FIVE HOURS OF TOPOLOGICAL FIELD THEORY

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Abstract. ...

0. Introduction

TODO: say more.

These are lecture notes I wrote for a minicourse on TQFT at UT Austin in summer 2019. These ideas have been drawn from a number of references, which I unfortunately have not added in yet. Beware of potential typos.

1. Definitions and generalities

TODOs: better exposition of tangential structures; address what a ξ-diffeomorphism is

1.1. Bordism.

Probably you already know the basic definitions, but let's refresh them.

Definition 1.1. Let M_0 and M_1 be closed n-manifolds. A *bordism* from M_0 to M_1 is a compact $(n+1)$-manifold X, a partition $\partial X = Y_0 \amalg Y_1$, and diffeomorphisms $\theta_i \colon Y_i \xrightarrow{\cong} M_i$. If there is a bordism from M_0 to M_1, we say M_0 and M_1 are *bordant*.

Importantly, the maps θ_i are part of the data. A few elementary observations:

- Bordism is *reflexive*: there's a bordism M to M, namely $M \times [0,1]$.
- Bordism is *symmetric*: if M and N are bordant, then N and M are bordant.
- Bordism is *transitive*: if X is a bordism from M to N, and Y is a bordism from N to P, then we can glue X and Y along N to obtain a bordism from M to P. (There are details I did not say here – if this is your first introduction to bordism, it may be helpful to fill them in.)

So bordism is an equivalence relation. The set of equivalence classes is denoted Ω_n.

The first step is to define an abelian group structure on Ω_n. One can take the disjoint union of bordisms: if X is a bordism from M_0 to M_1, and Y is a bordism from N_0 to N_1, then $X \amalg Y$ is a bordism from $M_0 \amalg N_0$ to $M_1 \amalg N_1$. Therefore disjoint union defines an operation on equivalence classes $\amalg \colon \Omega_n \times \Omega_n \to \Omega_n$. Disjoint union is associative and commutative (up to natural isomorphism), so this is associative and commutative, period. Moreover, the empty n-manifold is the identity for this operation, and we get an abelian group. Computing these abelian groups (and variants introduced below) was a classical problem in algebraic topology, and one of the first.

Remark 1.2 (Ring structure). The graded abelian group $\Omega_* := \bigoplus_{n\ge 0} \Omega_n$ has the structure of a ring, where multiplication is Cartesian product and the point is the identity. (Again, because this is defined on equivalence classes, there are some details to check here.) This structure is interesting in homotopy theory, but for whatever reason doesn't appear much in Segal-style TQFT. ◀

Often we care about manifolds with additional topological structure: orientations, principal G-bundles, etc. (We will not address geometric structures, such as metrics or connections.)

Definition 1.3. An *n-dimensional tangential structure* is a pointed space ξ_n and a pointed fibration $\alpha_n \colon \xi_n \to B\mathrm{O}_n$. An *$n$-manifold with ξ_n-structure* is an n-manifold M together with a lift of its classifying map $c_{TM} \colon M \to B\mathrm{O}_n$ across α_n.

Example 1.4. A large class of important examples arise from the data of a Lie group G_n and a representation $\rho_n\colon G_n \to \mathrm{O}_n$: taking classifying spaces, we get $B\rho_n\colon BG_n \to B\mathrm{O}_n$, whose associated tangential structure is called a *G_n-structure*. On an n-manifold M, a G_n-structure is equivalent to a choice of a principal G_n-bundle $P \to M$ and an isomorphism $P \times_{G_n} \mathrm{O}_n \xrightarrow{\cong} \mathcal{B}_{\mathrm{O}}(M)$. This includes many examples you may have already seen, such as orientations (SO_n), spin structures (Spin_n), and almost complex structures ($\mathrm{U}_{n/2}$). ◂

It's often useful to work independently of the dimension n. Let $\mathrm{O}_\infty := \mathrm{colim}_n\, \mathrm{O}_n$, which is a topological group.

Definition 1.5. A *stable tangential structure* is a pointed space ξ and a pointed fibration $\alpha\colon \xi \to B\mathrm{O}_\infty$. For any n, this defines an n-dimensional tangential structure $\xi_n \to B\mathrm{O}_n$ as the pullback of α across the map $B\mathrm{O}_n \to B\mathrm{O}_\infty$; a *$\xi$-structure* on an n-manifold M is then a ξ_n-structure.

Most examples of tangential structures are stable: we can form $B\mathrm{SO}_\infty$, $B\mathrm{Spin}_\infty$, $B\mathrm{Spin}^c_\infty$, $B\mathrm{Pin}^\pm_\infty$, etc., and so orientations, spin structures, spin^c structures, etc. can be defined in this way.

Remark 1.6. Two nonexamples:

- An almost complex structure is not the same thing as a $B\mathrm{U}_\infty$-structure (which is called a *stably almost complex structure*): a $B\mathrm{U}_\infty$-structure on M is a choice of a complex structure on $TM \oplus \underline{\mathbb{R}}^k$ for some k; in particular, M may be odd-dimensional.
- The r-fold connected covering map $\mathrm{SO}_2 \to \mathrm{SO}_2$, followed by the inclusion $\mathrm{SO}_2 \hookrightarrow \mathrm{O}_2$, defines a two-dimensional structure called an *r-spin structure*; since $\pi_1\mathrm{SO}_m = \mathbb{Z}/2$ if $m \geq 3$, this structure cannot stabilize. ◂

Remark 1.7. For some tangential structures ξ, we can make sense of the notion of a *ξ-diffeomorphism*: for example, if the tangential structure is a map to a smooth manifold X, we want a diffeomorphism which leaves the map invariant. If it's an orientation, we want orientation-preserving diffeomorphisms; for a general G-structure, we want data of a diffeomorphism $\varphi\colon M \to M$ and a G-equivariant diffeomorphism of the principal G-bundles of frames covering φ. Sometimes this is bigger than the ordinary diffeomorphism group – for example, spin manifolds have a spin diffeomorphism called *spin flip* which is trivial on the manifold but acts by -1 on the bundle of spin frames. Then one can take π_0 of the ξ-diffeomorphism group and obtain a ξ-mapping class group.

I'm not going to strive for maximal generality – when I discuss ξ-diffeomorphisms, I have in mind something like one of the above examples. ◂

Lemma 1.8. *A ξ-structure on a manifold M induces a ξ-structure on ∂M.*

That is, if $\dim M = n$, we have a ξ_n-structure on M, and this induces a ξ_{n-1}-structure on ∂M.

Exercise 1.9. Prove Lemma 1.8.

Crucial fact: there are two different ways to induce the ξ-structure on ∂M, stemming from the two trivializations of the normal bundle of $\partial M \hookrightarrow M$. The standard convention is to use the outward normal; hence, if we use the inward normal to define a ξ-structure on ∂M, we'll denote it $-\partial M$. For orientations, for example, $-\partial M$ carries the opposite orientation from ∂M.

Lemma 1.8 means we can make sense of bordisms of manifolds of ξ-structure.

Definition 1.10. Let M and N be closed n-manifolds with ξ-structure. A *bordism* X from M to N is a compact ξ-manifold X, an identification $\partial X \cong X_0 \amalg X_1$, and diffeomorphisms of ξ-manifolds $\theta_0\colon M \xrightarrow{\cong} X_0$ and $\theta_1\colon N \xrightarrow{\cong} -X_1$.

That minus sign is important!

Anyways, as before, bordism is an equivalence relation, and the equivalence classes in dimension n form an abelian group denoted $\Omega_n^{t\xi}$, under disjoint union.

Remark 1.11. We often, *but not always*, get a bordism ring $\Omega_*^{t\xi}$ as before, induced from Cartesian product. For example, this happens with orientations, spin structures, stably almost complex structures. But it's not always true that the product of ξ-manifolds admits a ξ-structure, e.g. this happens for pin$^+$ and pin$^-$ structures. ◂

1.2. **Bordism categories and topological field theories.** Fix a dimension n and a tangential (n-)structure ξ. When we checked that $\Omega_{n-1}^{t\xi}$ is an abelian group, we had to check the following (and also a few more axioms).

(1) Bordism is reflexive, meaning M is bordant to M, via the cylinder $M \times [0,1]$.
(2) Bordism is transitive (meaning if M is bordant to N and N is bordant to P, then M is bordant to P). We proved this by gluing bordisms: if X is a bordism from M to N and Y is a bordism from N to P, then $X \cup_Y N$ is a bordism from M to P.
(3) Disjoint union is associative and commutative up to natural isomorphism, and the empty $(n-1)$-manifold is a unit for it.

These allow us to extract a more elaborate object out of bordism: a symmetric monoidal category. I'm not going to get into the precise definition of a symmetric monoidal category here, but it consists of data of:

- a category $\mathcal{C}$,
- a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a unit $1 \in \mathcal{C}$, and also
- choices of natural isomorphisms ensuring that $\otimes$ is associative and commutative up to natural isomorphism, and that 1 is the unit for it (e.g. a natural isomorphism $1 \otimes - \Rightarrow \mathrm{id}$).

Similarly, we can ask for a functor F between symmetric monoidal categories to be symmetric monoidal, which involves additional data of natural isomorphisms $F(1) \cong 1$ and $F(x \otimes y) \cong F(x) \otimes F(y)$.

The point of all this is: a symmetric monoidal category is one where you can tensor stuff together, like in $\mathcal{V}ect_k, \otimes$, and $V \otimes W$ and $W \otimes V$ are identified. Symmetric monoidal functors are those which commute with the tensor product: $F(a \otimes b) \cong F(a) \otimes F(b)$.

Definition 1.12. The *bordism category* $\mathcal{B}ord_n^\xi$ is the symmetric monoidal category defined by the following data.

- The objects are closed $(n-1)$-manifolds with ξ-structure.
- The morphisms $M \to N$ are the diffeomorphism classes of[1] bordisms from M to N as ξ-manifolds. Composition is gluing of bordisms.
- The monoidal unit is the empty ξ-manifold.
- The monoidal product is disjoint union.

So, of course,

Definition 1.13 (Atiyah [Ati88], Segal [Seg88])**.** A *topological quantum field theory (of ξ-manifolds)* is a symmetric monoidal functor $Z\colon \mathcal{B}ord_n^\xi \to (\mathcal{V}ect_\mathbb{C}, \otimes)$.

