GST STUDENT SEMINAR: SPRING 2018

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Contents

1. The AGT correspondence with surface operators: 5/22/18

Today Shehper spoke about the AGT correspondence, following earlier lectures I wasn't in town for. The reference papers are [AGG+10] and [FGT16].

The AGT correspondence is a correspondence between

- the instanton partition function of a class-S field theory on a Riemann surface $\Sigma_{g,m}$ with gauge group a product of copies of SU_2, and
- conformal blocks of Liouville theory on $\Sigma_{g,m}$.

This requires choosing a pair-of-pants decomposition of $\Sigma_{g,m}$.

This arises from a compactification of the A_1 $(2, 0)$ 6D theory; the gauge group (specifically, the number of copies of SU_2) depends on the number of punctures and the genus in a way which can be seen from the Dynkin diagram. The 4D theory (i.e. the class-S theory) also has an SU_2^n flavor symmetry.

But today, we're going to focus on the AGT correspondence when surface operators are inserted. When you write down the 6D $(2, 0)$ supersymmetry algebra, it has central charges that suggest the possibility of 2D and 4D defects. That is, instead of scalar central charges, which correspond to worldlines of particles, these central charges live in higher-dimensional representations of the Poincaré algebra. It will also be possible to introduce these defects in the 4D theory — there's no *a priori* reason to do it, just that it's possible and interesting. The 2D defect will be realized by an M5-M2 brane system, and the 4D defect by an M5-M5 brane system.

Since an Md-brane is a $(d + 1)$-dimensional object, we can get a line operator in the 4D theory by taking the M2-brane to intersect C (in $C \times \mathbb{R}^4$, where the 6D theory is formulated) in a loop. Similarly, we can get a surface operator by having it intersect only at a single point; we'll call these operators of type A. For the 4D defects, have the two M5-branes intersect on $C \times \mathbb{R}^2 \subset C \times \mathbb{R}^4$; thus we obtain another kind of surface operator, called type B.

From string theory considerations that I'm not familiar with, one can deduce that type A operators correspond to type 2D-4D coupled systems, and type B operators to singularities in coupled fields.

- First, what's a 2D-4D coupled system? For concreteness, let's suppose the 4D theory is $\mathcal{N} = 2$ pure super-Yang-Mills, and suppose the surface operator $D \simeq \mathbb{R} \cdot \{x_0, x_1\} \subset \mathbb{R}^{1,3}$ (i.e. Minkowski space). Then, consider a 2D theory on D with flavor symmetry SU_2; the coupling is the idea that this is the same as the gauge symmetry in the bulk[1] Specifically, the "coupling" of the 2D-4D coupled system arises from adding a background connection for the 4D gauge group.

  The boundary theory can be any theory which makes this work; since SU_2 acts on $\mathbb{CP}^1$, we can take the σ-model with target $\mathbb{CP}^1$ on D, for example. There is a subtlety, though; you need to integrate out the x_2- and x_3-directions of the SU_2-connection. Conceptually, this seems reasonable, but there are details that have to

[1]There is a generalization where the flavor symmetry is replaced with another global symmetry.

be justified: why is it that when you restrict the connection to D, you get the global SU_2-symmetry of the boundary theory? Keep in mind that the SU_2 symmetry is a background symmetry, and is not gauged; for example, for the σ-model, the gauge group is U_1.

- Next, the type B defects, which we claimed are singularities in 4D fields. Choosing $D = \mathrm{span}_{\mathbb{R}}\{x_0, x_1\}$ again, let's introduce polar coordinates (r, θ) on $\mathrm{span}_{\mathbb{R}}\{x_2, x_3\}$. If A is a connection of the form $A = \alpha \mathrm{d}\theta + \cdots$, then there will be a singularity at the origin, as

$$\mathrm{d}\theta \sim \frac{1}{r}\,\mathrm{d}x \pm \frac{1}{r}\,\mathrm{d}y. \tag{1.1}$$

  Explicitly, the curvature is a δ-function: $F = 2\pi\alpha\delta_D$.

  If A is an SU_2-gauge field, then we take $\alpha \in \mathrm{U}_1$, for reasons which are unclear.

Our goal is to establish the AGT correspondence for both cases, and to understand if there's a relationship between the 4D theories with these two kinds of defects.

To do this, we'll have to modify the instanton partition function. For the type A defects, we'll take the 2D theory to be a σ-model with target $\mathbb{CP}^1$. Therefore we want to consider maps $\phi\colon \mathbb{R}^2 \to \mathbb{CP}^1$ which vanish at infinity,[2] hence extend to maps from the one-point compactification: $\phi\colon S^2 \to \mathbb{CP}^1$ which are 0 at the basepoint. When we take homotopy classes, we get $[S^2, \mathbb{CP}^1] \cong \mathbb{Z}$, which sends a map to its degree $m \in H_2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$. Let $\mathcal{M}_{k,m}$ denote the moduli space of solutions (to the instanton equations?) with instanton number k in 4D and soliton number m in 2D.

This theory arises as the low-energy theory of a U_1-gauge theory with two chiral superfields of type $\mathcal{N} = (2,2)$ in 2D, and monopoles in the UV flow down to maps $S^1_\infty \to \mathrm{U}_1$ with the prescribed winding number. It's stated in a somewhat different way in the physics, but the idea is to look at what happens to a large circle.

