2-LOCAL STRING BORDISM AND THE ADAMS SPECTRAL SEQUENCE

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I'm currently trying to learn how to compute string bordism and/or *tmf* of various things. This has some applications to questions in topology and mathematical physics. The goal of this short document is to exhibit a relatively simple example computation: there is big machinery and plenty of moving parts, but most things follow using formal structure, without too much elbow grease. The result is proven a different way and used by Crowley-Nagy [CN20] in order to understand the classification of four-(complex-)dimensional complete intersections up to diffeomorphism.

First, some quick background information on the objects and tools we will encounter today.

String bordism: *Bordism* of a space X computes abelian groups of closed n-manifolds with a map to X, modulo those which are boundaries. "Boundaries" means not just that the manifold bounds some compact $(n+1)$-manifold W, but also that the map extends across W.

Different notions of manifold (unoriented, oriented, etc.) yield different kinds of bordism. *String bordism* is the bordism of string manifolds. If M is a spin manifold, it has a canonical characteristic class $\lambda \in H^4(M;\mathbb{Z})$ such that 2λ is the first Pontrjagin class. A *string structure* on M is a trivialization of λ, similarly to how an orientation can be recast as a trivialization of the first Stiefel-Whitney class.

Topological modular forms: The abelian groups of string bordism can be collected into functors satisfying axioms much like the Eilenberg-Steenrod axioms for homology, but not quite. Accordingly, string bordism is called a *generalied homology theory*. Such things are represented by objects called "spectra," and the one for string bordism is denoted *MString*.

Topological modular forms are additional examples of spectra. There are a few different variants of the construction; we need the one called *tmf*, or "connective topological modular forms." This object has deep connections to geometry, number theory, and quantum field theory, but these are out of scope for today. Instead, what matters about *tmf* is the following two properties:

(1) In low dimensions, *tmf* is a very good approximation for *MString*.
(2) The Adams spectral sequence (see below) for computing *tmf*-homology is quite a bit simpler than the general case.

The Adams spectral sequence: This is a computational tool for computing various things in algebraic topology, typically homotopy groups or generalized homology groups. It has a fearsome reputation, and is indeed quite difficult in general, but the computation which is the point of this document takes advantage of a trick that greatly simplifies it. For the approach I take on Adams spectral sequence, Beaudry-Campbell [BC18] is an excellent reference, and has shaped how I think about these computations.

From here on, I'll focus on the computation, and less on background. The article of Beaudry-Campbell [BC18] explains most of the definitions that I don't.

1. The computation

Let $\xi \to \mathbb{CP}^1$ denote the tautological bundle and $X \coloneqq (\mathbb{CP}^1)^{\xi-2}$ its Thom spectrum; here we shift ξ to a rank-zero virtual bundle $\xi - 2$ so that the bottom cell of X is in degree zero.

Theorem 1.1. *The first few string bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{String}}(X) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{String}}(X) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{String}}(X) &\cong \mathbb{Z}\\
\widetilde{\Omega}_3^{\mathrm{String}}(X) &\cong \mathbb{Z}/12\\
\widetilde{\Omega}_4^{\mathrm{String}}(X) &\cong 0\\
\widetilde{\Omega}_5^{\mathrm{String}}(X) &\cong \mathbb{Z}/24\\
\widetilde{\Omega}_6^{\mathrm{String}}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_7^{\mathrm{String}}(X) &\cong 0\\
\widetilde{\Omega}_8^{\mathrm{String}}(X) &\cong \mathbb{Z}\oplus\mathbb{Z}/4\\
\widetilde{\Omega}_9^{\mathrm{String}}(X) &\cong 0\\
\widetilde{\Omega}_{10}^{\mathrm{String}}(X) &\cong \mathbb{Z}\oplus\mathbb{Z}/3.
\end{aligned}$$

Crowley-Nagy [CN20, Lemma 4.2] determine the torsion subgroup of $\widetilde{\Omega}_8^{\mathrm{String}}(X)$ using Toda brackets in the stable stem and the relationship between the stable stem (equivalent to stably framed bordism groups) and string bordism groups. We use something different: the Adams spectral sequence (at $p = 2$; for odd primes the problem is nearly trivial).

