HOW TO CALCULATE SPIN-U_2 BORDISM

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1. Introduction

Let $\mathrm{Spin}\text{-}\mathrm{U}_2 \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{U}_2$. We would like to compute the bordism groups of manifolds with spin-U_2 structures. Davighi-Lohitsiri [DL20, Appendix A] do this by expressing spin-U_2 structures as twisted spin structures; this implies spin-U_2 bordism groups are isomorphic to spin bordism groups of a certain Thom spectrum, and they use the Adams spectral sequence over $\mathcal{A}(1)$ to compute these spin bordism groups. Spin-U_2 structures also appear in Seiberg-Witten theory in mathematics, e.g. in [FL02, DW19], where they are called spin^u structures.

In this and a few other cases, one can take advantage of a trick to simplify the computation: spin-U_2 structures are in fact twisted spin^c structures, and the Adams spectral sequence for spin^c bordism can be simplified to work over $\mathcal{E}(1)$, a subalgebra of $\mathcal{A}(1)$.

The two purposes of this document are to show how to compute spin^c bordism groups with this case of the Adams spectral sequence and to compute spin-U_2 bordism groups. In §2, we show how to express spin-U_2 structures as twisted spin^c structures. In §3, we show spin-U_2 bordism has no p-torsion for odd primes p. Then, in §4, we explain how the computation of spin^c bordism modulo odd-primary torsion via the Adams spectral sequence simplifies to a computation over $\mathcal{E}(1)$; the argument is completely parallel to the use of the Adams spectral sequence over $\mathcal{A}(1)$ to compute spin bordism. Finally in §5, we use this method to compute spin-U_2 bordism groups through dimension 11.

2. Spin-U_2 bordism is twisted spin^c bordism

In this section, we show that spin-U_2 bordism is the spin^c bordism of something, so that we can use preexisting information about spin^c bordism to determine spin-U_2 bordism groups. This is an example of a "shearing argument," similar to others in the literature, though we tried to say it in a relatively geometric way. We give a fairly specific argument in terms of twisted spin^c structures; the ideas of this proof generalize to twisted orientations, twisted spin structures, etc.

Definition 2.1. Let $V \to X$ be a virtual vector bundle. An (X, V)*-twisted spin^c structure* on a vector bundle $E \to M$ is data of a map $f\colon M \to X$ and a spin^c structure on $E \oplus f^*V$. Two such structures are equivalent if the maps to X are homotopic and the spin^c structures are equivalent.

Let $V_{\mathrm{taut}} \to B\mathrm{Spin}^c$ denote the tautological stable vector bundle.

Lemma 2.2 (Shearing)**.** *The classifying space for (X, V)-twisted spin^c structures is $\phi\colon B\mathrm{Spin}^c \times X \to B\mathrm{O}$ with the map given by the vector bundle $-V_{\mathrm{taut}} \oplus V \to B\mathrm{Spin}^c \times X$.*

By "classifying space" we mean a space B with a map $B \to B\mathrm{O}$ such that equivalence classes of (X, V)-twisted spin^c structures on a vector bundle $E \to M$ are in natural bijection with homotopy classes of lifts of the classifying map $c_E\colon M \to B\mathrm{O}$ for E to B:

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ M & \xrightarrow[c_E]{} & B\mathrm{O}. \end{array} \tag{2.3}$$

The Yoneda lemma guarantees that the classifying space together with its map to $B\mathrm{O}$ is unique up to homotopy of these data.

Proof of Lemma 2.2. Given an (X, V)-twisted spin^c structure on $E \to M$, namely a map $f\colon M \to X$ and a spin^c structure on $E \oplus f^*V$, we obtain a map $M \to B\mathrm{Spin}^c \times X$ given by $(E \oplus f^*V, f)$, and using that homotopy classes of maps to $B\mathrm{Spin}^c$ are naturally identified with stable virtual spin^c vector bundles. Conversely, given a map $(\psi, f)\colon M \to B\mathrm{Spin}^c \times X$, take $\phi \circ \psi$ to obtain a spin^c vector bundle E', and let $E \coloneqq E' - f^*V$; then we have a canonical (X, V)-twisted spin^c structure on E. These two operations are inverses up to isomorphism, so every (X, V)-twisted spin^c structure on any vector bundle on any space pulls back from $B\mathrm{Spin}^c \times X$ in a unique way up to homotopy, which is what we wanted to prove. ⊠

The Pontrjagin-Thom theorem then implies

Corollary 2.4. *The bordism groups of manifolds with (X, V)-twisted spin^c structures are $\Omega_*^{\mathrm{Spin}^c}(X^V)$, where X^V is the Thom spectrum of $V \to X$.*

The main theorem of this section identifies spin-U_2 structures as twisted spin^c structures, where $X = B\mathrm{SO}_3$ and V is the tautological bundle.