Sometimes people use other target categories, e.g. $s\mathcal{V}ect_\mathbb{C}$, $\mathbb{Z}/2$-graded vector spaces with the tensor product using the Koszul sign rule. Another common example is the category of chain complexes, for the derived people in the audience.

Example 1.14 (Euler TQFT)**.** Fix $\lambda \in \mathbb{C}^\times$. We will define an n-dimensional TQFT $Z_\lambda\colon \mathcal{B}ord_n \to \mathcal{V}ect_\mathbb{C}$, called the *Euler TQFT*, one of the simplest possible examples. The state space of every closed $(n-1)$-manifold is $\mathbb{C}$, and given a bordism $X\colon Y \to Z$, $Z(X)\colon \mathbb{C} \to \mathbb{C}$ is multiplication by $\lambda^{\chi(X,Y)}$. Notice that composition of bordisms leads to additivity of the relative Euler characteristic, hence multiplication as we need. ◀

Now let's prove the first general theorem about TQFTs.

Theorem 1.15. *If $Z\colon \mathcal{B}ord_n^\xi \to \mathcal{V}ect_\mathbb{C}$ is a TQFT, then for any closed $(n-1)$-dimensional ξ-manifold M, $Z(M)$ is finite-dimensional.*

We'll need to discuss duality before we can prove this.

Definition 1.16 (Dold-Puppe [DP83])**.** Let $\mathcal{C}$ be a symmetric monoidal category. *Duality data* for an $x \in \mathcal{C}$ is an object $x^\vee \in \mathcal{C}$ and morphisms $e\colon x \otimes x^\vee \to 1$ and $c\colon 1 \to x^\vee \otimes x$, such that the following maps compose

[1] We must take diffeomorphism classes so that composition is associative on the nose (rather than up to some isomorphism). Diffeomorphisms are taken rel boundary.

to the identity:

$$x \xrightarrow{c\otimes \mathrm{id}_x} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{id}_x \otimes e} x \tag{1.17a}$$

$$x^\vee \xrightarrow{\mathrm{id}_{x^\vee}\otimes c} x^\vee \otimes x \otimes x^\vee \xrightarrow{e\otimes \mathrm{id}_{x^\vee}} x^\vee. \tag{1.17b}$$

If duality data exists for x, we call x *dualizable*, $x^\vee$ the *dual* of x, e *evaluation*, and c *coevaluation*.

It is common to denote evaluation and coevaluation with the bordism-like pictures in Figure 1, so that the

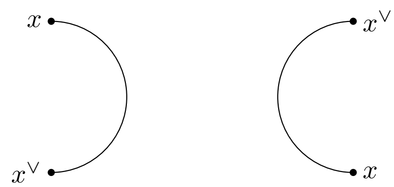

FIGURE 1. Evaluation (on left) and coevaluation (on right).

conditions in (1.17) admit a nice pictoral interpretation as in Figure 2. For reasons that may be apparent

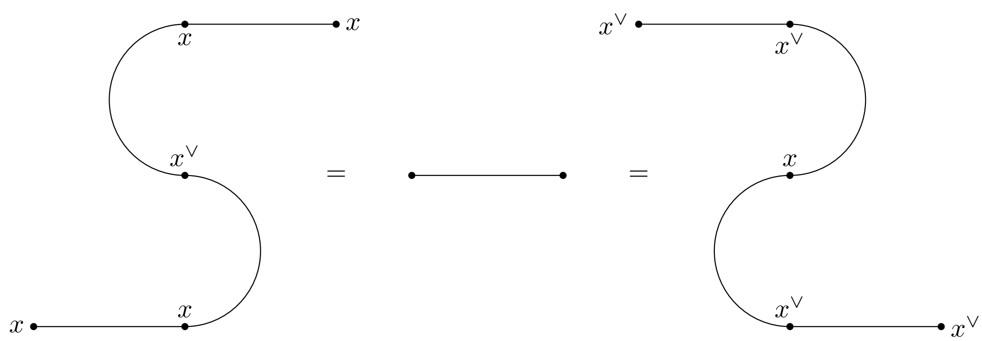

FIGURE 2. Left: the S-diagram, encoding (1.17a). Right: the Z-diagram, encoding (1.17b). These equalities are the conditions on duality data.

they are called the *S-diagram* and *Z-diagram*.

Remark 1.18. It looks like we made a choice, so why "the" dual? You can check that any two choices of duality data for x are isomorphic via a unique isomorphism. ◀

Example 1.19. In $\mathcal{V}ect_k$, dualizability is equivalent to being finite-dimensional: given a vector space V with duality data $(V^\vee, c, e)$, let $c(1) = \sum v_i \otimes v_i'$ for $v_i, v_i' \in V$; crucially, this is a finite sum. Using (1.17b), for any $x \in V$,

$$x = \sum e(x, v_i) v_i', \tag{1.20}$$

so $\{v_i'\}$ spans V. (In the other direction, we can construct duality data for any finite-dimensional vector space V by taking $V^\vee := \mathrm{Hom}_k(V, k)$, etc.).

The same argument works for $s\mathcal{V}ect_k$. ◀

Example 1.21. In $\mathcal{B}ord_n^\xi$, all objects are dualizable: the dual of M is $-M$, and evaluation and coevaluation are the cylinder bordisms, where both components are outgoing or incoming. The conditions on duality data follow from "Zorro's lemma," that the two bordisms in Figure 3 are diffeomorphic (this is the Z-diagram; the argument for the S-diagram is analogous). ◀

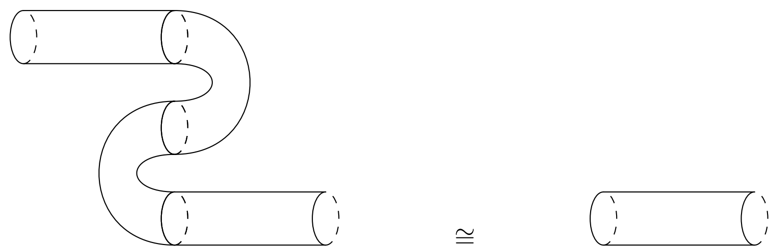

FIGURE 3. "Zorro's lemma," that these two bordisms are equivalent, shows that all objects of $\mathcal{B}ord_n^\xi$ are dualizable.

Proof of Theorem 1.15. We will prove the stronger result that if $Z\colon \mathcal{B}ord_n^\xi \to \mathcal{C}$ is a TQFT, where $\mathcal{C}$ is any symmetric monoidal category, then Z of any object is dualizable; this suffices by Example 1.19.

The notion of duality data is preserved by a symmetric monoidal functor, so since any object $M \in \mathcal{B}ord_n^\xi$ is dualizable, then $Z(M)$ must also be dualizable. ⊠

Definition 1.22. The *dimension* of a dualizable object x in a symmetric monoidal category $\mathcal{C}$ is $e \circ c \in \mathrm{End}_{\mathcal{C}}(1)$.

Example 1.23. In $\mathcal{V}ect_k^{\mathrm{fd}}$, under the identification $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$, this number agrees with $\dim V$. But in $s\mathcal{V}ect_k^{\mathrm{fd}}$, it's (TODOcheck) $\dim V^0 - \dim V^1$. ◂

To say this in a different way, if M is a closed $(n-1)$-manifold, then $Z(S^1 \times M) = \dim Z(M)$. Sometimes this is a useful calculational tool.

Next we'll generalize this a bit: given a closed $(n-1)$-dimensional ξ-manifold M and any n-dimensional TQFT of ξ-manifolds Z, we'll construct a canonical $\mathrm{MCG}^\xi(M)$-action on $Z(M)$, as long as we know what a ξ-diffeomorphism is.

Let $\varphi\colon M \to M$ be a ξ-diffeomorphism. The *mapping cylinder* of φ, denoted C_φ, is the bordism $M \to M$ which as a manifold is $M \times [0,1]$, glued by id at 0 and φ at 1.

Lemma 1.24. *If φ is isotopic to the identity,*[2] *then in the bordism category, $C_\varphi = \mathrm{id}$.*

Proof. It suffices to produce a diffeomorphism rel boundary from $\Phi\colon M \times [0,1] \xrightarrow{\cong} C_\varphi$. Let $f_t\colon I \to \mathrm{Diff}^\xi(M)$ be a path from id to φ; then $\Phi(x,t) := (f_t(x), t)$ works. ⊠

Therefore this is actually an $\mathrm{MCG}^\xi(M)$-action. Often these are interesting on their own (TODO: example).

Definition 1.25. The *mapping torus* of a $\varphi \in \mathrm{Diff}^\xi(M)$, denoted M_φ, is the ξ-manifold $[0,1] \times M/(0,x) \sim (1, f(x))$.

As above, the ξ-diffeomorphism type of M_φ doesn't depend on the path component of φ, so we can make the definition for $\varphi \in \mathrm{MCG}^\varphi(M)$.

Proposition 1.26. *For any $\varphi \in \mathrm{MCG}^\xi(M)$, $Z(M_\varphi) = \mathrm{tr}(Z(C_\varphi))$. That is, turning the mapping cylinder into a torus takes the trace of the action.*

This is an instance of a very general phenomenon: crossing with a circle (and maybe twisting something) in a TQFT is usually akin to some sort of trace.

Proof. Write M_φ as the following composition of bordisms: coevaluation, then $-\mathrm{id} \amalg C_\varphi$, then evaluation. If $\{e_i\}$ is a basis of $Z(M)$ and $\{e^i\}$ denotes the dual basis, then $Z(M_\varphi)$ factors as

$$1 \xmapsto{c} e^i \otimes e_i \xmapsto{\mathrm{id}\otimes Z(C_\varphi)} e^i \otimes Z(C_\varphi)(e_i) \xmapsto{e} e^i(Z(C_\varphi)(e_i)) = \mathrm{tr}(Z(C_\varphi)). \tag{1.27}$$ ⊠

Exercise 1.28. There is a *direct sum* of TQFTs $Z_1, Z_2\colon \mathcal{B}ord_n^\xi \to \mathcal{V}ect_{\mathbb{C}}$, defined as follows: if M is a connected $(n-1)$-manifold with ξ-structure, $(Z_1 \oplus Z_2)(M) := Z_1(M) \oplus Z_2(M)$, which therefore defines it on all $(n-1)$-dimensional ξ-manifolds (if M isn't connected, $(Z_1 \oplus Z_2)(M)$ is *not* $Z_1(M) \oplus Z_2(M)$!). Make sense of the direct sum on bordisms, and check that it's a TQFT.