Proof. We will work p-locally for each prime p; first, $p = 2$.

The Witten genus $MString \to tmf$ is 15-connected, so we can and will replace string bordism with tmf-homology [Hil09, Theorem 2.1]. There is an isomorphism [HM14, Mat16]

$$\widetilde{H}^*(tmf;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{A}(2)}\mathbb{F}_2, \tag{1.2}$$

where $\mathcal{A}(2)$ is the subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ generated by Sq^1, Sq^2, and Sq^4. This allows the use of a change-of-rings trick that greatly simplifies the E_2-page of the Adams spectral sequence calculating $\pi_*(tmf\wedge X)$:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\widetilde{H}^*(tmf;\mathbb{F}_2)\otimes\widetilde{H}^*(X;\mathbb{F}_2),\mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(H^*(X;\mathbb{F}_2),\mathbb{F}_2). \tag{1.3}$$

As $\mathcal{A}(2)$ is much smaller than $\mathcal{A}$, this is easier to compute.[1]

The Thom isomorphism says $\widetilde{H}^*(X;\mathbb{F}_2)$ has a single $\mathbb{F}_2$ summand in degrees 0 and 2, and vanishes in all other degrees. The $\mathcal{A}(2)$-module structure is determined by the Stiefel-Whitney classes of ξ. For degree reasons, the only possible nonzero Steenrod operation would be a Sq^2 from degree 0 to degree 2; because $w_2(\xi)\neq 0$, this Sq^2 is indeed nonzero.[2] Call this $\mathcal{A}(2)$-module $C\eta$.

Our next job is to determine $\mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(C\eta,\mathbb{F}_2)$. The simplest way to do this is to ask a computer: Hood Chatham and Dexter Chua have written a program to compute Ext over subalgebras of the Steenrod algebra, which has a web interface available at https://spectralsequences.github.io/rust_webserver/. Feed it the following file:

```
{
    "p": 2,
    "generic": false,
    "algebra_name": "adem",
    "profile": {
        "truncated": true,
        "p_part": [3, 2, 1]
    },
```

[1] If you're familiar with the trick to compute Ext over $\mathcal{A}(1)$ for spin bordism or *ko*-theory, the same thing is happening here, just over $\mathcal{A}(2)$ instead of $\mathcal{A}(1)$.

[2] Again, I'm not going into great detail; the relationship between Stiefel-Whitney classes and Steenrod squares in Thom spaces is well summarized by Beaudry-Campbell [BC18, §3.3].

```
        "type": "finite dimensional module",
        "algebra": ["milnor"],
        "gens": {
            "x0": 0,
            "x2": 2
        },
        "actions": [
            "Sq2 x0 = x2"
        ]
    }
```

This returns the Ext chart in Figure 1. The x-axis is $t-s$ and the y-axis is s, as usual. If you want to see

