ratio of the separation of the two points to the wavelength.

The next section is review and has been omitted.

20.6 Power, Intensity, and Decibels

A wave transfers energy from one point to another, and so has a given power P and an intensity $I = P/A$ (i.e. power per unit area). This leads to the relation for spheres: $I_1/I_2 = r_2^2/r_1^2$. This also leads to a relation between intensity and amplitude: $I \propto A^2$.

Loudness is measured by intensity, from the threshold of hearing on the quiet end to the threshold of pain on the loud end. Thus, one can define the decibel scale $\beta = 10\log(I/I_0)$, where $I_0 = 1.0 \cdot 10^{-12}\,\mathrm{W}/\mathrm{m}^2$.

20.7 The Doppler Effect

Conceptually, this is nothing new, but the formulas are reproduced here.

1. For a moving source s and a stationary observer o:
   - If the source approaches the observer, $f_+ = \frac{f_0}{1-v_s/v}$.
   - If the source recedes from the observer, $f_- = \frac{f_0}{1+v_s/v}$.
2. If the source is stationary and the observer is in motion:
   - $f_+ = (1 + v_0/v)/f_0$ if the observer approaches the source.
   - $f_- = (1 - v_0/v)/f_0$ if the observer recedes from the source.

21 Superposition

21.1 The Principle of Superposition

When two or more waves are simultaneously present at a point, the displacement at that point is equal to the sum of the individual displacements due to each individual wave. This allows waves to "pass through" each other.

21.2 Standing Waves

A standing wave is a vibration in which both ends are fixed, so the wave is in superposition with itself. This creates nodes and antinode; the former are the points that do not move, and the latter have the maximum amplitude. The nodes exist because they are sites of destructive interference and the antinodes have constructive interference.

A standing wave is given by the equation $D(x,t) = A(x)\cos\omega t$, where the amplitude is $A(x) = 2a\sin kx$. Notice that this is not a travelling wave; instead, each point oscillates in simple harmonic motion with frequency $f = \omega/2\pi$.

21.3 Transverse Standing Waves

Consider a transverse wave on a string which is fixed on both ends. At the boundary, it is flipped, though the amplitude (as well as the wavelength and frequency) remains unchanged.

Here, a boundary condition appears, that the displacements at the edges must be zero. With a bit of pushing math around, this solves to the wavelength of a standing wave that doesn't destroy itself when it sets up oscillations is $\lambda_m = 2L/m$, for $m \in \mathbb{Z}$. Then, the frequencies are $f = \frac{mv}{2L}$. The longest such frequency is f_1, the fundamental frequency, and the others are called harmonics. The possible standing waves are the normal modes of the string.

Note that m is the number of antinodes, not the nodes. Avoid off-by-one errors. Additionally, the fundamental node has wavelength equal to twice the length of the string.

Additionally, standing electromagnetic waves are lasers, which are pretty cool.

21.5 Interference in One Dimension

Interference is the term used to describe the pattern of two waves in superposition. While a standing wave (Section 21.2) is produced by interference between waves travelling in opposite directions, this section will address interference of waves traveling in the same direction.

Suppose two waves are sinusoidal, with the same frequency, amplitude, and direction (which will be represented as towards the positive x-axis). Then, the two wave equations are

$$D_1(x,t) = a\sin(kx - \omega t + \phi_{1,0})$$
$$D_2(x,t) = a\sin(kx - \omega t + \phi_{2,0}),$$

so the only thing that actually differs is the phase (and sometimes not even that).

Two waves that are aligned crest-to-crest (and therefore also trough-to-trough) are said to be in phase. When they are combined, they produce constructive interference. If the waves are exactly in phase, this is called maximum constructive interference.

Correspondingly, waves where crests align with troughs are called out of phase, and create destructive interference. If the two are exactly out of phase, the destructive interference leads to a net zero displacement, called perfect destructive interference.

One can define the phases of the two waves as $\phi_1(t) = kx_1 - \omega t + \phi_{1,0}$ (and ϕ_2 is defined analogously). Then, the phase difference between the two waves is $\Delta\phi = 2\pi\Delta x/\lambda + \Delta\phi_0$, in which the (horizontal) distance between the two sources, Δx, is called the path-length difference and $\Delta\phi_0 = \phi_{2,0} - \phi_{1,0}$ is called the inherent phase difference between the two sources.

Maximum constructive interference occurs when the path-length is an integral multiple of the wavelength, in which case $\Delta\phi = 2\pi m$, where $m = 0, 1, \ldots$, producing a wave of amplitude $2a$.

Conversely, destructive interference occurs when the path-length difference between two identical sources is a half-integer multiple of the wavelength: $\Delta\phi = 2\pi(m + 1/2)$, where $m = 0, 1, \ldots$, though two waves with no path-length difference can be out of phase if their sources are.

