FUN WITH $\mathcal{E}(1)$-MODULES: A COMPUTATION OF pin^c BORDISM

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Theorem 0.1 (Bahri-Gilkey [BG87a, BG87b]). *The first several pin^c bordism groups are*

$$\begin{aligned}
\Omega_0^{\mathrm{Pin}^c} &\cong \mathbb{Z}/2\\
\Omega_1^{\mathrm{Pin}^c} &\cong 0\\
\Omega_2^{\mathrm{Pin}^c} &\cong \mathbb{Z}/4\\
\Omega_3^{\mathrm{Pin}^c} &\cong 0\\
\Omega_4^{\mathrm{Pin}^c} &\cong \mathbb{Z}/8\oplus\mathbb{Z}/2\\
\Omega_5^{\mathrm{Pin}^c} &\cong 0\\
\Omega_6^{\mathrm{Pin}^c} &\cong \mathbb{Z}/16\oplus\mathbb{Z}/4\\
\Omega_7^{\mathrm{Pin}^c} &\cong 0.
\end{aligned}$$

The goal of this document is to prove this in a slightly unconventional way: reduce to computing *ku*-homology of something, which we do using the Adams spectral sequence and a change-of-rings trick. This is reminiscent of a better-known approach to the computation of low-dimensional spin bordism groups by reducing to computing *ko*-homology and using a change-of-rings trick to simplify the Adams spectral sequence, but here we work over a different subalgebra of the Steenrod algebra, called $\mathcal{E}(1)$. The upshot is that the overall structure of the argument is similar, but the details are different, and simpler.

We will assume familiarity with the approach in the case of *ko*.[1] That is the more standard approach — and indeed, you can prove Theorem 0.1 this way, as Beaudry and Campbell do [BC18, §5.6]. So why work over $\mathcal{E}(1)$? The computation is simpler and easier over $\mathcal{E}(1)$, which is a major boon; there is a disadvantage, though, in that more has to be done from scratch. Anyways, I worked this out and wrote it up because it might be useful for computing other spin^c bordism groups of spaces or spectra.

1. Reduction to the Adams spectral sequence over $\mathcal{E}(1)$

The first thing we need is that a pin^c structure on a vector bundle $V\to M$ is equivalent to a spin^c structure on $V\oplus\mathrm{Det}(V)$ that induces the canonical orientation. This is itself equivalent to a map $f\colon M\to B\mathrm{O}_1$ and a spin^c structure on $V\oplus f^*\sigma$, where σ is the tautological bundle: the orientation (part of the data) and a Riemannian metric (a contractible choice) identify $f^*\sigma$ and $\mathrm{Det}(V)$. Therefore[2] there is an equivalence

$$MTPin^c \simeq MTSpin^c\wedge(B\mathrm{O}_1)^{\sigma-1}. \tag{1.1}$$

The Thom spectrum $(B\mathrm{O}_1)^\sigma$ is often denoted MO_1, so we get $MTSpin^c\wedge\Sigma^{-1}MO_1$.

Seminal work of Anderson-Brown-Petersen [ABP67] determined the homotopy type of $MTSpin^c$ at the prime 2.[3] In general it is complicated, but for us the upshot is that there is a map (additively but not multiplicatively!)

$$MTSpin^c \longrightarrow ku\vee\Sigma^4 ku \tag{1.2}$$

which is a 2-primary equivalence in degrees 7 and below. Since $\Sigma^{-1}MO_1$ is 2-primary,[4] the same is true when we smash with $\Sigma^{-1}MO_1$.

Therefore to prove Theorem 0.1 it will suffice to establish

[1] If this is an incorrect assumption, Beaudry and Campbell [BC18] have written a detailed introduction to this method of calculation.

[2] I'm skipping some steps here. Hopefully I can come back and fill them in later. In any case, this is proven in a few different places, including [FH16].

[3] Away from 2, it was already known that $MTSpin^c\simeq MTSO\wedge(B\mathrm{U}_1)_+$.

[4] This follows because its mod p cohomology vanishes when p is an odd prime.

