TOPOLOGICAL AND GEOMETRIC METHODS IN QFT

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These notes were taken at the NSF-CBMS conference on topological and geometric Methods in QFT at Montana State University in summer 2017. Most of the lectures were given by Dan Freed. I live-TEXed these notes using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`; any mistakes in the notes are my own. Thanks to Nilay Kumar for most of the figures.

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Day 1. July 31

1. Dan Freed: Bordism and TFT

"Quantization is an art, not a functor."

The first lecture will be about topology, specifically bordism; we'll talk about the grand plan near the end.

Definition 1.1. Let Y_0 and Y_1 be closed d-manifolds. Then, Y_0 and Y_1 are *bordant* if there exists a compact $(d+1)$-manifold X such that $\partial X = Y_0 \amalg Y_1$.

FIGURE 1. A bordism between $(S^1)^{\amalg 3}$ and $(S^1)^{\amalg 2}$.

The empty set is a manifold of any dimension, and the disc is a bordism between S^1 and $\varnothing$.

Bordism is an equivalence relation: reflexivity and symmetry are apparent, and transitivity comes from gluing. The set of equivalence classes is a group under disjoint union, denoted Ω_d and called the *bordism group* of d-dimensional manifolds.

The idea of bordism dates back to Poincaré, who tried to use it to define a homology theory of maps of manifolds into a space. He ended up using simplices, and we got the homology we're familiar with.

Example 1.2. In dimension 0, a single point is not cobordant to an empty set. This comes from one of the most basic theorems in differential topology, that a compact 1-manifold has an even number of boundary points. However, two points are cobordant to an empty set, so the number of points mod 2 defines an isomorphism $\Omega_0 \to \mathbb{Z}/2$. ◀

Example 1.3. It's also true that $\Omega_2 \cong \mathbb{Z}/2$. The complete invariant is a nice exercise in differential topology à la Guilleman and Pollack: let $\operatorname{Det} TY$ denote the determinant line bundle of the tangent bundle of Y and s be a section of $\operatorname{Det} TY$ transverse to the zero section. If $S \coloneqq s^{-1}(0)$, then S is a codimension-1 submanifold of Y, and the mod-2 intersection number of S with itself defines an isomorphism $\Omega_2 \cong \mathbb{Z}/2$. ◀

Definition 1.4. A *bordism invariant* is a homomorphism $\Omega_d \to \mathbb{Z}$.

You can replace $\mathbb{Z}$ with other abelian groups, as we did above in Examples 1.2 and 1.3.

Example 1.5.

(1) One can consider bordism of oriented manifolds, with oriented cobordisms between them. This is again an abelian group, denoted $\Omega_d(\mathrm{SO})$. If $d = 4k$, the signature of the intersection pairing defines a bordism invariant $\Omega_{4k}(\mathrm{SO}) \to \mathbb{Z}$.

(2) Manifolds with a U_n-structure (we'll discuss these and other structures in a little bit) form a cobordism group called $\Omega_d(\mathrm{U})$. The Todd genus $\mathrm{td} : \Omega_{2k}(\mathrm{U}) \to \mathbb{Z}$ is a bordism invariant.

(3) Spin manifolds have an $\hat{A}$-genus $\hat{A} \colon \Omega_{4k}(\mathrm{Spin}) \to \mathbb{Z}$. ◀

The systematic investigation of genera and bordism invariants was undertaken by Hirzebruch. Notice that the bordism invariants $\operatorname{Hom}(\Omega_d, \mathbb{Z})$ is an abelian group.

We'll now do something called categorification, a specific example of a process that adds additional structure to things: sets or vector spaces are replaced with categories, and functions with functors. Throughout this lecture (and following lectures), let $n \coloneqq d + 1$.

Definition 1.6. The *bordism category* $\mathsf{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category specified by the following data.

- The objects are closed $(n-1)$-manifolds.
- The hom-set $\mathsf{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $Y_0 \to Y_1$.
- Composition is gluing of bordisms.
- The identity $\mathrm{id}_Y : Y \to Y$ is the cylinder $Y \times [0,1]$.
- The monoidal product is disjoint union.
- The monoidal unit is the empty set, regarded as an $(n-1)$-manifold.

There are many ways to think of categories, some more philosophical than others; we're in the business of treating them as algebraic structures like groups or rings. You might imagine a bunch of points with arrows between them. But unlike when we defined bordism groups, these bordisms now have a direction: each bordism X comes with a locally constant function $\partial X \to \{0,1\}$ choosing which boundary components are incoming and outgoing. Gluing must glue the outgoing component of one bordism to the incoming component of the other. Thus you might imagine each $(n-1)$-manifold M to have a *collar*, a neighborhood of it in these cobordisms diffeomorphic to $M \times [0,1)$, and cobordisms should respect this collar. You can think of this collar as an infinitesimal thickening in the direction of cobordisms.

We can apply the monoidal product (disjoint union) to both objects and morphisms. It's symmetric, meaning that there's a natural isomorphism $M \amalg N \cong N \amalg M$, which is the maximally symmetric tensor structure one can apply in this case. It's the categorification of the fact that Ω_d is an abelian group.

Our central definition is the categorification of Definition 1.4. We also need a categorification of $\mathbb{Z}$, and we choose $\mathsf{Vect}_\mathbb{C}$, the category of complex vector spaces and linear maps, and we'll choose $\otimes$ to be the monoidal structure (you could also choose $\oplus$, but we will not). The "decategorification" from $\mathsf{Vect}_\mathbb{C}$ to $\mathbb{Z}$ is the dimension.

Definition 1.7 (Atiyah [Ati88])**.** A *topological field theory* (TFT) is a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow (\mathsf{Vect}_\mathbb{C}, \otimes).$$

You could ask whether the bordism invariants we discussed lift; that they're integer-valued is an interesting hint, which Atiyah and Segal wondered about (leading to Dirac operators and all sorts of wonderful geometry). You may be wondering where the physics is, given the physics-sounding name of a topological field theory. We'll certainly get there.

The definition of a topological field theory is relatively new, stemming from attempty to understand Chern-Simons theory and related phenomena in the 1980s. As such, it's not as set in stone as other mathematical definitions, and we'll certainly consider variants along the way. So maybe it's better to think of Atiyah's definition as an axiom system, rather than a complete mathematical characterization of physical phenomena.

Topological field theories have stringent finiteness condition.

Definition 1.8. Let C be a symmetric monoidal category and $y \in \mathsf{C}$. *Duality data* for y is a triple $(y^\vee, e, c)$, where $y^\vee \in \mathsf{C}$ and $c\colon 1 \to y \otimes y^\vee$ and $e\colon y^\vee \otimes y \to 1$ are C-morphisms satisfying axioms called the *S-diagrams*. y is *dualizable* if it has duality data; then, $y^\vee$ is called its *dual*, e is called *evaluation*, and c is called *coevaluation*.

In $\mathsf{Vect}_\mathbb{C}$, $Y^\vee$ is the usual vector-space dual $\operatorname{Hom}(Y, \mathbb{C})$: evaluation applies a functional to a vector, and its adjoint is coevaluation. But this can only be written as a finite sum of basis vectors if Y is finite-dimensional. Thus a vector space is dualizable iff it's finite-dimensional.

Lemma 1.9. *Every object in* $\mathsf{Bord}_{\langle n-1,n\rangle}$ *is dualizable.*

Corollary 1.10. *Since a symmetric monoidal functor sends dualizable objects to dualizable ones,* $F(Y)$ *is a finite-dimensional vector space for any closed manifold* Y *and TFT* F*.*

Proof sketch of Lemma 1.9. Let Y be a closed $(n-1)$-manifold and $Y^\vee \coloneqq Y$. Then, evaluation will be the "outgoing cylinder" $Y \amalg Y \to \varnothing$, and coevaluation is the "incoming cylinder" $\varnothing \to Y \amalg Y$, and these satisfy the necessary axioms. ⊠

Figure 2. The evaluation (left) and coevaluation (right) morphisms in $\mathsf{Bord}_{\langle n-1,n\rangle}$.

That the state spaces are finite-dimensional is striking, and certainly not true for quantum mechanics and quantum field theory in general. So to get to physics we're going to have to leave the purely topological world.

There are many examples, some in Dan's lecture notes.

Example 1.11 (Finite gauge theory [DW90, FQ93])**.** Fix a finite group G, which we'll call the *gauge group* of this theory. Let $\mathrm{Bun}_G(S)$ denote the groupoid of principal G-bundles on a space S; that is, principal G-bundles on S form a category, but all morphisms are invertible. Since G is finite, these are Galois covering spaces of S with covering group G. You can imagine a groupoid with dots and arrows again, but this time every arrow is double-headed.

How should we turn this into a field theory? Principal G-bundles pull back, so given a cobordism $X\colon Y_0 \to Y_1$, we obtain a correspondence diagram

$$\begin{tikzcd} & \mathrm{Bun}_G(X) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathrm{Bun}_G(Y_0) & & \mathrm{Bun}_G(Y_1). \end{tikzcd}$$

This is highly nonlinear, yet a TFT is a linear thing. We'll linearize it by taking functions: if $\mathcal{G}$ is a groupoid, $\mathsf{Fun}_{\mathbb{C}}(\mathcal{G})$ denote the vector space of complex-valued functions on the set of isomorphism classes of $\mathcal{G}$. Since X, Y_0, and Y_1 are compact, their groupoids of principal G-bundles have finitely many isomorphism classes of objects, so we can both pull functions back and push them forward (summing over the fibers), hence defining a linear map

$$t_* \circ s^*\colon \mathsf{Fun}_{\mathbb{C}}(\mathrm{Bun}_G(Y_0)) \longrightarrow \mathsf{Fun}_{\mathbb{C}}(\mathrm{Bun}_G(Y_1)).$$

Thus we obtain a functor F_G, assigning Bun_G to objects and this push-pull formula to morphisms. To a closed n-manifold X (a bordism from $\varnothing$ to itself), we obtain the number $F_G(X) = \#\,\mathrm{Bun}_G(X)$, summing over the groupoid of bundles — but this is a groupoid, not a set, so we have to weight by the number of automorphism groups:

$$F_G(X) = \#\,\mathrm{Bun}(X) = \sum_{[P]\in\pi_0 \mathrm{Bun}_G(X)} \frac{1}{\#\,\mathrm{Aut}(P)}.$$

This already models the physical case: the principal G-bundles are examples of *fluctuating fields*, introduced to define the theory but summed over. The groupoid sum is a simple example of the path integral! ◀

The category of TFTs in dimension n, denoted $\mathsf{TFT}_n \coloneqq \mathrm{Hom}^{\otimes}(\mathsf{Bord}_{\langle n-1,n\rangle}, \mathsf{Vect}_{\mathbb{C}})$, has a composition law that's done pointwise: $(F_1\otimes F_2)(M) \coloneqq F_1(M)\otimes F_2(M)$, and similarly for bordisms. This will be useful when we try to classify TFTs, providing extra structure useful to us.

Tangential structures. We'll hear more about tangential structures from a geometric perspective later today. Right now, we'll adopt a more homotopical approach. We've just been talking about bare manifolds, but often one introduces additional structure: orientation, spin, and more. Tangential structures are a way to capture a large class of such structures (broadly, the topological ones).

The tangent bundle of an n-manifold M defines a classifying map $M \to B\mathrm{GL}_n(\mathbb{R})$, which lifts to a pullback

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & W_n \arrow[d] \\ M \arrow[r] & B\mathrm{GL}_n(\mathbb{R}). \end{tikzcd}$$

To define a tangential structure, we'll consider Lie group homomorphisms $\rho_n\colon H_n \to \mathrm{GL}_n(\mathbb{R})$ (e.g. inclusion of SO_n, projection down from Spin_n, and so forth). This lifts to a map $B\rho_n\colon BH_n \to B\mathrm{GL}_n(\mathbb{R})$. An

H_n*-structure* is a lift of the classifying map

$$\begin{tikzcd}
 & & B\rho_n^* W_n \arrow[dl] \arrow[d] \\
TM \arrow[urr, dashed] \arrow[r] \arrow[d] & W_n \arrow[d] & BH_n \arrow[dl, "B\rho_n"] \\
M \arrow[urr, dashed] \arrow[r] & B\mathrm{GL}_n(\mathbb{R}). &
\end{tikzcd} \tag{1.12}$$

For example, an SO_n-structure is the same thing as an orientation. You will have to reconcile this definition with the more familiar, geometric one.

Hence we have a general definition of what we need.

Definition 1.13. A *tangential structure* is a fibration $\rho\colon \mathcal{X}_n \to B\mathrm{GL}_n(\mathbb{R})$. An $\mathcal{X}_n$-structure on an n-manifold M is a lift of the classifying map along ρ as in (1.12).

For example, an orientation is specified by the map $B\mathrm{SO}_n \to B\mathrm{GL}_n(\mathbb{R})$, and if $\mathcal{X}_n = B\mathrm{GL}_n(\mathbb{R}) \times S$, you get cobordism of manifolds with a map to S.

Path of future lectures.

(1) Bordism and TFT, as we just saw.
(2) Quantum mechanics
(3) An axiom system for Wick-rotated quantum field theory.
(4) Another advantage of axiom systems is they allow you to consider classification theorems.
(5) We'll expand to variations on Definition 1.7, including in particular an extended notion of locality.
(6) Invertibility in TFT, and hence some stable homotopy theory.
(7) The Wick-rotated analogue of unitarity
(8) Extended positivity for invertible TFTs
(9) Non-topological invertible theories
(10) Computations for some electron systems in condensed-matter physics.

We're roughly following the material in [FH16], which will also be useful to keep in mind throughout the week.

2. Dave Morrison: Geometry and Physics: An Overview

> *"The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities."*
> *– Paul Dirac*

The title is an impossibly large topic to tackle in an hour, but we'll do what we can to introduce the interaction between geometry, topology, and physics in its modern form. It will be impressionistic and historical.

Maxwell's equations for electricity and magnetism are beautifully symmetric between electricity and magnetism — almost. We add a source term for the electricity term, an electron. But we don't for magnetism, because experimentalists have not discovered a magnetic analogue, a hypothetical *magnetic monopole*.

Dirac's monopoles. In 1931, Dirac asked, what if there was a magnetic monopole m? As an electrically charged particle moves in the presence of a magnetic monpole, there's a singularity if the path hits the monopole, and otherwise is locally constant, but can depend on the path. In particular, if two paths π_1 and π_2 differ only by going different ways around m, the difference in their actions is $I_2 - I_1 = \hbar eg$. In particular, if the particle travels in a loop ℓ, the action depends on the winding number $n(\ell)$ of the loop:

$$I_\ell = n(\ell)\hbar eg.$$

This is a topological invariant, and a discrete one: we exponentiate $e^{2\pi i I_\ell}$, hence $\hbar eg \in \mathbb{Z}$! This is the first instance of topology appearing in physics.

Dirac thought of this in a surprisingly prescient way, chopping up the integral into a lot of little pieces and integrating over paths, long before the notion of a path integral was ever dreamed of.

Interlude. The beginning of quantum field theory, as discovered by Schwinger, Dyson, Feynman, and Tomonaga, was understood reasonably well from the physical perspective, but they weren't able to put it on mathematical foundations. This was particularly true for Feynman's formalism of the path integral. Impressively, the theoretical methods they developed anyways managed to agree with experiment to a stunning degree of accuracy, coming to a zenith in quantum electrodynamics (QED).

As such, the physicists drifted away from mathematics: they couldn't and didn't use math to shore up their theoretical physics, and didn't need to in order to get amazingly accurate results. They abandoned Dirac's manifesto, and in a sense math and physics divorced until the 1970s.

Yang-Mills theory. Around the 1950s, Yang and Mills wrote down nonabelian gauge theory to understand elementary particles with nonabelian gauge symmetry (e.g. SU_2 or SU_3). This wasn't taken so seriously at first; it took an approach different from the S-matrix philosophy popular at the time. This lasted until about the 1970s, when t' Hooft and others quantized it and managed to make it predictive of the experiments coming from particle accelerators. This began the shift in popularity from the S-matrix-dominant perspective to the prevalence of gauge theory that exists today.

Gauge theory is the quantum theory of principal G-bundles and connections. Mathematicians had also been working on these, but in parallel, and so produced different words for the same concepts.[1] In the 1970s, Simons and Yang were both at Stony Brook, and realized after talking to each other that they had such different words for the same concepts, leading to a paper [WY75] of Wu and Yang that was a dictionary between the two fields!

The Atiyah-Singer index theorem. A third interesting interaction between geometry and physics is the Atiyah-Singer index theorem from the early 1960s. This was all developed in and with mathematics: principal G-bundles, characteristic classes, Dirac operators on manifolds, and more.

The physicists and mathematicians were brought together again by the theory of Yang-Mills instantons. For a Lie group G, one considers a principal G-bundle on a 4-manifold M and its curvature F. Then, one can take the Lie-algebra-valued trace: one is interested in the spaces of solutions related to

$$\mathcal{L} = \int_M \mathrm{tr}(F \wedge (\star F)). \tag{2.1}$$

To understand this properly, one need to understand both the mathematical and physical phenomena behind it. There's also interplay between Euclidean and Minkowski signature — one important input is action-minimizing solutions to Euclidean Yang-Mills in $\mathbb{R}^4$ that either vanish at infinity or have bounded growth of some sort.

The ADHM construction. Atiyah, Drinfel'd, Hitchin, and Manin [AHDM78], four mathematicians, found all of the solutions for $G = \mathrm{SU}_2$. This is impressive on its own, but they used some surprisingly fancy mathematics (Penrose's twistor transform and some algebraic geometry) that was previously not known to be connected to physics. Subsequently, Atiyah gave the Loeb lectures in the Harvard physics department, and this was big news: a mathematician was using geometry to talk to physicists! Even though the Harvard math and physics buildings were near each other, there hadn't been a lot of discussion between the two departments at the time, barring some more traditional mathematical study of PDEs arising in physics.

One surprising fact about these solutions is that even though we want the solutions to be strongly controlled at infinity, the connection does not need to be. You can get a topological invariant called the *instanton number* from the degree of a map from a large S^3 in $\mathbb{R}^4$ to $\mathrm{SU}_2 \cong S^3$. Since $\pi_3(\mathrm{SU}_2) = \pi_3(S^3) \cong \mathbb{Z}$, the homotopy class of this map, written as an integer k, is called the instanton number of the solution. You can also compute it geometrically:

$$8\pi k = \int \mathrm{tr}(F \wedge F).$$

ADHM constructed solutions with arbitrary instanton number.

[1] Both sets of words are still in vogue, even though the mathematicians and physicists are talking to each other again.

Since the Lagrangian (2.1) looks very similar to $k/8\pi$, and for a 2-form F, $\star F = \pm F$, you could ask whether your solutions are *self-dual* ($\star F = F$) or *anti-self-dual* ($\star F = -F$). It turns out there's always a decompositioin

$$F = F_{\mathrm{sd}} + F_{\mathrm{asd}},$$

and

$$\begin{aligned}\|F\|^2 &= \|F_{\mathrm{sd}}\|^2 + \|F_{\mathrm{asd}}\|^2\\ 8\pi^2 k &= \|F_{\mathrm{sd}}\|^2 - \|F_{\mathrm{asd}}\|^2,\end{aligned}$$

so the minimal-action solutions are either self-dual or anti-self-dual.

Anomalies. The next interaction between physics and mathematics arose in the study of anomalies. These are symmetries of the field theory that do not preserve the integration measure in the path integral. The fields are sections of some bundle built from the tangent bundle or spinor bundles (for fermionic theories), or self-dual fields. But in the case of spinor bundles, anomalies popped up.

This led to a question which looks very mathematical: suppose we have a bundle $E \to M \times S^1$, which we can understand as using a symmetry of M to glue $M \times [0,1]$. Choose a B such that $\partial B = M \times S^1$, and we want to extend this structure to B. The anomaly ends up stated in terms of characteristic classes and invariant polynomials of this structure on B. There are specific steps which determine how this acts on the measure, and if they don't vanish, the symmetry of the classical theory is not a symmetry of the quantum theory, and you have an anomaly. This is okay, but there are some where you really need the symmetry to be present at the quantum level, and for these checking the anomaly is an important and useful tool. This differential-geometric perspective on manipulations of the path integral is due to Zumino and collaborators.

In a spinor theory, matter is essentially a section of a spinor bundle tensored with a gauge bundle. Hence it's potentially subject to an anomaly, but one of the remarkable early discoveries in this field is that the anomaly cancels. When people generalized to supersymmetry, this anomaly vanishes for trivial reasons, and has interesting ramifications on 12-manifolds for the type IIB theory. This leads to the famous Green-Schwarz mechanism. In string theory, there are other ways for the anomalies to cancel.

Donaldson's work on Yang-Mills. The ADHM construction works on $\mathbb{R}^4$ and S^4; Donaldson generalized it to arbitrary compact 4-manifolds to produce remarkable results in topology. This is in some ways the opposite to Dirac's manifesto, taking physics and using it to understand mathematics. At least topology, this was probably the first time understanding flowed in that direction.

In 1988, Witten [Wit88] found a physical interpretation of Donaldson's solutions, but strangely, it didn't depend on the metric, leading to the definition of a topological field theory. From the perspective of something like quantum gravity, the absence of metric dependence is crazy, but it has been extremely useful. With more physics input, Seiberg and Witten took a new approach to the Donaldson-Witten TQFT [SW94a, SW94b] which has made some of the computations more straightforward.

These days, there's also the large overlap between the mathematics and physics of topological phases of matter, kicked off by Haldane and Wen's work. Wen was a string theorist before he did condensed-matter, which is probably where he picked up the perspective of geometric methods.

This ping-pong between math and physics is a great perspective to adopt, and hopefully future research in this area will continue to use input from math to understand physics and physics to understand math.

3. Dan Freed: An axiomatic system for quantum mechanics

First, Dan encouraged all of us to look at the notes he posts online: they contain lots more examples of TFTs, and exercises that will probably generate interesting discussion.

Axiom systems for quantum mechanics have been considered for a long time, starting with Dirac, but mathematicial physicists have considered myriad variations on these axioms. The ones we consider will be useful for considerations on Wick rotation that we'll see in later lectures.

We start with a Riemannian manifold (M, g) together with a potential function $V\colon M \to \mathbb{R}$. This at least seems to model a single particle moving on M, but if, e.g. $M = (\mathbb{R}^n)^k$, this system tracks k particles moving in $\mathbb{R}^n$.

We also have time $\mathbb{M}^1$, which is an affine space modeled on the Euclidean line $\mathbb{E}^1$.[2]

The *Lagrangian* of the system is a density representing the total energy of the system: if we let the system evolve from t_0 to t_1, we get a map $\phi\colon \mathbb{M}^1 \to M$ encoding the trajectory of the particle, and the Lagrangian is

$$L = \left(\frac{1}{2}|\dot{\phi}|^2 - \phi^* V\right) |\mathrm{d}t|.$$

From this we derive both classical and quantum physics. Classically, we apply the Euler-Lagrange equations (which in this case reduce to Newton's equations of motion) to determine which geodesics are permitted, leading to the solution space $\mathcal{N} \subset \mathrm{Map}(\mathbb{M}^1, M)$, which obtains a symplectic form from the Lagrangian density.

Quantum mechanics does something different, integrating over the trajectories. There's a space $\mathcal{S}$ of *states*, which are points of $\mathcal{N}$, or more generally probability distributions on $\mathcal{N}$. There's also a space $\mathcal{O}$ of observables. In general, $\mathcal{S}$ is a convex set containing the *pure states* $\mathcal{S}_0$ (the probability distributions concentrated at a point); the rest are called *mixed states*. The observables $\mathcal{O}$ form a complex vector space with a real structure, and in the same way that $\mathcal{N}$ acquires a symplectic form, $\mathcal{O}$ contains a Lie algebra $\mathcal{O}^\infty$; the bracket is called the *Poisson bracket*.

There will also be a particular observable $H \in \mathcal{O}_\mathbb{R}^\infty$ called the *Hamiltonian*. Observing an observable in a given state defines a map from $\mathcal{O}_\mathbb{R} \times \mathcal{S}$ to the space of probability measures on $\mathbb{R}$. One can take the expected value of such a measure, and this is the expected or average value of that observable in that stete. Moreover, the Hamiltonian defines a semigroup of automorphisms of $\mathcal{S}$ and $\mathcal{O}$, which describes the time evolution of this system. There are different perspectives on this, some of which are dual (e.g. the Heisenberg picture vs. the Schrödinger picture).

It turns out that, with this mathematical data, $\mathcal{O}$ is also an associative algebra, even a Poisson algebra, but there doesn't seem to be physical meaning to the multiplication. It's more helpful to think of $\mathcal{O}$ as a vector bundle over $\mathbb{M}^1$; given A_{t_i} in $\mathcal{O}_{t_i}$ (the fiber over time t_i), one can form the correlator $\langle A_{t_1} \cdots A_{t_k}\rangle$, which is an important invariant, often with physical meaning.

Using this, we can formulate an axiom system.

Definition 3.1. A *quantum system* is the following data.

- A complex Hilbert space $\mathcal{H}$.
- The Hamiltonian, a self-adjoint operator $H\colon \mathcal{H} \to \mathcal{H}$.
- The space of pure states $\mathcal{S}_0 = \mathbb{P}\mathcal{H}$, and the space of mixed states

$$\mathcal{S} = \{\rho\colon \mathcal{H} \to \mathcal{H} \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\}.$$

- The space of observables $\mathcal{O}_\mathbb{R}$, the self-adjoint operators on $\mathcal{H}$.
- Time evolution, a semigroup law

$$t \longmapsto U_t = e^{-itH/\hbar}\colon \mathcal{H} \longrightarrow \mathcal{H}.$$

The observation map comes from von Neumann's spectral theorem: given a self-adjoint operator A, one obtains a projection-valued measure π_A on the line. Hence the map sends A and ρ to the probability measyre

$$E \subset \mathbb{R} \longmapsto \mathrm{tr}(\pi_A(E) \circ \rho).$$

With our Riemannian manifold (M, g) as above, you should think of $\mathcal{H} = L^2(M)$ and $H = \Delta_g$.

Example 3.2 (Toric code)**.** This example is relevant to what we'll be thinking about this week. It was introduced by Kitaev [Kit03], albeit not quite in this form. Throughout, d denotes the space dimension.

Let Y be a closed manifold with the structure as a finite CW complex, i.e. finite sets of i-cells Δ^i for each $0 \leq i \leq d$. Let Y^i denote the i-skeleton, the cells of dimensions at most i; then $Y^0 \subset Y^1 \subset \cdots$, and this is a filtration. Let Δ^i denote the set of i-cells of Y.

