Symmetric Gapped States and Symmetry-Enforced Gaplessness in 3 Dimensions

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We establish a comprehensive framework for characterizing the infrared (IR) phases of a fermionic quantum theory in three spatial dimensions, based on its *quantum anomalies* associated with a finite symmetry. We uncover a fundamental dichotomy among these anomalies: the first class of anomalies can *always* be realized by symmetric gapped states, while the second class can *never* be realized by gapped states without breaking the given symmetry, establishing the phenomenon of symmetry-enforced gaplessness in these settings. Moreover, using the construction of symmetry extension, we construct the candidate gapped states that theories with the first class of anomalies can flow to in the IR. As an application, we provide concrete predictions of the candidate IR phases of (3+1)-dimensional gauge theories based on our results. Our results also suggest that systems with certain discrete chiral anomalies cannot be gapped out by adding arbitrary bosonic degrees of freedom.

Introduction. Symmetries [1] play a central role in modern physics [2], most notably within the Landau paradigm of phases and phase transitions. Nevertheless, it is recognized that symmetries alone are insufficient to characterize all possible quantum phases. Thus, considerable effort has been devoted to expanding the landscape beyond the traditional framework [3–11]. In this broader perspective, *quantum anomalies* [12], also referred to as *'t Hooft anomalies* [13], have emerged as a powerful diagnostic tool.

Quantum anomalies are commonly defined as obstructions to gauging a global symmetry [13, 14]. As such, their presence immediately implies that the corresponding quantum state cannot be completely trivial. Assuming that the symmetry remains unbroken in the infrared (IR), this basic constraint suggests that either the IR theory is gapless, or it realizes a nontrivial topological order.

It is natural to ask whether anomalies alone can provide further insight into the physical system. In particular, can anomalies be strong enough to rule out *all* symmetric gapped states, including those with intrinsic topological order? [15] This phenomenon is termed *symmetry-enforced gaplessness* [16] in Ref. [17], and has proven to be crucial in the understanding of deconfined quantum critical points [18] and extensively analyzed in e.g. [19–28]. Conversely, given a prescribed anomaly, can one systematically construct candidate gapped states that realize the given anomaly? These candidate states may naturally serve as the candidate IR phases for a (3+1)-dimensional gauge theory or a 3-dimensional lattice system. Thus, answering these questions may give exact nonperturbative predictions for these strongly-coupled systems.

For bosonic systems, an affirmative answer to the latter question is provided by the symmetry extension construction by Wang, Wen, and Witten [29] and many others [30–33], when the quantum anomaly takes value in ordinary cohomology [32]. In that framework, one first enlarges the symmetry group such that the anomaly becomes trivial upon extending to the larger group, and subsequently gauges part of the extended symmetry to obtain a gauge theory that faithfully realizes the original anomalous symmetry. This construction gives a remarkably general method for engineering symmetric gapped states that realize a given anomaly in bosonic systems.

In this work, we aim to generalize this framework to fermionic systems with finite symmetries. While attempts have been made to extend lattice [34–36] and path integral constructions to fermionic systems, both approaches present significant technical challenges. Nevertheless, at our disposal is a category-theory-based framework that has recently been developed in [37–43]. In the companion paper [43], we provide a systematic construction of fermionic topological orders for a given anomaly. The formal procedure is similar to the bosonic case, but one notable exception is that one replaces regular cohomology with *supercohomology* [44]. Despite the extensive use of mathematics in the companion paper, we present our results here with minimal mathematical input and explore their potential physical applications in greater detail. Moreover, we identify the precise condition under which such constructions are obstructed. This leads to a sharp refinement of the predictions from an anomaly in the form of symmetry-enforced gaplessness.

Our results have strong implications for interacting (3+1)d systems, especially systems with chiral fermions, where 't Hooft anomalies are ubiquitous. We show that these anomalies impose robust, nonperturbative constraints on the possible infrared phases. These constraints have direct implications for lattice realizations and Weyl semimetals, as well as for physics beyond the Standard Model.