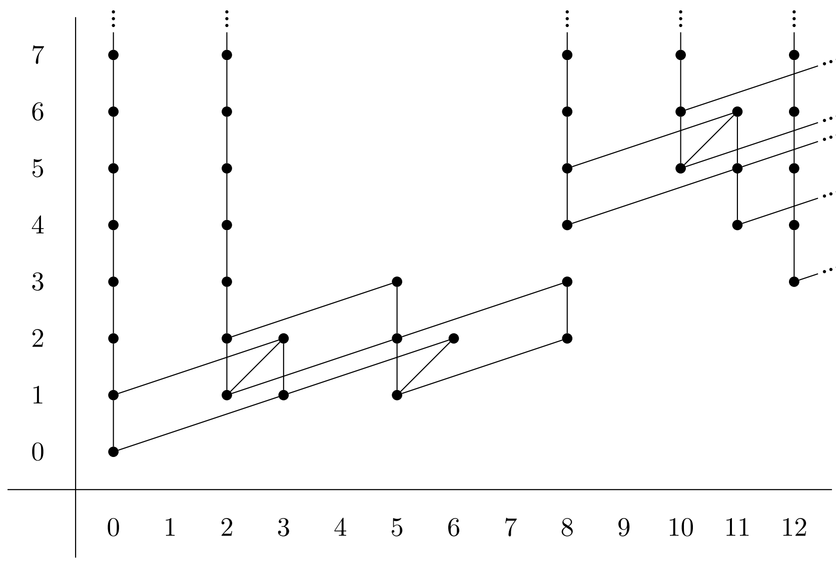

FIGURE 1. $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$, the E_2-page for our Adams spectral sequence calculation. The vertical lines indicate an h_0-action; the lines of slope 1/2 indicate an h_1-action, and the lines of slope 1/3 indicate an h_2-action. See the text for more information.

how you would make such a computation by hand, see §2.

In Figure 1, a dot at point $(t-s, s)$ indicates an $\mathbb{F}_2$ summand in $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$. This bigraded vector space carries an action by the algebra $H^{*,*}(\mathcal{A}(2)) := \mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$; the lines in the figure describe some of this action.

- The vertical lines represent action by an element called $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$.
- The diagonal lines with slope 1/2 represent action by an element $h_1 \in \mathrm{Ext}^{1,2}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$; $h_0h_1 = 0$ and $h_1^4 = 0$.
- The diagonal lines with slope 1/3 represent action by an element $h_2 \in \mathrm{Ext}^{1,4}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$; $h_0^3 h_2 = 0$, $h_1h_2 = 0$, $h_2^3 = 0$, and (important for later) $h_0^3 h_2 = 0$.

These elements do not generate all of $H^{*,*}(\mathcal{A}(2))$, but they're enough to see some of the structure. Moreover, they will solve some extension problems for us: $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is the E_2-page of the Adams spectral sequence for computing $(tmf^\wedge_2)_*$, and the action of $(tmf^\wedge_2)_*$ on $\pi_*(tmf \wedge X)^\wedge_2$ is compatible with the $H^{*,*}(\mathcal{A}(2))$-action at the level of E_2-pages of the respective Adams spectral sequences. In particular, h_0 is a permanent cycle and will converge to $2 \in (tmf^\wedge_2)_0 \cong \mathbb{Z}_2$; h_1 and h_2 are also permanent cycles and converge to elements called $\eta \in (tmf^\wedge_2)_1$ and $\nu \in (tmf^\wedge_2)_3$, respectively. The relations between the h_is above lift to relations with these elements, e.g. $2\eta = 0$ and $8\nu = 0$.

With this grading, the d_r differentials move one tick to the left and r ticks upwards. This already forces many differentials in the range shown to vanish. Almost all of the remaining differentials in the range displayed in Figure 1 also vanish! This is because they must commute with the $H^{*,*}(\mathcal{A}(2))$-action. In particular, if

$d_r(x) = y$, then $d_r(h_0^k x) = h_0^k y$. Hence, for any x and y, if $h_0^k x = 0$ but $h_0^k y \neq 0$, this means $d_r(x) \neq y$. So let's walk through the Ext chart.

- There are no elements in degree 1, so no differentials involving elements of degree 0.
- The fact indicated above about h_0-actions zeroes out all differentials from degree 3 to degree 2.
- There are no elements in degree 4, so no nonzero differentials to degree 3.
- The fact about h_0 indicated above rules out nonzero differentials from degree 6 to degree 5.

And so on — the lowest-degree differential that could possibly be nonzero is from degree 12 to degree 11. Therefore we know the E_∞-page in degrees 10 and below. There remains the question of extension problems, but many of them are completely determined by the $H^{*,*}(\mathcal{A}(2))$-action, as indicated above: action by h_0 lifts to multiplication by 2, for example, forcing $\pi_3 \cong \mathbb{Z}/4$ and $\pi_5 \cong \mathbb{Z}/8$. The infinite towers of multiplication by 2 in degrees 0, 2, and 10 hence lift to copies of $\mathbb{Z}_2$ (since X is a finite spectrum, these must have come from copies of $\mathbb{Z}$ before we 2-completed). The only remaining nontrivial extension appears in degree 8, where we see $\mathbb{Z}/4$ and $\mathbb{Z}_2$, but there could be a "hidden extension," in that the extension problem

