STACKS

ARUN DEBRAY
JUNE 28, 2018

These notes were taken in Adrian Clough's minicourse in Summer 2018. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Tom Gannon for fixing a typo.

CONTENTS

Day 1. June 18

I missed the morning and afternoon lectures for this day, which focused on the two main themes (descent and building new spaces from old) through examples.

Day 2. June 19

1. MORNING

We'll start with a review of some of yesterday's material.

Let X and R be manifolds, and $g : X \to R$ be a smooth map. Given a submersion $p : U \to X$, g lifts to a map $U \to R$. In this case, the pullback $U \times_X U \rightrightarrows U$ is a smooth manifold, and p is exactly the coequalizer of that diagram. This is a very simple instance of descent: we have an isomorphism between the space of smooth maps from X to R with the space of maps from U to R that coequalize $U \times_X U \rightrightarrows U$ (i.e. after composing with such a map, the two structure maps become equal).

More generally, let $\mathfrak{U}$ be an open cover of X by balls and $U := \coprod_{V \in \mathfrak{U}} V$. The story goes through with $U \times_X U \rightrightarrows U \to X$. But descent doesn't always hold: for example, if you try to lift a map $E \to X$ to U. You need additional data.

It's also useful to replace sets with groupoids. For example, if you want to consider descent data for principal G-bundles, then you can't consider the set of equivalence classes: the pullback map $\mathsf{Bun}_G(X) \to \mathsf{Bun}_G(U)$ isn't very interesting, because $\pi_0 \mathsf{Bun}_G(U)$ is trivial. Therefore in order to set up descent, we'll need to (1) consider groupoids, and (2) provide additional data on $\mathfrak{U}$ that makes interesting things possible. In this case the passage from injective to surjective to bijective is replaced with faithful to fully faithful to an equivalence of categories.

Recall that an *epimorphism* in a category C is a map $f : A \twoheadrightarrow B$ such that for any two maps $g, h \colon B \rightrightarrows C$ such that $g \circ f = h \circ f$, then $g = h$. In Set this corresponds to the usual notion of surjective.

Definition 1.1. An epimorphism $f : U \to X$ is an *effective epimorphism* if the pullback $U \times_X U$ exists and f is its coequalizer.

You can think of this as a categorified version of an equivalence relation.

It's also possible to go the other direction: given an equivalence relation $R \to U \times U$ on U and an epimorphism $U \twoheadrightarrow X$, you can ask whether $R \cong U \times_X U$.

In geometric sitations, it seems generally true that one glues by equivalence relations. This seems a little mysterious, but is one motivation for thinking about stacks in this way.

Now we turn to creating new spaces from old. In many situations you're considering a nice class of spaces (schemes, varieties) $\mathscr{C}$, but some reasonable construction starting with stuff in $\mathscr{C}$ leads you to something which does not exist in $\mathscr{C}$. For example, studying manifolds leads to asking about classifying spaces of principal G-bundles, which cannot be finite-dimensional; thus, you have to enlarge your category of spaces to get at them. Another example is fine moduli spaces in algebraic geometry, which often aren't varieties and sometimes aren't schemes.

The general approach is to (1) enlarge to a category $\widetilde{\mathscr{C}}$ of *all* objects that can reasonably be constructed from objects in $\mathscr{C}$ (i.e. by taking colimits and satiafying a descent condition), and then (2) restricting to a subcategory of nicer spaces. Depending on your application, there are different choices of nice subcategories for a given $\widetilde{\mathscr{C}}$.

Example 1.2. Let Cart be the category whose objects are open subsets of $\mathbb{R}^n$ and whose morphisms are smooth maps. Then, inside $\widetilde{\mathsf{Cart}}$, one could restrict to smooth manifolds, or more generally Fréchet manifolds, or more generally Frölicher spaces, or more generally diffeological spaces, depending on your application. ◀

To do this, we need some good candidate for $\widetilde{\mathscr{C}}$. One, which we denote $\widehat{\mathscr{C}}$, is the collection of *presheaves on* $\mathscr{C}$, i.e. contravariant functors $\mathscr{C} \to \mathsf{Set}$. We have the Yoneda embedding $\mathscr{C} \hookrightarrow \widehat{\mathscr{C}}$, and we might want to complete $\widehat{\mathscr{C}}$ under colimits to a map $\mathscr{C} \to \mathscr{D}$; $\mathscr{D}$ exists and is unique by a universal property. However, often coequalizers of the kinds we've been considering are not preserved by this construction. Here are two possible fixes for a category C and a collection τ of coequalizers we want to keep:

(1) Let $U \times_X U \rightrightarrows U \twoheadrightarrow X$ be in τ, and apply the Yoneda embedding to it, producing a diagram

$$\begin{tikzcd} \mathsf{C}(-,U) \times_{\mathsf{C}(-,X)} \mathsf{C}(-,U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathsf{C}(-,U) \arrow[r] \arrow[dr] & \mathsf{C}(-,X). \\ & & \mathrm{Coeq}(f,g) \arrow[u, "(**)"'] \end{tikzcd} \tag{1.3}$$

The fix will be to localize the morphisms $(**)$ in $\widehat{\mathsf{C}}$, Namely, let W denote the subcategory generated by these morphisms over all diagrams in τ. The localization is denoted $\widehat{\mathsf{C}}[\mathsf{W}]$, and has an obvious universal property: given a functor $F \colon \mathsf{C} \to \mathsf{D}$, such that D is cocomplete and F preserves colimits and all coequalizers in τ, there's a unique functor $\widehat{\mathsf{C}}[\mathsf{W}] \to \mathsf{D}$. This is nice, but localization is terrible (see Remark 1.8), so it might be good to try something else.