Exercise 1.29. Proposition 1.26 used that the target category is $\mathcal{V}ect_{\mathbb{C}}$. What about $s\mathcal{V}ect_{\mathbb{C}}$?

[2] This is also true in the case of pseudoisotopy.

2. Invertible TFTs and homotopy theory

TODO: fix definition of k-invariant

Today we will discuss the simplest class of examples of TFTs, and find that they can be described in terms of fairly classical algebraic topology: bordism invariants. There are different ways to prove this; the most general, due to Freed-Hopkins [FH16], goes through some homotopy theory (in particular the determination of the homotopy type of the bordism category). Today, I'll present a less general but somewhat more geometric approach due to Kreck-Stolz-Teichner (and closely related to the approach of Rovi-Schoembauer [RS18]).

Definition 2.1. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category and $x \in \mathcal{C}$. We say x is *invertible* if there is an object $x^{-1} \in \mathcal{C}$ and an isomorphism $x \otimes x^{-1} \xrightarrow{\cong} 1$.

The inverse x^{-1} is unique up to unique isomorphism, so long as it exists.

Definition 2.2. A *Picard groupoid* is a symmetric monoidal category which is a groupoid (so every morphism is invertible under composition) and such that every object is $\otimes$-invertible.

Given any symmetric monoidal category $\mathcal{C}$, the subcategory $\mathcal{C}^\times$ of invertible objects and invertible morphisms is a Picard groupoid. For example, $\mathcal{V}ect_{\mathbb{C}}^\times$ is the category of complex lines and nonzero linear maps between them.

Definition 2.3 (Freed-Moore [FM06])**.** A topological field theory $Z\colon \mathcal{B}ord_n^\xi \to \mathcal{V}ect_{\mathbb{C}}$ is *invertible* if it factors through $\mathcal{C}^\times \hookrightarrow \mathcal{C}$.

Equivalently:

- for any closed $(n-1)$-dimensional ξ-manifold M, $Z(M)$ is $\otimes$-invertible (so in $\mathcal{V}ect_{\mathbb{C}}$, a one-dimensional vector space) and for any bordism X, $Z(X)$ is invertible under composition.
- Alternatively, the symmetric monoidal category $\mathcal{TQFT}_n^\xi$ of TQFTs has objects symmetric monoidal functors $\mathcal{B}ord_n^\xi \to \mathcal{V}ect_{\mathbb{C}}$ and morphisms the symmetric monoidal natural transformations between them. The tensor product is "pointwise": $(Z_1 \otimes Z_2)(M) \coloneqq Z_1(M) \otimes Z_2(M)$, which physically corresponds to formulating two systems in the same material with no interactions. (In condensed-matter physics, this is called *stacking*). Anyways, the invertible objects of $\mathcal{TQFT}_n^\xi$ are precisely the invertible TQFTs.

The first step in the classification of invertible field theories is observing that we can complete $\mathcal{B}ord_n^\xi$ to a Picard groupoid $|\mathcal{B}ord_n^\xi|$ by formally adjoining inverses of every object (under $\otimes$) and every morphism (under composition), and that every symmetric monoidal functor $\mathcal{B}ord_n^\xi \to \mathcal{C}^\times$, where $\mathcal{C}^\times$ is a Picard groupoid, factors uniquely (up to natural isomorphism) through $|\mathcal{B}ord_n^\xi|$. This is closely analogus to the localization of a ring and its universal property.

The upshot is: *the abelian group of invertible maps $\mathcal{B}ord_n^\xi \to \mathcal{C}$ is equivalent to the one of all maps $|\mathcal{B}ord_n^\xi| \to \mathcal{C}^\times$.* So to classify invertible field theories, we "just" have to understand some Picard groupoids, which turns out to not be so bad.

(Notice that we've done different things to the domain and codomain: the Picard groupoid completion is a quotient, and $\mathcal{C}^\times \subset \mathcal{C}$ is a subcategory.)

A Picard groupoid $\mathcal{C}$ has three invariants.

- $\pi_0\mathcal{C}$, the set of isomorphism classes of objects. This is an abelian group under tensor product.
- $\pi_1\mathcal{C} \coloneqq \mathrm{Aut}(1_{\mathcal{C}})$. This is also an abelian group, which follows from Eckmann-Hilton (for the same reason π_1 of a topological group is abelian).
- The *k-invariant* $k\colon \pi_0\mathcal{C} \otimes \mathbb{Z}/2 \to \pi_1\mathcal{C}$, which is defined to be the composition

$$1_{\mathcal{C}} \xrightarrow{\theta} x \otimes x \xrightarrow{\sigma} x \otimes x \xrightarrow{\theta^{-1}} 1_{\mathcal{C}}, \tag{2.4}$$

  for any $x \in \mathcal{C}$ and $\theta\colon 1 \to x \otimes x$; one then checks that this doesn't depend on those choices. Here σ implements the map $x \otimes y \to y \otimes x$.

Theorem 2.5 (Hoàng[3] [Hoà75])**.** *(π_0, π_1, k) is a complete invariant of Picard groupoids.*

[3] Though her full name is Hoàng Xuân Sính, Hoàng is her family name. Sometimes I've seen this theorem attributed to Sính, which is confusing.

That is, if $\mathcal{C}$ and $\mathcal{D}$ have the same π_0, π_1, and k, there is an equivalence of symmetric monoidal categories between them.

Thus, it suffices to understand these data for $|\mathcal{B}ord_n^\xi|$ and $s\mathcal{V}ect_{\mathbb{C}}^\times$. The latter is fairly simple.

Proposition 2.6. *$\pi_0(s\mathcal{V}ect_{\mathbb{C}}^\times) \cong \mathbb{Z}/2$, $\pi_1(s\mathcal{V}ect_{\mathbb{C}}^\times) \cong \mathbb{C}^\times$, and the k-invariant is nontrivial (which determines it).*

Proof. Well π_0 is asking about isomorphism classes of $\otimes$-invertible super-vector spaces. This implies invertibility on the underlying vector space, so if V is invertible, it's one dimensional, hence isomorphic to either $\mathbb{C}$ or $\Pi\mathbb{C}$. Then, $\pi_1(\mathcal{C}) = \operatorname{Aut} \mathcal{C}(1)$, so we're asking for the group of automorphisms of $\mathbb{C}$, which is $\mathbb{C}^\times$.

The k-invariant is asking what the symmetry map $\Pi\mathbb{C}\otimes\Pi\mathbb{C} \to \Pi\mathbb{C}\otimes\Pi\mathbb{C}$ is. The whole point of the Koszul sign rule is that this sends $x\otimes y \mapsto -y\otimes x$, meaning the k-invariant, as a map $\pi_0 s\mathcal{V}ect_{\mathbb{C}}^\times \to \pi_1 s\mathcal{V}ect_{\mathbb{C}}^\times \cong \mathbb{C}^\times$, sends $\Pi\mathbb{C} \mapsto -1$. ⊠

The data for $|\mathcal{B}ord_n^\xi|$ requires another definition.

Definition 2.7 (Jänich, Karras-Kreck-Neumann-Ossa [KKNO73]). The *SKK group* SKK_n^ξ is the group completion of the monoid of diffeomorphism classes of n-dimensional ξ-manifolds under $\amalg$, modulo the following relation: let $X_0, \dots, X_3$ be compact n-dimensional ξ-manifolds together with identifications of their boundaries with some $(n-1)$-dimensional ξ-manifold Y; then, in SKK_n^ξ, we impose that

$$(X_0 \cup_Y X_1) \amalg (X_2 \cup_Y X_3) = (X_2 \cup_Y X_1) \amalg (X_0 \cup_Y X_3). \tag{2.8}$$

TODO: add picture.

The acronym "SKK," when translated from German to English, means "controlled cutting and pasting:" the original (different but equivalent) definition, these are manifold invariants whose values change in a controlled way under cutting along a codimension 1 submanifold and pasting back together in a possibly different way.

Theorem 2.9 (Galatius-Madsen-Tillmann-Weiss [GMTW09]). *The data for $|\mathcal{B}ord_n^\xi|$ is $\pi_0 = \Omega_{n-1}^\xi$, $\pi_1 = \mathrm{SKK}_n^\xi$, and the k-invariant is $M \mapsto M \times S^1$, where S^1 carries the ξ-structure induced by the Lie group framing.*

The difficult part is π_1; understanding why π_0 is the usual bordism group is a good way to check if you're following the definitions.

Anyways, here's the main result for today.

Corollary 2.10. *Taking the partition function defines an isomorphism of abelian groups $(\mathcal{TQFT}_n^\xi)^\times \to \operatorname{Hom}(\mathrm{SKK}_n^\xi, \mathbb{C}^\times)$.*

Proof. First, why is there even a map? A map of Picard groupoids $\mathcal{C} \to \mathcal{D}$ is equivalent data to abelian group maps $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ and $\pi_1\mathcal{C} \to \pi_1\mathcal{D}$ intertwining the k-invariant. In our setting, the partition function of an invertible TQFT is the map on π_1; using **??**, this is an SKK invariant valued in $\mathbb{C}^\times$. Similarly, the state space (or rather, its class in $\pi_0 s\mathcal{V}ect_{\mathbb{C}}^\times$) is a bordism invariant.

We didn't prove that $\pi_1|\mathcal{B}ord_n^\xi| \cong \mathrm{SKK}_n^\xi$, but it's also possible to give a direct proof that the partition function must be an SKK invariant. Consider the bordism TODO: an invertible TQFT Z assigns to it the map

$$\mathbb{C} \xrightarrow{Z(M_0)\otimes Z(M_2)} Z(Y)\otimes Z(Y) \xrightarrow{\ \sigma\ } Z(Y)\otimes Z(Y) \xrightarrow{Z(M_1)\otimes Z(M_3)} \mathbb{C}. \tag{2.11}$$

It suffices to show $Z(\sigma) = \mathrm{id}$. This is not *a priori* true; if $Z(Y)$ were odd, then it would be -1. But the degree of the state space is a bordism invariant as we noted, so $Z(Y) \cong Z(\varnothing)$ is even.