From this perspective, the correct instanton partition function is

$$Z = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}} q^k e^{itm} \int_{\mathcal{M}_{k,m}} \mathrm{dvol}. \tag{1.2}$$

The surface operators are inserted at a point $z \in \Sigma$, and z is related to t in some way. t is called an *FI parameter*, and appears as a term in the action:

$$S_{\mathrm{GLSM}} = \cdots + \int \mathrm{d}^2\widetilde{\theta}\,(-tS). \tag{1.3}$$

This has something to do with a weakly coupled system, and when it flows to the IR, the M5-branes flow to just a single M5-brane wrapping around the Seiberg-Witten curve in $\mathbb{R}^4$. This is not completely understood, but there is a lot of evidence for this argument.

The other side of the AGT correspondence, on conformal blocks of the Liouville theory, now has an insertion of a vertex operator $e^{-b/2\phi(z)}$ inserted. In thus case the four-point function

$$\langle V_1(0)V_2(1)V_3(q)V_4(\infty)\rangle \tag{1.4}$$

is replaced with

$$\langle V_1(0)V_2(1)V_b(z)V_3(q)V_4(\infty)\rangle, \tag{1.5}$$

where $V_3(z) \coloneqq e^{-b/2\phi(z)}$.

Now let's discuss what happens for the operators of type B. Suppose $A = \alpha\mathrm{d}\theta + \cdots$, so $F = 2\pi\alpha\delta_D + \cdots$. Then $\oint F \in 2\pi\mathbb{Z}$, and the instanton partition function is

$$Z^{\mathrm{inst}} \coloneqq \sum_{m\in\mathbb{Z}} \sum_{k=0}^{\infty} q_1^l k q_2^m \int_{\widetilde{\mathcal{M}}_{k,m}} \mathrm{dvol}. \tag{1.6}$$

There is a WZW model conformal field theory associated to the Kac-Moody algebra called affine $\mathfrak{sl}_2$; the AGT correspondence says that the instanton partition function should correspond to conformal blocks of this CFT. There are a bunch of subtleties going into this.

Anyways, we now have two kinds of surface defects, and get AGT correspondences with two different CFTs. One can ask whether these CFTs are related, or dually, whether these 4D theories are related.

[2] Or more accurately, which have a finite limit at infinity.

The answer is yes: if $Z^{\mathrm{WZW}}(x,\tau)$ denotes the WZW conformal block, where $x \in \mathrm{Bun}_{\mathrm{SU}_2}(\Sigma)$ and $\tau \in \mathbb{H}$, and Z^L denotes the Liouville conformal block, then

$$Z^{\mathrm{WZW}}(x,\tau) = \int \mathrm{d}u\, \kappa(x,y) Z^L(u,\tau). \tag{1.7}$$

This is an example of separation of variables! See the referenced papers for details; since conformal blocks are mathematically understood, there's a good chance this is rigorously proven!

For the relations between the 4D theories, we can rewrite the instanton partition function in terms of an *effective twisted superpotential* $\widetilde{W}$:

$$Z^{\mathrm{inst}} = \exp\left(-\frac{F}{\epsilon_1\epsilon_2} - \frac{\widetilde{W}}{\epsilon_1} + \cdots\right). \tag{1.8}$$

For the two theories we have two superpotentials $\widetilde{W}^L$ and $\widetilde{W}^{\mathrm{WZW}}$, and they're related by

$$\widetilde{W}^{\mathrm{WZW}}(a,u,\tau) = \widetilde{W}^L(a,u,\tau) + \widetilde{W}^{\mathrm{SOV}}(x,u,\tau). \tag{1.9}$$

In the IR, these two describe the same physics, hence one says they have IR duality.

2. Differential cohomology and gerbes via Čech cohomology: 6/1/18

Today, Ivan spoke about differential cohomology and gerbes from as concrete of a perspective as possible, following Hitchin [Hit99]; then he talked about higher abelian gauge theory and examples, including the usual Maxwell theory. If time permits, we'll also see U_1-gerbes with connection, which appear in the next nontrivial example.

Let M be a smooth n-manifold, and for $p \geq 0$ let $C_p(M)$ denote the abelian group of *singular p-chains*, i.e. the free abelian group on the set of smooth maps $\Delta^p \to M$, where Δ^p is the *p-simplex*. That is, a singular p-chain is a finite linear combination of such maps, where the coefficients are integers.[3] There's a boundary map $\partial\colon C_p(M) \to C_{p-1}(M)$; the kernel of this map is called the space of *p-cycles* and denoted $Z_p(M)$.

Definition 2.1. For $p \geq 1$, the p^{th} *Cheeger-Simons differential cohomology group*, denoted $\check{H}^p(M)$, is the subgrup $\chi \in \mathrm{Hom}_{\mathsf{Ab}}(Z_{p-1}(M), \mathrm{U}_1)$ such that there is an $\omega \in \Omega^p(M)$ such that for all $\sigma \in C_p(M)$,

$$\chi(\partial\sigma) = \exp\left(i\int_\sigma \omega\right).$$

We'll see that ω is uniquely determined from χ; if χ is a field, ω will represent its field strength, and hence will sometimes also be denoted F_χ. Here are a few other nice facts coming from this definition: ω is in fact a closed p-form, and has integer periods, meaning $i\int_\sigma \omega \in 2\pi\mathbb{Z}$ for all cycles σ. The abelian group of p-forms with integral periods is denoted $\Omega^p_{\mathbb{Z}}(M)$; de Rham cohomology with these forms recovers the torsion-free part of $H^*(M;\mathbb{Z})$.