Proposition 1.3. *For all $n \geq 0$,*

$$\widetilde{ku}_n(\Sigma^{-1}MO_1) \cong \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}/2^{n/2+1}, & n \text{ even.} \end{cases} \tag{1.4}$$

If one uses the Adams spectral sequence to compute this, the E_2-page is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\widetilde{H}^*(ku;\mathbb{F}_2) \otimes_{\mathbb{F}_2} \widetilde{H}^*(\Sigma^{-1}MO_1;\mathbb{F}_2), \mathbb{F}_2). \tag{1.5}$$

This seems large. But just as the E_2-page of the Adams spectral sequence for *ko*-theory simplifies to Ext over $\mathcal{A}(1)$, the E_2-page of the Adams spectral sequence for *ku*-theory simplifies to Ext over $\mathcal{E}(1) \coloneqq \langle Q_0, Q_1 \rangle$, where $Q_0 \coloneqq \mathrm{Sq}^1$ and $Q_1 \coloneqq \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. This is because $\widetilde{H}^*(ku;\mathbb{F}_2) \cong \mathcal{A}/\!\!/\mathcal{E}(1)$, and we can apply a change-of-rings theorem

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}/\!\!/\mathcal{E}(1) \otimes_{\mathbb{F}_2} M, N) \cong \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M, N). \tag{1.6}$$

So (1.5) simplifies to

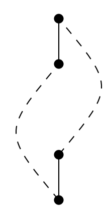
$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\widetilde{H}^*(\Sigma^{-1}MO_1;\mathbb{F}_2), \mathbb{F}_2), \tag{1.7}$$

which we spend the next few sections computing.

2. $\mathcal{E}(1)$ and some of its important modules

The module $\mathcal{E}(1)$ is four-dimensional over $\mathbb{F}_2$. Here's what it looks like.

(2.1)



When we draw $\mathcal{E}(1)$-modules, including the one above, the y-coordinate represents the degree. Straight solid lines indicate the action of $Q_0 = \mathrm{Sq}^1$ and curvy dashed lines indicate the action of $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$.

The following lemma is a consequence of Koszul duality.

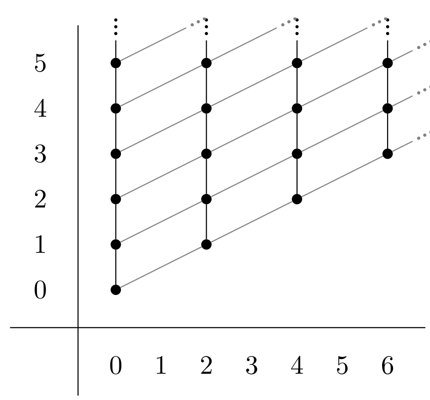
Lemma 2.2 ([BC18, Remark 4.5.4])**.** *Let R be a graded exterior algebra over $\mathbb{F}_2$ with generators $x_1, \dots, x_n$. Then there is an isomorphism of bigraded algebras $\mathrm{Ext}_R^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[y_1, \dots, y_n]$ with $|y_i| = (1, |x_i|)$.*

$\mathcal{E}(1)$ is an exterior algebra on Q_0 and Q_1 in degrees 1 and 3, respectively, so

$$\mathrm{Ext}_{\mathcal{E}(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, v_1], \tag{2.3}$$

where $h_0 \in \mathrm{Ext}_{\mathcal{E}(1)}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$ and $v_1 \in \mathrm{Ext}_{\mathcal{E}(1)}^{1,3}(\mathbb{F}_2, \mathbb{F}_2)$. Therefore the Adams diagram for $\mathbb{F}_2$ as an $\mathcal{E}(1)$-module is

(2.4)



Here, a vertical black line segment indicates multiplication by h_0, and a diagonal gray line segment indicates multiplication by v_1.

In Figure 1, we display the generators h_0 and v_1 as extensions of $\mathcal{E}(1)$-modules.

FIGURE 1. Left: h_0 is the class of the extension $0 \to \Sigma\mathbb{F}_2 \to \mathcal{E}(0) \to \mathbb{F}_2 \to 0$. Right: v_1 is the class of the extension $0 \to \Sigma^3\mathbb{F}_2 \to \mathcal{E}(1) /\!\!/ \mathcal{E}(0) \to \mathbb{F}_2 \to 0$.

Remark 2.5. The Adams spectral sequence for *ku* is multiplicative: the algebra structure on $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ (from the Yoneda product on extensions) tells you the ring structure on $(ku_*)^\wedge_2$. Likewise, the $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2)$-module structure on $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(H^*(X;\mathbb{F}_2);\mathbb{F}_2)$ (again by composing extensions) is compatible with the $(ku_*)^\wedge_2$-action on $ku_*(X)^\wedge_2$. All of this also applies to the Adams spectral sequence over $\mathcal{A}(1)$ and *ko*-theory.

