GEOMETRY AND STRING THEORY SEMINAR: SPRING 2018

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Contents

1. Comments on Global Symmetry, Anomalies, and Duality in $(2+1)d$: 1/24/18

Today's talk was given by Val Zakharevich, on the paper [BHS17].

Definition 1.1. Let A and B be UV theories which, under renormalization group flow, flow to the same IR theory C. Then we'll say that theories A and B are dual.

Example 1.2. Let $N_f \le N$. Then there is a conjectured duality between $\mathrm{SU}(2)_k$-Chern-Simons theory with N_f scalars, also known as *Wilson-Fischer theory*, and $\mathrm{U}(k)_{-N+N_f/2}$-Chern-Simons theory with N_f fermions. ◀

The paper [BHS17] computes the higher symmetries and anomalies of both sides of this duality and of several others; 't Hooft anomaly matching tells us that these should be the same.

This is related to our overarching goal of understanding QCD_4 with a single fermion ψ, which has a Lagrangian

$$\mathcal{L} = \mathrm{tr}(F \wedge \star F) + \overline{\psi}\not{D}\psi + m\overline{\psi}\psi, \tag{1.3}$$

where $m \in \mathbb{C}$ is a parameter whose phase diagram we're interested in. Let $\overline{m}$ denote the mass of the domain wall theory, which is a 3D QCD theory, so $\overline{m}$ is real. If m is real and negative, there's a phase transition: for $\overline{m} \ll 0$, the low energy theory is believed to be trivial, and for $\overline{m} \gg 0$ (m negative of larger magnitude), the low-energy theory is believed to be $\mathrm{SU}(N)$-Chern-Simons theory at level 1. The transition point, at $\overline{m} = 0$, should be described by $\mathrm{SU}(N)_{1/2}$ with a single fermion.

1.1. Level-rank duality.

Level-rank duality is the conjecture that $\mathrm{SU}(N)_k$-Chern-Simons theory and $\mathrm{U}(k)_{-N}$-Chern-Simons theory are isomorphic. A natural generalization is to consider $\mathrm{SU}(N)_k$ together with N_f scalar fields of mass m, where $m \in \mathbb{R}$ and $N_f < N$.

- If $m \ll 0$, the Higgs mechanism implies this should be the $\mathrm{SU}(N-N_f)_k$ theory.
- if $m \gg 0$, we should expect the $\mathrm{SU}(N)_k$ theory again.

On the dual side, let's consider $\mathrm{U}(k)_{-N-N_f/2}$ with N_f fermions of mass $m\in\mathbb{R}$.

- If $m \gg 0$, we expect to get $\mathrm{U}(k)_{-N+N_f/2}$.
- If $m \ll 0$, we expect to get $\mathrm{U}(k)_{-N}$, with Lagrangian shifted by N_f:

$$\mathcal{L} = \frac{-N+N_f}{4\pi}\operatorname{tr}\left(b\,\mathrm{d}b - \frac{2i}{3}b^3\right) + \overline{\psi}\not{D}\psi + m\psi\overline{\psi} + \text{c.c.} \tag{1.4}$$

Level-rank duality switches positive-mass scalars and negative-mass fermions, promising dualities between $\mathrm{SU}(N-N_f)_k \longleftrightarrow \mathrm{U}(k)_{-N+N_f/2}$ and $\mathrm{SU}(N)_k \longleftrightarrow \mathrm{U}(k)_{-N}$.

1.2. **Symmetries.** We now see what symmetries these theories have. First, $\mathrm{SU}(N)_k$ with N_f scalars. On a 3-manifold M, the fields are triples (P, A, Φ), where

- $P \to M$ is a principal $\mathrm{SU}(N)$-bundle with connection,
- A is a connection on P, and
- $\varphi \in \Gamma(P\times_{\mathrm{SU}(N)}(\mathbb{C}^N\otimes\mathbb{C}^{N_f}))$ is the N scalar fields.

The Lagrangian is

$$\mathcal{L}(A,\Phi) = \frac{k}{4\pi}\operatorname{tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A^3\right) + |D_A\varphi|^2 + m|\varphi|^2 + \lambda|\varphi|^4. \tag{1.5}$$

As usual, we have an $\mathrm{SU}(N)$-gauge symmetry, and there's also a $\mathrm{U}(N_f)$-symmetry acting on $\mathbb{C}^{N_f}$, in which $e^{2\pi i/N}\mathbf{1}$ acts by a gauge symmetry. Hence the global symmetry group (for these symmetries) is $\mathrm{U}(N_f)/(\mathbb{Z}/N)$.

Ansatz 1.6. Let G be a compact Lie group and k be a level for G, and let $\mathcal{L}_{G_k}$ denote the Lagrangian for Chern-Simons theory with group G and level k. Let

$$1 \longrightarrow G \xrightarrow{\ \rho\ } H \xrightarrow{\ \sigma\ } L \longrightarrow 1 \tag{1.7}$$

be a short exact sequence of Lie groups. Then, we take as an ansatz that coupling the G_k theory to a principal L-bundle (i.e. given a principal L-bundle $P\to M$, we sum over the groupoid of all principal H-bundles which quotient to L) produces a classical gauge theory for H with Lagrangian $\mathcal{L}_{\widetilde{k}}$ such that

$$\mathcal{L}_{G_k}(P_G, A_G) = \mathcal{L}_{\widetilde{k}}((P_G, A_G)\times_G H). \tag{1.8}$$

When G is finite (so we're in the setting of Dijkgraaf-Witten theory) this is studied in [KT14].

In our setting, (1.7) specializes to $G=\mathrm{SU}(N)$, $H = (\mathrm{SU}(N)\times U(N_f))/(\mathbb{Z}/N)$, and $L = U(N_f)/(\mathbb{Z}/N)$. Chern-Simons theories for G are labeled by $H^4(BG;\mathbb{Z})$, and the map $\rho\colon G\to H$ defines a pullback

$$\rho^*\colon H^4(BH;\mathbb{Z}) \longrightarrow H^4(BG;\mathbb{Z}).$$

Given a $k\in H^4(BG;\mathbb{Z})$, we want to know whether we can implement the theory with a global L-symmetry; hence we want to know whether $k\in\operatorname{Im}(\rho^*)$; the theory is anomalous iff this is not true.

If the theory is anomalous, we'd like to compute the anomaly. Suppose that we have a $\widehat{k}_{\mathbb{R}}\in H^4(BH;\mathbb{R})$ such that $\rho^*(\widehat{k}_{\mathbb{R}}) = k_{\mathbb{R}}$ (i.e. the image of k in real cohomology). Then, we can't eliminate the anomaly, but we can couple to a bulk theory: suppose that we can extend $(P,A)\to M$ to $(P_H, A_H)\to X$, where X is a compact 4-manifold with $\partial X = M$. Then, we have an action

$$\text{“}S_{\widehat{k}_{\mathbb{R}}}(A_H)\text{”}\colon ((P_H, A_H)\to X) \longmapsto \int_X \widehat{k}_{\mathbb{R}}(F_H), \tag{1.9}$$

where F_H is the curvature of A_H.

This depends on the choice of X and $P_H\to X$ extending P, but we can hope that the dependence goes away after exponentiating the action. Let X' be another compact 4-manifold bounding M, and let $(P_H', A')\to X'$ be another extension of (P,A). Let $\widehat{X} \coloneqq X\cup_M X'$; then, (P_H, A_H) and (P_H', A_H') glue to a principal G-bundle $\widetilde{P}_H\to\widetilde{X}$ with connection $\widetilde{A}$. Then we have that

$$e^{2\pi i S_{\widehat{k}_{\mathbb{R}}}(\widetilde{P}_H, \widetilde{A}_H)} = S_{\hat{k}}(\widetilde{P}_H\times_H L)\in\mathbb{R}/2\pi i\mathbb{Z} \tag{1.10}$$

for some $\hat{\hat{k}} \in H^4(BL;\mathbb{R}/\mathbb{Z})$. This $\hat{\hat{k}}$ tells us the anomaly, so we're interested in computing it. Ultimately, this comes from a question purely in algebraic topology: we have a big commutative diagram

$$\begin{array}{ccccc} H^4(BL;\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^4(BH;\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^4(BG;\mathbb{R}/\mathbb{Z}) \\ & & \uparrow & & \uparrow \\ & & H^4(BH;\mathbb{R}) & \longrightarrow & H^4(BG;\mathbb{R}) \\ & & \uparrow & & \uparrow \\ & & H^4(BH;\mathbb{Z}) & \longrightarrow & H^4(BG;\mathbb{Z}) \end{array} \tag{1.11}$$

Then we have $\hat{\hat{k}}$ in the upper left, $\widehat{k}_{\mathbb{R}}$ in the middle, and k in the lower right. In this case the anomaly theory is purely topological. The computation for the dual theory follows a similar story, but is harder.

To actually calculate this, you can use the Leray-Serre spectral sequence; $k \in H^4(BG;\mathbb{Z})$ transgresses to something in $H^5(BL;\mathbb{Z})$, which tells you which component $\hat{\hat{k}}$ is in.

2. On Gauging Finite Subgroups: 1/31/18

Today's talk was given by Dan Freed, on Tachikawa's paper [Tac17].

Let's start with classical electromagnetism on n-dimensional Minkowski spacetime $\mathbb{M}$. Choose an $A \in \Omega^1_{\mathbb{M}}/\mathrm{d}\Omega^0_{\mathbb{M}}$, and let $F_A = \mathrm{d}A$. Maxwell's laws tell us that

$$\begin{aligned} \mathrm{d}F_A &= 0 \\ \mathrm{d}{\star}F_A &= 0, \end{aligned}$$

but more generally we could let $\mathrm{d}F_A = j_B \in \Omega^3$, a *magnetic current*, and $\mathrm{d}{\star}F_A = j_E \in \Omega^{n-1}$, an *electric current*. If both $j_E$and j_B are nonzer, the theory has an anomaly.

We next consider the quantum theory, by doing some things such as Wick rotation, charge quantization, and downshifting the degree of A.[1] The Wick-rotated quantum theory is formulated on an oriented[2] Riemannian manifold X, and A is a map $X \to \mathbb{R}\mathbb{Z}$, or its exponentiated version $\lambda\colon X \to \mathbb{T}$.

If we introduce point charges $p_1, \dots, p_m \in X$ with charges $k_1, \dots, k_m \in \mathbb{Z}$,[3] then the electric current, inserted in the exponentiated action, is

$$\prod_{j=1}^m \lambda(p_j)^{k_j} = \exp\left(2\pi \sum_{j=1}^m i k_j A(p_j)\right), \tag{2.1}$$

which has degree n with $\mathbb{Z}$ coefficients.

The magnetic current is defined using a circle bundle $P \to X$ with connection Θ; one can think of λ as a section of P, and this data is used to define the kinetic term. This is a degree-2 term with $\mathbb{Z}$ coefficients.

If $L \coloneqq P \times_{\mathbb{T}} \mathbb{C}$ is the complex line bundle associated to P, then the *electric coupling* is

$$\prod_{i=1}^m \lambda(p_i)^{k_i} \in \bigotimes_{i=1}^m L_{p_i}^{\otimes k_i}. \tag{2.2}$$

From this perspective, the anomaly is "$\int_X j_B \cdot j_E$."

Remark 2.3. The term λ only exists if $P \to X$ is topologically trivializable.

This is akin to something that happens in topological field theory. Let Z denote the 4D oriented TQFT defined by summing the trivial theory over spin structures. Then $Z(\mathbb{CP}^2) = 0$, since $\mathbb{CP}^2$ admits no global spin structure. But if one varies the manifold in a family, interesting things may nonetheless happen. ◀

[1] TODO: I have no idea what just happened.

[2] One could impose time-reversal symmetry and study the theory more generally on unoriented manifolds, but for our purposes this will not be necessary.

[3] TODO: I may have gotten this wrong.

Remark 2.4. One could also replace $\mathbb{T}$ by any finite abelian group A. In this case a lot of things are still the same, though we don't choose a connection for P. In this case, the magnetic current lives in $H^1(X; A)$ rather than $H^2(X; \mathbb{Z})$ (and we could have thought of it as $H^1(X; \mathbb{T})$ for the $\mathbb{T}$-theory). However, the electric current lives in $H^n(X; A^\vee)$, where $A^\vee := \mathrm{Hom}(A, \mathbb{T})$ denotes the Pontrjagin dual of A.

We could think of the magnetic current as a map $X \to BA$; in this case the electric current is a map $X \to B^n A^\vee := K(A^\vee, n)$. In the rest of this talk, we will adopt this more abstract approach, but you should keep the rigid, geometric approach that we started with in mind for intuition or an example. ◀

From this perspective, the anomaly is a map

$$BA \times B^n A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

which is induced from the pairing $A \otimes A^\vee \to \mathbb{T}$. On a closed, oriented manifold X with an electric and magnetic current we can pull this back to X and integrate it; this is the partition function for the anomaly theory.

Gauging. Suppose T is some kind of theory (here we probably mean a Wick-rotated field theory on Riemannian manifolds, perhaps with extra structure and background fields), and suppose it has a (global) Γ-symmetry,[4] where Γ is a finite group. This means that we can couple the theory to Γ-bundles, formulating it on manifolds with the above data and a background principal Γ-bundle.

Sometimes this symmetry gets tangled up with other symmetries, e.g. if T is a σ-model to a space X and Γ is a symmetry of X. Then, depending on how we implement the symmetry, we might end up with sections of some associated bundle.

But if this is not the case (in a σ-model sense, if $B\Gamma$ splits off from the target), then we can sum over the maps to $B\Gamma$. This process is called *gauging*.

