arXiv:2611.24725v1 [math.AT] 18 Nov 2026

AN ATIYAH–HIRZEBRUCH TO ADAMS MAP AND THE HASTENED ADAMS SPECTRAL SEQUENCE FOR SUPERCOHOMOLOGY

ARUN DEBRAY, WEICHENG YE, AND MATTHEW YU

Abstract. We construct a map of filtered spectra inducing a map from the Atiyah–Hirzebruch spectral sequence (AHSS) to the Baker–Lazarev Adams spectral sequence of a ko-module M. Using this and a theorem of Bruner–Rognes, we construct a hastened Adams spectral sequence (HASS) for the Postnikov truncations of ko and the connective Adams summand. Then, motivated by questions in quantum field theory, topological phases of matter, and tensor category theory, we apply the HASS to *supercohomology*, the cohomology theory $I_{\mathbb{C}^\times}(\tau_{\leq 2}ko)$, and observe that in examples of interest it is easier to calculate with than either the AHSS or the usual Adams spectral sequence. We include a User's Guide to the HASS and several worked examples, as well as hastened Adams charts for many commonly encountered $\mathcal{A}(1)$-modules.

Contents

0. Introduction 1
1. Compatibility of the Atiyah–Hirzebruch and Adams filtrations 7
2. The hastened Adams spectral sequence 16
3. User's guide 27
4. Further directions 47
Appendix A. Two definitions of twisted supercohomology are equivalent 50
Appendix B. Ext and $\mathcal{Q}$ computations 61
References 68

0. Introduction

As our modern understanding of quantum phases and quantum anomalies has deepened, physicists and mathematicians have been led to explore stable homotopy calculations involving more exotic generalized cohomology theories [MZB+25, GJF19, Déc24b, JF25, DYY25, DHJF+24]. *Supercohomology*[1] [WG18, KT17, WG20] is a prominent example: it has been applied to understanding modular extensions [JF25] and quantum anomalies [DYY25, DY26, WW25, WWY26] in fermionic topological orders, mathematically modeled using tensor category theory.

For generalized cohomology theories (GCTs) like supercohomology, there are two standard tools available for making computations:

Date: November 19, 2026.
2020 *Mathematics Subject Classification.* 55T15, 55N20.

[1]Unfortunately, "supercohomology" is commonly used to refer to two closely related, but distinct, things: *restricted supercohomology*, as constructed by Freed [Fre08] and Gu–Wen [GW14], and *extended supercohomology*, as constructed by Wang–Gu [WG18, WG20] and Kapustin–Thorngren [KT17]. In this paper, "supercohomology" and *SH* always refer to extended supercohomology; when we briefly discuss restricted supercohomology in §4.1, we will use the notation *rSH*.

- the *Atiyah–Hirzebruch spectral sequence* (AHSS), which approximates $SH^*(X)$ in terms of $H^*(X; SH^*(\mathrm{pt}))$.
- The *Adams spectral sequence*, which uses some extra structure on $H^*(X; \mathbb{Z}/2)$ to compute $SH^*(X)$ (up to 2-completion).

In the mathematical physics literature, these tools are both routinely applied to a closely related GCT, denoted $\mathcal{U}^*_{\mathrm{Spin}}$ and called the *Anderson dual of spin bordism*; both tools are relatively easy to use and quite powerful. See, for example, [Cam17, BC18]. The Adams computations make use of a change-of-rings theorem, substantially simplifying the homological algebra calculations. For supercohomology, however, *neither* of these tools is easy to compute with. Specifically, the SH^* Adams spectral sequence generically has more differentials that mix with the "native" ko-Adams differentials in subtle ways, even after applying the change-of-rings simplification to the Adams spectral sequence.[2,3]

In this paper, we introduce another technique, called the *hastened Adams spectral sequence* (HASS), for supercohomology and some related theories. We show in examples that the HASS is tractable in examples of interest in physics and tensor category theory, and generally easier to use than the AHSS or the Adams spectral sequence.

Heuristically, the HASS is constructed by quotienting the ko-homology Adams spectral sequence by the corresponding spectral sequence for a Postnikov cover of ko, in such a way that taking the quotient eliminates these new, difficult differentials. It is not automatic that such a quotient exists or is well-behaved, but Bruner–Rognes [BR21] give a sufficient criterion. We prove their criterion in our setting by constructing a map from the Atiyah–Hirzebruch spectral sequence to the Adams spectral sequence for a ko-module. On the AHSS side, the analogue of Bruner–Rognes' criterion is vacuously true, and using our map we carry that over to the Adams side. Though our main goal in this project was to construct and use the HASS, we believe this AHSS-to-Adams map could be of independent interest.

Constructing a map between spectral sequences is a lot like constructing a natural transformation between GCTs: one must show compatibility with a large amount of algebraic data. For this reason, when constructing maps between GCTs, researchers often instead construct maps between their representing spectra: the category of spectra provides a homotopy theory for GCTs, allowing tools first developed in the homotopy theory of topological spaces to be applied to GCTs. This simplifies proofs and constructions in most cases. By analogy, we would like to construct our AHSS-to-Adams map in a well-behaved category describing some sort of homotopy theory of spectral sequences.

The category $\mathcal{F}il$ of *filtered spectra* fills this need.[4] The perspective to use $\mathcal{F}il$ to prove theorems about spectral sequences has been around in the homotopy theory, if somewhat implicitly, for a while, but recently has become quite prominent: see for example [BMS19, Hed20, Ari21, GIKR22, HRW22, Pst23a, Pst23b, Ait24, Ant24, AR24, BJM24, CD26, CDvN24, Ant25a, Ant25b, AP25a, AP25b, Car25, CDvN25, Cre25, HM25, QZ25, vN25, BHS26, Emm26, Pie26, PP26].

We construct the HASS in two steps. First, we construct a map Φ^{ko} from the filtered spectrum representing the AHSS of a ko-module M to the filtered spectrum representing Baker–Lazarev's ko-module Adams spectral sequence for M [BL01]. This map requires reindexing the filtration,

[2]We will give an example of this behavior in Proposition 2.6.

[3]As we discuss further in §3, it is more convenient to run the Adams spectral sequence for the Pontrjagin dual to supercohomology, then dualize afterwards. This does not change our qualitative conclusions about this spectral sequence in the introduction. We will also implicitly dualize in the HASS that we discuss below.

[4]We will define filtered spectra in Definition 1.1. The reader does not need to know the definition in order to read the introduction.

which means that the induced map of spectral sequences is not degree-preserving. We give analogous results for the connective Adams summand ℓ, which specializes to (the 2-localization of) ku at the prime 2.

The second step is to use Φ^{ko} to construct the HASS. Hastened Adams spectral sequences were introduced by Behrens–Hill–Hopkins–Mahowald [BHHM08], then studied in great generality by Bruner–Rognes [BR21, §12.6]. A HASS is associated to a cofiber sequence of spectra

$$X \xrightarrow{p} Y \xrightarrow{q} Z, \tag{0.1a}$$

where p has positive Adams filtration. The spectral sequence converges to the homotopy of Z. We apply the HASS to the Postnikov fiber sequences

$$\tau_{\geq(n+1)}ko \longrightarrow ko \longrightarrow \tau_{\leq n}ko, \tag{0.1b}$$

smashed over ko with a ko-module M. We use our Atiyah–Hirzebruch-to-Adams map Φ^{ko} to prove a key technical criterion which, by [BR21, Proposition 12.33], gives a usable description of the E_2-page. In examples, the HASS is about as difficult as the ordinary Adams spectral sequence for ko-homology. We also give analogous results with ℓ in place of ko. Supercohomology specifically is represented by the Pontrjagin dual of $\tau_{\leq 2}ko$ (see Definition 3.3); thus the $\tau_{\leq 2}ko$-homology and the supercohomology of a space or spectrum straightforwardly determine each other (Lemma 3.4), so in a slight abuse of notation we refer to the $\tau_{\leq 2}ko$ HASS as the supercohomology HASS.

Finally, we run the HASS to compute (twisted) supercohomology in several physically motivated examples. This includes the nontrivial twist of supercohomology over $B^2\mathbb{Z}/2$, which by work of Johnson-Freyd [JFR24] computes the sixteen-fold way of modular extensions of $s\mathcal{V}ect$ [Kit09, BGH+17, LKW17, JFR24]. We included these examples to demonstrate both how to use the HASS and that it is useful and tractable in applications.

Main results. One of our main results is the construction of the HASS for a class of theories including twisted $\tau_{\leq 2}ko$-homology, stated precisely in Theorem 2.26 below. Before reading these results, though, the reader may want to look at our "user's guide" to the HASS in §3. This guide includes the following.

- First, in §3.1, we summarize the steps of the procedure to use the HASS to compute $SH^*(X; a, b)$. The steps are similar to the computation of the corresponding twisted ko-homology groups of X, as described in [BC18] and [DY23a, §3.1], but instead of using Ext over $\mathcal{A}(1)$, one uses the functor $\mathcal{Q}$ defined in Theorem 2.26.
- In §3.2, we give $\mathcal{Q}(M)$ for thirteen frequently encountered $\mathcal{A}(1)$-modules M.
- In §3.3, we work out three examples of the HASS. In §3.3.1, we compute $SH^*(B\mathbb{Z}/2n)$ and $SH^*(B\mathbb{Z}/2n, 0, y)$, where $y \in H^2(B\mathbb{Z}/2n; \mathbb{Z}/2)$ is the unique nonzero class. In §3.3.2, we compute the twisted supercohomology of $K(\mathbb{Z}/2, 2)$, and relate it to Kitaev's sixteen-fold way [Kit06, §9]. In §3.3.4, we compute $SH^*(BQ_8)$.

Now let us discuss the specifics of our main theorems. We have two main results: the construction of the map from the Atiyah–Hirzebruch to the Adams spectral sequence for ko- and ℓ-modules (where ℓ is the connective Adams summand, see Lemma 1.26), and the existence and formula for the E_2-page of the HASS for the Postnikov truncation $\tau_{\leq n}ko$- and $\tau_{\leq n}\ell$-homology (see Definition 1.5). Dualizing the result for $\tau_{\leq 2}ko$, the corresponding HASS calculates supercohomology.

Before stating the main theorem, we need the following function to shift the degrees of the Atiyah–Hirzebruch filtration and the Adams filtration.

Definition 1.33. *For $x \in \mathbb{Z}$, write $x = 8n + k$ for $0 \leq k < 8$. Then define*

$$g(x) \coloneqq \begin{cases} 4n + k, & 0 \leq k \leq 3 \\ 4n + 3, & k = 4 \\ 4n + 4, & 5 \leq k \leq 7. \end{cases} \tag{0.2a}$$

Likewise, for each prime number p, define $g^c(p, -) \colon \mathbb{Z} \to \mathbb{Z}$ by

$$g^c(p, x) \coloneqq \left\lceil \frac{x}{2p-2} \right\rceil. \tag{0.2b}$$

(In the main text, we give a different, but equivalent, definition of g and g^c.)

Then, we have the following promised "Atiyah–Hirzebruch to Adams" map for the two corresponding filtrations, defined in Definitions 1.6 and 1.8, respectively.

For R an E_∞-ring spectrum, $\mathcal{F}il_R$ denotes the ∞-category of filtered R-module spectra. For example, the R-module Atiyah–Hirzebruch, resp. Adams filtrations (Definition 1.6, resp. 1.8) define functors $\mathrm{AH}^R_\bullet, \mathrm{Ad}^R_\bullet \colon \mathcal{M}od_R \to \mathcal{F}il_R$. The colimit of a filtered spectrum X is denoted $X_{-\infty}$.

Theorem 1.35.

(1) *There is a natural transformation of functors $\mathcal{M}od_{ko} \to \mathcal{F}il_{ko}$*

$$\Phi^{ko} \colon \mathrm{AH}^{ko}_\bullet \Longrightarrow \mathrm{Ad}^{ko}_{g(\bullet)} \tag{0.3a}$$

which, when composed with $(-)_{-\infty}$, is homotopy equivalent to 2-completion.

(2) *Let p be a prime. There is a natural transformation of functors $\mathcal{M}od_\ell \to \mathcal{F}il_\ell$*

$$\Phi^{\ell} \colon \mathrm{AH}^{\ell}_\bullet \Longrightarrow \mathrm{Ad}^{\ell}_{g^c(p,\bullet)} \tag{0.3b}$$

which, when composed with $(-)_{-\infty}$, is homotopy equivalent to p-completion.

From this, we have a map of spectral sequences, albeit one that shifts the grading.

Our proof of Theorem 1.35 is fairly specific to ℓ and ko. In §1.3, we indicate how it generalizes to any E_∞-ring spectrum which is *rising* (Definition 1.47), but this property does not seem to be especially common. (See, however, Conjecture 4.6.)

Corollary 1.37. *Let X be a space or spectrum. After regrading the Adams spectral sequence, there is a map of spectral sequences from the Atiyah–Hirzebruch spectral sequence for $ko_*(X)$ to the Adams spectral sequence for $ko_*(X)^\wedge_2$ (and likewise for ℓ at all primes); this map converges to the 2-completion map $ko_*(X)^\wedge_2$.*

Computations in examples suggest that this map explains a curious phenomenon that difficult differentials in the 2-primary Adams spectral sequence for $ko_*(X)$ tend to correspond to difficult differentials in the AHSS for $ko_*(X)$. We plan to study this map in detail in future work.

Afterwards, we build on the construction of Behrens–Hill–Hopkins–Mahowald [BHHM08] and Bruner–Rognes [BR21] (see Theorem 2.18 and Corollary 2.21), to construct the hastened Adams resolution and the associated HASS corresponding to the truncation of ko. The following theorem gives an explicit construction of the HASS for supercohomology.

Let $C_k \coloneqq \Sigma^{-k} H^*_{ko}(\tau_{\geq k} ko)$, which we describe explicitly in Lemma 1.51.

Theorem 2.26.

(1) *There is a map of $\mathbb{Z}^2$-graded $\mathrm{Ext}(\mathbb{Z}/2)$-modules*

$$f_{n-1} \colon \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(C_n) \longrightarrow \mathrm{Ext}^{s+g(n),t+g(n)+n}_{\mathcal{A}(1)}(\mathbb{Z}/2), \tag{0.4}$$

which is induced from the Postnikov cover map $\tau_{\geq n}\colon \tau_{\geq n} ko \to ko$.

(2) *There is a functor* $\mathcal{Q}_n^{*,*}$ *from* $\mathcal{A}(1)$*-modules to* $\mathbb{Z}^2$*-graded* $\mathrm{Ext}(\mathbb{Z}/2)$*-modules which commutes with direct sums and such that for all* $\mathcal{A}(1)$*-modules* M*, there is a long exact sequence*

(0.5)
$$\cdots \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(C_n \otimes M) \xrightarrow{f_{n-1}} \mathrm{Ext}^{s+g(n),t+g(n)+n}_{\mathcal{A}(1)}(M) \xrightarrow{\partial_{n-1}} \mathcal{Q}_n^{s,t}(M) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)}(C_n \otimes M) \xrightarrow{f_{n-1}} \cdots$$

(3) *Let* M *be a finite type* ko*-module.*[5] *Then the HASS for* $\tau_{\leq n} ko \wedge_{ko} M$ *converges strongly and has signature*

$$E_2^{s,t} = \mathcal{Q}_n^{s,t}(H^*_{ko}(M)) \Longrightarrow \pi_{t-s}(\tau_{\leq n} ko \wedge_{ko} M)^\wedge_2. \tag{0.6}$$

The map $\pi_*(M) \to \pi_*(\tau_{\leq n} ko \wedge_{ko} M)$ *lifts to a map from the ordinary Adams spectral sequence to the HASS which on the* E_2*-page is the map* ∂_n *from* (2.28).

An analogous result is true for truncations of ℓ at an arbitrary prime, and the proof is identical.

Though we prove the general result above, in applications we focus on the case $n = 2$, as $\tau_{\leq 2} ko$-homology is dual to supercohomology. As such, we will use $\mathcal{Q}$ to denote $\mathcal{Q}_2$. In [DYY25, Theorem E.6], we stated this special case of Theorem 2.26 without proof[6] and used it to make supercohomology calculations in (*ibid.*, Propositions E.20, E.23, and F.3). Thus, our proof of Theorem 2.26 in this paper supports those computations.

In practice, the HASS works quite similarly to the usual Adams spectral sequence, except that it replaces the Ext module in the Adams spectral sequence with the module given by the functor $\mathcal{Q}$ of the theorem. In the users' guide in Section 3, we compute $\mathcal{Q}$ for many common $\mathcal{A}(1)$-modules, then use them to give several example HASS calculations, motivated by problems arising in physics. In particular, we rederive the 16-fold way [Kit09, BGH+17, LKW17, JFR24] from the calculation of (twisted) supercohomology, after Johnson-Freyd's original proof [JFR24]. We learned from these examples that running the HASS to compute $\tau_{\leq 2} ko$-homology is generally no more difficult than running the corresponding ko-homology Adams spectral sequence.