Theorem 2.5. *Spin-U_2 structures are naturally equivalent to $(B\mathrm{SO}_3, V_{\mathrm{taut}})$-twisted spin^c structures.*

Proof. The first step is to identify

$$\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{U}_2 \cong (\mathrm{Spin}_n \times \mathrm{U}_1 \times \mathrm{SU}_2)/\{\pm 1\} \cong \mathrm{Spin}_n^c \times_{\{\pm 1\}} \mathrm{SU}_2. \tag{2.6}$$

These maps commute with the maps to SO_n. Therefore spin-U_2 structures are equivalent to spin^c-SU_2 structures.

To finish the proof, we will show that for $n \geq 1$, $\mathrm{Spin}_n^c \times_{\{\pm 1\}} \mathrm{SU}_2$ is the pullback

$$\begin{array}{ccc} \mathrm{Spin}_n^c \times_{\{\pm 1\}} \mathrm{SU}_2 & \xrightarrow{\phi} & \mathrm{Spin}_{n+3}^c \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ \mathrm{SO}_n \times \mathrm{U}_1 \times \mathrm{SO}_3 & \xrightarrow{j} & \mathrm{SO}_{n+3} \times \mathrm{U}_1. \end{array} \tag{2.7}$$

Here p_1 and p_2 take the quotient by $\{\pm 1\}$. The map j is the direct sum of the standard inclusion $\mathrm{SO}_n \times \mathrm{SO}_3 \hookrightarrow \mathrm{SO}_{n+3}$ as block matrices together with the identity on U_1. We will define ϕ in a moment. Once we show (2.7) is a pullback square, this will imply that a lift of a map f to $\mathrm{SO}_n \times \mathrm{U}_1 \times \mathrm{SO}_3$ across p_1 is equivalent to a lift of $j \circ f$ across p_2. The former is the data on transition functions of an oriented vector bundle $E \to M$, a complex line bundle $L \to M$, and a principal SO_3-bundle $P \to M$ to define a spin^c-SU_2 structure, and the latter is the data on transition functions to define a spin^c structure on the direct sum of $E \oplus V_P$, where V is the associated rank-3 vector bundle to P and L is the determinant line bundle of the spin^c structure.

To show (2.7) is a pullback, we first need to define ϕ. We do so by seeing that $\{\pm 1\} \subset \mathrm{Spin}_n^c \times_{\{\pm 1\}} \mathrm{SU}_2$ is sent to 0 under $j \circ p_1$, so a lift ϕ across p_2 exists; choose one. The pullback of p_2 is a double cover of $\mathrm{SO}_n \times \mathrm{U}_1 \times \mathrm{SO}_3$. If we pull back further to $\mathrm{SO}_n \times \mathrm{U}_1 \subset \mathrm{SO}_n \times \mathrm{U}_1 \times \mathrm{SO}_3$, we must obtain $\mathrm{Spin}_n^c \to \mathrm{SO}_n \times \mathrm{U}_1$, because the inclusion $\mathrm{SO}_n \times \mathrm{U}_1 \hookrightarrow \mathrm{SO}_{n+3} \times \mathrm{U}_1$ does not lift across p_2. And if we pull back to $\mathrm{SO}_3 \subset \mathrm{SO}_n \times \mathrm{U}_1 \times \mathrm{SO}_3$, we must obtain $\mathrm{SU}_2 \to \mathrm{SO}_3$, since j does not lift across $\mathrm{Spin}_3 \to \mathrm{SO}_3$ (equivalently, the tautological vector bundle on $B\mathrm{SO}_3$ is not spin). The only way to obtain both Spin_n^c and SU_2 in a double cover is for the pullback to be $\mathrm{Spin}_n^c \times_{\{\pm 1\}} \mathrm{SU}_2$. ⊠

Combining Theorem 2.5 with Corollary 2.4, we see that spin-U_2 bordism groups are isomorphic to the spin^c bordism groups of the Thom spectrum of $V_{\mathrm{taut}} - 3 \to B\mathrm{SO}_3$.[1] This spectrum is denoted $\Sigma^{-3} MSO_3$. So we've reduced to determining $\Omega_*^{\mathrm{Spin}^c}(\Sigma^{-3} MSO_3)$.

Remark 2.8. This result also follows directly from [DL20, (A.9)], which identifies the Thom spectrum for spin-U_2 bordism with $MTSpin \wedge \Sigma^{-2} MU_1 \wedge \Sigma^{-3} MSO_3$: combine their result with the equivalence [BG87a, BG87b]

$$MTSpin^c \xrightarrow{\simeq} MTSpin \wedge \Sigma^{-2} MU_1. \tag{2.9}$$ ◄

Wang [Wan08] considers a different notion of twisted spin^c bordism which is more general, but less geometric.