Now suppose $A := \Gamma$ is abelian. Then the gauged theory has a higher symmetry akin to electromagnetism, a $B^{n-2}A^\vee$ symmetry, and we can couple the theory to a background $B^{n-2}A^\vee$-field, which is exactly putting in the electric current. If you try to gauge this symmetry, you'll end up back where you started with, which is a kind of Fourier transform.

On a compact oriented manifold X, the electric coupling lives in $H^1(X; A) \times H^{n-1}(X; A^\vee)$; there's a product map

$$H^1(X; A) \times H^{n-1}(X; A^\vee) \longrightarrow H^n(X; \mathbb{T});$$

then we can evaluate on the fundamental class to obtain an element of $\mathbb{T}$, which is what one inserts into the action. This exhibits the two cohomology groups as Pontrjagin duals of each other, so the Fourier transform is an isomorphism between spaces of functions on them.

Turning to the material in the paper, let

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence of groups, where A is abelian. In particular, A is normal in Γ; we do not assume it is central. An example (in which A is not central) is

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

We consider the situation of a theory T with a Γ-symmetry, hence an A-symmetry, and we assume we can gauge A. What happens when we do this?

The new theory should have a G-symmetry, which arises as follows: given a principal G-bundle $Q \to X$, we can sum over pairs $(P \to X, \varphi)$, where $P \to X$ is a principal Γ-bundle and

$$\begin{array}{ccc} Q & \xrightarrow[\cong]{\ \varphi\ } & P/A \\ & {\scriptstyle G}\searrow \qquad \swarrow{\scriptstyle G} & \\ & X & \end{array}$$

is an isomorphism of principal G-bundles. In this case the magnetic current arises from the map $BG \to B^2A$ coming from extending the fiber sequence $BA \to B\Gamma \to BG$.

[4] "Global symmetry" is redundant, because there is no other kind of symmetry.

There's another symmetry associated to $B^{n-2}A^\vee$, which arises in a similar way as before. So, following what we did with electromagnetism, if you try to couple the theory to $BG \times B^{n-1}A^\vee$, two things might happen (and are not mutually exclusive).

(1) The two symmetries might interact nontrivially, producing an extension

$$B^{n-1}A^\vee \longrightarrow \mathcal{X} \longrightarrow BG.$$

(2) There may be an anomaly $\mathcal{X} \to B^{n+1}\mathbb{T}$.

For example, $\mathcal{X}$ might be $B\mathcal{G}$, where $\mathcal{G}$ is an extension of G by $B^{n-2}A^\vee$, so an extension of a group by a higher group.

First suppose Γ is anomaly-free, so that we can gauge it. Then, the anomaly for the theory coupled to G-symmetry is the compositions of the maps

$$BG \times B^{n-1}A^\vee \longrightarrow B^2A \times B^{n-1}A^\vee \longrightarrow B^{n+1}\mathbb{T},$$

where the first map comes from the connecting map $BG \to B^2A$ and the second is the Pontrjagin dual pairing.

Now suppose the Γ theory has an anomaly. There are different ways to produce anomalies, such as beginning with a map $MSO \wedge (B\Gamma)_+ \to B^{n+1}\mathbb{T}$, which produces a *gauge-gravity anomaly*, but let's begin with just $B\Gamma \to B^{n+1}\mathbb{T}$, or a *pure gauge anomaly*. This data is equivalent to a cohomology class $[\alpha] \in H^{n+1}(B\Gamma; \mathbb{T})$.

The presence or absence of the anomaly arises from a filtration on $H^{n+1}(B\Gamma;\mathbb{T})$ induced from the fiber sequence $BA \to B\Gamma \to BG$: BG has a cell structure, and we can ask whether a cohomology class over the basepoint extends over the n-skeleton. We will discuss what happens in the case when the cohomology class is in the last two pieces of the filtration, which are simpler.

To compute this, one uses the *Leray-Serre spectral sequence*

$$E_2^{p,q} \coloneqq H^p(G; H^q(A;\mathbb{T})) \Longrightarrow \mathrm{gr}_F H^{p+q}(\Gamma;\mathbb{T}).$$

Here $H^q(A;\mathbb{T})$ is a nontrivial G-module from the residual G-action induced by Γ. Now let's look at the two cases.

(1) Suppose $[\alpha]$ lives in the highest filtered part; then, there's an $\overline{\alpha}\colon BG \to B^{n+1}\mathbb{T}$ lifting α across the map $B\Gamma \to BG$. In this case, $\overline{\alpha}$ is the anomaly: if you gauge the A-symmetry, the theory couples to $BG \times B^{n-1}A^\vee$, and the anomaly is a sum of the electromagnetic anomaly and $\overline{\alpha}$. This corresponds to a transgression in the spectral sequence.

(2) If $[\alpha]$ lives in the next highest filtered part, we get something in $H^n(G; \underline{A}^\vee)$ (the underline representing a nontrivial G-action). The paper [Tac17] considers only the special case, where A is central and we get a bundle

$$B\Gamma \times_{BG} \mathcal{X},$$

where $B\Gamma \to BG$ is a BA-bundle and $\mathcal{X} \to BG$ is a $B^{n-1}A^\vee$ bundle. Then we have maps

$$BG \to B^2A \times B^nA^\vee \to B^{n+2}\mathbb{T},$$

and we consider the case where this composite is null. This means it lifts to a map

$$\beta\colon B\Gamma \times_{BG} \mathcal{X} \to B^{n+1}\mathbb{T}.$$

Then, we take $\beta|_{B\Gamma}$ is the anomaly in Γ. If this was the original anomaly in Γ, then the theory couples to the extension $B^{n-1}A^\vee \to \mathcal{X} \to BG$, corresponding to some kind of extension theory, and the anomaly for this theory is $\beta|_X$. Then you could gauge the subgroup $B^{n-1}A^\vee$ and go back to the original theory.

(3) Suppose $[\alpha]$ lives in the n^{th} piece of the filtration, and assume A is central. Then the map $B\Gamma \to B\mathbb{T}^{n+1}$ doesn't descend to BG, but does descend to a map $BG \to E$, where E fits into a sequence $B^{n+1}\mathbb{T} \to E \to B^nA^\vee$. This corresponds to an anomaly in a generalized cohomology theory, albeit a relatively simple one. We could try to use this to build $\mathcal{X}$. But you also have to add the electromagnetic anomaly, and this is a little bit unclear.

3. Theta, time-reversal, and temperature I: 2/7/18

Today's talk was given by Andy Neitzke, on the paper [GKKS17].

The goal of this paper is to say something new about the phases of 4D Yang-Mills theory with group $G = \mathrm{SU}_N$. There's no supersymmetry here.

Before we tackle this, however, let's look at a simpler example, a 2D U_1 gauge theory with N charged scalar fields z^i, $i = 1, \dots, N$, subject to the constraint that $\sum |z_i|^2 = 1$. This is equivalent data to a map to $\mathbb{CP}^{N-1}$, and hence this model is sometimes thought of as a σ model with target $\mathbb{CP}^{N-1}$. The U_1 gauge field is denoted a, and its curvature is F_a. The action on a surface Σ is

$$S = \int_\Sigma \sum_i |D_a z^i|^2 + \frac{i\theta}{2\pi}\int_\Sigma F_a + \frac{1}{g^2}\int_\Sigma F_a \wedge \star F_a. \tag{3.1}$$

Here θ is a real parameter, but we typically think of the θ theory as equivalent to the $\theta + 2\pi$ theory, because on a closed surface Σ, $\int_\Sigma F_a = 2\pi n \in 2\pi\mathbb{Z}$, so after exponentating, $e^{in\theta} = e^{in(\theta+2\pi)}$.

The first question you might ask is: what's the (global) symmetry of this theory? The answer is PSU_N, which might be a surprise: there's a U_N symmetry on the fields, but we gauged the U_1 subgroup, so only $\mathrm{PSU}_N \coloneqq \mathrm{U}_N/\mathrm{U}_1$ acts faithfully on the (gauge-invariant) local operators.

Example 3.2. The operator z^1 transforms nontrivially under the U_1 symmetry, hence is not gauge invariant. The operator $z^1\overline{z}^1$ is gauge invariant. ◀

The more modern way of saying that this theory has a PSU_N symmetry is to say that it couples to background PSU_N gauge fields. Let A_{bkgd} be a PSU_N-connection; we want to include A_{bkgd} in the theory such that if it's the trivial connection, we get back (3.1).

Somewhat like what we did last time, instead of U_1-connections, we'll consider lifts A of A_{bkgd} to a U_N-connection. Locally $A = a \cdot I_N + A_{\mathrm{bkgd}}$. When writing the action, the first term doesn't change much, but the θ term is interesting, since we no longer have a U_1-connection.

If $A_{\mathrm{bkgd}} = 0$, then $A = a \cdot I_N$, and $F_a = (1/N)\operatorname{tr}(F_A)$. Hence we'll replace F_a with $(1/N)\operatorname{tr}(F_A)$, even when $A_{\mathrm{bkgd}} \neq 0$. That is, the new θ term is

$$\frac{i\theta}{2\pi N}\int_\Sigma \operatorname{tr}(F_A). \tag{3.3}$$

We do something analogous for the kinetic term. But we pay a price — since we divided by N, this term is not invariant under $\theta \mapsto \theta + 2\pi$. Instead, it's invariant under $\theta \mapsto \theta + 2\pi N$. If we shift $\theta \mapsto \theta + 2\pi$, the exponentiated action changes by

$$\exp\left(\frac{2\pi i}{N}\int_\Sigma w_2(A_{\mathrm{bkgd}})\right). \tag{3.4}$$

In particular, the change only depends on the background field. Therefore this is also true for the partition functions:

$$Z(\theta + 2\pi, A_{\mathrm{bkgd}}) = Z(\theta, A_{\mathrm{bkgd}}) \cdot \exp\left(\frac{2\pi i}{N}\int_\Sigma w_2(A_{\mathrm{bkgd}})\right). \tag{3.5}$$

This exhibits a mixed anomaly between the shift symmetry $\theta \mapsto \theta + 2\pi$ of $\mathbb{Z}$ and the PSU_N symmetry. You can't gauge both of them at once.

Remark 3.6. From the perspective of symmetries as coupling to background bundles, we want to express this shift symmetry with a background $\mathbb{Z}$-bundle. The mixed anomaly is saying that θ becomes a section of an associated bundle, such that $e^{i\theta}$ is still a function to U_1, but not a constant function. ◀

Another option is to add some function of the background fields to the action; this is called a *counterterm*. One natural choice is, for some $p \in \mathbb{Z}$,

$$p\frac{2\pi i}{N}\int w_2(A_{\mathrm{bkgd}}) \in (2\pi i\mathbb{Z})/(2\pi i N\mathbb{Z}). \tag{3.7}$$

If we exponentiate this, it's an N^{th} root of unity, and therefore the theory with parameters θ and p should be equivalent to the theory with parameters $\theta + 2\pi$ and $p + 1$.

One thing we can do with this is implement time reversal, thought of as reversing the orientation of Σ. This acts on S by $\theta \mapsto -\theta$. In the absence of a background PSU_N field, this is only a symmetry when $\theta = 0$ or $\theta = 2\pi$, in which case the time-reversed theory is equivalent.

With a background field and the counterterm (3.7), time-reversal acts as $(\theta, p) \mapsto (-\theta, -p)$. Therefore we can get time-reversal symmetry for $\theta = 0$ and $p = 0$. And what about $\theta = \pi$? In this case, $(-\pi, -p) \simeq (\pi, -p+1)$, and to get this equivalent to (π, p), we need $p \equiv -p + q \bmod N$, i.e. $2p \equiv 1 \bmod N$. Therefore if N is even, there's no way to preserve time-reversal at $\theta = \pi$.

This means the theory has a mixed anomaly between time-reversal symmetry and a PSU_N symmetry. Anomaly matching means this tells us something about the infrared theory. There are two possibilities.

- The IR theory "has the same anomaly," in that it can be coupled to PSU_N background fields in such a way that, if we shift the background coupling by p, then $p \mapsto -1 + p$ under time reversal.
- At least one of the PSU_N or time-reversal symmetries is broken in the infrared. This is a little weird, and is an example of spontaneous symmetry breaking.

What actually happens is the second option, and specifically time-reversal symmetry is broken.[5] The theory is gapped, and therefore the IR theory should be an explicit topological field theory we could look at.

Back to Yang-Mills theory. Remember that we're really interested in Yang-Mills theory with $G = \mathrm{SU}_N$, whose action is

$$aS = \frac{1}{g^2}\int \mathrm{tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F). \tag{3.8}$$

Again, after exponentiating the action, we have an equivalence between the theory with parameter θ and the theory with parameter $\theta + 2\pi$. Now there's a $\mathbb{Z}/N$ one-form symmetry, or a $B\mathbb{Z}/N$ symmetry, so we should be able to couple the theory to background $B\mathbb{Z}/N$ fields. Whatever these are, they should have a characteristic class $[B] \in H^2(X; \mathbb{Z}/N)$.

As before, this can be done, and the price is that the theory is not periodic in θ. And as before, there's a specific term expressing that failure: assume N is even. When we shift $\theta \mapsto \theta + 2\pi$, e^S shifts by

$$\exp\left(\frac{2\pi i(N_1)}{2N}\int_X \mathfrak{P}_2(B)\right),$$

where $\mathfrak{P}_2\colon H^2(X;\mathbb{Z}/N) \to H^4(X;\mathbb{Z}/2N)$ is a cohomology operation called the *Pontrjagin square*. Therefore we can regard this as an element of $\mathbb{Z}/N$, and introduce a counter-term as above: $(\theta, p) \sim (\theta + 2\pi, p + N - 1)$. Therefore one can calculate that time-reversal symmetry cannot hold when N is even at $\theta = \pi$.