The map $F\colon \chi \mapsto (1/2\pi)F_\chi$ is a linear map $F\colon \check{H}^p(M) \to \Omega^p_{\mathbb{Z}}(M)$. There's also a *characteristic class map* $c\colon \check{H}^p(M) \to H^p(M;\mathbb{Z})$ sending $\chi \mapsto (1/2\pi)[F_\chi]$, much like the first Chern class.

$\check{H}^p(M)$ is uniquely characterized as the group that fits into the following short exact sequences.

$$0 \longrightarrow H^{p-1}(M;\mathrm{U}_1) \xrightarrow{i_1} \check{H}^p(M) \xrightarrow{F} \Omega^p_{\mathbb{Z}}(M) \longrightarrow 0 \tag{2.2a}$$

$$0 \longrightarrow \Omega^p(M)/\Omega^{p-1}_{\mathbb{Z}}(M) \xrightarrow{i_2} \check{H}^p(M) \xrightarrow{c} H^p(M;\mathbb{Z}) \longrightarrow 0. \tag{2.2b}$$

This means that if you come up with a group which admits injective maps as in (2.2a) and (2.2b), with the same quotients, then it must be $\check{H}^p(M)$. We're going to use this to identify $\check{H}^p(M)$ in a higher gauge theory.

Remark 2.3. There is a description of differential cohomology in terms of a cochain complex, but it's much more complicated (involving hypercohomology of a complex of sheaves), which is why it was left out. ◀

Anyways, physics!

Definition 2.4. A *higher abelian gauge theory* is a field theory in the classical sense (so, fields and equations of motion), and in particular a U_1-gauge theory whose gauge equivalence classes of fields are $\check{H}^p(M)$ for some p specific to the theory.

[3] The p-simplex is not a manifold; by a smooth map we mean a map from the p-simplex which extends to a smooth map of a neighborhood of the p-simplex in $\mathbb{R}^p$.

Example 2.5. The first interesting example is when $p = 2$. We claim that $\check{H}^2(M)$ is in bijection (natural with respect to pullback) with the set of equivalence classes of principal U_1-bundles on M with connection A (which is the electromagnetic potential of the theory), which is in bijection with the set of equivalence classes of Hermitian line bundles on M with connection ∇.

Let $\mathrm{Pic}_\nabla(M)$ denote the abelian group of isomorphism classes of Hermitian line bundles L on M with connection ∇. Then there are maps $F\colon \mathrm{Pic}_\nabla(M) \to \Omega^2_{\mathbb{Z}}(M)$ sending $F(L,\nabla) \mapsto F_\nabla$, and $c\colon \mathrm{Pic}_\nabla(M) \to H^2(M;\mathbb{Z})$ sending $(L,\nabla) \mapsto c_1(L)$.

These will be the maps in (2.2); now we need to define i_1 and i_2. The former is "the holonomy map of a flat connection," which sounds nonsensical but has meaning. Namely, $H^1(M;\mathrm{U}_1) \cong \mathrm{Hom}(H_1(M);\mathrm{U}_1)$ by the universal coefficient theorem, and the monodromy map of a flat connection gives you something in $\mathrm{Hom}(H_1(M),\mathrm{U}_1)$.[4] Conversely, if $P \in \mathrm{Hom}(H_1(M);\mathrm{U}_1)$, one can construct a line bundle $L \to M$ and a connection ∇ for L with $F_\nabla = 0$, and whose monodromy map is P. Therefore if you replace $\check{H}^2(M)$ with $\mathrm{Pic}_\nabla(M)$, (2.2a) is exact.

For (2.2b), the argument is similar: if you're in $\ker(c)$, the first Chern class of L is trivial, so L is trivial. Therefore up to gauge equivalence, ∇ can be represented by $\mathrm{d} + A$ for some one-form $A \in \Omega^1(M;i\mathbb{R})$. Thus if $[A] \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$, then $i_2([A]) := [\underline{\mathbb{C}}, \mathrm{d} + A]$. The quotient is needed for i_2 to be injective: up to gauge, monodromy suffices to distinguish line bundles, and integral periods aren't seen by this. Thus (2.2b) is exact when $\mathrm{Pic}_\nabla(M)$ replaces $\check{H}^2(M)$.

Therefore $\check{H}^2(M) = \mathrm{Pic}_\nabla(M)$. ◂

Example 2.6. The $p = 1$ case is less interesting: $\check{H}^1(M) \cong \Omega^0(M;\mathrm{U}_1)$. You can prove this in a similar way, showing that $\Omega^-(M;\mathrm{U}_1)$ fits into (2.2a) and (2.2b). In this case, F measures whether the function is locally constant, and c measures whether the function lifts to an $\mathbb{R}$-valued function. Explicitly, a $\chi \in \check{H}^1(M)$ is a homomorphism $Z_0(M) \to \mathrm{U}_1$, hence defines a map $\overline{\chi}\colon M \to \mathrm{U}_1$. Then one has $\mathrm{d}\overline{\chi} = i\overline{\chi}F_{\overline{\chi}}$, which allows us to define F, and $c\overline{\chi} = \chi^*[\mathrm{d}\theta]$, where $\mathrm{d}\theta \in \Omega^1(S^1)$ is the usual generator. One has to check that this all works out, of course. ◂

When $p = 3$, we get something new, a higher abelian gauge theory. Explicitly, $\check{H}^3(M)$ will be isomorphic to the set of equivalence classes of U_1-gerbes with connection over M. We're going to define what these things are, and what F and c are, from a Čech cohomology viewpoint.