3. Nothing interesting at odd primes

The point of this section is:

Proposition 3.1. *$\Omega_*^{\mathrm{Spin}^c}(\Sigma^{-3} MSO_3)$ has no p-torsion for any odd prime p.*

Proof. Consider the Atiyah-Hirzebruch spectral sequence with signature

$$E^2_{r,q} = H_r(\Sigma^{-3} MSO_3; \Omega_q^{\mathrm{Spin}^c}) \Longrightarrow \Omega_{r+q}^{\mathrm{Spin}^c}(\Sigma^{-3} MSO_3). \tag{3.2}$$

The presence of p-torsion in $\Omega_*^{\mathrm{Spin}^c}(\Sigma^{-3} MSO_3)$ would imply p-torsion in the E^∞-page of this spectral sequence, but we will show that there is no p-torsion on the E^∞-page using the following three facts.

1. $\Omega_q^{\mathrm{Spin}^c}$ is finitely generated and has no p-torsion.
2. $H_r(\Sigma^{-3} MSO_3; A)$ lacks p-torsion whenever A is a finitely generated abelian group without p-torsion.
3. The free summands in $\Omega_*^{\mathrm{Spin}^c}$ and $H_*(\Sigma^{-3} MSO_3)$ occur in even degrees.

[1]The use of $V_{\mathrm{taut}} - 3$, rather than just V_{taut}, is so that the Thom isomorphism (and therefore this equivalence too) does not shift degree.

Points (1) and (2) imply there is no p-torsion on the E^2-page of (3.2), so the only way to create p-torsion on the E^∞-page would be a differential between free summands. However, (3) rules out such differentials: free summands only occur in even total degree, and differentials lower total degree by 1.

Now to fulfill our promises. Point (1) is proven by Stong [Sto68, p. 336]. For (2): since the tautological bundle on $B\mathrm{SO}_3$ is oriented, there is a Thom isomorphism $H_*(B\mathrm{SO}_3; A) \cong H_*(\Sigma^{-3}MSO_3; A)$. Borel-Hirzebruch [BH59, §30.5] showed $H^*(B\mathrm{SO}_3;\mathbb{Z})$ lacks p-torsion; the universal coefficient theorem then implies that $H_*(B\mathrm{SO}_3; A)$ has no p-torsion. Finally, (3) is analogous: the part for $\Omega_*^{\mathrm{Spin}^c}$ is also in Stong [Sto68, p. 347], and for $H_*(\Sigma^{-3}MSO_3)$, use the Thom isomorphism and the universal coefficient theorem to reduce to showing that the free summands in $H^*(B\mathrm{SO}_3;\mathbb{Z})$ are in even degrees, which is also due to Borel-Hirzebruch [BH59, Proposition 30.3].[2] ⊠

Therefore it suffices to study the 2-primary part of $\Omega_*^{\mathrm{Spin}^c}(\Sigma^{-3}MSO_3)$. This falls to Adams spectral sequence methods, as we establish in the next section.

4. The Adams spectral sequence for 2-primary spin^c bordism

By a "2-primary isomorphism" we mean an isomorphism after tensoring with the ring $\mathbb{Z}_{(2)}$ of rational numbers with odd denominators. For finitely generated abelian groups, 2-primary isomorphism means the same number of free summands and the same number of $\mathbb{Z}/2^k$ summands for each k, but this definition knows nothing about $\mathbb{Z}/p^k$ summands when p is an odd prime. In the last section, we saw that spin-U_2 bordism has no odd-primary torsion, so 2-primary isomorphisms are good enough.

The Adams spectral sequence for 2-primary spin^c bordism is much simpler than the general 2-primary Adams spectral sequence, and the purpose of this section is to explain how it simplifies. The story is parallel to the simplification for spin bordism, so we first summarize how that story went.

(1) Anderson-Brown-Peterson [ABP67] provide a map

$$\Omega_*^{\mathrm{Spin}}(X) \longrightarrow ko_*(X) \oplus ko_{*-8}(X) \oplus ko_{*-10}(J \wedge X) \tag{4.1}$$

which for X a space or connective spectrum is a 2-primary isomorphism in degrees 11 and below, where ko is connective real K-theory and J is a not-too-complicated spectrum. (Anderson-Brown-Peterson also prove a more general splitting result in all degrees with similar terms.) Therefore to determine spin bordism, it suffices to compute ko-homology of X and $J \wedge X$.