21.6 The Mathematics Of Interference

Interference is useful in various applications, such as thin-film optical coatings, which help prevent reflections by creating destructive interference with them. This is a bit more convoluted because of indices of refraction, so the wavelengths of the waves might change.

21.7 Interference in Two and Three Dimensions

The mathematics of multidimensional interference are a bit more complicated, but it is important to view the wave fronts as in motion as expanding circles. If two or more wave fronts are in the same plan, they will produce patterns of interference, with points of constructive interference at the intersections of crest circles and destructive interference at trough circles. Interestingly, though, the points of constructive and destructive interference are unaffected by the motions of the waves themselves. The equations for the locations of the distance from the source are identical, which isn't too surprising.

All points of maximum constructive interference lie on a set of hyperbolas with the same foci (the sources). Oddly enough, individual portions of this family of curves are called antinodal lines, and the hyperbolas of destructive interference are called nodal lines. The displacement is always zero along these lines.

There are various ways to picture this sort of interference, such as three-dimensional graphs or contour plots. Color can help make the pictures clearer, too.

22 Wave Optics

22.4 Single-Slit Diffraction

Diffraction is how light interferes with itself after passing through narrow slits. If there is only one such slit, the diffraction is called single-slit diffraction.

If a viewing screen is placed behind the slit, a pattern of light and dark bands appears, with the largest, brightest band at the center (central maximum; the other maxima are called secondary maxima).

Much of the previous analysis of diffraction treated waves as point sources, but it will be helpful to think of the propagation of an extended wave front. According to Huygens' principle, each point on a wave front is the source of a spherical wave that spreads out at the wave speed, and eventually, the shape of the wave front is the line tangent to all of the wavelets. From this, the linear and spherical wave fronts make more sense.

This leads to analyses of wavelets travelling at different angles. Those that travel directly forward, perpendicular to the direction of the slit, all travel the same distance to the screen. Thus, they arrive in phase, and interfere constructively with each other, producing the wide, bright central maximum.

For points away from the center, the waves arrive out of phase, so there is some destructive interference, and the amount of constructive interference is given by whether things line up nicely. The dark fringes (maximum destructive interference) occur when $\theta_1 = p\lambda/a$, for $p \in \mathrm{N}$. Notice that $p = 0$ is explicitly not a dark fringe.

However, finding the maxima is not as simple, and destructive interference can still occur, making life unnecessarily complicated, and calculating their locations is slightly beyond the scope of the textbook.

Using the small-angle apprixomation, the positions of the dark fringes are at $y_p = p\lambda L/a$, for $p \in \mathrm{N}$, so the width of the central maximum is $w = 2\lambda L/a$. Thus, the width of the central maximum is twice the spacing between the dark fringes on either side. This means that the smaller the slit, the wider the central maximum, contrary to intuition.

22.5 Circular-Aperture Diffraction

This is a bit more mathematically convoluted than in the linear case, but there end up being a series of concentric dark and light circles, with the brightest (which corresponds to the widest band) at the center. The width of the central maximum is $w = 2L \tan\theta_1 \approx 2.44\lambda L/D$. This is slightly different than in the previous case.

This sort of diffraction is important for establishing the wave versus the ray model of light. Waves spread out differently behind a slit: the smaller the wavelength, the longer distance it takes for a wave to spread horizontally over an area. Thus, waves of small wavelength act more like beams.

The ray model can be used when the spreading due to diffraction is less than the size of the opening, since then there is no noticeable diffraction. The crossover length is $D \approx \sqrt{2.44\lambda L}$, which for visible light is about 1 mm; if the opening is significantly larger than D, use the ray model. Note that around 1 mm, it is ambiguous, but this edge case will be avoided for now.

22.6 Interferometers

Interference is helpful because lots of little ingenious applications allow for some very precise measurements. One example of this is the interferometer.

The basic idea behind an interferometer is to take a wave of an unknown wavelength and then split it. Make one of the two waves travel farther than the other, and then recombine them in a way that causes interference. After measuring the interference, one cna calculate the wavelength. If the interferometer has a slide that allows the difference in distance to change, the number of maxima oer unit length is $\Delta m = 2\Delta L/\lambda$, where ΔL is the change in length. This allows one to measure the wavelength accurately.

The Michelson interferometer is an optical interferometer (though one can use acoustical interferometers and such as well) that is widely used in experiments to measure wavelengths and other distances. The light wave is divided by a beam splitter, a partially reflective mirror, and then pushed around by that can be moved. The interference pattern is of concentric cicles.

One can use the interferometer to measure indices of refraction as a function of wavelength, since the number of wavelengths changes if the index of refraction does. This leads to $\Delta m = (n-1)2d/\lambda$, where λ is measured in a vacuum.

Interferometers also happen in holography, where the complex interference pattern of a 3-D object is recorded on a piece of film. But then, shining a reference beam through this film recreates the original wavefronts, reconstructing the light that was emitted by the object.