Conversely, given an SKK invariant, we can recover an invertible TQFT. This is because the k-invariant of $s\mathcal{V}ect_{\mathbb{C}}^\times$ is injective: it's the map $\mathbb{Z}/2 \cong \{\pm 1\} \hookrightarrow \mathbb{C}^\times$. Therefore as soon as we know the map on π_1, we get the map on π_0, because the following diagram must commute.

$$\begin{tikzcd} \pi_1|\mathcal{B}ord_n^\xi| \arrow[r] & \pi_1 s\mathcal{V}ect_{\mathbb{C}}^\times \\ \pi_0|\mathcal{B}ord_n^\xi| \arrow[r] \arrow[u, "k"] & \pi_0 s\mathcal{V}ect_{\mathbb{C}}^\times. \arrow[u, hook] \end{tikzcd} \tag{2.12}$$

⊠

Remark 2.13. You can play with pictures to see the k-invariant, but one interesting way is using homotopy theory: given any Picard groupoid $\mathcal{C}$, $\pi_*\mathbb{S}$ acts on $|N_\bullet\mathcal{C}|$ (as it does on any spectrum), and the k-invariant is multiplication by the Hopf element $\eta \in \pi_1\mathbb{S}$. When we look at bordism categories, we apply Pontrjagin-Thom, and η acts by its representative in framed bordism, which is S^1 with its Lie group framing. ◀

Do you know any SKK invariants? We can get most of them from bordism invariants.

Theorem 2.14 (Karras-Kreck-Neumann-Ossa [KKNO73])**.** *Sending a manifold to its bordism class descends to a homomorphism* $\mathrm{SKK}_n^\xi \to \Omega_n^\xi$*, under the assumption that* ξ*-structures make sense in dimension* $n+1$*.*

Proof. We prove something that's *a priori* stronger: let M_1 and M_2 be two compact ξ-manifolds with identifications $\theta_i\colon \partial M_i \xrightarrow{\cong} N$ and $\varphi, \psi \in \mathrm{Diff}^\xi(N)$. It suffices to show that $M_1 \cup_\varphi M_2$ and $M_1 \cup_\psi M_2$ are bordant: then we see the SKK relation by letting $M_1 := X_0 \amalg X_2$, $M_2 := X_1 \amalg X_3$, $N := Y \amalg Y$, φ be the identity, and ψ switch the two copies of Y.

Let's construct a bordism from $M_1 \cup_\varphi M_2$ to $M_1 \cup_\psi M_2$. Begin with $(M_1 \times [0,1]) \amalg (M_2 \times [0,1])$; then identify $(x,t) \sim (\varphi(x), t)$ for $x \in \partial M_1$ and $t \in [0,1/3]$ and $(x,t) \sim (\psi(x),t)$ for $x \in \partial M_1$ and $t \in [2/3,1]$; then smooth. See Figure 4 for a picture. ⊠

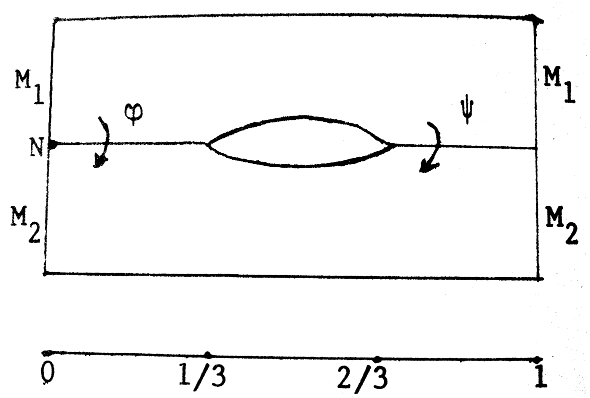

FIGURE 4. The bordism employed in the proof of Theorem 2.14. Source: [KKNO73].

So let's make some bordism invariants. *Characteristic numbers* are a nice class of bordism invariants: begin with cohomology classes canonically associated with the manifold and/or ξ-structure, and do something to get them in top degree. Then evaluate on the fundamental class to obtain a number. Here's an example; the proofs will generalize to many other situations.

Lemma 2.15. *The quantity* $\langle p_1(M)c_1(P), [M]\rangle$ *defines a (*$\mathbb{Z}$*-valued) bordism invariant of oriented* 6*-manifolds* M *with a principal* U_1*-bundle* $P \to M$*.*

In some situations, there's an easy proof using Stokes' theorem.

Proof. Suppose (M,P) bounds, i.e. there's a compact oriented 7-manifold W and a principal U_1-bundle $Q \to W$ with $Q|_M \cong P$. We can realize c_1 and p_1 as closed differential forms using Chern-Weil theory, and can choose them such that $c_1(P) = c_1(Q)|_M$ in Ω_M^2 and $p_1(M) = p_1(W)|_M$ in Ω_M^4. By Stokes' theorem,

$$\int_M c_1(P)p_1(M) = \int_W \mathrm{d}(c_1(P)p_1(M)) = 0. \tag{2.16}$$

One should also check additivity under disjoint union, but that part's easier. ⊠

But not all bordism invariants come from differential forms. Here's another proof that works in general, giving you, e.g., Stiefel-Whitney numbers for unoriented manifolds.

Proof. Again, let (W,Q) bound (M,P). The key fact is that $TW|_M = TM \oplus \nu$, where ν is the normal bundle for $i\colon M \hookrightarrow W$, and ν is trivial. The first Pontrjagin class is *stable*, so $p_1(E \oplus \underline{\mathbb{R}}) = p_1(E)$, and this means that $i^*p_1(W) = p_1(M)$, and since $Q|_M \cong P$, $i^*c_1(Q) = c_1(P)$. Now we can use the long exact sequence of a pair:

$$H^6(W) \xrightarrow{i^*} H^6(M) \xrightarrow{\delta} H^7(W,M), \tag{2.17}$$

so $p_1(M)c_1(P) \in \mathrm{Im}(i^*) = \ker(\delta)$.

Let $[W,M] \in H_7(W,M;\mathbb{Z})$ denote the fundamental class of the pair: under the connecting morphism $\partial\colon H_7(W,M) \to H_6(M)$, $[W,M] \mapsto [M]$. Lefschetz duality gives us a version of Stokes' theorem: if $x \in H^6(M)$, then

$$\langle x, \partial[W,M]\rangle = \langle \delta x, [W]\rangle. \tag{2.18}$$

Hence

$$\langle p_1(M)c_1(P), [M]\rangle = \langle p_1(M)c_1(P), \partial[W,M]\rangle = \langle \delta(p_1(M)c_1(P)), [W,M]\rangle = 0. \tag{2.19}$$ ⊠

In fact, if you're willing to play with twisted, generalized, or twisted generalized cohomology, the essence of the above argument still works, and you get even more bordism invariants. This is particularly useful for spin bordism and its variants.

Example 2.20 (The Arf theory)**.** The point of this extended example is to use this algebro-topological perspective to extract a lot of information about a particular invertible TQFT, the Arf theory, which is a 2D spin TQFT first defined by Gunningham [Gun16]. Though it's a toy model for us, it shows up in interesting places in physics and math.

To begin, let's talk about spin bordism.

- $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$. There are two non-bordant spin structures in S^1: the *bounding spin circle* S^1_b and the *nonbounding spin circle* S^1_{nb}. These can both be realized as the spin structure canonically induced from a framing: for S^1_b, choose the framing arising from $S^1 = \partial D^2$, and for the nonbounding one, use the Lie group framing.
- $\Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2$, and the *Arf invariant* is a complete invariant. It admits a few different descriptions, but here's one I like: a spin structure on a surface Σ induces spin structures on all embedded circles. This defines a function $q\colon H_1(\Sigma;\mathbb{Z}/2) \to \Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$, and the Arf invariant is whichever element of $\mathbb{Z}/2$ has a bigger preimage. (It turns out ties cannot occur.)

To the best of my knowledge, these bordism groups were first computed by Milnor [Mil63]; Kirby-Taylor [KT90, §2] have a nice proof. The Arf invariant was first studied by Atiyah [Ati71] and Johnson [Joh80].

Hence, the Arf invariant $(-1)^{\mathrm{Arf}(\Sigma,\mathfrak{s})} \in \{\pm 1\} \subset \mathbb{C}^\times$ is a spin bordism invariant, hence defines a 2D invertible spin TQFT $Z_{\mathrm{Arf}}\colon \mathcal{B}ord_2^{\mathrm{Spin}} \to s\mathcal{V}ect_{\mathbb{C}}$.

We know the partition function is just the Arf invariant again, but what about the state spaces? Let's compute. Because the k-invariant of $s\mathcal{V}ect_{\mathbb{C}}^\times$ is injective, we can recover $Z_{\mathrm{Arf}}(C)$ from $Z_{\mathrm{Arf}}(C \times S^1_{nb})$, as the k-invariant is $- \times S^1_{nb}$. So:

- $Z_{\mathrm{Arf}}(S^1_b \times S^1_{nb}) = 1$, because $S^1_b \times S^1_{nb}$ bounds $D^2 \times S^1_{nb}$. Therefore the image of the class of S^1_b under $\Omega_1^{\mathrm{Spin}} = \pi_0\mathcal{B}ord_2^{\mathrm{Spin}} \to \pi_0 s\mathcal{V}ect_{\mathbb{C}}^\times = \mathbb{Z}/2$ is 0, which is the class of $\mathbb{C}$ (an even line).
- $Z_{\mathrm{Arf}}(S^1_{nb} \times S^1_{nb}) = -1$. Therefore the image of the class of S^1_{nb} under $\Omega_1^{\mathrm{Spin}} \to \mathbb{Z}/2$ is nontrivial, so $Z_{\mathrm{Arf}}(S^1_{nb}) = \Pi\mathbb{C}$, an odd line.

So the fact that we have $s\mathcal{V}ect_{\mathbb{C}}$, rather than $\mathcal{V}ect_{\mathbb{C}}$, is important. ◀

Bordism invariants are a nice class of examples, but they don't capture everything.

Theorem 2.21 (Karras-Kreck-Neumann-Ossa [KKNO73])**.** *With the same assumption as in Theorem 2.14, the sequence*

$$\mathbb{Z} \xrightarrow{1\mapsto[S^n]} \mathrm{SKK}_n^\xi \longrightarrow \Omega_n^{t\xi} \longrightarrow 0 \tag{2.22}$$

is exact. If n is even, the first arrow is also injective. If n is odd, $[S^n] \in \mathrm{SKK}_n^\xi$ is 2-torsion, and nonzero iff there exists a closed $(n+1)$-dimensional ξ-manifold with odd Euler characteristic.