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In this paper, we provide an intuitive exposition of the main results in Ref. [43] and explore their physical implications. We explain how the framework of symmetry extension can be generalized to the fermionic setting. We sketch why this construction is always possible for one class of anomalies, whereas for the other, it implies that certain fermionic topological orders are forbidden. We then discuss the connection between our findings and longstanding problems, such as the IR fate of systems with chiral fermions.

Supercohomology and beyond-supercohomology anomaly. We begin with a lightning review of supercohomology and explain why it replaces ordinary cohomology, which classifies bosonic anomalies, in fermionic settings. Quantum anomalies are fundamentally linked to projective representations [45]. Given a symmetry group G, a projective representation of G is defined to be a function $\rho\colon G \to \mathrm{U}(n)$ for some positive integer n, which is a homomorphism up to a "correction term" $\omega(g,h) \in \mathrm{U}(1)$, in that for any $g,h \in G$ we have,

$$\rho(g) \cdot \rho(h) = \omega(g,h)\rho(gh). \tag{1}$$

Physically, we allow the presence of such nontrivial $\omega(g,h)$ taking values in $\mathrm{U}(1)$ because a vector in the physical Hilbert space represents the same physical state if it is multiplied by a $\mathrm{U}(1)$ factor. This explains the origin of the $\mathrm{U}(1)$ coefficient in the classification of in-cohomology anomalies in bosonic systems. In particular, anomalies for G-symmetries in (0+1)d quantum mechanical systems are exactly in one-to-one correspondence with projective representations of G, and are classified by $\mathrm{H}^2(G;\mathrm{U}(1))$ [46]. In n spatial dimensions with $n \geq 1$, various G-defects may carry nontrivial projective representations of G, and the associated in-cohomology anomalies are classified by $\mathrm{H}^{n+2}(G;\mathrm{U}(1))$ [47, 48].

In fermionic systems, because of the presence of the extra fermion parity symmetry and local fermion excitations, extra actions are possible. In the context of 3-dimensional fermionic topological order, on top of the multiplication by $\mathrm{U}(1)$ on the physical Hilbert space, we also have local fermion excitations and their "condensations" which physically correspond to the Kitaev chain. In the categorical language [38], these "symmetry actions" correspond to the Picard 2-groupoid $\mathbf{2sVect}^\times$, and together form a nontrivial 3-group [49], with the 0-form part $\mathrm{U}(1)$, 1-form part $\mathbb{Z}_2$, and 2-form part $\mathbb{Z}_2$. Therefore, roughly speaking, we should replace $\omega(g,h)$ with actions in the 3-group [50, 51], as explained in more detail in Appendix I [52]. The corresponding *generalized cohomology* theory after taking into account the extra action is precisely *supercohomology*, which we denote as $\mathrm{SH}^{n+2}(G)$ for n-dimensional systems. Thus, we should anticipate that supercohomology in the fermionic setting plays a role analogous to ordinary cohomology in the bosonic setting.

However, in both bosonic and fermionic contexts, in-cohomology or supercohomology anomalies do not coincide with the full 't Hooft anomalies. We review the classification of 't Hooft anomalies in fermionic settings in the end matter. We call these additional anomalies not captured by cohomology nor supercohomology *beyond-cohomology* or *beyond-supercohomology* anomalies, respectively. We will discuss the physical meaning of beyond-supercohomology anomalies in the *Symmetry-Enforced Gaplessness* paragraph.