(2) Stong [Sto63] showed that $H^*(ko;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ as a module over the Steenrod algebra $\mathcal{A}$.

(3) There is a general change-of-rings theorem, where if $\mathcal{B}$ is a graded Hopf algebra, $\mathcal{C} \subset \mathcal{B}$ is a graded Hopf subalgebra, and M and N are graded $\mathcal{B}$-modules, then there is a natural isomorphism

$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(\mathcal{B} \otimes_{\mathcal{C}} M, N) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{C}}^{s,t}(M, N). \tag{4.2}$$

[2] Brown [Bro82] explicitly computes $H^*(B\mathrm{SO}_3;\mathbb{Z})$, which provides another way to verify these claims about the cohomology of $B\mathrm{SO}_3$.

(4) Therefore, when computing (the 2-primary part of) $ko_*(X)$ with the Adams spectral sequence, the E_2-page simplifies to Ext over $\mathcal{A}(1)$, which is much easier to calculate.

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(ko\wedge X),\mathbb{Z}/2)\\ &\cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}\otimes_{\mathcal{A}(1)}\mathbb{Z}/2\otimes_{\mathbb{Z}/2}H^*(X;\mathbb{Z}/2),\mathbb{Z}/2)\\ &\overset{(4.2)}{\cong} \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2). \end{aligned}$$

Each step of this argument has an analogue for spin^c bordism.

(1) The analogue of the map to ko is a map [ABP67]

$$\Omega_n^{\mathrm{Spin}^c}(X)\longrightarrow ku_n(X)\oplus ku_{n-4}(X)\oplus ku_{n-8}(X)\oplus ku_{n-8}(X)\oplus H_{n-10}(X;\mathbb{Z}/2) \tag{4.3}$$

which for X a space or connective spectrum is a 2-primary isomorphism in degrees 11 and below.[3] Here ku is connective complex K-theory. So to compute spin^c bordism, we need ku-homology and ordinary mod 2 homology.

(2) Adams [Ada61] showed that $H^*(ku;\mathbb{Z}/2)\cong\mathcal{A}\otimes_{\mathcal{E}(1)}\mathbb{Z}/2$, where $\mathcal{E}(1)$ is the algebra generated by $Q_0=\mathrm{Sq}^1$ and $Q_1=\mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^1\mathrm{Sq}^2$.

(3) Using the change-of-rings theorem as above, the E_2-page of the Adams spectral sequence computing (the 2-primary part of) $ku_*(X)$ is

$$E_2^{s,t}=\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2). \tag{4.4}$$

It turns out that computing Ext over $\mathcal{E}(1)$ is easier than over $\mathcal{A}(1)$. The rest of this section shows a few examples. When we write $\mathrm{Ext}(M)$, we mean $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M,\mathbb{Z}/2)$. The Ext calculations use some tools that are discussed in more depth in, e.g., [BC18].

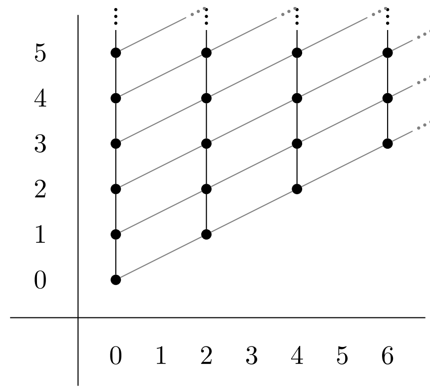
Example 4.5 ($\mathbb{Z}/2$)**.** The first example we want is $\mathrm{Ext}(\mathbb{Z}/2)$. This has an algebra structure, and acts naturally on $\mathrm{Ext}(M)$ for any $\mathcal{E}(1)$-module M. It is computed by Koszul duality: first, one observes that $\mathcal{E}(1)$ is an exterior algebra: $\mathcal{E}(1)=\mathbb{Z}/2[Q_0,Q_1]/(Q_0^2,Q_1^2)$, with $|Q_0|=1$ and $|Q_1|=3$. Therefore [BC18, Example 4.5.6]

$$\mathrm{Ext}_{\mathcal{E}(1)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2[h_0,v_1], \tag{4.6}$$

where $h_0\in\mathrm{Ext}^{1,1}$ and $v_1\in\mathrm{Ext}^{1,3}$. In Adams diagrams, we will draw h_0 as a vertical line and v_1 as a lighter diagonal line; for example, we draw $\mathrm{Ext}(\mathbb{Z}/2)$ in (4.7). As always, the x-axis is $t-s$ and the y-axis is s.

[3] In higher dimensions, there is an analogous decomposition with more summands. See Bahri-Gilkey [BG87a, p. 5] for more.