Along the way to our main results and examples, we proved the following results, which may be of independent interest.

- In Appendix A, we show that the definition of (twisted) supercohomology that arises from the study of fermionic fusion 2-categories agrees with a homotopical definition of supercohomology given as a twisted generalized cohomology theory à la the appoach of Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14a, ABG+14b].
- Sanford [San26, Theorem 5.10] proves that the kernel of the canonical map from the Witt group of $\mathcal{V}ect_{\mathbb{R}}$ to the Witt group of $s\mathcal{V}ect_{\mathbb{R}}$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is an $\mathbb{R}$-analogue of the sixteenfold way over $\mathbb{C}$. Following [JFR24, §4.4], we reinterpret this in Corollary 3.37 as saying a certain twisted $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$-cohomology group of $B^2\mathbb{Z}/2$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. We then prove this directly, giving another proof of Sanford's result. The bulk of this proof is identifying the homotopy type of $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ (Proposition 3.50), which was a folklore result. Our proof of Corollary 3.37 does not directly use the HASS, but does use our computations from §3.3.2.

[5] The finite-type assumption appears only for technical reasons, and is true in all examples we consider in this paper.

[6] In the arXiv v1 and v2 of that paper, there is a typo in the degree of the map f_2, which is called g_4 there; instead of shifting $(s,t) \mapsto (s+3, t+2)$, it should shift $(s,t) \mapsto (s+3, t+7)$. This typo did not propagate to affect any of our computations.

- In Theorem B.12 we compute $\mathcal{Q}(M)$ for several commonly occurring $\mathcal{A}(1)$-modules M. We used these calculations to draw the $\mathcal{Q}$ charts in Figure 4. Along the way to proving Theorem B.12, we make some $\mathrm{Ext}_{\mathcal{A}(1)}$ computations that may be of independent interest. For example, in Lemma B.4 we determine the algebra structure on $\mathrm{Ext}_{\mathcal{A}(1)}(H^*((B\mathbb{Z}/2)^{2\sigma-2}; \mathbb{Z}/2), \mathbb{Z}/2)$, which could be useful for studying the ring structure on Spin $\times_{\{\mathbb{Z}/2\}} \mathbb{Z}/4$ bordism (see [Cam17, §7.8]).

In Section 4 we conclude with some conjectures and further directions regarding the HASSes in this paper.

- In §4.1 we give some potential applications of the $\tau_{\leq 1}ko$-homology HASS (i.e. for restricted supercohomology).
- In §4.2 we conjecture that Φ^{ko} and Φ^{ℓ} are lax monoidal and that the HASS is multiplicative.
- In Conjectures 4.6, 4.8 and 4.9, we speculate on generalizations to connective self-conjugate K-theory and equivariant and motivic versions of ko and ku.

Outline. §1 is devoted to proving Theorem 1.35, which constructs the Atiyah–Hirzebruch-to-Adams maps Φ^{ko} and Φ^{ℓ}. First, in §1.1, we introduce our point of view spectral sequences from the point of view of filtered spectra, and describe the Atiyah–Hirzebruch and Adams filtrations and the induced spectral sequences. We finish this subsection by stating Theorem 1.35 and discussing what we learn, at the qualitative level, from the existence of the AHSS-to-Adams map. In §1.2, we prove Theorem 1.35 for ℓ by comparing the $H\mathbb{Z}_{(p)}$- and $H\mathbb{Z}/p$-based ℓ-module Adams spectral sequences. In §1.3, we prove the ko part of Theorem 1.35.

In §2, we prove our other main theorem, Theorem 2.26, constructing the HASS for $\tau_{\leq n}ko$- and $\tau_{\leq n}\ell$-modules. First, in §2.1, we work out the Baker–Lazarev ko-module Adams spectral sequence for $\tau_{\leq 2}ko$. In §2.2, we review work of Behrens–Hill–Hopkins–Mahowald [BHHM08] and Bruner–Rognes [BR21, §12] on hastened Adams spectral sequences in general. In §2.3, we state and prove Theorem 1.35, applying the general setup of §2.2 to truncations of ko and ℓ. We finish this section by working out the HASS for $\tau_{\leq 2}ko$ as a module over itself and observing that this spectral sequence is easier than the ordinary Adams spectral sequence from §2.1.

§3 is our user's guide. In §3.1 we walk through the steps of the method of computing twisted supercohomology with the HASS. Then, in §3.2, we give the charts for $\mathcal{Q}(-)$, the input to the HASS, for many commonly encountered $\mathcal{A}(1)$-modules. In §3.3, we give several examples of the HASS: $B\mathbb{Z}/2n$ in §3.3.1, $B^2\mathbb{Z}/2$ and the sixteen-fold way in §3.3.2, and BQ_8 in §3.3.4. In addition, we use our computations to study Sanford's real fourfold way [San26, Theorem 5.10] in §3.3.3.

In §4, we speculate on further directions and conjectures. In §4.1, we list three potential applications of the HASS for $\tau_{\leq 1}ko$-homology. In §4.2, we discuss generalizations of our work in a few different directions.

We conclude with two appendices. In Appendix A, we prove Theorem A.35, that two different definitions of twisted supercohomology are naturally isomorphic. The first definition, expressed using category theory and more closely tied to applications, appears in §A.1, and the second, defined with homotopy theory and better for computations, is given in §A.2. Then, in §A.3, we prove Theorem A.35. Appendix B contains all of the $\mathrm{Ext}_{\mathcal{A}(1)}$ and $\mathcal{Q}$ computations needed to support the $\mathcal{Q}$ charts in §3.2.

Acknowledgements. It is a pleasure to thank Prerna Dhankhar, Mike Hopkins, and Morgan Opie for helpful conversations. MY is supported by the EPSRC Open Fellowship EP/X01276X/1

and the Fund for Natural Science and the Adler Family Fund. No authors have competing interests to declare that are relevant to the content of this article.

Statement on AI use. We did not use AI tools to develop any ideas, proofs, or examples in this paper. We completed the main results and examples in 2025, before the most advanced LLM tools became available.

1. Compatibility of the Atiyah–Hirzebruch and Adams filtrations

1.1. **Dramatis personæ.** Here we review the language of filtered objects in an ∞-category and the examples that enter into the construction of the Atiyah–Hirzebruch and Adams spectral sequences. In particular, we prove that there is a natural map from the Atiyah-Hirzebruch spectral sequence to the Adams spectral sequence for ko or the connective Adams summand ℓ, in the sense of Corollary 1.37.

First, we introduce the Atiyah–Hirzebruch and Adams filtrations, which induce the corresponding spectral sequences in a sense we discuss below.

Definition 1.1 ([Lur17, Definition 1.2.2.9])**.** Let $\mathcal{C}$ be an ∞-category.

(1) A *filtered object* $C_\bullet$ of $\mathcal{C}$ is a contravariant functor from the poset category $(\mathbb{Z}, \leq)$ to $\mathcal{C}$. Equivalently, it is data of an object $C_s \in \mathcal{C}$ for each $s \in \mathbb{Z}$ and $\mathcal{C}$-morphisms $f_s \colon C_s \to C_{s-1}$.
(2) Assume $\mathcal{C}$ is stable. The *associated graded* of a filtered object $C_\bullet$ is the filtered spectrum $\mathrm{gr}(C)_\bullet$ defined by

$$\mathrm{gr}(C)_s \coloneqq \mathrm{cofib}(\mathrm{f}_{s+1} \colon C_{s+1} \to C_s). \tag{1.2}$$

Its structure maps $\mathrm{gr}(C)_{s+1} \to \mathrm{gr}(C)_s$ are zero.
(3) The *abutment* of C is $C_{-\infty} \coloneqq \mathrm{colim}_s C_s$ (which may or may not exist).

We will let $\mathcal{F}i\ell(\mathcal{C}) \coloneqq \mathcal{F}un(\mathbb{Z}^{op}, \mathcal{C})$ denote the ∞-category of filtered objects in $\mathcal{C}$. For the most part, our filtered spectra will have $C_s = 0$ for $s < 0$, and we will let $\mathcal{C}$ be the category $\mathcal{M}od_R$ of modules over an E_∞-ring spectrum R. In this case we abbreviate $\mathcal{F}i\ell(\mathcal{M}od_R)$ as $\mathcal{F}i\ell_R$.

For $X_\bullet \colon \mathbb{Z}^{op} \to \mathcal{C}$ a filtered spectrum, we will let $X_s \coloneqq X_\bullet(s)$. If $t \leq s$ we will use lowercase $\mathrm{f}_{s,t} \colon X_s \to X_t$ to denote the image under $X_\bullet$ of the unique morphism $t \to s$ in the category $\mathbb{Z}$ and $\mathrm{f}_s \coloneqq \mathrm{f}_{s,s-1}$.

For $s \in \mathbb{Z}$, $ev_s \colon \mathcal{F}i\ell(\mathcal{C}) \to \mathcal{C}$ denotes the "evaluation at s" functor $X_\bullet \mapsto X_s$.

Remark 1.3 (Reindexing). Let $C_\bullet$ be a filtered object of an ∞-category $\mathcal{C}$. We will sometimes need to reindex the filtration on $C_\bullet$, e.g. changing filtration s to filtration $s + k$ for some k. Unwinding the definitions, reindexing amounts to precomposing the functor $C \colon \mathbb{Z}^{op} \to \mathcal{C}$ with a functor $f \colon \mathbb{Z}^{op} \to \mathbb{Z}^{op}$. Since $\mathrm{Hom}_{\mathbb{Z}^{op}}(m, n)$ is empty if $m < n$ and a singleton if $m \geq n$, the data of such a functor f is precisely a nondecreasing function $f \colon \mathbb{Z} \to \mathbb{Z}$.

Suppose our nondecreasing function f is unbounded above. Then we may define $f^\dashv, f^\vdash \colon \mathbb{Z} \to \mathbb{Z}$ by

$$f^\dashv(m) \coloneqq \max\{n \mid m \geq f(n)\}, \qquad f^\vdash(m) \coloneqq \min\{n \mid f(n) \geq m\}. \tag{1.4}$$

One may directly check that $f^\dashv$ and $f^\vdash$ are nondecreasing, so that they define functors $\mathbb{Z}^{op} \to \mathbb{Z}^{op}$, and that $f^\dashv$, resp. $f^\vdash$, is left, resp. right adjoint to f.

Definition 1.5. Let $\tau_{\leq n}\mathcal{S}p$ be the full sub-∞-category of spectra whose homotopy groups vanish in degrees $n+1$ and above. The inclusion $\tau_{\leq n}\mathcal{S}p \hookrightarrow \mathcal{S}p$ has a left adjoint $\tau_{\leq n}\colon \mathcal{S}p \to \tau_{\leq n}\mathcal{S}p$, which we call *Postnikov n-truncation*.

Dually, let $\tau_{\geq n}\mathcal{S}p$ be the full sub-∞-category of spectra whose homotopy groups vanish in degrees $n-1$ and below. The inclusion $\tau_{\geq n}\mathcal{S}p \hookrightarrow \mathcal{S}p$ admits a right adjoint $\tau_{\geq n}\colon \mathcal{S}p \to \tau_{\geq n}\mathcal{S}p$, which we call the *Postnikov $(n-1)$-connected cover*.

See [Lur17, §1.2.1] for a discussion in the generality of stable ∞-categories, and (*ibid.*, Propositions 1.4.3.4 and 1.4.3.6) for why this applies to $\mathcal{S}p$ in particular.

For any spectrum E, the unit and counit of these adjunctions, resp., give rise to maps $\tau_{\leq n}\colon E \to \tau_{\leq n}E$ and $\tau_{\geq n}\colon \tau_{\geq n}E \to E$. By [Lur17, Remark 1.2.1.8] these maps participate in a fiber sequence $\tau_{\geq (n+1)}E \to E \to \tau_{\leq n}E$. Furthermore, by [Lur17, Proposition 7.1.1.13], $\tau_{\leq n}$ and $\tau_{\geq n}$ send R-modules to R-modules.

Definition 1.6. Let R be an E_∞-ring spectrum and M be an R-module. The *(R-module) Atiyah–Hirzebruch filtration* of M, denoted $\mathrm{AH}^R_\bullet(M)$, is the functor sending $n \mapsto \tau_{\geq n}(R) \wedge_R M$ and the unique map $m \to n$, $m \leq n$, to $\tau_{\geq n}(R) \wedge_R \mathrm{id}\colon \tau_{\geq n}(R) \wedge_R M \to \tau_{\geq m}(R) \wedge_R M$.

Remark 1.7. We call this the "Atiyah–Hirzebruch filtration" because it gives rise to a generalization of the Atiyah–Hirzebruch spectral sequence in the category of R-module spectra; see [EKMM97, Theorem IV.3.7]. If $M = R \wedge X$ for a spectrum X, the spectral sequence obtained from the filtered spectrum $\mathrm{AH}^R_\bullet(M)$ is naturally isomorphic to the usual Atiyah–Hirzebruch spectral sequence for the R-homology of X.[7]

Definition 1.8. Let $\theta\colon R \to S$ be a map of E_∞-ring spectra, $\overline{S}$ be the cofiber of θ, and $\alpha\colon \Sigma^{-1}\overline{S} \to R$ be the fiber of θ. The *S-based R-module Adams filtration* of an R-module M, denoted $\mathrm{Ad}^{R,S}_\bullet(M)$, is the functor sending $n \mapsto (\Sigma^{-1}\overline{S})^{\wedge_R n} \wedge_R M$, and the map $n \to n+1$ to

$$\alpha \wedge \mathrm{id}_{\mathrm{Ad}^{R,S}_n(M)}\colon \Sigma^{-1}\overline{S} \wedge_R (\Sigma^{-1}\overline{S})^{\wedge_R n} \wedge_R M \to R \wedge_R (\Sigma^{-1}\overline{S})^{\wedge_R n} \wedge_R M \simeq (\Sigma^{-1}\overline{S})^{\wedge_R n} \wedge_R M. \tag{1.9}$$

Often, $S = H\mathbb{Z}/p$ for a prime p and θ is the Postnikov 0-truncation of R followed by modulo p reduction. In this setting, we will write $\mathrm{Ad}^R_\bullet(M)$ instead of $\mathrm{Ad}^{R,H\mathbb{Z}/p}_\bullet(M)$. However, we will need the more general version of Definition 1.8 in §1.2 and §1.3.

Remark 1.10. Though Definition 1.8 is essentially a restatement of Baker–Lazarev's construction of their version of the Adams spectral sequence in [BL01, §2], often one sees the Adams filtration defined differently, as the Bousfield–Kan filtration [BK72, BK73a, BK73b, Bou03] associated to the cosimplicial R-module $S^{\wedge_R \bullet} \wedge_R M$, e.g. in van Nigtevecht [vN25, §2.5.2]. These two approaches coincide, and by Mathew–Naumann–Noel [MNN17, Proposition 2.14], this fact can be lifted to a natural equivalence of filtered R-modules between the two definitions of the Adams filtration. See van Nigtevecht [vN25, §2.5] for further discussion.

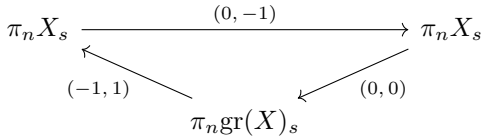
Definition/Theorem 1.11 ([vN25, Construction 2.31])**.** *Let X be a filtered spectrum.*

(1) *The* associated exact couple *of X is the data of the bigraded abelian groups*

$$A_{n,s}(X) := \pi_n X_s \qquad \textit{and} \qquad E_{n,s}(X) := \pi_n \mathrm{gr}(X)_s, \tag{1.12a}$$

[7] Atiyah–Hirzebruch [AH61, §2] defined their spectral sequence using the skeletal filtration on X. Maunder [Mau63, Theorem 3.3] showed that using the Postnikov filtration on R, as we do in Definition 1.6, leads to an isomorphic spectral sequence from the E^2-page onward. Antieau [Ant24, Proposition 9.2] lifts Maunder's theorem to a natural equivalence of filtered spectra.

together with the usual maps in the long exact sequence relating them:

(1.12b)


(Each map is labeled with its (n,s)-bidegree.)

(2) *The* underlying spectral sequence *of X, written $\{E^r_{n,s}(X),\, d^r\}_{r\geq 1}$, is the spectral sequence arising from the exact couple associated with X. By definition, this spectral sequence is of the form*

$$E^1_{n,s}(X) := \pi_n \mathrm{gr}(X)_s \implies \pi_n X_{-\infty} \tag{1.12c}$$

and the differential d^r has (n,s)-bidegree $(-1,r)$ for $r \geq 1$.

The "theorem" part of Definition/Theorem 1.11 is the claim that $(A_{n,s}(X), E_{n,s}(X))$ is an exact couple, and that $(E^r_{n,s}, d^r)$ is a spectral sequence.

In many but not all cases, this spectral sequence converges to $X_{-\infty}$.