Construction of fermionic gauge theories. Nontrivial gapped states usually take the form of finite gauge theories [53–56], and in 3 spatial dimensions these are the only possibilities. This motivates the Wang–Wen–Witten symmetry extension construction, which proceeds as follows in bosonic settings. We start with a symmetry group G, and consider the symmetry extension of G according to a short exact sequence

$$1 \to K \to H \xrightarrow{p} G \to 1, \tag{2}$$

such that for a given element $\alpha \in \mathrm{H}^{n+2}(G;\mathrm{U}(1))$, the pullback along p trivializes α in the sense that $p^*\alpha = 0 \in \mathrm{H}^{n+2}(H;\mathrm{U}(1))$. Thus, we can gauge the subgroup K in an H-symmetric state to obtain a K-gauge theory. The anomalous symmetry action of G on the obtained K gauge theory, a gapped topological order, then gives the prescribed anomaly α.

The symmetry extension construction in the bosonic setting is backed up by path integral formulations and lattice constructions [29]. However, generalizations of these constructions to the fermionic setting are much harder. Nevertheless, in Ref. [43], we interpret various pieces of the symmetry extension construction in the context of fusion 2-categories. Thereafter, the generalization to the fermionic setting is straightforward. [57] The central difference is precisely the difference of the "coefficient" being a $\mathrm{U}(1)$ action in the bosonic setting versus a 3-group action in the fermionic setting, as explained in the previous paragraph.

Formally, given a finite symmetry G [58], we just need to replace ordinary cohomology with supercohomology. Then for a specified element $\alpha \in \mathrm{SH}^{n+2}(G)$, if the pullback along p in Eq. (2) trivializes α in the sense that $p^*\alpha = 0 \in \mathrm{SH}^{n+2}(H)$, we can gauge the subgroup K in an H-symmetric state to obtain a K-gauge theory with an anomalous G-action corresponding to α.

Ref. [32] showed that, given any in-cohomology anomaly $\alpha \in \mathrm{H}^{n+2}(G;\mathrm{U}(1))$ of bosonic systems with *finite* G, it is always possible to obtain such H that trivializes α. In Ref. [43], we prove this is also true for 3-dimensional supercohomology anomalies, leading to the following result.

Theorem 1. *In 3 spatial dimensions, every quantum anomaly of a finite fermionic symmetry that is captured by supercohomology can be realized by a fermionic topological order using the symmetry extension construction.*

The proof proceeds by unfolding the supercohomology anomaly into three different layers, each of which is captured by ordinary cohomology. Then we just need to trivialize one layer at a time. We give a sketch of the proof in Appendix II [52], leaving the full mathematical details for Ref. [43].

Candidate states in the IR. Building on this general framework, we now examine specific symmetries and anomalies relevant to physical contexts, identifying the fermionic topological orders that realize each anomaly. We focus on cyclic symmetries that appear frequently in physics. The explicit computations for the anomaly classification are given in Ref. [43]. The procedure is straightforward:

1. For each symmetry group G, calculate the classification of the corresponding supercohomology anomaly.

2. For each generator of the supercohomology anomaly, calculate the minimal group H that trivializes the anomaly; then the gauge theory that realizes the given anomaly is simply the finite K-gauge theory with the gauge group K being the subgroup of H from Eq. (2).

The results are summarized in Table I. We will comment on their potential applications of these results in physics in the *Applications* paragraph.

G_f	Supercohomology	K-gauge theory
$\mathbb{Z}_{p^k} \times \mathbb{Z}_2^F$	$(\mathbb{Z}_{p^k}, \mathrm{DW})$	$\mathbb{Z}_p$
$\mathbb{Z}_{2^k} \times \mathbb{Z}_2^F$	$(\mathbb{Z}_{2^{k-1}}, \mathrm{DW})$	$\mathbb{Z}_{2^m}, m = \lceil \frac{k-1}{2} \rceil$
$\mathbb{Z}_4^F$	$(\mathbb{Z}_8, \mathrm{Maj})$	$\mathbb{Z}_4$
$\mathbb{Z}_{2^{k+1}}^F, k \geq 2$	$(\mathbb{Z}_{2^{k+1}}, \mathrm{GW})$	$\mathbb{Z}_{2^m}, m = \lceil \frac{k}{2} \rceil$
	$\oplus (\mathbb{Z}_2, \mathrm{Maj})$	$\mathbb{Z}_4$
$\mathbb{Z}_{2^{k+1}}^F \times \mathbb{Z}_2^T, k \geq 2$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_2, \mathrm{DW})$	$\mathbb{Z}_2$
	$\oplus (\mathbb{Z}_4, \mathrm{GW})$	$\mathbb{Z}_4 \times \mathbb{Z}_2$

