SPECTRAL SEQUENCES IN (EQUIVARIANT) STABLE HOMOTOPY THEORY

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MAY 18, 2017

Contents

1. The homotopy fixed-point spectral sequence: 5/15/17

Today, Richard spoke about the homotopy fixed-point spectral sequence in equivariant stable homotopy theory.

We'll start with the Bousfield-Kan spectral sequence (BKSS). One good reference for this is Guillou's notes [4], and Hans Baues [2] set it up in a general model category.

We'll work in sSet, so that everything is connective. Consider a tower of fibrations

$$\cdots \longrightarrow Y_s \xrightarrow{p_s} Y_{s-1} \xrightarrow{p_{s-1}} Y_{s-2} \longrightarrow \cdots \tag{1.1}$$

for $s \geq 0$, and let $Y \coloneqq \varprojlim Y_s$. Let F_s be the fiber of p_s.

Theorem 1.2 (Bousfield-Kan [3])**.** *In this situation, there is a spectral sequence, called the* ***Bousfield-Kan spectral sequence****, with signature*

$$E_1^{s,t} = \pi_{t-s}(F_s) \Longrightarrow \pi_{t-s}(Y).$$

If everything here is connective (which is not always the case in other model categories, as in one of our examples), this is first-quadrant. One common convention is to use the **Adams grading** $(t-s, s)$ instead of (s,t).

We can extend (1.1) into a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \\
& F_s \arrow[u, "i_s"'] & F_{s-1} \arrow[u, "i_{s-1}"'] & F_{s-2}, \arrow[u, "i_{s-2}"'] &
\end{tikzcd}$$

and hence into an exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\
& \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] & \pi_*(F_{s-2}), \arrow[u, "i_{(s-2)*}"] &
\end{tikzcd}$$

and the differentials are the compositions of the maps $\pi_*(F_s) \to \pi_*(Y_s) \to \pi_*(F_{s+1})$:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\
\arrow[from=r] & \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[l, "d_1"] \arrow[u, "i_{(s-1)*}"'] & \pi_*(F_{s-2}). \arrow[l, "d_1"] \arrow[u, "i_{(s-2)*}"'] & \arrow[l]
\end{tikzcd}$$

Taking homology, we'll get a differential d_2 that jumps two steps to the left, then d_3 three steps to the left, and so on. After you check that $\operatorname{Im}(d_r) \subset \ker(d_r)$, you can define $E_r^{s,s+1} := \ker(d_r)/\operatorname{Im}(d_r)$. Let $A_s := \operatorname{Im}(\pi_0(Y_{s+r}) \to \pi_0(Y_s))$, and let $Z_r^{s,s} := (i_s)_*^{-1}(A_s)$. Then, $E_{r+1}^{s,s} = Z_r^{s,s}/d_r(E_r^{s-r,s-r+1})$.

Remark. One important caveat is that for $i \le 2$, π_i does not produce abelian groups, but rather groups or just sets! This means that a few of the columns of this spectral sequence don't quite work, but the rest of it is normal, and the degenerate columns can still be useful. This is an example of a **fringed spectral sequence**. ◀

Bousfield and Kan cared about this spectral sequence because it allowed them to write down a useful long exact sequence, the $r^{\mathbf{th}}$ **derived homotopy sequence**: let $\pi_i Y^{(r)} := \operatorname{Im}(\pi_i(Y_{n+r}) \to \pi_i(Y_n))$; then, there's a long exact sequence

$$\cdots \longrightarrow \pi_{t-s-1}Y_{s-r-1}^{(r)} \longrightarrow E_{r+1}^{s,t} \longrightarrow \pi_{t-s}Y_s^{(r)} \xrightarrow{\delta} \pi_{t-s}Y_{s-1}^{(r)} \longrightarrow E_{r+1}^{s+r,t+r+1} \longrightarrow \pi_{t-s+1}Y_s^{(r)} \longrightarrow \cdots$$

You can do something like this in general given a spectral sequence, though you need to know how to obtain it from the exact couple.

Remark. When $r = 0$, $E_1^{s,t} = \pi_{t-s}(F_s)$, and so the first derived homotopy sequence is the long exact sequence of homotopy groups of a fibration. ◀

One nice application is to **Tot towers** ("Tot" for totalization).

Definition 1.3. Let $X^\bullet$ be a cosimplicial object in sSet. Then, its **totalization** is the complex

$$\operatorname{Tot}(X^\bullet) := \mathsf{sSet}(\Delta^\bullet, X^\bullet),$$

i.e.

$$\operatorname{Tot}_n(X^\bullet) := \mathsf{sSet}(\operatorname{sk}_n \Delta^\bullet, X^\bullet).$$

Here $(\operatorname{sk}_n \Delta^\bullet)^n := \operatorname{sk}_n \Delta^m$.

Then

$$\varprojlim \operatorname{Tot}_n(X^\bullet) = \operatorname{Tot}(X^\bullet),$$

reconciling the two definitions.

Exercise 1.4. In the Reedy model structure, $\operatorname{Tot}_n(X^\bullet) \to \operatorname{Tot}_{n-1}(X^\bullet)$ is a fibration.

Assuming this exercise, we can apply the Bousfield-Kan spectral sequence.

One place this pops up is that if $C, D \in \mathsf{C}$ and $X_\bullet \to C$ is a simplicial resolution in a simplicial category C,[1] then $\operatorname{Hom}_{\mathsf{C}}(X_\bullet, D)$ is a cosimplicial object, and this spectral sequence can be used to compute homotopically meaningful information about $\mathsf{sSet}(C, D)$.

We can use this formalism to derive the homotopy fixed point spectral sequence. Let G be a group, and X be a spectrum with a G-action. Then, the **homotopy fixed points** of X are

$$X^{hG} := F((EG)_+, X)^G,$$

i.e. the G-equivariant maps $(EG)_+ \to X$.[2] The bar construction gives us a simplicial resolution of $(EG)_+$, producing a cosimplicial object that can be plugged into the Bousfield-Kan spectral sequence. Specifically, we write $EG = B^\bullet(G, G, *)$, add a disjoint basepoint, and then take maps into X.

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[1] Meaning that after geometrically realizing, there's an equivalence.

[2] Notationally, this is the function spectrum of maps from $\Sigma^\infty(EG)_+$ to X, or you can use the fact that spectra are cotensored over spaces.