(2) The new basic idea is that if $X \in \widetilde{\mathsf{C}}$, then we should be able to recover X from $\widetilde{\mathsf{C}}(-,X)$ restricted to C. Thus we identify X and $\widetilde{\mathsf{C}}(-,X)$; that is, we want $\widetilde{\mathsf{C}} \subseteq \widehat{\mathsf{C}}$.

Now let's think about colimits. Let $F \colon \mathsf{J} \to \mathsf{A}$ be a diagram; then $\mathsf{A}(\varinjlim F, X) \cong \varprojlim \mathsf{A}(F,X)$, or in other words, $\varinjlim X$ represents the functor $\varprojlim \mathsf{A}(F,-)$. This means that $A \rightrightarrows B \to C$ is a coequalizer iff for all $X \in \mathsf{A}$,

$$\mathsf{A}(C,X) \longrightarrow \mathsf{A}(B,X) \rightrightarrows \mathsf{A}(A,X) \tag{1.4}$$

is an *equalizer* diagram (since $\mathsf{A}(-,X)$ is contravariant).

Now assume $P \in \widetilde{\mathsf{C}} \subseteq \widehat{\mathsf{C}}$ and $R \rightrightarrows U \twoheadrightarrow X$ is in τ. Then we can apply $\widehat{\mathsf{C}}(-,P)$ to obtain a diagram

$$\begin{tikzcd} \widehat{\mathsf{C}}(X,P) \arrow[r] \arrow[d, equal] & \widehat{\mathsf{C}}(U,P) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \widehat{\mathsf{C}}(R,P) \arrow[d, equal] \\ P(X) \arrow[r] & P(U) \arrow[r, shift left] \arrow[r, shift right] & P(R). \end{tikzcd} \tag{1.5}$$

Requiring this to be a coequalizer diagram is exactly a sheaf condition! For example, in the category of smooth manifolds, if $\mathfrak{U}$ is an open cover of X and $P\colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{Set}$ is a presheaf, then we can apply it to the diagram

$$\coprod_{U,V\in\mathfrak{U}} U\cap V \rightrightarrows \coprod_{U\in\mathfrak{U}} U \longrightarrow X \tag{1.6}$$

to obtain (assuming $\mathsf{C} \to \widetilde{\mathsf{C}}$ preserves coproducts)

$$\begin{tikzcd} P(X) \arrow[r] \arrow[d, equal] & P\left(\coprod_{U\in\mathfrak{U}} U\right) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & P\left(\coprod_{U,V\in\mathfrak{U}} U\cap V\right) \arrow[d, equal] \\ P(X) \arrow[r] & \prod_{U\in\mathfrak{U}} P(U) \arrow[r, shift left] \arrow[r, shift right] & \prod_{U,V\in\mathfrak{U}} P(U\cap V), \end{tikzcd} \tag{1.7}$$

and asking for this to be a coequalizer diagram is the usual notion of a presheaf being a sheaf (e.g. you can find this diagram in Vakil).

Remark 1.8. Localization of categories is technical and messy: for example, it's easy to write down an example where after the naïve notion of localization, there are collections of morphisms between objects that are so big as not to be sets. This is bad. One of the reasons people like model categories is because they avoid this problem. ◁

2. Afternoon

"Geometry is the art of giving funny names to morphisms."

These two different ways of fixing coequalizers turn out to be the same.

Now let's suppose that $\widetilde{\mathsf{C}}$ is the category of sheaves on C with respect to the coequalizers in τ (so the second approach).

Theorem 2.1. *The inclusion $\widetilde{\mathsf{C}} \hookrightarrow \widehat{\mathsf{C}}$ admits a left adjoint $a\colon \widehat{\mathsf{C}} \to \widetilde{\mathsf{C}}$, which is precisely the localization functor from the first approach.*

This implies in particular that the localization exists and is nice (e.g. you don't have to consider arbitrarily long zigzags in it, much like in a model category). This theorem is saying something deep.

Remark 2.2. If you do the ∞-version of this, with simplicial sets and/or ∞-groupoids, this recovers localization of model categories (or ∞-categories), producing simplicial sheaves from simplicial presheaves. ◁

Theorem 2.1 holds for arbitrary τ (probably), but we'll prove it under a reasonable assumption on τ, which is also a motivation to learn about sites.

Sites.