We'll consider the (discrete) groupoid of "relative principal G-bundles" $\mathrm{Bun}_G(Y^1, Y^0)$, pairs (P, s) where $P \to Y^1$ is a principal G-bundle and $s\colon Y^0 \to P|_{Y^0}$ is a section of P on the 0-skeleton. As a set, this is a product of copies of G indexed by the edges of Y.

[2] You might think the distinction between affine space and a vector space is fussy, but it's different to say "this lecture ends in an hour" and "this lecture ends at 1:00," especially since it ends at 3.

Now we can incorporate this system into our axiomatic framework. The complex Hilbert space of states is actually finite-dimensional:

$$\mathcal{H} \coloneqq \mathrm{Map}(\mathrm{Bun}_G(Y^1, Y^0); \mathbb{C}) \cong \bigotimes_{e\in\Delta^1} \mathrm{Map}(G, \mathbb{C}).$$

The Hamiltonian is

$$H \coloneqq \sum_{v\in\Delta^0} H_v + \sum_{f\in\Delta^2} H_f,$$

where H_v and H_f are terms corresponding to 0- and 2-cells respectively: given a vertex v, let $\varphi_v\colon \mathrm{Bun}_G(Y^1, Y^0) \to \mathrm{Bun}_G(Y^1, Y^0)$ send $(P, s) \mapsto P(P, s_v)$, where

$$s_v(v') = \begin{cases} s(v), v \neq v' \\ 1 + s(v), v = v'. \end{cases}$$

Then,

$$H_v\psi \coloneqq \frac{1}{2}(\psi - \varphi_v^*\psi),$$

and

$$H_f\psi \coloneqq \mathrm{Hol}_{\partial f}(P) \cdot \psi.$$

That is, take the holonomy of P around the boundary of f, which is either -1 or 1, and multiply by that.

From this definition, it's evident that $\operatorname{Spec} H \subset \mathbb{Z}^{\geq 0}$. The space of ground states is $\mathcal{H}_0 = \mathrm{Map}(\mathrm{Bun}_G(Y); \mathbb{C})$. Why is this? We have a correspondence diagram

$$\mathrm{Bun}_G(Y^1, Y^0) \longrightarrow \mathrm{Bun}_G(Y^1) \longleftarrow \mathrm{Bun}_G(Y);$$

if $H_v\psi = 0$, then $\varphi_v^*\psi = \psi$, so ψ cannot depend on the value of the section s at v; dually, if $H_f\psi = 0$, then $\psi = 0$ on all bundles P which have nontrivial holonomy around f. Thus, requiring $H_v\psi = 0$ for all v pushes us forward to $\mathrm{Bun}_G(Y^1)$, and requiring $H_f\psi = 0$ pulls us back to $\mathrm{Bun})G(Y)$. ◀

Relativity tells us that certain approximations of these systems are the same: since $\hbar$ has units of ML^2/T, then low-energy behavior is the same thing as long-time behavior, and using the speed of light c, which has units of L/T, then this is also the same thing as long-range (long-distance). Much of the interesting qualitative behavior of the system (e.g. ergodicity) fits into one of these paradigms, so understanding this behavior (e.g. via the space of ground states) is important, and is something we'll see later this week. One suprising phenomenon is that, though the toric code depends strongly on the lattice, its space of ground states is a purely topological invariant. This is expected behavior of *gapped systems*, those whose Hamiltonians have a gap between their two smallest eigenvalues. Another example of a gapped system is a particle moving on a compact Riemannian manifold, using spectral theory of the Laplacian; compactness is necessary here.

We want to consider families of systems, e.g. for classifying them. This involves forming a moduli space, a space parameterizing geometric objects. Here's a simple example.

Example 3.3. Let V be a real, two-dimensional vector space, so that $\mathrm{Sym}^2 V^*$ is the space of symmetric bilinear forms $V \times V \to \mathbb{R}$. Such a form has a signature: there's a cone of forms with signature ± 2, and the rest have signature 0, along with some degenerate forms Δ. Thus, the moduli space of nondegenerate bilinear forms is $\mathcal{M} \coloneqq \mathrm{Sym}^2 V^* \setminus \Delta$, and its set of connected components, also called the *deformation classes* for the original moduli problem, is given by the signature $\sigma\colon \pi_0\mathcal{M} \to \{-2, 0, 2\}$, and is a bijection. ◀

In general, you have to fix some discrete invariants: signature or Euler characteristic of a geometric object, dimension, etc.

We'll want to form a moduli space of quantum-mechanical systems and determine the deformation classes. In general, this is set up by fixing some data (e.g. dimension), then considering all systems and removing some singularities. The singularities are those where the Hamiltonians are gapless, and are *phase transitions* (exactly as in the phase transitions from ice to water to gas). There are two kinds: in a *first-order phase transition*, one of the eigenvalues is brought down to zero, but the spectrum is still discrete and even gapped: the dimension of the ground state jumps. In a *second-order phase transition*, the energy gap closes, and the ground state is part of the continuous spectrum. For water, all phase transitions are first-order except for the triple point, which is second-order.

So we throw out the phase transitions and, given a dimension d and a symmetry group I, we'd get a moduli space $\mathcal{M}(d, I)$ of lattice systems in dimension d with I-symmetry. We want to compute the set of deformation classes $\pi_0\mathcal{M}(d, I)$.

But there's a lot more to do yet — we haven't defined these lattice models, let alone the moduli space. More concretely, to attack this physical problem mathematically, we need to make a mathematical model F from it, and justify why we believe this is a good model for the physical problem. After this, we can prove theorems about F, then try to apply these theorems to the original problem.

Though we won't construct moduli space, we do get mathematical models and enough information to compute. The approach proceeds by producing a (not yet completely well-defined) map from $\mathcal{M}(d, I)$ to a moduli space of field theories $\mathcal{M}'(d + 1, H)$, where H is some other symmetry group. This map is expected to exist for physical reasons, and we can use $\mathcal{M}'(d + 1, H)$, which we understand better, to make progress on the original problem.

Wick rotation. Let's change gears a bit for the last few minutes.

Recall that time evolution defines for every point $t \in \mathbb{R}$ the unitary operator $U_t = e^{-itH/\hbar}$. Because the Hamiltonian H should be a positive definite operator, we can formally extend this to $\mathbb{C}_-$, the semigroup of complex numbers with nonpositive imaginary part. The function $t \mapsto e^{-it\lambda}$, $\lambda > 0$, conformally maps $\mathbb{C}_-$ into the (closed) unit disc. We end up with a holomorphic semigroup whose limit on the boundary is the unitary group, and it acts by "small" operators (in a sense that they're analytically easy to control). This is a problem-solving technique in much the same way that one uses contour integration to understand problems that are formulated entirely on the real line.

Now, if you look at the ray through $-i$, you get a real contracting semigroup $\tau \mapsto e^{-\tau H\hbar}$, whose "imaginary time" is easier to analytically understand. One might wonder whether restricting to imaginary time is sufficient to understand the system, and for quantum mechanics a little operator theory shows this to be the case. The axiom system we discuss in a few lectures uses Wick rotation in a crucial way.

Axioms for quantum mechanics. Let $\mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^\nabla)$ be the bordism category of oriented Riemannian 0-manifolds (with collars), and $\mathsf{tVect}_\mathbb{C}$ be the category of complex topological vector spaces. Then, one could try to think of quantum mechanics as a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^\nabla) \longrightarrow \mathsf{tVect}_\mathbb{C}.$$

How do we see this? We want to send $\mathrm{pt} \mapsto \mathcal{H}$, and the interval $[a, b]$ to time evolution by $\tau = -i(b - a)$, which is $e^{-\tau H/\hbar}\colon \mathcal{H} \to \mathcal{H}$. The observables also have a geometric interpretation: to observe at x, cut out a small ball around x, producing a bordism starting at the S^0 around x. Hence we get something roughly like $\mathcal{H}^* \otimes \mathcal{H}$, and evaluation defines the observable. (There are some missing words here: we really should let the neighborhood of x shrink to 0 and take a limit, and think about distributions on $\mathcal{H}$.) More generally, to calculate a Wick-rotated correlation function, excise several points, producing maps from $\mathcal{H}^k \otimes (\mathcal{H}^*)^k \to \mathbb{C}$, which gives you the correlation function in question (modulo the same caveats).

We'll generalize this to arbitrary functions to get the story for Wick-rotated quantum field theory in general, and then go back to discuss the relativistic physics that underlines it. For a good reference on all this, see Segal's lectures on this material from about five years ago.

4. Robert Bryant: Symmetries and G-structures

"Sorry... that's the only physics joke I'll make."

The idea for this lecture is that there is a whole collection of geometric structures: complex, almost complex, symplectic, almost symplectic, CR, and more, and we can treat them in a unified way that extends what you've learned about Riemannian geometry. The idea is to look at local invariants and symmetry groups. This perspective was known to Cartan a century ago, but the examples are often newer.

Throughout this lecture, we'll consider geometric structures on an m-manifold M. It'll often be useful to have an auxiliary vector space $\mathfrak{m}$ around, which is a real m-dimensional vector space which we'll think of as a generic tangent space to M.

The bundle of *principal coframes* $\pi\colon \mathscr{F}_M(\mathfrak{m}) \to M$ is the bundle whose fiber at an $x \in M$ is the space of isomorphism $u\colon T_xX \to \mathfrak{m}$. This space is a right $\mathrm{GL}(\mathfrak{m})$-torsor (hence a $\mathrm{GL}_m(\mathbb{R})$-torsor), where if $A \in \mathrm{GL}(\mathfrak{m})$, $u \circ A = A^{-1} \circ u$, so for any Lie subgroup $H \subseteq \mathrm{GL}(\mathfrak{m})$, we can consider a subspace $B \to M$ which is a principal right H-subbundle. An *H-structure* is a section of the bundle $\mathscr{F}_M(\mathfrak{m})/H$.

This formalism captures many different kinds of geometric structures on manifolds.

Example 4.1. Let q be a quadratic form on $\mathfrak{m}$ and $H = \mathrm{O}(\mathfrak{m}, q)$, the orthogonal group preserving q. Then, a point in the coframe bundle $u\colon T_xM \to B$ that's in a principal H-subbundle determines and is determined by a nondegenerate, smoothly varying quadratic form on TM, i.e. a section of $\mathrm{Sym}^2(T^*M)$. Thus, an H-structure is a Riemannian metric. ◁

Example 4.2. Now suppose $J_0\colon \mathfrak{m} \to \mathfrak{m}$ is a complex structure on $\mathfrak{m}$. If we take $H = \mathrm{GL}(J_0, \mathfrak{m})$, then choosing an H-subbundle $H \to B \to M$ is equivalent to choosing an almost complex structure on M. ◁

Example 4.3. Similarly, if $\beta \in \Lambda^2(\mathfrak{m}^*)$ is nondegenerate, then letting $H = \mathrm{Sp}(\beta, \mathfrak{m})$ (the symplectic group preserving this form) we find that H-structures are symplectic structures on M. ◁

The assignment $M \to \mathscr{F}_M(\mathfrak{m})$ is functorial: for diffeomorphisms $f\colon M_1 \to M_2$, we get a map $f_*\colon \mathscr{F}_{M_1}(\mathfrak{m}) \to \mathscr{F}_{M_2}(\mathfrak{m})$ which sends $u \mapsto u \circ (f'(\pi_{M_1}(u)))^{-1}$. This generalizes to H-structures as long as f preserves the H-structure.

The purpose of this talk will be to show why this is interesting and useful. We won't really talk about when H-structures exist: there are topological obstructions, and most even-dimensional manifolds aren't almost complex or symplectic. For a given manifold, it's often not easy to determine when an almost complex structure integrates to a complex structure.

However, homogeneous spaces provide a family of examples with H-structures. Let P be a closed subgroup of a Lie group G and $\eta\colon TG \to \mathfrak{g}$ be the left-invariant 1-form such that $\eta_e = \mathrm{id}_{\mathfrak{g}}$. If $\pi\colon G \twoheadrightarrow G/P$ is the quotient map, then its derivative maps $T_gG \to T_{gP}(G/P)$, and we get a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & T_g(gP) \arrow[r] \arrow[d, "\eta"] & T_gG \arrow[r] \arrow[d, "\eta"] & T_{gP}(G/P) \arrow[r] \arrow[d, "u(g)"] & 0 \\
0 \arrow[r] & \mathfrak{p} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{p} = \mathfrak{m} \arrow[r] & 0.
\end{tikzcd}$$

In this case, G/P has an H-structure, where $H = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{p}}(P) \subset \mathrm{Aut}(\mathfrak{m})$.

Example 4.4. Let $G = \mathrm{SU}(n+1)/\mathrm{U}(n)$, where we map $\mathrm{U}(n) \to \mathrm{SU}(n+1)$ through the map

$$A \longmapsto \begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix}.$$

Let $P = \mathrm{U}(n)$. Then, $G/P = \mathbb{CP}^n$, though this isn't an injective map: the kernel of the embedding is $Z(\mathrm{SU}(n+1)) = \mathbb{Z}/(n+1)$. Hence there's a fiber bundle $G/(\mathbb{Z}/(n+1)) \to B \to \mathbb{CP}^n$, and $\pi_1(B) \cong \mathbb{Z}/(n+1)$. ◁

This is an example where H isn't a subgroup of $\mathrm{GL}(\mathfrak{m})$, though it is a covering group of a subgroup of $\mathrm{GL}(\mathfrak{m})$. One might call these extended H-structures $\widetilde{H} \to H \hookrightarrow \mathrm{GL}(\mathfrak{m})$, where the first map is a finite cover, and we have an $\widetilde{H}$-bundle $\widetilde{B} \to M$ together with a map $\varphi\colon \widetilde{B} \to B \hookrightarrow \mathscr{F}_M(\mathfrak{m})$, where again the first map is a finite cover.

Example 4.5. There are two common choices of $\widetilde{H}$ common in physics: $\mathrm{Spin}(\mathfrak{m})$, which is a double cover of $\mathrm{SO}(\mathfrak{m})$; and $\mathrm{Pin}^+(\mathfrak{m})$ and $\mathrm{Pin}^-(\mathfrak{m})$, which are double covers of $\mathrm{O}(\mathfrak{m})$. ◁

You could use a compact Lie group fiber instead of a finite cover, and these are the more interesting cases, though a few things have to change. In general, using a finite cover at least doesn't really change this story with regards to calculating local symmetries or invariants.

Another fun example is $G = \mathrm{G}_2$ and $P = \mathrm{SU}_3$. In this case $G/P = S^6$, and you can use this to get an SU_3-structure on S^6. The inclusion $\mathrm{SU}_3 \hookrightarrow \mathrm{U}_3$ produces the standard almost complex strucure on S^6.

Distinguishing different H-structures locally. Though you might know how to do this for Riemannian geometry, we're going to talk about a uniform way to do this for all groups. The key topological information is the *soldering form*: if $\pi\colon \mathscr{F}_M(\mathfrak{m}) \to M$ is the projection map, then at a $u \in \pi^{-1}(x)$ in $\mathscr{F}_M(\mathfrak{m})$, then we're provided with an isomorphism $T_xX \to \mathfrak{m}$, so the projection map $T_u\mathscr{F}_M(\mathfrak{m}) \twoheadrightarrow T_xM$ defines a smoothly varying assignment to $\mathfrak{m}$, hence a smooth $\mathfrak{m}$-valued 1-form $\omega \in \Omega^1_{\mathscr{F}_M(\mathfrak{m})}(\mathfrak{m})$, called the soldering form. The same construction serves to define a soldering form ω_H for a manifold with H-structure.

Lemma 4.6. *Let $f\colon M_1 \to M_2$ be a diffeomorphism.*

(1) *Then, $f^*(\omega_2) = \omega_1$, where ω_i is the soldering form on M_i.*
(2) *If f is in addition an isomorphism of H-structures, then $f^*(\omega_{2,H}) = \omega_{1,H}$.*
(3) *If H is connected, the converse to (2) is true.*

Example 4.7. Let $H = \{e\}$, corresponding to a parallelization. Then, $\pi\colon B \to M$ is a diffeomorphism, and so $\omega\colon TM \to \mathfrak{m}$ is preserved by a unique $C\colon M \to \mathfrak{m} \otimes \Lambda^2\mathfrak{m}^*$ satisfying

$$\mathrm{d}\omega = C(\omega \wedge \omega),$$

or in indices,

$$\mathrm{d}\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k.$$

That is, if $f\colon M \to M$ satisfies $f^*\omega = \omega$, then $f^*(C) = C$. You can also check that if C' satisfies $\mathrm{d}C = C'(\omega)$, then $f^*(C') = C'$. These relate to the codimension of the symmetry group: the number of independent such functions is equal to the codimension of the symmetry group. ◀

This procedure allows you to discover what the symmetries of an H-structure are, which by Noether's theorem is a powerful tool for understanding conservation laws in physics.

So we've solved the problem in the trivial case. Great! Now we'll try to reduce nontrivial cases to the trivial cases, following Cartan.

Suppose we have $H \to B \to M$ and $\omega\colon TB \to \mathfrak{m}$ now has a kernel V, then vertical tangent space. Each fiber can be parallelized individually, because they're canonically identified with $\mathfrak{h}$ through left translation $\tau\colon V \to \mathfrak{h}$. So what we need to do is stitch these together into a form.

Definition 4.8. A *pseudo-connection* on B is an $\mathfrak{h}$-valued 1-form $\Theta\colon TB \to \mathfrak{h}$ such that over each $u \in B$, $\Theta|_{\ker(\omega_u)} = \tau_u$.

Cartan just calls this a connection, but because we haven't asked Θ to be H-equivariant, it's not quite what we're looking for.

Definition 4.9. A pseudo-connection Θ is a *connection* if $R_h^*(\Theta) = \mathrm{Ad}(h^{-1})(\Theta)$ for all $h \in H$. (Here R_h is right translation by h.)

This is the standard definition. But to make Cartan's algorithm work, we need to work with pseudo-connections (or restrict to semisimple groups).

First of all, pseudo-connections always exist (assuming M is paracompact and stuff like that), because connections always exist. But we don't just want some connection, we want one guaranteed to be preserved by our notion of equivalence, like C was in the framed case. This motivates us to write down the *structure equation* for a pseudo-connection Θ:

$$\mathrm{d}\omega = -\Theta \wedge \omega + C(\omega \wedge \omega), \tag{4.10}$$

where C depends on Θ. We want to find a way to choose Θ such that C is preserved by any isomorphism of H-manifolds. So if $\overline{\Theta} = \Theta - p\omega$ is some other pseudo-connection, where $p\colon B \to \mathfrak{h} \otimes \mathfrak{m}^*$ is any smooth map, then

$$\begin{aligned} -\overline{\Theta} \wedge \omega + \overline{C}(\omega \wedge \omega) &= -\Theta \wedge \omega + C(\omega \wedge \omega) \\ (\overline{C} - C)(\omega \wedge \omega) &= -(p\omega) \wedge \omega = (\delta p)(\omega \wedge \omega). \end{aligned}$$

To describe δ, observe that $\mathfrak{h} \subset \mathfrak{m} \otimes \mathfrak{m}^*$, and the composition

$$\mathfrak{h} \otimes \mathfrak{m} \hookrightarrow (\mathfrak{m} \otimes \mathfrak{m}^*) \otimes \mathfrak{m}^* \longrightarrow \mathfrak{m} \otimes \Lambda^2\mathfrak{m}^*$$

is our δ. If $\mathfrak{h}^{(1)} \coloneqq \ker(\delta)$ and $H^{0,2}(\mathfrak{h}) \coloneqq \mathrm{coker}(\delta)$, then we obtain an exact sequence

$$0 \longrightarrow \mathfrak{h}^{(1)} \longrightarrow \mathfrak{h} \otimes \mathfrak{m}^* \xrightarrow{\delta} \mathfrak{m} \otimes \Lambda^2\mathfrak{m}^* \longrightarrow H^{0,2}(\mathfrak{h}) \longrightarrow 0. \tag{4.11}$$

This is the key: the kernel and cokernel determine existence and uniqueness of connections satisfying the structure equation: the cokernel determines whether you can modify the pseudo-connection without modifying C, and the kernel controls existence.

Definition 4.12. The map $T\colon B \to H^{0,2}(\mathfrak{h})$ is called the *intrinsic torsion* of B.

For example, if $H = \mathrm{O}(n)$, then one can identify $\mathfrak{h} \subset \mathfrak{m}^* \otimes \mathfrak{m}^*$ with $\Lambda^2(\mathfrak{m}^*)$. This means δ is an isomorphism, so $\mathfrak{h}^{(1)}$ and $H^{0,2}(\mathfrak{h})$ vanish. This tells us something familiar.

Corollary 4.13 (Fundamental theorem of Riemannian geometry)**.** *On any Riemannian manifold (M, g), there exists a unique pseudo-connection Θ_0 such that $\mathrm{d}\omega = -\Theta_0 \wedge \omega$, and in fact Θ is a connection.*

Hence we get everything local in Riemannian geometry: (ω, Θ_0) is a canonical choice of coframings, and

$$\mathrm{d}\Theta_0 = -\Theta_0 \wedge \Theta_0 + R(\omega \wedge \omega),$$

for some $R\colon B \to \mathfrak{h} \otimes \Lambda^2\mathfrak{m}^*$. This is more familiarly known as the Riemann curvature tensor.

Geometrically, T is the obstruction to being able to choose a flat H-structure to first order, i.e. the first derivatives that don't vanish under changes of coordinates. The second-order terms show up in R. Moreover, if a metric is flat to second-order at every point (so $R = 0$), then it's flat.

Example 4.14. Let $H = \mathrm{Sp}(\beta, \mathfrak{m})$, where β is a nondegenerate 2-form on $\mathfrak{m}$. This defines an isomorphism $\mathfrak{m} \cong \mathfrak{m}^*$ allowing us to lower indices, so we can define $\mathfrak{h}^\flat := \mathrm{Sym}^2 \mathfrak{m}^* \subset \mathfrak{m}^* \otimes \mathfrak{m}^*$. Hence δ is a map

$$\delta\colon \mathrm{Sym}^2(\mathfrak{m}^*) \otimes \mathfrak{m}^* \longrightarrow \mathfrak{m}^* \otimes \Lambda^2\mathfrak{m}^*.$$

This is the exterior derivative of a degree-2 polynomial, which is linear. Your quadratic is the derivative of a cubic function, and so the kernel is $\mathfrak{h}^1 = \mathrm{Sym}^3(\mathfrak{m})$. The cokernel is $H^{0,2}(\mathfrak{m}) \cong \Lambda^3(\mathfrak{m}^*)$. So the obstruction to uniqueness is a 3-form, and the only one we have is $\mathrm{d}\beta$. Similarly, a nontrivial kernel means there's no way to choose a canonical connection. If β is closed, you can at least get a flat space, and Darboux's theorem offers a converse. This story is unusual: usually $\mathfrak{h}^{(1)} = 0$, and for semisimple groups, (4.11) splits equivariantly, so you can use this to choose canonical connections (e.g. for an almost Hermitian structure),[3] not just pseudo-connections, for most structures you will run across in real life.

In the symplectic case, $H^{1,2}(\mathfrak{m}) = 0$ implies $H^{*,2}(\beta) = 0$ for all orders: flatness to first order implies flatness to all orders. ◀

If you do this with a unitary group, you'll discover that it does not carry a unique connection.

If you do this with an extended H-structure $\widetilde{H} \to H$, then the invariants arise as pullbacks of those for H, and similarly for the coframe bundle.

Example 4.15. The simplest example where you need a pseudoconnection instead of a connection is in dimension 3 is the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix}.$$

This is an abelian group isomorphic to $\mathbb{R}^2$, but is not reductive as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. In this case, $\mathfrak{h}^{(1)} = 0$, but (4.11) does not split equivariantly, so you get a unique pseudo-connection which is not a connection in general.

Geometrically, this structure is the structure of a full flag $0 \subsetneq L_1 \subsetneq L_2 \subsetneq TM$ plus an isomorphism $L_2/L_1 \cong TM/L_2$. There are connections which match the intrinsic torsion, but they're not canonical, unless the intrinsic torsion vanishes, which it does not always do. ◀

5. Question session

"I'm not going to say what a quantum field theory is. Maybe tomorrow."

After the lectures, we had a discussion/question section about things that confused us. Today, that was gauge theory, classical bordism invariants, the Euler TQFT, Wick rotation, particles and symmetry groups in quantum mechanics, and Lagrangians. People then reviewed several of these topics.

[3] There are in fact multiple choices for canonical connections in the Hermitian case; they're all functorial for diffeomorphisms. Two examples include the Chern connection and the Kähler connection.

5.1. **Dan Freed: Classical bordism invariants and the Euler TQFT.** Any compact manifold Y has a well-defined Euler characteristic $\chi(Y)$. Is this a bordism invariant? Clearly not: the sphere (nonzero Euler characteristic) is bordant to an empty set (zero Euler characteristic). However, you can check that the mod 2 Euler characteristic does define a bordism invariant $\Omega_d \to \mathbb{Z}/2$. In general, this is not an isomorphism; in some dimensions, it's neither injective nor surjective. We might ask if it's possible to categorify this invariant into a TQFT.

Definition 5.1. The symmetric monoidal category of *super vector spaces* $\mathsf{sVect}_{\mathbb{C}}$ is the category given by the following data.

- The objects are complex vector spaces with a $\mathbb{Z}/2$-grading $V = V^0 \oplus V^1$. Equivalently, you could ask for an involution $\varepsilon\colon V \to V$; then, the $\mathbb{Z}/2$-grading comes from its (± 1)-eigenvalues.
- The maps are the even linear maps.
- The monoidal structure is the tensor product. But the symmetric monoidal structure uses the grading, the map $V \otimes V' \to V' \otimes V$ sending

$$v \otimes v' \longmapsto (-1)^{|v||v'|} v' \otimes v.$$

  Here, $|v|$ is 0 if it's in V^0 and 1 if it's in V^1.

The *dimension* of a super-vector space is $\dim V :- \dim V^0 - \dim V^1$.[4]

Thus, a super-vector space is a categorification of the Euler characteristic: instead of just the ranks of homology groups, we remember the groups themselves, sorting them into odd and even pieces.

But we can also turn it into a TQFT.

Example 5.2. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^\times$. Then, there is an n-dimensional TQFT called the *Euler TQFT*

$$\varepsilon_\lambda\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}}$$

which assigns $\mathbb{C}$ to every $(n-1)$-manifold and to any cobordism X multiplication by $\lambda^{\chi(X)}$. This is a simple and important example of a TQFT, and is invertible, in that it factors through the maximal subgroupoid of $\mathsf{Vect}_{\mathbb{C}}$, called $\mathsf{Line}_{\mathbb{C}}$, the groupoid of complex lines and nonzero maps; we'll discuss invertible field theories more later.