This theorem is suggesting that SKK invariants are just bordism invariants and the Euler characteristic. This is a good heuristic, but is not always true: for example, the *Kervaire semicharacteristic* of a $(4k+1)$-dimensional manifold [Ker56],

$$\kappa(M) \coloneqq \sum_{i=0}^{2k} b_i(M) \bmod 2, \tag{2.23}$$

is an oriented SKK invariant that's not a bordism invariant and independent of the Euler characteristic.

Remark 2.24. SKK groups and invariants have been studied under different names. The following concepts are all the same.

- The SKK groups.
- The *vector field bordism* groups studied by Reinhart [Rei63], which are hence also called *Reinhart bordism groups*.
- $\pi_0 MT\xi_n$, where $MT\xi_n$ is the *Madsen-Tillmann spectrum* associated to $\xi_n \to B\mathrm{O}_n$ [GMTW09]. This group is sometimes called a *Madsen-Tillmann bordism group*.

Proofs of these equivalences are given by Karras-Kreck-Neumann-Ossa [KKNO73, Theorem 4.4] and Bökstedt-Svane [BS14]. ◂

Exercise 2.25. Show that the Euler characteristic is an SKK invariant (hence recovering the Euler TQFT from yesterday). Show that it is *not* an unoriented or oriented bordism invariant.

Exercise 2.26. Consider the function on spin surfaces $(\Sigma, \mathfrak{s})$ with a principal $\mathbb{Z}/2$-bundle $P \to \Sigma$ which computes $(-1)^{\mathrm{Arf}(\Sigma, \mathfrak{s}+P)}$; here $\mathfrak{s}+P$ denotes the action of $\pi_0 \mathcal{B}un_{\mathbb{Z}/2}(\Sigma) = H^1(\Sigma; \mathbb{Z}/2)$ on the set of spin structures of Σ. Explicitly, you know a spin structure on a surface if you know the function $q_{\mathfrak{s}} \colon H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/2$ which assigns to an embedded circle 0 or 1 depending on whether its induced spin structure is bounding or nonbounding; then, acting on $\mathfrak{s}$ by P swaps the value of $q_{\mathfrak{s}}$ on any cycle C such that $P|_C$ is nontrivial.

1. Show that this is a bordism invariant, and hence defines an invertible TQFT of spin surfaces with a principal $\mathbb{Z}/2$-bundle.
2. What are the state spaces of this TQFT? (Note: this is invertible, so each state space is either an even line or an odd line.)

Exercise 2.27. What are π_0, π_1, k for $\mathcal{V}ect_{\mathbb{C}}^{\times}$? Which invertible TQFTs can we get if we use this target instead of $s\mathcal{V}ect_{\mathbb{C}}$?

3. The finite path integral

Yesterday was a bit homotopical, so I'll begin by reviewing the pieces that we'll need today, without getting into the homotopical details.

- If $\alpha \colon \Omega_n^{\xi} \to \mathbb{C}^{\times}$ is a bordism invariant, we can use homotopy theory to define an invertible TQFT $Z_\alpha \colon \mathcal{B}ord_n^{\xi} \to s\mathcal{V}ect_{\mathbb{C}}$ whose partition function $Z_\alpha(M) = \alpha(M)$, and Z_α is unique up to isomorphism.
- Integrating characteristic classes, or any natural cohomology classes you can define with the ξ-structure, defines bordism invariants. For example, if we wanted to consider bordism of 5-manifolds with a principal $\mathbb{Z}/2$-bundle P, we can take $\langle w_4(M) w_1(P), [M]\rangle \in \mathbb{Z}/2$, and take (-1) to that to land in $\{\pm 1\} \subset \mathrm{U}_1$.

Today, we'll use these and a tool called the "finite path integral" to construct a large class of examples of TQFTs, including all Dijkgraaf-Witten theories, which are fundamental examples in TQFT in a way that makes explicit computation of their partition functions and state spaces fairly straightforward.

Remark 3.1 (Inspiration from physics). The name "finite path integral" sure sounds like physics, and indeed it's telling a parallel story to one in quantum field theory: in a gauge theory, the fields include a connection on a principal G-bundle, where G is a Lie group. The path integral is over the space of connections, which generally doesn't make sense mathematically.

But if G is finite, everything is OK: each principal G-bundle has a unique connection, which is flat, and there are finitely many principal G-bundles, so the path integral reduces to a finite sum and becomes rigorous mathematics. The finite path integral is a bit more general, but this is the motivating – and most commonly studied – case. ◂

Next, we define the finite path integral construction.

Warning 3.2. Existing literature focuses almost exclusively on the special case where one sums over principal G-bundles. We will work in greater generality – which means there may not be references for some of the skipped proofs. The proofs from the case of principal G-bundles should generalize, but be especially aware of possible typos and mistakes.

Let $\phi\colon \xi' \to \xi$ be a map of n-dimensional tangential structures (so ϕ commutes with the maps down to $B\mathrm{O}_n$), which means that a ξ-structure on a manifold induces a ξ'-structure. The example to carry with you in this lecture is $B\mathrm{O}_n \times BG \to B\mathrm{O}_n$, where G is finite.

The finite path integral is a functor $\mathcal{TQFT}_n^{\xi} \to \mathcal{TQFT}_n^{\xi'}$ (and it's definitely not symmetric monoidal). I'll tell you first what it does to partition functions and state spaces, then give the construction in a setting where things are nicer (which will tell you how to make sense of it on bordisms). Let $Z^{c\ell}$ be a ξ-TQFT, and let Z denote the ξ'-TQFT obtained using the finite path integral.

For any ξ'-manifold X, let $G_{\xi/\xi'}(X)$ denote the space of *compatible* ξ-structures on X (i.e. those which induce the given ξ'-structure). This is the subspace of the space of all maps $X \to \xi$ whose composition with the map $\xi \to \xi'$ is the given map $X \to \xi'$. Therefore $\pi_0\mathcal{G}_{\xi/\xi'}$ is the set of isomorphism classes of compatible ξ-structures.

Definition 3.3. A space X has *finite total homotopy* if it has only finitely many nonzero homotopy groups, each of which is finite. We also require $\pi_0 X$ to be finite.

If M is a compact space and X has finite total homotopy, $[M, X]$ is a finite set.

We will assume that on every compact n- or $(n-1)$-manifold X, $G_{\xi/\xi'}(X)$ has finite total homotopy.

Example 3.4. For example, consider the projection map $B\mathrm{O}_n \times BG \to B\mathrm{O}_n$; a $(B\mathrm{O}_n \times BG)$-structure is data of a principal G-bundle, and the projection map just forgets the bundle. So the assumption is met if G is finite (so then there are only finitely many isomorphism classes of principal G-bundles over a compact space). Forgetting orientations and spin structures also meet this assumption, but the map of tangential structures $B\mathrm{Spin}_n^c \to B\mathrm{SO}_n$ does not: there are compact manifolds with infinitely many spin^c structures. ◀

Definition 3.5. The *homotopy cardinality* of a space X with finite total homotopy at a point x is defined to be

$$h(X, x) := \prod_{n=0}^{\infty} |\pi_n(X, x)|^{(-1)^n}. \tag{3.6}$$

Then, if M is any closed n-dimensional ξ'-manifold, its partition function is the weighted sum

$$Z(M) = \sum_{x \in \pi_0 G_{\xi/\xi'}(M)} h(G_{\xi/\xi'}(M), x) Z^{c\ell}(M, x). \tag{3.7}$$

Our assumption above ensures the sums and cardinalities in (3.7) are finite, and therefore we really get a number.

Next, the state space on a closed $(n-1)$-dimensional ξ'-manifold N. There is a vector bundle $L \to G_{\xi/\xi'}(N)$ whose fiber at a point x (which is a compatible ξ-structure on N) is the state space $Z^{c\ell}(N, x)$. This vector bundle carries a flat connection, which can be described in terms of its parallel transport maps: a path $\gamma\colon [0,1] \to \mathcal{G}_{\xi/\xi'}$ defines a bordism X_γ from $(N, \gamma(0))$ to $(N, \gamma(1))$ of ξ-manifolds, where the underlying manifold is $N \times [0,1]$ and the ξ-structure at time t is given by $\gamma(t)$. Then, the parallel transport map $L_{\gamma(0)} \to L_{\gamma(1)}$ is simply $Z^{c\ell}(X_\gamma)$. Now, the state space $Z(N)$ is the space of covariantly constant sections; our assumption guarantees this is finite-dimensional. In fact, it's free on the set of isomorphism classes of compatible ξ-structures x such that the connection has no holonomy in the connected component of $G_{\xi/\xi'}(N)$ containing x.

Remark 3.8. I don't want to give the general construction – there are enough details that I don't know solidly and don't know where to look up, making it likely that I'd get some important nuance wrong. ◀

Hence, for the construction, *we specialize to principal G-bundles*, i.e. $\xi = \xi' \times BG$. Then things simplify considerably: the spaces $G_{\xi/\xi'}$ have only π_0 and π_1, so we can replace them with their fundamental groupoids and forget about the connections.[4] In this case, $G_{\xi/\xi'}(X) = \mathcal{B}un_G(X)$, the groupoid of principal G-bundles on X and their morphisms (which are always invertible). This case is also worked out in a few references (TODO).

Definition 3.9. A line bundle on a groupoid X is a functor $X \to \mathcal{V}ect_{\mathbb{C}}^{\times}$, i.e. to every object $x \in X$ we associate some one-dimensional vector space V_x, and to every map $f\colon x \to y$ some nonzero linear map $V_f\colon V_x \to V_y$.

The space of sections of such a line bundle is its colimit, as a diagram $X \to \mathcal{V}ect_{\mathbb{C}}$; this is isomorphic to the free vector space on the set of isomorphism classes of X such that the maps associated to each automorphism are the identity.

Let $\mathcal{C}$ denote the category of spans of groupoids with vector bundles. That is, an object of $\mathcal{C}$ is a pair of a finite[5] groupoid X and a vector bundle $V \to X$, and a morphism (X_1, V_1) to (X_2, V_2) is data of

- a finite groupoid Y and maps $p_i\colon Y \to X_i$,
- a vector bundle $W \to Y$, and
- maps $\phi_i\colon p_i^* V_i \to W$.