Theorem 1.5. *If X is a spectrum with a G-action, there's a spectral sequence, called the* ***homotopy fixed-point spectral sequence****, with signature*

$$E_2^{p,q} = H^p(G, \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{hG}).$$

Example 1.6. The first example is really easy. Let k be a field, and consider the Eilenberg-Mac Lane spectrum Hk. Let G act trivially on k; we want to understand $\pi_*(Hk^{hG})$. The homotopy fixed-points spectral sequence is particularly simple:

$$E_2^{p,q} = H^p(G; \pi_q(Hk)) = \begin{cases} H^p(G;k) & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a single row,[3] all differentials vanish, and this is also the E_∞ page. So we just have to compute $H^p(G;k)$ for $k \geq 0$.

For example, if $G = \mathbb{Z}/2$ and $k = \mathbb{F}_2$, then $H^*(\mathbb{Z}/2;\mathbb{F}_2) = H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$. There are no extension issues, since there's only one nonzero term in each total degree. Thus,

$$\pi_{-p}(Hk^{hG}) = H^p(G;k).$$

If you let $G = \mathbb{Z}/2$ and k be any field of odd characteristic, then $H^*(\mathbb{Z}/2;k) = k$ in degree 0, so the homotopy groups of $Hk^{h\mathbb{Z}/2}$ are all trivial except for π_0, which is k. ◀

In the context of group actions on spectra, there's another spectral sequence called the Tate spectral sequence. If X is a genuine G-spectrum, there's a norm map $X_{hG} \to X^{hG}$ whose cofiber is called the **Tate spectrum** X^{tG}.[4] This is a generalization of Tate cohomology $\widehat{H}^p$ in group cohomology. Here, $X_{hG} \coloneqq (EG_+ \wedge X)_G$ is the **homotopy orbits** of X. Then, there is a spectral sequence, called the **Tate spectral sequence**, with signature

$$E_2^{p,q} = \widehat{H}^p(G; \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{tG}).$$

The similarities with the homotopy fixed point spectral sequence are no coincidence.

Example 1.7. Let C_2 act on S^1 by reflection. Then, $\pi_i(S^1)$ is trivial unless $i = 1$, in which case we get $\mathbb{Z}$. Hence,

$$E_2^{p,q} = \begin{cases} H^p(C_2;\mathbb{Z}), & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Under the isomorphism $\mathbb{Z}[C_2] \cong \mathbb{Z}[x]/(x^2-1)$, the $\mathbb{Z}[C_2]$-module structure on $\mathbb{Z}$ is the map $\mathbb{Z}[C_2] \to \mathbb{Z}$ sending $x \mapsto -1$, i.e. C_2 acts on $\mathbb{Z}$ through the nontrivial action. We'll let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ with this action, and $\mathbb{Z}$ denote the integers with the trivial C_2-action. To compute the group cohomology, we need to compute a free resolution $P_\bullet \to \mathbb{Z}$ as a *trivial* $\mathbb{Z}[C_2]$-module:

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x+1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{x\mapsto 1} \mathbb{Z} \longrightarrow 0.$$

Now we compute $\mathrm{Hom}_{\mathbb{Z}[C_2]}(P_\bullet, \mathbb{Z}_\sigma)$:

$$\cdots \longleftarrow \mathbb{Z} \xleftarrow{-2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{-2} \mathbb{Z}.$$

Taking homology, we conclude that

$$H^p(\mathbb{Z}/2, \mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p > 0 \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Since the spectral sequence degenerates at page 2, this is also the homotopy groups of $(S^1)^{hC_2}$. This *a priori* doesn't make any sense, though — $(S^1)^{hC_2}$ is a space, so cannot have negative-degree homotopy groups. But since this is a fringed spectral sequence, the stuff in negative degrees doesn't apply to the calculation

[3] It's a single row in the usual grading, and a single diagonal line with slope -1 in the Adams grading.

[4] The norm map is the spectral analogue of a more concrete construction: let M be a $\mathbb{Z}[G]$-module. Then, the assignment

$$m \longmapsto \sum_{g\in G} g \cdot m$$

lands in M^G, so it factors through orbits, defining a map $M_G \to M^G$.

of homotopy groups, and $q-p=0,1$ mixes together in a complicated way. In this case, it tells us that $\pi_0((S^1)^{hC_2})=\mathbb{Z}/2$ and higher homotopy groups vanish.[5] $\blacktriangleleft$

2. $KU^{hC_2}=KO$: 5/16/17

Our immediate goal is to apply the homotopy fixed-point spectral sequence to prove the following theorem. Let KU (resp. KO) denote the spectrum representing complex K-theory (resp. real K-theory).

Theorem 2.1. *Let C_2 act on KU by complex conjugation. Then, $KU^{hC_2}=KO$.*

Proof. By Bott periodicity,

$$\pi_*(KU)=\mathbb{Z}[\mu], |\mu|=2=\begin{cases}\mathbb{Z}, & \text{even degrees}\\ 0, & \text{odd degrees.}\end{cases}$$

The C_2-action on KU induces a C_2-action on $\mathbb{Z}[\mu]$, which sends $\mu\mapsto-\mu$. In particular, it's trivial in all dimensions except $q=4k+2$, where it's multiplication by -1.

In this scenario, the homotopy fixed point spectral sequence is not fringed, and has signature

$$E_2^{p,q}=H^p(C_2,\pi_q(KU))\Longrightarrow\pi_{q-p}(KU^{hC_2}).$$

So we need to compute some group cohomology. Let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module with the action multiplication by -1; in Example 1.7 we showed that

$$H^p(C_2;\mathbb{Z}_\sigma)=\begin{cases}\mathbb{Z}/2, & p\text{ odd}\\ 0, & p\text{ even.}\end{cases}$$

We also need to compute $H^*(C_2;\mathbb{Z})$ (i.e. with the trivial $\mathbb{Z}/2$-action). There are a few ways to do this: for example,

$$H^p(C_2;\mathbb{Z})=H^p(BC_2;\mathbb{Z})=H^p(\mathbb{RP}^\infty;\mathbb{Z})=\begin{cases}\mathbb{Z}, & p=0\\ \mathbb{Z}/2, & p>0\text{ even}\\ 0, & \text{otherwise.}\end{cases}\tag{2.2}$$

Alternatively, you can write down an explicit free resolution:

$$\cdots\longrightarrow\mathbb{Z}[C_2]\xrightarrow{\cdot(x+1)}\mathbb{Z}[C_2]\xrightarrow{\cdot(x-1)}\mathbb{Z}[C_2]\xrightarrow{x\mapsto 1}\mathbb{Z}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-,\mathbb{Z})$, we get

$$\cdots\longrightarrow\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{2}\mathbb{Z}\xrightarrow{0}\mathbb{Z},$$

obtaining the same cohomology groups as in (2.2).