Remark 1.13. Though we will use the ∞-category of filtered spectra as our setting to study the homotopy theory of spectral sequences, it is worth mentioning another perspective. Work of Brotherston, Circi, Egas Santander, Fu, Guan, Livernet, Muro, Rotzheim, and Whitehouse [MR19, CESLW20, FGLW22, Bro24, LW24a, LW24b, Bro25, CLW25] constructs and studies for each r model structures on filtered (multi)complexes in which weak equivalences are isomorphisms on the E_r-pages of the induced spectral sequences. See Aitken [Ait24, §3.2] for a comparison to the ∞-categorical point of view on filtered spectra.

Example 1.14 (Maunder [Mau63])**.** Choose a space X. Let's see what Definition/Theorem 1.11 looks like for the Atiyah–Hirzebruch filtration for an E_∞-ring spectrum R and the R-module $R \wedge X_+$. The cofiber of $\tau_{\geq(s+1)}R \to \tau_{\geq s}R$ is $\tau_{\leq s}\tau_{\geq s}R \simeq \Sigma^n H\pi_s(R)$. Smash with X to conclude that the associated graded of $\mathrm{AH}^R_\bullet(R \wedge X)$ has s^{th} object

$$\mathrm{gr}(\mathrm{AH}^R(R \wedge X))_s \simeq \Sigma^s H\pi_s(R) \wedge X, \tag{1.15}$$

whose homotopy groups are $H_*(X; \pi_s(R))$ shifted in degree by s. That is, in the language of Definition/Theorem 1.11, the associated spectral sequence is

$$E^1_{n,s} \cong \widetilde{H}_n(X; \pi_s(R)). \tag{1.16}$$

Now this looks like the E^2-page, not the E^1-page, of the Atiyah–Hirzebruch spectral sequence. What happened? The answer is that we may reindex the filtration, or equivalently reindex the spectral sequence, such that (1.16) happens on E^2, rather than E^1. See Antieau [Ant24, §7, §9] for further information.

In general, when R and M are bounded below, the spectral sequence associated to $\mathrm{AH}^R_\bullet(M)$ converges to its abutment, which is M: see [Ant24, §6].

Definition 1.17 ([EKMM97, Definition IV.1.7])**.** Let p be a prime number and R be an E_∞-ring spectrum with an E_∞-ring map $\theta\colon R \to H\mathbb{Z}/p$. The *$R$-module homology and cohomology* of an R-module M are

$$H^R_*(M) := \pi_*(M \wedge_R H\mathbb{Z}/p), \qquad H^*_R(M) := \pi_{-*}(\mathrm{Map}_R(M, H\mathbb{Z}/p)). \tag{1.18}$$

We write $H^R_* H$ for $H^R_*(H\mathbb{Z}/p)$, and analogously for $H^*_R H$.

Though p and θ are implicit in this notation, they will be clear from context wherever we use H^*_R and H^R_* in this paper.

Definition 1.19. Let $f\colon M \to N$ be an R-module map.

(1) We say f has *(R-module) Adams filtration $\geq s$* if there is a sequence of R-module maps

$$M = X_0 \xrightarrow{t_1} X_1 \xrightarrow{t_2} \cdots \xrightarrow{t_s} X_s = N \tag{1.20}$$

such that $(t_i)_*\colon H^R_*(X_{i-1}) \to H^R_*(X_i)$ is the zero map.

(2) If f has filtration $\geq s$ but not $\geq (s+1)$, we say f has *(R-module) Adams filtration s*.

We have now defined two Adams filtrations: on an R-module (Definition 1.8) and on the abelian group of homotopy classes of maps between two R-modules (Definition 1.19). When the domain is R, this filters $\pi_*(N)$. The content of the next lemma is that π_* of the filtration on N is the filtration on $\pi_*(N)$.

Lemma 1.21 (Adams [Ada74, Theorem III.15.1], Baker–Lazarev [BL01, §2])**.** *Let p be a prime. The filtration induced on $\pi_*(M) \cong \pi_*(\mathrm{Map}_R(R, M))$ by applying π_* to the filtered spectrum $\mathrm{Ad}^{R,H\mathbb{Z}/p}_n(M)$ defined in Definition 1.8 equals the Adams filtration on $\pi_*(M)$ defined in Definition 1.19.*

Suppose $\mathcal{C}$ is now a stable ∞-category with a t-structure and containing sequential colimits, such as $\mathcal{M}od_R$. If $X_\bullet$ is a filtered object of $\mathcal{C}$, then it gives rise to a spectral sequence in a manner generalizing the spectral sequence associated to a filtered chain complex; see [Lur17, §1.2.2].

Definition 1.22. The *Baker–Lazarev Adams spectral sequence* for an R-module M is the spectral sequence associated to the canonical 2-primary Adams filtration of M.

Theorem 1.23 (Baker–Lazarev [BL01, Theorems 2.1 and 2.3])**.** *Let R be as above and M be an R-module.*

(1) *Suppose R and M are bounded-below, cellular spectra of finite type. Then the Baker–Lazarev Adams spectral sequence converges to $\pi_*(M)^\wedge_p$ with the R-module Adams filtration.*

(2) *The E_2-page of the Baker–Lazarev Adams spectral sequence is*[8]

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{H^R_* H}(\mathbb{Z}/p, H^R_*(M)). \tag{1.24}$$

Baker–Lazarev (*ibid.*, §2) also discuss convergence of their spectral sequence. Using the criteria they provide, one can show that, for $R = ko$ or ℓ and M bounded below, their spectral sequence converges strongly to the abutment of $\mathrm{Ad}^R_\bullet(M)$.

Remark 1.25. Baker–Lazarev work in greater generality than we do, and therefore their version of (2) requires a flatness assumption. This assumption is always true in the setting we work in. In this generality, their spectral sequences converges to the homotopy groups of the "R-module $H\mathbb{Z}/p$-nilpotent completion" of M, which can be identified with the p-completion (see [DY23a, §2.2], which follows [Rav84, Example 1.16]).

We would like to specialize to real and complex connective K-theory. At odd primes, ku is decomposable, so we replace it with one of its summands.

[8]Here, "Ext" means the derived functor of the functor of *co*module homomorphisms. Some references will therefore write coExt or Coext to emphasize this.

Lemma 1.26 (Adams [Ada69, Lecture 4, Corollary 8], Johnson–Wilson [JW73, Proposition 2.7])**.** *For any prime p, there is a p-local spectrum ℓ and a p-local homotopy equivalence*

$$ku \simeq \bigvee_{i=1}^{p-1} \Sigma^{2i-2}\ell. \tag{1.27}$$

The spectrum ℓ admits an essentially unique E_∞-ring structure [BR08, Corollary 1.4] and is called the *connective Adams summand*. It is sometimes denoted $BP\langle 1\rangle$. When we refer to ℓ, we have implicitly chosen a prime p to localize at.

At $p = 2$, $ku \simeq \ell$. When p is odd, ko is also a sum of shifts of ℓ.

Remark 1.28. Adams [Ada69, Lecture 4, Corollary 8] proves a version of Lemma 1.26 for periodic K-theory; Johnson–Wilson [JW73, Proposition 2.7] explicitly prove the connective case, which they say was a folk theorem.

In this paper, the base ring spectrum R will either be ko or ℓ. For $R = ko$, we will restrict to $p = 2$. We first give some basic properties of these spectra that we will use many times.

Proposition 1.29. *There are isomorphisms of graded rings*

$$\pi_*(ko) \cong \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta\alpha, \alpha^2 - 4\beta) \tag{1.30a}$$

$$\pi_*(\ell) \cong \mathbb{Z}_{(p)}[v_1], \tag{1.30b}$$

where $|\eta| = 1$, $|\alpha| = 4$, $|\beta| = 8$, and $|v_1| = 2p - 2$.

The isomorphisms in Proposition 1.29 are not unique, but one can construct a preferred choice of isomorphism using the map from spin bordism (for ko), resp. (the connective cover of) Adams' choice of splitting in Lemma 1.26 (for ℓ).

Lemma 1.31 (Baker [Bak20, Lemma 5.1])**.**

1. *Let $\mathcal{A}(1)$ denote the subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ generated by Sq^1 and Sq^2. Then there is a unique isomorphism $H^*_{ko}H \cong \mathcal{A}(1)$ identifying the forgetful map $H^*_{ko} \to H^*_{\mathbb{S}}H \cong \mathcal{A}$ with the inclusion $\mathcal{A}(1) \hookrightarrow \mathcal{A}$.*
2. *For any prime p, let $\mathcal{E}(1)$ denote the subalgebra of the mod p Steenrod algebra $\mathcal{A}_p$ generated by the Bockstein β and the operation $Q_1 \coloneqq \beta\mathcal{P}^1 - \mathcal{P}^1\beta$. Then there is a unique isomorphism $H^*_\ell H \cong \mathcal{E}(1)$ identifying the forgetful map $H^*_\ell H \to H^*_{\mathbb{S}}H \cong \mathcal{A}_p$ with the inclusion $\mathcal{E}(1) \hookrightarrow \mathcal{A}_p$.*
3. *If $\mathcal{A}(1)_*$, resp. $\mathcal{E}(1)_*$ denote the dual Hopf algebras to $\mathcal{A}(1)$ and $\mathcal{E}(1)$, there are unique isomorphisms $H^{ko}_*H \cong \mathcal{A}(1)_*$ and $H^\ell_*H \cong \mathcal{E}(1)_*$ such that the base change maps $H^{\mathbb{S}}_*H \to H^{ko}_*H$, resp. $H^{\mathbb{S}}_*H \to H^\ell_*H$ are identified with the usual quotient maps $\mathcal{A}_* \to \mathcal{A}(1)_*$, resp. $(\mathcal{A}_p)_* \to \mathcal{E}(1)_*$.*

At $p = 2$, $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$.

The following lemma is standard, following for example from the analogous facts for the spaces $B\mathrm{O}$ and $B\mathrm{U}$.

Lemma 1.32. *ko and ℓ satisfy the hypotheses on R in item (1) of Theorem 1.23, i.e., they are bounded-below, cellular spectra of finite type.*

Definition 1.33. Let R be a connective spectrum. Define the function $g^R\colon \mathbb{Z} \to \mathbb{Z}$ as follows: $g^R(n) = m$ means that, counting upwards from zero, $\pi_m(R)$ is the n^{th} nonzero homotopy group of R.

We also let $g \coloneqq g^{ko}$ and $g^c(p,-) \coloneqq g^\ell$.

Explicitly, for $x \in \mathbb{Z}$, write $x = 8n + k$ for $0 \leq k < 8$. Then

$$g(x) \coloneqq \begin{cases} 4n + k, & 0 \leq k \leq 3 \\ 4n + 3, & k = 4 \\ 4n + 4, & 5 \leq k \leq 7. \end{cases} \tag{1.34a}$$

Likewise, for each prime number p,

$$g^c(p, x) \coloneqq \left\lceil \frac{x}{2p-2} \right\rceil. \tag{1.34b}$$

If $x \geq 0$, then in the $E_2 = E_\infty$-page of the 2-primary Adams spectral sequence for ko, the element of minimal Adams filtration in topological degrees $\geq t - s$ has filtration $g(t-s)$. The analogous fact is true for the p-primary Adams spectral sequence for ℓ and $g^c(p,-)$.

As mentioned in the introduction, we aim to produce a map from the Atiyah–Hirzebruch to the Adams filtrations.

Theorem 1.35.

(1) *There is a natural transformation of functors $\mathcal{M}od_{ko} \to \mathcal{F}i\ell_{ko}$*

$$\Phi^{ko} \colon \mathrm{AH}^{ko}_\bullet \Longrightarrow \mathrm{Ad}^{ko}_{g(\bullet)} \tag{1.36a}$$

which, when composed with $(-)_{-\infty}$, is homotopy equivalent to 2-completion.

(2) *Let p be a prime. There is a natural transformation of functors $\mathcal{M}od_\ell \to \mathcal{F}i\ell_\ell$*

$$\Phi^{\ell} \colon \mathrm{AH}^{\ell}_\bullet \Longrightarrow \mathrm{Ad}^{\ell}_{g^c(p,\bullet)} \tag{1.36b}$$

which, when composed with $(-)_{-\infty}$, is homotopy equivalent to p-completion.

This yields a somewhat curious corollary.

Corollary 1.37. *Let X be a space or spectrum. After regrading the Adams spectral sequence, there is a map of spectral sequences from the Atiyah–Hirzebruch spectral sequence for $ko_*(X)$ to the Adams spectral sequence for $ko_*(X)^\wedge_2$ (and likewise for ℓ at all primes); this map converges to the 2-completion map $ko_*(X)^\wedge_2$ (resp. to p-completion).*

We plan to investigate this map in future work. Specifically, we believe this explains a phenomenon we noticed studying Adams spectral sequences for computing twisted ko-homology: "difficult differentials" in the 2-primary Adams spectral sequence for $ko_*(X)$ tend to correspond to difficult differentials, sometimes of different lengths, in the AHSS for $ko_*(X)$, in that one vanishes if and only if the other does. Two examples of this phenomenon occur in [Ped17, §9] and [Deb21, §5.3].

Corollary 1.37 and the aforementioned examples offer a concrete answer to a frequently asked question in this area: *why, and to what extent, is the Adams spectral sequence so much more effective than the AHSS for ko-homology?* We have learned three things towards the answer.

(1) The Adams filtration makes multiplication by 2 visible in most cases thanks to the algebraic structure on the E_∞-page, but the AHSS doesn't. Thus, the Adams spectral sequence is much better at addressing extensions by 2.
(2) The comparison map from Corollary 1.37 suggests that Adams and Atiyah–Hirzebruch differentials could obey similar formulas. This is also something we see in examples: for example, higher differentials in both spectral sequences can be described using higher cohomology operations. See for example [BJ04, AS06].

(3) However, we do not expect Φ^{ko} to identify all differentials between these two spectral sequences. For example, the May–Milgram theorem [MM81] calculates differentials between h_0-towers in Adams spectral sequences. The corresponding elements in the AHSS collapse into a single bidegree, so there is not in general an analogue of the May–Milgram theorem available in the AHSS. In this case, we think these differentials will not be in the image of Φ^{ko}.

Moreover, some difficult differentials of the AHSS for $ko_*(X)$ in low degrees have been identified in [WG18, NRW+25], etc. It is interesting to see how these results about the AHSS can help with the differentials of the Adams spectral sequence.

Remark 1.38. Remark 1.3 implies that we can rephrase the natural transformation Φ^{ko} from Theorem 1.35 using the adjunction $g \dashv g^{\vdash}$ to obtain a natural transformation

$$(\Phi^{ko})^{\vdash} \colon \mathrm{AH}^{ko}_{g^{\vdash}(\bullet)} \Longrightarrow \mathrm{Ad}^{ko}_{\bullet}, \tag{1.39}$$

and analogously for ℓ using $(g^c)^{\vdash}$. That is, one can regrade the Atiyah–Hirzebruch filtration instead if one wants, though in practice we found the result to be cleaner when we lengthen the Adams filtration rather than compress the Atiyah–Hirzebruch filtration.

To obtain (1.39), we used a standard if slightly unintuitive fact in category theory, that given an adjunction $f \colon \mathcal{C}_1 \rightleftarrows \mathcal{C}_2 \colon g$ and another category $\mathcal{D}$, the precomposition functor

$$- \circ g \colon \mathcal{F}un(\mathcal{C}_2, \mathcal{D}) \longrightarrow \mathcal{F}un(\mathcal{C}_1, \mathcal{D}) \tag{1.40}$$

is *left* adjoint to $- \circ f$, as $\mathcal{F}un(-, \mathcal{D})$ is contravariant.

1.2. **Proof in the complex case.** In this subsection, we first prove part (2) of Theorem 1.35: the compatibility between the Atiyah–Hirzebruch and p-primary Adams filtrations for the Adams summand ℓ. When $p = 2$, this specializes to ku, which is why we call this the "complex case".

Recall that the Postnikov 0-truncation map $\tau_{\leq 0} \colon \ell \to H\mathbb{Z}_{(p)}$ realizes $H\mathbb{Z}_{(p)}$ as an E_∞-ℓ-algebra. Therefore, following Baker–Lazarev [BL01], we can consider the $H\mathbb{Z}_{(p)}$-based, ℓ-module Adams spectral sequence. Unlike the usual $H\mathbb{Z}_{(p)}$-based, $\mathbb{S}$-module Adams spectral sequence, this will be easy to compute.

Theorem 1.41. *Let* $\mathrm{Ad}^{\ell, H\mathbb{Z}_{(p)}}_{\bullet}$ *denote the* $H\mathbb{Z}_{(p)}$*-based,* ℓ*-module Adams filtration. Then there is a natural isomorphism* $\widetilde{\Phi}^{\ell} \colon \mathrm{AH}^{\ell}_{\bullet} \simeq \mathrm{Ad}^{\ell, H\mathbb{Z}_{(p)}}_{g^c(p,\bullet)}$ *of functors* $\mathcal{M}od_\ell \to \mathcal{F}i\ell_\ell$.