4. Theta, time-reversal, and temperature II: 2/14/18

Today, Andy Neitzke continued to speak on [GKKS17].

Today, instead of focusing on a toy model, we will look at the case of interest, 4D Yang-Mills theory with gauge group SU_N, whose action on X has a term of the form

$$\frac{i\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F).$$

Before coupling to background fields, the theory with parameter θ is (believed to be) equivalent to the theory with parameter $\theta + 2\pi$. We will write this as $\theta \simeq \theta + 2\pi$.

This theory has a $B\mathbb{Z}/N$ symmetry, i.e. a $\mathbb{Z}/N$ one-form symmetry; here $\mathbb{Z}/N$ arises as the center of SU_N. At $\theta = 0, \pi$, there's also time-reversal symmetry T: $T(\theta) = -\theta \simeq \theta$ iff $\theta = 0, \pi \bmod 2\pi$. When we try to implement both of them simultaneously, we'll discover a mixed anomaly.

This theory couples to a background field, a $\mathbb{Z}/N$-gerbe B, which is classified by its characteristic class $[B] \in H^2(X;\mathbb{Z}/N)$.[6]

Remark 4.1. In [GKKS17], this coupling is described explicitly, and they provide a nice model for the background $B\mathbb{Z}/N$-field, a pair (B, C) where $B \in \Omega^2(X)$ and C is the connection of a line bundle, such that $NB = F_C$, where F_C denotes the curvature of C. This is something kind of like $\mathbb{Z}/N$ de Rham differential forms, but instead of something being exact, we want it to be N-torsion up to exact things.

In this case, if $\lambda \in \Omega^1(X)$, the symmetry sends $B \mapsto B + \mathrm{d}\lambda$ and $C \mapsto C + N\lambda$. ◀

[5] More generally, any continuous symmetry cannot be spontaneously broken in 2D.

[6] If you don't really know what a gerbe is, that's OK; we're mostly just going to use its characteristic class.

You can also add counterterms depending purely on the background field B (or its characteristic class), such as

$$\frac{2\pi ip}{2N}\int \mathfrak{P}_2([B]), \tag{4.2}$$

where $\mathfrak{P}_2\colon H^2(X;\mathbb{Z}/N)\to H^4(X;\mathbb{Z}/2N)$ is the Pontrjagin square, a cohomology operation, and $p\in\mathbb{Z}/2N$, so that this makes sense after exponentiation.

The Pontrjagin square has an abstract definition, but in the explicit model described above,

$$\int \mathfrak{P}_2([B]) = \int \mathrm{d}C\wedge \mathrm{d}C. \tag{4.3}$$

Now there is an equivalence of theories

$$(\theta,p)\simeq(\theta+2\pi,p+N-1) \tag{4.4}$$

and time-reversal symmetry acts by $T(\theta,p)=(-\theta,-p)$.

So at $\theta=\pi$, we need $p=-p+N-1$ to implement both time-reversal and the $B\mathbb{Z}/N$ symmetry. If this is not the case (e.g. N is even), time-reversal symmetry is broken by coupling to $B\mathbb{Z}/N$ fields, exhibiting a mixed anomaly between these two symmetries.

This story has been an UV story so far; 't Hooft anomaly matching posits that there are IR consequences, which could include

(1) the vacuum supporting a TQFT which has the same anomaly (i.e. coupling the background fields in the same way),
(2) the theory is gapless (which is regarded as unlikely, and would imply a low-energy conformal field theory rather than a topological field theory),
(3) the theory is gapped, and the $B\mathbb{Z}/N$-symmetry is broken, or
(4) the theory is gapped, and the time-reversal symmetry is broken.

So we expect there's a TQFT (case (2) is considered unlikely), but its nature is unclear — is it invertible? Case (1) is also considered unlikely, especially for large N. Case (3) is also unlikely, which we'll say more about later, which implies that time-reversal symmetry is probably broken in IR. This implies at $\theta=\pi$, time-reversal symmetri is broken and there are two vacua, exchanged by T.

Remark 4.5. In a different theory, $\mathcal{N}=1$ supersymmetric Yang-Mills theory, it's known (at a physical level of rigor) that there are N vacua. By adding a mass term $m\in\mathbb{R}_+$, one can perturb the theory slightly, breaking supersymmetry. For $\theta\neq 0,\pi$, you get two vacua, but at $\theta=0,\pi$, there are two degenerate vacua and time-reversal symmetry is broken.

This parallel story is analogous to ours, and suggests that we're on the right track. ◀

You can draw a phase diagram for these theories in θ: at $\theta=\pi$, it looks like a quartic with two global minima, and elsewhere it looks like a quartic with one global minimum and two local minima. At $\theta=0,2\pi$, the quartic has a unique local minimum. So there must be a phase transition at π.

We promised to discuss why the $B\mathbb{Z}/N$ symmetry is unbroken, which has something to do with confinement. The theory has line defects with charges $(a,b)\in(\mathbb{Z}/N)^\vee\times\mathbb{Z}/N$ ($(\mathbb{Z}/N)^\vee\cong\mathbb{Z}/N$ abstractly, of course, but this illustrates how it arises). The first component is an electric charge, and the second is magnetic. But these aren't usual line defects: for $b=0$ they are, but for $b\neq 0$, we need additional data to formulate the defect on a closed 1-manifold ℓ, which is the topological data of a surface which bounds ℓ.

Remark 4.6. If we had started with PSU_N, the roles of a and b would be switched, with $a\neq 0$ defects arising as line defects on boundaries of surfaces. ◀

If W is a Wilson line given by a representation V of SU_N, then its charge (a,b) is $a=V|_{\mathbb{Z}/N}$ and $b=0$, though of as charges of "probe particles" we can insert in the theory.

The basic question here is: which line bundles are confined? Here, Q is confined iff a defect L of charge Q has

$$\langle L(\text{loop})\rangle\sim\exp(-\alpha\Delta X\Delta T) \tag{4.7}$$

asymptotically, i.e. the energy cost of creating the loop is linear in ΔX. Heuristically, this means that correlation functions with an insertion of L on a line vanish. The higher-symmetry way to say this is that the charge Q transforms nontrivially under the unbroken 1-form symmetry. The idea is that invariant plus transforming nontrivially forces it to be zero.

So the question of who's confined is related to the question of which symmetries remain unbroken in a given vacuum. There have been many numerical simulations of SU_N-Yang-Mills suggesting that at any θ, all Wilson lines are confined. In the language of higher symmetries, this implies the $B\mathbb{Z}/N$ symmetry is unbroken.

5. Theta, time-reversal, and temperature III: 2/21/18

Today, Fei Yan gave the third and last talk on [GKKS17], today focusing on temperature — the point is to use all the material that has been developed to study phases of 4D SU_N Yang-Mills theory.

The action of this theory is given by

$$S = \int_X \left(-\frac{1}{4g^2} \operatorname{tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2} \operatorname{tr}(F \wedge F) \right). \tag{5.1}$$

We've talked about a $B\mathbb{Z}/N$ symmetry arising because $\mathbb{Z}_N = Z(\mathrm{SU}_N)$, and, at $\theta = 0, \pi$, there's a time-reversal symmetry $T\colon \theta \mapsto -\theta$.

For even N, at $\theta = \pi$, there is a mixed anomaly between these two symmetries, and there is no counterterm which resolves this. For odd N at $\theta = \pi$, the anomaly can be resolved with a counterterm, but there is no counterterm that works for both $\theta = 0$ and $\theta = \pi$.

In the IR limit, one of these symmetries is spontaneously broken, and there's an argument that it's time-reversal symmetry for $\theta = \pi$ and both even and odd N. Hence we can draw a phase diagram at zero temperature, as we discussed last time, and there is a first-order phase transition at $\theta = \pi$.

Now we turn to finite temperature. That is, take a 3-manifold Y and formulate the theory on $Y \times S^1_\beta$ (here S^1_β is a circle with circumference β), where we regard S^1 as time. As $\beta \to \infty$, this recovers the zero-temperature case, and the canonical partition function at temperature $T := 1/\beta$ is given by the partition function of the theory on $Y \times S^1_\beta$, which justifies calling this the finite-temperature setting.[7]

Let's regard this as a 3D theory in Y. What are its symmetries?

- If $\theta \neq 0, \pi$, the $B\mathbb{Z}/N$ symmetry splits as $B\mathbb{Z}/N$ and $\mathbb{Z}/N$ symmetries (based on whether the one-form symmetry is in the Y direction or the S^1_β direction).
- If $\theta = 0, \pi$, we have $B\mathbb{Z}/N$ and $\mathbb{Z}/N$ symmetries as before, and the time reversal symmetry T passes to a global $\mathbb{Z}/2$ symmetry by reversing the orientation on S^1_β.[8]

Recall that in the 4D theory, we had line operators charged by $(a, b) \in (\mathbb{Z}/N)^\vee \times \mathbb{Z}/N$, and the genuine line operators are those with $b \equiv 0 \bmod N$ (i.e. not coupled to some surface), which are Wilson lines. When $b \not\equiv 0 \bmod N$, the line has to bound a surface. When we dimensionally reduce, we could wrap a genuine line operator around S^1, obtaining a point operator, or we could not wrap, getting a line operator. But if $b \neq 0$, things are slightly different: you could not wrap around the S^1, getting the same thing back, or we could wrap around S^1, getting a point operator attached to a line.

There are special local operators, called *Polyakov loops* P, which are Wilson loops wrapped around S^1. These detect confinement at finite temperature, which happens iff $\langle P \rangle = 0$. As you increase the temperature, the theory deconfines, which is a phase transition called (unsurprisingly) *the confinement-deconfinement phase transition.*

Akin to the mixed $B\mathbb{Z}/N$ and T symmetry in 4D, the 3D theory has a mixed anomaly between the $B\mathbb{Z}/N$ symmetry, the $\mathbb{Z}/N$ symmetry, and the $\mathbb{Z}/2$ symmetry. We've already studied this in the low-temperature limit, so let's turn to the high-temperature limit, where $\beta \ll \Lambda_{\mathrm{YM}}^{-1}$ (here $\Lambda_{\mathrm{YM}}^{-1}$ is the *dynamical scale* of the Yang-Mills theory). The high-temperature limit of Yang-Mills was studied in the 1980s, which is convenient for us.

Let $N = 2$. Then $\mathbb{Z}/2$ acts in two different ways; let $(\mathbb{Z}/2)^C$ denote the symmetry arising through the center of SU_2 and $(\mathbb{Z}/2)^T$ denote the symmetry coming from time-reversal on S^1. Let $g_{3d}^2 \sim g_{\mathrm{YM},4d}^2 \beta^{-1}$ be an SU_2 gauge field, and Φ be an adjoint scalar. Let

$$U = P e^{i \oint_{S^1} A} := e^{i\beta\Phi}, \tag{5.2}$$

[7] This is an instance of a very general fact about quantum mechanics: the partition function $Z = \operatorname{tr}(e^{-\beta H})$ is the partition function for the system at temperature $1/\beta$.

[8] There's another symmetry given by orientation-reversal on Y (after Wick rotation?), but it doesn't enter the discussion.

where $P = \operatorname{tr} U$, and pick a gauge

$$\Phi = \begin{pmatrix} \phi(x) & 0 \\ 0 & -\phi(x) \end{pmatrix}, \tag{5.3}$$

where $\phi \sim \phi + 2\pi/\beta$.

There are two minima of V_{eff}^{ϕ}: $\beta\phi = 0$ ($U = I$) and $\beta\phi = \pi$ ($U = -I$). In this case $(\mathbb{Z}/2)^c$ sends $\phi \mapsto \phi/\beta$ and $U \mapsto -U$, and the $B\mathbb{Z}/2$ and $(\mathbb{Z}/2)^T$ symmetries are unbroken (the latter when $\theta = 0, \pi$). The argument uses Polyakov loops: since $\operatorname{tr} U$ is the expectation of a Polyakov loop, but is nonzero, then the $(\mathbb{Z}/2)^C$ symmetry is broken. The argument is similar for general N.

To study phase transitions, let's organize this all into a table.

- The $B\mathbb{Z}/N$ symmetry is unbroken at all values of θ for the high-temperature and low-temperature cases.
- The $\mathbb{Z}/N$ symmetry is broken for all θ in the high-temperature case, and is preserved in the low-temperature case.
- The $(\mathbb{Z}/2)^T$ symmetry is unbroken at $\theta = 0, \pi$ in the high-temperature case, but is only unbroken at $\theta = 0$ in the low-temperature case.

This makes the mixed anomaly apparent: at $\theta = \pi$, in both the high-temperature and low-temperature cases, at least one symmetry is broken in the IR theory.

This tells us information about a two-dimensional phase diagram in θ and $T = 1/\beta$. We know at low temperature there's a phase transition at θ, and at some high temperature there's a confinement-deconfinement phase transition. For $N = 2$ confinement-deconfinement is second-order, and for $N > 2$ it's first-order.[9]

What happens when these two phase transitions meet? Let's specialize to $N = 2$ for a bit. To answer the question, we can gauge the $B\mathbb{Z}/2$ symmetry in 3D. A general fact about higher-form (global) symmetry is that if you gauge a discrete q-form symmetry, what you get has a $(d - q - 2)$-form symmetry. With $d = 3$ and $q = 1$, we expect a 0-form $\mathbb{Z}/2$-symmetry, which we'll call $(\mathbb{Z}/2)^B$.