TABLE I. Fermionic symmetries, their supercohomology anomalies in 3 dimensions, and the minimal gauge theory that realizes them. The second column classifies the anomalies by direct summands and indicates their generator layer: Majorana (Maj), Gu–Wen (GW), or Dijkgraaf–Witten (DW) (see the end matter). Green generators indicate a zero image in the full classification of 't Hooft anomalies. The third column gives the candidate states in terms of K-gauge theories with a finite gauge group K. In the first line, p is an odd prime.

Symmetry Enforced Gaplessness. Supercohomology does not capture the full 't Hooft anomalies in fermionic systems, analogous to the limitations of ordinary cohomology in bosonic systems. Thus, it is interesting to ask whether symmetric gapped states can capture these additional anomalies. The fusion 2-category picture already suggests that the answer is negative.

Theorem 2. *In 3 spatial dimensions, every quantum anomaly of a finite fermionic symmetry that cannot be captured by supercohomology cannot be realized by any fermionic gapped state without breaking the symmetry.*

We now give a heuristic argument of the physical picture behind Theorem 2. We start with an illustration of the physical picture of these beyond-supercohomology anomalies. Consider the $\mathrm{U}(1)^F$ symmetry of a single (3+1)-dimensional left-handed Weyl fermion χ [14],

$$\mathcal{L} = i\bar{\chi}\bar{\partial}\!\!\!/\chi, \tag{3}$$

where $\mathrm{U}(1)^F$ symmetry $e^{i\theta}$ acts on χ according to $\chi \to e^{i\theta}\chi$. The corresponding anomaly is classified by $\mathbb{Z} \oplus \mathbb{Z}$. In the language of standard quantum field theory [14], the first $\mathbb{Z}$ factor corresponds to the *pure* anomaly of global symmetries and is captured by supercohomology. In contrast, the second $\mathbb{Z}$ factor represents the mixed gravitational anomaly and lies beyond supercohomology. A single (3+1)-dimensional Weyl fermion carries a nontrivial value in both factors. While this can be seen via the standard triangle-diagram calculation, we demonstrate it here by analyzing the nontrivial U(1) vortex. Upon dimensional reduction, the U(1) vortex core hosts a (1+1)-dimensional chiral fermion. This U(1) vortex core is nontrivial: it exhibits both U(1) and gravitational anomalies. Consequently, the parent (3+1)-dimensional Weyl fermion must carry corresponding nontrivial anomalies in both factors. [59]

For anomalies in (3+1)d fermionic systems, the extra piece of data that lies beyond supercohomology is captured by a homomorphism $\rho\colon G \to \mathrm{U}(1)$ [60] (see the end matter). To understand the physical meaning of the homomorphism ρ, we also construct a G-vortex such that G symmetry acts on the extra 2 spatial dimension according to the homomorphism ρ. Then the beyond-supercohomology anomaly is captured by the fact that this G-vortex carries a nontrivial gravitational anomaly. However, the gravitational anomaly in (1+1)d suggests that the core of the G-vortex is always gapless. This gives a heuristic argument why, in the presence of beyond-supercohomology anomalies, there is always a gapless mode.

We provide a mathematical proof based on the arguments of Ref. [20] in Appendix II [52], with even greater details in Ref. [43]. This gives a comprehensive argument of Theorem 2 from both physical and mathematical perspectives.