Definition 2.3. Let C be a category.[1] Then a *coverage* on C is data of, for all $X \in \mathsf{C}$, a family $T_X = \{\{\pi_i\colon U_i \to X\}_{i\in I}\}$ of families of maps to X, such that given a map $f : X' \to X$ in C and a family $\{U_i \to X\}_{i\in I}$ in T_X, there is a family $\{U'_j \to X'\}_{j\in J}$ in $T_{X'}$ such that for all $j \in J$, there exists an $i \in I$ and a $g : U'_j \to U_i$ such that the diagram

$$\begin{tikzcd} U'_j \arrow[r, "g"] \arrow[d, "\pi'_j"] & U_i \arrow[d, "\pi_i"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{2.4}$$

commutes. A category together with a coverage is called a *site*.

Before we see some examples, we'll define some related notions.

[1] We'll assume C is small; this is not necessary for the definition, but will be true for all examples we consider.

Definition 2.5. Given a coverage $\mathbf{T}$ of C and a presheaf $\mathscr{F} \in \widehat{\mathsf{C}}$, let $\mathrm{Desc}_{\mathscr{F}}(\{U_i \to X\}_{i\in I})$ be the set of $(s_i) \in \prod_{i\in I} \mathscr{F}(U_i)$ such that for all $i, j \in I$ and diagrams $U_i \xleftarrow{f} V \xrightarrow{g} U_j$ such that $s_i|_V = s_j|_V$, $\mathscr{F}(f)(s_i) = s_i|_V$ and $\mathscr{F}(g)(s_j) = s_j|_V$.

This is a lot like a limit, but it's not clear how to make it exactly one nicely.

Definition 2.6. Given a category C with a coverage $\mathbf{T}$, a $\mathbf{T}$-*sheaf* $\mathscr{F}$ is a presheaf $\mathscr{F} \colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ such that for all $U \in \mathsf{C}$ and families $\{U_i \to U\}$ in $\mathbf{T}$, the natural map $\mathscr{F}(U) \to \mathrm{Desc}_{\mathscr{F}}(\{U_i \to U\})$ is an isomorphism.

Example 2.7.

(1) In manifolds, we could choose $\mathbf{T}$ to be coverings in the usual way, or jointly[2] surjective submersions (more in spirit of what we did yesterday), or *étale covers*, i.e. jointly surjective local homeomorphisms.
(2) Let C denote the category of open subsets of $\mathbb{R}^n$ (for all n) and smooth maps between them. Then one can once again consider covering maps, surjective submersions, or étale covers.
(3) Let Cart denote the category whose objects are $\mathbb{R}^n$ for each n and whose maps are smooth maps between them. In this case we can take surjective submersions but not open covers, because there are no coproducts.
(4) Let X be a space and C be the category of open subsets on X. Then we can take classical coverings, i.e. $\{U_i \to U\}$ is a covering iff $U \subset \bigcup U_i$. A $\mathbf{T}$-sheaf for this is a sheaf in the usual sense; this is because $\mathrm{Desc}_{\mathscr{F}}(\{U_i \subseteq U\})$ is exactly the collections of sections $\mathscr{F}(U_i)$ that agree on intersections. ◀

Looking into the last example in more detail, if you fix a manifold M, $\mathsf{Top}(M)^{\mathsf{op}} \hookrightarrow \mathsf{Man}^{\mathsf{op}}$ sending $U \mapsto U$, so we get a map from presheaves on Man to presheaves on $\mathsf{Top}(M)$, namely, restrict a presheaf $\mathscr{F}$ to open subsets of M.

Exercise 2.8. Show that a presheaf $\mathscr{F}$ on Man is a $\mathbf{T}$-sheaf (for the three choices of $\mathbf{T}$ we discussed above) iff for all manifolds M, $F|_M$ is a classical sheaf.

In fact, in Man, from any covering we can construct the classical descent situation, and we actually get an equivalence of categories $\widetilde{\mathsf{Man}} \simeq \widetilde{\mathsf{Cart}}$. This is why we want general coverages: in Cart we can't take coproducts, but we can in Man. This ultimately means that the kinds of spaces discussed earlier, namely Fréchet manifolds, Frölicher spaces, and diffeological spaces, can all be described as sheaves on Cart.

In general, suppose C admits pullbacks; then we have a similar story as for $\mathsf{Top}(X)$: descent means that given a map $f : V \to U$ and covers $\{V_i \to V\}$, $\{U_i \to U\}$, we get a diagram

(2.9)
$$\begin{tikzcd}
\coprod_{i,j} V_i \times_V V_j \arrow[r, "f''"] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{i',j'} U_{i'} \times_U U_{j'} \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_j V_j \arrow[r, "f'"] \arrow[d] & \coprod_{j'} U_{j'} \arrow[d] \\
V \arrow[r, "f"] & U.
\end{tikzcd}$$

Descent means that we should be able to reconstruct f just knowing f'' and f'. This looks a lot like descent for bundles, which seems important.