Well, almost – a morphism is an equivalence class of such data, for the same reason that morphisms in the bordism category are equivalence classes of bordisms. Composition is the usual thing in these categories, taking the pullback.

For any $y \in Y$, this morphism determines a linear map $\varphi(y)\colon V_1(p_1(y)) \to V_2(p_2(y))$ by a push-pull construction. Disjoint union of groupoids defines a symmetric monoidal structure on $\mathcal{C}$.

We next define the "quantization" functor $\Sigma\colon \mathcal{C} \to \mathcal{V}ect_{\mathbb{C}}$, which on to an object assigns

$$\Sigma\colon (X, V) \longmapsto \Gamma(V) := \varinjlim_{x \in X} V(x), \tag{3.10}$$

i.e. take the space of sections. Given a morphism (Y, W, ϕ_1, ϕ_2) as above, the maps $\varphi(y)$ for $y \in Y$ pass to the colimit to define a map

$$\widetilde{\varphi}\colon \pi_0 Y \to \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{3.11}$$

Then, Σ assigns to this morphism the linear map

$$\Sigma(Y, W) := \sum_{[y] \in \pi_0 Y} \frac{\widetilde{\varphi}(y)}{|\mathrm{Aut}(y)|} \in \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{3.12}$$

This functor is symmetric monoidal [**?**, Theorem 5.1].

Given a TQFT $Z^{c\ell}\colon \mathcal{B}ord_n^{\xi} \to \mathcal{V}ect_{\mathbb{C}}$, the functor $F_{Z^{c\ell}}\colon \mathcal{B}ord_n^{\xi'} \to \mathcal{C}$ sending

$$F_{Z^{c\ell}}\colon M \longmapsto \left(\mathcal{B}un_{\mathbb{Z}/2}(M), P \longmapsto Z^{c\ell}(M, P)\right) \tag{3.13}$$

is also symmetric monoidal [**?**, Theorem 3.9], and therefore the composition

$$Z\colon \mathcal{B}ord_n^{\xi'} \xrightarrow{F_{Z^{c\ell}}} \mathcal{C} \xrightarrow{\Sigma} \mathcal{V}ect_{\mathbb{C}} \tag{3.14}$$

is symmetric monoidal, i.e. a TQFT of ξ'-manifolds.

In the case where we're considering G-bundles, the formulas for the state spaces and partition functions simplify, which I'll discuss in the first example.

Remark 3.15. Terminology: $Z^{c\ell}$ is sometimes called the *classical theory* and Z the *quantum theory*, though this only sometimes is accurate from a physics standpoint (e.g. we never actually checked that $Z^{c\ell}$ is a classical field theory). One will also hear Z called the *gauged theory* or the *orbifold theory* or the *orbifoldization* of $Z^{c\ell}$. ◀

[4]This drastic simplification also applies in a few other settings, e.g. summing over spin structures compatible with a given orientation.

[5]Meaning finitely many isomorphism classes, each of which has finitely many automorphisms; sometimes this is called *essentially finite*.

Example 3.16 (Dijkgraaf-Witten theories)**.** These are very well-studied TQFTs that exist in any dimension, and are often one's first examples of TQFTs. They fit into the finite path integral paradigm, where we fix a finite group G and sum over principal G-bundles.

Fix a finite group G and a cohomology class $\alpha \in H^n(BG; \mathbb{R}/\mathbb{Z})$. From this data we will define an invertible n-dimensional TQFT $Z_\alpha^{c\ell}$ on oriented manifolds with a principal G-bundle, then sum over principal G-bundles to define a (generally noninvertible) oriented TQFT Z_α.

Our input data α defines a bordism invariant ϕ_α of oriented n-manifolds with a principal G-bundle $P \to M$, as follows. Let $f_P\colon M \to BG$ be the classifying map for P; then $\phi_\alpha(M, P) \in \mathbb{C}^\times$ is $\exp 2\pi i \langle f_P^*\alpha, [M]\rangle$.[6]

As we saw yesterday, there is an invertible TQFT $Z_\alpha^{c\ell}\colon \mathcal{B}ord_n^{\mathrm{SO}}(BG) \to \mathcal{V}ect_{\mathbb{C}}^\times$ whose partition function is ϕ_α.[7] Sometimes this is called *classical Dijkgraaf-Witten theory* with *Lagrangian* (or *action*) α.

Then, apply the finite path integral to obtain a TQFT $Z_\alpha\colon \mathcal{B}ord_n^{\mathrm{SO}} \to \mathcal{V}ect_{\mathbb{C}}$, which is called *(quantum) Dijkgraaf-Witten theory* with *Lagrangian* (or *action*) α.

First, suppose $\alpha = 0$. Then $Z^{c\ell}$ is the trivial TQFT, which has the upshot that all vector bundles appearing in the finite path integral procedure are trivial. The space of sections of a trivial bundle is just the space of functions on the set of isomorphism classes. Thus, for the $\alpha = 0$ theory,

- the state space on a closed $(n-1)$-manifold N is the space of functions on the set of isomorphism classes of principal G-bundles on N.
- The partition function on a closed n-manifold M is

$$Z_0(M) = \sum_{P \in \pi_0 \mathcal{B}un_G(M)} \frac{1}{|\mathrm{Aut}(P)|}. \tag{3.17}$$

What happens on a bordism? TODO: put the push-pull description here.

And what if $\alpha \neq 0$? In that case the partition function is

$$Z_\alpha(M) = \sum_{P \in \pi_0 \mathcal{B}un_G(M)} \frac{\exp(2\pi i \langle f_P^*\alpha, [M]\rangle}{|\mathrm{Aut}(P)|}. \tag{3.18}$$

The state space on a closed, oriented $(n-1)$-manifold N is the space of sections of the following vector bundle $L \to \mathcal{B}un_G(N)$.

- To every object assign $\mathbb{C}$.
- Given an automorphism $\varphi\colon P \to P$, which is a morphism in $\mathcal{B}un_G(N)$, let $P_\varphi \to S^1 \times N$ denote the *mapping torus* of φ, i.e. $[0,1] \times P/(0,x) \sim (1, \varphi(x))$. The action of φ on the line bundle is $Z_\alpha^{c\ell}(S^1 \times N, P_\varphi)$.

Thus the state space has as a basis those principal G-bundles such that $Z_\alpha^{c\ell}$ vanishes for all mapping tori.

Making this construction explicit on bordisms is pretty intricate, and I'll point the interested reader to Freed-Quinn for the details.

TODO: many references. ◀

Example 3.19 (Quinn's finite homotpy TQFT)**.** There's a reformulation of Dijkgraaf-Witten theory in terms of maps to another space which makes for an immediate generalization. Nonetheless, this class of TQFTs seems to have been rediscovered a few times.

For any topological group G, there is a space BG, unique up to homotopy, such that a principal bundle $P \to M$ is the same thing as a map $M \to BG$. That is: homotopy classes of maps $[M, BG]$ are in natural bijection with isomorphism classes of principal G-bundles, and moreover $\mathrm{Aut}(P \to M)$ is isomorphic to $\pi_1(\mathrm{Map}(M, BG), P)$.

So why not replace BG with an arbitrary topological space X? Well – X might be too big (remember, state spaces must be finite-dimensional). But once you impose this constraint, anything goes.

Now, we mimic Dijkgraaf-Witten theory: fix a cohomology class $\alpha \in H^n(X; \mathbb{R}/\mathbb{Z})$. This defines a bordism invariant of oriented n-manifolds together with a map to X (where bordism means the map to X must extend across the bulk): the value on $(M, f\colon M \to X)$ is $\langle f^*\alpha, [M]\rangle$. This is a bordism invariant for the same reason as above, and therefore defines an invertible TQFT $Z_\alpha^{c\ell}\colon \mathcal{B}ord_n^{\mathrm{SO}}(X) \to \mathcal{V}ect_{\mathbb{C}}$. The domain category is the

[6]These are not all of the bordism invariants of oriented n-manifolds with a principal G-bundle, just a large and interesting class of them.

[7]To be precise, we got that the target is $s\mathcal{V}ect_{\mathbb{C}}^\times$. But one can show that this theory factors through $\mathcal{V}ect_{\mathbb{C}}^\times \hookrightarrow s\mathcal{V}ect_{\mathbb{C}}^\times$.

bordism category of oriented manifolds with a map to X. Then perform the finite path integral over maps to X to obtain a TQFT $Z_\alpha\colon \mathcal{B}ord_n^{\mathrm{SO}} \to \mathcal{V}ect_{\mathbb{C}}$.

Dijkgraaf-Witten theories are nice, but their dependence on things other than π_1 can be a little mysterious. Having TQFTs which more explicitly depend on higher homotopy groups is helpful; for example, these TQFTs were used by Kreck-Teichner to prove that (a large class of) TQFTs are not complete manifold invariants in dimension ≥ 5.

Before we turn to the next example, here's some words that you may have heard.

- From a physics standpoint, the fields in this TQFT are the maps to X. In general field theory, a *sigma model* with *target* X is a theory whose fields include a map to X.
- Suppose the only nonvanishing homotopy groups of X are π_1 and π_2. Then this construction reproduces the *Yetter model*, which is generally defined in a more combinatorial way. $\blacktriangleleft$

Example 3.20 (Bosonization and fermionization)**.** In physics, there is a procedure called the *Jordan-Wigner transform* which exchanges two-dimensional fermionic systems with bosonic systems with a $\mathbb{Z}/2$ symmetry. In this example, we discuss the TQFT analogue, a procedure for exchanging 2D spin TQFTs and 2D TQFTs on oriented surfaces with a principal $\mathbb{Z}/2$-bundle.

Let α be the invertible TQFT from TODO: it's defined on the bordism category of spin surfaces with a principal $\mathbb{Z}/2$-bundle, and its partition function is $\alpha(\Sigma, \mathfrak{s}, P) = (-1)^{\mathrm{Arf}(\Sigma, \mathfrak{s}+P)}$. Hopefully this feels like a canonical pairing between spin structures and principal $\mathbb{Z}/2$-bundles. We call α the *Jordan-Wigner kernel.*

Given a 2D spin TQFT $Z_f\colon \mathcal{B}ord_2^{\mathrm{Spin}} \to s\mathcal{V}ect_{\mathbb{C}}$, we produce its *bosonization*, a 2D $\mathrm{SO} \times \mathbb{Z}/2$ TQFT, as follows: first tensor with α, then sum over spin structures. Conversely, given a 2D $\mathrm{SO} \times \mathbb{Z}/2$ TQFT Z_b, we obtain its *fermionization*, a 2D spin TQFT, by tensoring with α and summing over principal $\mathbb{Z}/2$-bundles.