This theory satisfies gluing (i.e. is a functor) because the Mayer-Vietoris formula controls how the Euler characteristic changes when you glue across a common boundary. ◀

If you try to generalize this to other bordism invariants, you'll run into a roadblock: the Euler characteristic can be easily defined on a manifold with boundary, but this is less true for things like the signature, Todd genus, etc.

You can also try to use other characteristic-class invariants. For example, you could define for a closed 2-manifold X, the action $\alpha(X) = (-1)^{\langle w_1(X)^2, [X]\rangle}$, but by the Wu formula $w_1^2 = w_2$, so this is again an Euler theory ε_{-1}!

But there are interesting theories. Consider an oriented 8-dimensional invertible theory $\alpha_\lambda\colon \mathsf{Bord}_{\langle 7,8\rangle}(\mathrm{SO}) \to \mathsf{Line}_{\mathbb{C}}$ which to a closed 8-manifold assigns the number

$$\alpha_\lambda(X) := \lambda^{\langle p_1(X)^2 - 17p_2(X), [X]\rangle},$$

which comes from its signature.

5.2. **Andy Neitzke and Dave Morrison: What is gauge theory?** There's a whole mathematical subject called gauge theory, which roughly means anything related to principal G-bundles and equations on them. This was kicked off by the work of Donaldson. A typical problem is to study the moduli space of G-bundles on M with connection whose curvature F_∇ satisfies $F_\nabla = \star F_\nabla$.

In physics, gauge theory is a quantum field theory, which is in a sense a quantization of the above; we want to integrate over the principal G-bundles. The trick is, we need to mod out by gauge equivalence, but somehow this whole story is a little unphysical: a modern perspective on quantum field theory eschews the perspective of "quantize a classical field theory" as possibly missing some or having artifacts. The gauge equivalences come up in computations, but the gauge symmetry somehow isn't observable, and this is

[4] This agrees with the abstract notion of dimension in a symmetric monoidal category.

fundamental. In particular, there are theories which have two descriptions, one gauge-theoretic and one not, so the gauge-theoretic description cannot be invariant.

This was kicked off because in the past 10 or 15 years or so, people have found theories that we can't write down in terms of Lagrangians. These came from other kinds of physics (e.g. string theory). Maybe we'll find better ways of writing down Lagrangians, but people are wondering whether there's a completely different way to characterize these theories which will make the whole business of Lagrangians historical and quaint. Of course, nobody knows how this might go. There are arguments that it's not even possible to write down Lagrangians for them (e.g. that they don't have couplings), but their arguments aren't watertight.

Related to this is the question whether every quantum field theory has a Poisson bracket on its observables. The answer is no, because quantum field theories aren't determined by their point operators.

One example is a six-dimensional example with a self-dual 3-form. This self-dual 3-form is what makes it hard to write a Lagrangian description. However, if you formulate this on $S^1 \times \mathbb{R}^{1,4}$ and do a Fourier expansion on the S^1 and make an effective theory, this turns into an ordinary theory: the self-dual 2-form becomes a connection valued in the Lie algebra of a nonabelian Lie group. But we don't know how to promote this to a self-dual 2-form in 6 dimensions.

5.3. **Andy Neitzke: Why are particles representations of the Poincaré group?** So you have a quantum field theory, and hence a huge Hilbert space acted on by the Poincaré group. There's one state invariant under the action of the Poincaré group, which is (by definition) the vacuum. There's another subspace corresponding to one-particle states, and these are irreducible representation: each type of particle is an irreducible component, because any two universes containing a single particle of the same kind are related by a transformation of the Poincaré group. Such representations are almost always infinite-dimensional. There are lots of other representations (multiple-particle states and so on), of course.

You can also think of the vacuum and one-particle states as the discrete part of the spectrum, and the multi-particle states as a continuum.

In a topological theory, everything is finite-dimensional, so there aren't really particles. Extended notions of locality remember some facts about the excitations, though.

Day 2. August 1

6. Dan Freed: An Axiom System for Wick-rotated QFT

Yesterday, we defined an axiom system for topological field theory, as symmetric monoidal functors

$$F\colon \mathsf{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathsf{Vect}_{\mathbb{C}}.$$

We then described an axiom system for Wick-rotated quantum mechanics, which considered Riemannian manifolds and topological vector spaces:

$$F\colon \mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}^{\nabla}) \longrightarrow \mathsf{tVect}_{\mathbb{C}}.$$

To define an axiom system for Wick-rotated quantum field theory, we simply combine them.

Definition 6.1 (G. Segal)**.** A *Wick-rotated quantum field theory* is a symmetric monoidal functor

$$F\colon \mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^{\nabla}) \longrightarrow \mathsf{tVect}_{\mathbb{C}},$$

where $\mathcal{X}_n^{\nabla}$ is a geometric analogue of a tangential structure.

This seems surprisingly sparse, but works surprisingly well.

We're not going to precisely define the structures $\mathcal{X}_n^{\nabla}$, but instead give several examples.

- Tangential structures like we discussed yesterday: orientation, spin structure, pin^+-structure, etc.
- More geometric structures such as a Riemannian metric, a conformal structure, a volume form, a principal K-bundle with connection (where K is a compact Lie group), and so on.
- Maps to a space M, sections of a fiber bundle, and so on.

In physics, these are all called fields. In this context, they're *background fields*: unlike the fluctuating fields we considered yesterday, they are not integrated over. Though we won't define fields precisely, the key requirement is a sheaf condition: you must be able to glue local data of a field into global data, and all of the examples above satiafy that condition.

One particular special case is when H_n is a Lie group and $\rho_n\colon H_n \to \mathrm{GL}_n(\mathbb{R})$ is a map of Lie groups. For topological field theory, it's important for this map to have finite fibers, though there are some examples for which this doesn't hold, e.g. $H_n = \mathrm{Spin}_n^c$.

We'll see several examples in a later lecture, and for some of them it could be useful to try to fit them into this axiomatic framework, at least heuristically. Until then, we'll show how to extract the usual ingredients of a QFT from this axiom system.

- Let Y be a closed $(n-1)$-manifold with $\mathcal{X}_n^\nabla$-structure, so it's an object in $\mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla)$. Since $\mathcal{X}_n^\nabla$ is not just a topological structure, we need to consider Y with a two-sided collar. This makes composition (gluing) make sense, and allows you to think of everything as n-dimensional. The collar represents an infinitesimal slice of time, and Y represents space. You might imagine undoing the Wick rotation to obtain something with Lorentz signature and quantizing to produce a state space, which is some topological vector space, and this is what $F(Y)$ is.
- If X is an n-manifold and $x \in X$, we can ask about the quantities near x that can be measured. Physicists call these *local observables*, but you can also call them *point observables*. To see these from the functorial perspective, let $S_\varepsilon(x)$ denote the sphere of radius ε around x in X. Then, $S_\varepsilon(x)$ is a manifold with $\mathcal{X}_n^\nabla$-structure, so $F(S_\varepsilon(x))$ is a vector space of data "near x." To make it "at x," we take the inverse limit:
$$\mathcal{O}_x = \varprojlim_{\varepsilon\to 0} F(S_\varepsilon(x)).$$
- Correlation functions have a similar description: a k-point correlation function $\Phi\colon \mathcal{O}_{x_1}\otimes\cdots\otimes\mathcal{O}_{x_k} \to \mathbb{C}$ comes as the inverse limit as $\varepsilon \to 0$ of this data of spheres of radius ε around $x_1,\dots,x_k$. We can think of $X \setminus (B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_k))$ as a bordism from $S_\varepsilon(x_1) \amalg\cdots\amalg S_\varepsilon(x_k) \to \varnothing$, and applying F and taking the limit produces Φ.

If the theory doesn't depend on the metric, meaning it's conformal or topological, you can ignore the limit, because $F(S_\varepsilon(x))$ doesn't depend on ε. This would mean that the space of operators is a state space, a phenomenon called *operator-state correspondence.*

Generally, though, these theories are not scale-independent. Rescaling everything by some constant defines a functor from $\mathsf{Bord}_{\langle n-1,n\rangle}(\mathcal{X}_n^\nabla)$ to itself, and this is the action of the renormalization group. One would like for this to have short-range or long-range limits; the short-range limit, if it exists, would still be scale-independent, and hence a conformal field theory. The long-range limit will be useful in the classification of phases.

Let $\mathbb{M}^n$ denote Minkowski spacetime, an affine space modeled on $\mathbb{R}^{1,n-1}$, which acts on $\mathbb{M}^n$ by translations. $\mathbb{R}^{1,n-1}$ is $\mathbb{R}^n$ with the Lorentz metric $x^0 = ct$ and coordinates $x^1,\dots,x^{n-1}$ with metric
$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - \cdots - (\mathrm{d}x^{n-1})^2.$$
There is a short exact sequence
$$1 \longrightarrow \mathbb{R}^{1,n-1} \longrightarrow \mathrm{Isom}(\mathbb{M}^n) \longrightarrow \mathrm{O}_{1,n-1} \longrightarrow 1.$$
The middle group has four connected components, and $\pi_0\,\mathrm{Isom}(\mathbb{M}^n) \cong \mathbb{Z}/2\times\mathbb{Z}/2$. One of these copies of $\mathbb{Z}/2$ asks whether a given isomorphism preserves or reverses orientation; the other asks whether it preserves or reverses time. There's a group called the *Poincaré group* $\mathcal{P}_n$, which is a double cover of the component of $\mathrm{Isom}(\mathbb{M}^n)$ containing the identity; this is thought of as the symmetry group of the theory.

For non-relativistic quantum field theory, we had a semigroup law $\mathbb{R} \to \mathrm{U}(\mathcal{H})$, telling us how time evolution acts on the state space by unitary operators. In the relativistic case, we have additional symmetries, so ask for a map $U\colon \mathcal{P}_n \to \mathrm{U}(\mathcal{H})$. We can also ask for our Hilbert space to be $\mathbb{Z}/2$-graded: $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. In physics, one says the statistics of particles is controlled by this splitting; $\mathcal{H}^0$ is for bosons, and $\mathcal{H}^1$ is for fermions.

Inside $\mathcal{P}_n$, there's a copy of translations $\mathbb{R}^{1,n-1}$. In the dual picture $(\mathbb{R}^{1,n-1})^*$, the two directions are energy and momentum; there's a lightcone in the energy direction coming from the Lorentz-signature metric on $(\mathbb{R}^{1,n-1})^*$, and we can precisely say that positive-energy means being in the positive (upper) part of the lightcone C_+^*.

There are multiple approaches to axiomatizing quantum field theory; both the Schrödinger and Heisenberg approaches to quantum mechanics generalize to quantum field theory; Heisenberg's approach becomes in the modern picture the theory of factorization algebras applied to quantum physics, but we're focusing on the Schrödinger formalism.

Wick rotation begins with the observation that $U|_{\mathbb{R}^{1,n-1}}$ is the boundary value of a contracting holomorphic semigroup on $\mathbb{R}^{1,n-1} \oplus \sqrt{-1}\mathbb{R}^{1,n-1}_{-} \subset \mathbb{C}^n$, the generalization of the lower half of the plane we discussed yesterday.

Positive energy allows you to extend this to a complexified domain $\mathcal{D}$, a complexification of $\mathbb{M}^n$, and once in $\mathcal{D}$, we can restrict to a Euclidean space $\mathbb{E}^n$. Thus we obtain a positive definite metric. This may have felt vague, but there's a mathematically rigorous theory underlying this.

Symmetry groups. The symmetry group $\mathcal{G}_n$ of our relativistic quantum field theory must act on $\mathbb{M}^n$ by isometries. Thus we know we have a homomorphism $q\colon \mathcal{G}_n \to \mathrm{Isom}(\mathbb{M}^n)$. This is not how it's usually thought about: especially in older groups, one reads that the Poincaré group is a subgroup of $\mathcal{G}_n$, but from our perspective it's more natural as a quotient. Relativistic invariance of the theory means the identity component $\mathrm{Isom}(\mathbb{M})^0$ is in $\mathrm{Im}(q)$.

We're going to make three assumptions on q. Some of these are strict and throw out interesting theories.

(1) If $K := \ker(q)$, we ask that K is a compact Lie group. Segal considered some noncompact groups, and there are interesting examples, but we're not going to consider them. There are also other kinds of symmetries we're ignoring: both supersymmetries (those that exchange the grading, and make K into a super-group), and higher symmetries (more homotopical things, making K into a 2-group or 3-group).

(2) $\mathbb{R}^{1,n-1}$ should lift to a normal subgroup of $\mathcal{G}_n$. This is in line with Klein's *Erlangen* program: we want translation-invariance in our theory, and therefore ask for translations in our symmetry group. With this assumption, we can define $G_n := \mathcal{G}_n/\mathbb{R}^{1,n-1}$.

Since $\mathcal{G}_n$ contains $\mathrm{Isom}(\mathbb{M}^n)^0$, which is noncompact, $\mathcal{G}_n$ isn't compact. But after the quotient by translations, we have an exact sequence

$$1 \longrightarrow K \longrightarrow G_n \longrightarrow O_{1,n-1}.$$

Wick rotation first tells us to complexify this, producing a morphism of group extensions for $\mathcal{D}$:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, hook] & G_n \arrow[r] \arrow[d, hook] & O_{1,n-1} \arrow[d, hook] \\ 1 \arrow[r] & K(\mathbb{C}) \arrow[r] & G_n(\mathbb{C}) \arrow[r] & \mathrm{O}_n(\mathbb{C}). \end{tikzcd}$$

The top row is the real forms of the groups in the bottom row that fix a Lorentz metric. But when we restrict to $\mathbb{E}^n$, we choose different real forms, those fixing a Euclidean metric:

$$\begin{tikzcd} 1 \arrow[r] & K(\mathbb{C}) \arrow[r] & G_n(\mathbb{C}) \arrow[r] & \mathrm{O}_n(\mathbb{C}) \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] \arrow[u, hook] & \mathrm{O}_n. \arrow[u, hook] \end{tikzcd}$$

Notably, all the groups on the bottom row are compact. Compact Lie groups are rigid, and so we can try to enumerate the possibilities.

The first question is to determine the image of ρ_n. Because $\mathrm{O}_{1,n-1}$ contains the image of $\mathrm{Isom}(\mathbb{M}^n)^0$, but $\mathrm{O}_n(\mathbb{C})$ and O_n only have two connected components, one can show that $\mathrm{Im}(\rho_n)$ is either SO_n or O_n. This is determined by whether the system has time-reversal symmetry, which is a particularly important symmetry for condensed-matter systems.

Let's write down some analogues of SO_n and Spin_n.

Definition 6.2. Let SH_n and $\widetilde{SH}_n$ be the Lie groups that fit into the following pullback diagrams:

$$\begin{tikzcd} \widetilde{SH}_n \arrow[r] \arrow[d] & \mathrm{Spin}_n \arrow[d] \\ SH_n \arrow[r] \arrow[d] & \mathrm{SO}_n \arrow[d, hook] \\ H_n \arrow[r, "\rho_n"] & \mathrm{O}_n. \end{tikzcd}$$

The following theorem says, in a few ways, that the symmetry group splits.

Theorem 6.3 ([FH16]). *Assume $n \geq 3$.*

(1) *If $\mathfrak{h}_n$ is the Lie algebra of H_n and $\mathfrak{k}$ is that of K, there's a splitting*

$$\mathfrak{h}_n \cong \mathfrak{o}'_n \oplus \mathfrak{k}$$

together with a Lie algebra isomorphism

$$\dot{\rho}_n \colon \mathfrak{o}'_n \xrightarrow{\cong} \mathfrak{o}_n.$$

(2) $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$, *and there's a $k_0 \in K$ with $k_0^2 = 1$ such that*

$$SH_n \cong (\mathrm{Spin}_n \times K)/\langle(-1, k_0)\rangle.$$

(3) *There's a canonical map $\mathrm{Spin}_n \to H_n$ sending $-1 \mapsto k_0$.*

This is an analogue of the Coleman-Mandula theorem.

There's also a stabilization theory which says that these symmetry groups fit into families: thus we can say spin theory, pin$^-$-theory, etc., rather than a Spin_n-theory, Pin_n^--theory, and so on.

Theorem 6.4 ([FH16]). *For each $m \geq n$, there is a compact Lie group H_m and homomorphisms $i_m \colon H_m \hookrightarrow H_{m+1}$ and $\rho_m \colon H_m \to \mathrm{O}_m$ fitting into a commutative diagram*

$$\begin{tikzcd}
H_n \arrow[r, hook, "i_n"] \arrow[d, "\rho_n"] & H_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\rho_{n+1}"] & H_{n+2} \arrow[r, hook] \arrow[d, "\rho_{n+2}"] & \cdots \\
\mathrm{O}_n \arrow[r, hook] & \mathrm{O}_{n+1} \arrow[r, hook] & \mathrm{O}_{n+2} \arrow[r, hook] & \cdots
\end{tikzcd}$$

such that for each m, $\ker(\rho_m) = K$ and each square is a pullback square.

We can therefore for the colimit $\rho \colon H \to \mathrm{O}$, and H_n is the pullback of ρ along the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{O}$.

This stable version of the symmetry group is called the (H, ρ) *symmetry type*, and similarly we speak of the unstable version, the (H_n, ρ_n) *symmetry type*. In Robert Bryant's lecture, we considered maps $\rho_n \colon H_n \to \mathrm{GL}_n(\mathbb{R})$ with finite fibers, but we're looking at Lorentz symmetry, and hence some things become nicer: the real form after Wick rotation is compact, so ρ_n factors through the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$.

Definition 6.5. Let (H_n, ρ_n) be a symmetry type and X be a smooth manifold. Then, a *differential H_n-structure* on X is the data (P, Θ, φ), where $P \to X$ is a principal H_n-bundle, Θ is a connection on P, and $\varphi \colon \mathcal{B}(X) \to \rho(P)$ is an isomorphism carrying Θ to the Levi-Civita connection.

We'll thus consider Wick-rotated theories with differential H_n-structure. The relevant bordism category is denoted $\mathsf{Bord}_{\langle n-1,n\rangle}(H_n^\nabla)$.

This axiom system is radical from the physics perspective: we've restricted to compact manifolds, and therefore one might guess this precludes its use to study long-range behavior. But as we'll see, this is not true.

7. Max Metlitski: Spin systems

Condensed-matter physicists study something that could be called "quantum zoology," classifying and understanding states and quantum matter. Today we're going to focus on bosonic/spin systems; in the real world you want electrons and hence fermionic systems, but the generalization is not as hard as you'd expect.

We'll start with a lattice system in any dimension; there are *sites* (dots in the lattice, or 0-cells in a implicial structure). This is in a manifold which represents space; there is no time present in this system yet. At each site i, there's a Hilbert space $\mathcal{H}_i$, which is a copy of $\mathbb{C}^m$. The total Hilbert space for the problem is

$$\mathcal{H} \coloneqq \bigotimes_{i \in \Delta^0} \mathcal{H}_i.$$

You can think of this as an approximation of the continuum system, a quantum field theory, which is why we can get away with finite-dimensional Hilbert spaces. For the purposes of low-energy physics, this approximation is especially useful.

The Hamiltonian is a local sum: at each site i, there is some operator O_i which acts on finitely many sites in the vincinity of i (and acting by the identity on all other sites).[5] We'd like these operators to all be the same, in that if φ is an automorphism of the manifold and lattice carrying $i \mapsto j$, then $\varphi^* O_j = O_i$. Then, the Hamiltonian is

$$H = \sum_{i \in \Delta^0} O_i.$$

Example 7.1. Suppose $m = 2$, so at each site i we have a qubit $\mathbb{C}^2 = \operatorname{span}\{e_1, e_2\}$ (the standard basis). One choice for the operators O_i produces the Hamiltonian

$$H = -h \sum_{i \in \Delta^0} \sigma_i^x - J \sum_{\substack{e \in \Delta^1 \\ \partial e = \{i,j\}}} \sigma_i^z \sigma_j^z \tag{7.2}$$

for some $h, J > 0$.

- The first term is some kind of magnetic term.
- The second term is a spin-spin interaction between nearest neighbors.

The σ_i-terms are *Pauli operators*, the generators of $\mathfrak{su}_2$:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Writing σ_i^x means applying σ^x to the $\mathbb{C}^2$ at the site i. These satisfy the commutation relations

$$[\sigma_i^a, \sigma_j^b] = 2i\epsilon^{abc}\delta_{ij}.$$

$\blacktriangleleft$

Let's impose periodic boundary conditions, so this system is on a torus of length $\vec{L}$ (i.e., $\vec{L} = (L_1, \dots, L_d)$, and the length in the x_i-coordinate is L_i). To understand the excitations, consider the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle.$$

Suppose that there's a gap between the smallest eigenvalue E_0 (corresponding to the ground states) and the second-smallest E_1 (corresponding to the lowest-energy excitations), as in Figure 3. Moreover, this is stable under refinement, in that

$$\lim_{L \to \infty} (E_1 - E_0) = \Delta > 0.$$

Thus, this gap is not an artifact of the discretization, but is inherent in the system.

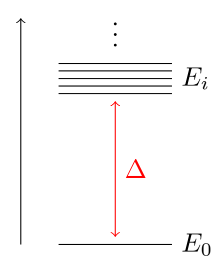

FIGURE 3. The spectrum of a nondegenerately gapped Hamiltonian.

An example ground state $|0\rangle$ is when all of the sites have the same spin, and an example of an excited state with lowest possible energy $|1\rangle$ is when all of the sites but one have the same spin, and the remaining site has opposite spin. When $J = 0$, the energy gap between $|0\rangle$ and $|1\rangle$ is $2h$, which is clearly independent of the length; if you "turn on J" (meaning make it a small positive number), this gap persists, though now it's $\Delta = 2h + O(J)$. This is hard to rigorously prove.

The other possibility is that the first p eigenvalus $E_0, \dots, E_{p-1}$ are 0, and then E_p is nonzero, with a gap Δ, as in Figure 4. This is called a *gapped, degenerate* system. We ask that

$$\lim_{L \to \infty} (E_\alpha - E_\beta) = 0$$

[5] You can generalize these to those that decay quickly, but we're not going to worry about this today.

for $\alpha, \beta \in \{0, \ldots, p-1\}$, and that

$$\lim_{L \to \infty} (E_p - E_0) = \Delta.$$

This can arise accidentally, e.g. if you have two eigenvalues whose values don't coincide generally. If you perturb this system, it returns to a nondegenerate gapped system, and hence is the less interesting case. For example, if $h = 0$, $|0\rangle$ has all spins pointing up, and $|1\rangle$ has all spins pointing down, and both of these have the same energy. The next state $|2\rangle$ will look like $\downarrow\downarrow\uparrow\uparrow\uparrow \cdots$, and its energy is $E_2 = E_0 + 4J$. But if you perturb by a magnetic field in the z-direction, producing

$$H = -J \sum_{\partial e = \{i,j\}} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^z$$

for some small g, E_0 is no longer equal to E_1.

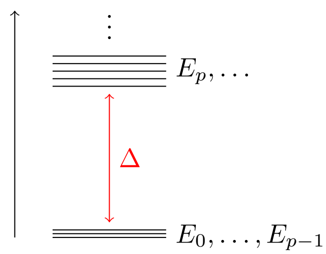

FIGURE 4. The spectrum of a degenerately gapped Hamiltonian.

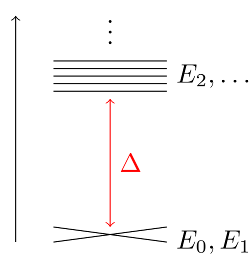

FIGURE 5. A degenerately gapped system can arise "accidentally," in that a small perturbation in some parameter (here on the x-axis) produces a nondegenerately gapped system.

Definition 7.3. Two degenerate ground states α and β are *locally indistinguishable* if for any local operator A_i,

$$\lim_{L \to \infty} \langle \alpha \mid A_i \mid \beta \rangle = C_a \delta_{\alpha\beta},$$

i.e. it's diagonal.

The toric code has local indistinguishability: in dimension d on a torus, its degeneracy (dimension of the space of ground states) is 2^d, and these states cannot be locally distinguished. More generally, this is true for (intrinsically) topologically ordered states. The examples that we know, and which we think might be all examples, are anyon models. In (spacetime) dinension 3, these are well-understood: the Levin-Wen construction [LW05] produces such a model from a modular tensor category C. The idea is that if you have two anyons in this model and braid them once around each other, the wavefunction changes by some number which is dictated by the data of C. This seems bizarre, but is realized in nature by the fractional quantum Hall state.

This is the zoology: you have a general picture of what can and can't happen.

The third possibility for the spectrum is that it's gapless: $\lim_{L \to \infty}(E_\alpha - E_0) = 0$ for infinitely many α. Often, the low-energy field theory is a conformal field theory.

Phases. We've talked a little bit about perturbing the Hamiltonian. When does this change the physics of the system?

Definition 7.4. Two Hamiltonians H_0 and H_1 belong to the same *gapped phase* if there is a smooth path of Hamiltonians $H(t)$ for $t \in [0,1]$ such that $H(0) = H_0$, $H(1) = H_1$, and $H(g)$ is gapped for every g.

We allow degenerately gapped systems.

There is a privileged phase, called the *trivial phase* or *product phase*, represented by the Hamiltonian (7.2) for $J = 0$. More generally, if $J \ll h$, the Hamiltonian belongs to the trivial phase. The name "product phase" highlights that each site is in the same state ψ, i.e. $|0\rangle = |\psi\rangle \otimes \cdots \otimes |\psi\rangle$.

Definition 7.4 is a nice definition, but doesn't allow us to change the Hilbert space. Let's stabilize it by permitting one to throw away degrees of freedom corresponding to a trivial phase. Namely, if $H_0 \colon \mathcal{H}_0 \to \mathcal{H}_0$ and $H_1 \colon \mathcal{H}_1 \to \mathcal{H}_1$ are two systems such that H_0 is in the trivial phase, we can couple them together (also called "stacking") and get a new system $H \coloneqq H_0 + H_1$ acting on $\mathcal{H} \coloneqq \mathcal{H}_0 \otimes \mathcal{H}_1$. We say that H_1 and H are in the same phase.[6]

When $\mathcal{H}_0$ and $\mathcal{H}_1$ are states for different lattices, defining coupling is a little more complicated, but can still be done.

Invertible gapped phases. Let H be a gapped Hamiltonian that is not in the trivial phase.

Definition 7.5. H is in an *invertible phase* if there's a Hamiltonian $\overline{H}$ such that the system produced by coupling H and $\overline{H}$ together is in the trivial phase, i.e. $H + \overline{H}$ acting on $\mathcal{H} \otimes \overline{\mathcal{H}}$ is in a trivial phase.