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_8(\mathit{tmf} \wedge X)_2^\wedge \longrightarrow \mathbb{Z}/4 \longrightarrow 0 \tag{1.4}$$

need not split. To show it does in fact split, let's see what goes wrong if we assume it doesn't. Let $x \in \pi_5(\mathit{tmf} \wedge X)$ have Adams filtration 1; that is, its image in the associated graded of $\pi_*(\mathit{tmf} \wedge X)$ under the Adams filtration (namely the E_∞-page of the Adams spectral sequence) is the nonzero element in bidegree $(t-s, s) = (5, 1)$. The h_2- and h_0-actions on the E_∞-page, together with the fact that (1.4) doesn't split, imply that $8\nu x \neq 0$, and in fact the image of $8\nu x$ in the E_∞-page is the nonzero element in bidegree $(8, 5)$. However, in $\pi_* \mathit{tmf}_2^\wedge$, $8\nu = 0$, so this is a contradiction. This finishes the 2-local calculation, telling us the free and 2-torsion summands of the answer; what remains is to determine the odd-primary torsion.

Another way to construct X is as the homotopy cofiber of the stabilized Hopf fibration $\eta^s\colon \mathbb{S}^1 \to \mathbb{S}^0$. That is, the Hopf fibration is a map $\eta\colon S^3 \to S^2$, and applying Σ^∞ gives us the map η^s between spectra. Then, the cofiber of η^s is weakly equivalent to X. There is an isomorphism $\pi_1\mathbb{S} \cong \mathbb{Z}/2$, and η^s is the generator, which implies that if p is an odd prime, then after p-completing, $\eta^s \simeq 0$! This implies $(X)_p^\wedge \simeq (\mathbb{S}^0)_p^\wedge \vee (\mathbb{S}^2)_p^\wedge$.

The axioms of a generalized homology theory imply wedge sums are sent to direct sums, and $\Omega_k^{\mathrm{String}}(\mathbb{S}^n) \cong \Omega_{k-n}^{\mathrm{String}}$, so

$$\widetilde{\Omega}_n^{\mathrm{String}}(X)_p^\wedge \simeq (\Omega_n^{\mathrm{String}})_p^\wedge \oplus (\Omega_{n-2}^{\mathrm{String}})_p^\wedge. \tag{1.5}$$

The only odd-primary torsion in $\Omega_*^{\mathrm{String}}$ below degree 12 is two $\mathbb{Z}/3$ summands, one in degree 3 and one in degree 10 [Gia71, HR95]. Therefore the odd-primary torsion in $\widetilde{\Omega}_*^{\mathrm{String}}(X)$ below degree 12 consists of three $\mathbb{Z}/3$ summands in degrees 3, 5, and 10. ⊠

2. Obtaining the E_2-page by hand

It is possible to compute much of Figure 1 by hand, especially in low degrees.

2.1. **A long exact sequence in Ext.** The key fact is that a short exact sequence of $\mathcal{A}(2)$-modules induces a long exact sequence in Ext. And $C\eta$ fits into a short exact sequence

$$0 \longrightarrow \Sigma^2\mathbb{F}_2 \longrightarrow C\eta \longrightarrow \mathbb{F}_2 \longrightarrow 0, \tag{2.1}$$

which we draw in Figure 2. (The curvy line indicates a Sq^2-action.) The induced long exact sequence in Ext

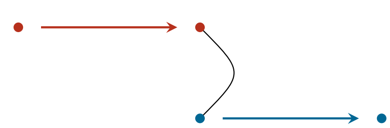

Figure 2. The extension (2.1).

is traditionally depicted in a grid, plotted by bidegree $(t-s, s)$ just like Adams E_2-pages. This means the boundary map goes one square to the left and one square upward. We draw this in Figure 3. See [BC18, §4.6.2] for more information on this method of computation and some worked examples over $\mathcal{A}(1)$.