This ties to the stuff we saw this morning: in $\widetilde{\mathsf{C}}$, for $\{U_i \to U\}$ in T_U, $\coprod \mathsf{C}(-, U_i) \to \mathsf{C}(-, U)$ becomes an effective epimorphism!

Day 3. **June 20**

3. Morning

When we studied descent, we concluded that the morphisms we want to descend along are gluing morphisms, which are often effective epimorphisms: these are epimorphisms $U \to X$ which are equivalent data (via pullback $U \times_X U \rightrightarrows U$) to equivalence relations.

[2]For a family, this means that the map from the disjoint union of all of them is surjective.

Then we discussed categories C of geometric objects (what a "geometric object" means depends on the application) for which we want to expand to more general objects, while preserving a collection τ of coequalizers. We suggested two ways to do this, one by localizing $\widehat{\mathsf{C}}$ in a way to get τ back, and another was to take a subcategory of $\widehat{\mathsf{C}}$, and it turns out that these two approaches are equivalent, and the localization is well behaved.

Then we discussed coverages and sheaves; in this case, τ isn't quite effective epimorphisms, but certain covering maps, and we force them to be effective epimorphisms in the presheaf category.

Today, we'll discuss more about Grothendieck topologies. The first part is technical, but it has to be: if you try to read this stuff somewhere else it will still be technical.

Let $\mathbf{T}$ be a coverage on C. Then we can ask two questions about it.

(1) Are there any aditional coverings $\mathfrak{U}$ that we get "for free," i.e. for which any $\mathbf{T}$-sheaf satisfies the sheaf condition for $\mathfrak{U}$?
(2) Given a $\{U_i \to U\}$ in T_U, can we add more morphisms to it such that the same presheaves are $\mathbf{T}$-sheaves?

Analogy 3.1. Once again, we'll compare to the localization of rings.

Given a subset S of a ring R, we can form the localization $S^{-1}R$; if S_m denotes the multiplicative closure of S, then $S^{-1}R \cong S_m^{-1}R$; in fact, you can even take the saturated closure S_s of S, and the localization is the same.

The point is: in this setting we can add extra stuff to S and get the same localization. And we obtain a bijection between saturated subsets of R and localizations of R.

Theorem 3.2. *Let C be a category and $\mathsf{S} \subset \widehat{\mathsf{C}}$ be a subcategory. Then, S is the category of $\mathbf{T}$-sheaves for some coverage $\mathbf{T}$ iff inclusion $\mathsf{S} \hookrightarrow \widehat{\mathsf{C}}$ admits a left exact left adjoint.*

So we want a bijection between such subcategories and some kinds of coverages, which are what we'll eventually call Grothendieck topologies.

Definition 3.3. A coverage $\mathbf{T}$ is *maximal* if it is not possible to add new objects or new morphisms to it without changing the category of $\mathbf{T}$-sheaves.

Theorem 3.4. *There is a bijection between the subcategories $\mathsf{S} \subset \widehat{\mathsf{C}}$ whose inclusion admits a left exact left adjoint and maximal coverages.*

We'll prove this with a sequence of lemmas. The first is not terribly surprising.

Lemma 3.5. *Let C be a category with a coverage $\mathbf{T}$. For all $U \in \mathsf{C}$ and $P \in \widehat{\mathsf{C}}$, P satisfies the sheaf condition with respect to $\{\mathrm{id}\colon U \to U\}$.*

Lemma 3.6. *Let $U \in \mathsf{C}$ and $\{V_j \to U\}$ be a covering not necessarily in T_U. If there exists a $\{U_i \to U\} \in T_U$ such that for all j there's an i and a commutative diagram*

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X, & \end{tikzcd} \tag{3.7}$$

then $\mathbf{T}$-sheaves satisfying the sheaf condition for $\{V_j \to U\}$ are $T \cup \{\{V_j \to U\}\}$-sheaves.

Example 3.8. Consider on Man the coverage of usual coverings by open sets, and suppose you want to add all jointly surjective submersions. Then the hypothesis of Lemma 3.6 certainly holds: locally a submersion $S \twoheadrightarrow X$ looks like a projection[3], so you can locally choose a section $\mathbb{R}^m \to S$, where $m = \dim X$; composing with the submersion produces a covering map locally, which suffices. $\blacktriangleleft$

Lemma 3.9. *If C has pullbacks, $\{U_i \to U\}$ is in T_U, and for all i, $\{U_{ij} \to U_i\}$ is in T_{U_i}, then all $\mathbf{T}$-sheaves are $\{\{U_{ij} \to U_i \to U\}\} \cup \mathbf{T}$-sheaves.*

Remark 3.10. This is stated in the Elephant without pullbacks, but Zhen Lin Low found a counterexample. $\blacktriangleleft$

The next result is a corollary of Lemma 3.6.

Corollary 3.11. *If C has pullbacks, then given $U \to V$ and $\{V_j \to V\}$ in T_V, all $\mathbf{T}$-sheaves are $\mathbf{T} \cup \{\{V_j \times_V U \to U\}\}$-sheaves.*

[3] In fact, for topological manifolds this is often taken as the definition of a submersion.