However, this does *not* happen everywhere. It does happen when $\theta \neq 0, \pi$, so we obtain a $(\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^T$ synmmetry. At $\theta = 0$, we have $(\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^B \times (\mathbb{Z}/2)^T$, and at $\theta = \pi$, we get a D_8 symmetry, arising through the extension

$$1 \longrightarrow (\mathbb{Z}/2)^C \times (\mathbb{Z}/2)^T \longrightarrow D_8 \longrightarrow (\mathbb{Z}/2)^B \longrightarrow 1. \tag{5.4}$$

The appearance of this D_8 is a bit of a surprise, and has something to do with the anomaly. There are two proofs present in [GKKS17].

Let c be a generator of the $(\mathbb{Z}/2)^C$ symmetry, and define b and t similarly. The Polyakov loop is given by the loop with charge $(1, 0)$. We're also interested in the twisted sectors $A := (0, 1)$ and $B := (1, 1)$ — A generates the $B\mathbb{Z}/2$ symmetry, which we've wrapped around the circle, and similarly with B.

To see why we get a D_8, we'll compute how c, b, and t act on A and B.

- Explicitly, c is the non-identity central element of SU_2. Thus it sends $A \mapsto A$ and $B \mapsto -B$.
- Since both A and B were twisted operators attached to a line, b maps $A \mapsto -A$ and $B \mapsto -B$.
- Since t comes from time-reversal, we recall that $\theta \mapsto \theta + 2\pi$ shifts your line operators by $(a, b) \mapsto (a + b, b)$. The upshot is that t exchanges A and B.

Therefore $tct = cb$, $tbt = b$, and $bc = bc$, so if you think of $A, B, -A, -B$ as the vertices of a square, you get all of the symmetries of the square from c and b, and therefore get a D_8.

We want to use this to understand the phase diagram. The analysis of this D_8-action indicates that $(\mathbb{Z}/2)^B$ must be spontaneously broken near the intersection of the two phase transitions.

- Below the confinement-deconfinement transition at $\theta < \pi$, we have two vacua spanned by A and B, with c acting by $A \mapsto -A$.
- Below the confinement-deconfinement transition at $\theta > \pi$, we have the same, but with $cb = bc$ acting by $B \mapsto -B$.
- Above the confinement-deconfinement transition, these two vacua separate into four vacua $(\pm A, \pm B)$.

[9] This was concluded using lattice simulations; it's not clear if there's a continuum argument for it.

If the two phase transitions didn't intersect, you would be able to show that the D_8 symmetry is completely unbroken, so they must intersect, and the $\theta = \pi$ phase transition comes up to meet the confinement-deconfinement one. For $N = 2$, at least, it goes slightly higher, but in general we don't know whether it just meets it.

Remark 5.5. Some of these arguments are definitely true for large N, but require an assumption on the breaking of the D_8 symmetry for smaller N. You could relax that assumption and end up with a more exotic phase diagram. ◀

6. 2-groups: 2/28/18

These are Arun's prepated notes for his talk in GST, on 2-groups. Today, all categories are small categories.

6.1. **2-groups and crossed modules.**

There are many ways to define 2-groups and a web of equivalences between them. We'll discuss a few of them in this part of the talk.

Various notions of 2-groups were introduced and compared by Whitehead [Whi46], Mac Lane-Whitehead, Brown-Spencer [BS76], Hoàng [Hoà75], Joyal-Street, and Baez-Lauda [BL04]. The definition given here, following Baez-Lauda, encodes the philosophy that a group is a monoid in which every element is invertible.

Definition 6.1. A *monoidal category* $(\mathsf{C}, \otimes, 1, \alpha, \ell, r)$ is:

- a category C,
- a functor $\otimes\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$,
- an *identity object* $1 \in \mathsf{C}$,
- and natural isomorphisms

$$\alpha_{x,y,z}\colon (x\otimes y)\otimes z \xrightarrow{\cong} x\otimes(y\otimes z), \tag{6.2a}$$

$$\ell_x\colon 1\otimes x \xrightarrow{\cong} x, \tag{6.2b}$$

$$r_x\colon x\otimes 1 \xrightarrow{\cong} x, \tag{6.2c}$$

subject to two coherence conditions that we won't write down (but can be found in [BL04, §2]).

In general, categorification turns conditions into data: associativity is implemented by choosing α, and similarly with identity. Here's another example.

Definition 6.3. Let $(\mathsf{C}_1, \otimes_1)$ and $(\mathsf{C}_2, \otimes_2)$ be monoidal categories. A *monoidal functor* is a functor $F\colon \mathsf{C}_1 \to \mathsf{C}_2$ together with natural isomorphisms $F(x)\otimes_2 F(y) \cong F(x\otimes_1 y)$ and $F(1_{\mathsf{C}_1}) \cong 1_{\mathsf{C}_2}$ such that (some coherence conditions in [BL04, §2]).

There is a similar notion of a *monoidal natural transformation*; again see [BL04, §2].

Definition 6.4. A *2-group* $\mathbb{G}$ is a monoidal category such that for every object $x \in \mathbb{G}$, there's a $y \in \mathbb{G}$ such that $x\otimes y \cong 1$ and $y\otimes x \cong 1$.

A morphism of 2-groups is a monoidal functor. We also have monoidal natural transformations, which means that there's a *2-category* of 2-groups. Approximately what this means is that there objects and morphisms as usual, but given morphisms $f, g\colon x \to y$, there can be "2-morphisms" $H\colon f \Rightarrow g$.[10]

Example 6.5. Since 2-groups are supposed to describe mixed 0-form and 1-form symmetries, they should specialize to ordinary groups if one of the symmetries is trivial.

(1) Given a group G, let G be the category whose objects are the elements of G and with only identity morphisms. Group multiplication defines a monoidal structure on G, making it into a 2-group. Heuristically, we've forgotten about the level-1 information, leaving just level-0 information.
(2) Dually, given a group G, we can consider the category BG with a single object and $\mathrm{Hom}_{BG}(*, *) = G$ (as a set, then with composition defined by group multiplication). This also defines a 2-group (e.g. the associator uses the associativity of G), which we can think of as having a trivial level-0 part and G for its level-1 part. ◀

[10] Making this precise, and even precisely defining 2-categories, requires some effort. We will avoid such questions while noting that good solutions exist.

An alternative perspective on 2-groups is to switch the role of the group and the category. If C is a small category, let C_0 denote its set of objects and C_1 its set of morphisms. Then we have two maps $s, t\colon C_1 \rightrightarrows C_0$ sending an arrow to its source, resp. target, and a map $i\colon C_0 \to C_1$ sending $x \mapsto \mathrm{id}_x$. It's possible to encode the axioms of a category (composition, identity, etc.) into properties of s, t, and i.

If $\mathbb{G}$ is a 2-group with objects G_0 and G_1, then G_0 and G_1 are groups under tensor product, and s, t, and i are group homomorphisms, so in some sense the diagrams that encode $(\mathbb{G}, \otimes)$ are formulated entirely in the category of groups! Such a diagram is called an *internal category* in Grp.

However, we now have access to more structure: we can define a *Lie 2-group* to be a pair of Lie groups G_0 and G_1 and Lie group homomorphisms $s, t, \colon G_1 \to G_0$ and $i\colon G_0 \to G_1$ satisfying the relations encoding associativity, etc.

Proposition 6.6. *This construction extends to an equivalence of 2-categories between 2-groups and internal categories in* Grp*.*

See [FB02, §3] for a proof. This in particular implies that every diagram of groups and group homomorphisms (G_0, G_1, i, s, t) satisfying the axioms defining a category arises in this way from a 2-group $\mathbb{G}$.

Example 6.7. Let X be a topological space with basepoint x. The *fundamental 2-group* of X, denoted $\pi_{1,2}(X, x)$ is the 2-group whose objects are loops in X based at x and whose morphisms are equivalence classes of homotopies between paths (where two homotopies are equivalent if they are themselves homotopic). The monoidal structure arises by composition of paths. Every 2-group arises in this way. ◀

6.2. **Classifying spaces and 2-gauge fields.** A third perspective on 2-groups in algebraic topology is that they describe spaces with only two nontrivial homotopy groups, through their classifying spaces. This leads to notions of principal 2-bundles and connections on them (though we won't have a lot to say about this). Fix a 2-group $\mathbb{G}$ (i.e. a discrete 2-group: we want G_0 and G_1 to be discrete).

We will describe a connected space $B\mathbb{G}$ such that $\pi_1(B\mathbb{G}) \cong H_1 := \pi_0\mathbb{G}$ and $\pi_2(B\mathbb{G}) \cong H_2 := \mathrm{Aut}(1_{\mathbb{G}})$. One such choice is $K(H_0, 1) \times K(H_1, 2)$,[11] but this is wrong: heuristically, it's telling us that the two symmetries don't mix at all. In particular, a map from a space X to this space is data of a principal H_0-bundle and an H_1-gerbe, corresponding in physics to an H_0 symmetry and a BH_1-symmetry, but they don't mix at all.

Instead, the mixing of these two symmetries is encoded by making $B\mathbb{G}$ a fiber bundle over $K(H_0, 1)$ with fiber $K(H_1, 2)$. We specify the fiber bundle $p\colon B\mathbb{G} \to K(H_0, 1)$ by its homotopy cofiber, a map $k\colon K(H_0, 1) \to \Sigma K(H_1, 2) \simeq K(H_1, 3)$ called the *k-invariant* of the space $B\mathbb{G}$. Namely, the associator of $\mathbb{G}$ defines a map $H_1 \times H_1 \times H_1 \to H_2$, which is a cocycle for $H^3(H_0; H_1) = H^3(BH_0; H_1)$. Since $BH_0 \simeq K(H_0, 1)$ and cohomology is represented by maps into Eilenberg-Mac Lane spaces, we have a natural identification $H^3(H_0; H_1) \cong [K(H_0, 1), K(H_1, 3)]$ sending the associator of $\mathbb{G}$ to the k-invariant of $B\mathbb{G}$.

$B\mathbb{G}$ is an example of a *homotopy 2-type*, i.e. a homotopy type with only two nontrivial homotopy groups. This generalizes the fact that if G is a discrete group, $\pi_i(BG)$ is only nontrivial when $i = 1$.

Proposition 6.8. *Every homotopy 2-type is the classifying space of some 2-group, and there is an equivalence of 2-categories between 2-groups and homotopy 2-types.*

If G is an (ordinary) compact Lie group, its classifying space BG is a moduli space for principal G-bundles, meaning that every principal G-bundle $P \to X$ is the pullback of the universal bundle $EG \to BG$ along a map $X \to BG$, together with a uniqueness condition. We would like something similar to be true for 2-groups, but this is a place in which categorification makes things much harder: this is spelled out by Bartels [Bar04], Baez-Scheiber [BS04], and Baez-Stevenson [BS08], but the interactions between category theory and topology were complicated enough that I didn't figure out anything useful.

As a substitute, if $\mathbb{G}$ is finite, we can use the Postnikov tower of $B\mathbb{G}$ to describe principal $\mathbb{G}$-bundles. Here's what we want.

- Over any space X, isomorphism classes of principal $\mathbb{G}$-bundles are in bijection with $[X, B\mathbb{G}]$.
- If the 2-group symmetry is really just an G_0-symmetry for some group G_0, principal $\mathbb{G}$-bundles should be the same as principal G_0-bundles.

[11] Here, $K(G, n)$ is an *Eilenberg-Mac Lane space*, which we've been more frequently denoting B^nG in this seminar; the notation $K(G, n)$ is more common in algebraic topology.

- If the 2-group symmetry is really just a BG_1-symmetry, then principal $\mathbb{G}$-bundles should be the same as G_1-gerbes.

We also want the k-invariant to appear somehow.

Given a map $X \to B\mathbb{G}$, we can compose with the projection and obtain a map $X \to BH_0$, so we get a principal H_0-bundle P. But we can also pull back $B\mathbb{G} \to BH_0$ to X, producing a fiber bundle with fiber $K(H_1, 2)$, i.e. an H_1-gerbe Q. The k-invariant provides a constraint on how these are related — unfortunately, I wasn't able to figure out how this goes, but it's going to go something like this: given three loops ℓ_1, ℓ_2, ℓ_3 in X with a common basepoint, they have monodromies $h_1, h_2, h_3 \in H_0$ for P. The k-invariant, as a cocycle for group cohomology, defines a $g = k(h_1, h_2, h_3) \in H_1$, and this should be some kind of monodromy around a higher-dimensional sphere for Q that's related to ℓ_1, ℓ_2, and ℓ_3.

6.3. **2-group symmetries in physics.** In the remainder of this talk we will discuss examples of 2-group symmetries in physics.

Example 6.9 (The Yetter model)**.** For any finite 2-group $\mathbb{G}$, there's a TQFT called the Yetter model with a $\mathbb{G}$-symmetry, defined in much the same way as Dijkgraaf-Witten theory.[12] This model was developed by Yetter [Yet93], Birmingham-Rakowski [BR96], and Mackaay [Mac99], and is a special case of a general construction of Quinn (TODO: cite).

Fix a finite 2-group $\mathbb{G}$ and a cohomology class $\alpha \in H^n(B\mathbb{G}; \mathbb{R}/\mathbb{Z})$. Then α defines a characteristic class for principal $\mathbb{G}$-bundles: if $P \to M$ is a principal $\mathbb{G}$-bundle, it defines up to homotopy a classifying map $f_P \colon M \to B\mathbb{G}$; we let $\alpha(P) \coloneqq f_P^*\alpha \in H^n(M; \mathbb{R}/\mathbb{Z})$.