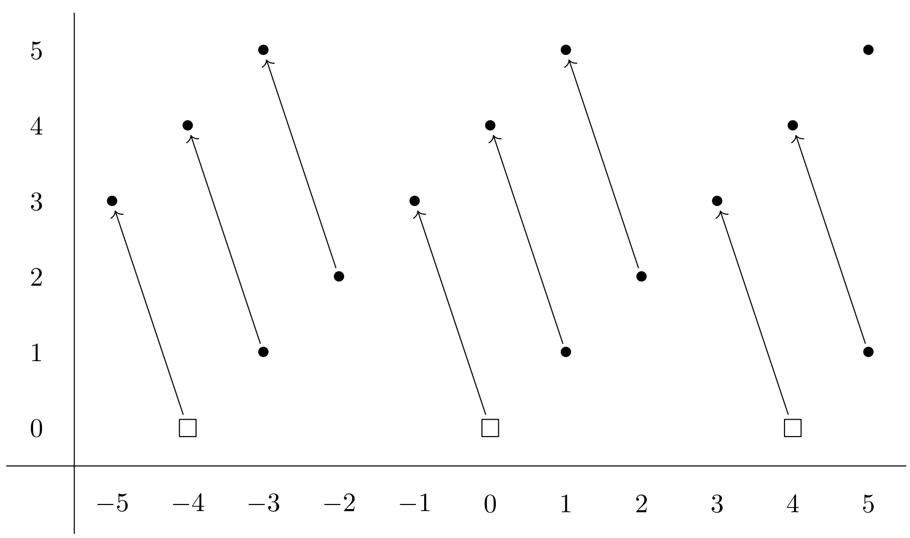
[5] Thanks to Tyler Lawson for this insight.

In the usual grading, the E_2 page therefore looks like this:

4	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
3						
2		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
1						
0	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	
-1						
-2		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
	0	1	2	3	4	5

In particular, there are no differentials on the E_2 page.

Regrading by the Adams grading $(p, q) \mapsto (q - p, p)$, let $\bullet$ denote a $\mathbb{Z}/2$ and $\square$ denote a $\mathbb{Z}$; then, the spectral sequence is



We'd like to get the Bott song out of this: $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, \mathbb{Z}, \ldots$. This means that some, but not all, of the d_3 differentials have to vanish. The multiplicative structure helps us here by translating the differentials.

We'll follow the geometric approach of Heard-Stojanoska [5]. Let ρ be the regular representation of C_2, i.e. a direct sum of the trivial and the sign representation. Let $S^{2\rho}$ denote the one-point compactification of $\rho \oplus \rho$ as a C_2-space (a space with a C_2-action). We'll write down a morphism of C_2-spectra

$$\Sigma^\infty S^{2\rho} \longrightarrow KU,$$

which, using the functoriality of the spectral sequence, induces a morphism of spectral sequences

$$\begin{tikzcd} H^p(C_2; \pi_*(\Sigma^\infty S^{2\rho})) \Longrightarrow \pi_*((\Sigma^\infty S^{2\rho})^{hC_2}) \arrow[d] \\ H^p(C_2; \pi_*(KU)) \Longrightarrow \pi_*(KU^{hC_2}). \end{tikzcd} \tag{2.3}$$

This will capture enough of the structure to see a few of the differentials, and periodicity and the multiplicative structure handle the rest.

As a C_2-space, $S^{2\rho}$ is the sphere with C_2-action reflection across a plane through the origin. This is homeomorphic to $\mathbb{CP}^1$ with the conjugation action.

There's an equivariant cell structure on S^ρ: a trivial 0-cell (trivial orbit), a trivial 1-cell, and a 2-cell $D^2 \times C_2$ with C_2 switching the two components. $\mathbb{CP}^1$ includes into $\mathbb{CP}^\infty = B\mathrm{U}_1$, which maps to $B\mathrm{U}$, which maps to KU, and all of these maps are C_2-equivariant, so we have a map

$$v_1 \colon S^\rho \xrightarrow{\cong} \mathbb{CP}^1 \hookrightarrow B\mathrm{U}_1 \longrightarrow B\mathrm{U} \longrightarrow KU.$$

We can compose this with the pinch map $S^{2\rho} \to S^\rho$ (making this equivariant requires a little thinking, but is OK), so we obtain a map $v_1^2 \colon S^{2\rho} \to KU$.

The homotopy groups of $S^{2\rho}$ are hard to compute, of course, but we can compute $(S^{2\rho})^{hC_2}$ using Spanier-Whitehead duality. If X is a spectrum, its **Spanier-Whitehead dual** is $DX \coloneqq F(X, \mathbb{S})$, the spectrum of maps $X \to \mathbb{S}$. This also works in the equivariant setting: there's a genuine G-equivariant sphere spectrum, also denoted $\mathbb{S}$, and if X is a G-spectrum, $DX = F(X, \mathbb{S})$, with G acting on the function spectrum through conjugation.