Fix a good cover $\mathfrak{U}$ of M indexed by an ordered set J.

Definition 2.7. The *Čech p-cochain group* $C^p(\mathfrak{u} : \mathrm{U}_1)$ is the abelian group of sets of collections of smooth maps

$$g_{\alpha_1\ldots\alpha_{p+1}}\colon U_{\alpha_1} \cap \cdots \cap U_{\alpha_{p+1}} \to \mathrm{U}_1$$

indexed over all $(p+1)$-fold intersections in $\mathfrak{U}$; the group structure is pointwise multiplication.

There is a coboundary operator $\delta\colon C^p(\mathfrak{U};\mathrm{U}_1) \to C^{p+1}(\mathfrak{U};\mathrm{U}_1)$: if $g = \{g_{\alpha_1\ldots\alpha_{p+1}}\}$, then

$$(\delta g)_{\alpha_1\ldots\alpha_{p+2}} = \prod_{i=1}^{p+2} g^{(-1)^i}_{\alpha_1\ldots\widehat{\alpha}_i\ldots\alpha_{p+2}}, \tag{2.8}$$

If we were using additive notation (e.g. with a target $\mathbb{R}$ instead of U_1) then this is take the alternating sum of evaluating g on the $(p+1)$-fold intersection where $U_{\alpha_{p_i}}$ is removed.

Then we proceed as usual: a *Čech cocycle* is a cochain with $\delta g = 1$, and a *Čech coboundary* is a cocycle in the image of δ. Then the p^{th} *Čech cohomology group* is the group of cocycles modulo coboundaries, and is denoted $H^p(M;C^\infty(\mathrm{U}_1))$.[5]

When $p = 0$, 0-cocycles are locally constant functions, because given a cocycle $f\colon U_\alpha \to \mathrm{U}_1$, $(\delta f)_{\alpha\beta} = f_\alpha f_\beta^{-1}$, so we can glue f_α and f_β. Therefore $H^0(M;\mathrm{U}_1) = \Omega^0(M,\mathrm{U}_1)$.

A 1-cocycle is a collection of maps $g = \{g_{\alpha\beta}\}$, where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{U}_1$, subject to the condition $1 = (\delta g)_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\beta\gamma}g^{-1}_{\alpha\gamma}$. This is the cocycle condition for the transition functions for a line bundle, and since they land in U_1, this is a Hermitian line bundle. Therefore a Čech 1-cocycle determines a Hermitian line bundle. If you mod out by coboundaries, you recover equivalence classes of Hermitian line bundles.

The next step, $p = 2$, will have to do with gerbes. There are various other definitions of gerbes, such as those involving sheaves of categories, but using Čech cohomology will make it easier for us to work with them.

Definition 2.9. A U_1-*gerbe* on a manifold M, denoted $\mathcal{G} \to M$, is a class in $H^2(M;C^\infty(\mathrm{U}_1))$.

[4]Normally, you would consider maps $\pi_1(M) \to \mathrm{U}_1$, but since U_1 is abelian, this factors through the abelianization of $\pi_1(M)$, which is $H_1(M)$.

[5]As the notation might suggest, Čech cohomology can be described more generally, with coefficients in a *sheaf* of abelian groups. Our notation comes from this perspective, but ultimately we won't need to worry about that level of generality.

That is, it's defined to be a collection of smooth functions $g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}_1$ such that

$$(\delta g)_{\alpha\beta\gamma\delta} \coloneqq g_{\beta\gamma\delta} g_{\alpha\gamma\delta}^{-1} g_{\alpha\beta\delta} g_{\alpha\beta\gamma}^{-1} = 1. \tag{2.10}$$

Remark 2.11. You might be uncomfortable requiring a chocie of cover, but this is just like defining smooth functions on a smooth manifold: when aking a definition, you use the maximal atlas of a manifold for the sake of avoiding choices, but when computing with stuff, you choose some atlas with only a few charts. ◀

Next we'll define the trivialization of a gerbe, again by analogy with the Čech perspective on line bundles. Suppose f is a trivialization of a line bundle. Then, it's equivalent data to a section, which we can specify by functions $f_\alpha : U_\alpha \to \mathrm{U}_1$ for all $\alpha \in J$ together with transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{U}_1$ such that $\{g_{\alpha\beta}\}$ is a Čech coboundary. In particular, given two trivializations f and f', there's a global function $h: U_\alpha \cup U_\beta \to \mathrm{U}_1$ which is f_α/f'_α on U_α and f_β/f'_β on U_β, because on $U_\alpha \cap U_\beta$, $f_\alpha/f'_\alpha = f_\beta/f'_\beta$: the two trivializations are related by something in $\Omega^0(U_1 \cup U_2, \mathrm{U}_1)$. So the point is: trivializations realize $g_{\alpha\beta}$ as a coboundary in $U_1 \cup U_2$.

For gerbes, we'll do this one level up.