We believe this is not a new theorem, but we were unable to find a statement or proof in the literature.

Proof. Let M be an ℓ-module. The functors $\mathrm{AH}^{\ell}_{\bullet}$ and $\mathrm{Ad}^{\ell, H\mathbb{Z}_{(p)}}_{g^c(p,\bullet)}$ are both defined by constructing the respective filtrations on ℓ, then levelwise smashing with M; therefore it suffices to construct an equivalence between the filtered ℓ-modules $\mathrm{AH}^{\ell}_{\bullet}(\ell)$ and $\mathrm{Ad}^{\ell, H\mathbb{Z}_{(p)}}_{g^c(p,\bullet)}(\ell)$. We will describe these filtrations and see that they are equivalent.

Recall from Proposition 1.29 that $\ell_* \cong \mathbb{Z}_{(p)}[v_1]$ with $|v_1| = 2p-2$. There is an ℓ-module equivalence $\Sigma^{2p-2}\ell \simeq \tau_{\geq (2p-2)}\ell$ (this is Bott periodicity if $p = 2$), and if $n \not\equiv 0 \bmod 2p-2$, $\pi_n(\ell) = 0$. Therefore the Atiyah–Hirzebruch filtration is easy to describe.

- If $n \not\equiv 0 \bmod 2p-2$, then $\tau_{\geq n} \colon \tau_{\geq n}\ell \to \tau_{\geq (n-1)}\ell$ is an ℓ-module equivalence.
- If $n \equiv 0 \bmod 2p-2$, then $\tau_{\geq n} \colon \tau_{\geq n}\ell \to \tau_{\geq (n-1)}\ell$ can be identified via the above equivalences with $\tau_{\geq n} \colon \tau_{\geq n}\ell \to \tau_{\geq (n-2p-2)}\ell$, which by v_1-periodicity is $v_1 \colon \Sigma^n \ell \to \Sigma^{n-2p+2}\ell$.

The $H\mathbb{Z}_{(p)}$-based Adams filtration is not much harder. The zeroth piece is ℓ, by definition, and the first is the fiber of $\tau_{\leq 0} \colon \ell \to H\mathbb{Z}_{(p)}$. By inspecting homotopy groups, the fiber is the Postnikov $(2p-3)$-connected cover, which, as mentioned above, is the map $v_1 \colon \Sigma^{2p-2}\ell \to \ell$. Thus, the first piece of the filtration is $\Sigma^{2p-2}\ell$ and the whole story repeats to imply the n^{th} step of the filtration is $v_1 \colon \Sigma^{n(2p-2)}\ell \to \Sigma^{(n-1)(2p-2)}\ell$. Thus, after reindexing the filtration with $g^c(p, \bullet)$, we obtain the Atiyah–Hirzebruch filtration. □

Proof of Theorem 1.35, part (2). For an E_∞-ring spectrum R and a map of E_∞-R-algebra spectra $\psi \colon S_1 \to S_2$, Baker–Lazarev's construction of the S_i-based, R-module Adams filtration is manifestly functorial in the bases S_i. Apply this with $R = \ell$, $S_1 = H\mathbb{Z}_{(p)}$, $S_2 = H\mathbb{Z}/p$, and $\psi = (\operatorname{mod} p)$ to obtain a natural transformation

$$\Phi^\ell \colon \mathrm{AH}^\ell_\bullet \xrightarrow[(1.41)]{\widetilde{\Phi}^\ell} \mathrm{Ad}^{\ell, H\mathbb{Z}_{(p)}}_{g^c(p,\bullet)} \xrightarrow{(\operatorname{mod} p)_*} \mathrm{Ad}^{\ell}_{g^c(p,\bullet)}. \tag{1.42}$$

□

Remark 1.43. Theorem 1.41 also implies that the $H\mathbb{Z}_{(p)}$-based ℓ-module Adams spectral sequence for $\ell \wedge X$ has a tractable E_2-page. This may be surprising, as the analogous fact over the sphere spectrum is well-known to be false (see for example [BP25, §1.1]), and when $p = 2$ it is also false over ko (see below).

One can verify that the E_2-page of the $H\mathbb{Z}_{(p)}$-based ℓ-module Adams spectral sequence is tractable directly in the standard way, by showing that the algebra $\pi_{-*}\mathrm{Map}_\ell(H\mathbb{Z}_{(p)}, H\mathbb{Z}_{(p)})$ of ℓ-module $H\mathbb{Z}_{(p)}$-cohomology operations is a flat $\mathbb{Z}_{(p)}$-module (see Baker–Lazarev [BL01, Theorem 2.1], who remark that their proof follows that of Adams [Ada74, Theorem III.15.1]). To check this, use the universal coefficient spectral sequence [EKMM97, Theorem IV.4.1]

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathbb{Z}_{(p)}[v_1]}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \Longrightarrow \pi_{-*}\mathrm{Map}_\ell(H\mathbb{Z}_{(p)}, H\mathbb{Z}_{(p)}). \tag{1.44}$$

Using Koszul duality,

$$\mathrm{Ext}^{*,*}_{\mathbb{Z}_{(p)}[v_1]}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[\varpi]/(\varpi^2), \qquad \varpi \in \mathrm{Ext}^{1, 2p^n - 2}, \tag{1.45}$$

so the spectral sequence (1.44) collapses, implying the algebra of ℓ-module $H\mathbb{Z}_{(p)}$ cohomology operations is exterior on a single generator, hence a free $\mathbb{Z}_{(p)}$-module as promised.

The story does not generalize to 2-local ko: Baker [Bak20, Example 5.5] exhibits torsion in the algebra $\pi_{-*}\mathrm{Map}_{ko_{(2)}}(H\mathbb{Z}_{(2)}, H\mathbb{Z}_{(2)})$.

Remark 1.46. Let $k(n)$ denote the connective Morava K-theory spectrum at the implicit prime p; then $\pi_*(k(n)) \cong \mathbb{Z}/p[v_n]$, with $|v_n| = 2p^n - 2$, and the fiber of the Postnikov 0-truncation map $\tau_{\leq 0} \colon k(n) \to H\mathbb{Z}/p$ can be identified with $v_n \colon \Sigma^{2p^n-2}k(n) \to k(n)$, in a higher analogue of Bott periodicity. Thus the proof of Theorem 1.41 generalizes straightforwardly to prove that the Atiyah–Hirzebruch and $H\mathbb{Z}/p$-Adams filtrations of a $k(n)$-module are naturally equivalent, hence that the induced Atiyah–Hirzebruch and $H\mathbb{Z}/p$-Adams spectral sequences are isomorphic after a regrading. This is a known theorem: see the discussion in [KL20], which inspired our proof of Theorem 1.41.

1.3. **Proof in the real case.** In this subsection, we finish the proof of Theorem 1.35 by proving the case of ko at the prime 2. We will prove this as a corollary of a more general result.

Definition 1.47. Let R be a connective E_∞-ring spectrum and S be an E_∞-R-algebra spectrum. We call R *rising with respect to* S if for all $n > 0$, either $\pi_n(R) = 0$ or, after applying $- \wedge_R S$, the Postnikov cover map $\tau_{\geq n} \colon \tau_{\geq n}R \to \tau_{\geq (n-1)}R$ is null-homotopic as an S-module map.

Proposition 1.48. *ko is rising with respect to $H\mathbb{Z}/2$.*

The *rising* property is essential in our proof in the following sense.

Theorem 1.49. *Let R be a connective E_∞-ring spectrum and S be an E_∞-R-algebra spectrum. If R is rising with respect to S, there is a natural transformation of functors $\mathcal{M}od_R \to \mathcal{F}i\ell_R$*

$$\Phi^R \colon \mathrm{AH}_\bullet^R \Longrightarrow \mathrm{Ad}^{R,S}_{g^R(\bullet)} \tag{1.50}$$

which is the S-nilpotent completion [Bou79, §5] *on abutments, where $g^R \colon \mathbb{Z} \to \mathbb{Z}$ is the degree of the n^{th} nonzero homotopy group of R, given in Definition 1.33.*

When $S = H\mathbb{Z}/p$, the $H\mathbb{Z}/p$-nilpotent completion is the usual p-completion [Rav84, Example 1.16].

Lemma 1.51 (Baker [Bak20, §5])**.**

(0) $H^*_{ko}(\tau_{\geq 0}ko) = H^*_{ko}(ko) \cong \mathbb{Z}/2$. *Thus by Bott periodicity,* $H^*_{ko}(\tau_{\geq 8}ko) \cong \Sigma^8\mathbb{Z}/2$.
(1) $H^*_{ko}(\tau_{\geq 1}ko) \cong \Sigma Q$, *where* $Q \coloneqq \mathcal{A}(1)/(\mathrm{Sq}^2)$.
(2) $H^*_{ko}(\tau_{\geq 2}ko) \cong \Sigma^2 J$, *where* $J \coloneqq \mathcal{A}(1)/(\mathrm{Sq}^3)$. *Since* $\pi_3(ko) = 0$, *this is also true for* $\tau_{\geq 3}ko$.
(4) $H^*_{ko}(\tau_{\geq 4}ko) \cong \Sigma^4\mathring{O}$, *where* $\mathring{O} \coloneqq \mathcal{A}(1)/(\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^3)$. *Since* $\pi_k(ko)$ *vanishes for* $k = 5, 6, 7$, *this is also true for* $\tau_{\geq k}ko$ *for those* k.

Thus, in particular, for all $n \geq 0$, $H^*_{ko}(\tau_{\geq n}ko)$ is a cyclic $\mathcal{A}(1)$-module.

See Figures 4a, 4e, 4f and 4g for pictures of these $\mathcal{A}(1)$-modules.

We will need the following analogue of the universal coefficient theorem.

Lemma 1.52 (Relative universal coefficient theorem)**.** *Let R be an E_∞-ring spectrum with an E_∞-ring map $\theta \colon R \to H\mathbb{Z}/p$. Then for an R-module M, there is a natural isomorphism*

$$H^*_R(M) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}/p}(H^R_*(M), \mathbb{Z}/p). \tag{1.53}$$

Like for the ordinary universal coefficient theorem, naturality implies (1.53) is an isomorphism of H^*_RH-modules, where the H^*_RH-module structure on $\mathrm{Hom}_{\mathbb{Z}/p}(H^R_*(M), \mathbb{Z}/p)$ is the one defined to be dual to the H^R_*H-comodule structure on $H^R_*(M)$.

Proof. The universal coefficient spectral sequence [EKMM97, Theorem IV.4.1]

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\pi_*(H\mathbb{Z}/p)}(H^R_*(M), \pi_*(H\mathbb{Z}/p)) \Longrightarrow H^*_R(M) \tag{1.54}$$

(see also [BL01, (3.1)]) is concentrated in degree 0 because $\pi_*(H\mathbb{Z}/p)$ is a field. Thus it collapses to imply the isomorphism in the lemma statement. □

The proof of the following lemma is likewise straightforward from the relative Künneth spectral sequence [EKMM97, Theorem 4.1].

Lemma 1.55 (Relative Künneth formula)**.** *With p, R, and M as in Lemma 1.52, and N another R-module, there is a natural isomorphism*

$$H^*_R(M \wedge_R N) \xrightarrow{\cong} H^*_R(M) \otimes_{\mathbb{Z}/p} H^*_R(N). \tag{1.56}$$

Using the Hopf algebra structure on H^*_RH, (1.56) upgrades to an H^*_RH-module isomorphism.

Proof of Proposition 1.48. We will show that $\tau_{\geq n}$ is the zero map on H^{ko}_*, which implies that $\mathrm{id}_{H\mathbb{Z}/2} \wedge_{ko} \tau_{\geq n} \simeq 0$, because a map of $H\mathbb{Z}/2$-modules is determined up to homotopy by its behavior on homotopy groups. By Lemma 1.52 we may equivalently use H^*_{ko} instead of H^{ko}_*. Real Bott periodicity

produces a ko-module equivalence $\Sigma^8 ko \simeq \tau_{\geq 8} ko$ which is commutative with respect to the multiplication by the Bott class; thus for all $k \geq 0$ and $n > 0$, $\tau_{\geq(8k+n)} \colon \tau_{\geq(8k+n)} ko \to \tau_{\geq(8k+n-1)} ko$ is Σ^{8k} of $\tau_{\geq n} \colon \tau_{\geq n} ko \to \tau_{\geq(n-1)} ko$. Therefore, it suffices to prove the theorem for $n \leq 8$. And since $\pi_n ko$ vanishes for $n = 3, 5, 6, 7$, we need only consider the four cases $n = 1, 2, 4, 8$.

Recall from Lemma 1.51 the $\mathcal{A}(1)$-modules $H^*_{ko}(\tau_{\geq n} ko)$ for these values of n. All four of these $\mathcal{A}(1)$-modules are cyclic, so the $\mathcal{A}(1)$-module map $(\tau_{\geq n})^* \colon H^*_{ko}(\tau_{\geq(n-1)} ko) \to H^*_{ko}(\tau_{\geq n} ko)$ is determined by its value on the generator of the domain. But in all four cases we need, namely $n = 1, 2, 4, 8$, $H^{n-1}_{ko}(\tau_{\geq n} ko) = 0$, so the map must vanish. □

Proof of Theorem 1.49. As in the proof of Theorem 1.41, it suffices to construct a map between the respective filtrations of R itself, as the filtrations on a more general R-module M are defined by smashing the filtrations on R levelwise with M.

We induct on the filtration degree n; the base case is the observation that in degree 0, both filtrations are just R. Thus, assume that we have constructed a map $\phi_{n-1} \colon \tau_{\geq(n-1)} R \to \mathrm{Ad}^{R,S}_{g^R(n-1)}(R)$. We would like to construct a map $\phi_n \colon \tau_{\geq n} R \to \mathrm{Ad}^{R,S}_{g^R(n)}(R)$ such that the following diagram commutes.

$$\begin{array}{ccc} \tau_{\geq n} R & \xrightarrow{\phi_n} & \mathrm{Ad}^{R,S}_{g^R(n)}(R) \\ \downarrow{\scriptstyle \tau_{\geq n}} & & \downarrow{\scriptstyle \alpha} \\ \tau_{\geq(n-1)} R & \xrightarrow{\phi_{n-1}} & \mathrm{Ad}^{R,S}_{g^R(n-1)}(R). \end{array} \tag{1.57}$$

If $\pi_n(R) \cong 0$, this is vacuous: $g^R(n) = g^R(n-1)$, so the vertical maps in (1.57) are both the identity, up to the natural isomorphism in the Postnikov cover. Thus, assume $\pi_n(R) \ncong 0$. In this case, to show that a map ϕ_n as desired exists, we could equivalently show that $\phi_{n-1} \circ \tau_n$ vanishes after mapping to the cofiber of α, which is by definition $S \wedge_R \mathrm{Ad}^{R,S}_{g^R(n-1)} R$. This is base change along $R \to S$, hence also factors through the base changes of $\tau_{\geq n}$ and ϕ_{n-1} along $R \to S$:

$$\begin{array}{ccccc} & & \tau_{\geq n} R & \xrightarrow{\phi_n} & \mathrm{Ad}^{R,S}_{g^R(n)}(R) \\ & \swarrow{\scriptstyle \mathrm{id} \wedge_R S} & \downarrow{\scriptstyle \tau_{\geq n}} & & \downarrow{\scriptstyle \alpha} \\ S \wedge_R \tau_{\geq n} R & & \tau_{\geq(n-1)} R & \xrightarrow{\phi_{n-1}} & \mathrm{Ad}^{R,S}_{g^R(n-1)}(R) \\ & \searrow{\scriptstyle 0} & \downarrow{\scriptstyle \mathrm{id} \wedge_R S} & & \downarrow{\scriptstyle \mathrm{cofib}(\alpha)} \\ & & S \wedge_R \tau_{\geq(n-1)} R & \xrightarrow[\mathrm{id}_S \wedge \phi_{n-1}]{} & S \wedge_R \mathrm{Ad}^{R,S}_{g^R(n-1)}(R) \end{array} \tag{1.58}$$

Since R is rising with respect to S, $\mathrm{id}_S \wedge_R \tau_{\geq n} \simeq 0$, so the map to the cofiber vanishes, allowing us to choose a lift ϕ_n as desired. □

One can prove that ℓ is rising with respect to $H\mathbb{Z}/p$ in essentially the same way as for ko, giving another proof of the complex case of Theorem 1.35.

2. The hastened Adams spectral sequence

The main result of this section is Theorem 2.26, which calculates the E_2-page of the HASS for all Postnikov truncations $\tau_{\leq n} ko$ of ko. As we discuss in Definition 3.3, $\tau_{\leq 2} ko$ is Pontrjagin dual to supercohomology, so it is straightforward to infer twisted supercohomology groups from

the $\tau_{\leq 2}ko$-homology HASS (Lemma 3.4). For this reason, whenever we refer to the "HASS for supercohomology," we mean the $\tau_{\leq 2}ko$-homology HASS, leaving the duality implicit.