So by adjointness, $(S^{2\rho})^{hC_2} \cong (DS^{-2\rho})^{hC_2}$. This is homotopic to $D((S^{-2\rho})_{hC_2})$, because

$$F(X, Y)^{hC_2} \simeq F(X_{hC_2}, Y).$$

This is saying that an equivariant function is determined by its value on the orbits, though turning this into a proof requires turning it into a statement about ordinary orbits and fixed points:

$$F(X, Y)^{hC_2} \simeq F(EC_2, F(X, Y))^{C_2} \simeq F(EC_{2+} \wedge X, Y)^{C_2} \simeq F(X_{hC_2}, Y).$$

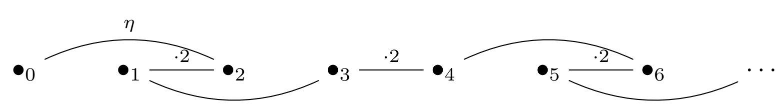
Cool. So

$$(S^{2\rho})^{hC_2} \simeq D((S^{-2\rho})_{hC_2}) \simeq D(\Sigma^{-2}(S^{-2\sigma})_{hC_2}).$$

Let $\mathbb{RP}^\infty_{-n}(\mathbb{RP}^\infty)^{-n\xi}$, i.e. the Thom spectrum of the virtual vector bundle $-n\xi$, where ξ is the tautological bundle over $\mathbb{RP}^\infty$. Thus

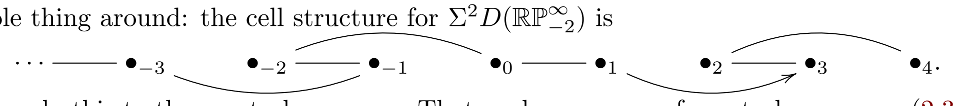
$$(S^{2\rho})^{hC_2} \simeq D(\Sigma^{-2}(\mathbb{RP}^\infty_{-2})) = \Sigma^2(D(\mathbb{RP}^\infty_{-2})).$$

Thom spectra have nice cell structures. This one has a cell in every dimension $n \geq 0$ with attaching maps as follows:



Here $\eta \colon S^3 \to S^2$ is the Hopf fibration.

Applying Σ^{-2} to this shifts the cells downward by 2 degrees. Then, applying Spanier-Whitehead duality flips the whole thing around: the cell structure for $\Sigma^2 D(\mathbb{RP}^\infty_{-2})$ is



Now we can apply this to the spectral sequence. That we have a map of spectral sequences (2.3) means that it commutes with differentials. In particular, the d_3 emerging from $(4, 0)$ (in the Adams page) witnesses the attaching map η for the 4-cell. Moreover, the Adams spectral sequence, which encodes cohomology operations, acts on these spectral sequences, and this map is equivariant for this action.[6]

That the map (2.3) is nonzero comes from the fact that η lives to the E_∞ page of the Adams spectral sequence, hence defines a nonzero element of the stable stem. The rest of the argument follows from the analysis of a single differential and the multiplicative structure. For example, because adding a 4-cell to finish $\mathbb{RP}^\infty_{-n}$ kills η, then the d_3 emerging from $(4, 0)$ is zero. ⊠

[6]To do this, you should localize and complete at 2, so we only see 2-torsion phenomena, but this suffices.

3. The Adams spectral sequence: 5/17/17

Today, Ernie spoke about the Adams spectral sequence. Everything today is stable.[7] The goal is to recover $[X, Y]$, the stable homotopy classes of maps $X \to Y$, from $H^*(X)$ and $H^*(Y)$.

Let's say we have a map $f\colon X \to Y$. Is it the zero map? If so, $f^*\colon H^*(Y) \to H^*(X)$ is also the zero map. How about the converse: if $f^* = 0$, is f null-homotopic? Another criterion is that $f \simeq 0$ iff $Cf \simeq \Sigma X \vee Y$. That is, the mapping cone of f is formed from attaching the suspension of X to Y — if f is null-homotopic, we may as well attach X to Y by crushing X to a point, an dvice versa.

What does this criterion say about cohomology? The cofiber sequence

$$X \xrightarrow{f} Y \longrightarrow Cf$$

induces a long exact sequence in cohomology

$$\cdots \longleftarrow H^*(X) \xleftarrow{f^*} H^*(Y) \longleftarrow H^*(Cf) \longleftarrow H^{*-1}(X) \xleftarrow{f^*} H^{*-1}(Y) \longleftarrow \cdots$$

but $f^* = 0$, so this breaks up into short exact sequences

$$0 \longrightarrow H^{k-1}(X) \longrightarrow H^k(Cf) \longrightarrow H^k(Y) \longrightarrow 0.$$

So understanding whether f is null-homotopic involves studying extensions of $H^*(Y)$ by $H^{*-1}(X)$. We have a tool for understanding this, namely $\operatorname{Ext}^*(H^*(X), H^*(Y))$. But we need this to respect some natural operations called stable cohomology operations.

Stable cohomology operations. For the rest of this lecture all spaces are localized and completed at $p = 2$, and $\mathbb{F}_2$-coefficients for cohomology are implicit.

Definition 3.1. A **stable cohomology operation** is a natural transformation $H^*(-, \mathbb{F}_2) \to H^{*+i}(-; \mathbb{F}_2)$ that commutes with suspension.

These form a graded algebra over $\mathbb{F}_2$ called the **Steenrod algebra** $\mathcal{A}$, which is characterized by the following axioms.

(1) $\mathcal{A}$ is generated by the **Steenrod squares** $\mathrm{Sq}^i\colon H^*(-; \mathbb{F}_2) \to H^{*+i}(-; \mathbb{F}_2)$ for $i \ge 0$.
(2) $\mathrm{Sq}^0 = \mathrm{id}$.
(3) If $x \in H^*(X)$ and $i > |x|$, then $\mathrm{Sq}^i x = 0$.
(4) If $i = |x|$, then $\mathrm{Sq}^i x = x^2$.
(5) If δ is the connecting homomorphism for the long exact sequence of a pair, then $\mathrm{Sq}^i \delta = \delta \mathrm{Sq}^i$.
(6) The **Cartan formula**:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$

(7) The **Adem relations**:[8] if $a < 2b$, then

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-a}{a-2c} \mathrm{Sq}^{a+b-c}\mathrm{Sq}^c. \tag{3.2}$$

These axioms define a graded $\mathbb{F}_2$-algebra. Computing these in practice means understanding binomial coefficients mod 2. There's a nice trick to compute this which uses binary expansions.[9]

Example 3.3. Let's use the Adem relations to compute $\mathrm{Sq}^1\mathrm{Sq}^1$:

$$\mathrm{Sq}^1\mathrm{Sq}^1 = \binom{1-0-1}{1-2(0)} \mathrm{Sq}^2\mathrm{Sq}^0 = \binom{0}{-1}\mathrm{Sq}^2 = 0.$$

◀

[7]There is an unstable Adams spectral sequence that uses unstable cohomology operations to compute the unstable homotopy groups of $\mathrm{Map}(X, Y)$, but the stable Adams spectral sequence is hard and the unstable Adams spectral sequence is even harder.