Definition 2.12. Let $\mathcal{G} = \{g_{\alpha\beta\gamma}\}$ be a U_1-gerbe. A *trivialization* of $\mathcal{G}$ over $U_\alpha \cup U_\beta \cup U_\gamma$ is a representation of $\{g_{\alpha\beta\gamma}\}$ as a coboundary in $U_\alpha \cup U_\beta \cup U_\gamma$, i.e. a choice of $\{f_{\alpha\beta}, f_{\beta\gamma}, f_{\gamma\alpha}\}$ such that

$$g_{\alpha\beta\gamma} = f_{\beta\gamma} f_{\alpha\beta} f_{\gamma\alpha}.$$

Two trivializations f and f' over $U_\alpha \cup U_\beta \cup U_\gamma$ are related by a line bundle: let $h_{\alpha\beta} \coloneqq f_{\alpha\beta}/f'_{\alpha\beta}$, and define $h_{\beta\gamma}$ and $h_{\gamma\alpha}$ similarly. Then since $g = \delta f = \delta f'$, then

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = 1, \tag{2.13}$$

so these are the transition functions for a Hermitian line bundle on $U_\alpha \cup U_\beta \cup U_\gamma$.

Remark 2.14. There is a notion of "higher gerbes:" the next step up represents elements of $H^3(M; C^\infty(\mathrm{U}_1))$, and a two trivializations of such a higher gerbe are related by a gerbe. These correspond to 2-form U_1-symmetries. ◀

Now we'll talk about connections on gerbes. Again we'll start by seeing how to realize connections on line bundles from the Čech perspective. Specifically, if L is a line bundle with connection ∇ and transition functions $g_{\alpha\beta}$, then $\nabla|_{U_\alpha} = \mathrm{d} + A_\alpha$ for some 1-form A_α, and the cocycle condition forces

$$A_\alpha = A_\beta + g_{\alpha\beta}^{-1} \mathrm{d} g_{\alpha\beta} \tag{2.15}$$

on $U_\alpha \cap U_\beta$. Then, the curvature is just $F|_{U_\alpha} = \mathrm{d}A_\alpha$. By Chern-Weil theory, $F/2\pi i$ has integer periods; conversely, any closed 2-form F such that $F/2\pi i$ has integer periods is the curvature of some connection; you can choose A_α such that $\mathrm{d}A_\alpha = F|_{U_\alpha \cap U_\beta}$, and then check the transition functions.

Now we'll do this for gerbes. Everything is shifted up one, so a connection is locally a 2-form whose curvature is a 3-form. Let $G \in \Omega^3(M; i\mathbb{R})$ with $\mathrm{d}G = 0$ and $(1/2\pi)G \in \Omega^3_{\mathbb{Z}}(M; i\mathbb{R})$, which we'll think of as the curvature. Locally we can choose $F_\alpha \in \Omega^2(U_\alpha, i\mathbb{R})$ with $G|_{U_\alpha} = \mathrm{d}F_\alpha$ such that $F_\alpha - F_\beta = \mathrm{d}A_{\alpha\beta}$ for $A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta, i\mathbb{R})$ such that on triple intersections,

$$A_{\alpha\beta} + A_{\beta\gamma} - A_{\alpha\gamma} = \mathrm{d} f_{\alpha\beta\gamma} \tag{2.16}$$

for some $f_{\alpha\beta\gamma} \in \Omega^0(U_\alpha \cap U_\beta \cap U_\gamma, i\mathbb{R})$, and this defines a cocycle which is a gerbe. Then $A = \{A_{\alpha\beta}\}$ defines its connection.

The analogues of the maps in (2.2a) and (2.2b) are the map $F: \check{H}^3(M) \to \Omega^3_{\mathbb{Z}}(M)$ sending a gerbe to $1/2\pi$ times its curvature, and $c: \check{H}^3(M) \to H^3(M; \mathbb{Z})$ sending a gerbe to $1/2\pi$ times the equivalence class of its connection.

3. Tame punctures and $\mathcal{N} = 2$ dualities: 6/5/18

Today Shehper spoke about dualities between 4D $\mathcal{N} = 2$ superconformal field theories; as my keyboard was broken, I did not take notes. Sorry about that.

4. : 6/15/18

Today, Shehper spoke about a paper of Chacaltana-Distler-Tachikawa [CDT13]. I was out of town and therefore couldn't take notes. Sorry about that.

These are lecture notes for Arun's talk on functorial TQFT and invertible TQFT from a mathematical perspective. The definitions are familiar to some of the audience members, so the goal of the talk is to have something to say about why these definitions are useful.

5.1. **Tangential structures.** Different kinds of QFTs and TQFTs are formulated on manifolds with different additional data (e.g. a spin structure, a principal bundle and connection, etc.). Tangential structures provide a unified way to understand this data.

Recall that for every topological group G, there is a *classifying space* BG, unique up to homotopy equivalence, such that for any space X, homotopy classes of maps $X \to BG$ are in natural bijection with isomorphism classes of principal G-bundles $P \to X$.

Let M be a closed n-manifold with a Riemannian metric and $B_{\mathrm{O}} \to M$ denote the principal O_n-bundle whose fiber at x is the O_n-torsor of orthonormal bases of T_xM. Thus we have a canonical map $f_M \colon M \to B\mathrm{O}_n$; in fact, since the space of metrics is contractible, the homotopy type of this map doesn't depend on the metric, so it's topological information about M.