As an appetizer for why we use the HASS, in §2.1 we explain in an example the complications due to differentials that arise when looking at truncations of ko. In §2.2 we introduce the concept of a hastened Adams resolution. From this resolution we obtain a corresponding HASS, and we then present the form of its E_2-page.

2.1. **The Baker–Lazarev Adams spectral sequence for supercohomology.** In §1.1, we introduced the Baker–Lazarev Adams filtration (Definition 1.8) for ko-modules and briefly discussed the induced spectral sequence. In this subsection, we work through this spectral sequence for the $\tau_{\leq 2}ko$-homology of a point. As mentioned above, and as we discuss in more detail in Lemma 3.4, this is Pontrjagin dual to $SH^*(\mathrm{pt})$.

We discussed the Baker–Lazarev Adams spectral sequence in the abstract in §1.1, so let us start by making it more concrete. Combining Theorem 1.23 and Lemma 1.31, we learn that the E_2-page of the Baker–Lazarev Adams spectral sequence for a ko-module M is

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{Z}/2, H^{ko}_*(M)), \tag{2.1a}$$

since $\mathcal{A}(1)_* \cong H^{ko}_* H$. We find it more intuitive to dualize and work with $H^*_{ko}H \cong \mathcal{A}(1)$ and the $\mathcal{A}(1)$-module structure on $H^*_{ko}(M)$ rather than the $\mathcal{A}(1)_*$-comodule structure on $H^{ko}_*(M)$. If M is finite type, as is the case for all examples in this paper and in [DYY25, DYY26b], we can invoke Lemma 1.52 to do this: see [Ada69, Sal26]. Thus in this case the E_2-page is

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_{ko}(M), \mathbb{Z}/2). \tag{2.1b}$$

In this form, the ko-module Baker–Lazarev Adams spectral sequence may be familiar from prior work employing it, such as in [DY23a, Kur25, CDDH26, DKS26, DYY26a]; more generally, as discussed in [DY23a, §2.1], if $M = ko \wedge X$, (2.1b) coincides with the E_2-page of the usual Adams spectral sequence for M via the change-of-rings theorem, and there are many examples of such computations in the literature, including but not limited to [ABP69, Gia73a, Gia73b, Wil73, Gia76, Per78, Bay94, Sto94, Yu95, BB96, Fra11, Pea14, Cam17, Ped17, BC18, Guo18, WW19, DL20, GOP+20, KPMT20, BR21, Deb21, DL21a, DL21b, FH21, LOT21, LT21, LY22, DL23, BGHR24, DDHM24, BDD+25, DW25, Kne25, She25, Dav26, GLWZ26, KKSW26]. We point the reader to Beaudry–Campbell's guide [BC18] for an excellent introduction to this spectral sequence; in what follows we will assume familiarity with the material in that paper.

For $M = ko$, $H^*_{ko}(ko) \cong \mathbb{Z}/2$ (Lemma 1.51), and we get the following E_2-page.

Theorem 2.2 (Liulevicius [Liu62, Theorem 3])**.** *There is an isomorphism of bigraded $\mathbb{Z}/2$-algebras*

$$\mathbb{E} \coloneqq \mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, h_1, v, w]/(h_0h_1, h_1^3, vh_1, h_0^2w + v^2), \tag{2.3}$$

where $h_0 \in \mathrm{Ext}^{1,1}$, $h_1 \in \mathrm{Ext}^{1,2}$, $v \in \mathrm{Ext}^{3,7}$, and $w \in \mathrm{Ext}^{4,12}$.

The algebra $\mathbb{E}$ acts naturally on all Baker–Lazarev Adams spectral sequences over the ring ko, as the filtering of the ko-action on any ko-module. The class $h_0 \in \mathbb{E}$ detects products with $2 \in ko_0$, h_1 detects $\eta \in ko_1$, v detects $\alpha \in ko_4$, and w detects the Bott class $\beta \in ko_8$ (see Proposition 1.29).

We will also repeatedly use the following computation of $\mathrm{Ext}_{\mathcal{A}(1)}(\mathring{\mathcal{O}})$, where $\mathring{\mathcal{O}}$ is the $\mathcal{A}(1)$-module we defined in Lemma 1.51, ultimately because $\tau_{\leq 2}ko$ is the cofiber of the Postnikov cover map $\tau_{\geq 4}ko \to ko$ and $H^*_{ko}(\tau_{\geq 4}ko) \cong \Sigma^4\mathring{\mathcal{O}}$ (Lemma 1.51).

Lemma 2.4 ([BR21, Figure 2.5])**.** *There is an $\mathbb{E}$-module isomorphism*[9]

$$\operatorname{Ext}_{\mathcal{A}(1)}(\mathring{\mathcal{O}}) \cong \mathbb{E}\{\delta, \epsilon\}/(h_1\delta, v\delta + h_0^2\epsilon, w\delta + v\epsilon), \tag{2.5}$$

with $\delta \in \operatorname{Ext}^{0,0}$ and $\epsilon \in \operatorname{Ext}^{1,5}$.

We will use this to show:

Proposition 2.6.

(1) *The E_2-page of the Baker–Lazarev ko-module Adams spectral sequence for $\tau_{\leq 2}ko$ is isomorphic as $\mathbb{E}$-modules to*

$$\mathbb{E} \cdot 1 \oplus \mathbb{E} \cdot \{\delta, \epsilon\}/(h_1\delta, v\delta + h_0^2\epsilon, w\delta + v\epsilon), \tag{2.7}$$

where $1 \in \operatorname{Ext}^{0,0}$, $\delta \in \operatorname{Ext}^{0,5}$, and $\epsilon \in \operatorname{Ext}^{1,10}$.

(2) *If $r \geq 2$ and $r \neq 3$, $d_r = 0$. On the E_3-page, $d_3(1) = 0$, $d_3(\delta) = v \cdot 1$, and $d_3(\epsilon) = w \cdot 1$.*

The classes δ and ϵ are responsible for the new d_3-differentials that are absent in the ordinary ko-Adams spectral sequence. This is a serious problem for our goal of computing $(\tau_{\leq 2}ko)_*(BG, a, b)$ for various groups G and twisting data (a, b): if there are differentials in the Baker–Lazarev Adams spectral sequence for $ko_*(BG, a, b)$, they interact with the d_3s we found in Proposition 2.6 in ways that significantly complicate the computation. There are plenty of small groups G and twisting data (a, b) such that the Adams spectral sequence for $ko_*(BG, a, b)$ has differentials in low degrees, including several examples in [Bay94, BB96, Fra11, Cam17, Ped17, GOP+20, Deb21, DL21a, DGL22, LY22, BGHR24, DDHM24, BDD+25, DYY25, Kne25, Dha26].

We turn to the HASS in this section to solve this problem. In particular, we will modify this family of Adams spectral sequences in a way that cleanly separates the differentials coming from $ko_*(BG, a, b)$ from those differentials coming from $\tau_{\leq 2}ko$. The resulting spectral sequences will be simpler in general.

The broad outline of Proposition 2.6, and its proof, applies in the same way to $\tau_{\leq n}ko$ for $n > 1$, as well as (slightly more broadly) to $\tau_{\leq n}\ell$. As a first step, we must calculate $H^*_{ko}(\tau_{\leq 2}ko)$.

Lemma 2.8. *There is an $\mathcal{A}(1)$-module isomorphism $H^*_{ko}(\tau_{\leq 2}ko) \cong \mathbb{Z}/2 \oplus \Sigma^5\mathring{\mathcal{O}}$.*

Proof. Essentially by definition, $H^*_{ko}(ko) \cong \mathbb{Z}/2$, which necessarily has trivial $\mathcal{A}(1)$-action. By Lemma 1.51, there is an isomorphism of $\mathcal{A}(1)$-modules $H^*_{ko}(\tau_{\geq 3}ko) \cong \Sigma^4\mathring{\mathcal{O}}$. The cofiber sequence

$$\tau_{\geq 3}ko \longrightarrow ko \xrightarrow{\tau_{\leq 2}} \tau_{\leq 2}ko \tag{2.9}$$

induces a long exact sequence on H^*_{ko}; exactness uniquely characterizes the $\mathcal{A}(1)$-module structure on $H^*_{ko}(\tau_{\leq 2}ko)$. □

Proof of Proposition 2.6. Lemma 2.8 implies that to prove part (1) of the proposition statement, it suffices to compute $\operatorname{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2 \oplus \Sigma^5\mathring{\mathcal{O}})$ as an $\mathbb{E}$-module. Ext commutes with direct sums, and $\operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(\Sigma^n M) \cong \operatorname{Ext}^{s,t+n}_{\mathcal{A}(1)}(M)$ for an $\mathcal{A}(1)$-module M, so to compute the E_2-page of the spectral sequence, it suffices to know $\operatorname{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2)$ and $\operatorname{Ext}_{\mathcal{A}(1)}(\mathring{\mathcal{O}})$, which we gave in Theorem 2.2 and Lemma 2.4 respectively.

On to part (2). For all $r \geq 2$, $d_r(1) = 0$ for degree reasons; since differentials commute with the $\mathbb{E}$-action on the E_r-page, to figure out the rest of the spectral sequence all we have to do is compute $d_r(\delta)$ and $d_r(\epsilon)$.

[9] Adams–Priddy [AP76, §3] compute $\operatorname{Ext}(\mathring{\mathcal{O}})$ as a bigraded vector space, and the $\mathbb{E}$-module structure is implicit in their method of proof.

This spectral sequence converges to $\pi_*(\tau_{\leq 2}ko)^\wedge_2$, so $\mathbb{Z}_2$ in degree 0, $\mathbb{Z}/2$ in degrees 1 and 2, and 0 in all other degrees. In particular, all classes with $t-s \geq 2$ must vanish before the E_∞-page. This uniquely determines all differentials: the class $v \in E_2^{3,7}$ must be in the image of a differential, but the only way for this to happen is for $d_3(\delta) = v$. $\mathbb{E}$-linearity of differentials then implies $d_3(v\delta) = v^2$; since $v\delta = h_0^2\epsilon$ and $v^2 = h_0^2 w$, if $d_3(\epsilon) = x$ then $h_0^2 x = h_0^2 w$. The map $h_0^2 \colon E_2^{4,12} \to E_2^{6,14}$ is injective, so $x = w$. To summarize, the spectral sequence is completely determined by the $\mathbb{E}$-action and the differentials $d_3(\delta) = v$, $d_3(\epsilon) = w$, and $d_r(1) = 0$ for all r. We draw the E_3-page with these differentials in Figure 1, left, and the $E_4 = E_\infty$-page in Figure 1, right. □

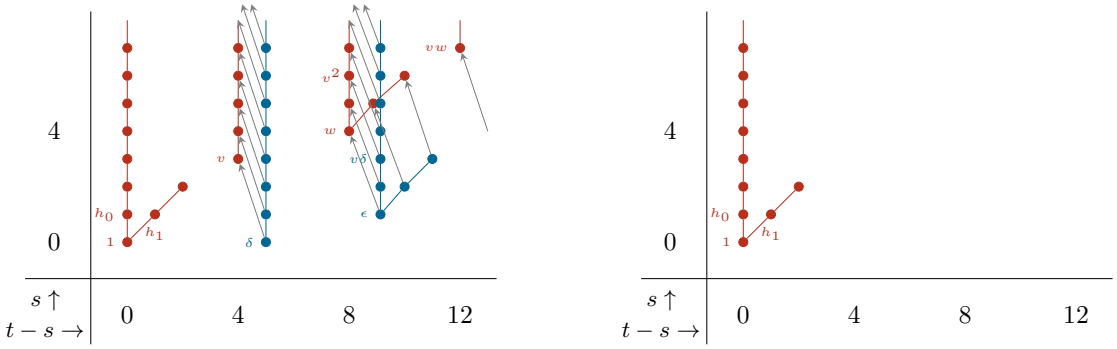


FIGURE 1. Left: the $E_2 = E_3$-page of the Baker–Lazarev Adams spectral sequence for $\tau_{\leq 2}ko$, from Proposition 2.6. Right: the $E_4 = E_\infty$-page.

2.2. **Hastened Adams spectral sequences.** The computation in Proposition 2.6 suggests that the ordinary Baker–Lazarev Adams filtration is not well adapted to truncated theories. Given an R-module map $f \colon M \to N$, let Cf denote the cofiber of f. We will use the idea of Behrens–Hill–Hopkins–Mahowald [BHHM08] to modify the Adams resolution on Cf so that the information coming from the truncation is directly input into the resolution. This will be done in Definition 2.10. The resulting spectral sequence converges to $\pi_*(Cf)^\wedge_2$, and we calculate the E_2-page in Theorem 2.18.

Let R be an E_∞-ring spectrum with an E_∞-ring map $R \to H\mathbb{Z}/p$ for some prime p. Recall from Definition 1.8 the notation $\mathrm{Ad}^R_\bullet(M)$ for the $H\mathbb{Z}/p$-based, R-module Adams filtration on an R-module M. For $s \geq 0$, we will let $\alpha_M \colon \mathrm{Ad}^R_{s+1}(M) \to \mathrm{Ad}^R_s(M)$ denote the map between adjacent entries in this filtration. The values of s and p will be clear from context when needed, and if M is clear from context, we will abbreviate α_M by α.

Definition 2.10 (Behrens–Hill–Hopkins–Mahowald [BHHM08, §3], Bruner–Rognes [BR21, Definition 12.31])**.** Given an R-module map $f \colon M \to N$, let Cf denote the cofiber of f. Suppose that there exists a lift $g \colon M \to \mathrm{Ad}^R_m(N)$ of f, i.e. there is a homotopy of R-module maps $\alpha^m_N \circ g \simeq f$. The *hastened Adams resolution* of Cf is the filtered R-module $(W_\bullet, \phi_\bullet)$ defined by

$$W_s \coloneqq \begin{cases} \mathrm{cofib}(\alpha_N^{m-s} \circ g), & s \leq m \\ \mathrm{Ad}^R_s(Cf), & s \geq m \end{cases}. \tag{2.11}$$

For $s = m$ these cases agree.

The map ϕ_s equals α_{Cf} for $s \geq m$; for $s \leq m$, define ϕ_s to be the map $C(\alpha_N^{m-s} \circ g) \to C(\alpha_N^{m-s-1} \circ g)$ induced by α_N. (See [BR21, (12.6)].)

A priori, the hastened Adams resolution depends on both f and g; in fact, the dependence on g is very weak, but this construction very much depends on f.

Definition 2.12 (Hastened Adams spectral sequence [BHHM08, Proposition 3.1], [BR21, Definition 12.31])**.** The *hastened Adams spectral sequence* for Cf is the spectral sequence associated to the hastened Adams resolution.

Let $\mathcal{D}(H^R_*H)$ denote the derived ∞-category of graded H^R_*H-comodules; the objects of this ∞-category are chain complexes of graded H^R_*H-comodules, and the homotopy equivalences are the grading-preserving quasi-isomorphisms. For an object $L_\bullet$ of $\mathcal{D}(H^R_*H)$, we let $\Sigma^t L_\bullet$ denote the chain complex with each entry $(\Sigma^t L)_i$ equal to the graded H^R_*H-module $\Sigma^t(L_i)$, and let $L_\bullet[s]$ denote the shift with respect to the chain complex grading.