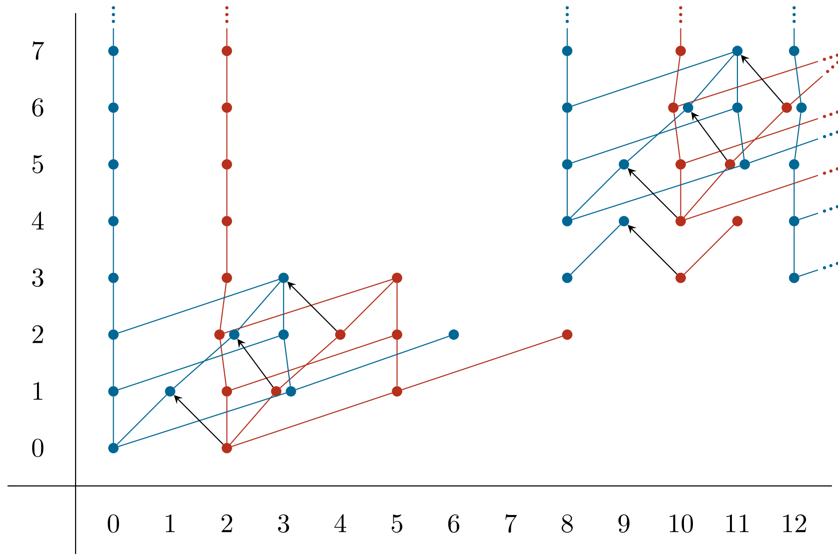

FIGURE 3. The long exact sequence in Ext associated to (2.1). The "differentials" of bidegree $(-1, 1)$ are the boundary maps. In the text we discuss how to determine whether they vanish.

The seven boundary maps in range may or may not be zero. To determine this, we use that they commute with the $H^{*,*}(\mathcal{A}(2))$-action.

- First, there is no nonzero $\mathcal{A}(2)$-module morphism $C\eta \to \Sigma^2\mathbb{F}_2$. Therefore $\mathrm{Ext}^{0,2}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2) = \mathrm{Hom}_{\mathcal{A}(2)}(C\eta, \Sigma^2\mathbb{F}_2) = 0$. Thus the boundary map emerging from bidegree $(2, 0)$ is nonzero.
- Acting by h_1, this implies the boundary maps emerging from bidegrees $(3, 1)$ and $(4, 2)$ are also nonzero.
- Acting by the generator of $H^{4,12}(\mathcal{A}(2)) \cong \mathbb{Z}/2$ carries the first three boundary maps to the ones emerging from bidegrees $(10, 4)$, $(11, 5)$, and $(12, 6)$, so those boundary maps are all nonzero.
- Acting by the generator of $H^{3,11}(\mathcal{A}(2)) \cong \mathbb{Z}/2$ carries the boundary map emerging from bidegree $(2, 0)$ to the last remaining boundary map, emerging from bidegree $(10, 3)$, so that boundary map is nonzero.

We got lucky here; because $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$ is a cyclic $H^{*,*}(\mathcal{A}(2))$-module, we didn't need as much information as usual to calculate boundary maps. In any case, when we account for the boundary maps, we obtain $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$ and *some* of the $H^{*,*}(\mathcal{A}(2))$-module structure, drawn in Figure 4. There can be "hidden extensions" which are invisible to the long exact sequence. Indeed, comparing with Figure 1, there are several. We draw them with dashed lines.

2.2. **Computing hidden extensions in the E_2-page.** Determining these hidden extensions is usually not straightforward. In the proof of Theorem 1.1, we only need the hidden h_0-action from bidegree $(8, 2)$ to bidegree $(8, 3)$, so I'll focus on that one.

One way to compute hidden extensions is to represent elements of $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$ as actual length-s extensions of $\mathcal{A}(2)$-modules; given such an extension with equivalence class $x \in \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$, and an extension y from $\mathbb{F}_2$ to itself (interpreted as an element of $\mathrm{Ext}^{s',t'}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathcal{A}(2))$), there is a recipe [BC18, §4.2] for gluing these extensions into an extension representing the class $y \cdot x \in \mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$. This is called the *Yoneda product*. The details are somewhat involved and I'm not going to go into them, but here are a few useful facts.