In our case, $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by h_0 and v_1. In ku_*, h_0 represents $2 \in (ku^\wedge_2)_0 \cong \mathbb{Z}_2$, and therefore the h_0-action on the E_∞-page of the Adams spectral sequence for X lifts to multiplication by 2 in $ku_*(X)^\wedge_2$. Similarly, v_1 represents the Bott element $\beta \in (ku^\wedge_2)_2 \cong \mathbb{Z}_2$, so the v_1-action on the E_∞-page lifts to action by β on $ku_*(X)^\wedge_2$. Everything in the Adams spectral sequence over $\mathcal{E}(1)$ is linear with respect to the actions of h_0 and v_1, which is sometimes useful. ◂

Let $\mathcal{E}(0) \coloneqq \langle Q_0 \rangle \subset \mathcal{E}(1)$. Then $\mathcal{E}(0)$ is an exterior algebra on the single generator Q_0 in degree 1, so Lemma 2.2 calculates

$$\mathrm{Ext}^{*,*}_{\mathcal{E}(0)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0], \tag{2.6}$$

with $|h_0| = (1,1)$. By the change-of-rings theorem, this is also $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\mathcal{E}(1)/\!\!/\mathcal{E}(0), \mathbb{F}_2)$. In Figure 2 we display $\mathcal{E}(1)/\!\!/\mathcal{E}(0)$ and its Adams diagram.

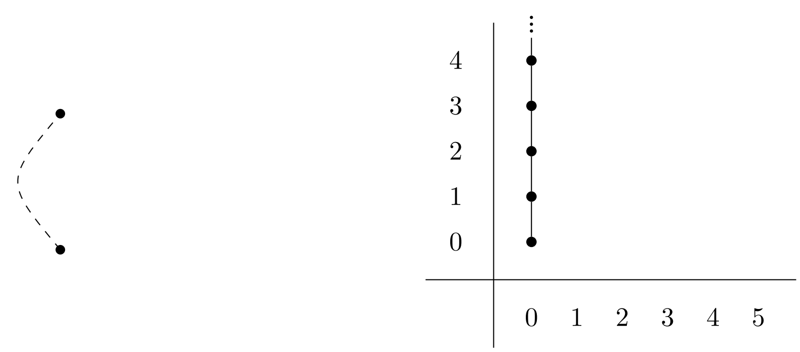


FIGURE 2. The $\mathcal{E}(1)$-module $\mathcal{E}(1)/\!\!/\mathcal{E}(0)$ (left) and $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\mathcal{E}(1)/\!\!/\mathcal{E}(0), \mathbb{F}_2)$ (right).

Alternatively, you can calculate this directly with a minimal resolution of $\mathcal{E}(1)/\!\!/\mathcal{E}(0)$, as in Figure 3, which shows that we have a single $\mathbb{F}_2$ summand in $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\mathcal{E}(1)/\!\!/\mathcal{E}(0), \mathbb{F}_2)$ when $s = t = k$, for each $k \geq 0$, and no other summands. The h_0-action is nontrivial because successive steps in the resolution are linked by Q_0.

3. Computing the E_2-page

In this section, we determine the $\mathcal{E}(1)$-module structure on $H \coloneqq \widetilde{H}^*(\Sigma^{-1}MO_1;\mathbb{F}_2)$. Then we compute its Ext by repeatedly using the fact that a short exact sequence of $\mathcal{E}(1)$-modules induces a long exact sequence of Ext groups. MO_1 is the Thom spectrum of the tautological bundle $\sigma \to B\mathrm{O}_1$. Therefore the Thom isomorphism tells us its cohomology as a graded $\mathbb{F}_2$-vector space, and the Stiefel-Whitney classes of σ determine the $\mathcal{A}$-module structure. Specifically, $H^*(B\mathrm{O}_1;\mathbb{F}_2) \cong \mathbb{F}_2[x]$ with $|x| = 1$. When we apply the Thom isomorphism theorem, the Thom class U would have degree 1 for MO_1; for $\Sigma^{-1}MO_1$ it's downshifted to degree 0. Thus, as a graded vector space,

$$H = \widetilde{H}^*(\Sigma^{-1}MO_1;\mathbb{F}_2) \cong \mathbb{F}_2 \cdot \{U, Ux, Ux^2, \ldots\}. \tag{3.1}$$