Applications. Our results have manifest consequences for constraining dynamics of quantum theories in 3 spatial dimensions. As our first application, consider gauge theories with N_f flavors of Dirac fermions transforming in some representation R of the gauge group $\mathcal{G}$, with the Lagrangian taking the standard form [14]

$$\mathcal{L} = \sum_{i=1}^{N_f} i\bar{\psi}_i \not{D}_R \psi_i, \quad \not{D}_R = \gamma^\mu(\partial_\mu + iA_\mu^a t_R^a), \qquad (4)$$

where ψ_i are charged Dirac fermions, A_μ^a are $\mathcal{G}$ gauge fields and t_R^a are the generators of the Lie algebra of $\mathcal{G}$ in the representation R. If the fermion parity $(-1)^F$ is not part of $\mathcal{G}$, local fermions are present, and our discussions are applicable. Importantly, we also allow the possibility of adding symmetry-preserving interactions, and ask whether it is possible to open up a gap in the IR in the *entire* phase diagram under consideration. Moreover, we ask what the candidate topological order realizations are if we do open up the gap. This is reminiscent of "symmetric mass generation" [4, 61] when no 't Hooft anomaly is present, and is sometimes referred to as "topological mass generation". We provide our answers solely from identifying the global symmetries and their corresponding 't Hooft anomalies. The answers serve as a direct consequence of Theorem 1, 2, and Table I.

Classically, there is a chiral $\mathrm{U}(1)^F$ symmetry rotating the fermions, but the ABJ anomaly reduces this symmetry to

$$\mathrm{U}(1)^F \to \mathbb{Z}^F_{4N_f \cdot T(R)} \qquad (5)$$

where $T(R)$ is the Dynkin index of the representation. The classification of their 't Hooft anomalies and their corresponding supercohomology groups are listed in Table III. The exact value of the 't Hooft anomaly associated with the theory in Eq. (4) should be $2 \cdot N_f \cdot \dim(R)$ times the generator of the $p + ip$ layer, where $\dim(R)$ is the dimension of the representation R, plus an extra contribution from the Dijkgraaf-Witten layer [62].

We collect the results for gauge groups $\mathcal{G}$ and their representations R that we consider in Table II, including their global symmetries, associated 't Hooft anomalies, and candidate IR phases. In two of these examples, the associated 't Hooft anomalies fall into the set of beyond-supercohomology anomalies. Consequently, the IR must be either gapless or spontaneously break the chiral symmetry, independent of any symmetry-preserving interactions one might add. In the other two examples, the IR is gappable, and the candidate IR phases can be directly read off from Table I. In particular, for the $\mathrm{SO}(2N+1)$ gauge theory with charged fermions in the vector representation, we predict that the IR can never be trivially gapped and cannot even be the simplest $\mathbb{Z}_2$ gauge theory.

These results have direct consequences for physics beyond the Standard Model (BSM) [24, 63–66], where gauged chiral fermions play a significant role. It has been proposed in Ref. [66] that the apparent incompleteness of the Standard Model—most notably the quantum anomalies arising from the absence of observed right-handed neutrinos—can be theoretically resolved by augmenting the SM with a non-perturbative topological sector. The specific topological orders investigated in this work provide a concrete candidate for such a sector, thereby supporting the consistency of a fully gauged, anomaly-free formulation of the Standard Model without requiring conventional sterile neutrinos. These possibilities are analyzed further in Wang et al's more recent work [64, 67].