Lemma 2.13. *The map $\alpha^m\colon \mathrm{Ad}^R_m(N) \to N$ induces a natural homotopy class of morphisms in $\mathcal{D}(H^R_*H)$ of the form*

$$\underline{\alpha}^m\colon \Sigma^m H^R_*(\mathrm{Ad}^R_m(N))[-m] \longrightarrow H^R_*(N). \tag{2.14}$$

Proof. We prove the lemma for the case $m = 1$ and then iterate to get the general case. Consider the long exact sequence on H^R_* induced by the fiber sequence $\mathrm{Ad}^R_1(N) \to N \to N \wedge_R H$:

$$\cdots \to H^R_{k+1}(\mathrm{Ad}^R_1(N)) \xrightarrow{0} H^R_{k+1}(N) \to H^R_{k+1}(N \wedge_R H) \xrightarrow{\partial} H^R_k(\mathrm{Ad}^R_1(N)) \xrightarrow{0} H^R_k(N) \to \cdots \tag{2.15a}$$

Since $\mathrm{Ad}^R_1(N) \to N$ is the zero map on H^R_*, (2.15a) degenerates into a short exact sequence of the form

$$0 \longrightarrow H^R_{*+1}(N) \longrightarrow H^R_{*+1}(N \wedge_R H) \longrightarrow H^R_*(\mathrm{Ad}^R_1(N)) \longrightarrow 0. \tag{2.15b}$$

That is, we have produced an H^R_*H-comodule extension of $H^R_*(\mathrm{Ad}^R_1(N))$ by $H^R_{*+1}(N) \cong \Sigma^{-1}H^R_*(N)$, or equivalently of $\Sigma H^R_*(\mathrm{Ad}^R_1(N))$ by $H^R_*(N)$, which is classified up to homotopy by its class in

$$\mathrm{Ext}^1_{H^R_*H}(\Sigma H^R_*(\mathrm{Ad}^R_1(N)), H^R_*(N)) \cong \mathrm{Hom}_{\mathcal{D}(H^R_*H)}(\Sigma H^R_*(\mathrm{Ad}^R_1(N))[-1], H^R_*(N)). \tag{2.16}$$

□

At this point, assume $g_*\colon H^R_*(M) \to H^R_*(\mathrm{Ad}^R_m(N))$ is injective. Consider the map $\underline{g} \coloneqq \underline{\alpha}^m \circ g$, which is a morphism in $\mathcal{D}(H^R_*H)$ of the form

$$\underline{g} \coloneqq \underline{\alpha}^m \circ g\colon \Sigma^m H^R_*(M)[-m] \longrightarrow H^R_*(N). \tag{2.17}$$

Theorem 2.18 (Behrens–Hill–Hopkins–Mahowald [BHHM08, §3], Bruner–Rognes [BR21, Proposition 12.33])**.** *Assume that $g_*\colon H^R_*(M) \to H^R_*(\mathrm{Ad}^R_m(N))$ is injective and let $\mathcal{Q}_\bullet$ denote the cofiber in $\mathcal{D}(H^R_*H)$ of the map $\underline{g}$ from (2.17). Then, the E_2-page of the hastened Adams spectral sequence from Definition 2.12 is*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{D}(H^R_*H)}(\mathbb{Z}/p, \mathcal{Q}_\bullet), \tag{2.19}$$

and this spectral sequence converges strongly to $\pi_*(Cf)^\wedge_p$ with the R-module Adams filtration.

As always, with finite type hypotheses, we may pass to cohomology. The derived ∞-category of H^*_RH-modules, $\mathcal{D}(H^*_RH)$, is defined analogously to $\mathcal{D}(H^R_*H)$. Analogously to the map in (2.17), we get a morphism in $\mathcal{D}(H^*_RH)$

$$\overline{g}\colon H^*_R(N) \longrightarrow \Sigma^{-m} H^*_R(M)[m]. \tag{2.20}$$

Corollary 2.21. *Assume M and N are finite type and that $g^*\colon H^*_R(N) \to H^*_R(M)$ is surjective. Let $\mathcal{Q}^\bullet$ denote the cofiber in $\mathcal{D}(H^*_R H)$ of the map $\overline{g}$ from (2.20); then the E_2-page of the hastened Adams spectral sequence from Definition 2.12 is*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{D}(H^*_R H)}(\mathcal{Q}^\bullet, \mathbb{Z}/p). \tag{2.22}$$

Remark 2.23 (The synthetic point of view). A great deal of recent work on the Adams spectral sequence uses what have come to be called "synthetic techniques;" in this remark, we briefly explain the connection to the HASS.

For E an *Adams-type ring spectrum* (see [Pst23b, §1.1] and [Ada74, Condition III.13.3]; this includes $H\mathbb{Z}/p$ for all primes p), Pstrągowski [Pst23b] constructs a symmetric monoidal ∞-category $\mathcal{S}yn_E$ and lax monoidal functors

$$\mathcal{S}p \xrightarrow{\nu} \mathcal{S}yn_E \xrightarrow{\sigma} \mathcal{F}i\ell(\mathcal{S}p) \tag{2.24}$$

whose composite is the E-based, $\mathbb{S}$-module Adams filtration (Definition 1.8). Just as filtered spectra refine spectral sequences, objects of $\mathcal{S}yn_E$ refine and generalize E-based Adams spectral sequences, and $\mathcal{S}yn_E$ provides a useful setting for proving things about these Adams spectral sequences.

The functor ν does not preserve cofiber sequences (see [vN25, Proposition 4.9(2)]). Therefore, if $f\colon X \to Y$ is a map of spectra and $C \in \mathcal{S}yn_E$ is the cofiber of $\nu(f)$, $\sigma(C)$ is a filtered spectrum that, heuristically speaking, ought to correspond to something like an Adams spectral sequence, but in general will not be the Adams spectral sequence for the cofiber Cf of f. Instead, as explained in [LWX24, Remark 3.21] (see also [vN25, §4.3.4] and [CD26, §2.2]), the spectral sequence associated to the filtered spectrum $\sigma(C)$ is precisely the HASS of Cf!

To connect this approach to the HASSes appearing in our paper, one would want a generalization of $\mathcal{S}yn_E$ and its connection to the Adams filtration to a category of synthetic R-module spectra affiliated with Baker–Lazarev's R-module Adams filtration. To our knowledge, this has not appeared in the literature, though we believe it is likely that the approach for $R = \mathbb{S}$ generalizes. Patchkoria–Pstrągowski [PP26] construct synthetic analogues in a very general setting including module spectra.

Suppose such a generalization existed, in the form of a stable ∞-category $\mathcal{S}yn^{ko}_{H\mathbb{Z}/2}$ and lax symmetric monoidal functors $\nu\colon \mathcal{M}od_{ko} \to \mathcal{S}yn^{ko}_{H\mathbb{Z}/2}$ and $\sigma\colon \mathcal{S}yn^{ko}_{H\mathbb{Z}/2} \to \mathcal{F}i\ell_{ko}$ such that $\sigma\circ\nu \simeq \mathrm{Ad}^{ko}$ and, like over $\mathbb{S}$, σ admits a right adjoint ρ.[10] Then it would be possible to lift Theorem 1.35 to synthetic spectra as follows. Use Remark 1.38 to rephrase the AHSS-to-Adams map Φ^{ko} as a natural transformation $(\Phi^{ko})^\vdash\colon \mathrm{AH}^{ko}_{g^\vdash(\bullet)} \Rightarrow \mathrm{Ad}^{ko}_\bullet$, i.e.

$$(\Phi^{ko})^\vdash \in \mathrm{Hom}_{\mathcal{F}un(\mathcal{M}od_{ko}, \mathcal{F}i\ell_{ko})}(\mathrm{AH}^{ko}\circ g^\vdash, \mathrm{Ad}^{ko}). \tag{2.25a}$$

Since $\mathrm{Ad}^{ko} \simeq \sigma\circ\nu$, we may use the $\sigma \dashv \rho$ adjunction to port $(\Phi^{ko})^\vdash$ to a natural transformation

$$\Phi^{ko}_{\mathrm{Syn}} \in \mathrm{Hom}_{\mathcal{F}un(\mathcal{M}od_{ko}, \mathcal{S}yn^{ko}_{H\mathbb{Z}/2})}(\rho(\mathrm{AH}^{ko}\circ g^\vdash), \nu), \tag{2.25b}$$

that is, a natural transformation $\Phi^{ko}_{\mathrm{Syn}}\colon \rho(\mathrm{AH}^{ko}_{g^\vdash(\bullet)}) \Rightarrow \nu$ in the setting of synthetic ko-modules. It would be interesting to give a direct construction of $\widetilde{\Psi}$ and see whether this could provide an

[10] Since ν is lax symmetric monoidal, for any ko-module M, $\nu(M)$ has a natural $\nu(ko)$-module structure, suggesting that $\mathcal{S}yn^{ko}_{H\mathbb{Z}/2}$ could be the ∞-category of $\nu(ko)$-modules in $\mathcal{S}yn_{H\mathbb{Z}/2}$. However, this appears to be the wrong definition: it lifts the action of the Adams spectral sequence for ko on the ordinary Adams spectral sequence for M. In general, the usual and Baker–Lazarev Adams spectral sequences do not coincide, e.g., for $tmf_{(3)}$ as a module over itself: compare Hill's computation [Hil07, §2] of the latter with Culver's computation [Cul21] of the former.

another conceptual explanation for the existence of the HASSes we constructed in this paper, or whether such a construction could shed light on Conjecture 4.4.

2.3. **Applying the HASS to truncations of ko.** In this section, we prove the following main theorem about the constructions of the HASS for truncations of ko. Let $C_k \coloneqq \Sigma^{-k} H^*_{ko}(\tau_{\geq k} ko)$. We describe these $\mathcal{A}(1)$-modules explicitly in Lemma 1.51; in particular, the desuspension by k puts the lowest-degree nonzero element of C_k in degree 0.

Theorem 2.26.

(1) *There is a map of $\mathbb{Z}^2$-graded $\mathrm{Ext}(\mathbb{Z}/2)$-modules*

$$f_{n-1} \colon \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(C_n) \longrightarrow \mathrm{Ext}^{s+g(n),t+g(n)+n}_{\mathcal{A}(1)}(\mathbb{Z}/2), \tag{2.27}$$

which is induced from the Postnikov cover map $\tau_{\geq n} \colon \tau_{\geq n} ko \to ko$.

(2) *There is a functor $\mathcal{Q}^{*,*}_n$ from $\mathcal{A}(1)$-modules to $\mathbb{Z}^2$-graded $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2)$-modules which commutes with direct sums and such that for all $\mathcal{A}(1)$-modules M, there is a long exact sequence*

$$\cdots \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(C_n \otimes M) \xrightarrow{f_{n-1}} \mathrm{Ext}^{s+g(n),t+g(n)+n}_{\mathcal{A}(1)}(M) \xrightarrow{\partial_{n-1}} \mathcal{Q}^{s,t}_n(M) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)}(C_n \otimes M) \xrightarrow{f_{n-1}} \cdots \tag{2.28}$$

(3) *Let M be a finite type, bounded-below ko-module.*[11] *Then the HASS for $\tau_{\leq n} ko \wedge_{ko} M$ converges strongly and has signature*

$$E_2^{s,t} = \mathcal{Q}^{s,t}_n(H^*_{ko}(M)) \Longrightarrow \pi_{t-s}(\tau_{\leq n} ko \wedge_{ko} M)^\wedge_2. \tag{2.29}$$

The map $\pi_*(M) \to \pi_*(\tau_{\leq n} ko \wedge_{ko} M)$ lifts to a map from the ordinary Adams spectral sequence to the HASS which on the E_2-page is the map ∂_n from (2.28).

Notice the shift in n in part (3) – to calculate the E_2-page of the HASS for $\tau_{\leq n} ko \wedge_{ko} M$, we use the map f_n with domain $\mathrm{Ext}(C_{n+1})$!

An analogous result is true for truncations of ℓ at an arbitrary prime, and the proof is identical. In both cases, the proof amounts to verifying the hypotheses in Corollary 2.21 and then making explicit the details of the calculation of the E_2-page of the HASS.

Remark 2.30. The reader may be wondering – where on Earth did $g(n)$ and $n + g(n)$ come from in Theorem 2.26? To answer this question, we first clarify what we mean by "$[f_{n-1}]$ is induced from the Postnikov cover map" in Item 3. Specifically, f_{n-1} is the composition of the following maps.

(1) First, the map obtained by applying $\mathrm{Ext}_{\mathcal{A}(1)}$ to the Atiyah–Hirzebruch-to-Adams map

$$\tau_{\geq n} ko \longrightarrow \mathrm{Ad}^{ko}_{g(n)}(ko) \tag{2.31a}$$

that we built in Theorem 1.35. This map preserves (s,t)-bidegrees of Ext classes.

(2) The second map uses the fact that the Ext of the m^{th} Adams cover of M gives rise to the filtration-m elements of the Adams spectral sequence for M. Specifically, we get a map

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_{ko}(\mathrm{Ad}^{ko}_m(M))) \longrightarrow \mathrm{Ext}^{s+m,t+m}_{\mathcal{A}(1)}(H^*_{ko}(M)). \tag{2.31b}$$

Now we put these together. By definition, $H^*_{ko}(\tau_{\geq n} ko) \cong \Sigma^n C_n$, so we have an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(C_{n+1}) \xrightarrow{\cong} \mathrm{Ext}^{s,t+n}_{\mathcal{A}(1)}(H^*_{ko}(\tau_{\geq n} ko)). \tag{2.32a}$$

[11] The finite-type assumption appears only for technical reasons, and is true in all examples we consider in this paper. Likewise, all examples we consider in this paper are bounded below.

Now apply $\mathrm{Ext}^{s,t+n}_{\mathcal{A}(1)}$ to (2.31a):

$$\mathrm{Ext}^{s,t+n}_{\mathcal{A}(1)}(H^*_{ko}(\tau_{\geq n}ko)) \to \mathrm{Ext}^{s,t+n}_{\mathcal{A}(1)}(H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}ko)). \tag{2.32b}$$

Now apply (2.31b) with $m = g(n)$ and again shifted by n in the t-direction:

$$\mathrm{Ext}^{s,t+n}_{\mathcal{A}(1)}(H^*_{ko}(\mathrm{Ad}^{ko}_{m}(ko))) \longrightarrow \mathrm{Ext}^{s+g(n),t+n+g(n)}_{\mathcal{A}(1)}(H^*_{ko}(ko)), \tag{2.32c}$$

and $H^*_{ko}(ko) \cong \mathbb{Z}/2$ (Lemma 1.51). The map f_{n-1} is the composition of (2.32a), (2.32b), and (2.32c).

The construction of Item 3 follows the discussion in the previous section closely. Specializing to truncations of ko, this reduces to verifying the following two hypotheses in Theorem 2.18, as well as their analogues with ℓ in place of ko.

(1) For all $n \geq 0$, the Postnikov n-connected covering map $\tau_{\geq n}ko \to ko$ lifts to the $g(n)^{\text{th}}$ Adams cover $\mathrm{Ad}^{ko}_{g(n)}(ko)$ of ko.
(2) The induced map $H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko)) \to H^*_{ko}(\tau_{\geq n}ko)$ is surjective.

In fact, we already took care of item (1) in Theorem 1.35. Thus, we will focus on part (2) in this subsection, which we restate as the following proposition.

Proposition 2.33. *Let $\gamma_n \colon \tau_{\geq n}ko \to \mathrm{Ad}^{ko}_{g(n)}(ko)$ be any ko-module map whose composition with the $g(n)$-fold Adams cover $\alpha_n \colon \mathrm{Ad}^{ko}_{g(n)}(ko) \to ko$ is the Postnikov n-connected covering map $\tau_{\geq n} \colon \tau_{\geq n}ko \to ko$. Then $\gamma_n^* \colon H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko)) \to H^*_{ko}(\tau_{\geq n}ko)$ is surjective.*

To reiterate, by Theorem 2.18, this and Theorem 1.35 prove Theorem 2.26.

To get started, we should identify the $\mathcal{A}(1)$-modules that γ_n^* maps between. We have already seen $H^*_{ko}(\tau_{\geq n}ko)$ in Lemma 1.51: if $n = 4k+\ell$, with $0 \leq \ell < 4$, $H^*_{ko}(\tau_{\geq n}ko)$ is isomorphic to $\Sigma^{8k}\mathbb{Z}/2$ for $\ell = 0$, $\Sigma^{8k+1}Q$ for $\ell = 1$, $\Sigma^{8k+2}J$ for $\ell = 2$, and $\Sigma^{8k+4}\tilde{C}$ for $\ell = 3$.

Definition 2.34. Inductively define the *seagull module* Υ_k as follows: let $\Upsilon_1 \coloneqq \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$. Then one can directly verify that, up to isomorphism of extensions, there is a unique nonsplit $\mathcal{A}(1)$-module extension

$$0 \longrightarrow \Sigma^{4k-4}\Upsilon_1 \longrightarrow \Upsilon_k \longrightarrow \Upsilon_{k-1} \longrightarrow 0. \tag{2.35}$$

Adamyk [Ada23] introduced the name "seagulls" for these modules, as well as the seagull-like notation Υ_k; see (*ibid.*, Figure 1) for pictures of Υ_k for some small k.

Definition 2.36 (Milgram [Mil75, Definitions 1.6, 1.8, and 1.10])**.** Let $k \geq 0$ and $j \in \mathbb{Z}/4$ and define the $\mathcal{A}(1)$-module

$$C_{j,k} \coloneqq \begin{cases} \Sigma^{4k}\mathbb{Z}/2, & j = 0 \\ \Sigma^{4k}\tilde{C}, & j = 1 \\ \Sigma^{4k-2}J, & j = 2 \\ \Sigma^{4k-3}Q, & j = 3. \end{cases} \tag{2.37}$$

Then, the reader may verify that, up to isomorphism of extensions, there is a unique nonsplit $\mathcal{A}(1)$-module extension

$$0 \longrightarrow \Upsilon_k \longrightarrow Q_{j,k} \longrightarrow C_{j,k} \longrightarrow 0. \tag{2.38}$$

The middle term of this extension is called a *Milgram module*. We refer to the submodule Υ_n as the *seagull submodule* and the quotient $C_{j,k}$ as the *cyclic quotient* of $Q_{j,k}$.