The idea is to let U_i be the pullback of $V_j \to V$ by $U \to V$.

Now we can define the notion of a Grothendieck pretopology; it is presented in a way that suggests a strong resemblance to the usual notion of covering maps of topological spaces.

Definition 3.12. Let C be a category with pullbacks. A coverage τ is called a *Grothendieck pretopology* if

(1) for all $U \in \mathsf{C}$, $\{\mathrm{id}\colon U \to U\} \in \tau_U$,
(2) for all $U \to V$ and $\{U_j \to V\} \in \tau_V$, the "restriction" $\{U_j \times_V U \to U\} \in \tau_U$.
(3) the conclusion of Lemma 3.9 holds.

Theorem 3.13. *Let C be a category with pullbacks and $\mathbf{T}$ be a coverage on C; then, the intersection of all Grothendieck pretopologies $\tau \supset \mathbf{T}$ is again a Grothendieck pretopology with the same sheaves as $\mathbf{T}$.*

4. Afternoon

Definition 4.1. Let $U \in \mathsf{C}$. A covering $\mathfrak{U}$ of U is called a *sieve* if for all $U' \to U$ in $\mathfrak{U}$ and all $U'' \to U'$, the composition $U'' \to U$ is in $\mathfrak{U}$.

So there's an obvious way to complete any covering to a sieve.

Lemma 4.2. *For any $U \in \mathsf{C}$, there's a bijection between the sieves on U and the subpresheaves of $\mathsf{C}(-,U)$.*

Proof sketch. Given a sieve S on U, you can obtain a presheaf sending U' to the set of all maps $U' \to U$ in S; the sieve property guarantees functoriality.

Conversely, a subpresheaf of $\mathsf{C}(-,U)$ identifies a certain subset of maps $U' \to U$ for each U', such that composition still lands in the corresponding subset, which is exactly the sieve condition. ⊠

We will hence identify a sieve S with the subpresheaf of $\mathsf{C}(-,U)$ it corresponds to. Then, given a map $f : U' \to U$, let f^*S denote the pullback

$$\begin{tikzcd} f^*S \arrow[r] \arrow[d] & S \arrow[d, hook] \\ \mathsf{C}(-,U) \arrow[r, "f_*"] & \mathsf{C}(-,U). \end{tikzcd} \tag{4.3}$$

Thinking about what pullbacks in Set are, f^*S is the presheaf

$$V \longmapsto \{g : V \to U' \mid g \circ f \in S(U)\}. \tag{4.4}$$

Lemma 4.5. *Let $\mathfrak{U}$ be a coverin gof U and S be the sieve it generates. Let P be a presheaf; then,*

$$\mathrm{Desc}_P(\mathfrak{U}) \cong \varprojlim_{(V \to U) \in S} P(V).$$

Proof sketch. There is a natural map $\mathrm{Desc}_P(\mathfrak{U}) \to \varprojlim P(V)$ (restrict each s_i to V) — it's injective when you pass to $\prod P(V)$, hence must have been injective. For surjectivity, there's an explicit inverse: things in the limit are indexed by larger diagrams, and we can restrict to the smaller triangles in $\mathrm{Desc}_P(\mathfrak{U})$. ⊠

Lemma 4.6. *Let $\mathbf{T}$ be a coverage and $\mathbf{T}'$ be the completion of $\mathbf{T}$ in the sense that all coverings are completed to sieves; then, $\mathbf{T}'$-sheaves are precisely $\mathbf{T}$-sheaves.*

Lemma 4.7. *Let $U \in \mathsf{C}$, $P \in \widehat{\mathsf{C}}$, and S be a sieve on U. Then, P is a sheaf with respect to S iff*

$$\widehat{\mathsf{C}}(\mathsf{C}(-,U),P) \longrightarrow \widehat{\mathsf{C}}(S,P)$$

is a bijection.

The proof is an exercise, but the key step is to show that $\widehat{\mathsf{C}}(S,P) \cong \varprojlim_{(V \to U) \in S} P(V)$.

Lemma 4.8. *With the same notation in the previous lemma, for all sieves $S' \subseteq \mathsf{C}(-,U)$ containing S, if P is a sheaf for S, then it's also a sheaf for S'.*

Definition 4.9. A coverage is *sifted* if all of its constituent coverages are sieves.

Lemma 4.10. *Let $\mathbf{T}$ be a sifted coverage on C. Then, for all $U \in \mathsf{C}$, all $f : U \to V$ in C, and all $S \in T_U$, all $\mathbf{T}$-sheaves are also sheaves with respect to f^*S.*

Now we arrive at the definition of a Grothendieck topology!

Definition 4.11. A sifted coverage τ is a *Grothendieck topology* if it satisfies the following axioms.

(1) Coverings are closed under pullback.
(2) For all $U \in \mathsf{C}$, the sieve associated to $\{\mathrm{id}\colon U \to U\}$ is in τ_U.
(3) For any $U \in \mathsf{C}$, $R \in \tau_U$, and $S \subset \mathsf{C}(-, U)$, and for all $f : V \to U$ in R with f^*S in τ_V, we have $S \in \tau_U$.