[8]These are named after José Ádem, and hence sometimes written the **Ádem relations**.

[9]You can also use Alex Kruckman's calculator: http://pages.iu.edu/~akruckma/adem/.

If X is a space, $H^*(\Sigma X)$ is an $\mathcal{A}$-module[10] in a way compatible with the grading: $\mathrm{Sq}^i \in \mathcal{A}$ acts through the cohomology operation Sq^i.

To actually construct the Steenrod squares, you can look at $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$ (since they're stable operations, suspension gives you the other $K(\mathbb{Z}/2, n)$s) and construct them there.

We care about these operations because of their role in the Adams spectral sequence.

Theorem 3.4 (Adams [1])**.** *There is a spectral sequence, called the* ***Adams spectral sequence****, whose signature is*[11]

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\widetilde{H}^*(X), \widetilde{H}^*(Y)) \Longrightarrow \pi_{t-s} F(\Sigma^\infty X, \Sigma^\infty Y)_2^\wedge.$$

Remark. This can be generalized: for example, we can replace X and Y with connective spectra. There's also a version called the **Adams-Novikov spectral sequence** [8] for generalized cohomology theories in place of $\widetilde{H}^*$. There is no version of this over $\mathbb{Z}$ instead of $\mathbb{F}_2$, however. ◂

We're going to mostly focus on the special case where $X, Y = S^0$, so that we obtain a spectral sequence converging to the 2-completed stable homotopy groups of the spheres:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s} \mathbb{S}_2^\wedge.$$

Remark. By $Z_2^\wedge$, we mean **completing** the space Z at 2. This is a crazy operation $-_2^\wedge \colon \mathsf{Top} \to \mathsf{Top}$ (or $\mathsf{Sp} \to \mathsf{Sp}$) whose effect on homotopy and homology is to complete at the prime 2, as algebras and modules. The net effect is to kill all torsion except for 2-torsion. ◂

Multiplication by 2 defines a map $(\cdot 2) \colon S^0 \to S^0$ which is not null-homotopic, and whose cofiber $C(2)$ is the desuspension of the space given by taking the mapping cone of S^1 attached to itself by a map of degree 2. This is $\mathbb{RP}^2$, so $C(2) = \Sigma^{-1}\mathbb{RP}^2$, and we know its (reduced) cohomology: there's an $\mathbb{F}_2$ in degree 0 and an $\mathbb{F}_2$ in degree 1, and the action of the Steenrod algebra is trivial except for Sq^0, which is the identity, and Sq^1, which carries the degree-0 part to the degree-1 part. This is sometimes depicted graphically as in Figure 1.

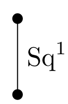
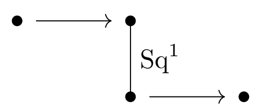


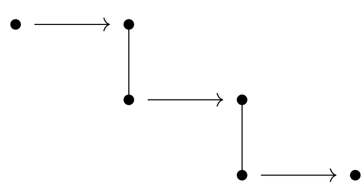
FIGURE 1. The cohomology of $C(2) = \Sigma^{-1}\mathbb{RP}^2$ as a module over the Steenrod algebra.

Now, there's a short exact sequence of maps $\Sigma\mathbb{F}_2 \to H^*(C(2)) \to \mathbb{F}_2$ (i.e. mapping in with degree 1, mapping out with degree 0), which defines an element of $\mathrm{Ext}_{\mathcal{A}}^1(\mathbb{F}_2, \Sigma\mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$:



But since the Sq^1-action is nontrivial on $H^*(C(2))$, then this is not equivalent to the trivial extension $\Sigma\mathbb{F}^2 \to \Sigma\mathbb{F}^2 \oplus \mathbb{F}_2 \to \mathbb{F}_2$, which defines $0 \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$. The punchline is that Sq^1 detects multiplication by 2.

Because extensions have a composition (since Ext is derived Hom), which is gluing two extensions into a longer extension, we can use this to conclude that, for example, multiplication by 4 is detected by the following extension.

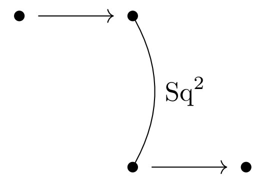


[10] It's *not* an $\mathcal{A}$-algebra, because of the Cartan formula.

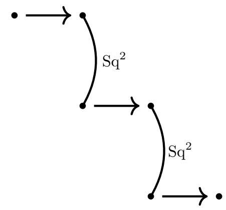
[11] One uses s, t instead of p, q so that you can say the Adams spectral sequence is a work of ecstasy. This would work better if we were over $\mathcal{C}$ instead of $\mathcal{A}$.

This quickly gets way too complicated to write down explicitly, but there is one more nice explicit one: Sq^2 detects the Hopf element $\eta\colon S^4 \to S^3$ through the class of the following extension.

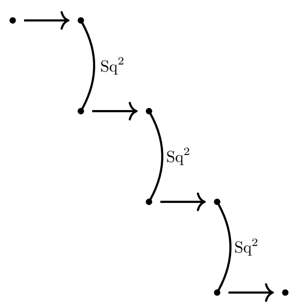
(3.5)



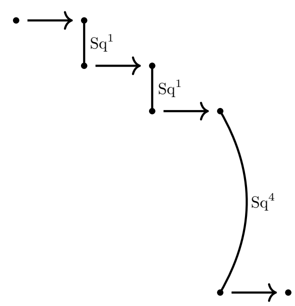
Hence, by multiplicativity, the following extension detects η^2.



There are also some relations: for example, in $\mathrm{Ext}^{3,6}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, the following two extensions are the same.



and



We achieve something concrete by restricting to subalgebras of $\mathcal{A}$. Let $\mathcal{A}(1)$ denote the subalgebra generated by Sq^0, Sq^1, and Sq^2. There's a nice picture for this: TODO.