The nice thing about invertible gapped phases is that they form an abelian group, under the group operation of stacking. The identity is the trivial phase, and the inverse of H is $\overline{H}$ as above.

Another nice thing about (nondegenerately gapped) invertible phases is that they have a unique ground state on every closed manifold, in particular on the torus. This is because when you stack H and $\overline{H}$, it deforms to a trivial phase, which has a single ground state, so since we deformed without closing the energy gap, there has to only be one ground state before deformation. In particular, the anyon models are generally not invertible.

In the real world, though, we can't really impose periodic boundary conditions, and thus we have to consider systems with boundaries. There are lots of choices for terminating the Hamiltonian at the boundary, leading to notions of edge modes. At least for invertible $(2+1)D$ systems (without symmetry), no matter how you terminate the boundary, it's gapless, which is somewhat disconcerting. In other words, for infinitely many $\alpha > 0$, it's possible to get $|\alpha\rangle = A|0\rangle$ for a local operator A living at the boundary. So you end up with a conformal field theory on the boundary that's chiral (with left and right central charges, whose difference vanishes mod 8). In $(1+1)D$, by contrast, the chiral central charge is always 0. One says that (back in $(2+1)D$) the boundary is anomalous. This is sort of the meaning of an anomaly: an anomalous system is one that can only live on the boundary of a higher-dimensional bulk.

The groups of invertible phases in low dimensions have been calculated, and some are given in Table 1. The generators for the nonzero groups are known:

- The generator of the group of fermionic phases in $(1+1)D$ is the Majorana wire [Kit01], which has been realized physically [MZF$^+$12, DYH$^+$12, DRM$^+$12, FVHM$^+$13, RLF12].
- The generator of the group of fermionic phases in $(2+1)D$ is the $p + ip$ superconductor. Twice the generator is in the same phase as the $\nu = 1$ integer quantum Hall effect.
- The generator of the group of bosonic phases in $(2+1)D$ is the $\nu = 8$ integer quantum Hall effect.

	$(1+1)D$	$(2+1)D$	$(3+1)D$
bosons	0	$\mathbb{Z}$	0
fermions	$\mathbb{Z}/2$	$\mathbb{Z}$	0

TABLE 1. Groups of invertible phases. These have been calculated in several different ways, and one reference is [FH16], which also computes examples with symmetry groups.

[6] When we say $H_0 + H_1$, we mean more precisely $H_0 \otimes 1 + 1 \otimes H_1$, which does actually act on $\mathcal{H}_0 \otimes \mathcal{H}_1$.

8. Dan Freed: Classification theorems

One advantage of axiom systems is that they allow you to classify all objects satisfying the axioms. Sometimes there's a uniqueness result, e.g. $\mathbb{R}$ is the unique complete ordered field (up to isomorphim). Other times, uniqueness fails, such as for knot theory, but the results are still interesting. In this lecture, we're going to discuss some classification theorems for field theories in low dimensions; most will be topological, but one won't.

Morse theory. First let's recall a little bit of Morse theory. Let M be a manifold M, and recall that a function $f\colon M \to \mathbb{R}$ is *Morse* if every critical point of f is nondegenerate (has a nonsingular Hessian).

Lemma 8.1 (Morse)**.** *If $p \in M$ is a critical point of f, then in a neighborhood of p,*

$$f = (x^1)^2 + \cdots + (x^{n-r})^2 - (x^{n-r+1})^2 - \cdots - (x^n)^2 + f(p).$$

In the above situation, we say the *index* of p is r. This means that the neighborhood of p looks like $D^r \times D^{n-r}$, and in particular is a bordism from $S^{r-1} \times D^{n-1}$ to $D^r \times S^{n-r-1}$. One can use this to make a *Kirby graphic* for f, plotting the critical values and the indices of their respective critical points, as in Figure 7. If each critical value is the image of a unique critical point, f is called *excellent.* The preimage of the neighborhood of such a critical point is called an *elementary bordism*; using an excellent Morse function, any bordism is a composition of elementary bordisms. Inside the space of all functions on a bordism X, the functions which are not excellent Morse functions are a singular set, so we can always make this decomposition.

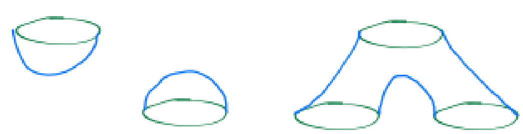

Figure 6. The four elementary oriented bordisms in 2D. From left to right, these are denoted u, $u^\vee$, m, and $m^\vee$.

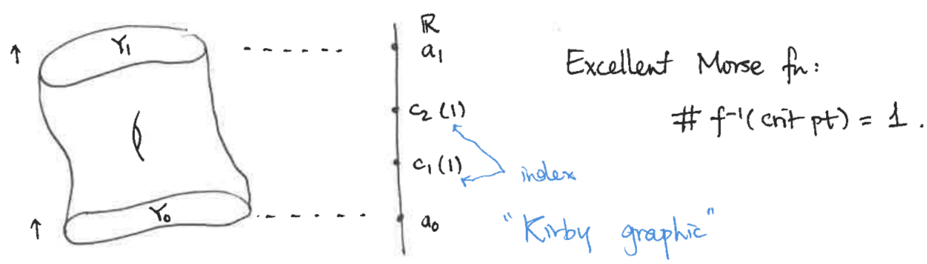

Figure 7. The Kirby diagram for an excellent Morse function.

This allows us to make a generators-and-relations presentation of the bordism category, and therefore classify TFTs. We'll have to understand what happens at the walls, though, which is the domain of Cerf theory.

Definition 8.2. Let $f\colon M \to \mathbb{R}$ be smooth. Then, f has a *birth-death singularity* at p if it can be written in local coordinates as

$$f = (x^1)^3 + (x^2)^2 + \cdots + (x^{n-r})^2 - (x^{n-r+1})^2 - \cdots - (x^n)^2 + f(p).$$

Definition 8.3. Let $f\colon M \to \mathbb{R}$ be smooth.

- f is *good of Type α* if it has a single birth-death singularity and is otherwise excellent Morse.
- f is *good of Type β* if it's excellent Morse except that two critical points have the same critical value.

Theorem 8.4 (Cerf [Cer70]). *Any two excellent Morse functions for a bordism X can be connected by a path f_t of smooth functions that are excellent Morse or good of types α or β, and the non-excellent functions are isolated.*

This differential topology is crucial in the proofs of harder classification theorems such as the cobordism hypothesis [BD95, Lur09].

The algebraic structure we end up getting can be defined over any field, though we're only going to care about $\mathbb{C}$.

Definition 8.5. A *commutative Frobenius algebra* (A, τ) over a field k is a finite-dimensional unital commutative associative k-algebra A together with a map $\tau\colon A \to k$ such that the bilinear map $A \times A \to k$ sending $(x, y) \mapsto \tau(xy)$ is nondegenerate.

Example 8.6.

(1) Let G be a finite group and $A = \mathrm{Map}(G, \mathbb{C})^G$, the central functions in $\mathbb{C}[G]$. This is an algebra under the convolution
$$(f_1 * f_2)(x) = \sum_{x_1 \cdot x_2 = x} f_1(x_1) f_2(x_2).$$
The unit is δ_e, assigning 1 to e and 0 to all other conjugacy classes. The trace is
$$\tau(f) = \frac{f(e)}{|G|}.$$
This A is a commutative Frobenius algebra.

(2) Let M be a compact, oriented manifold. Then, $A = H^*(M; \mathbb{C})$ is a Frobenius algebra with trace evaluation with the fundamental class $[M]$. ◀

Another example of a Frobenius algebra appears in the definition of Khovanov homology.

The classification result below is one of the oldest results in TFT; it was probably first written down by Dijkgraaf, but the proof we provide is due to Moore-Segal.

Theorem 8.7 (Dijkgraaf [Dij89]). *Let $F\colon \mathsf{Bord}_{\langle 1,2\rangle}(\mathrm{SO}_2) \to \mathsf{Vect}_k$ be a TFT. Then, $F(S^1)$ is a commutative Frobenius algebra. Conversely, if A is a commutative Frobenius algebra, there's a TFT F unique up to isomorphism such that $F(S^1) = A$.*

Proof sketch. Given a TFT F, let $A = F(S^1)$ as a k-vector space. Multiplication is F applied to the bordism m in Figure 6, τ is $F(u^\vee)$, and the unit is $1 = F(u)(1) \in A$. You have to check associativity and commutativity, but that follows because the two bordisms corresponding to $m(–, m(–, –))$ and $m(m(–, –), –)$ are diffeomorphic, and similarly for commutativity.

The converse is the harder part. Given a Frobenius algebra A, we can set $F(S^1) := A$. Since $\mathrm{Diff}^{\mathrm{SO}}(S^1) \simeq \mathrm{SO}_2$, which is in particular connected, there's no ambiguity in doing so, even though we didn't pick an orientation. Since any closed 1-manifold is a finite disjoint union of copies of S^1, this determines F on objects. Then, $F(m)$ is the multiplication map, and $F(m^\vee)$ is the comultiplication $A \to A \otimes A$ sending $x \mapsto x x_i \otimes x^i$, where $\{x_i\}$ is a basis for A.

Now, we want to understand what's happening on cobordisms, which is where Cerf theory comes in to show that everything is well-defined. There are four kinds of birth-death singularities that can occur, and by drawing a picture for what happens, you can show these have already been accounted for. ⊠

Figure 8. The four kinds of birth-death singularities that can arise, and how they are resolved.

Now we'll turn to non-topological theories, specifically $(1,2)$-theories depending on an area form, classified by Segal. There's a basic theorem of Moser which reduces the classification of volume forms to the classification of volumes.

Theorem 8.8 (Moser)**.** *Let M be a compact, connected, oriented manifold (possibly with boundary) and ω_0 and ω_1 be two volume forms on M. If*

$$\int_M \omega_0 = \int_M \omega_1,$$

then there is a diffeomorphism $\varphi\colon M \to M$ such that $\varphi^*\omega_1 = \omega_0$.

In general, if X is disconnected, we have one number for every connected component.

Theorem 8.9 (Segal)**.** *There's an equivalence between two-dimensional theories depending on an area form and triples (A, τ, ε_t), where*

- *A is a commutative topological algebra,*
- *$\tau\colon A \to \mathbb{C}$ is a nondegenerate trace, and*
- *$\varepsilon_t \in A$ is a family of elements converging to 1 as $t \to 0$ and such that $\varepsilon_{t_1}\varepsilon_{t_2} = \varepsilon_{t_1+t_2}$ and multiplication by ε_t is trace-class.*

This is roughly an infinite-dimensional version of a Frobenius algebra, though the topology makes things complicated. We might not have a unit on the nose, though, as in the following example.

Example 8.10. If G is a compact Lie group with a bi-invariant metric, we can take $A = C^\infty(G)^G$ with convolution and $\tau(f) = f(1)$ and $\varepsilon_t = e^{-t\Delta\delta_e}$. That is, we mollify the δ-function at e, since it isn't actually an element of A. According to Theorem 8.9, this defines a 2D quantum field theory, called *2D Yang-Mills theory*, which has the action

$$\int_X F_A \wedge \star F_A.$$

When G is finite, we recover the TFT from Theorem 8.7.

We want to look at low-energy and high-energy limits. These correspond to $t \to \infty, 0$ respectively. There are issues as $t \to 0$, because what you get should only depend on the topology of M (since the area form goes to 0), but it has infinite-dimensional state spaces. This doesn't fit into the functorial TFT paradigm, so instead one restricts to cobordisms with at least one incoming component, destroying the need for finite-dimensionality.

The $t \to \infty$ low-energy limit is a little nicer: it's an Euler theory that depends only on the total volume of G. ◀

There are many other classification theorems, e.g. by Stolz, Teichner, and their collaborators, which often use supersymmetry. But we'll give one last one, which is relatively simple: it's in dimension 1, which should be easier than dimension 2.

First, we'll consider the bordism-theoretic analogue.

Theorem 8.11. *Let M be a monoid. Then, the evaluation map*

$$ev\colon \mathrm{Hom}_{\mathsf{Mon}}(\Omega_0(\mathrm{SO}), M) \longrightarrow M$$

sending $F \mapsto F(\mathrm{pt}_+)$ is an isomorphism.

The classification theorem looks very similar, but is categorified. Let C be a symmetric monoidal category, and let

$$\mathsf{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(\mathsf{C}) := \mathrm{Hom}^{\otimes}(\mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1), \mathsf{C}).$$

Though we said these are categories, the axioms of a TFT guarantee that these are groupoids, so what we have is more like a space of TFTs.

Theorem 8.12 (Cobordism hypothesis, 1-dimensional version)**.** *Let $(\mathsf{C}^{fd})^\sim$ denote the maximal subgroupoid of the category of fully dualizable objects in C. Then, the map $F \mapsto F(\mathrm{pt}_+)$ defines an equivalence of groupoids*

$$\mathsf{TFT}^{\mathrm{SO}}_{\langle 0,1\rangle}(\mathsf{C}) \longrightarrow (\mathsf{C}^{fd})^\sim.$$

9. Andy Neitzke: Examples of QFT

This lecture is about examples: quantum field theories that aren't topological, but also their relations to topological ones. There are two broad ways of going from a nontopological theory to a topological one: the first is taking a low-energy limit (as we've seen in a few previous lectures), or Witten's notion of a topological twist. This was the first way TQFTs were constructed, e.g. for Donaldson-Witten theory. The structures in the end are the same, but the journey goes through very different terrain.

Example 9.1 (Free scalar field)**.** The free massive scalar field of mass $m \geq 0$ is the first example of a quantum field theory one learns in school. This works in any spacetime dimension n, which is unusual — most interesting quantum field theories exist in a specific dimension, and the higher you go, the harder it is to write down interesting ones. For example, there are no known interacting quantum field theories in dimension greater than 6. The free scalar field theory is a free theory, though.

This theory can be formulated on a Riemannian or Lorentzian manifold; we'll work in Riemannian signature, and specifically formulate it on $\mathbb{R}^n$ with the usual metric. We're interested in the local operators in this theory, which are all point operators. These are generated by a single operator $\mathcal{O}$. For distinct points $x_1, \dots, x_k \in \mathbb{R}^n$, we want to compute a correlation function,[7] a piece of basic data

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_k)\rangle = ?$$

We'll say that if nothing is inserted, you get $\langle 1\rangle = 1$, and if a single operator is present, $\langle \mathcal{O}(x)\rangle = 0$. What's interesting is the two-point correlation function

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = G(x_1, x_2),$$

which is the Green's function for $-\Delta + m^2$, where Δ is the Hodge Laplacian:

$$(-\Delta_x + m^2)G(x,y) = \delta(x-y).$$

This is a function of $x_1 - x_2$.

The three-point correlation function is again zero, but the four-point function is nontrivial:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = G(x_1,x_2)G(x_3,x_4) + G(x_1,x_3)G(x_2,x_4) + G(x_1,x_4)G(x_2,x_3).$$

The Feynman diagrams for this are nice; see Figure 9.

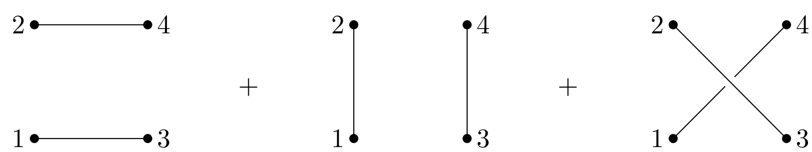

Figure 9. The Feynman diagram for a four-point correlation function.

One can formally write this with a path integral:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{\int_{\mathcal{C}} \mathcal{D}\phi\, \phi(x_1)\phi(x_2)e^{-S(\phi)}}{\int_{\mathcal{C}} \mathcal{D}\phi\, e^{-S(\phi)}}.$$

The denominator exists only for the normalization. Here, $\mathcal{C} = \{\phi \colon \mathbb{R}^n \to \mathbb{R}\}$ and the action is

$$S(\phi) = \int_{\mathbb{R}^n} \left(\|\mathrm{d}\phi\|^2 + m^2\phi^2\right) \mathrm{d}V.$$

Usually, this path integral doesn't make rigorous mathematical sense, but in this case, because the action is qnadratic, this can be made rigorous.

You can use this to approximate the physics of this system at short and long distances: the short-distance approximation is

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{\|x-y\|^{n-2}},$$

[7] We're actually considering normalized correlation functions.

and the long-distance approximation is

$$\langle \mathcal{O}(x), \mathcal{O}(y)\rangle \sim \begin{cases} \dfrac{1}{\|x-y\|^{n-2}}, m = 0 \\ e^{-m\|x-y\|}, m > 0. \end{cases}$$

A codimension 1 manifold has a Hilbert space of states. For $\mathbb{R}^{n-1}$, this Hilbert space is a direct sum of the vacuum, the single-particle states $\mathcal{H}_1$, the two-particle states $\mathcal{H}_1^{\otimes 2}$, the three-particles $\mathcal{H}_1^{\otimes 3}$, and so forth. Thus, $\mathcal{H}_{\mathbb{R}^{n-1}} = \mathrm{Sym}^* \mathcal{H}_1$. This is not something you can expect in general theories. Here, $\mathcal{H}_1$ is an irreducible representation of $\mathrm{ISO}_{1,n-1}$ introduced from the trivial representation of SO_{n-1} or SO_{n-2}. If $p_i \in \mathfrak{iso}_{1,n-1}$ is the translation generator, then $M = \sum_i p_i^2$ is the *Casimir operator* for this theory.

If you look at the spectrum for this theory, there are two cases, depending on whether there's a mass.

- If $m > 0$, then the lowest-energy state after the ground state is at m, then at $4m^2$, and then there are more. So this is a gapped system.
- If $m = 0$, there are energy levels all the way down to the vacuum; this is a gapless system.

As has been said before, one can expect to obtain a topological field theory from a gapped system. This is the action of the renormalization group flow, an action of the semigroup $\mathbb{R}_+$ on the space of QFTs. Namely, $t \in \mathbb{R}_+$ rescales all distances by e^t. In particular, it maps the free scalar theory with mass m to the free scalar theory with mass $e^t m$. Thus it flows from small positive numbers to larger ones, with $m = 0, \infty$ as fixed points.

- $m = 0$ is a free massless scalar theory. The action of $\mathbb{R}_+$ on this theory by automorphisms is nontrivial, sending $\mathcal{O} \mapsto e^{(n-2/2)t}\mathcal{O}$ but not changing the theory.
- At $m = \infty$, the correlation functions are dying off faster and faster, and renormalization can't compensate. The physical interpretation is that the correlation functions are made closer and closer to zero, hence the local operator itself is being crushed out of existence. The mass gap is getting bigger and bigger, so the only physics left is that of the vacuum. So this particular TQFT is probably trivial — though it would be interesting to calculate this explicitly. ◀

It's generally believed by physicists that if you apply renormalization group flow to a gapped theory, you'll arrive at a topological theory in the limit $t \to \infty$. This means a lot more than we've considered: you have to consider the theory on all manifolds in the given dimension. To set this up and dodge counterexamples, one must be careful about which field theories this ansatz applies to.

For any quantum field theory, the question as to whether it's gapped or not is very important. If it is gapped, the second question is: what TQFT do you get by flowing to the infrared? This involves setting up the theory, with the same action and same path integral, on an arbitrary compact Riemannian manifold of the same dimension.

Example 9.2 (Yang-Mills theory [YM54])**.** This 4-dimensional theory depends on more data: fix a compact semisimple Lie group G and a coupling $\tau \in \mathbb{H}$ (the upper half-plane). Conventionally one writes it as a real part and an imaginary part:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

Strictly speaking, g is usually called the coupling.

The configuration space is now $\mathcal{C}$, the space of principal G-bundles with connection on $\mathbb{R}^4$, modulo gauge equivalence, and the action is

$$S = \frac{1}{2g^2}\int \mathrm{tr}\|F\|^2 + \frac{i\theta}{2\pi}\int \mathrm{tr}\, F \wedge F.$$

If G is abelian, this is quadratic, and so this is a free theory much like the previous one. In fact, it's scale-invariant, like the theory of the massless free scalar field. It's still interesting, but a very simple theory to compute in. It's also not gapped. For $G = \mathrm{U}_1$, this is the theory describing electromagnetism, and the fact that we can see photons at low energies is related to the fact that you have a massless particle and energy levels all the way down to the vacuum.

If G is nonabelian, things are different: this is not a quadratic function (coming from the difference in how you write the curvature $F = \mathrm{d}A + A \wedge A$), and is an interacting theory. It is widely, perhaps universally believed, that in this case the theory is gapped: to prove this rigorously is a Millenium Prize problem. There are empirical reasons to believe this, including experimental evidence, such as the fact that the gluons in the SU_3 piece of the Standard Model don't appear at low energies and numerical approximations of the lattice model. For N large, holography provides evidence for SU_N, but this might not be true for SU_2 or SU_3.

If it is gapped, you can further ask which topological field theory you would get. It's not clear what this would be, or even whether it's invertible.

There are supersymmetric variants of Yang-Mills theory for which we do know it's gapped, but it's also not entirely clear what the low-energy theory is. $\blacktriangleleft$

The other way to produce a topological theory out of a quantum field theory is a topological twist. Recall that in a TFT, the n-point function $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\rangle$ must be independent of the metric on M and the positions of the x_i. In a general quantum field theory, the dependence on the metric is nonzero, but an operator $T_{\mu\nu}$ called the *energy-momentum tensor* measures how it changes:

$$\frac{\delta}{\delta g_{\mu\nu}(x)}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\rangle = \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)T_{\mu\nu}(x)\rangle.$$

So if you can produce a way to make $T = 0$, you wind up with a topological field theory (or more generally, $T_{\mu\nu}\sim g_{\mu\nu}$). The low-energy limit we've been discussing is one way to do this, but there's another way. This arises in the context of supersymmetry, where there's a supergroup symmetry, and its Lie superalgebra has an odd generator Q with $Q^2 = 0$.

This implies

$$\langle Q\mathcal{O}\cdots\mathcal{O}\rangle + \langle \mathcal{O}Q\mathcal{O}\cdots\mathcal{O}\rangle + \cdots + \langle \mathcal{O}\cdots\mathcal{O}Q\rangle = 0.$$

Since $Q^2 = 0$, it makes sense to think cohomologically: if we restrict to Q-closed operators, i.e. those where $Q\mathcal{O}_i = 0$, then the n-point function $\langle \mathcal{O}_1\cdots\mathcal{O}_n\rangle$ only depends on the cohomology classes $[\mathcal{O}_i]\in\ker(Q)/\operatorname{Im}(Q)$. If in addition $T_{\mu\nu}$ is Q-exact, then these functions are independent of the metric! This is how Witten [Wit88] described Donaldson theory as a TQFT, starting from $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. The process of changing a theory so that this works, which doesn't affect the theory on flat space but does on curved manifolds, is called twisting.

10. Question session

Today, the things that came up in the discussion session:

- What is the classifying space of principal G-bundles with connection, BG_∇?
- What can you infer about a quantum field theory from the TQFT that is its low-energy limit? How about for a lattice model?
- What *is* quantum field theory?
- What are some typical quantities/qualitative properties one can extract from a QFT?
- What is a Berry phase?

10.1. **Andy Neitzke and Dan Freed: What does it mean to have a QFT?**

There are many different ways to describe a QFT: a Lagrangian description or a description relative to some other system, or through symmetries and local operators, or more. The Lagrangian formulation is a more well-known one: there's a space of fields, an action functional, and the path integral. There are ways to obtain the data of a functorial QFT (not completely rigorously) for manifolds with or without boundary. One way to study it is to form the space of classical solutions, which is a symplectic space, and then follow a prescribed formula to geometrically quantize it and determine the theory. But this is not easy either. To compute the vector space of states, think about an $(n-1)$-manifold as an n-manifold with one infinitesimal direction. Then there's a wave equation to solve. It would be nice to be able to Wick-rotate bordisms from Euclidean to Lorentzian signature directly, and this is ongoing work of Kontsevich-Segal.

But more generally, mathematicians and physicists ask about what QFT is not just to dot the is and cross the ts, but to gain physical understanding. One physicist who likes to think about this is Nathan Seiberg, who has given multiple interesting talks with the title "What is quantum field theory?"[8] His thesis is that there are many starting points and many coincidences, but we still don't know what the right starting point or the universal picture is.

[8] http://scgp.stonybrook.edu/video_portal/video.php?id=389.

10.2. **Many people: What quantities can you obtain from a QFT?** As we've discussed, you can get state spaces and correlation functions. The phenomenologists take the correlation functions and Fourier-transform them, then do a little more processing, to obtain an amplitude measurement described in momenta. This is something you can apply to an experiment in a particle accelerator, and see whether the experimental probabilities match the theoretical ones.

Laboratory scientists deal with something called *effective field theory*, which is only required to be defined at medium and long distances; we'd like to extend them to short distances/high energy, but there are sometimes problems with this, suggesting we need a new way to think about this. In practice, the techniques that have so far been written down don't always apply. So in general, an effective field theory is something you can obtain from a QFT.

In the specific case of two-dimensional conformal field theory, it's possible to get a lot of previously-known mathematical objects out, such as operator algebras or conformal blocks or things. Sometimes CFT is studied by mathematicians for this reason.

The correlation functions for a QFT come in an enormous variety of flavors: point observables, loop observables, and in general an observable for every submanifold of your ambient space, and conceivably one could compute these given a trace or expectation. Related to loops are things called *Wilson line operators* in gauge theory, which computes holonomy around a loop. This depends on the perimeter and area of the loop, but the way in which they do is an important physical invariant of your theory, determining whether or not it's confined.

10.3. **Dan Freed: what kinds of local operators can one obtain from functorial QFT?** We've seen how to get point operators from the functorial QFT perspective, by excising a small sphere and thinking of it as a bordism. What about other operators, such as monopole operators? You can also do this with functorial QFT, it turns out. Start by thinking about the symmetry type $(H_n, \rho_n \colon H_n \to \mathrm{O}_n)$. For example, if $H_n = \mathrm{SO}_n \times \mathbb{T}$ and ρ_n is projection onto SO_n, then the inclusion $\mathrm{SO}_n \hookrightarrow \mathrm{O}_n$, then an H_n-structure on a manifold is an orientation plus a principal $\mathbb{T}$-bundle and a connection (for a quantum field theory and/or differential H_n-structure).

If the bundle is trivial, you can think of the monopole operator as extending into a bulk and acting there. But, e.g., for $n = 3$, there are nontrivial principal $\mathbb{T}$-bundles which do not extend over the bulk, and in this case the monopole operator has to extend in a singular way to the bulk. So the symmetry of the theory carries a lot of information about it.