23 Ray Optics

23.1 The Ray Model of Light

This model is a further simplification of the wave model, though it can still be very useful, particularly in the macroscopic world where interference isn't as noticeable.

A light ray is a path of light traveling at speed $v = c/n$ in a straight line (where n is the index of refraction of the material). Light rays do not interfere — they can cross. They interact with matter by reflection, refraction, scattering, or absorption, and otherwise are not stopped.

Objects, which are sources of light, may be reflective or self-luminous. In the latter case, such as the sun, these objects make light using some other source of evergy; in the former case, as in the moon, they reflect light emitted by some other object. Light rays can emerge from all directions of a point source, but they are often considered as a parallel bundle instead. This latter case is the limiting behavior of light emitted from a very distant object. In order to represent significant rays, one would draw a ray diagram.

One can pass light through a small hole called an aperture. This is used in a projector-like device called a camera obscura, and the final size of the object can be calculated with similar triangles.

23.2 Reflection

In addition to diffuse reflection, where light shines off an irregularly shaped object in many directions, specular reflection is reflection on a flat, smooth surface. This obeys the familiar laws of reflection: $\theta_i = \theta_f$, etc. (Even in diffuse reflection, this law is obeyed, but it is not guaranteed that this angle will vary nicely with small variations in position.)

The most simple example of reflection is the plane mirror, in which the distance between the object and the mirror is the same as that between the (virtual) image and the mirror. However, a mirror doesn't preserve asymmetries, but instead flips them.

23.3 Refraction

When light encounters a boundary between two transparent materials, some of it reflects, but some of it enters the transparent material. The reflecting light acts as in the previous section, but the path of the transmitted light is refracted, so that it changes direction slightly. Refraction is goverened by Snell's law: if θ_1 and θ_2 are the angles of the light ray before and after refraction, respectively, with n_1 and n_2 the indices of refraction of the materials, then $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

The index of refraction is a measurement of how much light slows doen in a substance: $n = c/v$, where v is the velocity of light in that substance. Thus, $n_{\text{vacuum}} = 1$ and $n > 1$ for any other medium (e.g. $n = 1.50$ for glass).

However, if the light is refracted sufficiently strongly, then ther eis no refracted beam; using Snell's law, one would have calculated that $\sin\theta_2 > 1$, which is impossible. Experimentally, this coincides with a lack of a refracted beam of light. All of the light then reflects out of the medium, in a process called total internal reflection. The critical angle past which this happens is $\theta_c = \sin^{-1}(n_2/n_1)$. Thus, there is no critical angle at all (and therefore no total internal reflection) when $n_2 > n_1$. This has applications in binoculars, but also in fiber optics, where total internal reflection is used to reduce loss of light along the cable.

23.4 Image Formation by Refraction

Refraction can change the percieved location of objects across a boundary between different media. The canonical example is the illusion caused by water viewed from air — distances appear to be shortened.

The equation for this, solving via Snell's law and a small amount of angle-chasing, is $s' = s\tan\theta_1/\tan\theta_2$. However, the angle between any of these rays and the optical axis is very small, so these rays are called paraxial rays, and using the small-angle approximation $s' = sn_2/n_1$. Since s' is independent of θ_1, then the image is well-defined.

23.5 Color and Dispersion

The index of refraction of some transparent materials varies with the wavelength of the light. This leads to the prism effect: white light entering a prism will be dispersed out into a spectrum. This is the origin of the rainbow.

Blue skies (and red subsets) are explained slightly differently: light reflected off particularly small particles reflects differently based on wavelength: blue light is more likely to be reflected by particles of air. This is called Rayleigh scattering, and means that the longer red wavelengths are left over after most of the blue light has scattered (e.g. at sunset, when the sunlight travels through more of the atmosphere).

23.6 Thin Lenses: Ray Tracing

Most of this is review: thin lenses are assumed to be perfectly flat. A lens can be converging or diverging, and has an associated focal length. Images can be real or virtual, etc.

Sign conventions are helpful: for a real image, $s > 0$, and for a virtual image, $s' < 0$.

23.7 Thin Lenses: Refraction Theory

The purpose of this section is to derive the thin lens equation, which is just $1/s + 1/s' = 1/f$. However, remembering sign conventions is also helpfull the radius of curvature is positive when the lens is convex towards the object.

The lens-maker's equation is also helpful; if n is the index of refraction of a lens and R_1, R_2 are its radii of curvature on different sides, then $1/f = (n-1)(1/R_1 - 1/R_2)$.

23.8 Image Formation with Spherical Mirrors

Mirrors aren't so different from lenses, after all. They can be concave or convex, though only the former can produce a real image.

The radius of curvature is given by $2f = R$.

The mirror equation is the same as the lens equation, though in this case it is important to invert magnification before using it.

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