The Yetter model is the n-dimensional, oriented TQFT with a fluctuating $\mathbb{G}$-gauge field P and whose action is $\alpha(P)$. Thus its partition function on a closed n-manifold M is summed over $\mathrm{Bun}_{\mathbb{G}}(M)$ using the "2-groupoid measure," so that

$$Z_{\mathbb{G},\alpha}(M) = \text{``}\int_{\mathrm{Bun}_{\mathbb{G}} M} e^{i\pi\langle \alpha(P), [M]\rangle}\, \mathrm{d}P\text{''} = \sum_{P \in \pi_0 \mathrm{Bun}_{\mathbb{G}}(M)} \frac{|^2\mathrm{Aut}(P)|}{|\mathrm{Aut}(P)|} \exp(i\pi\langle \alpha(P), [M]\rangle). \tag{6.10}$$ ◂

Remark 6.11. I wanted to say something about the QED-like theories with 2-group symmetries discussed in [CDI18], but ran out of time, and also didn't completely understand the examples given. ◂

7. Phases of 4D QCD, I: 3/7/18

"I'm about to say what you're about to say, but in a better way."

Today, Jacques Distler spoke about phases of QCD theory in 4D. We'll follow Seiberg's convention that dymanical fields are lowercase and background fields are uppercase. In this convention, the Yang-Mills action that we've been learning about is

$$S_E = \int_X \mathrm{tr}\left(-\frac{1}{4g^2} f \wedge \star f + \frac{i\theta}{8\pi^2} f \wedge f\right). \tag{7.1}$$

Here a is an $\mathrm{SU}(N_c)$ connection and f is the field strength (the curvature of a). Here g^2 isn't really a parameter — it's not invariant under renormalization, and it will be traded for a dimensionful parameter Λ. We've discussed a $B\mathbb{Z}/N_c$ symmetry acting on Wilson lines in this theory, along with a time-reversal symmetry when $\theta = 0, \pi$.

Next, we'll introduce Weyl fermions, which are Grassmann-valued sections of $S_+(X)$, and eventually couple them to background fields. One important takaeaway is that the $\mathbb{Z}/N_c$ centers of the $\mathrm{U}(N_f)_L$ and $\mathrm{U}(N_f)_R$

	$\mathrm{U}(N_f)_L$	$\mathrm{U}(N_f)_R$	$\mathrm{SU}(N_c)$
ψ	N_f	1	N_c
$\widetilde{\psi}$	1	$\overline{N}_f$	$\overline{N}_c$

Table 1. TODO: I don't know what this means.

are identified, so we really have a symmetry by something like $\mathrm{U}(N_f)_L \times_{\mathbb{Z}/N_c} \mathrm{U}(N_f)_R$.

[12] This is not the same thing as the Crane-Yetter model!

The action is

$$S_{\text{fermions}} = \int_X \psi^\dagger i\sigma\cdot D\psi + \widetilde{\psi}^\dagger i\sigma \cdot D\widetilde{\psi}. \tag{7.2}$$

Using this, one can calculate that

$$b_1 = \frac{1}{3}(-11N_c + 2N_f), \tag{7.3}$$

and if you only take 1-loop terms, you get an expression for Λ.

$$\frac{1}{g^2(m)_{\text{1-loop}}} = -\frac{b_1}{8\pi^2}\log\Big(\frac{\mu}{\Lambda}\Big). \tag{7.4}$$

When $N_f \geq (11/2)N_c$ this theory is IR free. There's an N_* such that if $N_* \leq N_f < (11/2)N_c$, you get a conformal field theory in the IR, and for $N < N_*$ the theory is confining. This was discovered by Banks-Zaks [BZ82], who showed it for large N_c; it's believed to persist for finite N_c. The precise value of N_* is unknown, but we do know that

$$N_* = \frac{34}{13}N\bigg(1 + \frac{3}{13N_c} + O\Big(\frac{1}{N_c^2}\Big)\bigg). \tag{7.5}$$

We will assume $N_f < N_*$. For large N_c, one can send $N_c \to \infty$ holding $\lambda = g^2N_C$ fixed, and trade it for Λ as before. In this case $1/N_c^2$ is a small parameter.

The global $\mathrm{U}(N_f)_L \times_{\mathbb{Z}/N_c} \mathrm{U}(N_f)_R$ symmetry we discussed above is anomalous. We can describe it concretely via its anomaly polynomial

$$\frac{1}{48\pi^3}\operatorname{tr}(\mathscr{F}^3) = \frac{N_c}{48\pi^3}\big(\operatorname{tr}(F_L^3) - \operatorname{tr}(F_R^3)\big) + \frac{1}{16\pi^3}(\operatorname{tr}(F_L) - \operatorname{tr}(F_R))\operatorname{tr}(F^2). \tag{7.6}$$

In particular, if $\alpha\colon \mathrm{U}(N_f)_L \times_{\mathbb{Z}/N_c} \mathrm{U}(N_f)_R \to \mathrm{U}(1)_A$ sends

$$\alpha\colon g_L, g_R \longmapsto \det(g_L)\overline{\det g_R}, \tag{7.7}$$

then α is surjective and (7.6) explicitly breaks the $\mathrm{U}(1)_A$ symmetry. Given an $e^{i\beta} \in \mathrm{U}(1)_A$, we get an identification of the theory with parameter θ and the theory with parameter $\theta - N_f\beta$.

We can extract more insight if we turn on a mass term

$$S_m = \int_X \psi\widetilde{\psi} + \text{h.c.}. \tag{7.8}$$

In this case $e^{i\beta} \in \mathrm{U}(1)_A$ provides an identification $(\theta, m) \simeq (\theta - N_f\beta, e^{i\beta}m)$. This theory also breaks the $\mathrm{U}(N_f)_L \times_{\mathbb{Z}/N_c} \mathrm{U}(N_f)_R$ symmetry down to the diagonal, which will be important later.

Let $\overline{\theta} := \theta + N_f \arg(m)$. Then the physics of the theory only depends on $\overline{m}^{N_f} = m^{N_f}e^{i\theta} = |m|^{N_f}e^{i\overline{\theta}}$.

As usual, we're going to do 't Hooft anomaly matching, but unlike the fancy anomalies we've been seeing this semester with their fancy low-energy topological field theories with the same symmetries, this can be attacked by older methods. The first term tells us that even though the theory confines, it cannot be gapped (here $N_f > 1$). Hence in the IR, $G := \ker(\alpha)$ must be spontaneously broken to $H = \mathrm{U}(N_f)_\Delta/\mathbb{Z}/N_c$, where $\mathrm{U}(N_f)_\Delta$ denotes the diagonal inside $\mathrm{U}(N_f)_L \times_{\mathbb{Z}/N_c} \mathrm{U}(N_f)_R$. There must be propagating degrees of freedom, which are massless Goldstone bosons (or *pions*). These are described on X by a σ-model into G/H, or more generally, sections of a G/H-bundle over X.

In our case, G/H is noncanonically isomorphic to $\mathrm{SU}(N_f)$, so we can introduce a dimensionless $\mathrm{SU}(N_f)$-valued field Σ and write the action as

$$S = \int \frac{f_\pi^2}{2}\operatorname{tr}\big(D_\mu\Sigma^\dagger D^\mu\Sigma\big) + S_{\text{anom}} + \dots, \tag{7.9}$$

where $f_\pi \sim 4\pi\Lambda$ and S_{anom} contains an explicit breaking of G-gauge invariance, but is itself H-gauge-invariant. For $m \neq 0$, there's an explicit breaking of $G \to H$, but for $|m| \ll \Lambda$, the σ-model description is still good.

The mass term is now

$$S_{\text{mass}} = \int K\operatorname{tr}\big(m\Sigma + m^*\Sigma^\dagger\big), \tag{7.10}$$

where $K \sim \Lambda_{\text{QCD}}^3$, and the minimum is at $\Sigma = \pi$. Thus the pions are massive. For $|m| \gg \Lambda_{\text{QCD}}$, we could have integrated out the fermions, and therefore the IR physics reduces to pure Yang-Mills theory.

Now let's suppose N_c is large. In this case, the $\mathrm{U}(1)_A$-symmetry breaking that comes from the anomaly changes. Instead, the full $\mathrm{U}(N_f)\times_{\mathbb{Z}/N_c}\mathrm{U}(N_f)$ symmetry is broken to H, which implies there's N_f^2, rather than N_f^2-1, bosons.

The cheap, but not completely correct, argument as to why: in Euclidean signature, the Schwarz inequality implies that

$$\int \operatorname{tr}(f\wedge\star f)\geq\left|\int\operatorname{tr}(f\wedge f)\right|, \tag{7.11}$$

with equality when $f=\pm\star f$. This can be used to show that

$$\exp\left(-\frac{1}{4g^2}\int\operatorname{tr}(f\wedge\star f)\right)\leq\exp\left(-\frac{2\pi^2}{\lambda}N_c|\nu|\right). \tag{7.12}$$

This argument is a bit too naïve: the mass of the additional Goldstone boson $m_\eta^2\sim 1/N_c$, so it does go away, but at a different rate.

Remark 7.13. Some of these particles, e.g. pions, really exist in the real world! Part of the reason these kinds of questions were originally studied in the 1970s was to understand how the masses, etc. of these particles (and therefore the physics of the real world) depended on the initial parameters of the theory. ◂

So you can write down the action as

$$S=\frac{f_\pi^2}{2}\int\left(\operatorname{tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right)+\cdots+\frac{c_1}{N_c}\left(\operatorname{tr}(\Sigma^\dagger\partial_\mu\Sigma)\right)^2+c_2\Lambda\left(m\operatorname{tr}\Sigma+m^*\operatorname{tr}\Sigma^\dagger\right)-\frac{c_3\Lambda^2}{N_c}(\theta+i\log\det\Sigma)^2\right). \tag{7.14}$$

This looks a little crazy – the logarithm is multivalued, and θ is supposed to be an angle, hence only well defined mod 2π. Fortunately, these issues cancel themselves out, because the ambiguity that different branches of the logarithm give you is $2\pi n$ for some $n\in\mathbb{Z}$.

If you minimize (7.14), you get

$$V(\Sigma)=\frac{f_\pi^2}{2}\left(-c_2\Lambda|m|\operatorname{tr}(\Sigma+\Sigma^\dagger)+\frac{c_3\Lambda^3}{N_c}(\theta+i\log\det\Sigma)^2\right). \tag{7.15}$$

Specializing to $N_f=1$, we get the potential for η', where $\Sigma=e^{i\theta'/f_\pi}$:

$$V(\eta')=f_\pi^2\Lambda\left(-c_2|m|\cos(\eta'/f_\pi)+\frac{c_3\Lambda}{2N_c}(\theta-\eta'/f_\pi)^2\right). \tag{7.16}$$

This provides a concrete interpretation to the lines in the phase diagrams for QCD_4 that we've been seeing.[13]

Now let's try to minimize this. We want

$$0=c_2|m|\sin v-\frac{c_3\Lambda}{N_c}(\theta-v), \tag{7.17}$$

where $v:=(\eta'/f_\pi)_{\min}$. Therefore

$$m_{\eta'}^2=\left(1+\frac{c_1}{N_c}\right)^{-1}\left(c_2|m|\Lambda\cos v+\frac{c_3\Lambda^2}{N_c}\right). \tag{7.18}$$

In the limit $|m|\to 0$, $v\approx\theta$, so

$$m_{\eta'}^2=\frac{c_3A^2}{N_c}+c_2|m|\Lambda\cos\theta+\dots. \tag{7.19}$$

This is another way to see the phase diagram that we've been discussing (with the same caveat) — we're arguing for large N_c, but there's an argument is that it should persist for finite N_c as well.

We've seen that for pure Yang-Mills theory at $\theta=\pi$, there's two vacua with a domain wall between them, and a $B\mathbb{Z}/N_c$ symmetry with an 't Hooft anomaly coming from the bulk, hence cannot be a trivial theory on the domain wall. Instead, it's some 3D TQFT, specifically $\mathrm{SU}(N)_1$-Chern-Simons theory (this is in [GKKS17]).

[13] Well, we were looking at $N_f=0$, where there's no scalar potential. But for N_c large enough, the phase diagrams for $N_f=0$ and $N_f=1$ look very similar, so we were drawing something which looks like this potential, and in this case it makes sense.

Near $m_0 := |m|e^{i\theta}$, the low-energy physics is described by an effective Lagrangian for $\phi = \eta' - f_\pi v$, with potential

$$V(\phi) = c\Lambda(\overline{m} + m_0)\phi^2 - c'\phi^4 + \cdots, \tag{7.20}$$

and this implies we get a second-order phase transition, with IR trivial domain wall. Therefore at some $m_t < 0$, there's a phase transition between these domain walls, while nothing happens on the bulk. The answer argued for in [GKKS17] is an $\mathrm{SU}(N)_{1/2}$-Chern-Simons theory with a fermion ψ.

8. Phases of 4D QCD, II: 3/28/18

Today Mario Martone spoke. I missed the first 10 minutes, which was mostly review of previous talks.

As usual we have the $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ global symmetry, as well as time-reversal $T\colon \theta \mapsto -\theta$, which is a symmetry for $\theta = 0, 2\pi$. There are several phases: if $N_f > 11N/2$, then the theory is IR-free; if $N_{\mathrm{CFT}} \le N_f < 11N/2$, there's a nontrivial fixed point, and if $N_f < N_{\mathrm{CFT}}$, the theory confines with confining scale Λ_{QCD}. Below this scale, there's chiral symmetry breaking, and even for $M = 0$, the $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ symmetry is broken to $\mathrm{U}(N_f)$. In this case, the theory is described by a nonlinear σ-model, and the relevant degrees of freedom are the Goldstone bosons $G \to H$.

Today, we're going to do a more careful analysis of this phase. We'll start by thinking about the $\mathrm{U}(1)_A$ anomaly, and how it implies that the theory only depends on a single parameter $m\exp(i\theta/N_f)$. When $m = 0$, the theory is completely independent of θ, but in general this is not true.

Explicitly, the $\mathrm{U}(1)_A$ anomaly is that an $e^{i\beta} \in \mathrm{U}(1)_A$ sends the theory with $(\theta, m) \mapsto (\theta - N_f\beta, e^{i\beta}m)$.