We draw the Milgram modules, their seagull submodules, and their cyclic quotients in Figure 2.

Proposition 2.39 (Milgram [Mil75, Corollary 1.15])**.** *For all j_1, k_1, j_2, k_2, there is a finite-rank free $\mathcal{A}(1)$-module F_{j_1,k_1,j_2,k_2} and an $\mathcal{A}(1)$-module isomorphism*

$$Q_{j_1,k_1} \otimes Q_{j_2,k_2} \cong Q_{j_1+j_2,k_1+k_2+\delta} \oplus F_{j_1,k_1,j_2,k_2}, \tag{2.40}$$

where $j_1 + j_2$ is interpreted modulo 4 and

$$\delta = \begin{cases} -1 & \text{if } \{3\} \subseteq \{j_1, j_2\} \subseteq \{2, 3\} \\ 1 & \text{if } \{1\} \subseteq \{j_1, j_2\} \subseteq \{1, 2\} \\ 0, & \text{else.} \end{cases} \tag{2.41}$$

Corollary 2.42. *For a positive integer k, write $k = 4\ell + m$ for $m \in \{1, 2, 3, 4\}$ and $\ell \in \mathbb{Z}$. Let $\delta_{m,4} = 1$ if $m = 4$ and $\delta_{m,4} = 0$ otherwise. Then there is a finite-rank, free $\mathcal{A}(1)$-module F_k and an $\mathcal{A}(1)$-module isomorphism*

$$H^*_{ko}(\mathrm{Ad}^{ko}_k(ko)) \cong Q_{4-k,2\ell+\delta_{k,4}+1} \oplus F_k. \tag{2.43}$$

See also [Mil75, Theorem 2.4] and [DGM81, Lemma 3.13], which are closely related.

Proof. Since $\mathrm{Ad}^{ko}_k(ko)$ is a smash product over ko of k copies of $\Sigma^{-1}\overline{H}$, the ko-module Künneth theorem (Lemma 1.55) allows us to induct. The base case is the fact that $H^*_{ko}(\Sigma^{-1}\overline{H}) \cong Q_{3,1}$, which can be verified using the long exact sequence in H^*_{ko} associated to the fiber sequence $\Sigma^{-1}\overline{H} \to ko \to H$. The inductive step is Proposition 2.39, together with the fact that if F is a free $\mathcal{A}(1)$-module and M is any $\mathcal{A}(1)$-module, $F \otimes M$ is free. □

For convenience, we let $\vec{v}(n) \coloneqq (4 - g(n), 2\ell + \delta_{g(n),4} + 1)$, where $g(n) = 4\ell + m$, $\ell \in \mathbb{Z}$, and $1 \leq m \leq 4$. Then $H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko)) \cong Q_{\vec{v}(n)} \oplus F_{g(n)}$.

By directly comparing Lemma 1.51 and Definition 2.36, we observe:

Lemma 2.44. *The cyclic quotient $C_{\vec{v}(n)}$ of $Q_{\vec{v}(n)}$ is* graded *isomorphic to $H^*_{ko}(\tau_{\geq n} ko)$.*

This is not a coincidence.

Lemma 2.45. *For any choice of the isomorphism $H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko)) \cong Q_{\vec{v}(n)} \oplus F_{g(n)}$ from Corollary 2.42 and any grading-preserving $\mathcal{A}(1)$-module map $\gamma^*\colon H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko)) \to H^*_{ko}(\tau_{\geq n} ko)$, the restriction of γ^* to $Q_{\vec{v}(n)}$ is either zero or surjective.*

Proof. Let $\psi(n) \coloneqq 2\ell + \delta_{g(n),4} + 1$, where $\ell = \lfloor g(n)/4 \rfloor$. As an $\mathcal{A}(1)$-module, $\Upsilon_{\psi(n)}$ is generated by classes in degrees $4\psi(n) - 4$ and below. By Lemma 1.51, $H^*_{ko}(\tau_{\geq n} ko)$ is $(4\psi(n) - 4)$-connected, so the restriction of γ to the seagull submodule $\Upsilon_{\psi(n)} \subset Q_{\vec{v}(n)}$ vanishes and $\gamma|_{Q_{\vec{v}(n)}}$ descends to the cyclic quotient $C_{4-g(n),2\ell+\delta_{g(n),4}+1}$. By Lemma 2.44, $C_{\vec{v}(n)} \cong H^*_{ko}(\tau_{\geq n} ko)$, so by inspection the only two options for a grading-preserving $\mathcal{A}(1)$-module map out of $C_{\vec{v}(n)}$ are 0 or an isomorphism. Pulling back to $Q_{\vec{v}(n)}$, these options become 0 or surjectivity. □

By Margolis' theorem for ko-modules [DKS26, Theorem 5.25] (a sharpening of the original Margolis theorem [Mar74]), any $\mathcal{A}(1)$-module splitting of the form (2.43) lifts to a ko-module splitting

$$\mathrm{Ad}^{ko}_{g(n)}(ko) \xrightarrow{p} \overline{\mathrm{Ad}}^{ko}_{g(n)}(ko) \vee M_n, \tag{2.46}$$

where $H^*_{ko}(\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)) \cong Q_{\vec{v}(n)}$, M_n is ko-module equivalent to a sum of shifts of $H\mathbb{Z}/2$, and $H^*_{ko}(M_n) \cong F_{g(n)}$, such that $H^*_{ko}(p)$ is the splitting of $H^*_{ko}(\mathrm{Ad}^{ko}_{g(n)}(ko))$ chosen. Thus, in what

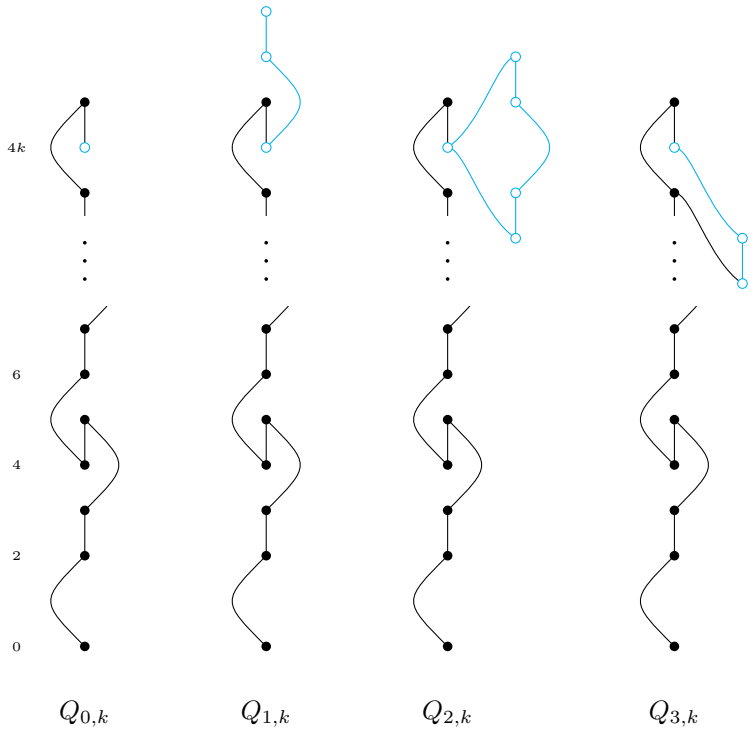


FIGURE 2. The Milgram modules $Q_{j,k}$ (Definition 2.36). Each module is an extension of the cyclic $\mathcal{A}(1)$-module $C_{j,k}$ (in blue, empty vertices) by the seagull module Υ_k (in black, filled vertices).

follows, choose such a splitting, and let p_i denote the projection from $\mathrm{Ad}^{ko}_{g(n)}(ko)$ onto the i^{th} factor of the direct summand.

Lemma 2.47. *Let $c \in \pi_n(ko)$ be nonzero. Then c, interpreted as a ko-module map $c\colon \Sigma^n ko \to ko$, does not factor as a ko-module map through any ko-module M which is ko-module equivalent to a sum of shifts of $H\mathbb{Z}/2$.*

Proof. Since c is a ko-module map, it commutes with the Bott class β, so in particular $\beta c\colon \Sigma^{n+8} ko \to \Sigma^8 ko$ is nonzero on π_* (as, tautologically, all nonzero elements of $\pi_n ko$ induce nonzero maps $\pi_{*+n} ko \to \pi_*(ko)$). However, β is the zero map on $\pi_*(H\mathbb{Z}/2)$, hence also on $\pi_*(M)$, so if c factors through M, βc is the zero map on π_*. And a map $\Sigma^n ko \to ko$ is nonzero only if it is nonzero on homotopy groups. □

Because $\Sigma^n ko$ is $(n-1)$-connected, c factors as a ko-module map as a composition

$$\widetilde{c}\colon \Sigma^n ko \longrightarrow \tau_{\geq n} ko \tag{2.48}$$

followed by the Postnikov cover $\tau_{\geq n}\colon \tau_{\geq n} ko \to ko$. This factorization is unique up to homotopy of ko-module maps. And as we showed in Theorem 1.35, $\tau_{\geq n}$ factors through the Adams cover $\alpha_{g(n)}\colon \mathrm{Ad}^{ko}_{g(n)}(ko) \to ko$.

Corollary 2.49. *The composition*

$$\Sigma^n ko \xrightarrow{\widetilde{c}} \tau_{\geq n} ko \xrightarrow{\gamma_n} \mathrm{Ad}^{ko}_{g(n)}(ko) \xrightarrow{p} \overline{\mathrm{Ad}}^{ko}_{g(n)}(ko) \vee M_n \xrightarrow{\alpha_{g(n)}} ko, \tag{2.50}$$

which is ko-module equivalent to the map c as discussed above, factors though the $\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)$ summand of $\mathrm{Ad}^{ko}_{g(n)}(ko) \simeq \overline{\mathrm{Ad}}^{ko}_{g(n)}(ko) \vee M_n$.

Proof. For $i = 1, 2$, let $f_i \coloneqq p_i \circ \gamma_n \circ \widetilde{c}$, so that $p \circ \gamma_n \circ \widetilde{c} \simeq f_1 \vee f_2$. We wish to prove that $c \simeq \alpha_n|_{\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)} \circ f_1$, so assume this is *not* true. Then $c - \alpha_n|_{\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)} \circ f_1$ is a nontrivial ko-module map $\Sigma^n ko \to ko$ that factors through a sum of shifts of $H\mathbb{Z}/2$, which by Lemma 2.47 cannot happen. □

Lemma 2.51. *For all $k \geq 0$, the classes $\beta^k \in \pi_{8k} ko$, $\beta^k \eta \in \pi_{8k+1} ko$, $\beta^k \eta^2 \in \pi_{8k+2} ko$, and $\beta^k a \in \pi_{8k+4} ko$ have Adams filtrations equal to $4k$, $4k+1$, $4k+2$, and $4k+3$ respectively. In particular, of these classes, the class in degree n has Adams filtration $g(n)$.*

Proof sketch. This is standard: it follows from the fact that they are detected by the classes w^k, $w^k h_1$, $w^k h_1^2$, and $w^k v$ on the Adams E_∞-page, which have the corresponding s-degrees. □

This means, for example, it is possible to write $\beta\eta$ as a composition of five ko-module maps which are all 0 on H^*_{ko}, but it is not possible to write it as a composition of six such ko-module maps.

Proposition 2.52. $(p_1 \circ \gamma_n)^* \colon H^*_{ko}(\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)) \to H^*_{ko}(\tau_{\geq n} ko)$ *is nonzero.*

Proof. Suppose instead that $(p_1 \circ \gamma_n)^* = 0$. By construction, $\alpha_{g(n)}$ is a composition of $g(n)$ ko-module maps $h_{g(n)} \circ \cdots \circ h_1$, each of which induces the zero map on H^*_{ko}; thus the same is true if we restrict h_1 to $\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)$.

Let c be a generator of $\pi_n(ko)$; thus, if $n = 8k$, $8k+1$, $8k+2$, or $8k+4$, respectively, c is $\pm\beta^k$, $\beta^k\eta$, $\beta^k\eta^2$, or $\pm\beta^k a$, and by Lemma 2.51, c has Adams filtration $g(n)$. (Since ± 1 are indistinguishable in any $\mathbb{Z}/2$-vector space, for any map f, f and $-f$ have equal Adams filtrations.) By Corollary 2.49, c factors as the following composition of ko-module maps:

$$c \simeq h_{g(n)} \circ \cdots \circ h_1|_{\overline{\mathrm{Ad}}^{ko}_{g(n)}(ko)} \circ (p_1 \circ \gamma_n \circ \widetilde{c}), \tag{2.53}$$

and by hypothesis, each of these $g(n)+1$ maps is the zero map on H^*_{ko}. This implies that c has Adams filtration at least $g(n)+1$, which contradicts Lemma 2.51. □

Proof of Proposition 2.33. By Lemma 2.45 and Proposition 2.52, γ_n^* is surjective when restricted to a summand of $\mathrm{Ad}^{ko}_{g(n)}(ko)$. Thus, the entire map is also surjective. □

Thus, we have finished the proof of Theorem 2.26.

Example 2.54. We use the HASS to compute $\tau_{\leq 2} ko_*(\mathrm{pt})$ and compare with the usual Baker–Lazarev Adams spectral sequence computation, which we did in Proposition 2.6. The difference is that the map f_2 from Theorem 2.26 replaces the d_3 differentials – in other words, we have isolated the information those differentials contained from the rest of the spectral sequence computation.

We will use Theorem 2.26 with $n = 2$. Throughout this example, and the next section, $\mathcal{Q}^{s,t} \coloneqq \mathcal{Q}_2^{s,t}$. We need to calculate $\mathcal{Q}(H^*_{ko}(ko \wedge_{ko} \mathrm{pt}_+))$, i.e. $\mathcal{Q}(\mathbb{Z}/2)$.

Lemma 2.55.

(1) *The $\mathbb{E}$-module map*

$$f_2 \colon \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\widetilde{\mathcal{O}}) \longrightarrow \mathrm{Ext}^{s+3,t+7}_{\mathcal{A}(1)}(\mathbb{Z}/2). \tag{2.56}$$

sends $\delta \mapsto v$ and $\epsilon \mapsto w$, where δ, ϵ are as in Lemma 2.4.

(2) *Let* $s\mathbb{E} \coloneqq \mathcal{Q}(\mathbb{Z}/2)$. $s\mathbb{E} \cong \mathbb{Z}/2[h_0, h_1]/(h_0h_1, h_1^3)$, *with* $h_0 \in \mathcal{Q}^{1,1}$ *and* $h_1 \in \mathcal{Q}^{1,2}$.

Proof. Recall the $\mathbb{Z}^2$-graded $\mathbb{Z}/2$-algebra $\mathbb{E} \coloneqq \mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{Z}/2)$ from Theorem 2.2. By part (2) of Theorem 2.26, $\mathcal{Q}(\mathbb{Z}/2)$ fits into a long exact sequence (2.28) with f_2. Here we used that $g(2) = 2$ (Definition 1.33) and $C_3 \cong \Sigma^4\mathring{\mathcal{O}}$ (Lemma 1.51).

Recall from Lemma 2.4 that, as an $\mathbb{E}$-module, $\mathrm{Ext}(\mathring{\mathcal{O}})$ is generated by two classes $\delta \in \mathrm{Ext}^{0,0}$ and $\epsilon \in \mathrm{Ext}^{1,5}$, so f_2 is uniquely determined by its values on δ and ϵ. In particular, $f_2(\delta) \in \mathrm{Ext}^{3,7}(\mathbb{Z}/2)$ and $f_2(\epsilon) \in \mathrm{Ext}^{4,12}(\mathbb{Z}/2)$; looking at the structure of $\mathrm{Ext}(\mathbb{Z}/2)$ given in Theorem 2.2, we see that $f_2(\delta)$ is either 0 or v, and $f_2(\epsilon)$ is either 0 or w. The relations $v\delta = h_0^2\epsilon \neq 0$ and $v^2 = h_0^2 w \neq 0$ in $\mathrm{Ext}(\mathring{\mathcal{O}})$ (Lemma 2.4) and $\mathrm{Ext}(\mathbb{Z}/2)$ respectively imply that $f_2(\delta) = v$ iff $f_2(\epsilon) = w$.

It is traditional to draw long exact sequences of bigraded Ext groups as Adams charts (so, graded by $(t-s, s)$ such that the boundary map has bidegree $(-1, 1)$ in the horizontal, resp. vertical directions. See [BC18, Figures 24, 26, 33, 36, and 41] for some examples. In our case, we would like to think of f_2 as the boundary map, so that the "middle term" of the long exact sequence (2.28) is $\mathcal{Q}(\mathbb{Z}/2)$. To do this, we should shift $\mathrm{Ext}(\mathcal{Q})$ five units to the right and two upwards. We draw f_2 in this way in Figure 3, left.