There's a bijective correspondence between Grothendieck topolgoies on C and subcategories $S \subset \widehat{\mathsf{C}}$ with left exact left adjoints.

Day 4. **June 26**

5. Morning

Suppose we start with a category C and a coverage τ. We'll think of the objects of C as spaces (often, they actually are spaces, or manifolds, or schemes, etc.). We want to build all of the objects that we can from C while preserving τ, and we can do this either using a subcategory of sheaves or a localization of presheaves. These approaches are equivalent (Theorem 2.1), which further implies the localization is really nice.

Then we saturated τ, adding all possible coverings which induce the same sheaves, along with some morphisms, leading to the notion of a Grothendieck topology. Different saturated localizations induce different Grothendieck topologies, and (Theorem 3.2) there's a bijection between subcategories of $\widehat{\mathsf{C}}$ whose inclusion admits a left exact left adjoint, and Grothendieck topologies on C.

Recall that a *site* is a category together with a Grothendieck topology.

Definition 5.1. A *Grothendieck topos* is a category equivalent to a category of sheaves on a site.

Toposes are used in several different fields (logic, algebraic geometry, category theory) and so different people may use words such as "elementary topos" or "Grothendieck topos" with different meanings.

Remark 5.2. A category is *presentable* if it's cocomplete and every object is the colimit of objects in a small category. Grothendieck toposes are presentable, and conversely an elementary topos is a Grothendieck topos iff it's presentable. In some sense, elementary toposes can be "too small" (not admitting all colimits) or "too big" (containing objects which aren't colimits over small categories), though the latter examples are weirder.

However, it is weird that we went from a site to sheaves on the site: in an attempt to do geometry, we went from one kind of thing to another kind of thing. But it is true that a category is a Grothendieck topos iff it's presentable and, given the *canonical topology* (i.e. the coverage given by jointly surjective collections of maps), every sheaf is representable (that is, the Yoneda map $\mathsf{C} \hookrightarrow \widetilde{\mathsf{C}}$ is an equivalence). ◂

Every epimorphism in a Grothendieck topos is effective, which makes sense: that's what we were trying to do.

You can think of this as a completion: you want to build spaces out of old spaces, and you've gotten all the ones you can. If you try to do it again, one of the above equivalences shows you get back what you started with.

The left exact left adjoint property makes a lot of basic proofs easy.

Proposition 5.3. *Colimits are preserved by pullbacks in a topos. That is, let C be a Grothendieck topos, $X \in \mathsf{C}$, and $\widetilde{\mathsf{C}}/X$ denote the slice category (objects with a map to X). Let $A\colon \mathsf{J} \to \widetilde{\mathsf{C}}/X$ be a functor. Given an $f_{ij}\colon A_i \to A_j$ in J, we have a commutative diagram*

$$\begin{tikzcd} & \varinjlim_{\mathsf{J}} A & \\ A_i \arrow[ur] \arrow[rr, "f_{ij}"] \arrow[dr] & & A_j \arrow[ul] \arrow[dl] \\ & X. & \end{tikzcd} \tag{5.4}$$

Given a map $g\colon Y \to X$, we can pull the objects in (5.4) back along g to obtain another diagram

$$\begin{tikzcd} & Y \times_X \varinjlim_{\mathsf{J}} A & \\ Y \times_X A_i \arrow[ur] \arrow[rr] \arrow[dr] & & Y \times_X A_j \arrow[ul] \arrow[dl] \\ & Y. & \end{tikzcd} \tag{5.5}$$

The proposition is that $Y \times_X \varinjlim_{\mathsf{J}} A$ is the colimit of the pullback of A, and the maps in the above diagram to it are its structure maps.

Proof sketch. First, one proves it for the "final topos" Set, where it's explicit. Using this, and the fact that colimits of presheaves are pointwise, the result follows for $\widehat{\mathsf{C}}$. Now, we can compute colimits and finite limits of $\widetilde{\mathsf{C}}$ in $\widehat{\mathsf{C}}$, as long as we sheafify afterwards: the adjoint to inclusion (which is sheafification) is a left adjoint, so it commutes with colimits, and is left exact, so it commutes with finite limits. ⊠

Why is Set the final topos?

Definition 5.6. A *geometric morphism* between two toposes C and D is a map C → D which has a left exact left adjoint.

Toposes, geometric morphisms, and natural transformations form a 2-category,[4] and the final object is Set as the topos of presheaves on a point. Therefore if you like geometry (you like geometry… right?), you can think of this as a point (the final object in things like Top, Man, Sch, etc.).

Remark 5.7. Here we can say something about the original motivation of all this stuff. We start with (C,τ), such as affine schemes and the Zariski topology if you like algebraic geometry, or Cart with the topology of open sets if you like differential geometry. We then want more spaces, because answers to questions we can pose with just (C,τ) fall outside of it, so we pick them out of $\widetilde{\mathsf{C}}$.