Suppose M is oriented. Then we know which bases are positively oriented, and we can throw out the other ones, leaving a principal SO_n-bundle of positively oriented orthonormal bases. This induces a map $M \to B\mathrm{SO}_n$ whose composition with the "forgetful" map $B\mathrm{SO}_n \to B\mathrm{O}_n$ (induced from the inclusion $\mathrm{SO}_n \hookrightarrow \mathrm{O}_n$) recovers f_M. So one could define an orientation to be a lift of the map $M \to B\mathrm{O}_n$ to $B\mathrm{SO}_n$. Alternatively, one could do this for all n at once: let O denote the union[6] of the sequence of topological groups $\mathrm{O}_1 \subset \mathrm{O}_2 \subset \cdots$, which is also a topological group, and define SO similarly. Then the map $\mathrm{O}_n \hookrightarrow \mathrm{O}$ and the bundle of orthonormal frames define a map $M \to B\mathrm{O}$, and an orientation is equivalent data to a lift of this across $B\mathrm{SO} \to B\mathrm{O}$.

This example motivates the general notion of tangential structures.

Definition 5.1. (definition of a tangential structure)

Remark 5.2. (for $BG \to B\mathrm{O}_n$, this is reduction of the structure group to G) ◀

The point of all this is that a given QFT or TQFT may depend on different kinds of data, either as structure on spacetime needed as input to the theory (e.g. an orientation, a spin structure), or as a background field. In cases where this information is topological (so, not a Riemannian metric or a conformal structure), it can be described by a tangential structure.

Example 5.3 (G-structures)**.** In view of **??**, one class of examples comes from G-structures: a Lie group G_n and a map $\rho \colon G_n \to \mathrm{O}_n$ (or, a family of Lie groups $G_n \hookrightarrow G_{n+1} \hookrightarrow \cdots$ if we don't want to carry around the dimension).

- As already discussed, for $G_n = \mathrm{SO}_n$, an SO-structure is the same thing as an orientation.
- A spin structure is the same thing as a G-structure for the map $\mathrm{Spin}_n \twoheadrightarrow \mathrm{SO}_n \hookrightarrow \mathrm{O}_n$.
- Some QFTs at first appear to require a spin structure, but don't require an orientation (e.g. there is a reflection or time-reversal symmetry). One example is the Majorana chain. In this case, the tangential structure in question is a G-structure for $G = \mathrm{Pin}_n^-$ or Pin_n^+, two double coverings of O_n which contain Spin_n. ◀

Example 5.4 (Background gauge and higher gauge fields)**.** Consider the (B, f)-structure $B\mathrm{O} \times BG \to B\mathrm{O}$ which is projection onto the first factor; then, a (B, f)-structure on a manifold M is a choice of a map $M \to BG$, i.e. a principal G-bundle. This does not come with data of a connection (that's "geometric" as opposed to "topological" — connections cannot be described by a (B, f)-structure). Therefore if G is discrete, a theory with a background G-symmetry is one that can be formulated on manifolds with a $(B\mathrm{O} \times BG)$-structure. This works just as well with $B\mathrm{SO} \times BG$, $B\mathrm{Spin} \times BG$, etc.

More generally, suppose one of your fields is a map to a space X. You can describe this by the (B, f)-structure $B\mathrm{O} \times X \to B\mathrm{O}$: a lift of $M \to B\mathrm{O}$ to this is exactly a choice of a map $M \to X$. Equivalence classes of this structure are determined by homotopy classes of maps to X.

Now suppose your theory has a 1-form symmetry that you've managed to describe as a background higher gauge field, e.g. a gerbe for a finite abelian group A (finite so we can ignore connections). Well, I don't know what gerbes are, but they're classified by maps to $B^2A = K(A, 2)$ (the latter notation is more common in algebraic topology), so if a theory has a background 1-form A-symmetry, it can be formulated on manifolds with (B, f)-structure for $B\mathrm{O} \times K(A, 2) \to B\mathrm{O}$ (or $B\mathrm{SO}$, etc.). This also works for k-form symmetries and $K(A, k+1)$, which is easier for me than other ways of thinking about higher gerbes. ◀

[6] Rigorously, this is actually a colimit, though union is good intuition.

Example 5.5 (2-groups)**.** We've been thinking about 2-group symmetries in this seminar. One way to describe a 2-group $\mathbb{G}$ is as data (G, A, ρ, k), where G is a group, A is an abelian group, $\rho\colon G \to \mathrm{Aut}(A)$ is an action, and $k \in H^3(BG; A)$ is called the *k-invariant.*

Background 2-group symmetries also fit into the framework of a tangential structure: there is a classifying space $B\mathbb{G}$ for $\mathbb{G}$, uniquely characterized up to weak homotopy equivalence by the following information:

- $\pi_1(B\mathbb{G}) = G$ and $\pi_2(B\mathbb{G}) = A$.
- There's a fibration $\pi\colon B\mathbb{G} \to BG$ with fiber $K(A, 2)$; this induces an action of $\pi_1(B\mathbb{G}) = G$ on $\pi_2(B\mathbb{G}) = A$, which must coincide with ρ.
- The cofiber of π, $C(\pi)\colon BG \to K(A, 3)$,[7] defines the k-invariant: $[BG, K(A, 3)] = H^3(BG; A)$, and this must send $C(\pi) \mapsto k$.

Like for ordinary symmetries, there's a notion of principal $\mathbb{G}$-bundles, and isomorphism classes of these are identified with homotopy classes of maps to $B\mathbb{G}$.