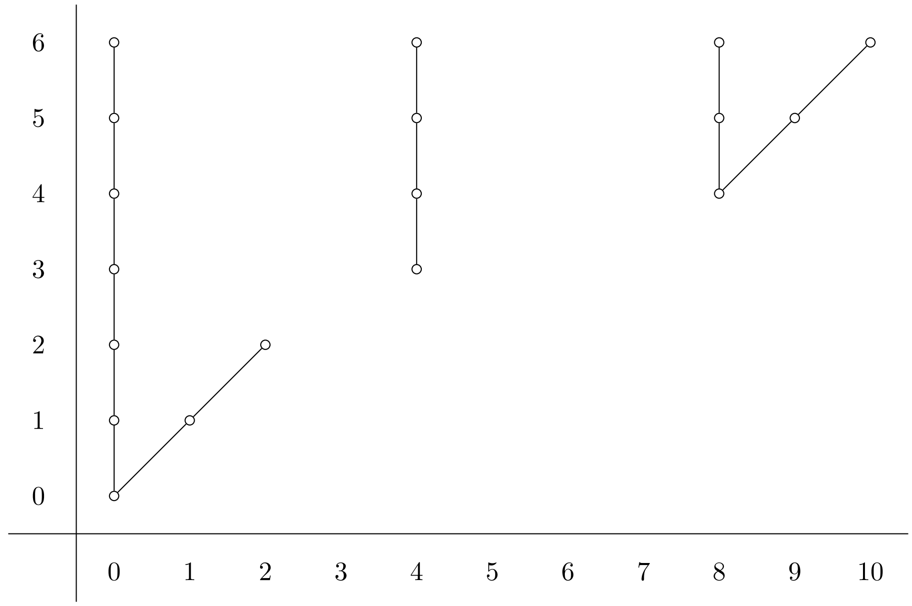
By the change-of-rings theorem,

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}^{*,*}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2, \mathbb{F}_2),$$

and there's a theorem that, as an $\mathcal{A}$-module, $H^*(ko) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2$, where ko denotes real connective K-theory.

You can explicitly compute a projective resolution of $\mathbb{F}_2$ as an $\mathcal{A}(1)$-module and use it to compute Ext, and from this recover the 8-fold periodicity of $H^*(ko)$.

If you adopt the Adams grading $(t-s,s)$, the E_2 page of the spectral sequence over $\mathcal{A}(1)$ looks like



So when you look at this spectral sequence, you see that

- it's 8-fold periodic: $E_2^{t-s,t} = E_2^{t-s+8,t+4}$, and
- there are only a few possible differentials, and they all vanish: the only possible ones go from $(1,1)$ to $(0,1+r)$ (and periodic translations thereof), but if they're nontrivial, then composing them with Sq^1 must also produce something nontrivial, a differential from $(2,2)$ to $(1,2+r)$, which must be zero (as the $(1,2+r)$-term is zero).

Thus the E_∞ page, and the 2-completed homotopy groups of ko are $\mathbb{Z}_2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}_2, 0, 0, 0, \dots$, which is exactly the Bott song for real K-theory.

Some of the elements in the E_2 page have names, e.g. the one in Adams degree $(4,3)$ is called v_1^2, even though the element v_1 (in the Adams spectral sequence for $\mathbb{S}$) is zero for ko.

A short exact sequence of $\mathcal{A}$-modules induces a long exact sequence of Ext groups. For example, let M have a copy of $\mathbb{F}_2$ in degrees 0 and 2, joined by a Sq^2, so as in (3.5), it fits into a short exact sequence

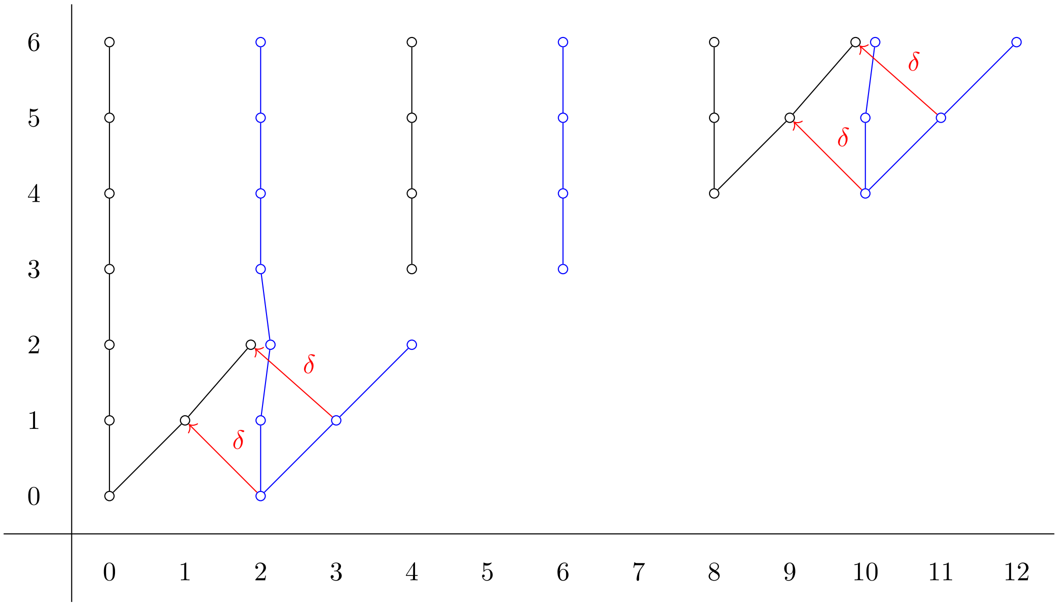
$$0 \longrightarrow \Sigma^2\mathbb{F}_2 \longrightarrow M \longrightarrow \mathbb{F}_2 \longrightarrow 0,$$

and therefore defines a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{s-1,\bullet}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s,\bullet}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,\bullet}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,\bullet}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots$$

We're being agnostic about which algebra we're working over: for concreteness we're going to work with $\mathcal{A}(1)$, but it also works over $\mathcal{A}$.

Let's use this to compute $\mathrm{Ext}^{\bullet,\bullet}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$: in the following, the ordinary circles come from $\mathrm{Ext}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, and the blue ones come from $\mathrm{Ext}^{*,*}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2)$. The "differentials" are the connecting morphism δ.



Let ku denote connective complex K-theory. Then, it's a theorem that $H^*(ku) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} M$, so by exactness, we kill the domain of δ we obtain $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$, and hence the E_2 page for the homotopy groups of ku. The differentials all vanish for the same reason as before, so the 2-completed homotopy groups of ku are $\mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \cdots$, which recovers Bott periodicity for complex K-theory. (Of course, Bott periodicity was implicit in the proof that $H^*(ku) \simeq \mathcal{A} \otimes_{\mathcal{A}(1)} M$).