For example, for schemes we usually use locally ringed spaces instead of $\widetilde{\mathsf{Aff}}$; similarly, for smooth manifolds, you can formulate differential geometry for smooth manifolds starting with locally ringed spaces (this is done in a book by Görtz), rather than the category of presheaves; for topological manifolds one works in Top. ◁

Here's a schema for definitions: given a site (C,τ) and a class $\mathbb{P}$ of morphisms, a *geometric space* (w.r.t $\mathbb{P}$, unless $\mathbb{P}$ is clear from context) is a sheaf X on C, such that for all coverings $\{U_i \to X\}$ with U_i representable (i.e. in C) which are jointly epimorphic, $U_i \to X$ is in $\mathbb{P}$. This recovers examples we care about, including:

- For C = Cart and $\mathbb{P}$ open immersions, geometric spaces are manifolds. (After all, things covered by open subsets of $\mathbb{R}^n$ are exactly manifolds.)
- If $\mathsf{C} = \mathsf{AffSch}_R$, where R is a commutative ring, and $\mathbb{P}$ is open immersions, geometric spaces are schemes over R.
- If $\mathsf{C} = \mathsf{AffSch}_R$, but $\mathbb{P}$ is either étale or smooth maps, geometric spaces are the things called *algebraic spaces*.

6. Afternoon

Proposition 6.1. *The functor $F\colon \mathsf{Man}_n \to \widetilde{\mathsf{Cart}}_n$ sending*

$$X \longmapsto (U \longmapsto \mathsf{Man}_n(U,X))$$

is fully faithful.

We also haven't discussed open immersion yet; this is related to characterizing the image of F.

Definition 6.2. Let $\widetilde{\mathsf{C}}$ be a site, and let $f\colon X \to Y$ be a morphism in $\widetilde{\mathsf{C}}$. Then f is *representable* if for all $U \in \mathsf{C}$ and maps $g\colon U \to Y$, the pullback $X \times_Y U$ (which exists in $\widetilde{\mathsf{C}}$) is in C.

So the idea is that if you only care about representable objects, this pullback exists.

[4] This sounds intimidating, but really this is a 2-category for the same reason Cat is, i.e. there are objects, morphisms, and natural transformations; you don't need to engage with 2-category theory to parse this sentence.

Definition 6.3. Consider the site on Cart_n given by open immersions. Then an *open immersion* in $\widetilde{\mathsf{Cart}}_n$ is a map $X \to Y$ that is representable and such that for all $U \in \mathsf{Cart}_n$ and maps $f : U \to Y$, $X \times_Y U \to U$ is an open immersion (in the usual sense).

This is an instance of a common theme: to define something for sheaves, define it for objects of C and then check it when pulled back by representable morphisms.

Example 6.4. Let X and Y be manifolds, thought of as sheaves on Cart_n. Then, an open immersion is a map which, when restricted to any chart of $U \subset X$ whose image lies in a chart $V \subset Y$, is an open immersion. Therefore this is the same thing as being an open immersion in the familiar sense, i.e. with the topology on X and Y. ◂

In particular, a sheaf on Cart_n is a manifold iff it's a geometric space.

Let G be a discrete group acting on an $X \in \mathsf{Man}_n$, so that the quotient $X/G \in \widetilde{\mathsf{Cart}}_n$, as the coequalizer of $X \times G \rightrightarrows X$, where one map is the action map and the other is projection.

Proposition 6.5. *Let G and X be as above.*

(1) *If G acts freely on X, then $X \twoheadrightarrow X/G$ is a local homeomorphism in $\widetilde{\mathsf{Cart}}_n$.*
(2) *If G acts freely and totally discontinuously on X, then $X/G \in \mathsf{Man}_n$.*

The first case includes things we don't usually consider nice, e.g. $\mathbb{R}/\mathbb{Q}$. This is definitely not a manifold, but as a sheaf it's locally homeomorphic to $\mathbb{R}$, which is in some sense a principal $\mathbb{Q}$-bundle over it.

For an example of something really nice, $\mathbb{R} \times \mathbb{Z} \rightrightarrows \mathbb{R} \to S^1$ all lives in Man_n. This is the classifying space for principal $\mathbb{Z}$-bundles, in that a principal $\mathbb{Z}$-bundle is the same data as a map $X \to S^1$ (given by pulling back $\mathbb{R} \to S^1$). You can think of the total space as being partitioned into circles, and X is the set of such circles. This is the kind of thing we wanted to have with coverages, so it's definitely something important to have — and since things go badly when you try this in Top, we know Top is not a good ambient category for Man.

Some people also care about algebraic geometry.

Proposition 6.6. *Let R be a commutative ring. The map $F : \mathsf{Sch}_R \to \widetilde{\mathsf{Aff}}$ sending $X \mapsto \mathsf{Sch}_R(-, X)$ is fully faithful.*

We can define *open immersions* of schemes analgously as for manifolds, and they agree with the usual definition.