10.4. **Dan Freed: What is the classifying space for principal G-bundles with connection?** We want this classifying space BG_∇ to be, well, a space such that $[M, BG_\nabla]$ is naturally isomorphic to the set of isomorphism classes of principal G-bundles $P \to M$ with connection. But one can prove no such space exists. So one has to widen the notion of "space" to make this work. For a reference, see [FH13].

A related question is what represents differential forms. It has the same problem, as you want to define a space B such that $\mathrm{Map}(S, B) \cong \Omega^j(S)$, but this doesn't work unless $j = 0$. Instead, you only have the functor Ω^j, and that's what you think of as the "space" BG_∇, and there are ways to make this feel a bit geometric. For more detail, check out [FH13].

Day 3. August 2

11. Dan Freed: Extended Field Theory

> *"There's some Bureau of Standards sphere in Washington, and we have to compare it with our sphere."*

Today's two lectures will focus on extensions and variations of the axioms of QFT. They apply to QFT in general, but by focusing on topological field theories we can obtain a clearer understanding of them. In this lecture, we'll study an extended notion of locality.

Recall that any bordism is a composition of elementary bordisms (e.g. those in Figure 6 for oriented, 2-dimensional bordisms). Therefore it's possible to compute the partition function of a closed n-manifold by cutting it into elementary bordisms. Suppose X decomposes as $X_- \colon \varnothing \to Y$ and $X_+ \colon Y \to \varnothing$. Then, if ξ^i is a basis of $F(Y)$, there are $a^i, b_i \in \mathbb{C}$ such that

$$\begin{aligned} F(X_-) &= a^i \xi_i \in F(Y) \\ F(X_+) &= b_i \xi^i \in F(Y)^*. \end{aligned} \tag{11.1}$$

By functoriality this means

$$F(X) = \sum_i a^i b_i.$$

But physics tells us that field theories have more locality than in just one direction: it should be possible to

`placeholder`

Figure 10. Cutting a manifold in codimension 1.

do the same thing to Y. A Morse function divides it again into elementary bordisms in one dimension lower, say $Y_- : \varnothing \to Z$ and $Y_+ : Z \to \varnothing$. We'd like to say something analogous to (11.1): there should be things $c_i \in F(Z)$ and vector spaces A^i, B_i such that

$$\begin{aligned} F(Y_-) &= A^i c_i \in F(Z) \\ F(Y_+) &= B_i c^i \\ F(Y) &= \bigoplus_i A^i \otimes B_i. \end{aligned}$$

This isn't quite rigorous yet, but it seems like $F(Z)$ needs to be a category.

More concretely, say $n = 2$. Now, a decomposition of the elementary bordisms as elementary bordisms (with corners) in one dimension lower produces two different directions of composition. Implementin this in the bordism category produces a higher category $\mathsf{Bord}_{\langle 0,1,2\rangle}$.

`placeholder`

Figure 11. Cutting a surface into elementary bordisms in two directions.

Example 11.2. As another instance of these kinds of categorical structures, suppose S is a space. Then, its connected components $\pi_0 S$ form a set. You can extract more information from S, however: let $\pi_{\le 1} S$ denote the category whose objects are points of S and whose morphisms $\mathrm{Hom}_{\pi_{\le 1}}(x, y)$ are the homotopy classes of paths from x to y. (We need to restrict to homotopy classes so that composition is associative). Since all paths are reversible, this is a groupoid, and is called the *fundamental groupoid* of S; its isomorphism classes are the set $\pi_0 S$, and the automorphism group of an $x \in S$ is $\pi_1(S, x)$.

We can upgrade this to a higher structure: keep track of the entire set of paths from x to y, and the homotopies between them, but keeping track of these homotopies only up to homotopy. This produces a *2-groupoid* $\pi_{\le 2} S$, called the *fundamental 2-groupoid* of S, whose objects are paths of S, morphisms between x and y are all paths between them, and whose 2-morphisms (the second composition law) are homotopies of paths. In order to make composition associative, we must only take these homotopies up to homotopy. ◀

Remark 11.3. It's possible to keep track of more and more homotopies between homotopies, producing higher groupoids $\pi_{\le k} S$ and even $\pi_{\le\infty} S$, the *fundamental* ∞*-groupoid* of S. Grothendieck's homotopy hypothesis says this remembers all of the homotopical information of X; one can make this precise, or just remember the notion that "∞-groupoids are the same data as spaces." Certainly, spaces are usually easier to work with! ◀

As soon as one wants to work with homotopies of homotopies or multiple composition laws, higher categories are pretty much inevitable. We're not going to be precise about our definitions, as this would be difficult and unilluminating. Nonetheless, here are some of the players.

- For every $m \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$, there is a notion of an *m-category*, where we have objects, morphisms, 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on up to m (or continuing forever if $m = \infty$). These are required to be associative only up to higher morphisms, though making this precise can be a challenge.

- If $0 \le n \le m$, an *(m, n)-category* is an m-category whose k-morphisms are invertible whenever $k > n$. For example, an $(m, 0)$-category is also called an m-groupoid.

These notions of higher category theory snuck into quantum field theory and topological field theory through considerations of extended locality for Chern-Simons theory: how do you determine the value of a partition function on X when you've cut X in two different directions?

Definition 11.4. The *bordism n-category* Bord_n is the symmetric monoidal n-category defined by the following data.

- The objects are 0-manifolds.
- The 1-morphisms are bordisms between 0-manifolds.
- The 2-morphisms are bordisms between 1-morphisms (so 2-manifolds with corners).
- The 3-morphisms are bordisms between the 2-morphisms, and so on.
- The n-morphisms are *isotopy classes* of bordisms of $(n-1)$-morphisms.

As before, the symmetric monoidal structure arises from disjoint union.

There are a few variants of this notion.

- As before, you can define Bord_n with various kinds of tangential structure, by requiring all manifolds and bordisms to have orientation, spin, a principal G-bundle, etc.
- You can also keep going, defining the n-morphisms to be bordisms of $(n-1)$-morphisms, the $(n+1)$-morphisms to be isotopies of n-morphisms, $(n+2)$-morphisms to be diffeomorphisms of the $(n+1)$-morphisms, and so on. This is useful for considering families of manifolds, and the result is a bordism (∞, n)-category.

A field theory with this notion of extended locality should be a symmetric monoidal functor out of Bord_n. But what should the codomain be? $\mathsf{Vect}_{\mathbb{C}}$ does not generalize to a symmetric monoidal (∞, n)-category as easily. There are a few constraints told to us by physics.

- We still want partition functions to be complex numbers, so we can ask for some sort of $\mathbb{C}$-linearity.
- Fermionic and bosonic statistics suggest that we consider something generalizing super-vector spaces.

But we won't always do this, and there are useful constructions that don't meet these criteria.

Definition 11.5. Let C be a symmetric monoidal (∞, n)-category. Then, an *extended field theory* with values in C is a homomorphism (i.e. a symmetric monoidal functor)

$$F\colon \mathsf{Bord}_n(\mathcal{X}_n) \longrightarrow \mathsf{C}.$$

Example 11.6. One somewhat nonphysical choice for C is $\mathsf{Bord}_{n+1}(\mathcal{X}_{n+1})$ — but this is useful. The *dimensional reduction* of a theory is its composition with

$$- \times S^1 \colon \mathsf{Bord}_n(\mathcal{X}_n) \longrightarrow \mathsf{Bord}_{n+1}(\mathcal{X}_{n+1}),$$

which is symmetric monoidal and hence meets the definition. This has been studied in practice [CWZM97, HKM08, Fre09]. ◂

Example 11.7. For something that looks more like a field theory, fix a finite group G and consider finite gauge theory from Example 1.11. The theory F_G sends $S^1 \mapsto \mathrm{Map}(G, \mathbb{C})^G$, the class functions, and on an oriented surface of genus g, the partition function is

$$F_G(X_g) = \sum_{[W] \text{ irreps of } G} \left(\frac{\dim W}{\#G}\right)^{2-2g}.$$

You can generalize this to other fields, but in modular characteristic G-representations are not semisimple and these formulas do not work out so cleanly.

We'd like to extend this theory. The first thing we need is a symmetric monoidal 2-category, and there are two natural choices.

(1) One choice is $\mathsf{Cat}_{\mathbb{C}}$, the 2-category specified by the following data:
  - the objects are $\mathbb{C}$-linear categories,
  - the 1-morphisms $\mathrm{Hom}(\mathsf{C}, \mathsf{D})$ are the $\mathbb{C}$-linear functors $\mathsf{C} \to \mathsf{D}$, and
  - the 2-morphisms $\mathrm{Hom}(F, G)$ are the $\mathbb{C}$-linear natural transformations $F \Rightarrow G$.

The symmetric monoidal structure is coproduct. In this case, the extended theory

$$\widehat{F}_G\colon \mathsf{Bord}_{\langle 0,1,2\rangle} \longrightarrow \mathsf{Cat}_{\mathbb{C}}$$

sends the positively oriented point to Rep_G, the category of complex representations of G.

(2) A second possiblity is the *Morita 2-category* $\mathsf{Alg}_{\mathbb{C}}$:
- the objects are $\mathbb{C}$-algebras A,
- the 1-morphisms $\mathrm{Hom}(A, B)$ are the (B, A)-bimodules ${}_BM_A$. The composition is tensor product: given ${}_CN_B$ and ${}_BM_A$, one can form the (C, A)-bimodule ${}_CN \otimes_B M_A$.
- The 2-morphisms are the intertwiners (bimodule homomorphisms).
- The symmetric monoidal structure is $\otimes_{\mathbb{C}}$.

A 1-morphism from A to B is classically known as a *Morita equivalence*, hence the name of this category. One can also consider a variant on this category where algebras and modules are $\mathbb{Z}/2$-graded, a "super-Morita 2-category."

In this case, the extension of finite gauge theory assigns to the positively oriented point the algebra $\mathrm{Map}(G, \mathbb{C})$, with the convolution product.

Both of these categories are deloopings of $\mathsf{Vect}_{\mathbb{C}}$, in that $\mathrm{Hom}_{\mathsf{C}}(1_{\mathsf{C}}, 1_{\mathsf{C}}) \simeq \mathsf{Vect}_{\mathbb{C}}$ naturally. They are different, but there is a functor $\mathsf{Alg}_{\mathbb{C}} \to \mathsf{Cat}_{\mathbb{C}}$ sending $A \mapsto {}_A\mathsf{Mod}$, its category of left A-modules, and this identifies our two extensions of F_G. ◂

Remark 11.8. There's a variant of this theory where one chooses a central extension $\widetilde{G}$ of G by $\mathbb{T}$ and assigns to a point the category of projective representations of G that extend to representations of $\widetilde{G}$. ◂

In physics, this notion was present in a different form from the 1970s, using extended operators. We've already seen point operators created by shrinking an excised sphere around a point, and considering the bordism $x\colon S^{n-1} \to \varnothing$. But there are also higher-dimensional operators. Let L be a line in X; we can do something similar by removing a small tubular neighborhood, and by locality, it should be possible to cut the line up and compute this on a link of the line, which is an S^{n-2}. In this case, we would obtain a category of operators, and in higher dimensions, you get higher categorical structures. If you work this out in 3-dimensional finite gauge theory, you can see Wilson and t' Hooft operators in this way.

Algebraic structures. You can think of 1-dimensional TFT as encoding multiplication of square matrices: a bordism from the point to itself defines a matrix, and a bordism from four points to two points colliding two together and keeping the others separate is matrix multiplication. This is associative, because the two bordisms that would represent $(-\cdot(-\cdot-))$ and $((-\cdot-)\cdot-)$ are diffeomorphic, but it's not commutative, reflecting the associativity but noncommutativity of matrix multiplication.

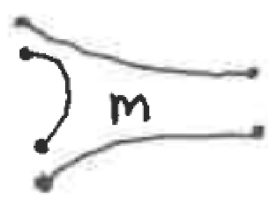

FIGURE 12. Matrix multiplication in 1-dimensional TFT.

In two dimensions, you get a partial notion of commutativity: the algebraic structure comes from the pair-of-pants bordism, and so local operators are associative and partly commutative, but not entirely: it matters how many times you've wound one around the other. In higher dimensions, there are stronger and stronger versions of commutativity, all encoded by various operadic structures. You can do similar things with higher operators.

Next, you might wonder: we saw that point operators are closely related to states. Can we do the same thing with line operators? The answer is yes: we can think of a line L as the Wick-rotated worldline of a particle, so the category you get is the category of particles. This means that an extended low-energy effective field theory sees more of the total theory (continuum or lattice) than you think: this category of particles can be seen by the low-energy theory and does correspond to the lower-energy excitations in the full theory (here we're assuming the system is gapped).

12. Dan Freed: Invertible Field Theories and Stable Homotopy Theory

In this section, we'll discuss extended structure on invertible field theories, both topological and non-topological. This will bring us to the world of stable homotopy theory, and in this lecture we'll explain why.

The symmetric monoidal structure on the bordism n-category is evident even at the 0-categorical level: we have bordism groups, not bordism sets. If we restrict to invertible objects, we find symmetric monoidal groupoid structures.

Definition 12.1. A *Picard groupoid* is a symmetric monoidal groupoid in which every object is invertible under the tensor product.

Example 12.2.

(1) The groupoid of lines over a field k is a Picard groupoid. You can also form the Picard groupoid of super-lines over k, which is the maximal subgroupoid of sVect_k.

(2) The *Brauer groupoid* is a Picard 2-groupoid, the maximal sub-2-groupoid of Alg_k. If we use superalgebras, this is called the *Brauer-Wall groupoid*, and its group of objects over $\mathbb{C}$ is $\mathbb{Z}/2$ but over $\mathbb{R}$ is $\mathbb{Z}/8$, which ties to all sorts of interesting things such as Bott periodicity. ◀

Definition 12.3. A field theory $F\colon \mathsf{Bord}_n(\mathcal{X}_n) \to \mathsf{C}$ is *invertible* if it factors through $\mathsf{C}^\times$, the maximal subgroupoid of C.

In the analogous story for rings, a map $R \to S^\times \hookrightarrow S$ should factor through a quotient of R, and we expect the same thing to happen here: there should be a universal quotient $|\mathsf{Bord}_n(\mathcal{X}_n)|$ which is a Picard (n-)groupoid, and F should factor through a map

$$\alpha\colon |\mathsf{Bord}_n(\mathcal{X}_n)| \longrightarrow \mathsf{C}^\times.$$

Forgetting the Picardness for a moment, we should be able to use the homotopy hypothesis to move to a map of spaces, and then we can figure out what the symmetric monoidal structure does for us.

Recall that we understand the domain, but not the codomain. We can therefore determine $|\mathsf{Bord}_n(\mathcal{X}_n)|$, which is science, but determining $\mathsf{C}^\times$ is art — easier art, because we only need the Picard groupoid, not the entire n-category.

So we have several questions to address in this lecture.

(1) What extra structure does the symmetric monoidal structure on a TFT induce on the spaces $|\mathsf{Bord}_n(\mathcal{X}_n)|$ and $\mathsf{C}^\times$?

(2) How do we obtain reasonable choices for $\mathsf{C}^\times$?

As a first step, let's look at oriented 1-dimensional theories: we should have a map

$$i\colon \mathsf{Bord}_1(\mathrm{SO}_1) \longrightarrow |\mathsf{Bord}_1(\mathrm{SO}_1)|.$$

A Picard (1-)groupoid $\mathcal{G}$ has three invariants that completely characterize it up to symmetric monoidal equivalence.

- $\pi_0\mathcal{G}$ has the structure of an abelian group thanks to the symmetric monoidal product.
- $\pi_1(\mathcal{G}, 1)$ is a nonabelian group; for every object x, $\pi_1(\mathcal{G}, x) \cong \pi_1(\mathcal{G}, 1)$ by conjugating through $-\otimes x^{-1}$.
- The *k-invariant* $\pi_0\mathcal{G} \otimes \mathbb{Z}/2 \to \pi_1(\mathcal{G}, 1)$, which is the composition

$$1 \xrightarrow{\otimes(y\otimes y)} y \otimes y \xrightarrow{\sigma_{y,y}} y \otimes y \xrightarrow{\otimes(y\otimes y)^{-1}} 1,$$

where $\sigma_{y,y}$ is id for the identity of $\mathbb{Z}/2$ and transposition for the nonidentity element.

Example 12.4. We can explicitly calculate these invariants for $|\mathsf{Bord}_1(\mathrm{SO}_1)|$. The first key observation is that if f_1, f_2, g are morphisms in $\mathsf{Bord}_1(\mathrm{SO}_1)$ such that if $f_1 \circ g = f_2 \circ g$, then $i(f_1) = i(f_2)$, because all morphisms in a groupoid are invertible.

We can therefore compute:

- $\pi_0|\mathsf{Bord}_1(\mathrm{SO}_1)| \cong \mathbb{Z}$, which is the number of points, counted with orientation.
- $\pi_1(|\mathsf{Bord}_1(\mathrm{SO}_1)|, \varnothing) \cong \mathbb{Z}/2$, which comes from the fact that $\varnothing \sim (S^1)^{\amalg 2}$, and there's a nontrivial map $\alpha\colon \mathsf{Bord}_1(\mathrm{SO}_1) \to \mathsf{sLine}_\mathbb{C}$ sending the positively oriented point to an odd line and $S^1 \mapsto -1$, so it must be a nontrivial element of π_1.

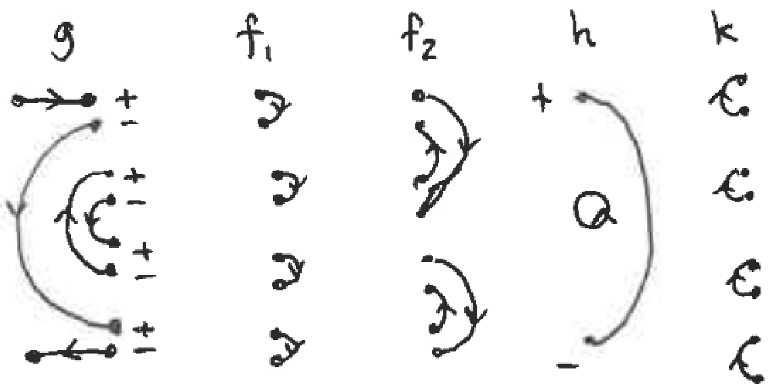

FIGURE 13. Generating morphisms in $\mathsf{Bord}_1(\mathrm{SO}_1)$.

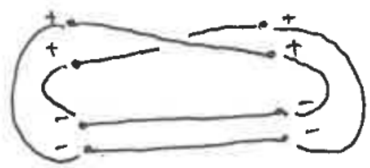

FIGURE 14. $|\mathsf{Bord}_1(\mathrm{SO}_1)|$ has a nontrivial k-invariant.

- Finally, the k-invariant is nontrivial. ◀

The extra structure guaranteed by a symmetric monoidal product turns this space into an infinite loop space. This is a pointed space $T_0 \cong \Omega T_1 \cong \Omega^2 T_2 \cong \cdots$. This adds a lot of structure: $\pi_0 T_0 \cong \pi_1 T_1 \cong \pi_2 T_2$, and therefore $\pi_0 T_0$ is an abelian group (as is π_1, for similar reasons). This is a structure on a space, just as a symmetric monoidal product is a structure on a category.

To turn this condition into data, we make the following definition.

Definition 12.5. A *spectrum* T is the data of spaces $\{T_q\}_{q\in\mathbb{Z}}$ together with maps $s_q\colon \Sigma T_q \to T_{q+1}$.

Since Σ and Ω are adjoints, this is the same data as maps $t_q\colon T_q \to \Omega T_{q+1}$. Suspension comes from crossing with an interval, so we get a direction of time. An Ω*-spectrum* is one in which these maps t_q are homeomorphisms, so that T_0 is an infinite loop space. Any spectrum can be completed into an Ω-spectrum, so infinite loop spaces can be recovered from spectra. (The two notions are in fact equivalent.)

The 1-truncation of a spectrum is exactly a Picard groupoid: it has abelian π_0 and π_1, plus the k-invariant tying the two together.

The point is: an invertible field theory is the data of a map of spectra. This lands us in stable homotopy theory.

Thinking back to Example 12.4, we ask: what's a spectrum E such that $\pi_0 E \cong \mathbb{Z}$, $\pi_1 E \cong \mathbb{Z}/2$, and there's a nontrivial k-invariant? One choice is the *sphere spectrum* S^0, for which $T_q = S^q$, and the bonding maps are the identifications $\Sigma S^q \cong S^{q+1}$. $\pi_0 S^0 = \mathbb{Z}$, given by the degree of a map $S^n \to S^n$, and $\pi_1 S^0 = \mathbb{Z}/2$, generated by the Hopf fibration $\eta\colon S^3 \to S^2$. This has infinite order, but its suspension $\Sigma\eta\colon S^4 \to S^3$ has order 2. Finally, you can check that the k-invariant is nontrivial.

Given a spectrum T, one can pass to its homotopy groups $\bigoplus_q \pi_q T$, which is a $\mathbb{Z}$-graded abelian group. Negative homotopy groups are possible. Thus a spectrum is a refinement of a $\mathbb{Z}$-graded abelian group, but the k-invariants assembling the graded pieces together are additional data.

Now we want to apply this to $\alpha\colon |\mathsf{Bord}_n(\mathcal{X}_n)| \to \mathsf{C}^\times$, and determine what spectra correspond to the domain and codomain. There's a lot to say here, and we won't be able to say all of it. First some more examples of invertible field theories.

Example 12.6.

(1) The Euler theory extends: given a $\mu \in \mathbb{C}^\times$, we have an invertible field theory e_μ which to a closed n-manifold X, assigns $\varepsilon_\mu(X) = \mu^{\chi(X)}$, to a closed $(n-1)$-manifold Y assigns $\mathbb{C}$, and is trivial in all higher codimensions.
(2) Kervaire defined an invariant for $n = 4\ell + 1$ given by exponentiating the *Kervaire semicharacteristic*:

$$\kappa(x) := (-1)^{\sum_i b_{2i}(X)},$$

where b_j is the j^{th} Betti number of X. This defines an extended invertible field theory $\mathsf{Bord}_n \to \mathsf{sLine}_{\mathbb{C}}$ in the same way as the Euler theory, and in dimension 1 is the theory we used to prove $\pi_1|\mathsf{Bord}_n(\mathrm{SO}_1)|$ is nontrivial. ◂

Theorem 12.7. *Let $\alpha\colon \mathsf{Bord}_{\langle n-1,n\rangle}(\mathrm{SO}_n) \to \mathsf{sLine}_{\mathbb{C}}$ be an invertible field theory.*
(1) *If $\alpha(S^n) = 1$, then $[\alpha]\colon \Omega_n(\mathrm{SO}_n) \to \mathbb{C}^\times$ sending $X \mapsto \alpha(X)$ is a bordism invariant.*[9]
(2) *In general, $[\alpha]\colon \Omega_n^{\mathrm{MT}}(\mathrm{SO}_n) \to \mathbb{C}^\times$ defines a cobordism invariant in the same way.*

We need to define this more exotic bordism theory.

Definition 12.8. A closed n-manifold X is *MT null bordant*[10] if there exists a compact $(n+1)$-manifold W such that $\partial W = X$ and a rank-n vector bundle $E \to W$ such that $E|_X \cong TX$ and E is stably equivalent to TW.

So this is a stricter condition that the usual notion of bounding ("Thom null bordance").

The basic link between manifolds and homotopy theory is the *Pontrjagin-Thom construction*, beautifully exposited in Milnor's little book on differential topology. Pontrjagin considered a map of spheres $f\colon S^{n+q} \to S^q$. Suppose $p \in S^q$ is a regular value of f and $X := f^{-1}(p)$, which is a closed n-manifold in S^{n+q}. Pulling back T_pS^q, we obtain a normal framing on X^n. This construction extends to an isomorphism

$$[S^{n+q}, S^q] \xrightarrow{\cong} \Omega_n^{\mathrm{fr}}(S^{n+q}),$$

and we can replace S^{n+q} with any closed manifold M. The inverse map has a beautiful description: given a nroamlly framed manifold X, we take a tubular neighborhood and collapse everything else outside of it, which crushes to a q-sphere; this map is called *Pontrjagin-Thom collapse*.

The colimit as $q \to \infty$ has a geometric description as embedding in high-dimensional affine spaces, so we end up with a stable version: $\pi_n S^0 \cong \Omega_n^{\mathrm{fr}}$: the framed cobordism groups are the stable homotopy groups of the spheres.

Generally manifolds aren't framed, so if we embed it in a big sphere (using Whitney's theorem), the normal bundle isn't framed. Instead it determines a classifying map to $B\mathrm{GL}_q(\mathbb{R})$ and do a Pontrjagin-Thom collapse to the classifying bundle over $B\mathrm{GL}_q(\mathbb{R})$, producing the Thom space of $B\mathrm{GL}_q(\mathbb{R})$ for this bundle. Thus we get a map from S^{n+q} to this Thom space. Stabilizing, we get a map of spectra from S^0 to an object called the *(unoriented) Thom spectrum MO*. In particular, $\Omega^n \cong \pi_n MO$. You can do this for any tangential structure, obtaining different Thom spectra.

There's a corresponding construction of *Madsen-Tillman spectra $MT\mathcal{X}_n$* whose homotopy groups are Madsen-Tillman bordism groups. These are the domain spectra.

Theorem 12.9 (Galatius-Madsen-Tillman-Weiss [GMTW09])**.** *There's an equivalence of spectra $|\mathsf{Bord}_n(H_n)| \simeq \Sigma^n MTH_n$.*

So we know the domain for our invertible field theories. What about the codomain?

We want to promote the invariant $[\alpha]\colon \Omega_n^{\mathrm{MT}}(H_n) \to \mathbb{C}^\times$ to a spectrum map $\alpha\colon \Sigma^n MTH_n \to E$ for some E. Well, $\mathrm{Hom}(\Omega_n^{\mathrm{MT}}(H_n), \mathbb{C}^\times)$ is the Pontrjagin dual to $\Omega_n^{\mathrm{MT}}(H_n)$, so we should consider a spectrum-level analogue of the Pontrjagin dual.

There is a spectrum $I\mathbb{C}^\times$ with the universal property that for every spectrum E,

$$[E, I\mathbb{C}^\times] \cong \mathrm{Hom}_{\mathsf{Ab}}(\pi_0 E, \mathbb{C}^\times).$$

Thus our choice for codomain is $\Sigma^n I\mathbb{C}^\times$, and our ansatz is: *an invertible TFT with symmetry group H_n is a morphism of spectra*

$$\alpha\colon \Sigma^n MTH_n \longrightarrow \Sigma^n I\mathbb{C}^\times.$$

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[9]The hypothesis is necessary: consider the Kervaire theory in dimension 1.

[10]The MT stands for Tangential Thom (because M is used for general Thom spectra) or for Madsen-Tillman [MT01], who first considered this theory. It could also stand for Montana.