It's almost true that fermionization and bosonization are inverse operations – going one direction, then the other, is equivalent to tensoring with an Euler TQFT. This should be thought of as akin to the factor of $(2\pi)^d$ in the Fourier transform: there are various ways to sweep it under the rug, but you can't make it go away completely, and somehow that's not very important. $\blacktriangleleft$

Exercise 3.21. One can make sense of Dijkgraaf-Witten theory on unoriented manifolds by choosing a cohomology class $\alpha \in H^n(BG; \mathbb{Z}/2)$, rather than using $\mathbb{R}/\mathbb{Z}$ coefficients, since every closed manifold has a unique mod 2 fundamental class. Recalling that $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$ with $|x| = 1$, consider the 3D Dijkgraaf-Witten theory with gauge group $\mathbb{Z}/2$ and cohomology class x^3. Show that on an orientable surface Σ, the state space has dimension $2^{|H^1(\Sigma;\mathbb{Z}/2)|}$, and on an unoriented surface, it has dimension $2^{|H^1(\Sigma;\mathbb{Z}/2)|-1}$. TODO: double check

Exercise 3.22. If $Z_f\colon \mathcal{B}ord_2^{\mathrm{Spin}} \to s\mathcal{V}ect_{\mathbb{C}}$ is a 2D spin TQFT, let Z_b denote its bosonized theory, which is a 2D TQFT of oriented manifolds with a principal $\mathbb{Z}/2$-bundle. Compute the state spaces Z_b assigns to S^1 with its trivial and nontrivial principal $\mathbb{Z}/2$-bundles.

Conversely, given a 2D TQFT of oriented manifolds with a principal $\mathbb{Z}/2$-bundle, can you compute what its fermionization assigns to S^1_b and S^1_{nb}?

4. State sums and the Turaev-Viro-Barratt-Westbury TQFT

In math, we often make definitions by adding some additional structure, then checking that our definition doesn't depend on that structure. In low-dimensional topology, one extremely common way to do this is for this additional data to be combinatorial, e.g. a triangulation, a knot diagram, or a Heegaard splitting. Then, one proves that any two choices of data for the same underlying object are related by a finite sequence of a small list of moves (e.g. Reidemeister moves for knot diagrams), so that checking on those moves suffices to prove invariance. For example, this approach is employed in the definitions of things such as the Jones polynomial and its friends (e.g. HOMFLY-PT, Khovanov homology, Reshetikhin-Turarev invariants), as well as in Heegaard Floer homology.

State-sum models are examples of TQFTs defined in this way.[8] One begins with some sort of category $\mathcal{C}$ with enough structure to make sense of irreducible objects (e.g. the category of representations of a finite group), and a manifold M with a triangulation. A *coloring* of M by $\mathcal{C}$ is an assignment from the edges of M to the (isomorphism classes of) irreducible objects in $\mathcal{C}$. One attaches some *weight*, which is a number, to

[8]Though not all TQFTs defined according to this general philosophy are state-sum models.

the coloring; then the invariant is computed by summing the weights over all colorings. One checks this is actually an invariant by computing what happens on a set of moves called *Pachner moves*, which relate all triangulations of the same manifold in a different dimension.

Today, we'll study a nice class of examples of state-sum models, the Turaev-Viro-Barratt-Westbury (TVBW) models (TODO: references), which are 3D oriented TQFTs.

Remark 4.1. Often this is just called the Turaev-Viro model, but the original paper of Turarev and Viro considers a narrower class of examples, where the spherical fusion category is the Drinfeld double of $\mathcal{U}_q(\mathfrak{sl}_2)$; then Barratt-Westbury generalized it to all spherical fusion categories. Hence it is most accurate to call it the TVBW model.

Subsequently, Balsam-Kirrilov described the TVBW model as an extended TQFT, and followup work of Balsam discussed its relationship to the Reshetikhin-Turaev model, the other important class of 3D TQFTs. We will follow Balsam-Kirrilov's construction of this model. ⋖

The input to the TVBW model is something called a spherical fusion category.

Definition 4.2. A $\mathbb{C}$-*linear* structure on a category $\mathcal{C}$ is a vector space structure on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathcal{C}$, such that composition is a linear map.

Given a $\mathbb{C}$-linear monoidal category (where we always ask that $\otimes$ is bilinear on morphisms), we can make sense of irreducible and indecomposable objects just as in representation theory: $x \in \mathcal{C}$ is *irreducible* or *simple* if it cannot be written as $x \cong y \oplus z$ in a nontrivial way, and *indecomposable* if it can't be written as an extension of two nontrivial elements. TODO: verify that

If all objects in $\mathcal{C}$ are finite direct sums of simple ones, we call $\mathcal{C}$ *semisimple*.

Definition 4.3. A *fusion category* is a semisimple $\mathbb{C}$-linear monoidal category such that

(1) every object has duals,
(2) there are finitely many isomorphism classes of simple objects, and
(3) the unit is simple.

Definition 4.4. A *pivotal structure* on a fusion category $\mathcal{C}$ is a monoidal, $\mathbb{C}$-linear natural isomorphism $\psi\colon \mathrm{id} \Rightarrow (-)^{**}$. This is *spherical* if the two numbers

$$d_+(x) := 1 \xrightarrow{c} x \otimes x^* \xrightarrow{\psi\otimes\mathrm{id}} x^{**} \otimes x^* \xrightarrow{e} 1 \tag{4.5a}$$

$$d_-(x) := 1 \xrightarrow{c} x^* \otimes x^{**} \xrightarrow{\mathrm{id}\otimes\psi^{-1}} x^* \otimes x \xrightarrow{e} 1 \tag{4.5b}$$

coincide.

For brevity we will refer to this as a spherical fusion category. We will use the notation

$$\langle x_1, \ldots, x_n \rangle := \mathrm{Hom}_{\mathcal{C}}(1, x_1 \otimes \cdots \otimes x_n). \tag{4.6}$$

The pivotal structure tells us this is invariant under cyclic permutations, i.e. $\langle x_1, \ldots, x_n \rangle = \langle x_2, \ldots, x_n, x_1 \rangle$.[9] Also, let $S(\mathcal{C})$ denote the set of isomorphism classes of simple objects, and choose a representative for each class.

Given $x \in \mathcal{C}$, let $\dim(x) := d_\pm(x)$, which we call the *dimension* of x. This is not always a natural number! If $x \in S(\mathcal{C})$, $\dim x \neq 0$. Also, let

$$D := \left(\sum_{x\in S} (\dim x)^2\right)^{1/2}. \tag{4.7}$$

$\dim x$ is some complex number, so we have to choose a square root, and we do. It's a theorem that this is nonzero.

We will work with a slightly more general class of combinatorial structures than triangulations, called *polytope decompositions*; speaking approximately, we'll also allow simplices with more faces. See Balsam-Kirrliov, §3, for precise details. A *combinatorial n-manifold* is an n-manifold with a polytope decomposition (today $n = 2, 3$).

[9] To say this precisely: fixing n, $\langle \ldots \rangle$ is a functor, and that "=" is a natural isomorphism.

On a general compact 2- or 3-manifold with boundary, any two polytope decompositions are related by a finite series of *Pachner moves* and their inverses. There are three such moves: deleting a vertex (Figure 5), deleting an edge (Figure 6), and deleting a face (Figure 7).

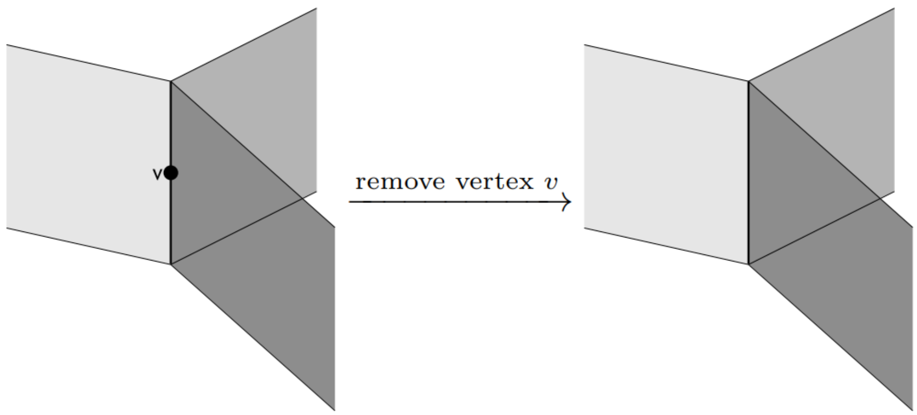

FIGURE 5. The first Pachner move. Figure from Balsam-Kirrilov.

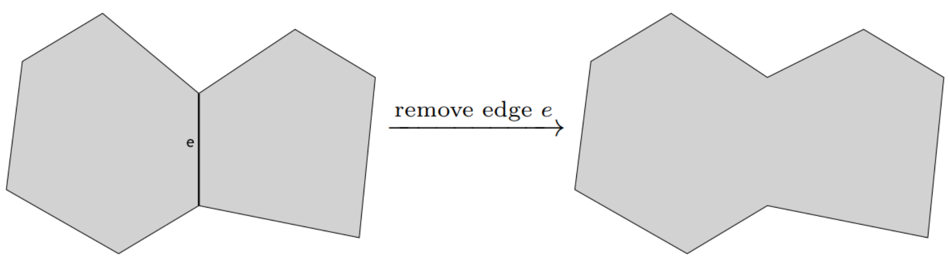

FIGURE 6. The second Pachner move. Figure from Balsam-Kirrilov.

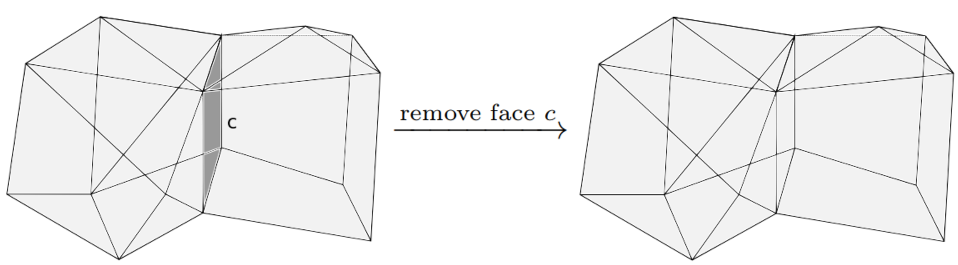

FIGURE 7. The third Pachner move. Figure from Balsam-Kirrilov. (This move is unnecessary on 2-manifolds.)