Proof. First, we have to check that the functor of points of X (i.e. $\mathsf{Sch}_R(-, X)$) is a sheaf. This is essentially because maps between schemes can be computed locally on the source. That is, given an affine open cover $\mathfrak{U}$ of an affine scheme Y and maps $f_U : U \to X$ for each $U \in \mathfrak{U}$, such that for all $U, V \in \mathfrak{U}$, $f_U|_{U\cap V} = f_V|_{U\cap V}$, we can glue them together to a map $f : Y \to X$. This follows from the definition of a map of schemes.

Now we need to argue this embedding F is fully faithful. Suppose we have maps $f, g : X \to Y$ of schemes such that $F(f) = F(g)$. Let $U \hookrightarrow X$ be an affine open subscheme; then $F(f)|_U = F(g)|_U$, so $f|_U = g|_U$, since we understand affine schemes. Then we can do this for every open, and glue. ⊠

Day 5. **June 27**

7. Morning

Let C be a category with a Grothendieck topology $\widetilde{\mathsf{C}}$. It turns out that the map $\mathsf{C} \to \widetilde{\mathsf{C}}$ is not always fully faithful! This is a bit of a rude surprise.

Definition 7.1. A Grothendieck topology is *subcanonical* if the map $\mathsf{C} \to \widetilde{\mathsf{C}}$ is fully faithful.

Equivalently, one asks that for all $X \in \mathsf{C}$, $\mathsf{C}(-, X)$ is a sheaf. In practice, Grothendieck toposes that you care about will probably be subcanonical.

Proposition 7.2. *Suppose C admits finite limits and finite coproducts, and that its Grothendieck topology is generated by finite coverings. Then the topology is subcanonical iff for all such coverings $\{U_i \to U\}_{i=1}^n$, each $\coprod_{i=1}^n U_i \to U$ is an effective epimorphism.*

This proposition is less useful than you might expect: we've been assuming C is small, and small categories generally don't have many (co)limits. For example, a small category with all colimits is a poset category! The proof is short.

Remark 7.3. Let κ be a cardinal; then, one can generalize Proposition 7.2 to the setting where C has all coproducts indexed by (at most) κ, and its Grothendieck topology is generated by coverings with cardinality (at most) κ. ◂

Proof sketch of Proposition 7.2. The main point of the proof is that $\mathfrak{U} := \{U' \to U\}$ is an effective epimorphism iff for all $X \in \mathsf{C}$, $\mathsf{C}(-,X)$ satisfies the sheaf condition with respect to $\mathfrak{U}$. That is, if

$$U' \times_U U' \rightrightarrows U' \longrightarrow U \tag{7.4}$$

is a coequalizer diagram, then as we've seen before

$$\mathsf{C}(U,X) \longrightarrow \mathsf{C}(U',X) \rightrightarrows \mathsf{C}(U' \times_U U', X) \tag{7.5}$$

is an equalizer diagram, and conversely. So we can let $U' = \coprod_{U \in \mathfrak{U}} U$; then, we're asking for (7.4) to be a coequalizer diagram, hence for (7.5) to be an equalizer diagram. But this is exactly the sheaf condition. ⊠

On the category of schemes (or affine schemes, or schemes over a commutative ring R, or affine schemes over $R\dots$), we're used to taking the Zariski topology (you might say "the Zariski site"). There are a bunch of other useful ones.

Definition 7.6. Consider the Grothendieck pre-topology on AffSch given by open coverings $\mathfrak{U} = \{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ such that

(1) $\mathfrak{U}$ is finite,
(2) the induced maps $A \to A_i$ is *faithfully flat*,[5] and
(3) the map $A \to \prod_i A_i$ is also faithfully flat.

The Grothendieck topology defined by saturating this pretopology is called the *fpqc topology*.[6]

Exercise 7.7. Show that condition (3) is redundant; this is a nice bit of commutative algebra.

Proposition 7.8. *The fpqc site is subcanonical.*

This is the finest topology people usually think about on schemes; then the smooth topology is coarser, the étale topology is even coarser (though it has the same sheaves as the smooth topology), and the Zariski topology is the coarsest yet.

Proving that fpqc is subcanonical means precisely that one has to show that the maps $\coprod \operatorname{Spec} A_i \to \operatorname{Spec} A$ are effective epimorphisms. But we can pass to rings, where one has to show that if $B := \prod A_i$, the diagram

$$A \longrightarrow B \rightrightarrows B \otimes_A B \tag{7.9}$$

is an equalizer. This is also fairly formal; one checks that it suffices to show this after applying $- \otimes_A B$:

$$B \longrightarrow B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B. \tag{7.10}$$

Then one produces a section of the first map (which is basically multiplication), and then it's only a few steps.

[5] This means that the map $A \to A_i$ makes A_i into a faithfully flat A-module. This means that A_i is *flat*, i.e. $- \otimes_A A_i$ is exact, and it's *faithful*, meaning if $B \not\cong 0$, then $A_i \otimes_A B \not\cong 0$.

[6] This is from French, and stands for "faithfully flat and quasicompact."