It is also interesting to analyze Weyl fermions with the Lagrangian in Eq. (3). To introduce nontrivial dynamics, we can couple an extra bosonic field with the Weyl fermion χ, and thus go beyond chiral symmetries. As a minimal example, consider a single left-handed chiral fermion χ coupled to a bosonic field ϕ which transforms as a vector under some $\mathbb{Z}_4$ action. The combined system exhibits a $G_f = \mathbb{Z}_4 \times \mathbb{Z}_2^F$ symmetry. The simplest nontrivial symmetric coupling that we can introduce is

$$\mathcal{L} = g\phi^2(\chi^T i\sigma_2 \chi + h.c.) \qquad (6)$$

The classification of the 't Hooft anomaly for G_f is $\mathbb{Z}_4$. Moreover, the theory has a nontrivial 't Hooft anomaly, taking the value $1 \in \mathbb{Z}_4$, i.e., a beyond-supercohomology anomaly as shown in Table III. From Theorem 2, we immediately conclude that Eq. (6) or any symmetry-preserving interaction cannot gap out the theory without breaking G_f. It is worth noting that the $\mathbb{Z}_4 \times \mathbb{Z}_2^F$ symmetry cannot be embedded into a chiral $\mathrm{U}(1)^F$ symmetry, and is hence beyond the discrete chiral symmetry discussed in the previous paragraph. More generally, a direct consequence of Ref. [62] is that beyond-supercohomology anomalies always persist after adding arbitrary bosonic degrees of freedom, which may completely change the symmetry group of the whole theory. Theorem 2 then immediately suggests that a discrete chiral anomaly that lies beyond supercohomology cannot be gapped out even after adding arbitrary bosonic degrees of freedom.

In condensed matter systems, Weyl fermions emerge as low-energy quasiparticles near discrete band-crossing points in *Weyl semimetals* [68, 69]. Crucially, lattice symmetries may act nontrivially on these nodes, permuting the Weyl cones and generating nontrivial beyond-supercohomology anomalies. A direct consequence of our results is a strict constraint on the phase diagram: if beyond-supercohomology anomalies are present, the system cannot be fully gapped without explicitly or spontaneously breaking the underlying lattice symmetries. Future numerical studies on lattice models would be valu-

Gauge group $\mathcal{G}$	Representation R	Symmetry	't Hooft anomaly	Gappable	Candidate Realization
$\mathrm{SU}(2N+1)$	**fund**	$\mathbb{Z}_2^F$	trivial	True	Trivial gapped state
$\mathrm{SO}(2N+1)$	**vec**	$\mathbb{Z}_4^F$	$2(2N+1) \in \mathbb{Z}_{16}$	True	$\mathbb{Z}_4$ gauge theory
$\mathrm{SU}(2)$	**3**	$\mathbb{Z}_8^F$	$(22,0) \in \mathbb{Z}_{32} \oplus \mathbb{Z}_2$	False	Gapless States
F_4	**fund**	$\mathbb{Z}_{12}^F$	$4 \in \mathbb{Z}_{144}$	False	Gapless States

TABLE II. Examples of gauge theories with gauge groups $\mathcal{G}$, and $N_f = 1$ Dirac fermion in representations R. The column titled "Symmetry" shows the global symmetries of these theories. The fourth column details the full 't Hooft anomaly classification for these symmetries, alongside the exact anomaly value for each theory. The last two columns indicate whether the anomaly can be saturated by a gapped or gapless theory, and what is the candidate IR realization from Table I.

able to corroborate these theoretical constraints.

Summary. In this Letter, we established a systematic framework for constructing (3+1)d fermionic topological orders that saturate prescribed global anomalies. By generalizing the symmetry-extension procedure to the fermionic setting, we identify a fundamental dichotomy: anomalies residing within supercohomology can be fully saturated by gapped topological orders, while those that lie beyond-supercohomology necessitate symmetry-enforced gaplessness. This result provides a rigorous criterion for determining the IR fate of strongly coupled UV theories based solely on their symmetry properties.