As far as applications to physics go, there's some magic, as expressed in Table 2. The $\mathbb{C}^\times$ and μ_2 in dimensions

q	$\pi_q S^0$	$\pi_q I\mathbb{C}^\times$
3	$\mathbb{Z}/24$	0
2	$\mathbb{Z}/2$	0
1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{C}^\times$
-1	0	μ_2
-2	0	μ_2
-3	0	μ_{24}

TABLE 2. Small homotopy groups of the spheres and $I\mathbb{C}^\times$.

0 and -1 give you the invariants of the Picard groupoid $\mathsf{sLine}_\mathbb{C}$ and with -2 gives you the Brauer-Wall groupoid, suggesting that this is actually the right codomain to make these invariants work.

Day 4. **August 3**

13. Dan Freed: The Wick-rotated Version of Unitarity

"Is it clear. . . ? It's clear to me."

Yesterday, we got to the point where we defined, or at least motivated, an extended notion of locality and field theories. In the invertible case, we ended up in topology, and made an ansatz that an invertible field theory is a map of spectra.

Let's recall that if (H_n, ρ_n) is a symmetry type, H_n is a compact Lie group and $\rho_n\colon H_n \to \mathrm{O}_n$ is a Lie group homomorphism. By relativistic concerns, the image of ρ_n is either O_n (in the presence of time-reversal symmetry) or SO_n (in its absence). Our ansatz was that an invertible extended TFT of symmetry type (H_n, ρ_n) is a map of spectra $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$. The symmetry type (H_n, ρ_n) has a stabilization (H, ρ), and we will use this information.

Let's talk a little more about these Madsen-Tillman spectra MTH_n. A spectrum is a sequence of spaces, so we need to describe those spaces. For simplicitly, we'll fix $H_n = \mathrm{O}_n$ and $\rho_n = \mathrm{id}$, but versions of this description work for general H_n.

Recall that the *Grassmannian* $\mathrm{Gr}_q(\mathbb{R}^{n+q})$ is the manifold of q-planes through the origin in $\mathbb{R}^{n+q}$. There is a short exact sequence of vector bundles

$$0 \longrightarrow S_q \longrightarrow \underline{\mathbb{R}}^{n+q} \longrightarrow Q_n \longrightarrow 0,$$

where S_q is the tautological bundle: a point in $\mathrm{Gr}_q(\mathbb{R}^{n+q})$ is a q-dimensional subspace $V \subset \mathbb{R}^{n+q}$, and the fiber over the point V is the vector space V. Then, Q_n is the fiberwise quotient of the trivial bundle by S_q; its rank is n.

The quotient of $\mathbb{R}^{n+q}$ at a point V is an n-dimensional vector space, and this varies smoothly, defining an isomorphism $\mathrm{Gr}_q(\mathbb{R}^{n+q}) \to \mathrm{Gr}_n(\mathbb{R}^{n+q})$, and this exchanges S_q and Q_q, and Q_n and S_n.

Including $\mathbb{R}^{n+q} \hookrightarrow \mathbb{R}^{n+q+1}$ as the first $n+q$ coordinates defines a map $i\colon \mathrm{Gr}_q(\mathbb{R}^{n+q}) \to \mathrm{Gr}_{q+1}(\mathbb{R}^{n+q+1})$ defined by summing each subspace with $\mathbb{R}\cdot e_{n+q+1}$. The pullback of $S_{q+1} \to \mathrm{Gr}_{q+1}(\mathbb{R}^{n+q+1})$ along this map is S_q, so we get a pullback diagram

$$\begin{tikzcd} S_q \arrow[r] \arrow[d] & S_{q+1} \arrow[d] \\ \mathrm{Gr}_q(\mathbb{R}^{n+q}) \arrow[r, "i"] & \mathrm{Gr}_{q+1}(\mathbb{R}^{n+q+1}). \end{tikzcd}$$

This shows that if you take the Thom spaces of these vector bundles, you get structure maps $\Sigma\tau(S_q) \to \tau(S_{q+1})$, hence defining a spectrum, which is a Thom spectrum called MO_n. Taking the limit in n defines MO.

The Madsen-Tillman spectrum MTO_n does the same thing with the quotient bundle: inclusion $\mathbb{R}^{n+q} \hookrightarrow \mathbb{R}^{n+q+1}$ defines a map $j\colon \mathrm{Gr}_n(\mathbb{R}^{n+q}) \to \mathrm{Gr}_n(\mathbb{R}^{n+q+1})$ just by including each subspace, and this pulls Q_{q+1}

back to Q_q. We again get a commutative diagram

$$\begin{tikzcd} Q_q \arrow[r] \arrow[d] & Q_q \arrow[d] \\ \mathrm{Gr}_n(\mathbb{R}^{n+q}) \arrow[r, "j"] & \mathrm{Gr}_n(\mathbb{R}^{n+q+1}). \end{tikzcd}$$

Hence we get structure maps between their Thom spaces, and the resulting spectrum is denoted MTO_n.

This construction stabilizes: the map $i\colon \mathrm{Gr}_n(\mathbb{R}^{n+q}) \to \mathrm{Gr}_{n+1}(\mathbb{R}^{n+q+1})$ pulls Q_q back to Q_q, and therefore defines a map $\Sigma^n MTO_n \to \Sigma^{n+1} MTO_{n+1}$, so it's possible to take the colimit of these maps; the resulting spectrum is called MTO.

MTO carries information about a *tangential* O_n-structure, but MO carries information about a *normal* O_n-structure. A tangential O_n-structure uniquely determines a normal one, so $MTO \simeq MO$, but this is not true for more general H_n-structures: the standard example is that Pin^+ and Pin^- are interchanged.

The colimit construction means there's a map $a_n\colon \Sigma^n MTH_n \to MTH$.

Definition 13.1. An invertible field theory $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ is called *stable* if it factors through $a_n\colon \Sigma^n MTH_n \to MTH$.

Since the maps $\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1}$ are fibrations, the obstructions to stability will lie in their (homotopy) fibers.

Lemma 13.2. *There is a fibration*

$$\Sigma^n (BH_{n+1})_+ \longrightarrow \Sigma^n MTH_n \longrightarrow \Sigma^{n+1} MTH_{n+1}.$$

Here, BH_{n+1} is the base of the universal family of H_n-spheres

$$\begin{tikzcd} S^n \arrow[r] & BH_n \arrow[d] \\ & BH_{n+1}. \end{tikzcd}$$

That is, H_n and H_{n+1} act on S^n through the maps to O_n and O_{n+1}.

The codomain. The spectrum $I\mathbb{C}^\times$ is a Pontrjagin dual in the world of spectra; this concept was introduced by Brown and Comenetz [BC76]. In abelian groups, there are two kinds of duality: the *Pontrjagin dual*

$$A^\vee := \mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{C}^\times)$$

and the module-theoretic dual

$$A^* := \mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Z}).$$

These are related through the *exponential exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{e^{2\pi i(-)}} \mathbb{C}^\times \longrightarrow 0. \tag{13.3}$$

If M is a smooth manifold, (13.3) induces a long exact sequence in cohomology.

$$0 \longrightarrow H^1(M;\mathbb{Z}) \longrightarrow H^1(M;\mathbb{C}) \xrightarrow{\exp} H^1(M;\mathbb{C}^\times) \xrightarrow{\beta_{\mathbb{Z}}} H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{C}) \longrightarrow \cdots \tag{13.4}$$

Since $\mathbb{C}^\times$ is abelian, $H^1(M;\mathbb{C}^\times)$ can be interpreted as the group of isomorphism classes of line bundles, or principal $\mathbb{C}^\times$-bundles with a flat connection. This is because these are identified with $\mathrm{Hom}_{\mathsf{Grp}}(\pi_1(M), \mathbb{C}^\times)$, and because $\mathbb{C}^\times$ is abelian, this factors through the abelianization.

So far everything is discrete, but $H^1(M;\mathbb{C}^\times)$ inherits a topology from $\mathbb{C}^\times$, and hence is a Lie group $\mathcal{A}$, and this allows for a beautiful reinterpretation of (13.4): $H^1(M;\mathbb{C})$ becomes identified with the Lie algebra $\mathfrak{a}$ of $\mathcal{A}$, and the map between them is the exponential map, which fits into a long exact sequence

$$0 \longrightarrow \pi_1\mathcal{A} \longrightarrow \mathfrak{a} \xrightarrow{\exp} \mathcal{A} \longrightarrow \pi_0\mathcal{A} = H^2(M;\mathbb{Z})_{\mathrm{tors}} \longrightarrow 0.$$

This is an instance of a more general long exact sequence in homological algebra for any abelian group A:

$$0 \longrightarrow A^* \longrightarrow \mathrm{Hom}(A, \mathbb{C}) \longrightarrow A^\vee \xrightarrow{\pi_0} \mathrm{Ext}^1(A, \mathbb{Z}) \longrightarrow 0.$$

This arises from applying the $\operatorname{Ext}^*(A,-)$ functor to the short exact sequence (13.4), using the fact that $\operatorname{Ext}^1(A,\mathbb{C}) = 0$.

Now we bring this into the world of spectra. We can define $I\mathbb{C}^\times$ to satisfy the universal property

$$\pi_q \operatorname{Map}(B, I\mathbb{C}^\times) = (\pi_{-q}B)^\vee$$

and prove this satisfies Brown representability, hence defines a spectrum $I\mathbb{C}^\times$. If you make this same construction with $\mathbb{C}$, you can show you obtain the Eilenberg-Mac Lane spectrum $I\mathbb{C} = H\mathbb{C}$. Then, define $I\mathbb{Z}$ to be the homotopy fiber of the exponential map $\exp\colon I\mathbb{C} \to I\mathbb{C}^\times$. This is called the *Anderson dual* to $\mathbb{Z}$, and was indeed first studied by Anderson. See Table 3 for a few homotopy groups of these spectra; because the sphere spectrum is connective, $I\mathbb{C}^\times$ and $I\mathbb{Z}$ are *coconnective*: all of their positive homotopy groups vanish. Of course, $H\mathbb{C}$ has a single nontrivial homotopy group, $\pi_0 H\mathbb{C} = \mathbb{C}$.

q	$\pi_q S^0$	$\pi_q I\mathbb{C}^\times$	$\pi_q I\mathbb{Z}$
4	0	0	0
3	$\mathbb{Z}/24$	0	0
2	$\mathbb{Z}/2$	0	0
1	$\mathbb{Z}/2$	0	0
0	$\mathbb{Z}$	$\mathbb{C}^\times$	$\mathbb{Z}$
-1	0	μ_2	0
-2	0	μ_2	μ_2
-3	0	μ_{24}	μ_2
-4	0	0	μ_{24}
-5	0	0	0

TABLE 3. Small homotopy groups of $I\mathbb{C}^\times$ and $I\mathbb{Z}$.

We're interested in computing the group of invertible field theories, $\pi_0 \operatorname{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$, which should correspond to using $\mathbb{C}^\times$ with the discrete topology, but if we care about deformation classes, we should use the complex topology on it. We can avoid this by observing that a deformation factors through $\Sigma^n I\mathbb{C}$, and hence taking deformation classes defines a group homomorphism

$$[\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times] \longrightarrow [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}].$$

Thus we have a classification result.

Theorem 13.5 ([FH16])**.** *There is a one-to-one correspondence between the deformation classes of invertible topological field theories of symmetry type (H_n, ρ_n) and the torsion subgroup of $[\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}]$.*

If you want to consider noninvertible field theories, it ought to be possible to take the entire subgroup, not the torsion subgroup. This theorem has been useful in work of Freed, Hopkins, Telemann, and their collaborators; see [FKS17] for one example.

Unitarity from the homotopical viewpoint. For applications in mathematics, what we've done is already good, but if we want to solve actual problems in physics, the theories we consider must satisfy locality and unitarity. By using extended field theory, we have fully incorporated locality, but what about unitarity in this extended, Wick-rotated context?

Recall that we started with Lorentzian signature $\mathbb{M}^n$, then passed to a complex half-plane $\mathcal{D}$, then to Wick-rotated, Euclidean quantum field theory. Reconstruction theorems justify this by allowing calculations in the Euclidean theory to determine what goes on in the Lorentzian theory, and then one can place it on curved Riemannian manifolds.

The Wick-rotated analogue of unitarity is called *reflection positivity*. These are two different things: reflection is data and positivity is a condition. It's important to keep data and conditions separate, though many books skip over this point. In a later lecture, we will consider an extended notion of positivity, which will also be data.

Unitarity in Minkowski signature is evident in relativistic QFT: the symmetry group acts unitarily, through a map $\mathcal{G}_n \to \mathrm{U}(\mathcal{H})$. When we pass to $\mathbb{E}^n$, it provides data of a hyperplane Π splitting $\mathbb{E}^n$ into two pieces, $\mathbb{E}^n_+$ and $\mathbb{E}^n_-$. Let $\sigma : \mathbb{E}^n \to \mathbb{E}^n$ denote reflection across this hyperplane; we want σ to act by complex conjugation.

This is not part of the symmetry group, but means we can consider the Hilbert space of states attached to the $(n-1)$-manifold Π with a given orientation $\mathfrak{o}$; call this space $\mathcal{H}_{(\Pi,\mathfrak{o})}$.

Precisely, we want an isomorphism

$$\mathcal{H}_{(\Pi,-\mathfrak{o})} \xrightarrow{\cong} \overline{\mathcal{H}}_{(\Pi,\mathfrak{o})}$$

for which all observable quantities (e.g. collections of point operators) $\langle\mathcal{O}\rangle_{\mathbb{E}^n_+} \in \mathcal{H}_{(\Pi,\mathfrak{o})}$ are complex-conjugated under reflection:

$$\langle\sigma\mathcal{O}\rangle_{\mathbb{E}^n_-} = \overline{\langle\mathcal{O}\rangle}_{\mathbb{E}^n_+}.$$

This is data (isomorphism) and a condition (what it does).

Positivity asks that if you apply $\mathcal{O}$ and its conjugate, the result is nonnegative:

$$\langle\mathcal{O}\sigma\mathcal{O}\rangle_{\mathbb{E}^n} \geq 0.$$

This is a condition.

We have a few more things to do here:

- Understand what's going on for general H_n-structures, which isn't too complicated.
- Placing this on curved manifolds, which will be an essential step in formulating it functorially.
- Figuring out the extended analogue of positivity, which is straightforward but does bring us to equivariant spectra.

14. Agnès Beaudry: Computations in Homotopy Theory

Theorem 13.5 tells us that deformation classes of invertible field theories can be computed by homotopy classes of maps between spectra, and in this lecture we'll learn how to actually compute them.

First, though, a brief simplification: after a couple steps, one learns that

$$[MTH, \Sigma^{n+1}I\mathbb{Z}]_{\mathrm{tor}} \cong (\pi_n MTH)_{\mathrm{tor}}.$$

There is a machine for computing this called the *Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow (\pi^s_{t-s}X)\otimes\mathbb{Z}_2.$$

Here,

- Ext is the derived functors of something, which we'll go into later. Thus, it's singly graded, which is the s-grading.
- $H^*(X;\mathbb{Z}/2)$ is also singly graded, corresponding to the t-grading above.
- X is a space or spectrum; if it's a space, we need to use its reduced cohomology. $\pi^s_i X$ is the i^{th} stable homotopy group of X (the same as ordinary homotopy groups for spectra).
- $\mathcal{A}$ is a graded algebra called the *Steenrod algebra*; we'll say more about it soon.
- $\mathbb{Z}_2$ is the 2-adic integers. Thus we retain information about $\mathbb{Z}$- and $\mathbb{Z}/2$-summands, and lose everything else.

The Steenrod algebra is a graded, noncommutative algebra over $\mathbb{F}_2$. Associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

there's a long exact sequence in cohomology

$$\cdots \longrightarrow H^*(\text{–};\mathbb{Z}/4) \longrightarrow H^*(\text{–};\mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^1} H^{*+1}(\text{–};\mathbb{Z}/2) \longrightarrow H^{*+1}(\text{–};\mathbb{Z}/4) \longrightarrow \cdots \tag{14.1}$$

which is natural and commutes with the suspension isomorphism $H^*(X) \cong H^{*+1}(\Sigma X)$.

Definition 14.2. A *stable cohomology operation* is a natural transformation between cohomology groups that commutes with the suspension isomorphism. The *Steenrod algebra* $\mathcal{A}$ is the algebra of all stable cohomology operations between $\mathbb{Z}/2$ cohomology, with multiplication given by composition.

$\mathcal{A}$ is a noncommutative algebra, but admits a description as a tensor algebra modulo relations over *Steenrod squares*

$$\mathrm{Sq}^i\colon H^*(\text{–};\mathbb{Z}/2) \longrightarrow H^{*+i}(\text{–};\mathbb{Z}/2)$$

for $i \geq 0$. We learned what Sq^1 is, and $\mathrm{Sq}^0 = 1$, but the rest are hard to describe. The relations between the Steenrod squares are called *Ádem relations*, and are unenlightening to describe in generality, but here are some examples:

$$\begin{aligned}
\mathrm{Sq}^1\mathrm{Sq}^1 &= 0\\
\mathrm{Sq}^1\mathrm{Sq}^2 &= \mathrm{Sq}^3\\
\mathrm{Sq}^2\mathrm{Sq}^3 &= \mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1.
\end{aligned}$$

Combining the latter two relations, we see that

$$\mathrm{Sq}^5 = \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2,$$

and this generalizes.

Fact. $\mathcal{A}$ is generated, as an algebra, by Sq^{2^n} for $n \geq 0$. $\blacktriangleleft$

Example 14.3. As a ring,

$$H^*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^3), \quad |x| = 1.$$

That is, there's a single generator in degree 1 whose cube is zero. This comes from a CW structure on $\mathbb{RP}^2$: its 1-skeleton is an S^1, but then you attach a 2-cell along a degree-2 map. This is described by a diagram

$$\begin{tikzcd}
\partial D^2 \arrow[r, hook] \arrow[d, "\cdot 2"] & D^2 \arrow[d]\\
S^1 \arrow[r] & \mathbb{RP}^2.
\end{tikzcd}$$

We can describe this (both the cohomology ring and the CW structure) by a diagram, as in Figure 15. The proof that $\mathrm{Sq}^1 x = x^2$ comes directly from (14.1), and says geometrically that the 2-cell is attached by multiplication by 2. $\blacktriangleleft$

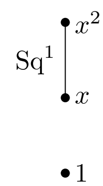

FIGURE 15. The cohomology of $\mathbb{RP}^2$: $\mathrm{Sq}^1 x = x^2$.

Example 14.4. There's a very similar story for $\mathbb{CP}^2$.

$$H^*(\mathbb{CP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^3), \quad |y| = 2,$$

and there's a 0-cell, a 2-cell, and a 4-cell, and the 4-cell is attached by the Hopf map η:

$$\begin{tikzcd}
\partial D^4 \arrow[r, hook] \arrow[d, "\eta"] & D^4 \arrow[d]\\
\mathbb{CP}^1 \arrow[r] & \mathbb{CP}^2.
\end{tikzcd}$$

In this case, $\mathrm{Sq}^2 y = y^2$, and Sq^2 witnesses the fact that the 4-cell is attached by the Hopf map. The picture is very similar, and given in Figure 16; generally one draws Sq^1 as a straight line, and Sq^i for $i > 1$ as increasingly curvier lines. $\blacktriangleleft$

For computing, the following formulas may be useful.

(1) $\mathrm{Sq}^0 x = x$.
(2) If $|x| = n$ (i.e. $x \in H^n$), then $\mathrm{Sq}^n(x) = x^2$.
(3) if $|x| < n$, then $\mathrm{Sq}^n(x) = 0$.

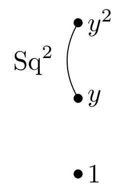

FIGURE 16. The cohomology of $\mathbb{CP}^2$: $\mathrm{Sq}^2 y = y^2$.

(4) The *Cartan formula*

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$

Example 14.5. The cohomology of $\mathbb{RP}^\infty$ is particularly simple: $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$, with $|x| = 1$. The formulas we just wrote down actually completely force the $\mathcal{A}$-module structure on $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$:

$$\mathrm{Sq}^n(x^m) = \binom{m}{n} x^{m+n},$$

where the binomial coefficient is interpreted modulo 2. Thus Steenrod squares of all indices appear. ◀

Jonathan Campbell [Cam17] has written a great exposition on how to calculate with the Steenrod algebra and elaborating on the calculations in [FH16].

Definition 14.6. Let $\mathcal{A}(n)$ denote the subalgebra of $\mathcal{A}$ generated by $\mathrm{Sq}^0, \mathrm{Sq}^1, \dots, \mathrm{Sq}^{2^n}$.

Hence $\mathcal{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$, and it is only 8-dimensional, which is nice. It has a nice pictoral description in Figure 17. This truncated spectrum makes it easier to make calculations. Here's an example.

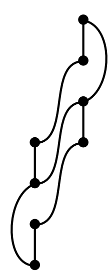

FIGURE 17. The algebra $\mathcal{A}(1)$: the vertical stratification is the degree, the straight lines are Sq^1, and the curvy lines are Sq^2.

Definition 14.7. Let *ko* denote the *connective real K-theory spectrum*, i.e. a truncation of *KO* (real *K*-theory) that removes all negative homotopy groups.

This means that

$$\pi_* ko = \pi_*(B\mathrm{O}) = \begin{cases} \mathbb{Z}, & * = 0, 4 \bmod 8 \text{ and } * \geq 0 \\ \mathbb{Z}/2, & * = 1, 2 \bmod 8 \text{ and } * \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{14.8}$$

ko has a nice structure over the Steenrod algebra:

$$H^*(ko) = \mathcal{A}/\mathcal{A}(1) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2. \tag{14.9}$$

You can use this to prove (14.8), though it's not the only way.

Calculations for Madsen-Tillman spectra. We want to compute $\pi_* MTH$.

Fact. There is a homotopy equivalence
$$MTH \simeq MSpin \wedge X,$$
where X is a suspension of a familiar Thom spectrum. ◀

For example, if $H = \mathrm{Pin}^-$, then
$$MTPin^- \cong MPin^+ \cong MSpin \wedge \Sigma^{-1} MO(1) \simeq MSpin \wedge \Sigma^{-1}\mathbb{RP}^\infty.$$
Concretely, by $\Sigma^{-1}MO(1)$, we mean the Thom spectrum of the tautological bundle over $B\mathrm{O}_1 \simeq \mathbb{RP}^\infty$.

Using the Künneth formula, we can conclude
$$H^*(MTH) \cong H^*(MSpin) \otimes H^*(X),$$
and if we only care about low degrees (as is the case in physics, where we care the most about physically realizable dimensions), there's a convenient reduction available.

Theorem 14.10 (Anderson-Brown-Peterson [ABP66])**.** *As $\mathcal{A}$-modules, $H^*(MSpin) \cong H^*(ko)$ for $* < 8$.*

And by (14.9), we know what the right-hand side is.

Now we can simplify the input term to the Adams spectral sequence, at least in low degrees: for $t - s < 8$,
$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MTH;\mathbb{Z}/2),\mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}/\mathcal{A}(1) \otimes H^*(X;\mathbb{Z}/2),\mathbb{Z}/2).$$
By a change-of-rings formula, we get Ext over a much simpler algebra:
$$\cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2).$$
This change-of-rings isomorphism a consequence of a more familiar fact, that $\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M,N)$.

The next step is to compute Ext by forming a resolution as usual. The s-degree increases and each term is a graded $\mathbb{F}_2$-vector space, in fact a projective $\mathcal{A}(1)$-module (free modules suffice), graded in the t-degree:
$$0 \longleftarrow H^*(X) \longleftarrow \mathcal{A}(1) \otimes_{\mathbb{F}_2} Q_0 \longleftarrow \mathcal{A}(1) \otimes_{\mathbb{F}_2} Q_1 \longleftarrow \mathcal{A}(1) \otimes_{\mathbb{F}_2} Q_2 \longleftarrow \cdots$$
where each Q_i is a graded $\mathbb{F}_2$-vector space (in the t-grading), so these are indeed free $\mathcal{A}(1)$-modules. Let
$$Q^i \coloneqq \mathrm{Hom}_{\mathcal{A}(1)}(\mathcal{A}(1) \otimes_{\mathbb{F}_2} Q_i, \mathbb{Z}/2).$$
Then, Ext is calculated as
$$\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) = H^s(Q^\bullet).$$
One can choose these Q_i to be a *minimal resolution*, meaning the maps in the complex $Q^\bullet$ are all 0, and therefore for such Q_i,
$$\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \cong Q^s.$$
The point is, you can do this yourself.

Example 14.11. Let's apply this to $H\mathbb{Z}$, the Eilenberg-Mac Lane spectrum which represents integral cohomology. We know the answer should be
$$(\pi_{t-s} H\mathbb{Z}) \otimes \mathbb{Z}_2 = \begin{cases} \mathbb{Z}_2, & t - s = 0 \\ 0, & \text{otherwise.} \end{cases}$$
As an $\mathcal{A}$-algebra, $H^*(H\mathbb{Z};\mathbb{Z}/2) \cong \mathcal{A}/\mathcal{A}(1) \otimes \mathcal{A}(1)/\mathrm{Sq}^1$. Hence
$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}/\mathcal{A}(1) \otimes \mathcal{A}(1)/\mathrm{Sq}^1, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}(1)}(\mathcal{A}(1)/\mathrm{Sq}^1, \mathbb{Z}/2),$$
and you can compute a minimal resolution by drawing a picture. The point is, this minimal resolution is

`placeholder`

FIGURE 18. A minimal resolution of $\mathcal{A}(1)/\mathrm{Sq}^1$. This is four-dimensional, with elements in degrees 0, 2, 3, and 5, and Sq^2 connecting 0 and 2 and 3 and 5, and Sq^1 connecting 2 and 3.

periodic with a small period, and we can conclude that

$$\operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(\mathcal{A}(1)/\mathrm{Sq}^1, \mathbb{Z}/2) \cong \begin{cases} \mathbb{F}_2, & t = s, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if you draw the Adams spectral sequence, it cannot have nontrivial differentials: all elements are on the $s = 0$ line. Therefore all you get is $\varprojlim \mathbb{Z}/2^k = \mathbb{Z}_2$ when $t - s = 0$ and 0 otherwise. ◀

15. Dan Freed: Reflection Positivity in Invertible Field Theory

We've decided that, after fixing a symmetry type (H_n, ρ_n), invertible field theories are given by maps $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$, and to obtain deformation classes, we compose with the map $\Sigma^n I\mathbb{C}^\times \to \Sigma^{n+1} I\mathbb{Z}$.