- h_0 is realized by the extension $0 \to \Sigma\mathbb{F}_2 \to C2 \to \mathbb{F}_2 \to 0$, where $C2$ denotes the $\mathcal{A}(2)$-module with two $\mathbb{F}_2$ summands in degrees 0 and 1, connected by a Sq^1. We draw this in Figure 5.
- h_1 is realized by $0 \to \Sigma^2\mathbb{F}_2 \to C\eta \to \mathbb{F}_2 \to 0$, drawn in Figure 2.
- h_2 is realized by the extension $0 \to \Sigma\mathbb{F}_2 \to C\nu \to \mathbb{F}_2 \to 0$, where $C\nu$ denotes the $\mathcal{A}(2)$-module with two $\mathbb{F}_2$ summands in degrees 0 and 4, connected by a Sq^4. We draw this in Figure 6.

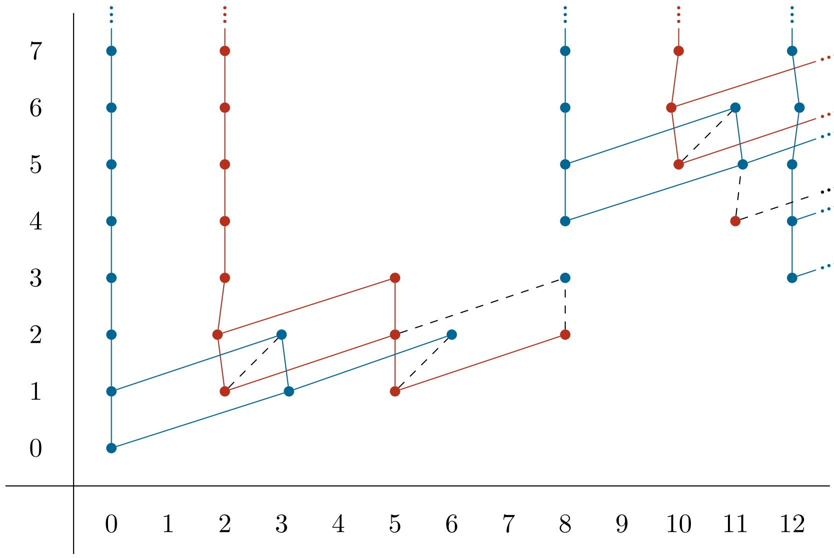

FIGURE 4. Determining "hidden extensions" in $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$.

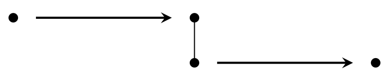

FIGURE 5. The extension $0 \to \Sigma\mathbb{F}_2 \to C2 \to \mathbb{F}_2 \to 0$ realizing $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$.

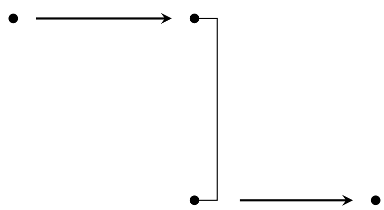

FIGURE 6. The extension $0 \to \Sigma^4\mathbb{F}_2 \to C\nu \to \mathbb{F}_2 \to 0$ realizing $h_2 \in \mathrm{Ext}^{1,4}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$.

Now determining a hidden extension. If x is the nonzero element in bidegree $(8, 2)$, we will produce an extension representing $h_0 x$ and argue that it's not equivalent to a split extension (which represents 0).