Therefore a field theory with a background $\mathbb{G}$-symmetry is one which makes sense on (B, f)-manifolds for the (B, f)-structure $B\mathrm{O} \times B\mathbb{G} \to B\mathrm{O}$ (or $B\mathrm{SO}$, $B\mathrm{Spin}$, etc.). ◀

I'll say more about this later, but one reason to care about these is the idea that the low-energy effective theory of a QFT has "the same symmetries" as the general theory. This means that, if your UV theory is formulated on manifolds with some (B, f)-structure (plus additional data, such as a Riemannian metric, a connection, …), then the low-energy effective theory should make sense on manifolds with the same (B, f)-structure. This principle will also apply to anomalies when thought of as invertible TQFTs: an anomalous QFT formulated on manifolds with (B, f)-structure has for its anomaly an invertible TQFT on manifolds with (B, f)-structure in one dimension higher.

5.2. **Functorial TQFT.** Let M be a manifold with boundary. Then, a (B, f)-structure on M induces a (B, f)-structure on ∂M, because $TM|_{\partial M} \cong T(\partial M) \oplus \underline{\mathbb{R}}$ (splitting off the normal vectors), so the classifying maps $\partial M \to B\mathrm{O}$ for $TM|_{\partial M}$ and $T(\partial M)$ are homotopic. Therefore we can define a *cobordism* between closed (B, f)-manifolds M and N to be a compact (B, f)-manifold X together with an isomorphism $\partial X \cong M \amalg N$ of (B, f)-manifolds, and therefore define a cobordism category $\mathrm{Bord}_n^{(B,f)}$ of (B, f) n-manifolds, which is a symmetric monoidal category under disjoint union, and define a *topological field theory* of (B, f)-manifolds to be a symmetric monoidal functor $Z\colon \mathrm{Bord}_n^{(B,f)} \to \mathrm{Vect}_{\mathbb{C}}$ as usual.

Great, but why?

- If N is a codimension 1 (B, f)-manifold, $Z(N)$ is a complex vector space which we think of as the state space of the theory formulated on N.
- If M is a closed (B, f) n-manifold, then $Z(M)\colon \mathbb{C} \to \mathbb{C}$ is multiplication by some complex number, which we think of as the partition function of M.
- Now suppose M is a (B, f) n-manifold with boundary. In this case, the partition function of a TQFT on M requires a choice of boundary condition, which is an element of the states on ∂M. Regarding M as a cobordism $\partial M \to \varnothing$, we obtain $Z(M)\colon Z(\partial M) \to \mathbb{C}$: given a boundary condition, we get a number, which is the partition function for that boundary condition. If you have multiple boundary components N_1 and N_2, you can ask: as I vary the boundary condition on N_1, how does the partition function $Z(N_2) \to \mathbb{C}$ change? This means you're describing a map $Z(N_1) \to Z(N_2)^{**} = Z(N_2)$, which is exactly what a cobordism should tell you.
- Gluing of cobordisms is related to locality: you should be able to compute the partition function of a manifold M by chopping M up into pieces, computing the theory on those pieces, and then assembling the data together.
- The fact that disjoint unions go to tensor product is a fact from quantum mechanics: the Hilbert space of states of two separate systems A and B is $\mathcal{H}_A \otimes \mathcal{H}_B$.

Remark 5.6. The fact that Z is topological is encoded in the fact that a (B, f)-structure does not include a metric, connection, or any other geometric information. One can define cobordism categories of Riemannian manifolds, etc. and try to define quantum field theories in that way; though this doesn't Just Work the same way it does for TQFT, it's still a useful perspective to keep in mind. I learned (probably from Dan Freed) that thinking about QFT concepts from this perspective provides a good foothold. ◀

Example 5.7 (The Euler TQFT)**.** Fix a $\mu \in \mathbb{C}^\times$. Then the *Euler TQFT* for μ is the functor $Z_\mu\colon \mathrm{Bord}_n^{\mathrm{O}} \to \mathrm{Vect}_{\mathbb{C}}$ sending all objects to $\mathbb{C}$ and a cobordism X to the map $\mathbb{C} \to \mathbb{C}$ which is multiplication by $\mu^{\chi}(X)$.

[7]This is true when ρ is trivial. Otherwise we have to use twisted cohomology.

If X is a cobordism from M to N and Y is a cobordism from N to P, then $\chi(Y \circ X) = \chi(Y) + \chi(X)$, so this is in fact a functor. Since $\chi(X \amalg Y) = \chi(X) + \chi(Y)$, this is symmetric monoidal, hence a TQFT. ◂

Example 5.8. (Dijkgraaf-Witten theory, maybe?) ◂

Remark 5.9. (Extended TQFT) ◂

Let $Z_1, Z_2 \colon \mathrm{Bord}_n^{(B,f)} \to \mathrm{Vect}_{\mathbb{C}}$ be two TQFTs. Then we can take their tensor product: $Z_1 \otimes Z_2(M) := Z_1(M) \otimes Z_2(M)$ for any object or morphism M.

5.3. **Invertible field theories.**

Definition 5.10. An *invertible topological field theory* Z is one which is $\otimes$-invertible, i.e. there is a TQFT Z' such that $Z \otimes Z'$ is isomorphic to the trivial theory (which assigns $\mathbb{C}$ to every object and id to every morphism).

The Euler TQFT is invertible, with inverse $Z_{\mu^{-1}}$.