If we add positivity, we pass from $\Sigma^n MTH_n$ to MTH, considering homotopy classes of maps $MTH \to \Sigma^{n+1} I\mathbb{Z}$, leading us to consider classical Thom spectra. When we incorporate reflection, we need to incorporate stable homotopy theory, but extended positivity will return us to nonequivariant spectra.

Recall that reflection positivity is the Wick-rotated version of unitarity, specifying on a Euclidean space $\mathbb{E}^n$ a hyperplane Π and the condition that reflecting operators across Π changes the correlation functions by complex conjugation. One form of positivity says that if you include both a set of operators and its reflection, the correlation functions are nonnegative. This is part of the axioms of Euclidean field theory, and one of the ingredients in the reconstruction theorem allowing us to understand the Lorentzian case from the Euclidean one.

We want to understand this within the homotopical framework for invertible field theory, and considering a general symmetry group H_n. Reflection symmetry is not yet part of H_n, so we need to enlarge it.

Theorem 15.1 ([FH16])**.** *There is a canonical extension*

$$1 \longrightarrow H_n \xrightarrow{j_n} \widehat{H}_n \longrightarrow \{\pm 1\} \longrightarrow 1$$

such that there is a commutative diagram

$$\begin{tikzcd} H_n \arrow[r, "j_n"] \arrow[d, "\rho_n"] & \widehat{H}_n \arrow[d, "\widehat{\rho}_n"] \\ \mathrm{O}_n \arrow[r, "j_n"] & \mathrm{O}_n \times \{\pm 1\} \end{tikzcd}$$

and a stabilization

$$\begin{tikzcd} \{\pm 1\} \times H \arrow[r] \arrow[d] & \widehat{H} \arrow[d] \\ \mathrm{O}_1 \times \mathrm{O} \arrow[r, "\oplus"] & \mathrm{O}. \end{tikzcd}$$

The idea is that $\widehat{H}_n$ is H_n plus the hyperplane reflection reflection symmetry, and the uniqueness result means we know what the answer is. For some simple examples, this answer is given in Table 4.

	H_n	$\widehat{H}_n$
bosons only, no time-reversal	SO_n	O_n
fermions, no time-reversal	Spin_n	Pin^+_n
bosons, time-reversal	O_n	$\mathrm{O}_n \times \{\pm 1\}$
fermions and time-reversal	Pin^+_n	$\widehat{\mathrm{Pin}}^+_n$
fermions and time-reversal	Pin^-_n	$\widehat{\mathrm{Pin}}^-_n$

Table 4. Incorporating reflection into various symmetry groups. The $\widehat{\mathrm{Pin}}^\pm_n$ groups are certain double covers of $\mathrm{Pin}^\pm_n$; see [FH16] for more details.

We've been thinking of quantum field theories as maps $F\colon \mathsf{Bord}_{\langle n-1,n\rangle} \to \mathsf{Vect}_\mathbb{C}$, but reflection positivity means that orientation-reversal must go to complex conjugation. Orientation-reversal is an involution β_B on

$\mathsf{Bord}_{\langle n-1,n\rangle}$, and complex conjugation is an involution β_C on $\mathsf{Vect}_{\mathbb{C}}$: on objects and morphisms, it complex-conjugates the action of $\mathbb{C}$. These are now symmetric monoidal categories with involution, and we require F to preserve this structure, but now it's data.

Definition 15.2. Let C be a symmetric monoidal category with an involution $\beta\colon \mathsf{C} \to \mathsf{C}$. Then, a *Hermitian structure* on a $c \in \mathsf{C}$ is an isomorphism $\beta c \to c^\vee$ satisfying certain axioms.

If $\mathsf{C} = \mathsf{Vect}_{\mathbb{C}}$ and β is complex conjugation, this is an isomorphism $\overline{V} \to V^*$, or equivalently a sesquilinear map $\overline{V} \otimes V \to \mathbb{C}$. This is the model case for the general definition.

We also want to explicitly understand the involution on $\mathsf{Bord}_{\langle n-1,n\rangle}(H_n)$.

Definition 15.3. Suppose X is an H_n-manifold, specified by a principal H_n-bundle $P \to X$ and an isomorphism $\rho_n(P) \cong \mathcal{B}_{\mathrm{O}}(X)$ (the orthonormal frame bundle) of principal O_n-bundles. Then, $j_n(P)$ is a principal $\widehat{H}_n$-bundle. The *opposite H_n-manifold* to X is the one with principal H_n-bundle $j_n(P) \setminus P \to X$.

This generalizes orientation-reversal to other H_n-structures. Some of the others are familiar: the O_n-case doesn't do anything, and this also specializes to the usual notion of an opposite spin structure. For the two pin structures, this is a different pin structure defined by tensoring with the orientation bundle.

Definition 15.4. A *reflection structure* on a TFT $F\colon \mathsf{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathsf{Vect}_{\mathbb{C}}$ is equivariance data for F under taking the opposite H_n-structure and complex conjugation.

Lemma 15.5. *If $Y \in \mathsf{Bord}_{\langle n-1,n\rangle}(H_n)$, then there's a canonical Hermitian structure $\beta Y \to Y^\vee$.*

This says that orientation-reversal is the same thing as reversing the arrow of time. But this in particular means that every object in $\mathsf{Bord}_{\langle n-1,n\rangle}(H_n)$ has a canonical Hermitian structure. Hence, $F(Y) \in \mathsf{Vect}_{\mathbb{C}}$ has a Hermitian structure, specifically a Hermitian metric, for all Y. This concretely comes from the cylinder $[0,1] \times Y$ with both ends as incoming components: this is a cobordism $Y \amalg Y^\vee \to \varnothing$, which defines a map $F(Y) \otimes \overline{F(Y)} \to \mathbb{C}$, and an S-diagram lemma proves this is nondegenerate.

Definition 15.6. A reflection structure on F is *positive* if the Hermitian form $F(Y)$ is positive definite for all Y.

So a reflection structure on F with positivity is the functorial version of reflection positivity in Wick-rotated QFT.

Exercise 15.7. Show that the Euler TFT ε_μ has a reflection structure iff $\mu^2 \in \mathbb{R}$, and has reflection positivity iff $\mu \in \mathbb{R}$.

Definition 15.8. Let X be a compact H_n-manifold with boundary. Then, its *double* ΔX is the (closed) H_n-manifold

$$\Delta X = e_{\partial X}(\beta X, X).$$

That is, we view X as a bordism $\varnothing \to \partial X$, and βX as a bordism $\partial X \to \varnothing$. Evaluation is the cylinder $\partial X \times [0,1]$ with X on one side and βX on the other.

You might be used to thinking of this as gluing X to its opposite H_n-structure along its boundary, and indeed that's what this construction is doing.

Example 15.9. The homogeneous space $H_{n+1}/H_n \cong S^n$ is a double. It's worth working this out in the case $n = 1$ and $H = \mathrm{Spin}$, in which case you get the statement that the bounding spin circle is a double. In general, ΔX is H_{n+1}-null bordant, and in particular they're all bordant to this double S^n. ◂

If F has a positive reflection structure, then $F(\Delta X) \geq 0$, because it's F applied to an evaluation on Y, which is exactly its Hermitian form.

Invertible field theories and reflection positivity. Let's see how this works from the homotopy theory perspective.

Theorem 15.10. *An invertible field theory $\alpha\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ is stable iff $\alpha(S^n) = 1$.*

This follows from the obstruction theory mentioned earlier; there's only a single obstruction, which is $\alpha(S^n)$.

But we're interested in deformation classes of theories, which are maps to $\Sigma^n I\mathbb{Z}$, and we want these theories to have reflection structure. This will entail promoting our spectra to spectra with involution, and maps to intertwiners. Equivariant stable homotopy theory is a big subject, and we can't do all of it, but here are some facts for a spectrum X with an involution τ, or (naïve) $\mathbb{Z}/2$-spectrum[11]

- The involution defines a grading-preserving involution on $\pi_* X$.
- Given any spectrum T, we can consider the $\mathbb{Z}/2$-spectrum i^*T which has T as its underlying spectrum and a trivial involution.
- Given a real representation V of $\mathbb{Z}/2$, we can take its one-point compactification, as a $\mathbb{Z}/2$-space, and obtain a sphere with involution S^V, called a *representation sphere*. Another way to obtain an equivariant spectrum from a nonequivariant spectrum T is to consider $S^{N(1-\sigma)} \wedge i^*T$ for any $N \geq 0$, where 1 is the trivial representation and σ is the sign representation.

If C is a Picard groupoid, then duality defines a canonical involution on it sending $c \mapsto c^\vee$ and $f\colon c' \to c$ to $f^\vee\colon (c')^\vee \to c^\vee$. Since Picard groupoids model spectra, we can ask what duality corresponds to.

Lemma 15.11. *If T is the (nonequivariant) spectrum representing the Picard groupoid C, then $S^{\sigma-1} \wedge i^*T$ is the $\mathbb{Z}/2$-spectrum representing C with the duality involution.*

Now suppose

$$\alpha\colon \Sigma^n MTH_n \longrightarrow \Sigma^n I\mathbb{C}^\times$$

is an extended invertible field theory. We'd like to implement reflection positivity, and our hand is forced.

There's a $\mathbb{Z}/2$-spectrum MTH_n^β refining MTH_n with the "opposite H_n-structure" involution. This is not particularly accessible, but its stabilization is nice.

Proposition 15.12. *There is a weak equivalence of $\mathbb{Z}/2$-spectra*

$$\operatorname*{colim}_{n\to\infty} \Sigma^n MTH_n^\beta \xrightarrow{\simeq} S^{1-\sigma} \wedge i^* MTH.$$

The codomain again requires a choice, but we have some information at hand: we know that its homotopy groups with involution should correspond to the involutions on the Picard and Brauer-Wall groups of $\mathbb{C}$, and $S^{\sigma-1} \wedge i^* I\mathbb{C}^\times$ works.

Next we need to implement positivity. There's a naïve possibility, that $\alpha(S^n) > 0$, which actually implies stability. But if you want to think about deformation classes, the map $\Sigma^n I\mathbb{C}^\times \to \Sigma^n I\mathbb{Z}$ coming from the exponential exact sequence is $\mathbb{Z}/2$-equivariant, so we're looking at the space of equivariant maps

$$S^{1-\sigma} \wedge i^*\Sigma^n MTH \longrightarrow S^{1-\sigma} \wedge i^*\Sigma^{n+1} I\mathbb{Z}.$$

These involutions match, so these maps are the same as maps with the trivial involution, which can be understood nonequivariantly:

$$\mathrm{Map}^{\mathbb{Z}/2}(i^*B, i^*I) \cong \mathrm{Map}(B, I) \times \mathrm{Map}(B, \Sigma^{-1}(S^{1-\sigma} \wedge i^*I)^{h\mathbb{Z}/2}). \tag{15.13}$$

This is deformation classes; for actual theories, you can't duck out of equivariance so easily.

But what's interesting about (15.13) is that it's product of the space of deformation classes of theories we considered without reflection positivity with deformation classes of theories in one dimension lower. The partition function of this theory on an $(n-1)$-manifold Y is an element of $\mathbb{Z}/2$ which tells us whether the Hermitian structure on our original theory is positive or negative. And there's data all the way down. You can prove that the deformation classes you get are given by these maps, and that you hit the torsion classes. It's kind of magical that the naïve notion of positivity splits off the extended notion, and it would be nice to know how this works in the noninvertible case.

We'll next discuss the effective theory of a general condensed-matter system and an example calculation for electron systems that physicists have understood another way.

[11] There are multiple notions of equivariant spectrum considered in stable homotopy theory, called naïve and genuine.

16. Sam Gunningham: The Arf Theory

"I told myself I wouldn't write out this diagram, but I couldn't help myself."

The Arf theory is an example of the theories we've been talking about: it's a 2D invertible topological field theory of spin manifolds, first written down in [Gun16]. We'll formulate it as a symmetric monoidal functor of 2-categories

$$F_{\mathrm{Arf}}\colon \mathsf{Bord}_2(\mathrm{Spin}_2) \longrightarrow \mathsf{sAlg}_{\mathbb{C}}.$$

The key is a classical invariant of a spin 2-manifold $p(\Sigma)$ called its *Arf invariant*, or *Atiyah invariant* or *parity*. On a closed spin surface Σ, $F_{\mathrm{Arf}}(\Sigma) = (-1)^{p(\Sigma)}$.

As we've been discussing, since this theory is invertible it factors through the maximal subgroupoid $\mathsf{sAlg}_{\mathbb{C}}^\times$ of $\mathsf{sAlg}_{\mathbb{C}}$, which is a Picard 2-groupoid whose classifying spectrum is homotopy equivalent to $\Sigma^2 I\mathbb{C}^\times$, and factors through the groupoid completion $|\mathsf{Bord}_2(\mathrm{Spin}_2)| \simeq \Sigma^2 \mathit{MTSpin}_2$. This is closely related to the classical Atiyah-Bott-Shapiro orientation [ABS64], a map $\widehat{A}\colon \mathit{MSpin} \to \mathit{KO}$, and there are explicitly defined maps $\Sigma^2 \mathit{MTSpin}_2 \to \mathit{MSpin}$ and $\mathit{KO} \to \Sigma^2 I\mathbb{C}^\times$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma^2 \mathit{MTSpin}_2 \arrow[r, "F_{\mathrm{Arf}}"] \arrow[d] & \Sigma^2 I\mathbb{C}^\times \\
\mathit{MSpin} \arrow[r, "\widehat{A}"] & \mathit{KO}. \arrow[u]
\end{tikzcd}$$

You could even take this to be a *definition* of the Arf theory. Similar ideas work for other orientations in stable homotopy theory, and you get, e.g. theories of massive free fermions in physics. The two vertical maps are:

- Stabilization defines a map $\mathit{MTH}_n \to \mathit{MTH}$, and $\mathit{MTSpin} \simeq \mathit{MSpin}$, so we get the left-hand map $\Sigma^2 \mathit{MTSpin}_2 \to \mathit{MSpin}$.
- Finally, KO and $\Sigma^2 I\mathbb{C}^\times$ look very similar after truncation, and the only issue to overcome is the map on π_0, which is a map $\mathbb{Z}/2 \to \mathbb{C}^\times$, which is the usual character valued in $\{\pm 1\}$.

The Arf theory, or at least something closely related to it, appears in condensed-matter physics as the low-energy theory of the Majorana chain [Kit01, KT17].

Spin structures and Clifford algebras. The theory of spin manifolds is closely related to that of Clifford algebras. Let V be a finite-dimensional vector space with a quadratic form q; then, one can define its *Clifford algebra* $\mathrm{C}\ell(V,q)$ to be the algebra generated by V together with the relations $v^2 = -q(v)$ for all $v \in V$, and if $e_1, \ldots, e_n$ is a basis for V, $e_i e_j = -e_j e_i$ and $e_i^2 = 1$. The Lie group generated by unit-length vectors in this Clifford algebra is called $\mathrm{Spin}(V,q)$, which is a connected double cover of $\mathrm{SO}(V,q)$ (in particular, it's not the orthogonal group).

A spin structure is a lift of the classifying map of the frame bundle $\mathcal{B}_{\mathrm{O}}\colon M \to B\mathrm{O}_n$ across the diagram

$$\begin{tikzcd}
 & B\mathrm{Spin}_n \arrow[d] \\
 & B\mathrm{SO}_n \arrow[d] \\
M \arrow[ruu, dashed] \arrow[ru, dashed] \arrow[r, "\mathcal{B}_{\mathrm{O}}"'] & B\mathrm{O}_n.
\end{tikzcd}$$

A lift just to $B\mathrm{SO}_n$ is an orientation.

A little more geometrically, if M is spin and $\mathcal{B}_{\mathrm{Spin}_n}$ is its bundle of spin frames, $S_M \coloneqq \mathcal{B}(\mathrm{Spin}_n)X \times_{\mathrm{Spin}_n} \mathrm{C}\ell_n$, which is a Clifford module bundle, and conversely a spin structure on a manifold M is a graded, invertible, $(\mathrm{C}\ell(TM), \mathrm{C}\ell_n)$-bimodule bundle S_M.

Example 16.1. Let's look at dimension 0. Then, $\mathrm{C}\ell(T\mathrm{pt}) = \mathrm{C}\ell_0 = \mathbb{R}$, so we're asking for a graded, invertible $(\mathbb{R},\mathbb{R})$-bimodule. If $S_M = \mathbb{R}$ is a line in even grading, then we get the "positively oriented" spin structure usually denoted pt_+, and if $S_M = \Pi\mathbb{R}$ is a line in odd grading (sometimes also written $\mathbb{R}[1]$), we get the "negatively oriented" spin structure pt_-.

However, there is a nontrivial spin diffeomorphism $\alpha_{\mathrm{pt}_+}\colon \mathrm{pt}_+ \to \mathrm{pt}_+$ which is trivial on the point, but acts by -1 on $S_{\mathrm{pt}_+} = \mathbb{R}$ (in even grading). ◂

Example 16.2. Suppose $M = S^1$. Then, there are two spin structures, because there are two choices for the Clifford bundle S_M.

- You can take S_M to be the Möbius bundle $E \to S^1$, which determines the spin structure on S^1 induced from its inclusion as the unit circle in $\mathbb{R}^2$ (with the canonical spin structure on $\mathbb{R}^n$). Thus it bounds a spin structure on the disc, and is called the *bounding spin structure* on S^1, denoted S^1_b.
- If you take S_M to be the trivial bundle $\underline{\mathbb{R}} \to S^1$, you get a *nonbounding spin structure* on S^1, denoted S^1_{nb}. This is the induced spin structure coming from the immersion $S^1 \to \mathbb{R}^2$ as a figure-8. ◂

Given any spin structure on a manifold, there's an important differential operator called the Dirac operator. Inclusion defines a map

$$\Gamma(M; S_M) \longrightarrow \Gamma(M; T^*M \otimes S_M),$$

and $\Gamma(M; T^*M)$ acts on $\Gamma(M; S_M)$ by differentiating sections. Thus we have a composition

$$\begin{tikzcd}
\Gamma(M; S_M) \arrow[rr, "D_M"] \arrow[dr] & & \Gamma(M; S_M) \\
& \Gamma(M; T^*M \otimes S_M) \arrow[ur] &
\end{tikzcd}$$

which is linear but has a Leibniz rule, and is called the *Dirac operator* for M.

Definition 16.3. The kernel of the Dirac operator is called the space of *harmonic spinors* on M.

This is related to the Atiyah-Bott-Shapiro orientation.

Example 16.4. Thinking back to Example 16.2, the space of harmonic spinors for the bounding spin structure on S^1 is trivial, and the space of harmonic spinors for the nonbounding spin structure for S^1 is one-dimensional. ◂

Returning to the Arf theory, its value on 1-manifolds is given by

$$F_{\mathrm{Arf}}(S^1_b) = \mathbb{C}$$
$$F_{\mathrm{Arf}}(S^1_{nb}) = \Pi\mathbb{C},$$

i.e. in even and odd grading, respectively.

Now suppose Σ is a closed spin 2-manifold. The spin structure determines an orientation, so we may choose a complex structure on Σ respecting this orientation, making it a Riemann surface. In this case, the Dirac operator is $\overline{\partial}$, so $S^+_\Sigma \to \Sigma$ is a holomorphic line bundle, and $S^+_\Sigma \otimes S^+_\Sigma$ is the canonical bundle. This implies that the space of harmonic spinors is identified with $H^0_{\mathrm{Hol}}(\Sigma; S^+_\Sigma)$. Atiyah [Ati71] provides an excellent reference for this perspective.

A line bundle squaring to the canonical bundle is often called a *θ-characteristic*, and if Σ has genus $g \geq 0$, it has 2^{2g} θ-characteristics. Alternatively, on a general manifold M which admits a spin structure, the set of isomorphism classes of spin structures is a torsor for $H^1(M; \mathbb{Z}/2)$, and we know this has cardinality 2^{2g} for $M = \Sigma$.

Definition 16.5. The *parity* $p(\Sigma, L)$ of a spin structure $L = S_M$ on a surface Σ is $\dim H^0_{\mathrm{Hol}}(\Sigma; L) \bmod 2$.

So now we know what the Arf theory assigns to 1- and 2-manifolds. To pt_+ and pt_- we assign the invertible superalgebra $C\ell_1$ (over $\mathbb{C}$); the fact that these agree is suggestive that this extends to a theory of pin structures, and indeed it extends to a Pin^--theory.

The Arf invariant. The Arf invariant is an invariant of a quadratic form on an $\mathbb{F}_2$-vector space. Specifically, we care about $H_1(\Sigma; \mathbb{F}_2)$ for a closed surface Σ. This is actually a symplectic vector space, and the symplectic form is given by the intersection pairing.

In most situations, a nondegenerate symmetric bilinear form is the same thing as a quadratic form. But in characteristic 2, this is a symplectic form, not a quadratic form, and in general there might be multiple quadratic forms defining the same bilinear form. However, a spin structure on Σ is the extra data needed to turn this symplectic form into a quadratic form q.

The simplest description of the Arf invariant of q is: on $V \setminus 0$, does q hit 0 or 1 more often? The Arf invariant is the "more popular" number. Alternatively, the symplectic group acts on $H_1(\Sigma; \mathbb{F}_2)$, and there are two orbits; the Arf invariant is 0 if you're in the same orbit as the trivial quadratic form, and 1 otherwise. Thus, in some sense there are only two kinds of quadratic forms, and the Arf invariant tells them apart.

Lemma 16.6. *The parity of a spin surface Σ is equal to the Arf invariant of the quadratic form it induces on its first homology.*

Another fun calculation is that if $C \hookrightarrow \Sigma$ is an embedded curve, $q(C)$ is 0 if $C \cong S^1_b$ and 1 if $C \cong S^1_{nb}$ as spin curves.

The Atiyah-Bott-Shapiro orientation. We claimed that the Arf TQFT is a shadow of the Atiyah-Bott-Shapiro orientation $\widehat{A}\colon \mathit{MSpin} \to \mathit{KO}$. This orientation (a morphism of spectra that preserves multiplicative structure) is related to the index of the Dirac operator on a spin manifold.

Whether or not you're comfortable with the notion of a morphism of spectra, it should induce a morphism of homotopy groups

$$\pi_*(\mathit{MSpin}) \longrightarrow \pi_*(\mathit{KO}).$$

The domain is also written Ω_*^{Spin}, the spin bordism groups! The codomain is an 8-periodic sequence of abelian groups

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots$$

It's traditional to sing this sequence to the tune of "Twinkle, twinkle, little star," in which case it's called the *Bott song*.

In each dimension there's a $\pi_i(\mathit{KO})$-valued invariant of the Dirac operator, and we saw hints of the $\mathbb{Z}/2$-valued ones in dimensions 1 and 2. These stitch together into the Atiyah-Bott-Shapiro orientation, though making this a map of spectra is a little more work, and teasing out the multiplicative structure, producing a morphism of E_∞-ring spectra, was done much more recently [Joa04].

For another example, there's a map of spectra $\mathit{MSO} \to H\mathbb{Z}$, which tracks all the possible Thom isomorphisms you can make for oriented manifolds. This is orientation in the usual sense: an oriented manifold has a Thom isomorphism for $\mathbb{Z}$-cohomology and a Thom isomorphism determines an orientation (and hence Poincaré duality and all the other nice things orientation buys us), so this map of spectra tracks orientations in the usual sense, justifying the name "orientation" for more general things such as $\widehat{A}$. In particular, these things give you a way of integrating, e.g.

$$H^n(M;\mathbb{Z}) \xrightarrow{\int_M} H^0(\mathrm{pt};\mathbb{Z}) \cong \mathbb{Z}$$

on an oriented manifold.

Similarly, if M is a spin manifold, one obtains an integration map

$$\mathit{KO}^n(M) \longrightarrow \mathit{KO}^0(\mathrm{pt}) \cong \mathbb{Z},$$

but since $\mathit{KO}^*(\mathrm{pt})$ has infinitely many nontrivial homotopy groups, we can ask what happens to $1 \in \mathit{KO}^0(M)$ under the map $\mathit{KO}^0(M) \to \mathit{KO}^{-n}(\mathrm{pt}) = \pi_n(\mathit{KO})$. This is an example of a *Gysin map*.

Just as the Arf TQFT is associated to the Atiyah-Bott-Shapiro orientation, there's a "higher" orientation $\mathit{MString} \to \mathit{TMF}$, and it could be interesting to consider TQFTs coming from this orientation.

17. Question session

17.1. **Aaron Mazel-Gee: what is a spectrum?** We've defined a (pre)-spectrum to be a sequence of spaces X_k together with bonding maps $\Sigma X_n \to X_{n+1}$ for $n \in \mathbb{Z}$. It's a *spectrum* if the adjoint maps $X_n \to \Omega X_{n+1}$ are equivalences.

The homotopy theories of prespectra and spectra are the same, but we give them different names because spectra are more convenient to work with, but many important constructions arise as prespectra. Fortunately, the forgetful functor from spectra to prespectra has a left adjoint L, called *spectrification*, which turns prespectra into spectra with the same important properties (homotopy groups, cohomology theory, etc.). For example, here's a functor $\mathsf{Top}_* \to \mathsf{preSp}$ called the *suspension spectrum*, sending

$$X \longmapsto \left\{\Sigma^n X, \Sigma(\Sigma^n X) \overset{\cong}{\to} \Sigma^{n+1} X\right\},$$

but we get a prespectrum, and we need to spectrify it to obtain a spectrum. (Here Top_* is the category of based topological spaces and basepoint-preserving maps.)

For example, if M is an abelian group, the *Eilenberg-Mac Lane spectrum* HM is the spectrum whose n^{th} space is $K(M,n)$, and whose bonding maps are the adjoints of the homotopy equivalences $K(M,n) \simeq \Omega K(M,n+1)$. This is a spectrum on the nose; we don't need to spectrify it! Then,

$$HM^n X \coloneqq [\Sigma^n \Sigma^\infty X, HM]_{\mathsf{preSp}} \cong [X, K(M,n)]_{\mathsf{Top}} \cong H^n(X;M).$$

Another perspective is that spectra capture stable homotopy types: after suspending something enough for the Freudenthal suspension theorem to apply, homotopy theory displays stable phenomena, and spectra capture this notion.

Spectra are also useful to represent generalized cohomology theories.

Definition 17.1. A *generalized cohomology theory* is a collection of functors $\{h^n \colon \mathsf{Top}_*^{\mathsf{op}} \to \mathsf{Ab}\}$ satisfying the *Eilenberg-Steenrod axioms*:

(1) h^n is invariant under weak homotopy equivalence.
(2) Given a pushout diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

there is a long exact sequence

$$\cdots \longrightarrow h^n(U \cup V) \longrightarrow h^n(U) \oplus h^n(V) \longrightarrow h^n(U \cap V) \longrightarrow h^{n+1}(U \cup V) \longrightarrow \cdots .$$

(3) There is a natural isomorphism $h^n(X) \cong h^{n+1}(\Sigma X)$.
(4) For any collection I,

$$h^n\left(\bigvee_{i \in I} X_i\right) \cong \prod_{i \in I} h^n(X_i).$$

These encode some kind of locality of the things defining your theory, especially if they have geometric meaning.