(4.7)



The $\mathrm{Ext}(\mathbb{Z}/2)$-action on $\mathrm{Ext}(M)$ provides information about extensions on the E_∞-page of the Adams spectral sequence; h_0 lifts to multiplication by 2 (just as it did for $\mathcal{A}(1)$) and v_1 lifts to multiplication by the Bott element $\beta \in ku_2$. ◂

Example 4.8 ($\mathcal{E}(1)$)**.** Since $\mathcal{E}(1)$ is a free $\mathcal{E}(1)$-module of rank 1, $\mathrm{Ext}(\mathcal{E}(1))$ has a single $\mathbb{Z}/2$ in degree $(0,0)$ and vanishes in all other degrees. ◂

Example 4.9 (Upside-down question mark)**.** Let $\rotatebox[origin=c]{180}{?}$ denote a $\mathcal{E}(1)$-module which is as a vector space generated by three elements x, y, and z, with $|x| = 0$, $|y| = 2$, and $|z| = 3$, such that $Q_0 x = z$ and $Q_1 y = z$. (All unspecified Q_0- and Q_1-actions vanish for degree reasons.) This module is sometimes called the *upside-down question mark*. See Figure 1, left, for a picture. We will compute its Ext in two different ways.

The first stage of a minimal resolution of $\mathbb{Z}/2$ (see [BC18, §4.4]) is the unique nonzero map $\mathcal{E}(1) \to \mathbb{Z}/2$; the kernel of this map is $\Sigma\rotatebox[origin=c]{180}{?}$. That is, in a minimal resolution of $\mathbb{Z}/2$

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}/2, \tag{4.10}$$

there must be an isomorphism $P_0 \cong \Sigma\rotatebox[origin=c]{180}{?}$; chopping off $\mathbb{Z}/2$ from (4.10) gives us a minimal resolution for $\Sigma\rotatebox[origin=c]{180}{?}$. Therefore for $s, t \geq 0$ there is an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\rotatebox[origin=c]{180}{?}) \xrightarrow{\cong} \mathrm{Ext}^{s+1,t+1}_{\mathcal{E}(1)}(\mathbb{Z}/2) \tag{4.11}$$

equivariant for the $\mathrm{Ext}(\mathbb{Z}/2)$-actions on both sides. See Figure 1, right, for a picture of $\mathrm{Ext}(\rotatebox[origin=c]{180}{?})$.

Alternatively, we can calculate $\mathrm{Ext}(\rotatebox[origin=c]{180}{?})$ using the fact that a short exact sequence of $\mathcal{E}(1)$-modules induces a long exact sequence in Ext groups (see [BC18, §4.6]). Let $N_2 := \mathcal{E}(1) \otimes_{\langle Q_0\rangle} \mathbb{Z}/2$; then N_2 is generated as a vector space by elements x_0 and x_3 in degrees 0 and 3 respectively, with $Q_1 x_0 = x_3$, and all other Q_i-actions equal to 0. Because $\langle Q_0\rangle$ is a sub-Hopf algebra of $\mathcal{E}(1)$, the change-of-rings theorem implies

$$\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(N_2, \mathbb{Z}/2) \cong \mathrm{Ext}^{s,t}_{\langle Q_0\rangle}(\mathbb{Z}/2, \mathbb{Z}/2), \tag{4.12a}$$

and Koszul duality calculates [BC18, Example 4.5.5]

$$\cong \mathbb{Z}/2[h_0]. \tag{4.12b}$$

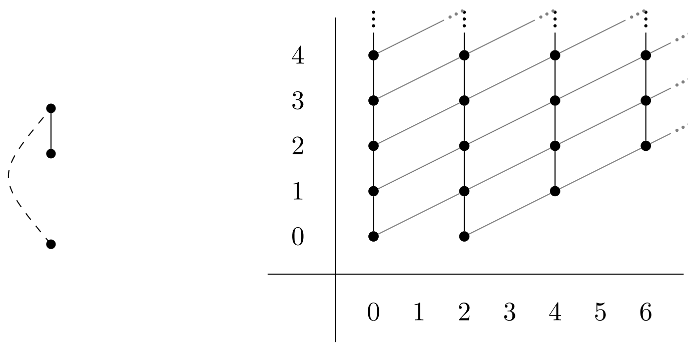


Figure 1. Left: ¿, the "upside-down question mark" $\mathcal{E}(1)$-module. Right: Ext(¿).