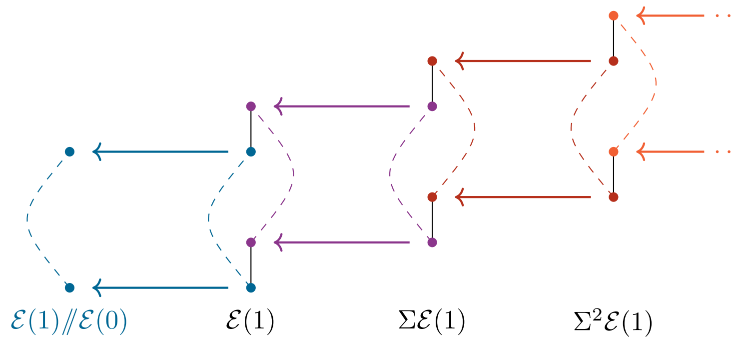


FIGURE 3. The beginning of a minimal resolution for $\mathcal{E}(1)/\!\!/\mathcal{E}(0)$.

The $\mathcal{A}$-action on $H^*(B\mathrm{O}_1;\mathbb{F}_2)$ is determined by that of the generator x, and the axiomatic properties of the Steenrod squares imply $\mathrm{Sq}(x) = x + x^2$. For $\Sigma^{-1}MO_1$, one can calculate $\mathrm{Sq}^k(Ux^n)$ by the rule $\mathrm{Sq}^k(U) = Uw_k(\sigma)$, then applying the Cartan formula for the Steenrod squares of a product. When you do this, you'll find that

$$Q_0(Ux^k) = \begin{cases} Ux^{k+1}, & k \text{ even} \\ 0, & k \text{ odd}; \end{cases} \tag{3.2}$$

$$Q_1(Ux^k) = \begin{cases} Ux^{k+3}, & k \text{ even} \\ 0, & k \text{ odd}. \end{cases} \tag{3.3}$$

We draw H as an $\mathcal{E}(1)$-module in Figure 4.

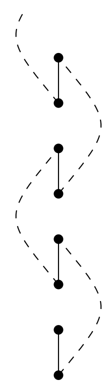


FIGURE 4. The $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(\Sigma^{-1}MO_1;\mathbb{F}_2)$.

This module is a little bit complicated, but we can determine the E_2-page by repeatedly using the trick that a short exact sequence $0 \to L \to M \to N \to 0$ of $\mathcal{E}(1)$-modules induces a long exact sequence in Ext. The way to visualize this is to graph $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(L,\mathbb{F}_2)$ and $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(N,\mathbb{F}_2)$ on the same Adams chart; then, the boundary map has bidegree $t-s=-1$, $s=1$. Boundary maps are linear with respect to the $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{F}_2,\mathbb{F}_2)$-action, which often simplifies computations.

As a first example, consider the $\mathcal{E}(1)$-module M_2 which is the extension of $\Sigma^2\mathcal{E}(1)/\!\!/\mathcal{E}(0)$ by $\mathcal{E}(1)/\!\!/\mathcal{E}(0)$ depicted on the left-hand side of Figure 5. The right-hand side displays $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(M_2,\mathbb{F}_2)$; the boundary maps all vanish for degree reasons. We will show the claimed v_1-actions in Lemma 3.4.

Lemma 3.4. *For all s, $v_1\colon \mathrm{Ext}^{s,s}_{\mathcal{E}(1)}(M_2,\mathbb{F}_2) \to \mathrm{Ext}^{s+1,s+3}_{\mathcal{E}(1)}(M_2,\mathbb{F}_2)$ is an isomorphism.*

Proof. h_0-linearity means it suffices to show this for $s=0$. To do this, we'll check that the extension obtained by acting on the nontrivial element of $\mathrm{Ext}^{0,0}_{\mathcal{E}(1)}(M_2,\mathbb{F}_2)$ by v_1 is nonzero.