Remark 8.1. We'd like to mix this with time-reversal symmetry. If we put these together, we get a time-reversal symmetry squaring to the fermion counting operator, rather than 1. This means that it should be able to make sense of this theory, with this symmetry, on Pin^+ manifolds. The time-reversal symmetry would switch left-handed fermions and right-handed fermions.[14] ◂

Time-reversal acts by $\theta \mapsto -\theta$ and $m \mapsto \overline{m}$, so in particular we have to assume $\theta = 0$ or $\theta = \pi$. We can write the Lagrangian in terms of a field $U\colon X \to G/H$; then

$$\mathcal{L} = \frac{f_\pi^2}{2}\operatorname{tr}(\partial U \partial U^\dagger) - V, \tag{8.2}$$

where

$$V := -\frac{1}{2} f_\pi^2 \Lambda m e^{i\theta/N_f} \operatorname{tr} U + \text{c.c.} \tag{8.3}$$

This comes from (I think?) $\langle \psi \mid \widetilde{\psi}\rangle = f_\pi^2\Lambda$. The $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ symmetry sends $U \mapsto M_1^\dagger U M_2$, where M_1 is in the first copy of $\mathrm{U}(N_f)$ and M_2 is in the second copy.

At $\theta = 0$, we have

$$V = -mN_f f_\pi^2 \Lambda \cos\left(\frac{2\pi k}{N_f}\right), \tag{8.4a}$$

so $k = 0$. For $\theta = \pi$,

$$V = -mN_f f_\pi^2 \Lambda \cos\left(\frac{\pi(2k+1)}{N_f}\right), \tag{8.4b}$$

so $k = 0$ or $k = -1$ (two different vacua, with a domain wall, as we saw before in the simpler case).

Therefore $\theta \mapsto \theta + 2\pi$ can be implemented by

$$U \longmapsto \exp 2\pi i/N_f U. \tag{8.5}$$

For $\theta = 0$, $T\colon U \mapsto U^\dagger$; for $\theta = \pi$, $U \mapsto \exp(-2\pi i/N_f)U^\dagger$.

If $U = \mathbf{1}$ and $\theta = 0$, then time-reversal symmetry isn't spontaneously broken. But at $\theta = \pi$, there are two vacua which T permutes, which are $U = \mathbf{1}$ and $U = \exp(-2\pi i/N_f)\mathbf{1}$. Therefore T *is* spontaneously broken.

There is a 3D theory on the domain walls between these two vacua at $\theta = \pi$.

TODO: I'm not sure what happened next, but if we want U to be diagonal in this 3D theory, we can assume that $k = N_f - 1$, and U is diagonal with diagonal entries $e^{i\alpha}, \dots, e^{i\alpha}, e^{i\beta}$. This breaks $\mathrm{SU}(N_f)$ to $S(\mathrm{U}(1) \times \mathrm{U}(N_f - 1))$. In particular, we get a σ-model into $\mathbb{CP}^{N_f - 1} = \mathrm{SU}(N_f)/S(\mathrm{U}(1) \times \mathrm{U}(N_f - 1))$.

[14] In that case, the anomaly theory should be some explicit 5D Pin^+ invertible TQFT, right? Which one is it?

This is in accordance with results about 3D Chern-Simons theory that we'll talk about next time: we'll see the theory is $\mathrm{SU}(N)_{1-N_f/2}$ Chern-Simons theory together with N_f fermions.

9. QCD$_4$ with $1 < N_f < N_{\mathrm{CFT}}$: 4/4/18

Today, Mario Martone spoke again, a continuation of last time's talk.

The restriction $1 < N_f < N_{\mathrm{CFT}}$ means the IR theory is confined. If we turn on a mass term, the theory depends not exactly on θ, nor on the mass $m > 0$, but on the parameter $\widetilde{m} := m^{N_f} e^{i\theta} \in \mathbb{C}^\times$. Namely, we have the following three cases.

- If $\widetilde{m} \in \mathbb{R}$ and is positive, time-reversal symmetry is unbroken (this is the case $\theta = 0$), and there's a single vacuum.
- If $\widetilde{m} \in \mathbb{R}$ and is negative, time-reversal symmetry is spontaneously broken (this is the case $\theta = \pi$). This means there are two vacua, and time-reversal symmetry exchanges them.
- If $\widetilde{m} \notin \mathbb{R}$, time-reversal symmetry is explicitly broken.

Remark 9.1. Strictly speaking, this is an approximation of the actual phase diagram; it becomes more accurate for $N_f \gg 0$. ◀

For the case $\widetilde{m} < 0$, we might ask if there's a domain wall between the two vacua. This is some 3D theory, though it sometimes only makes sense as a boundary theory.

In 3D, we have some dualities between Chern-Simons-matter theories. For $N_f \le 2k$, the $\mathrm{SU}(N)_{-k}$-Chern-Simons theory with N_f fermions is dual to the $|U(k+N_f/2)_N$ theory with N_f scalars. For $2k < N_f \le N_*(N,k)$ (for some N_*), this is still a conjecture, and N_* should have something to do with N_{CFT}.

One interesting aspect of this is that on the bosonic side of the theory, the $\mathrm{U}(k + N_f/2)$ theory, with positive masses not too large, does something weird aroiund $N_f \approx 2k$. For smaller N_f, it flows to a TQFT, namely $\mathrm{SU}(N)_{k+N_f/2}$, but for larger N_f, it flows to a nonlinear sigma model with target $\mathrm{U}(N_f)/(\mathrm{U}(N_f/2 + k) \times \mathrm{U}(N_f/2 - k))$, together with a Wess-Zumino term Γ.

This additional phase should also appear on the fermionic side, and it's conjectured to; see [KS18, BHS17].

Okay, back to 4D. Recall that the global symmetries for QCD$_4$ in the UV is $G_{UV} = \mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R \times \mathbb{Z}/2$, where the $\mathbb{Z}/2$ is time-reversal; the $\mathrm{U}(1)_A$-symmetry is anomalous, as we saw last month. There is an energy level Λ such that for energy levels below Λ, the theory confines, and in the IR is a nonlinear σ-model with target $\mathrm{SU}(N_f)$, so the G_{UV} symmetry breaks to $\mathrm{U}(N_f)_V \times \mathbb{Z}/2$.

For $\theta = \pi$ and $m \ll \Lambda$, there are two vacua, $U_1 = \mathbf{1}$ and $U_2 = e^{2\pi i/N_f}\mathbf{1}$, and time-reversal exchanges them. The $\mathrm{U}(N_f)_V$ symmetry is unbroken. We have a Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{2}\Big(\mathrm{tr}(\partial U \partial U^\dagger) - m\Lambda e^{i\theta/N_f}\,\mathrm{tr}\,U + \text{c.c.}\Big). \tag{9.2}$$

We want a $\widetilde{U}^{(t)}$ interpolating between the two vacua, so it's U_i at time $t = i-1$. Specifically, we want $\widetilde{U}^{(t)}$ to be diagonal, with diagonal entries $\exp(i\alpha_j(t))$. Since $\det \widetilde{U}^{(t)} = 1$, $\sum \alpha_i(t) = 0 \bmod 2\pi$, and we have some other information about the components:

$$\begin{gathered}\alpha_1^{(t)} = \alpha_2^{(t)} = \cdots = \alpha_k^{(t)}\\ \alpha_{k+1}^{(t)} = \alpha_{k+2}^{(t)} = \cdots = \alpha_{N_f}^{(t)}.\end{gathered}$$

At 1, we have

$$\begin{gathered}\alpha_1^{(1)} = \alpha_2^{(1)} = \cdots = \alpha_k^{(1)} = -\frac{2\pi}{N_f}\\ \alpha_{k+1}^{(1)} = \alpha_{k+2}^{(1)} = \cdots = \alpha_{N_f}^{(1)} = -\frac{2\pi}{N_f} + \frac{2\pi}{N_f - k} \equiv -\frac{2\pi}{N_f} \bmod 2\pi.\end{gathered}$$

Therefore we can conclude $N_f - k = 1$.

This means that for $m \ll \Lambda$, we have a nonlinear σ-model with target $\mathrm{SU}(N_f)/S(\mathrm{U}(N_f - 1) \times \mathrm{U}(1)) \cong \mathbb{CP}^{N_f - 1}$, which is nice.

For $m \gg \Lambda$, we should get exactly pure 4D Yang-Mills at $\theta = \pi$, which has $\mathrm{SU}(N)_1$-Chern-Simons theory on its domain wall.

10. A symmetry-breaking scenario for QCD_3: 4/11/18

Today, Shehper spoke about [KS18]. We consider $\mathrm{SU}(N)$ gauge theory with N_f fermions in the fundamental representation and with Chern-Simons level k. We can think of this as starting with some bare level $k_{\mathrm{bare}} \in \mathbb{Z}$; integrating out a massive fermion with positive mass does not affect the level, and integrating out a fermion of negative mass decreases the level by 1. This is because

$$\int D\psi\, D\overline{\psi}\, e^{-\overline{\psi} i \not{D} \psi \pm |m| \overline{\psi}\psi} = \exp\biggl(-\frac{1}{2}\mathrm{CS}(A) \pm \frac{1}{2}\mathrm{CS}(A)\biggr), \tag{10.1}$$

so to ensure the action is properly quantized, we need to decrease the level for negative mass. Therefore $k = k_{\mathrm{base}} - N_f/2$. After integrating, $k \mapsto k \pm \mathrm{sign}(m) N_f/2$.

Now we have a bunch of (conjectured?) dualities for various values of m, where $N_f \le 2k$ and $k \neq 0$.

- If $m \gg 0$, we have $\mathrm{SU}(N)_{k+N_f/2}$-Chern-Simons theory and $\mathrm{U}(k+N_f/2)_{-N}$-Chern-Simons theory.
- If $m \ll 0$, we have $\mathrm{U}(k-N_f/2)_{-N}$-Chern-Simons and $\mathrm{SU}(N)_{k-N_f/2}$-Chern-Simons.
- The theories $\mathrm{SU}(N)_k$ with N_f fermions and $\mathrm{U}(k+N_f/2)_{-n}$ with N_f scalars are IR dual.

As the mass decreases (?), these three dualities should change at some critical points. The conjecture is that the critical point for the two massive theories is the same, and the two theories have the same phase diagram/

We'd like to extend this conjecture to the regime where $2k < N_f < N_*(N,k)$ for some N_*. The problem with this is that for large negative m^2, the $\mathrm{U}(k+N_f/2)$ theory with N_f scalars has a σ-model phase in IR when $N_f > 2k$: we have $v \sim \mu D^i_j D^j_i$, where

$$D^i_j = \sum_{s=1}^{N_f} \phi^{is}\overline{\phi}_{js} - m\delta^i_j, \tag{10.2}$$

where $i,j = 1,\dots,k+N_f/2$. To minimize this, we need to look for $k+N_f/2$ orthogonal vectors in $\mathbb{C}^{N_f}$, so the result is a σ-model with target

$$\mathcal{M}(N_f,k) := \mathrm{Gr}(k+N_f/2, N_f) \cong \frac{\mathrm{U}(N_f)}{\mathrm{U}(N_f/2+k)\times\mathrm{U}(N_f/2-k)}. \tag{10.3}$$

Therefore, Komargodski-Seiberg [KS18] conjecture that for $N_* > N_f > 2k$, there is a σ-model phase for $\mathrm{SU}(N)_k$ with N_f fermions with target space $\mathcal{M}(N_f,k)$. For small $|m|$, the theory is strongly coupled, and the phase diagram appears to be a TQFT for large $|m|$ and a σ-model near the origin.

The reason this happens is symmetry breaking: $\psi\psi^\dagger$ is a diagonal matrix with $N_f/2+k$ entries of one number, and $N_f/2-k$ of another. This breaks the symmetry from $\mathrm{U}(N_f)$ to $\mathrm{U}(N_f/2+k)\times\mathrm{U}(N_f/2-k)$.

If we add a Wess-Zumino term $\pm N\Gamma$ (here Γ is the Wess-Zumino term), the phase diagram looks a little more complicated, according to another conjecture. Here $2k < N_f < N_*$.

- For $m \gg 0$, we have the duality between the $\mathrm{SU}(N)_{k+N_f/2}$ and $\mathrm{U}(N_f/2+k)_{-N}$ theories.
- Below that, there's the critical point for the $\mathrm{U}(N_f/2+k)_{-N}$ theory with N_f scalars, which is the same as a critical point for the $\mathrm{SU}(N)_k$ theory with N_f fermions.
- Below that there's a σ-model with target $\mathcal{M}(N_f,k)$, and with an $N\Gamma$ term.
- Below that there's the other critical point of the $\mathrm{SU}(N)_k$ theory with N_f fermions, which should be the same as that of the $\mathrm{U}(N_f/2-k)_N$ theory with N_f scalars.
- Below that, we have the duality between the $\mathrm{U}(N_f/2-k)_N$ and $\mathrm{SU}(N)_{k-N_f/2}$ theories.

Let's look at the $\mathcal{M}(N_f,k)$ σ-model when $N_f/2-k=1$, so the target is $\mathbb{CP}^{N_f-1}$. In this case, if π is in the bi-fundamental representation for $\mathrm{U}(N_f/2+k)\times\mathrm{U}(N_f/2-k)$,

$$\mathcal{L}_{\mathrm{Kinetic}} \sim \mathrm{tr}\bigl(\partial\pi\partial\pi^\dagger(1+\pi\pi^\dagger)^{-1} - \partial\pi\pi^\dagger(1+\pi\pi^\dagger)^{-1}\pi\partial\pi^\dagger(1+\pi\pi^\dagger)^{-1}\bigr), \tag{10.4}$$

and the Kähler potential is

$$K = \mathrm{tr}\log(1+\pi^\dagger\pi). \tag{10.5}$$

For $k \neq 0$, the theory is time-reversal invariant, and for $k=0$, there's a $\mathbb{Z}/2$-symmetry exchanging the two $\mathrm{U}(N_f/2)$ subgroups.