We care about this because ¿ is an extension of $\Sigma^2\mathbb{Z}/2$ by N_2, as in Figure 2, left. Therefore we can calculate $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}($¿$,\mathbb{Z}/2)$ using the induced long exact sequence in Ext, as in Figure 2, right.[4] ◀

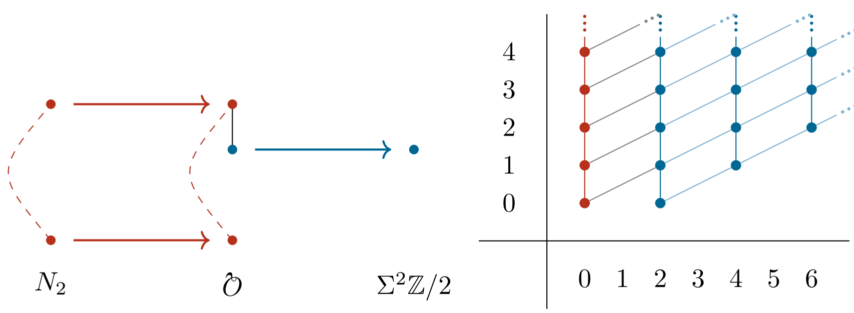


Figure 2. Computing $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}($¿$,\mathbb{Z}/2)$ (right) by fitting ¿ into an extension $0 \to N_2 \to$ ¿ $\to \mathbb{Z}/2 \to 0$. The gray v_1-actions cannot be detected by this method and must be computed a different way. See Example 4.9 for more information.

One important operation on $\mathcal{E}(1)$-modules is degree shift: $\Sigma^k M$ is the $\mathcal{E}(1)$-module M with the grading of all elements of M increased by k. This has the effect of shifting the Ext groups k units in the t direction.

5. Computation of $ku_*(\Sigma^{-3}MSO_3)$ in low degrees

Our first step is to determine the $\mathcal{E}(1)$-module structure on $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$. Beaudry-Campbell [BC18, Example 3.4.8] determine the $\mathcal{A}(1)$-module structure in low degrees. We

[4]To show that the indicated v_1-actions are indeed nontrivial, one can use h_0-linearity to reduce to checking this for the summand in bidegree $(0,0)$, where it amounts to explicitly computing what extension one obtains by acting on the nontrivial map ¿ $\to \mathbb{Z}/2$, which represents the nontrivial element in $\mathrm{Ext}^{0,0}_{\mathcal{E}(1)}($¿$,\mathbb{Z}/2) = \mathrm{Hom}_{\mathcal{E}(1)}($¿$,\mathbb{Z}/2)$, by an extension representing v_1, and observing that the resulting extension does not split. See [BC18, §4.2].

will follow their line of reasoning to determine the $\mathcal{E}(1)$-module structure in degrees 12 and below, though our arguments generalize to higher dimensions.

Here are the steps of the computation of the $\mathcal{E}(1)$-module structure on $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$.

(1) Determine the ring structure on $H^*(B\mathrm{SO}_3;\mathbb{Z}/2)$.
(2) Determine the action of the Steenrod squares on $H^*(B\mathrm{SO}_3;\mathbb{Z}/2)$.
(3) Use the Thom isomorphism, and the way it interplays with the Steenrod algebra action, to learn the $\mathcal{E}(1)$-action on $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$.

We proceed in the same way as Beaudry-Campbell, but recall the steps here in case it is helpful. First, the cohomology ring of $B\mathrm{SO}_3$.

Theorem 5.1 (Borel [Bor53, Proposition 23.1])**.** *$H^*(B\mathrm{SO}_n;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2,\dots,w_n]$, with $|w_i| = i$.*

These are the Stiefel-Whitney classes of the tautological vector bundle on $B\mathrm{SO}_n$, so the Wu formula [Wu50] computes their Steenrod squares:

$$\mathrm{Sq}^i(w_j) = \sum_{k=0}^{i} \binom{j-i+k+1}{k} w_{i-k}w_{j+k}. \tag{5.2}$$

Using this we can write down the Q_i-action on $H^*(B\mathrm{SO}_3;\mathbb{Z}/2)$. Now we need to transfer this information to $\Sigma^{-3}MSO_3$.

Theorem 5.3 (Thom isomorphism [Tho54, §II.2])**.** *If $V \to X$ is a virtual vector bundle, $H^*(X^V;\mathbb{Z}/2)$ is a free $H^*(X;\mathbb{Z}/2)$-module on a single generator U in degree* $\mathrm{rank}(V)$*.*

The generator U is called the *Thom class.* We will use this isomorphism to write elements of $H^*(X^V;\mathbb{Z}/2)$ as Ux, where $x \in H^*(X;\mathbb{Z}/2)$.