There are a few other $\mathcal{A}(1)$-modules for which working the Adams spectral sequence is instructive, such as $\mathcal{A}(0)$ (generated by Sq^0 and Sq^1), the **question mark**, and the **Spanish question mark**. The rest are either too simple or way too complicated for non-machine calculations.

But you can also run the first piece of the Adams spectral sequence for $\mathbb{S}$, and therefore recover the stable homotopy groups of the spheres in low degrees! We did so, assuming as input the E_2 page for $s - t \leq 15$ (which I was unable to live-TEX in time). From there, you can use the structure of the spectral sequence to compute which differentials do and don't vanish, and so we get

- ${}_2\pi_0\mathbb{S} = \mathbb{Z}_2$,
- ${}_2\pi_1\mathbb{S} = \mathbb{Z}/2$,
- ${}_2\pi_2\mathbb{S} = \mathbb{Z}/2$,
- ${}_2\pi_3\mathbb{S} = \mathbb{Z}/8$,
- ${}_2\pi_4\mathbb{S} = 0$,
- ${}_2\pi_5\mathbb{S} = 0$,
- ${}_2\pi_6\mathbb{S} = \mathbb{Z}/2$,
- ${}_2\pi_7\mathbb{S} = \mathbb{Z}/16$,
- ${}_2\pi_8\mathbb{S} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
- ${}_2\pi_9\mathbb{S} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
- ${}_2\pi_{10}\mathbb{S} = \mathbb{Z}/2$,
- ${}_2\pi_{11}\mathbb{S} = \mathbb{Z}/8$,
- ${}_2\pi_{12}\mathbb{S} = 0$,
- ${}_2\pi_{13}\mathbb{S} = 0$,
- ${}_2\pi_{14}\mathbb{S} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and
- ${}_2\pi_{15}\mathbb{S} = \mathbb{Z}/32 \oplus \mathbb{Z}/2$.

All of this was known from the E_2 page and only two differentials! Further computations have been worked out, and this is known to about $t - s = 60$. Over other primes, things are much less busy, and so more is known.

In some sense, this is amazing: with algebraic information about Ext and only slightly more information involving the spectral sequence, we can get a whole bunch of homotopy groups of the spheres. But it's also terrible to work out.

4. The May spectral sequence: 5/18/17

"I apologize in advance: every squiggle is going to be a ξ."

Today, Ernie spoke about the May spectral sequence. Once again, we're always working over the prime 2.

Last time, we disucssed the Adams spectral sequence, and how it uses $\operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ to compute stable homotopy groups of the spheres. This is cool, but it's extremely complicated. Traditionally, you can use a bar complex to make some computations, but this quickly gets far too big to control. The May spectral sequence can be used to get a handle on $\operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Milnor observed that the dual of the Steenrod algebra is somewhat simpler. It's a Hopf algebra whose coproduct contains information about the product on $\mathcal{A}$, which we saw last time was complicated (3.2).

Theorem 4.1 (Milnor [7])**.** *As a Hopf algebra, the dual Steenrod algebra is*

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots],$$

where the coproduct $\psi\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ *is determined by*

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{2^k} \otimes \xi_k.$$

These ξ_n are called the **Milnor basis** for $\mathcal{A}_*$. These days, people often use a different set of generators

$$h_{i,n} \coloneqq \xi_i^{2^n} \qquad \text{and} \qquad h_n \coloneqq h_{1,n} = \xi_1^{2^n}.$$

The dual of h_n is the element $h^n \in \operatorname{Ext}_{\mathcal{A}}^{2^n,0}(\mathbb{F}_2, \mathbb{F}_2)$ which shows up when calculating the stable stem. You can calculate the product of any combination of ξ_i^n in terms of these $h_{i,n}$: for example,

$$\xi_i^5 \xi_j^7 = \xi_i^{2^0+2^2} \xi_j^{2^0+2^1+2^2} = h_{i,0}h_{i,2}h_{j,0}h_{j,1}h_{j,2}. \tag{4.2}$$

Definition 4.3. Grade $\mathcal{A}_*$ by defining $|h_{i,n}| \coloneqq 2i-1$ (the **May grading**). Then, the **May filtration** on $\mathcal{A}_*$ defines $F_p\mathcal{A}_*$ to be the submodule of elements of degree at most p.

For example, using (4.2),

$$|\xi_i^5 \xi_j^7| = 2(2i-1) + 3(2j-1).$$

We have a filtered object, and hence can run a spectral sequence. But we now have three gradings: the external and internal gradings on $\operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, and the May filtration.

We can decompose the coproduct into homogeneous pieces under this filtration:

$$\psi(\xi_n) = \underbrace{\xi_n \otimes 1}_{2i-1} + \underbrace{\sum_{0<k<n} \xi_{n-k}^{2^k} \otimes \xi_k}_{2i-2} + \underbrace{1 \otimes \xi_n}_{2i-1}.$$

Therefore in the associated graded,

$$\psi(\xi_n) = \xi_n \otimes 1 + 1 \otimes \xi_n.$$

For a general Hopf algebra, such a generator is called a **primitive generator**; this means that $\operatorname{gr} \mathcal{A}_*$ has only primitive generators. This implies a nice fact about its Ext:

Theorem 4.4. *Let H be a graded Hopf algebra over $\mathbb{F}_2$ with a set of primitive generators S. Then,*

$$\operatorname{Ext}_H^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[S].$$

In particular, because $\operatorname{gr} \mathcal{A}_*$ has only primitive generators,

$$\operatorname{Ext}_{\operatorname{gr} \mathcal{A}_*}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{i,n} \mid i \geq 1, n \geq 0].$$

The bigrading on $\operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, plus the May grading, induces a *tri*graded spectral sequence.