Example 5.11. (Invertible TQFT from a cobordism invariant)
Examples: classical Dijkgraaf-Witten theory, Arf theory, Arf-Brown theory ◂

There are several different ways invertible TQFTs appear in physics.

One occurrence has something to do with discrete θ-parameters and/or setting the quantum integrand. Approximately speaking, suppose in a QFT we want to integrate over a space of fields that's not connected. Then there are different ways to weight the connected components: the partition function can be written as a sum

$$Z(M) = \sum_{\nu} \alpha_\nu(M) Z'(M, \nu) \tag{5.12}$$

for some weights α_ν. Different choices of α_ν give us a family of theories.

If $\alpha_\nu(M) \in \mathrm{U}_1$, then they should be the partition functions of an invertible TQFT of the same dimension. Its symmetry type should include the symmetries of Z, but also the data of ν.

Mathematically, you might think of this as $Z = \alpha_\nu \otimes Z'$.

Example 5.13. One place this can occur is in Chern-Simons theories with gauge group O_N, as explained by Córdova-Hsin-Seiberg [CHS18]. The quantum theory is (at a physical level of rigor) a theory of spin manifolds summing over O_N-connections. In general, there are multiple nonisomorphic principal O_N-bundles on a spin 3-manifold, so there are different ways we can sum: for example, we could weight by any cobordism invariant of spin 3-manifolds M with a principal O_N-bundle $P \to M$, including $\exp(i\pi\langle w_1(P)w_2(P), [M]\rangle) \in \{\pm 1\}$ or the Arf-Brown $\mathbb{Z}/8$-invariant of the Poincaré dual to $w_1(P)$. In each case, there is an invertible TQFT of spin manifolds together with a principal O_N-bundle (so the (B, f)-structure is $B\mathrm{Spin} \times B\mathrm{O}_N$) whose partition function is this cobordism invariant.[8]

Therefore the classification of these Chern-Simons theories picks up an additional $\mathbb{Z}/2 \times \mathbb{Z}/8$ factor, and this ends up important when studying level-rank duality. ◂

This is also discussed by Freed-Moore [FM06].

Another application of invertible TQFTs is the classification of SPTs in condensed-matter physics. This is sort of a tautology: SPT phases are precisely those whose low-energy effective theories are invertible TQFTs.[9] But the interesting thing is that we're looking at deformation classes, rather than isomorphism classes, of invertible TQFTs: if you can connect two invertible theories by a path, they're believed to have the same physics. (so you basically get cobordism invariants/Euler TQFTs go away)

A third application is to anomalies, as espoused by Freed-Telemann.

- A TQFT Z relative to another TQFT α is a natural transformation $\tau_{\le n}\alpha \to 1$. The upshot is that the partition function of Z on M is not a number, but an element of the complex line $\alpha(M)$, and this may not be trivialized.
- One application is that anomalous theories can (always? sometimes?) be realized as relative theories for an invertible theory in one dimension higher. This is the idea of "anomaly inflow."

In many cases, it appears that the partition function of the anomaly theory is a cobordism invariant. I don't know why this is true in general, but it leads to nice classifications.

Example 5.14. Free fermions and the Arf/Arf-Brown anomaly theories. ◂

[8]More generally, it's possible to lift any $\mathbb{C}^\times$-cobordism invariant of (B, f)-manifolds to an invertible TQFT of (B, f)-manifolds; this follows from the Freed-Hopkins classification, and is also described explicitly in a recent paper of Yonekura.

[9]Much of this previous sentence isn't quite rigorous, and making it so is a difficult open problem.

(say something about 't Hooft anomaly matching and (B, f)-structures)

Remark 5.15. Say something about the homotopical classification of TQFTs. ◀

If time, say something about the relationship between invertible TQFTs and cobordism invariants.

References

[AGG+10] Luis F. Alday, Davide Gaiotto, Sergei Gukov, Yuji Tachikawa, and Herman Verlinde. Loop and surface operators in $\mathcal{N} = 2$ gauge theory and Liouville modular geometry. *Journal of High Energy Physics*, 2010(1):113, Jan 2010. https://arxiv.org/pdf/0909.0945.pdf. 1

[CDT13] Oscar Chacaltana, Jacques Distler, and Yuji Tachikawa. Nilpotent orbits and codimension-2 defects of 6d $\mathcal{N} = (2, 0)$ theories. *International Journal of Modern Physics A*, 28(03n04):1340006, 2013. https://arxiv.org/pdf/1203.2930.pdf. 5

[CHS18] Clay Córdova, Po-Shen Hsin, and Nathan Seiberg. Global symmetries, counterterms, and duality in Chern-Simons matter theories with orthogonal gauge groups. *SciPost Phys.*, 4:21, 2018. https://arxiv.org/pdf/1711.10008.pdf. 8

[FGT16] Edward Frenkel, Sergei Gukov, and Jörg Teschner. Surface operators and separation of variables. *Journal of High Energy Physics*, 2016(1):179, Jan 2016. https://arxiv.org/pdf/1506.07508.pdf. 1

[FM06] Daniel S. Freed and Gregory W. Moore. Setting the quantum integrand of M-theory. *Communications in Mathematical Physics*, 263(1):89–132, Apr 2006. https://arxiv.org/pdf/hep-th/0409135.pdf. 8

[Hit99] Nigel Hitchin. Lectures on special Lagrangian submanifolds. 1999. https://arxiv.org/pdf/math/9907034.pdf. 3