If you additionally add the *dimension axiom* that $h^n(\mathrm{pt}) = 0$ unless $n = 0$, you end up concluding that h^n is ordinary reduced cohomology $\widetilde{H}^*(-;h^0(\mathrm{pt}))$, but in the absence of this axiom, there are many interesting generalized cohomology theories, including various cobordism theories and K-theory.

Anyways, one major reason to care about spectra is that they keep track of generalized cohomology theories.

Theorem 17.2 (Brown representability)**.** *Let $\{h^n\}$ be a generalized cohomology theory. Then, there is a spectrum E and natural isomorphisms*

$$h^n \xrightarrow{\cong} [\Sigma^\infty(\text{–}), \Sigma^n E]$$

that are compatible with the connecting morphisms in the long exact sequences of a pair.

Examples: the spectrum KO represents real K-theory, and KU represents complex K-theory. HM represents ordinary cohomology with coefficients in the abelian group M.

Yet another perspective on spectra are as derived versions of abelian groups. This perspective can get pretty homotopical. Let C be a (suitably nice) ∞-category, which we're going to think of as a homotopy theory, and as something which behaves somewhat like a smooth manifold. Given an $X \in \mathsf{C}$, we can form its *overcategory* $\mathsf{C}_{/X}$, whose objects are maps $f_Y \colon Y \to X$ and whose morphisms are commutative triangles

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "f_Y"'] & & Z \arrow[dl, "f_Z"] \\ & X. & \end{tikzcd}$$

If R is a commutative ring, Quillen defined an isomorphism $\mathsf{Mod}_R \cong \mathsf{Ab}(\mathsf{CRing}/R)$, the abelian group objects in the commutative rings over R, through the notion of a *square-zero extension*: $M \mapsto R \ltimes M$, which is $R \oplus M$ as an abelian group with the multiplication

$$(r_1, m_1) \cdot (r_2, m_2) \coloneqq (r_1 r_2, r_1 m_2 + r_2 m_1).$$

One good example is $\mathbb{R}[\varepsilon]/(\varepsilon^2)$, which is a square-zero extension that makes algebraic the notion of working up to $O(\varepsilon^2)$.

In higher category theory, there's a dictionary (called "nonabelian derived category") sending sets to spaces and R-modules to chain complexes.[12] This dictionary sends abelian groups to spectra.

As C was supposed to resemble a smooth manifold, given a "point" (object in C), you should be able to define a tangent space $T_x\mathsf{C}$, which is the derived version of Quillen's construction: the category of spectrum objects in $\mathsf{C}_{/X}$.

Example 17.3. If $\mathsf{C} = \mathsf{Top}$ and $X = \mathrm{pt}$, the "tangent category to the category of spaces" at pt is the category of spectrum objects in Top, which is the category of spectra. ◀

So, in this sense, spectra are derived versions of spaces, or even the tangent space to Top.

This is as yet a little circular, until we define a spectrum object. If C has finite limits and colimits (which Top certainly does), we can define the *loop space* of a $Y \in \mathsf{C}$ to be the pullback

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & *_{\mathsf{C}} \arrow[d] \\ *_{\mathsf{C}} \arrow[r] & Y. \end{tikzcd}$$

For $\mathsf{C} = \mathsf{Top}$, this recovers the usual notion of loop space, which is reassuring.

Definition 17.4. A *spectrum object* in C is a collection of C-objects $\{Y_k\}$ for $k \geq 0$ such that Y_k is the loop space object of Y_{k-1} for each k.

More generally, the tangent category to Top at a space X is a kind of "spectra over X," and this is made rigorous in the subject of parameterized homotopy theory.

Day 5. **August 4**

18. Dan Freed: Non-topological Invertible Theories

> *"Topology is geometry for the farsighted. And sometimes you see exactly what you need to see to solve a problem..."*

Today, we're first going to review the introduction of reflection positivity as involutions on the domain and codomain categories for a field theory.

On $\mathsf{Vect}_{\mathbb{C}}$, we have a few choices for our involutions: complex conjugation $V \mapsto \overline{V}$, linear duality $V \mapsto V^*$, or their combination $V \mapsto \overline{V}^*$. But this is contravariant, and we wanted it to be covariant. So let's restrict to invertible morphisms, sending $f \mapsto (\overline{f}^*)^{-1}$, which now is covariant.

Let $\gamma\colon \mathsf{Vect}_{\mathbb{C}}^{\times} \to \mathsf{Vect}_{\mathbb{C}}^{\times}$ denote this involution; there's a forgetful map $(\mathsf{Vect}_{\mathbb{C}}^{\times})^{\gamma} \to \mathsf{Vect}_{\mathbb{C}}^{\times}$; the data of a fixed point is a Hermitian form with symmetry group $\mathrm{U}_{p,q}$, and the forgetful map forgets this form. We'd like a homotopy section of this map, meaning a way to get from $\mathrm{GL}_n(\mathbb{C})$ symmetry to $\mathrm{U}_{p,q}$ symmetry. In the invertible case, at least, this is possible: we can choose a Hermitian metric with U_1-symmetry for each complex line, and life is good.

We also discussed how to model this by spectra. If I is a spectrum, then we used i^*I to denote the (naïve) $\mathbb{Z}/2$-spectrum with the trivial involution. Its homotopy fixed points are

$$(i^*I)^{h\mathbb{Z}/2} \simeq \mathrm{Map}(\mathbb{RP}^\infty_+, I) = I \times \mathrm{Map}(\mathbb{RP}^\infty, I).$$

The forgetful functor $i^*I \mapsto I$ can be identified with projection onto the first factor of $(i^*I)^{h\mathbb{Z}/2}$, so it has a clear homotopy section $s\colon I \dashrightarrow I \times \mathrm{Map}(\mathbb{RP}^\infty, I)$.

Example 18.1. Suppose $I = \Sigma^2 H\mathbb{Z}$. This is the spectrum that models complex lines, because $[X, \Sigma^2 H\mathbb{Z}] = H^2(X;\mathbb{Z})$ naturally, and this is naturally identified with the group of isomorphism classes of line bundles. If we induce it up to a $\mathbb{Z}/2$-spectrum,

$$(i^*\Sigma^2 H\mathbb{Z})^{h\mathbb{Z}/2} = \Sigma^2 H\mathbb{Z} \times \mathrm{Map}(\mathbb{RP}^\infty, \Sigma^2 H\mathbb{Z}),$$

[12] The full derived category of Set, using something called the Dold-Kan correspondence, is the category of spectra, which is another perspective: they want to be abelian, but don't quite cut it.

and taking π_0 of the second half, we get $H^2(\mathbb{RP}^\infty) \cong \mathbb{Z}/2$. This has something to say about Hermitian metrics on complex line bundles: there are two components, positive (corresponding to 0) and negative (corresponding to 1). ◀

Onwards to today. Our goal is to use mathematics to attach physics problems, in a three-step process.

(1) Take a problem P and find a good mathematical formulation F.
(2) Within F, prove theorems T.
(3) Apply T to the physical problem P.

We've done the first two so far this week; today is all about the third. However, it's never quite so clean, and often reasoning and research flows upstream as well. Today there will be conjectures as well as theorems inspired by several of these steps, and there is plenty of good mathematics to be done.

Today we're going to consider a lattice model in spatial dimension d with internal symmetry group I (a compact Lie group). We've seen some examples of these already: the toric code, the Ising model, and there are many more. We'd like to imagine there's a moduli space $\mathcal{M}(d, I)$ of these systems, but we had to throw out phase transitions: $\mathcal{M}(d, I)$ should be the space of gapped systems of this type.

Our problem P is to compute π_0 of this moduli space. We haven't defined what a lattice system is, though! So we can't quite define $\mathcal{M}(d, I)$ — this is a very interesting question, but we can make progress on computing its space of conncted components without knowing the entire story. We use two physical principles, which are not (yet) mathematical theorems, to move forward.

(1) The deformation class of a physical system (in our case, the component of $\mathcal{M}(d, I)$ it lives in) is determined by its long-range behavior. In a relativistic system, this also means long-time and low-energy.
(2) The long-range behavior of a gapped system is well-approximated by a topological field theory. There's a basic dichotomy in quantum systems determined by whether there's a gap between the two smallest eigenvalues, and we've seen a few times so far that gapped phases have interesting topological behavior.

The second statement has a caveat: it's not entirely true, and this lecture will explain this.

Example 18.2. For a toy model, let $\mathcal{M}_d$ denote the moduli space of Riemannian manifolds of dimension d. We don't yet know what this moduli space is, but it has a good chance of being a mathematical object, because we at least know what a Riemannian manifold is.[13] Some points would be "bad," just like phase transitions where the energy gap closes.

In this case, the Hamiltonian is replaced with the Laplace operator, so we throw out the noncompact Riemannian manifolds, and consider the moduli space of Riemannian manifolds. Noncompact manifolds would allow transitions such as splitting one circle into two. If you want to compute $\pi_0\mathcal{M}_d$, it will be the set of diffeomorphism classes of d-dimensional manifolds.

There are analogues of our two physical principles here: the deformation class of a compact Riemannian manifold is determined by its long-range behavior, because locally all smooth manifolds look the same. The second is about determining a Riemannian manifold from its spectrum. This is the subject of a famous paper by Kac called "Can one hear the shape of a drum?" [Kac66]. In this case, the answer is "not quite," but it's still a useful invariant. ◀

Assuming these physical principles, taking the long-range effective theory should define a map

$$\{\text{moduli of gapped lattice models}\} \longrightarrow \{\text{moduli of topological field theories}\}.$$

We don't know how to define most of the things in that equation, but we hope to be able to define a map

$$\pi_0\mathcal{M}(d, I) \longrightarrow \pi_0\{\text{moduli of TFTs}\}.$$

Physical principles tell us this map should be injective, and there's also evidence that it should be bijective: there is work in physics by Walker and collaborators (e.g. [BBJ+16]) to reconstruct a lattice model from its low-energy effective field theory using state-sum constructions.

Splitting $\mathbb{R}^n = \mathbb{R} \oplus \mathbb{R}^{n-1}$, you get a sequence of maps

$$\mathrm{O}_1 \times \{e\} \hookrightarrow \mathrm{O}_1 \times \mathrm{O}_{n-1} \hookrightarrow \mathrm{O}_n.$$

[13]But Riemannian manifolds have isometries, so this would be a moduli stack, but that's not important right now.

Let I_n be the pullback of $H_n \to \mathrm{O}_n$ under this sequence, e.g. if $H_n = \mathrm{O}_n$, then $K = \{e\}$ is the kernel of ρ_n, and $I = \mathbb{Z}/2$.

Anyways, we need to discuss the caveat about a system being determined by a topological low-energy theory.

Example 18.3. Let's consider 3D Yang-Mills theory together with a Chern-Simons term. This requires data of a compact Lie group G, and the Chern-Simons term also needs a level $\lambda \in \mathbb{Z}$, and the action is

$$gF_A \wedge (\star F_A) + \mathrm{CS}(\lambda),$$

where $\mathrm{CS}(\lambda)$ is the Chern-Simons form with level λ. This is a gapped system, and the long-range system sends $g \to 0$, and hence should flow to pure Chern-Simons theory. This is believed by physicists to be true, and Witten suggested that mathematically formalizing it would be both physically interesting and mathematically feasible.

Supposing this is correct, which it probably is, what is this pure Chern-Simons theory? We're starting with a quantum field theory

$$\mathrm{YM} + \mathrm{CS}\colon \mathsf{Bord}_{\langle 2,3\rangle}(\mathrm{SO}^{\nabla}) \longrightarrow \mathsf{tVect}_{\mathbb{C}}.$$

Pure Chern-Simons theory is a well-studied topological field theory, but to make it a purely topological field theory, something strange happens. To formulate it as a topological field theory, one needs some kind of framing, but relativistic invariance tells us that the symmetry group we get is either SO_n or O_n. So for a framing to appear at the low-energy limit would be very strange!

Witten [Wit89] studied this for $G = \mathrm{U}_1$, and showed that the low-energy limit does not introduce a framing, so to study the theory we need to introduce a framing from the beginning. Then the low-energy theory is not quite topological: it's the tensor product of a purely topological Chern-Simons theory and an invertible theory which tracks the anomaly. This is how one can make sense of the theory in the absence of a framing: each piece depends on a framing, but the total theory doesn't. ◂

This is the caveat: we need to accept topological theories which could be tensored with non-topological invertible theories. If we specialize to the invertible theories with this issue, then we're looking at invertible theories in general, which need not be topological.

We can study invertible non-topological theories in a similar way, and fortunately most of the same formalism applies, albeit with some theorems replaced with conjectures.

Example 18.4. Choose a $k \in \mathbb{Z}$ and consider an invertible theory of oriented 0-manifolds together with a principal $\mathbb{T}$-bundle with connection

$$\alpha_k\colon \mathsf{Bord}_{\langle 0,1\rangle}(\mathrm{SO}_1 \times \mathbb{T}^{\nabla}) \longrightarrow \mathsf{Line}_{\mathbb{C}}.$$

Over the oriented point, we have a circle Q. Let $\lambda_k\colon \mathbb{T} \to \mathbb{C}^\times$ send $z \mapsto z^k$, so we can consider the line $\lambda_k(Q)$, the associated line bundle (over a point, hence just a line) to the $\mathbb{T}$-torsor Q, which is what α_k assigns to pt_+.

Given a bordism $[a, b]$, we want a map $L_k(P_0) \to L_k(P_1)$, and this is exactly parallel transport determined by the connection. So this is an invertible field theory, but not topological, as it depends on the connection.

We want to think of this as a map of spectra. The $\mathbb{T}$-symmetry means we want it to specifically be a map

$$MSO \wedge B\mathbb{T}_+ \longrightarrow \Sigma^1 MTSO_1 \wedge B\mathbb{T}_+ \xrightarrow{\alpha} \Sigma^2 H\mathbb{Z} \longrightarrow \Sigma^3 I\mathbb{Z}.$$

Hence this computes $H^2(B\mathbb{T}; \mathbb{Z}) \cong \mathbb{Z}$, and this is exactly the integer k we began with. So if we don't pass to the torsion subgroup, we're able to see non-topological invertible field theories. ◂

Example 18.5. For a variation, consider a smooth manifold M together with a principal $\mathbb{C}^\times$-bundle $P \to M$ with a flat connection Θ. We'll consider oriented 0- and 1-manifolds together with maps to M. By pullback, this allows us to define a theory like α_k from Example 18.4; because Θ is flat, we expect this to be topological. In this case, such an invertible field theory would be defined by a map

$$MSO \wedge M_+ \longrightarrow \Sigma^1 H\mathbb{C}^\times,$$

and the group of homotopy classes of these maps is

$$[MSO \wedge M_+, \Sigma^1 H\mathbb{C}^\times] \cong H^1(M; \mathbb{C}^\times) \cong H^2(M; \mathbb{Z})$$

through the Bockstein homomorphism, which hits the torsion subgroup. This all fits into the formalism we developed over the last few days and is great.

But you can also imagine a theory in which Θ isn't flat, and the story still goes through. We get something in $H^2(M;\mathbb{Z})$, but it isn't an invertible theory, and isn't a torsion element.

One way to understand this is through differential cohomology as introduced by Cheeger-Simons [CS85]. The idea is that secondary invariants (e.g. of curvature) manifest geometric phenomena that appear in these non-topological theories. The seminal paper on this is by Hopkins-Singer [HS05]. ◂

Given a spectrum B, the universal coefficient theorem provides for us the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\pi_n B, \mathbb{Z}) \longrightarrow [B, \Sigma^{n+1} I\mathbb{Z}] \longrightarrow \mathrm{Hom}(\pi_{n+1} B, \mathbb{Z}) \longrightarrow 0.$$

Taking π_0 defines a map

$$[B, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}(\pi_n B, \mathbb{C}^\times) \longrightarrow \mathrm{Ext}^1(\pi_n B, \mathbb{Z}).$$

The torsion part of $[B, \Sigma^{n+1} I\mathbb{Z}]$ is seen by the Ext term, and corresponds to topological theories. The free part is seen by the quotient. This leads to the following conjecture; the infrastructure (e.g. for generalized differential cohomology refining $I\mathbb{Z}$) is not currently set up for it to be a theorem.

Conjecture 18.6 ([FH16])**.** The set of deformation classes of invertible, reflection-positive field theories with symmetry type (H_n, ρ_n) is in one-to-one correspondence with $[MTH, \Sigma^{n+1} I\mathbb{Z}]$.

Compare with Theorem 13.5. We'll put this to the test in the next lecture.

19. Dan Freed: Computations in Condensed-Matter Systems

Conjecture 18.6 calculates the deformation classes of invertible, reflection-positive theories with a given symmetry type as $\pi_0 \mathrm{Map}_0(MTH, \Sigma^{n+1} I\mathbb{Z})$, but if we want to know the entire moduli space, we ought to be able to remove the π_0, so we obtain an entire homotopy type (though this is even more of a conjecture). Nonetheless, this might be a useful ground for testing finer information about the moduli space of field theories. The conjecture on deformation classes will probably be true once the right definitions are posed.

To test this conjecture, we'll apply it to electron systems in condensed-matter. This is a good test, because it's far from where we began with quantum field theory, and there are preexisting classifications. Many authors including Kitaev classified free systems and considered the map from free to interacting theories.

We don't know how to state that an electron system is free in the bordism model. Perhaps it has something to do with factoring through a symplectic vector space, but we don't know. The classification of free systems is an algebraic problem; in condensed-matter it's used to write down lattice systems, and we'll use it to determine relativistic field theories.

The first question is: what symmetry types can exist? Electrons have electric charge, which is an integer, and Noether's theorem tells us this conserved quantity corresponds to a symmetry of the system. This is a U_1-symmetry, and charge conjugation is an involutive symmetry. These might interact in different ways, and there's three possible choices for $K = \ker(\rho_n)$: $\{\pm 1\}$, $\mathbb{T}$, and SU_2, the units of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ respectively. There are other ways to motivate this: you're writing down a real representation of a spin group, and its commutant is one of these three groups. There are yet other ways of motivating it; however you do so, these are the three internal symmetry groups we consider.

The second condition is the choice of a $k_0 \in K$ squaring to 1. This is central in the spin group, and tells us what bosons and fermions are. We ask that it's $-1 \in \{\pm 1\}$, $-1 \in \mathbb{T}$, and $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in \mathrm{SU}_2$. This tells us that if a state has even charge, it's bosonic, and if it has odd charge, it's fermionic. This is called the *spin-charge relation.*

With this constraints, we have a classification problem, of the Lie groups H_n fitting into a short exact sequence

$$1 \longrightarrow K \longrightarrow H_n \longrightarrow J_n \longrightarrow 1,$$

where J_n is either SO_n or O_n. This is called a "tenfold way;" there are several other tenfold ways in this field, though this one comes from relativistic considerations rather than quantum-mechanical ways. You can split this in a few different ways: $3+3+4$ from the three choices for K, and $8+2$ coming from real and complex K-theory. For the latter see Table 5; the two missing groups are Pin^c and Spin^c. The condensed-matter physicists have a description of these symmetry groups in terms of Cartan's types of homogeneous spaces.

The division algebras in Table 5 generalize the use of Clifford algebras to define spin groups, in the following sense.

s	H	K	D
4	$\mathrm{Spin}\times_{\mu_2}\mathrm{SU}_2$	SU_2	$\mathbb{H}$
3	$\mathrm{Pin}^+\times_{\mu_2}\mathrm{SU}_2$	SU_2	Cliff_3
2	$\mathrm{Pin}^+\ltimes_{\mu_2}\mathbb{T}$	$\mathbb{T}$	Cliff_2
1	Pin^-	μ_2	Cliff_1
0	Spin	μ_2	$\mathbb{R}$
-1	Pin^+	μ_2	Cliff_{-1}
-2	$\mathrm{Pin}^-\ltimes_{\mu_2}\mathbb{T}$	$\mathbb{T}$	Cliff_{-2}
-3	$\mathrm{Pin}^-\times_{\mu_2}\mathrm{SU}_2$	SU_2	Cliff_{-3}

TABLE 5. One presentation of the (complex piece of the) 10-fold way of the classification of possible symmetries. Here H is the (stabilized) symmetry group, K is the kernel of the map $H_n \to \mathrm{O}_n$, and D is a corresponding super-division algebra. $\mathrm{Pin}^{\pm}\ltimes_{\mu_2}\mathbb{T}$ is also called $\mathrm{Pin}^{\tilde{c}\pm}$; $\mathrm{Pin}^{\tilde{c}+}$ was first introduced in [Met15].

Proposition 19.1. *Using the notation in Table 5, there's an embedding*

$$H_n(s) \longrightarrow (\mathrm{Cliff}_n \otimes D(s))^0.$$

This allows you to study all of these groups geometrically, with Dirac operators and such, which is a good way to simplify many questions about them.

Now we need some fermionic fields, which will come from spinor[14] representations $H_n(s)\to \mathrm{Aut}(\mathbb{S})$, where $\mathbb{S}$ is some real vector space. For intuition, you should be thinking about $s=0$. Let $G_n(s)$ be a Lorentz form of $H_n(s)$, e.g. $\mathrm{Spin}_{1,n-1}$ for $s=0$. There's another piece of magic here: the existence of a pairing

$$\Gamma\colon \mathbb{S}\otimes\mathbb{S}\longrightarrow \mathbb{R}^{1,n-1}$$

which is $G_n(s)$-equivariant. So inside $\mathrm{Sym}^2\,\mathbb{S}$, there's a copy of the translation representations. Moreover, there's a sense in which this is positive definite: if you look at the lightcone C of timelike vectors in $\mathbb{R}^{1,n-1}$, then for every $s\in\mathbb{S}$, $\Gamma(s,s)\in\overline{C}_+$. This crucially requires Lorentz signature, and fails for every other signature.

This miraculous algebra is fascinating — is it the reason we live in Lorentz signature? But we can use it to write down the kinetic energy term in the Lagrangian.

Theorem 19.2. *Given $\mathbb{S}$ and Γ, one can obtain a $\mathrm{Cliff}_{n-1,1}\otimes D(s)$-module structure on $\mathbb{S}\oplus\mathbb{S}^*$, and this data conversely determines $\mathbb{S}$ and Γ.*

The magical embedding allows us to treat all cases simultaneously, and this tells us no information has been lost.

Next, we ask whether there's a mass term.

Proposition 19.3. *Mass terms in this framework correspond to extension to another Clifford generator e with $e^2=-1$.*

This is linear algebra, and means you can look at the Clifford modules for $\mathrm{Cliff}_{n-1,1}\otimes D(s)$ modulo those which have a mass term, and Atiyah-Bott-Shapiro [ABS64] showed this is a KO group. Anomalies do appear here, though.

We're interested in the mass terms, however, and their long-range behavior.

Conjecture 19.4. The long-range theory is trivial if it's possible to extend to another Clifford generator.

So taking the long-range theory defines a map from these Clifford modules to invertible theories, and conjecturally they vanish if these are modules over a certain extension of the Clifford algebra. This is again a Clifford algebra, and taking the long-range theory is a function out of this group.

The group itself is $\pi_{3-s-n}KO$, which is periodic by the Bott song. Kitaev [Kit09] used this to form a periodic table of topological insulators.[15]

[14] We mean by a "spinor representation" that the $G_n(s)$-representation extends to a Clifford module.

[15] The remaining two cases are classified by complex K-theory, with an analogous story.

Stitching all this together, we arrive at another conjecture.

Conjecture 19.5. Given massive free fermion data (modulo the conjecturally trivial theories) $y \in \pi_{3-s-n}KO$, the deformation class of the long-range effective theory, which is invertible, is the composition

$$MTH(s) \xrightarrow{\widehat{A}} \Sigma^{-1}KO \xrightarrow{ff^*} \Sigma^{n+1}I\mathbb{Z},$$

where the first map is the Atiyah-Bott-Shapiro orientation, shifted by the division algebras described in Table 5, and ff is a "transpose map" arising from the Anderson self-duality of KO [HS14]:

$$\pi_{3-s-n}KO \xrightarrow{\cong} [\Sigma^{-s}KO, \Sigma^{n+1}I\mathbb{Z}],$$

and ff^* is this map applied to y.

(Here there were a whole bunch of tables organizing these symmetry groups.)

Now we have to make some computations, which use the Adams spectral sequence as previously discussed. You end up with several abelian groups organizing these phases in low dimensions. You can see Bott periodicity for KO appearing in the classification of the free fermion phases, and Anderson duality in the groups of deformation classes of topological phases.

Many of these computations have been done elsewhere in condensed-matter or high-energy literature through other methods. For example, in dimension $3+1$ with Pin^+-symmetry, the group is $\mathbb{Z}/16$; this is well-known, and many papers have been written about it (e.g. reasons why 16 times any such phase vanishes), e.g. Wang-Senthil [WS14], who make this classification without reference to Anderson duals, the Adams spectral sequence, etc.

For $\mathrm{Pin}^{\tilde{c}+}$ (time reversal squares to the fermion-counting number), one has a $\mathbb{Z}/2$ in dimensions 3 and 4 for the free-fermion groups, which turns out to be the Kane-Mele-Fu invariant that has been of considerable interest. Metlitski [Met15] asked about the classification of 4-dimensional $\mathrm{Pin}^{\tilde{c}+}$-phases versus bordism computations, which were also studied (through non bordism-theoretic means) in [WPS14].

The computations for symmetry group $\mathrm{Pin}^+ \times_{\mu_2} \mathrm{SU}_2$ also confirm a conjecture of Wang-Senthil [WS14] that was obtained in non-homotopical ways. But there also also new predictions of topological phases that might not yet be in the condensed-matter literature. Jonathan Campbell's paper [Cam17] includes even more computations, which have also been checked against the physics literature.

Wrapping up. We've covered a lot this week. The first thing we needed in order to do anything is an axiom system. And the fact that these methods agree with very different calculations is, well, not a controlled experiment, but certainly is a good sign that the axiom system works: functorial TFT captures something essential, just as "complete ordered field" somehow captures the essentials of calculus. The fact that Wick rotation and the restriction to compact manifolds captures long-range behavior is another *a priori* surprise, and we have evidence that it works.

Extended TFT was also essential: if you don't do this, you obtain different answers, and so the extensions of the axioms are also important.

Finally, the basic idea of the two physical principles turning a lattice system into a long-range theory has been bolstered by these verifications.

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