Since $V_{\mathrm{taut}} - 3$ is rank 0, $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2) \cong H^*(B\mathrm{SO}_3;\mathbb{Z}/2)\cdot U$, with U in degree 0. However, this isomorphism does not commute with Steenrod squares. Instead, the Sq^i-action on the cohomology of the Thom spectrum for $V \to X$ can be computed using the following rules [Tho52, §III].

(1) $\mathrm{Sq}^i(U) = Uw_i(V)$.
(2) The Cartan formula can be used to compute $\mathrm{Sq}^i(Ux)$, where $x \in H^*(X;\mathbb{Z}/2)$. Thus, for example,

$$\mathrm{Sq}^1(Ux) = Uw_1(V)x + U\mathrm{Sq}^1(x) \tag{5.4a}$$

$$\mathrm{Sq}^2(Ux) = Uw_2(V)x + Uw_1(V)\mathrm{Sq}^1(x) + U\mathrm{Sq}^2(x). \tag{5.4b}$$

For $\Sigma^{-3}MSO_3$, $w_1(V_{\mathrm{taut}}) = 0$ and $w_2(V_{\mathrm{taut}}) = w_2$. So in principle we have all the information we need to compute the $\mathcal{E}(1)$-module structure on $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$. For example, $\mathrm{Sq}^1(U) = Uw_1(V_{\mathrm{taut}}) = 0$; $\mathrm{Sq}^2(U) = Uw_2(V_{\mathrm{taut}}) = Uw_2$, and $\mathrm{Sq}^1\mathrm{Sq}^2(U) = \mathrm{Sq}^1(Uw_2) = U(w_1(V_{\mathrm{taut}})w_2 + \mathrm{Sq}^1(w_2))$; since $w_1(V_{\mathrm{taut}}) = 0$ and the Wu formula computes $\mathrm{Sq}^1(w_2) = w_3$,[5] then $\mathrm{Sq}^1\mathrm{Sq}^2(U) = Uw_3$. Thus $Q_0(U) = 0$, $Q_1(U) = \mathrm{Sq}^2\mathrm{Sq}^1(U) + \mathrm{Sq}^1\mathrm{Sq}^2(U) = Uw_3$, and $Q_0(Uw_2) = Uw_3$, so the subspace spanned by $\{U, Uw_2, Uw_3\}$ is an $\mathcal{E}(1)$-submodule of $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$ isomorphic to $\mathcal{\mathring{O}}$.

[5]The Wu formula computes $\mathrm{Sq}^1(w_2) = w_1w_2 + w_3$, but in the cohomology of $B\mathrm{SO}_3$, $w_1 = 0$, so we just get w_3.

Continuing in this way, one finds that

$$H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2) \cong \widetilde{\mathcal{O}} \oplus \Sigma^4\widetilde{\mathcal{O}} \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^8\widetilde{\mathcal{O}} \oplus \Sigma^9\mathcal{E}(1) \oplus \Sigma^{11}\mathcal{E}(1) \oplus \Sigma^{12}\widetilde{\mathcal{O}} \oplus P, \tag{5.5}$$

where P contains no elements of degrees 12 or below. We only want to compute spin-U_2 bordism in degrees 11 and below, so we can and do ignore P. See Figure 3, left, for a picture of these summands.

Now using what we learned in Examples 4.8 and 4.9, we can draw the E_2-page of the Adams spectral sequence converging to $ku_*(\Sigma^{-3}MSO_3)$ in Figure 3, right. The d_r differential decreases $t-s$ (the x-coordinate) by 1 and increases s (the y-coordinate) by r. Therefore the only possible differentials in the range displayed are the ones emerging from Ext groups corresponding to $\Sigma^k\mathcal{E}(1)$ summands.