For $N_f/2-k=1$, the Wess-Zumino term is

$$\mathcal{L}_{WZ} \sim N\int \mathrm{d}^3x\, \epsilon^{\mu\nu\rho}\bigl(\partial_\mu\pi^\dagger\cdot\pi\bigr)\bigl(\partial_\nu\pi^\dagger\cdot\partial_\rho\pi\bigr). \tag{10.6}$$

For the $|U(1)_N$ theory with N_f scalars, the action on a spin manifold is

$$\frac{N}{4\pi} b\,\mathrm{d}b + \frac{1}{2\pi} B\,\mathrm{d}b, \tag{10.7}$$

where b is a background gauge field and B is a fluctuating gauge field. In this case, $\mathrm{d}b = \omega + \cdots$, where ω is the Kähler 1-form on $\mathbb{CP}^{N_f-1}$. Hence if $M = \partial W$ for a compact 4-manifold W, we can reexpress this as

$$\frac{N}{4\pi}\int_w \omega\wedge\omega + \frac{1}{2\pi}\int_M \omega\wedge B. \tag{10.8}$$

Since $H^4(\mathbb{CP}^{N_f-1})$ has one generator but $H^4(\mathcal{M}(N_f,k))$ has two generators, we get an additional constraint on the theory: it's a gauged linear σ-model with a $\operatorname{tr} F\wedge \operatorname{tr} F$ term.

Since $\pi_2(\mathcal{M}(N_f,k)) \cong \mathbb{Z}$, there are topological solitons, which are called *skyrmions*. One claim is that these are the same as baryons, as the quantum numbers agree, and another is that these are the same as monopoles.

11. Abelianization of classical complex Chern-Simons theory: 4/18/18

Today Andy spoke about a long-running project with Dan Freed, albeit with a new spin enriched by our perspective this semester.

If you take one thing away from this talk, let it be this: a relationship between a topological field theory with a global $\mathrm{GL}_1(\mathbb{C})$-symmetry on a manifold $\widetilde{M}$, i.e. it couples to a background principal $\mathrm{GL}_1(\mathbb{C})$-bundle; and a topological field theory with a global $\mathrm{GL}_2(\mathbb{C})$-symmetry on M, where there is a double cover $\widetilde{M}\to M$.[15] These theories are very simple: they're invertible, though hopefully it's true in a more general setting than that.

You might expect such a relationship from string theory, where it's not uncommon to have a relationship between branes on a k-fold cover and a U_k-symmetry on the base space.

The invertible theories we're going to study are *classical Chern-Simons theories.* Given a compact spin 3-manifold M and a $\mathrm{GL}_k(\mathbb{C})$-connection ∇ on M, this TQFT associates a nonzero complex number $\mathrm{CS}(M,\nabla)$. If $\nabla = \mathrm{d} + A$, where $A\in\Omega^1_M(\mathfrak{gl}_k(\mathbb{C}))$, this is

$$\mathrm{CS}(M,\nabla) = \exp\biggl(\frac{1}{4\pi i}\int_M \operatorname{tr}\biggl(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\biggr)\biggr). \tag{11.1}$$

For a nontrivial principal bundle, the formula can be more complicated. There are similar formulas for $\mathrm{SL}_k(\mathbb{C})$, $\mathrm{PSL}_k(\mathbb{C})$, and sometimes on manifolds with boundary: in general, this formula isn't gauge-invariant, unless $\nabla|_{\partial M}$ is strictly upper triangular. Usually, we're used to integrating over all connections in Chern-Simons theory, but here we're not.

Remark 11.2.

(1) The critical points of $\mathrm{CS}(M,\nabla)$ are the flat connections.
(2) Suppose M is hyperbolic. Then it comes with a flat $\mathrm{PSL}_2(\mathbb{C})$-connection ∇^{hyp} arising from the description of M as a quotient of $\mathbb{H}^3$ (which comes with a connection) by a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Then the Chern-Simons invariant of $(M,\nabla^{\mathrm{hyp}})$ captures its hyperbolic volume (which by hyperbolic rigidity is a topological invariant):

$$\mathrm{CS}(M,\nabla^{\mathrm{hyp}}) = \exp\biggl(-\frac{\mathrm{vol}(M)}{2\pi} + i(\dots)\biggr). \tag{11.3}$$

So there's a canonical complexification of the volume of a hyperbolic manifold, and it's a stronger invariant than the hyperbolic volume.
(3) Suppose M can be triangulated. Then, many flat connections can be described in terms of the triangulation, with $(k^3-k)/6$ parameters $\chi_i\in\mathbb{C}^\times$, and in this case they also provide an explicit formula for the Chern-Simons invariant:

$$\mathrm{CS}(M,\nabla) = \exp\biggl(\frac{1}{2\pi i}\sum \mathrm{Li}_2(\chi_i)\biggr). \tag{11.4}$$

[15]There are analogues for k-fold covers $\widetilde{M}\to M$ and a theory with $\mathrm{GL}_k(\mathbb{C})$-symmetry on M.

Here Li_2 is the dilogarithm. This is work of many people, starting with Thurston, Dupont, Neumann, Dimofte-Gabella-Goncharov, ...

For example, if M is the complement of the figure-8 knot, it admits a triangulation by two tetrahedra, which leads to

$$\mathrm{CS}(M, \nabla^{\mathrm{hyp}}) = \exp\left(\frac{1}{2\pi i}\left(\mathrm{Li}_2(e^{\pi i/3}) + \mathrm{Li}_2(e^{\pi i/3})\right)\right). \tag{11.5}$$

◀

These facts are all somewhat mysterious, and part of the purpose of this talk is to shed some light on them using the duality mentioned at the beginning.

Specialized to flat connections, the Chern-Simons invariant fits into an invertible spin TQFT with $\mathrm{GL}_k(\mathbb{C})$ global symmetry.[16] This means it assigns higher-categorical information in lower dimensions: to a closed 2-manifold it assigns a 1-dimensional vector space; to a closed 1-manifold it assigns something like a Picard category, and to a point it should assign. . . something, but it's not quite clear what. And on a compact manifold M with boundary ∂M, the Chern-Simons invariant of M is an element of the line $\mathrm{CS}(\partial M)$.

Deformed Chern-Simons theory. Now we consider what is almost a manifold $\widetilde{M}$ with a flat $\mathrm{GL}_1(\mathbb{C})$-connection $\widetilde{\nabla}$, except at some singular points $H_n \in \widetilde{M}$: usually, the boundary of an ε-neighborhood of a point is a sphere, but at each H_i we'll ask for it to be a torus. On this torus, $\widetilde{\nabla}$ will have nontrivial holonomy: around one fundamental cycle, it has holonomy $\chi \in \mathbb{C}^\times$, and around the other, it has holonomy $1-\chi$.[17] You can think of this as collapsing a small framed loop in your manifold to a point.

So now we have a whole new category of manifolds with dimension at most 3 and these singularities; we think of the singularities as living in codimension 3, so they are not present on lower-dimensional manifolds.

There is a natural reason to think about these almost-manifolds: the connections $\widetilde{\nabla}$ arise as critical points of a deformed $\mathrm{GL}_1(\mathbb{C})$-Chern-Simons action: given a spin 3-manifold M with embedded loops L_i,

$$S = \frac{1}{4\pi i}\int_M A \wedge \mathrm{d}A + \frac{1}{2\pi i}\sum_i \mathrm{Li}_2(\underbrace{\mathrm{hol}_{L_i}\nabla}_{\chi_i}). \tag{11.6}$$

If in a neighborhood of a point x¡ $\nabla = \mathrm{d} + A$, the variation of this action is

$$\frac{\delta S}{\delta A(x)} = \frac{1}{2\pi i}F(x) + \frac{1}{2\pi i}\frac{\partial\,\mathrm{Li}_2(\chi_i)}{\partial \chi_i}\frac{\partial \chi_i}{\partial A(x)}, \tag{11.7}$$

where F denotes curvature. So this contains a δ-function supported on the loops L_i. Then you get a $\log(1-\chi_i)\delta_{L_i}$-term, which justifies the unexpected holonomy condition at the singularities.

Anyways, in this case, there is a well-defined Chern-Simons invariant $\widetilde{\mathrm{CS}}(\widetilde{M}, \widetilde{\nabla}) \in \mathbb{C}^\times$. This is nontrivial: the deformed action (11.6) is not gauge-invariant, and one has to choose a branch cut for the dilogarithm function. But it turns out these two cancel out.

Remark 11.8. This looks a lot like adding a Wilson line, but it isn't: you'd need to add a $\log(\chi)$ to (11.6). ◀

For $\mathrm{GL}_k(\mathbb{C})$, one can deform the Chern-Simons action again, but the abelianized theories end up equivalent. The universality of the $\mathrm{GL}_1(\mathbb{C})$ theory is an interesting facet of the deformed theories.

Remark 11.9. This deformation of Chern-Simons theory arises naturally in the discussion of the *topological string* (A-type). In this case, we consider our 3-manifold M as a Lagrangian submanifold of a symplectic 6-manifold N. You can formulate the A-model topological string on N (mathematically, this is akin to studying Gromov-Witten theory on N) with a D-brane on $\widetilde{M}$.

Witten argued that the string field theory (low-energy field theory of this topological string) is Chern-Simons theory on $\widetilde{M}$ plus deformations coming from holomorphic discs with boundary on $\widetilde{M}$. Then Ooguri-Vafa figured out that an isolated holomorphic disc contributes $\mathrm{Li}_2(\chi)$, suggesting (11.6), at least when the loops are unlinked. ◀

[16]Since a flat $\mathrm{GL}_k(\mathbb{C})$-connection is equivalent data to a principal $\mathrm{GL}_k(\mathbb{C})$-bundle with the *discrete* topology, this really is topological.

[17]So we need a little extra data to determine a basis of H_1 of the torus neighborhood, but this is OK.

Now, suppose M is ideally triangulated. Then we can produce a branched double cover $\widetilde{M} \to M$, with an H_i in each tetrahedron, and a map from singular flat $\mathrm{GL}_1(\mathbb{C})$-connections on $\widetilde{M}$ to honest flat $\mathrm{SL}_2(\mathbb{C})$-connections on M, such that $\mathrm{CS}(\widetilde{M}, \widetilde{\nabla}) = \mathrm{CS}(M, \nabla)$, and this extends to an equivalence of topological field theories!

Let's describe this in more detail. ∇ comes with a convenient gauge-fixing: on each face of the tetrahedron, we can put some "walls" from the vertices to the branch point for $\widetilde{M} \to M$. If $\nabla = \mathrm{d} + A$, then A is diagonal away from three cuts (from the branch point to the edges), and the walls, where it also has a nice description. With this description, you can explicitly construct $\widetilde{\nabla}$ by removing the unipotent matrices on the walls. Ultimately, unipotent matrices don't contribute to the Chern-Simons action, which leads to the equivalence of topological field theories.

This has something to say about the mysterious shape parameters and dilogarithms: they are just shape parameters of $\widetilde{\nabla}$ and pieces of the deformed Chern-Simons action, respectively. Thus, the appearance of dilogarithms comes from the fact that lots of connections can be described in this way.

12. 2-group symmetries, I: 4/25/18

Today, Sebastian spoke about the more mathematical aspects of 2-group symmetries in quantum field theory, following Benini-Córdova-Hsin [BCH18], and Kapustin-Thorngren [KT13] for additional details.

Recall that a group can be understood as a category with a single object, and such that all morphisms are invertible. This motivates the following generalization.

Definition 12.1. A *2-group* is a weak[18] 2-category with a single object, such that all 1-morphisms are weakly invertible and all 2-morphisms are invertible.

That is, a 2-group is a group object in the category of groupoids.

Concretely, we can represent 2-groups as *crossed modules*, data (G_0, G_1, t, α), where $t\colon G_1 \to G_0$ and $\alpha\colon G_0 \to \mathrm{Aut}(G_1)$ are group homomorphisms, subject to the conditions that

(1) $t(\alpha(g_0)(g_1)) = g_0 t(g_1) g_0^{-1}$, and
(2) $\alpha(t(g_1))(g_1') = g_1 g_1' g_1^{-1}$.

These arise because of interactions between horizontal and vertical composition in the 2-category. They imply in particular that the image of t is a normal subgroup of G_0.

Example 12.2.

(1) Let G_0 be a Lie group and $t\colon G_1 \to G_0$ be a covering map. Then we can define

$$\alpha(g_0) := g_1 \longmapsto \widetilde{g}_0 g_1 \widetilde{g}_0^{-1}, \tag{12.3}$$

where $\widetilde{g}_0 \in t^{-1}(g_0)$ (the choice is arbitrary).

(2) The simplest nontrivial example has $G_0 = G_1 = \mathbb{Z}/4$, $t(n) := 2n$, and $\alpha(n)(m) := (-1)^n m$.[19] ◂

Many 2-group symmetries arise in physics in the following way: there is a symmetry group acting on a quantum system, which has an unbroken subgroup G_0. $\mathrm{Im}(t) \lhd G_0$ is the confined part, and $\operatorname{coker} t$ is the low-energy gauge group. The kernel of t is called the magnetic gauge group (though its interpretation in terms of electromagnetism might depend on what dimension you're in.)