- The class in $(8, 2)$ is in the image in the Ext long exact sequence of a class $\alpha \in \mathrm{Ext}^{2,10}_{\mathcal{A}(2)}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Ext}^{2,8}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$. As far as extensions go, this means we'll obtain an extension $0 \to \Sigma^8\mathbb{F}_2 \to P_2 \to P_1 \to \mathbb{F}_2 \to 0$; to obtain the corresponding extension $0 \to \Sigma^8\mathbb{F}_2 \to P_2' \to P_1' \to C\eta \to 0$, we let $P_2' = P_2$; for P_1, we stick $C\eta$ onto the bottom of P_1.
- Let's describe this α. In $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$, $\alpha = h_2^2 \cdot 1$, and we said above that h_2 is represented by the extension with $C\nu$ in it depicted in Figure 6. Let's feed this to the Yoneda product. The upshot is that we get two copies of $C\nu$, as in Figure 7, and therefore our desired representative of $\mathrm{Ext}^{2,10}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$ is drawn in Figure 8.
- Now we want to show that h_0 times this class is nontrivial. That's another Yoneda product, this time with $C2$, and we obtain Figure 9. We need to show the Ext class this represents is nonzero. Suppose it were equal to zero; then, there would be a length-3 extension of $C\eta$ by $\Sigma^{11}\mathbb{F}_2$ which splits, and $\mathcal{A}(2)$-equivariant maps from our extension to this one (or vice versa) intertwining the maps in the extensions. Such a commutative diagram would imply an Ádem relation $\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^4\mathrm{Sq}^2 = 0$ — but this is not the case. Instead, $\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^4\mathrm{Sq}^2 = \mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 \neq 0$. This means Figure 9 represents the generator of $\mathrm{Ext}^{3,11}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$, so we have detected the hidden h_0-extension we need.

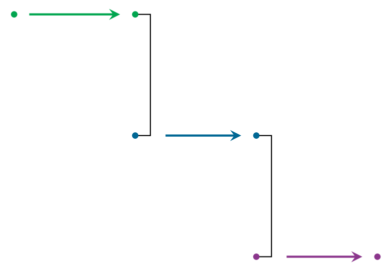

FIGURE 7. An extension $\alpha = h_2^2 \in \mathrm{Ext}^{2,8}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$.

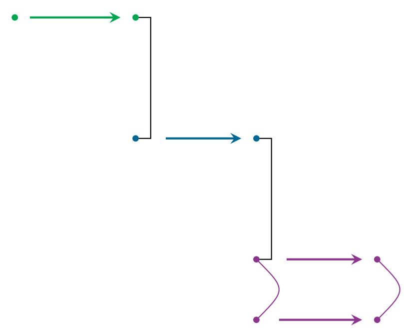

FIGURE 8. An extension representing the nonzero element of $\mathrm{Ext}^{2,10}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$.

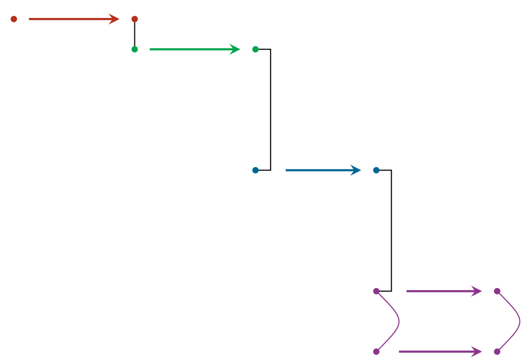

FIGURE 9. An extension representing an element of $\mathrm{Ext}^{3,11}_{\mathcal{A}(2)}(C\eta, \mathbb{F}_2)$. To establish the hidden h_0-extension we need in the proof of Theorem 1.1, we need to show this element is nonzero.

We also should check that hidden extensions which could exist but don't, in fact don't exist. There aren't many of them in range, and we fortunately can rule them all out with relations in $H^{*,*}(\mathcal{A}(2))$.

- Is there a nonzero h_0-action from $(8, 3)$ to $(8, 4)$? No, because $h_2 h_0^3 = 0$.
- Is there a nonzero h_1-action from $(10, 4 + k)$ to $(11, 5 + k)$ where $k = 0, 1, 2$? No, because $h_0 h_1 = 0$.

Remark 2.2. Fang-Klaus [FK96] and Fang-Wang [FW10] take up some related questions, computing things such as $\Omega_*^{\mathrm{String}}((\mathbb{CP}^\infty)^{\xi-2})$ in low degrees for various complex line bundles $\xi \to \mathbb{CP}^\infty$. They use the Adams spectral sequence over $\mathcal{A}(2)$, and have similar applications in mind. ◀

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