Now we begin defining the TVBW theory. Fix a spherical fusion category $\mathcal{C}$. First the state space on a closed, oriented surface Σ.

Definition 4.8. A *coloring* or *labeling* of a combinatorial n-manifold M is an assignment of a simple object $\ell(e) \in S$ to each oriented edge $e \in \Delta^1(M)$, such that $\ell(\overline{e}) = \ell(e)^*$ (here $\overline{e}$ is e with the opposite orientation).

Given a labeling and an oriented 2-cell f of M, define the space

$$H(f,\ell) \coloneqq \langle \ell(e_1), \ldots, \ell(e_k) \rangle, \tag{4.9}$$

where $e_1, \ldots, e_k$ are the oriented edges in ∂M, in the order specified by the orientation. (Well, to be precise, the orientation specifies only clockwise or counterclockwise, but since (4.9) is invariant under cyclic permutations, up to natural isomorphism, then this is fine.)

Suppose Σ is a closed, oriented, combinatorial 2-manifold. Given a labeling ℓ of Σ, define

$$H(\Sigma,\ell) \coloneqq \bigoplus_{f \in \Delta^2(\Sigma)} H(f,\ell) \tag{4.10}$$

and

$$H(\Sigma) \coloneqq \bigoplus_{\text{labelings } \ell} H(\Sigma,\ell). \tag{4.11}$$

(Here we sum over equivalence classes of labelings.) Hence $H(-\Sigma) = H(\Sigma)^*$. This is not yet the state space – we'll return to it.

Next, given a 3-cell f, we'll define $Z(f,\ell) \in H(\partial f, \ell)$. The polytope decomposition on M induces one, called Π, on $\partial f \cong S^2$; then let $\Pi^\vee$ be the dual polytope decomposition (TODOmaybe a picture). Orient its edges arbitrarily.

A labeling of Π induces a labeling $\ell^\vee$ of $\Pi^\vee$, as follows: if $e \in \Delta^1(\Pi)$, it intersects its dual edge $e^\vee \in \Delta^1(\Pi^\vee)$ at a single point. In the ambient orientations, as you travel forward along e, if $e^\vee$ is moving from right to left, let $\ell^\vee(e^\vee) \coloneqq \ell(e)$, anmd if $e^\vee$ is moving from left to right, let $\ell^\vee(e^\vee) \coloneqq \ell(e)^\vee$.

Choose for every $C \in \Delta^2(\partial f, \Pi) = \Delta^0(\partial f, \Pi^\vee)$ some $\varphi_C \in H(C,\ell)^*$; then we obtain a "string diagram:" remove a point from ∂f to make $\Delta^{\le 1}(\partial f, \Pi^\vee)$ into an oriented planar graph, with some sort of tensor φ_C at each vertex $C^\vee$; edges are to be interpreted (I think) as contractions of a vector and a covector. The axioms of a spherical category guarantee the number we obtain at the end once we contract everything (each edge appears twice with opposite orientations) doesn't depend on how we made this planar.

This procedure defines a function

$$\bigotimes_{C \in \Delta^2(\partial f)} H(C,\ell)^* = H(\partial f, \ell)^* \longrightarrow \mathbb{C}, \tag{4.12}$$

i.e. an element of $H(\partial f,\ell)^{**}$, which is canonically isomorphic to $H(\partial f, \ell)$. Let $Z(f,\ell)$ be the image of (4.12) under this map.

If M is a closed, oriented combinatorial 3-manifold, we have a map

$$\bigotimes_{f \in \Delta^3(M)} H(\partial f,\ell) = H(\partial M,\ell) \otimes \bigotimes_{c \in \Delta^2(M \setminus \partial M)} H(c,\ell) \otimes \underbrace{H(c,\ell)^*}_{=H(-c,\ell)} \xrightarrow{\mathrm{id}_{H(\partial M,\ell)} \otimes \bigotimes_c ev} H(\partial M, \ell), \tag{4.13}$$

where $ev \colon V \otimes V^* \to \mathbb{C}$ is the evaluation map. Apply the map (4.13) to the vector

$$\bigotimes_{f \in \Delta^3(M)} Z(f,\ell) \in \bigotimes_{f \in \Delta^3(M)} H(\partial f, \ell), \tag{4.14}$$

and call the result $Z(M,\ell)$. Finally, we sum this (with a normalization) over all weightings, to define

$$Z_{\mathcal{C}}(M) \coloneqq \frac{1}{D^{2v(M)}} \sum_{\text{labelings } \ell} Z(M,\ell) \prod_{e \in \Delta^2(M)} \dim(\ell(e))^{n(e)} \in Z_{\mathcal{C}}(\partial M). \tag{4.15}$$

where

- D is as in (4.7),
- $v(M)$ is the number of interior vertices of M plus 1/2 the number of vertices of ∂M, and
- $n(e) = 1$ for interior edges and 1/2 for boundary edges.

Now we can go back and define the state space. Consider

$$Z_{\mathcal{C}}(\Sigma \times I) \in Z(\Sigma) \otimes Z(-\Sigma) = Z(\Sigma)^* \otimes Z(\Sigma) = \mathrm{Hom}(Z(\Sigma), Z(\Sigma)). \tag{4.16}$$

This is a projection, and we define $Z_{\mathcal{C}}(\Sigma)$ to be its kernel.[10]

[10] Along the way, we had to choose a triangulation on I, but the projection turns out not to depend on that data.

We can think of this as a TQFT as follows.

- We already have defined a vector space associated to a closed, oriented surface (with a triangulation – though it turns out to not depend on the triangulation).
- If M is a closed 3-manifold, $Z_{\mathcal{C}}(M)$ is an element of $Z_{\mathcal{C}}(\varnothing)$, which is isomorphic to $\mathbb{C}$ kind of vacuously. So we get a partition function.
- More generally if M is an oriented bordism from Σ_0 to Σ_1, it comes with an identification $\partial M \cong -\Sigma_0 \amalg \Sigma_1$, and therefore

$$Z_{\mathcal{C}}(M) \in Z_{\mathcal{C}}(\partial M) = Z_{\mathcal{C}}(\Sigma_0)^* \otimes Z_{\mathcal{C}}(\Sigma_1) = \mathrm{Hom}(Z_{\mathcal{C}}(\Sigma_0), Z_{\mathcal{C}}(\Sigma_1)). \tag{4.17}$$

  That is, this construction defines a map $Z_{\mathcal{C}}(\Sigma_0) \to Z_{\mathcal{C}}(\Sigma_1)$, as we wanted.

But, there are many things yet left to check!

- Does disjoint union map to tensor product? (Yes, and this isn't that hard.)
- Why are all of these invariant under the Pachner moves?
- Why does a cylinder act by the identity?
- Why does gluing of bordisms correspond to composition?

These aren't all that complicated to check (except the second one, which is a little more involved), but I will nonetheless punt on it and encourage you to read the proofs in Balsam-Kirrilov.

Example 4.18 (Untwisted Dinkgraaf-Witten theory)**.** Fix a finite group G. We can recover untwisted Dijkgraaf-Witten theory from this model where $\mathcal{C}$ is the spherical fusion category of vector bundles on the set G. The monoidal structure is convolution:

$$(E \otimes F)_g = \bigoplus_{h_1 h_2 = g} E_{h_1} \otimes F_{h_2}. \tag{4.19}$$

You can take duals in the usual sense of vector bundles (i.e. fiberwise); the simple objects are the "skyscrapers" S_g whose fiber at one group element g is one-dimensional and whose fibers elsewhere vanish; and the skyscraper at the identity is the unit.

There's a pivotal structure induced in the usual way: the natural isomorphism $\mathcal{V}ect_{\mathbb{C}} \to \mathcal{V}ect_{\mathbb{C}}$ sending $V \mapsto V^{**}$ applied fiberwise, and this is spherical.

Now let's try to compute the state space associated to a surface. A coloring by $\mathcal{V}ect_G$ is an assignment of a skyscraper on each oriented edge, so equivalently an element of G on each oriented edge, with $\ell(e) = \ell(\overline{e})^{-1}$.

Lemma 4.20. *If $x_1, \dots, x_n$ are simple objects of $\mathcal{V}ect_G$, then $\langle x_1, \dots, x_n\rangle = 0$ unless $x_1 \cdots x_n = 1$, in which case it's $\mathbb{C}$.*

The idea of the proof is that the tensor product of skyscrapers at g and h is a skyscraper at gh, and then the version of Schur's lemma in this setting.

Great, so we're summing over labelings such that the product around any face is the identity. You can think of these as principal G-bundles together with a trivialization on the 0-skeleton of M as follows (TODO, but as usual). This is $H(\Sigma)$. The projection $Z_{\mathcal{C}}(\Sigma \times I)$ averages over the action of the (big) gauge group $\mathrm{Map}(\Delta^0(M), G)$ (where a δ-function with value g on a vertex acts on all the edges emerging from that vertex by g, and all edges terminating at that vertex by g^{-1}), i.e. shuffling all the trivializations on the 0-skeleton. Therefore the kernel is things which are invariant, i.e. functions that don't depend on the trivialization. So we recover $\mathbb{C}[\mathcal{B}un_G(M)]$ as expected. (Addressing the partition function is a bit more involved.) ◀

Remark 4.21. There is another spherical fusion category associated with G, namely $\mathcal{R}ep_G$. The TVBW theory it defines is canonically isomorphic to the theory for $\mathcal{V}ect_G$; Freed-Telemann describe this as a sort of electromagnetic duality. ◀

Here, if I have time, I can mention what's conjectured by Douglas, Schommer-Pries, and Snyder: pivotal fusion categories should define combed TQFTs, and spherical fusion categories should define oriented TQFTs.

Exercise 4.22. For any spherical fusion category $\mathcal{C}$, show that $Z_{\mathcal{C}}(S^2)$ is 1-dimensional.

5. Relationship with topological phases of matter

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