$\mathrm{Ext}^{0,0}_{\mathcal{E}(1)}(M_2,\mathbb{F}_2)$ is the group of degree-0 $\mathcal{E}(1)$-module homomorphisms $M_2 \to \mathbb{F}_2$. There is one nontrivial one, which maps the degree-0 summand onto $\mathbb{F}_2$ and kills everything else. Acting on this by an element of

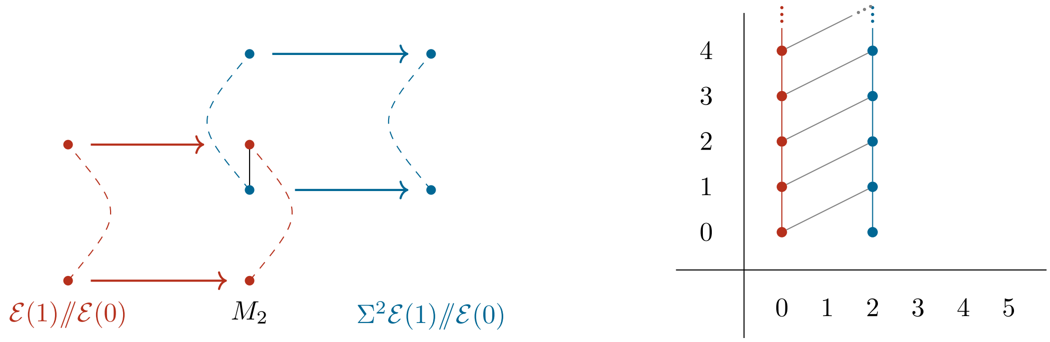


FIGURE 5. An extension of $\mathcal{E}(1)$-modules (left), which computes $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(M_2, \mathbb{F}_2)$ (right) in terms of the Exts of the sub and the quotient. However, the $\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2)$-action is not fully determined by this extension, and we show the claimed v_1-actions in Lemma 3.4.

$\mathrm{Ext}^{1,t}_{\mathcal{E}(1)}(\mathbb{F}_2, \mathbb{F}_2)$, represented by some explicit extension $0 \to \Sigma^t\mathbb{F}_2 \to M \to \mathbb{F}_2 \to 0$, produces the extension

$$0 \longrightarrow \Sigma^t\mathbb{F}_2 \longrightarrow M \times_{\mathbb{F}_2} M_2 \longrightarrow \mathbb{F}_2 \longrightarrow 0 \tag{3.5}$$

in $\mathrm{Ext}^{1,t}_{\mathcal{E}(1)}(M_2, \mathbb{F}_2)$ [BC18, §4.2]. We want to show that for v_1, represented by the extension in Figure 1, this pullback N is nontrivial. Drawing N is a little unwieldy, but we can describe it algebraically: it is a five-dimensional $\mathbb{F}_2$-vector space, generated by elements x_0, x_2, x_3, x_3', and x_5 with degrees $|x_i| = i$ and $|x_3'| = 3$. The $\mathcal{E}(1)$-action on N is: $Q_0x_2 = x_3$, $Q_1x_0 = x_3 + x_3'$, $Q_1x_2 = x_5$. There is an extension $0 \to \Sigma^3\mathbb{F}_2 \to N \to M_2 \to 0$ where the first map sends the generator to x_3' and the second is the quotient by x_3'. This is a nontrivial extension, so the v_1-action is nonzero. ⊠

We can iterate this, attaching on more copies of $\Sigma^{2k}\mathcal{E}(1)/\!\!/\mathcal{E}(0)$ by Q_0s to obtain taller and taller $\mathcal{E}(1)$-modules. Doing this infinitely many times yields a module called M_∞, and the arguments above tell us $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(M_\infty, \mathbb{F}_2)$; both are displayed in Figure 6.

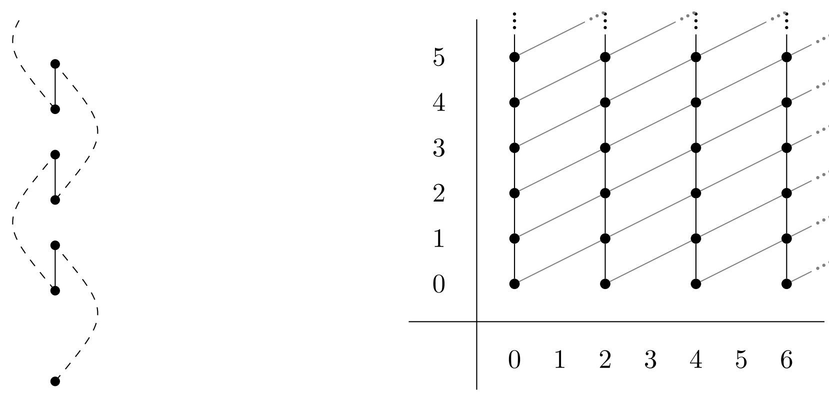


FIGURE 6. Left: the $\mathcal{E}(1)$-module M_∞. Right: $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(M_\infty, \mathbb{F}_2)$.