Several future directions are of immediate interest. First, we plan to extend our framework to continuous and time-reversal symmetries; in particular, for continuous symmetries, the criterion for symmetry-enforced gaplessness is expected to be notably more intricate and interesting [17, 25, 27]. Second, we will investigate the field-theoretic mechanisms by which parameter tuning drives the theory from candidate topological orders to chiral symmetry-breaking states [61, 70]. Moreover, our results provide a pathway for constructing lattice models for chiral theories with anomalous discrete or continuous symmetries [71–73]. We anticipate that the anomaly constrains how these symmetries act on the lattice in a specific non-on-site manner [71]. Finally, we wish to examine the relevance of our results to proposals for beyond-Standard-Model physics [66] in greater detail.

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compatibility with gauge invariance at the quantum level. In contrast, quantum anomalies under our consideration are always associated with *global symmetries*. These quantum anomalies should be thought of as an intrinsic property of the theory and do not suggest that the theory is pathological. Moreover, while quantum anomalies can naturally be canceled by realizing the theory on the boundary of a symmetry-protected topological (SPT) state, the system under study does not necessarily require such a boundary realization, as long as the symmetries responsible for the anomalies are either emergent or non-on-site.

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End Matter

In the end matter, we clarify the mathematical formulation of fermionic symmetries and the classification of their associated 't Hooft anomalies.

A fermionic system can be characterized by its symmetry action, which forms a group G_f. Beyond the group structure of G_f, two additional pieces of data are important: (1) a map $\rho\colon G_f \to \mathbb{Z}_2$ such that the symmetry element is antiunitary or unitary if the image under ρ is 1 or 0, respectively, and (2) a central $\mathbb{Z}_2$ subgroup $\langle(-1)^F\rangle \subset G_f$ in the kernel of ρ generated by fermionic parity. This motivates describing the fermionic symmetry using the following three pieces of data:

- a (bosonic) symmetry group $G \coloneqq G_f/\langle(-1)^F\rangle$ acting on the bosonic degrees of freedom;
- a class $s \in \mathrm{H}^1(G;\mathbb{Z}_2)$, corresponding to ρ;
- a class $\omega \in \mathrm{H}^2(G;\mathbb{Z}_2)$, classifying the extension of G by the $\mathbb{Z}_2$ subgroup $\langle(-1)^F\rangle$ to get G_f.

In the main text, sometimes we use the bosonic symmetry group G to refer to a fermionic symmetry, with its associated s and ω twists implicit. In particular, most symmetries we consider are unitary symmetries, and hence the corresponding values of s are trivial. When referring to fermionic symmetry groups, we will use superscript F to indicate that part of the symmetry is generated by fermion parity $(-1)^F$.

According to the mechanism of anomaly inflow, the full quantum anomaly associated with G_f or G is classified by fermionic symmetry-protected topological (SPT) phases in one dimension higher. The construction and classification of these SPT phases is given by the decoration of invertible states of lower dimensions [60]. In physically relevant dimensions, there are four layers of data:

- the $p+ip$ layer originating from decorating the $p+ip$ topological superconductor,
- the Majorana layer originating from decorating the Kitaev chain,
- the Gu–Wen layer originating from decorating complex fermions, and
- the Dijkgraaf–Witten layer originating from decorating bosonic SPTs protected by G-symmetry.

The mathematical details of these data can be found in Refs. [60, 78, 79]. In n spatial dimensions, these different layers together form a group that we denote as $\mho^{n+1}(G)$. In particular, the $p+ip$ layer is given by an element in $\mathrm{H}^{n-2}(G;\mathbb{Z})$. For quantum anomalies in 3 spatial dimensions, the relevant n equals 4. Interestingly, $\mathrm{H}^2(G;\mathbb{Z}) \cong \mathrm{H}^1(G;\mathrm{U}(1))$, and hence elements in $\mathrm{H}^2(G;\mathbb{Z})$ are canonically associated to a homomorphism $\rho\colon G \to \mathrm{U}(1)$. The remaining three layers constitute the data of supercohomology, denoted as $\mathrm{SH}^{n+1}(G)$.