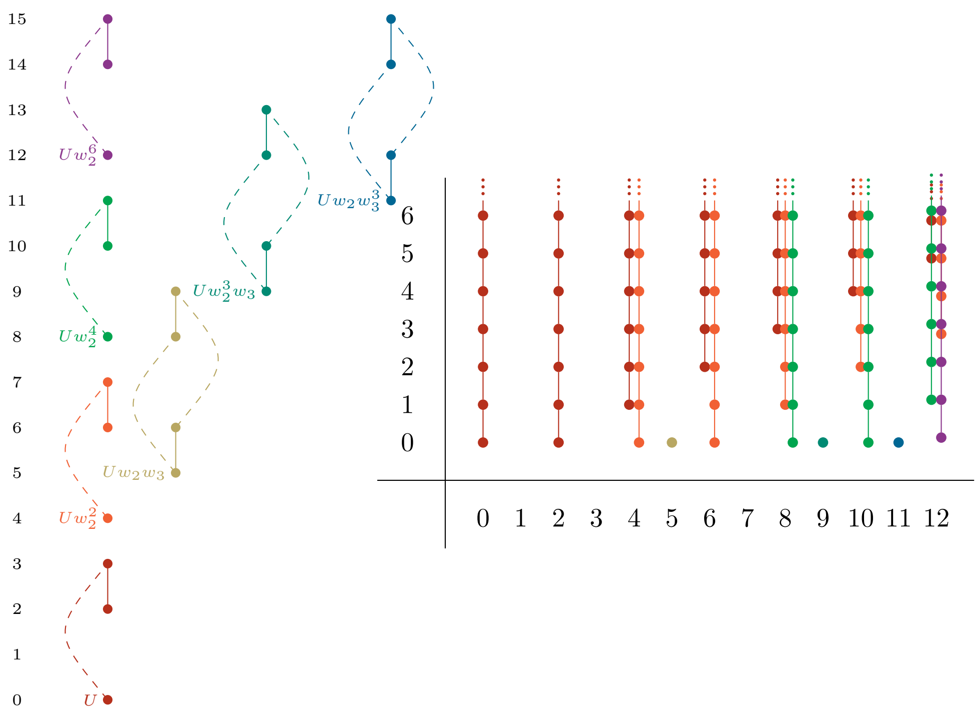


Figure 3. Left: $H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$ as a module over $\mathcal{E}(1)$ in low degrees. The pictured summand contains all elements in degrees 12 and below. Right: the Adams spectral sequence computing $ku_*(\Sigma^{-3}MSO_3)$. v_1-actions are hidden for legibility.

It is a theorem of Margolis [Mar74] that free summands in cohomology split off summands at the level of spectra.[6] This implies that the submodule of $\mathrm{Ext}(H^*(\Sigma^{-3}MSO_3;\mathbb{Z}/2))$ coming from the free $\mathcal{E}(1)$-module summands does not admit any nonzero differentials. Therefore the spectral sequence collapses.

[6]Margolis' result is for free summands as an $\mathcal{A}$-module. Passing through the change-of-rings theorem implies that free $\mathcal{E}(1)$-summands in $H^*(X;\mathbb{Z}/2)$ split pieces off of $ku\wedge X$, which is what we're after.

There are also no extension problems: each infinite tower linked by h_0-actions in even degrees lifts to a free cyclic summand in ku-theory. Thus we read off

$$\begin{aligned}
ku_0(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}\\
ku_1(\Sigma^{-3}MSO_3) &\cong 0\\
ku_2(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}\\
ku_3(\Sigma^{-3}MSO_3) &\cong 0\\
ku_4(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}^2\\
ku_5(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}/2\\
ku_6(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}^2\\
ku_7(\Sigma^{-3}MSO_3) &\cong 0\\
ku_8(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}^3\\
ku_9(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}/2\\
ku_{10}(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}^3\\
ku_{11}(\Sigma^{-3}MSO_3) &\cong \mathbb{Z}/2.
\end{aligned}$$

Combining this with (4.3), we obtain $\Omega_*^{\mathrm{Spin}^c}(\Sigma^{-3}MSO_3)$, i.e. spin-$\mathrm{U}_2$ bordism groups.

$$\begin{aligned}
\Omega_0^{\text{Spin-U}_2} &\cong \mathbb{Z}\\
\Omega_1^{\text{Spin-U}_2} &\cong 0\\
\Omega_2^{\text{Spin-U}_2} &\cong \mathbb{Z}\\
\Omega_3^{\text{Spin-U}_2} &\cong 0\\
\Omega_4^{\text{Spin-U}_2} &\cong \mathbb{Z}^3\\
\Omega_5^{\text{Spin-U}_2} &\cong \mathbb{Z}/2\\
\Omega_6^{\text{Spin-U}_2} &\cong \mathbb{Z}^3\\
\Omega_7^{\text{Spin-U}_2} &\cong 0\\
\Omega_8^{\text{Spin-U}_2} &\cong \mathbb{Z}^7\\
\Omega_9^{\text{Spin-U}_2} &\cong (\mathbb{Z}/2)^2\\
\Omega_{10}^{\text{Spin-U}_2} &\cong \mathbb{Z}^7 \oplus \mathbb{Z}/2\\
\Omega_{11}^{\text{Spin-U}_2} &\cong \mathbb{Z}/2.
\end{aligned}$$

We needed $H_k(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$ in degrees 0 and 1; the universal coefficient theorem implies this is the same dimension as $H^k(\Sigma^{-3}MSO_3;\mathbb{Z}/2)$, which we know by the Thom isomorphism and Theorem 5.1 above: $H^0(\Sigma^{-3}MSO_3;\mathbb{Z}/2) \cong \mathbb{Z}/2$ generated by U and H^1 vanishes.

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