As depicted in Figure 7, left, H is an extension of M_∞ by $\Sigma\mathbb{F}_2$, so we can use the same method to calculate $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(H, \mathbb{F}_2)$, as depicted in Figure 7, right.

The boundary maps commute with action by h_0 and v_1, so in Figure 7, the boundary map indicated by the black arrow determines all the other boundary maps (gray arrows): either all are nontrivial, or none are. We can quickly deduce the black arrow is nontrivial by noticing there are no nonzero $\mathcal{E}(1)$-module maps $\varphi\colon H \to \Sigma\mathbb{F}_2$: if such a map existed, $Q_0(\varphi(U)) = \varphi(Q_0(U))$, which must be nonzero in order for φ to be nontrivial, but then $\varphi(U) \neq 0$ too, and there are no nonzero elements in that degree. Thus $\mathrm{Ext}^{0,1}_{\mathcal{E}(1)}(H, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathcal{E}(1)}(H, \Sigma\mathbb{F}_2) = 0$, so the black arrow must be an isomorphism. Therefore $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(H, \mathbb{F}_2)$ is given in Figure 8.

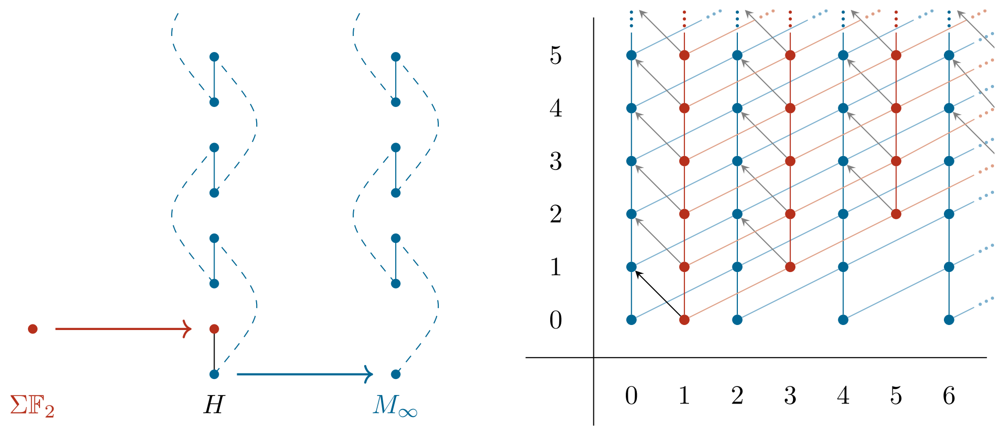


FIGURE 7. Computing $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(H, \mathbb{F}_2)$ via the long exact sequence of Ext groups induced from the short exact sequence of $\mathcal{E}(1)$-modules (pictured on the left).

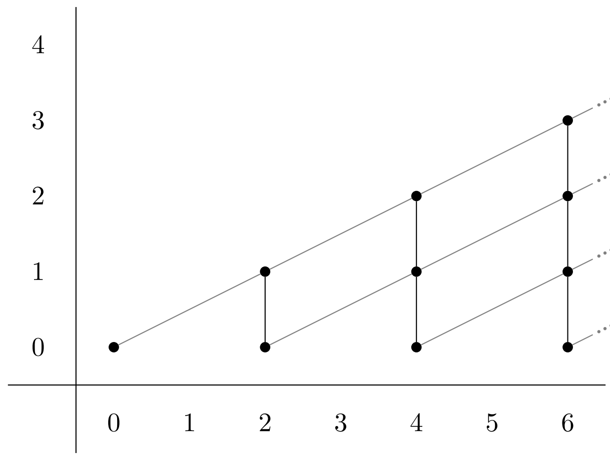


FIGURE 8. $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(H, \mathbb{F}_2)$.

4. THE ku-HOMOLOGY OF $\Sigma^{-1}MO_1$

The hard work is behind us. Looking at the E_2-page in Figure 8, there can be no nonzero differentials for degree reasons. The h_0-action gives you multiplication by 2, which means there can be no hidden extensions, and we've finished proving Proposition 1.3, hence also Theorem 0.1. Thanks for reading.

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