Moreover, there are natural maps between supercohomology $\mathrm{SH}^{n+1}(G)$, the classification of fermionic SPTs $\mho^{n+1}(G)$, and the data of the $p+ip$ layer $\mathrm{H}^{n-2}(G;\mathbb{Z})$,

$$\mathrm{SH}^{n+1}(G) \xrightarrow{i} \mho^{n+1}(G) \xrightarrow{p} \mathrm{H}^{n-2}(G;\mathbb{Z}), \tag{7}$$

which is exact in the middle entry, i.e., 't Hooft anomalies are either given by supercohomology anomalies, i.e., the image of i, or beyond-supercohomology anomalies, whose image under p is nontrivial.

As a concrete example, consider $G_f = \mathbb{Z}_4^F$. In 3 spatial dimensions, the classification of the full 't Hooft anomaly is $\mathbb{Z}_{16}$. The classification of the supercohomology anomaly is $\mathbb{Z}_8$, which is just the even elements of $\mathbb{Z}_{16}$. In contrast, the odd elements of $\mathbb{Z}_{16}$ exactly constitute the set of beyond-supercohomology anomalies. We can embed this symmetry into the $\mathrm{U}(1)^F$ symmetry and think of it as part of a continuous chiral symmetry. This symmetry is discussed in Refs. [20, 24, 43, 62, 80, 81].

As another example, consider $G_f = \mathbb{Z}_4 \times \mathbb{Z}_2^F$. In 3 spatial dimensions, the classification of the full 't Hooft anomaly is $\mathbb{Z}_4$. The classification of the supercohomology anomaly is $\mathbb{Z}_2$, which are again just even elements of $\mathbb{Z}_4$, and the odd elements of $\mathbb{Z}_4$ constitute the set of beyond-supercohomology anomalies. We discuss a field theory that realizes this anomaly in the main text. This symmetry is also discussed in Refs. [62, 80, 81].

For the reader's convenience, for all the symmetries we consider in Table I, we summarize the classification of supercohomology and full 't Hooft anomalies in Table III. Their calculation can be found in Refs. [43, 60, 62].

G_f	Supercohomology	Full 't Hooft anomaly
$\mathbb{Z}_{3^k} \times \mathbb{Z}_2^F$	$(\mathbb{Z}_{3^k}, \mathrm{DW})$	$\mathbb{Z}_{3^{k-1}} \oplus \mathbb{Z}_{3^{k+1}}$
$\mathbb{Z}_{p^k} \times \mathbb{Z}_2^F, p \geq 5$	$(\mathbb{Z}_{p^k}, \mathrm{DW})$	$\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$
$\mathbb{Z}_2 \times \mathbb{Z}_2^F$	trivial	trivial
$\mathbb{Z}_{2^k} \times \mathbb{Z}_2^F, k \geq 2$	$(\mathbb{Z}_{2^{k-1}}, \mathrm{DW})$	$\mathbb{Z}_{2^{k-2}} \oplus \mathbb{Z}_{2^k}$
$\mathbb{Z}_4^F$	$(\mathbb{Z}_8, \mathrm{Maj})$	$\mathbb{Z}_{16}$
$\mathbb{Z}_{2^{k+1}}^F, k \geq 2$	$(\mathbb{Z}_{2^{k+1}}, \mathrm{GW}) \oplus (\mathbb{Z}_2, \mathrm{Maj})$	$\mathbb{Z}_{2^{k+3}} \oplus \mathbb{Z}_{2^{k-1}}$
$\mathbb{Z}_{2^{k+1}}^F \times \mathbb{Z}_2^T, k \geq 2$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_2, \mathrm{DW}) \oplus (\mathbb{Z}_4, \mathrm{GW})$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$

TABLE III. Fermionic symmetries we consider and the classification of their supercohomology anomalies and 't Hooft anomalies in 3 spatial dimensions. For supercohomology, we also provide the layer in which the generator for that group resides. The green generator indicates that its image in the full classification of 't Hooft anomalies under Eq. (7) is zero.