Theorem 4.5 (May [6])**.** *There exists a spectral sequence, called the* ***May spectral sequence****, with signature*

$$E_1^{\bullet,\bullet,\bullet} = \mathbb{F}_2[h_{i,n} \mid i \geq 1, n \geq 0] \Longrightarrow \operatorname{Ext}_{\mathcal{A}}^{\bullet,\bullet}(\mathbb{F}_2, \mathbb{F}_2).$$

Moreover, all of the differentials come from differentials on the cobar complex induced on $\operatorname{gr} \mathcal{A}_*$*.*

There are explicit formulas for early differentials: for example,

$$d_1(h_{i,n}) = \sum_{k=0}^{n} h_{i-k,n+k} h_{k,n}. \tag{4.6}$$

This comes from the first differential in the cobar complex, plus (4.2).

To make things simpler, let's look at a truncation: $\mathcal{A}(1)_*$ is generated by $\{h_{01}, h_{11}, h_{20}, h_{12}\}$: h_{01} corresponds to Sq^1, h_{11} to Sq^2, and h_{20} to $\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. If we run the May spectral sequence for $\mathcal{A}(1)_*$, we'll recover $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$, which will tell us about the cohomology of ko as we saw last time.

The E_1 page will be a polynomial algebra in h_{01}, h_{11}, h_{20} and h_{12}, by Theorem 4.4. Since $h_{i,j} \in E_1^{i,2^j,2i-1}$, the whole E_1 plane is products of only a few dots:

	-1	0	1	2
2	h_{20}			
1		h_{10}	h_{11}	h_{12}.
0		1		

Using (4.6), we can write down what d_1 does:

$$d_1(h_{20}) = h_{20}h_{00} + h_{11}h_{10} + h_{02}h_{20} = h_{11}h_{10} + 2h_{20} = h_{11}h_{10}.$$

The curious degree of d_1 is because it's in the May filtration. But this means the E_2 page is freely generated by h_{12}, h_{11} and h_{10} with $h_{10}h_{11}$ killed off.

$$E_2^{\bullet,\bullet,\bullet} = \langle h_{10}, h_{11}, h_{12}\rangle/(h_{11}h_{10} = 0).$$

It starts like this and keeps going: For the whole dual Steenrod algebra, here are some results for $t - s \leq 13$.

- The E_2 page is generated by the following elements:

$$\begin{aligned} h_j &= h_{1,j} \in E_2^{1,2^j,1} \\ b_{i,j} &= h_{i,j}^2 \in E_2^{2,2^j 2(i-1),2^i} \\ x_7 &= h_{20}h_{21} + h_{11}h_{30} \in E_2^{2,9,4}, \end{aligned}$$

  with relations induced by (4.2)

$$\begin{aligned} h_j h_{j+1} &= 0 \\ h_2 b_{2,0} &= h_0 x_7 \\ h_2 x_7 &= h_0 b_{21}. \end{aligned}$$

Warning: there may be typos, especially in indices and gradings. Things got confused during the talk.

So we can draw a picture of the beginning of the E_2 page of the May spectral sequence, as in Figure 2.

The E_∞ page is, of course, the E_2 page for the Adams spectral sequence. Some differentials:

$$\begin{aligned} d_r(h_j) &= 0 \\ d_2(b_{2,j}) &= h_j^2 h_{j+2} + h_{j+1}^3 \\ d_2 x_7 &= h_0 h_2^2 \\ d_2(b_{30}) - &= h_1 b_{21} + h_3 b_{20} \\ d_4(b_{20}) &= h_0^4 h_3. \end{aligned} \tag{4.7}$$

Where do these formulas come from? Ravenel tells us that (4.7) comes from a calculation in the cobar complex, namely

$$d\big([\xi_2 \mid \xi_2] + [\xi_1^2 \mid \xi_1\xi_2] + [\xi_2\xi_1^2 \mid \xi_1]\big) = [\xi_1^2 \mid \xi_1^2 \mid \xi_1^2] + [\xi_1^4 \mid \xi_1 \mid \xi_1].$$

This is scary, but the point is that you need nothing else.

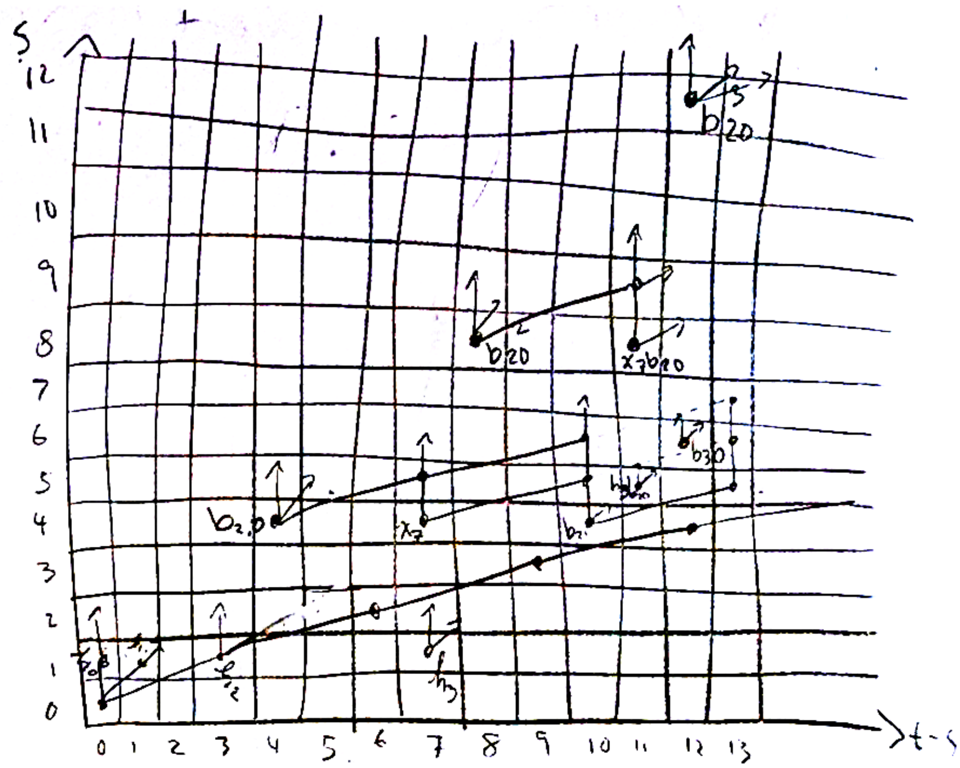


FIGURE 2. The E_2 page for the May spectral sequence, for $t - s \leq 13$. This is written down in Ravenel's book.

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