In physics, we will only know our 2-groups up to equivalence. This means we can use different models for them which determine 2-groups up to equivalence, but not necessarily isomorphism. For example, [BCH18] use the following, slightly different model for finite 2-groups: a 2-group is the data (G, A, ρ, β), where

- G is a finite group and A is a finite abelian group,
- $\rho\colon G \to \mathrm{Aut}(A)$ is a group homomorphism, and
- $\beta \in H^3(BG; A)$.

[18] We're not going to worry much about strictness issues, and they won't appear in obvious ways in the physics, but if you like categorical things, you might need to be careful about them.

[19] The classifying space of this 2-group has homotopy groups $\pi_1 = \mathbb{Z}/2$ and $\pi_2 = \mathbb{Z}/2$, and k-invariant $\mathrm{Sq}^2\colon K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}/2, 3)$. This is why there can't be anything interesting with G or H smaller: one of the homotopy groups would have to vanish.

Given a crossed module, we can obtain $G \coloneqq \operatorname{coker}(t)$ and $A \coloneqq \ker(t)$, and obtain ρ from α. Seeing β is a little harder, but the idea is that $H^3(BG; A)$ classifies *double extensions*, i.e. exact sequences

$$1 \longrightarrow A \longrightarrow H' \xrightarrow{t'} G' \longrightarrow G \longrightarrow 1, \tag{12.4}$$

such that (G', H', t', α') is a crossed module. Then, β corresponds to the sequence

$$1 \longrightarrow \ker(t) \longrightarrow G_1 \xrightarrow{t} G_0 \longrightarrow \operatorname{coker} t \longrightarrow 1. \tag{12.5}$$

Example 12.6. Let (X, x_0) be a pointed, connected space. Then there is a 2-group associated to X, called its *fundamental 2-group*, where G_0 is the group of loops based at x_0 under composition and G_1 is the group of equivalence classes of homotopies between loops. Passing to the above description, we obtain $G = \pi_1(X, x_0)$, $A = \pi_2(X, x_0)$, ρ as the monodromy action of π_1 on π_2, and $\beta \in H^3(B\pi_1, \pi_2)$ is the *Postnikov invariant* or *k-invariant*, which is a standard construction in algebraic topology. ◀

Now we'll discuss zero- and one-form symmetries in physics. Let M be a d-manifold with a *good cover* $\mathfrak{U} = \{V_i \mid i \in I\}$, indexed by an ordered set $i \in I$, such that all intersections $V_J \coloneqq \bigcap_{j \in J} V_j$ for all $J \subset I$ are either empty or contractible. Given this cover, we can define a simplicial complex with a vertex j inside V_j, and k-cells corresponding to k-fold intersections.

Zero-form symmetries for a group G can be understood in terms of *symmetry defects*, which are unitary operators U_g for $g \in G$ supported on codimension-1 manifolds X_{d-1}. These act on local operators: as a local operator $\mathcal{O}$ moves through X_{d-1}, it becomes $U_g\mathcal{O}$. These can couple to a flat G-bundle, which is described simplicially by $A_{ij} \in G$ over all edges $i \to j$, subject to the *cocycle condition* $A_{ij}A_{jk} = A_{ik}$.

One-form symmetries for a group A, necessarily abelian, admit a similar description. We have unitary operators W_a indexed by $a \in A$, which are supported on codimension-2 manifolds Y_{d-2}. These act on line operators $L(\ell)$:

$$\langle W_a L(\ell) \cdots \rangle = e^{2\pi i \Theta(a)} \langle L(\ell) \cdots \rangle, \tag{12.7}$$

where $\Theta \in \widehat{A} \coloneqq \operatorname{Hom}(A, \mathbb{R}/\mathbb{Z})$ is the Pontrjagin dual group.

One-form symmetries couple to A-gerbes, which can be described by combinatorial data $B_{ijk} \in A$ on triple intersections V_{ijk}, subject to the constraint

$$(dB)_{ijk\ell} \coloneqq B_{jk\ell} - B_{ik\ell} + B_{ij\ell} - B_{ijk} = 0. \tag{12.8}$$

Now let's consider a 2-group symmetry, with zero-form part G, one-form part A, and some kind of interaction, which arises by the map $\rho\colon G \to \operatorname{Aut}(A)$. First let's suppose $\beta = 0$. In this case, when a one-form operator W_a on a codimension-2 submanifold passes through X_{d-1} acting by U_g, it turns into $W_{\rho_g(a)}$. This can couple to pairs of a flat G-bundle an an A-gerbe, where now

$$(d_A B)_{ijk\ell} \coloneqq \rho_g(A_{ij})B_{jk\ell} - B_{ik\ell} + B_{ij\ell} - B_{ijk} = 0. \tag{12.9}$$

If $\beta \neq 0$, we can see it physically from a picture where four codimension-1 sheets meet, and are labeled by g, h, k, and ghk. In this setting, if a one-form operator W_a moves around here, there may be an issue caused by a lack of associativity: what we get by gluing g and h, then k might be different than what we get by gluing h and k, then gluing g with that. This is mediated by $\beta(g, h, k)$, which is a cocycle in cohomology with twisted coefficients, which means that

$$d_\rho\beta(g, h, k, \ell) \coloneqq \rho_g\beta(h, k, \ell) - \beta(gh, k, \ell) + \beta(g, hk, \ell) - \beta(g, h, k\ell) + \beta(g, h, k) = 0. \tag{12.10}$$

This looks like an anomaly, but isn't really: it's not coming from anomaly inflow from a bulk, but is rather some kind of connection on a principal 2-bundle.

The equation (12.10) admits a geometric description in terms of five sheets meeting at two intersection points. Then we get a description of what kinds of pairs of fields we can couple to, which is similar to (12.9) but with $d_A B \coloneqq A^*\beta$, where $A\colon M_d \to BG$ is the map classifying the principal bundle.

13. 2-group symmetries, II: 5/2/18

Today, Shehper spoke about the physics side of the last talk, on 2-group symmetries in physics. Recall that a finite 2-group is specified by data $(G, H, \rho, [\beta])$, where G and H are finite groups, H is abelian, $\rho\colon G \to \operatorname{Aut}(H)$ is a group homomorphism, and $[\beta] \in H^3(BG; H)$ is called the Postnikov class.

From a physics perspective, G is the zero-form symmetry group of a 2-group symmetry and H is the one-form symmetry group. The zero-form symmetries can act on particles charged under the one-form symmetry, and ρ specifies this action. The Postnikov class is the most difficult to interpret from this perspective; it arises as a codimension-2 defect that appears in an *F-move* (an associator for $(gh)k$ to $g(hk)$ of some sort).

One example that will make the appearance of the Postnikov class clearer is quantum field theory with a symmetry group Γ, where Γ is an extension of G by an abelian group A:

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{13.1}$$

This induces a group action $G \to \mathrm{Aut}(A)$, and also defines a cohomology class $[\omega] \in H^2(BG; A)$. These suffice to recover (13.1) up to equivalence. The extension defines a product rule on Γ. As a set, it's $A \times G$, but ω twists the product such that

$$(0, g) \cdot (0, h) = (\omega(g, h), gh). \tag{13.2}$$

Here we have to choose a cocycle representative for $[\omega]$, but this choice ends up not mattering: we end up with isomorphic Γ. Physically, (13.2) can be thought of as expressing that in a collision between particles charged as g and h under the G-symmetry produces particles charged under gh and $\omega(g, h)$, and $\omega(g, h)$ is a codimension-1 defect (foreshadowing the codimension-2 defect we're going to see for a 2-group symmetry).

Next let's step this up to 2-groups. Let H be a finite abelian group and BH denote its classifying space. Then we can regard $G = (G, 0, 1, 0)$ and $BH = (1, H, 1, 0)$ as 2-groups and consider an extension of 2-groups

$$1 \longrightarrow BH \longrightarrow \underline{\Gamma} \longrightarrow G \longrightarrow 1, \tag{13.3}$$

which is determined by the Postnikov class $[\beta] \in H^3(BG; H)$. This is *not* a sequence of groups (even though BH and G are groups)!

Remark 13.4. Another way to interpret this is through topology: there is a notion of a classifying space $B\underline{\Gamma}$ associated to a 2-group $\underline{\Gamma}$, and in this setting (13.3) means a fibration of pointed topological spaces

$$B^2H \longrightarrow B\underline{\Gamma} \longrightarrow BG. \tag{13.5}$$

◀

We can think of $\underline{\Gamma}$ as having a twisted product, akin to (13.2), in which for $g, h, k \in G$,

$$(0, (gh)k) = (\beta(g, h, k), g(hk)), \tag{13.6}$$

where β is a cocycle representative for $[\beta]$.

Example 13.7. Let's consider U(1)-Chern-Simons theory at level k together with N_f scalars of charge $q > 1$, formulated on a 3-manifold M. The charge is quantized, so $q \in \mathbb{Z}$; assume $k = q\ell$ for some $\ell \in \mathbb{Z}$. This has multiple zero-form symmetries:

- $G := \mathrm{U}(N_f)/(\mathbb{Z}/\ell)$, the *faithful symmetry*.
- $G^n := \mathrm{SU}(N_f) \times \mathrm{U}(1)_M$, the *naïve symmetry*, where $\mathrm{U}(1)_M$ has current

$$j^\mu := \frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}. \tag{13.8}$$

Here's why the faithful symmetry arises. For a monopole $\mathcal{M}$ with monopole number 1, Chern-Simons coupling implies a charge of k (with respect to the gauge group U(1)). Let

$$\mathcal{N} := \mathcal{M}\phi_1 \cdots \phi_p \phi_1^{\mathrm{T}} \cdots \phi_{p+\ell}^{\mathrm{T}}. \tag{13.9}$$

Under the $Z(\mathrm{SU}(N_f))$ symmetry, $\mathcal{N}$ transforms as

$$\mathcal{N} \longmapsto e^{-2\pi i\ell/N_f}\mathcal{N}. \tag{13.10}$$

Since $\mathcal{M}$ has charge 1 under $\mathrm{U}(1)_M$, then $\gamma := (e^{2\pi i/N_f}, e^{2\pi i\ell/N_f}) \in \mathrm{SU}(N_f) \times \mathrm{U}(1)_M$ leaves $\mathcal{N}$ invariant.

Let $\mathrm{U}(1)_M^\ell$ denote the ℓ-fold cover of $\mathrm{U}(1)_M$. Then we can write γ as $(e^{2\pi i/N_f}, e^{2\pi i/N_f}) \in \mathrm{SU}(N_f) \times \mathrm{U}(1)_M^\ell$, so we have a "less naïve" $\mathrm{U}(N_f)$-symmetry, and therefore a faithful symmetry of $\mathrm{U}(N_f)/(\mathbb{Z}/\ell)$.

There is a one-form $B\mathbb{Z}/q$ symmetry: given $e^{2\pi i/q} \in \mathbb{Z}/q$ and a Wilson loop L, we have

$$\left\langle e^{i\oint_L A} e^{iQ\oint_L A}\right\rangle = e^{2\pi iQ/q}\left\langle e^{iQ\oint_L A}\right\rangle. \tag{13.11}$$

Geometrically, you can think of this as having a principal U(1)-bundle with connection A and a principal U(1)-bundle with flat connection whose holonomies are q^{th} roots of unity (so, really, a principal $\mathbb{Z}/q$-bundle induced along $\mathbb{Z}/q \hookrightarrow \mathrm{U}_1$).

When we couple to a background G-gauge field, we're going to obtain a 2-group global symmetry. Under $\mathbb{Z}/\ell \subset \mathrm{U}(N_f)$, $\phi_i \mapsto e^{2\pi i/\ell}\phi_i$. (Here ϕ_i is one of the scalar fields.) Under $\mathbb{Z}/q\ell \subset \mathrm{U}(1)_{\mathrm{dyn}}$, $\phi_i \mapsto e^{2\pi i q/(q\ell)}\phi_i$. This suggests we can identify these two actions, meaning we should couple to bundles with structure group

$$\frac{\mathrm{U}(N_f) \times \mathrm{U}(1)_{\mathrm{dyn}}}{\mathbb{Z}/\ell}. \tag{13.12}$$

We have an action of $e^{2\pi i/q} \in \mathbb{Z}/q\ell$ on Wilson lines.

There are two possible cases depending on the bundle $P \to M$ in question.

(1) If the $\mathrm{U}(N_f)/(\mathbb{Z}/\ell)$-bundle can be lifted to a $\mathrm{U}(N_f)$-bundle, then its characteristic class $[w_2^{(\ell)}] = 0$ in $H^2(M;\mathbb{Z}/\ell)$ and $\int F \in 2\pi\mathbb{Z}/q\ell$.

(2) If not, then $[w_2^{(\ell)}] \in H^2(M;\mathbb{Z}/\ell)$ is nonzero and $\int F \in 2\pi n/q\ell$.

Case (2) is the general case, in that $[w_2^{(\ell)}] \in H^2(BG;\mathbb{Z}/\ell)$ is nonzero. In this case, $\mathbb{Z}/q$-gerbe fields transform nontrivially under G-gauge transformations, which is ultimately because the Postnikov class $[w_2^{(\ell)}]$ doesn't vanish. More specifically, associated to the short exact sequence

$$1 \longrightarrow \mathbb{Z}/q \longrightarrow \mathbb{Z}/q\ell \longrightarrow \mathbb{Z}/\ell \longrightarrow 1 \tag{13.13}$$

there's a Bockstein $B\colon H^2(BG;\mathbb{Z}/\ell) \to H^3(BG;\mathbb{Z}/q)$ and $[\beta] = B[\omega_2^{(\ell)}]$. In particular, we have a 2-group symmetry for $\underline{\Gamma} = (G, \mathbb{Z}/q, \rho, [\beta])$, where ρ arises from the action of $\mathbb{Z}/q\ell$ on the scalars. ◄

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