The reader can check that if $f_2(\delta) = v$, so that $f_2(\epsilon) = w$, then exactness of (2.28) calculates $\mathcal{Q}(\mathbb{Z}/2)$ in agreement with the lemma statement. Thus we are done if we can derive a contradiction from the assumption $f_2(\delta) = 0$. Assuming $f_2(\delta) = 0$, then $f_2(\epsilon) = 0$ and therefore the entire map f_2 vanishes, implying an $\mathbb{E}$-module isomorphism

$$\mathcal{Q}(\mathbb{Z}/2) \cong \mathbb{E} \oplus \Sigma^{2,7}\mathrm{Ext}(\mathring{\mathcal{O}}). \tag{2.57}$$

The bigraded suspension uses the bigrading (s, t) (so that, like above, we shifted $\mathrm{Ext}(\mathring{\mathcal{O}})$ five units to the right and two upwards in the $(t-s, s)$ bigrading). (2.57) and our knowledge of $\mathbb{E}$ and $\mathrm{Ext}(\mathring{\mathcal{O}})$ from Theorem 2.2 and Lemma 2.4 respectively would then imply that $\mathcal{Q}(\mathbb{Z}/2)$ contains h_0-towers in topological degrees 4 and 5, generated by v and δ in filtrations 3 and 2 respectively, and there are no other nonzero classes in topological degrees 3, 4, and 5.

Now run the HASS. By part (3) of Theorem 2.26, the E_2-page is $\mathcal{Q}(\mathbb{Z}/2)$, and this spectral sequence converges to $\pi_*(\tau_{\leq 2}ko)^\wedge_2$. Since $\pi_4(\tau_{\leq 2}ko) \cong 0$, all classes in the h_0-tower generated by v must be the sources or targets of differentials. For $r \geq 2$, $E_2^{3-r,8-r} = \mathcal{Q}^{3-r,8-r}(\mathbb{Z}/2) \cong 0$, so $d_r \colon E_r^{3-r,8-r} \to E_r^{3,7}$ is the zero map and cannot kill v. Likewise, $d_r(v) \in E_r^{3+r,6+r}$, but $E_2^{3+r,6+r} = \mathcal{Q}^{3+r,6+r}(\mathbb{Z}/2) \cong 0$, so v is a permanent cycle, giving the desired contradiction. □

We draw $s\mathbb{E}$ in Figure 3, right. By part (3) of Theorem 2.26, this is the E_2-page of the HASS for $\tau_{\leq 2}ko_*(\mathrm{pt})$. This spectral sequence immediately collapses for degree reasons, just as the usual Adams spectral sequence for $ko_*(\mathrm{pt})$ does, and there are no hidden extensions.

3. User's guide

In this section, we outline the general procedure for using the HASS to compute twisted supercohomology groups and give several examples. To start with, we need to define supercohomology.

Definition 3.1 (Brown–Comenentz [BC76])**.** Let X be a spectrum. The *Pontrjagin dual of the sphere*, denoted $I_{\mathbb{C}^\times}$, is the spectrum representing the generalized cohomology theory

$$X \longmapsto \mathrm{Hom}(\pi_{-*}(X), \mathbb{C}^\times). \tag{3.2}$$

The *Pontrjagin dual* of X is $I_{\mathbb{C}^\times}X \coloneqq \mathrm{Map}(X, I_{\mathbb{C}^\times})$.

Brown–Comenetz used $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{C}^\times$; the resulting theory is mostly the same.

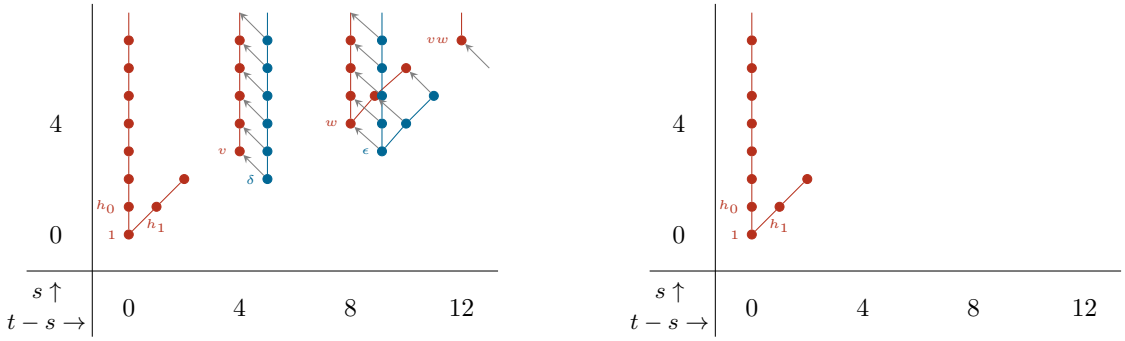


FIGURE 3. Left: the map f_2 from (2.26), part (1). We calculate this map, in particular showing $f_2(\delta) = v$ and $f_2(\epsilon) = w$, as part of the proof of Lemma 2.55. Right: $s\mathbb{E} \coloneqq \mathcal{Q}(\mathbb{Z}/2)$, calculated in Lemma 2.55. This is the $E_2 = E_\infty$-page of the HASS for $\tau_{\leq 2}ko_*(\mathrm{pt})$; we run this spectral sequence in Example 2.54. Compare these two charts to the ones in Figure 1 to compare the HASS (here) with the ordinary Baker–Lazarev Adams spectral sequence (there).

Definition 3.3 (Wang–Gu [WG18, WG20], Kapistin–Thorngren [KT17])**.** *Supercohomology* is the spectrum $SH \coloneqq I_{\mathbb{C}^\times}(\tau_{\leq 2}ko)$.

Thus SH defines a generalized cohomology theory on spaces X in the usual way. We will also need to consider twisted supercohomology, with the twist given by $a \in H^1(X; \mathbb{Z}/2)$ and $b \in H^2(X; \mathbb{Z}/2)$. We define twisted supercohomology in Appendix A. For applying the HASS, what one needs to know is the following lemma.

Lemma 3.4. *For any spectrum X, there is a natural isomorphism*

$$\pi_{-n}(I_{\mathbb{C}^\times}X) \xrightarrow{\cong} \mathrm{Hom}(\pi_n(X), \mathbb{C}^\times). \tag{3.5}$$

Thus, in particular, $SH^n(X, a, b)$ is the Pontrjagin dual group to $(\tau_{\leq 2}ko)_n(X, a, b)$.

This is a standard lemma that follows directly from Definition 3.1; see [BC76, §1]. Because of Lemma 3.4, we will use the HASS to compute twisted $\tau_{\leq 2}ko$-homology, then trust the reader to apply Pontrjagin duality to obtain the corresponding twisted supercohomology groups.

3.1. **Summary of the procedure.** The purpose of this section is to show how to use the HASS that we constructed in the previous section. The procedure follows the calculation of the Adams spectral sequence over ko closely.

The computation naturally separates into two pieces: at the prime 2 and away from 2.[12] The latter is easier, so we do it first.

Lemma 3.6. *For any space X, $a \in H^1(X; \mathbb{Z}/2)$, and $b \in H^2(X; \mathbb{Z}/2)$, the map $H^*(X; \mathbb{C}^\times_a) \to SH^*(X, a, b)$ induces an isomorphism on odd-primary torsion subgroups.*

Here $\mathbb{C}^\times_a$ is the abelian group $\mathbb{C}^\times$ with $\pi_1(X)$-action given by the cohomology class a, interpreted as a homomorphism $a\colon \pi_1(X) \to \{\pm 1\}$.

Proof. Applying Lemma 3.4, it suffices to show that $\tau_{\leq 0}\colon \tau_{\leq 2}ko \to H\mathbb{Z}$ induces an isomorphism on odd-primary subgroups for (X, a, b)-twisted homology. That is, we will show that $\tau_{\leq 0}$ is an

[12] See for example [DDHM24, §10.2].

equivalence away from the prime 2. This is because its fiber is the spectrum $\tau_{\leq 2}\tau_{\geq 1}ko$, whose homotopy groups are 0 away from degrees 1 and 2, where they are isomorphic to $\mathbb{Z}/2$. Thus after inverting 2, these homotopy groups are zeroed out, so away from the prime 2, $\tau_{\leq 0}$ has contractible fiber and thus is an equivalence of spectra, finishing the proof. □

At the prime 2, we will use the HASS to calculate $(\tau_{\leq 2}ko)_n(X,a,b)_{(2)}$ or, dually, $SH^n(X,a,b)_{(2)}$ for a space X. As with the usual Adams spectral sequence, the starting data is a module over the Steenrod algebra $\mathcal{A}(1)$.

Definition 3.7. Let X be a space, $a \in H^1(X;\mathbb{Z}/2)$, and $b \in H^2(X;\mathbb{Z}/2)$. Let $H^*_{a,b}(X;\mathbb{Z}/2)$ be the $\mathcal{A}(1)$-module which, as a $\mathbb{Z}$-graded $\mathbb{Z}/2$-vector space, is isomorphic to a free $H^*(X;\mathbb{Z}/2)$-module on a single degree-0 generator U, but with $\mathcal{A}(1)$-action given by

$$\begin{aligned} \mathrm{Sq}^1(Ux) &= U(ax + \mathrm{Sq}^1(x)) \\ \mathrm{Sq}^2(Ux) &= U(bx + \mathrm{Sq}^1(a)\mathrm{Sq}^1(x) + \mathrm{Sq}^2(x)). \end{aligned} \tag{3.8}$$

It is not a priori obvious that the action we defined in (3.8) satisfies the Adem relations and therefore actually defines an $\mathcal{A}(1)$-action, but this is true and is proven in [DY23a, Lemma 2.38(3)]. If there is a vector bundle $V \to X$ such that $a = w_1(V)$ and $b = w_2(V)$, this follows from the Wu formula and the Thom isomorphism, but such a V does not in general exist.

Lemma 3.9 ([DY23a, Theorem 2.39(3)])**.** *With (X,a,b) in Definition 3.7, let $M^{ko}h_{a,b}$ be the ko-module Thom spectrum of the map $h_{a,b}\colon X \to B\mathrm{GL}_1(ko)$ defined in Remark A.27. Then there is a natural $\mathcal{A}(1)$-module isomorphism*

$$H^*_{ko}(M^{ko}h_{a,b}) \xrightarrow{\cong} H^*_{a,b}(X;\mathbb{Z}/2). \tag{3.10}$$

By Theorem A.35, $SH^*(X,a,b) = \pi_{-*}(I_{\mathbb{C}^\times}(\tau_{\leq 2}ko \wedge_{ko} M^{ko}h_{a,b}))$. Thus, by Theorem 2.26, part (3), the E_2-page of the HASS computing $SH^*(X,a,b)$ is $\mathcal{Q}(H^*_{ko}(M^{ko}h_{a,b}))$, and therefore Lemma 3.9 calculates the $\mathcal{A}(1)$-module we want to apply $\mathcal{Q}$ to.

The steps of the HASS are as follows.

(1) Using Lemma 3.9, compute the $\mathcal{A}(1)$-module structure on $H^*_{ko}(M^{ko}h_{a,b})$. In practice, this is usually a direct sum of suspensions of modules appearing in the list in Definition 3.11.
(2) Draw the E_2-page of the HASS: for each $\mathcal{A}(1)$-module summand M, determine $\mathcal{Q}(M)$. Then direct sum all of these $\mathcal{Q}(M)$ charts together. These $\mathcal{Q}$ charts play the same role for the HASS that Ext charts play for the ordinary Adams spectral sequence.
(3) Compute the differentials and resolve any hidden extensions.
(4) Using Lemma 3.4, $SH^k(X,a,b) \cong \mathrm{Hom}(\tau_{\leq 2}ko_k(X,a,b), \mathbb{C}^\times)$.

Part (3) may be trivial or difficult, depending on the specific problem the HASS has been applied to. In practice, it tends to be about as difficult as the corresponding Adams spectral sequence for twisted ko-homology, which is usually tractable in examples.

3.2. **Ext charts.** In this section, we describe the Ext charts $\mathcal{Q}(M)$ for frequently encountered $\mathcal{A}(1)$-modules M. As described in the previous subsection, these charts give the input to the hastened Adams spectral sequence computing $\tau_{\leq 2}ko$-homology.

This section contains the following information:

(1) A theorem describing each $\mathcal{Q}(M)$ as a module over $s\mathbb{E} \coloneqq \mathcal{Q}(\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0,h_1]/(h_0h_1, h_1^3)$ (see Lemma 2.55).
(2) A figure displaying M and $\mathcal{Q}(M)$.

We defer proofs to Appendix B.

We begin by defining and naming the $\mathcal{A}(1)$-modules we will consider. Names will mostly follow [BC18, Deb21].

Definition 3.11 (Names of common $\mathcal{A}(1)$-modules)**.**

(1) $B \coloneqq \mathcal{A}(1) \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$ (called the *bow* in [BGHR24]).
(2) $N_1 \coloneqq \mathcal{A}(1)/(\mathrm{Sq}^2, \mathrm{Sq}^2\mathrm{Sq}^1)$.
(3) $\mathcal{O} \coloneqq \mathcal{A}(1)/(\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^3)$ as in Lemma 1.51 (often called the *upside-down question mark*).
(4) $J \coloneqq \mathcal{A}/(\mathrm{Sq}^3)$ as in Lemma 1.51 (called the *Joker* by Adams).
(5) $Q \coloneqq \mathcal{A}/(\mathrm{Sq}^2)$ as in Lemma 1.51 (often called the *question mark*).
(6) $R_2 \coloneqq Q_{3,1}$ (called the *elephant* in [BM26]).
(7) $\Upsilon_1 \coloneqq \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ as in Definition 2.34 (called the *seagull* in [Ada23]).
(8) $P \coloneqq \Sigma^{-1}\widetilde{H}^*(B\mathbb{Z}/2; \mathbb{Z}/2)$.
(9) $R_0 \coloneqq H^*((B\mathbb{Z}/2)^{1-\sigma}; \mathbb{Z}/2)$, where $\sigma \to B\mathbb{Z}/2$ denotes the tautological line bundle.
(10) $R_1 \coloneqq H^*((B\mathbb{Z}/2)^{2\sigma-2}; \mathbb{Z}/2)$.
(11) R_3 is the unique non-free $\mathcal{A}(1)$-module summand in $P \otimes P$.
(12) R_5 is the unique non-free $\mathcal{A}(1)$-module summand in $P \otimes P \otimes P$.
(13) $R_6 \coloneqq P \otimes C\eta$.

Remark 3.12. If R is an E_∞-ko-algebra, there is an induced $\mathbb{Z}/2$-coalgebra structure on $H^*_{ko}(R)$. This applies to $B \cong H^*_{ko}(ku)$ [Bak20, (5.12)], $\Upsilon_1 \cong H^*_{ko}(H\mathbb{Z})$ [Bak20, §5], and $R_1 \cong H^*_{ko}(ko \wedge (B\mathbb{Z}/2)^{2\sigma-2})$. Thus in these cases one obtains $\mathbb{E}$-algebra structures on $\mathrm{Ext}_{\mathcal{A}(1)}$ and $s\mathbb{E}$-algebra structures on $\mathcal{Q}$ of these $\mathcal{A}(1)$-modules.

Theorem 3.13. *Let M be one of the $\mathcal{A}(1)$-modules named in Definition 3.11. Then $\mathcal{Q}(M)$ is described in Figure 4.*

We give more detail in Theorem B.12, including explicit presentations of $\mathcal{Q}(M)$ for these M and a description of the image of $\partial_2 \colon \mathrm{Ext}_{\mathcal{A}(1)}(M) \to \mathcal{Q}(M)$. In particular, Theorem 3.13 follows from the proof of Theorem B.12 that we give in Appendix B.

3.3. **Examples.** We now use the $\mathcal{Q}$ charts worked out above to give some example computations of twisted supercohomology.

3.3.1. *Cyclic groups.* Recall that if n is even, $H^*(B\mathbb{Z}/n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]/(x^2 + ky)$, where $|x| = 1$, $|y| = 2$, and $k = 1$ if $n = 2$ and $k = 0$ otherwise.

We compute $SH^*(B\mathbb{Z}/n)$ and $SH^*(B\mathbb{Z}/n, 0, y)$ for all even n.

Proposition 3.14. *Let n be even. Then the untwisted and $(0, y)$-twisted supercohomology groups of $B\mathbb{Z}/n$ are given in Table 1.*

In low degrees, these computations have been done before by the following authors, usually using the Atiyah–Hirzebruch spectral sequence.

- $SH^k(B\mathbb{Z}/2^n)$: Wang–Gu [WG18, Table II] for $k \leq 3$, Décoppet [Déc24b, Example 4.13] for $k = 4$, and [DYY25, Propositions E.11 and E.16] for $k = 5$.
- $SH^k(B\mathbb{Z}/2^n, 0, y)$: Wang–Gu [WG20, Table III] and Zhang–Wang–Yang–Qi–Gu [ZWY+20] for $k \leq 3$ and [DYY25, Propositions E.20 and E.23] for $k = 4, 5$.

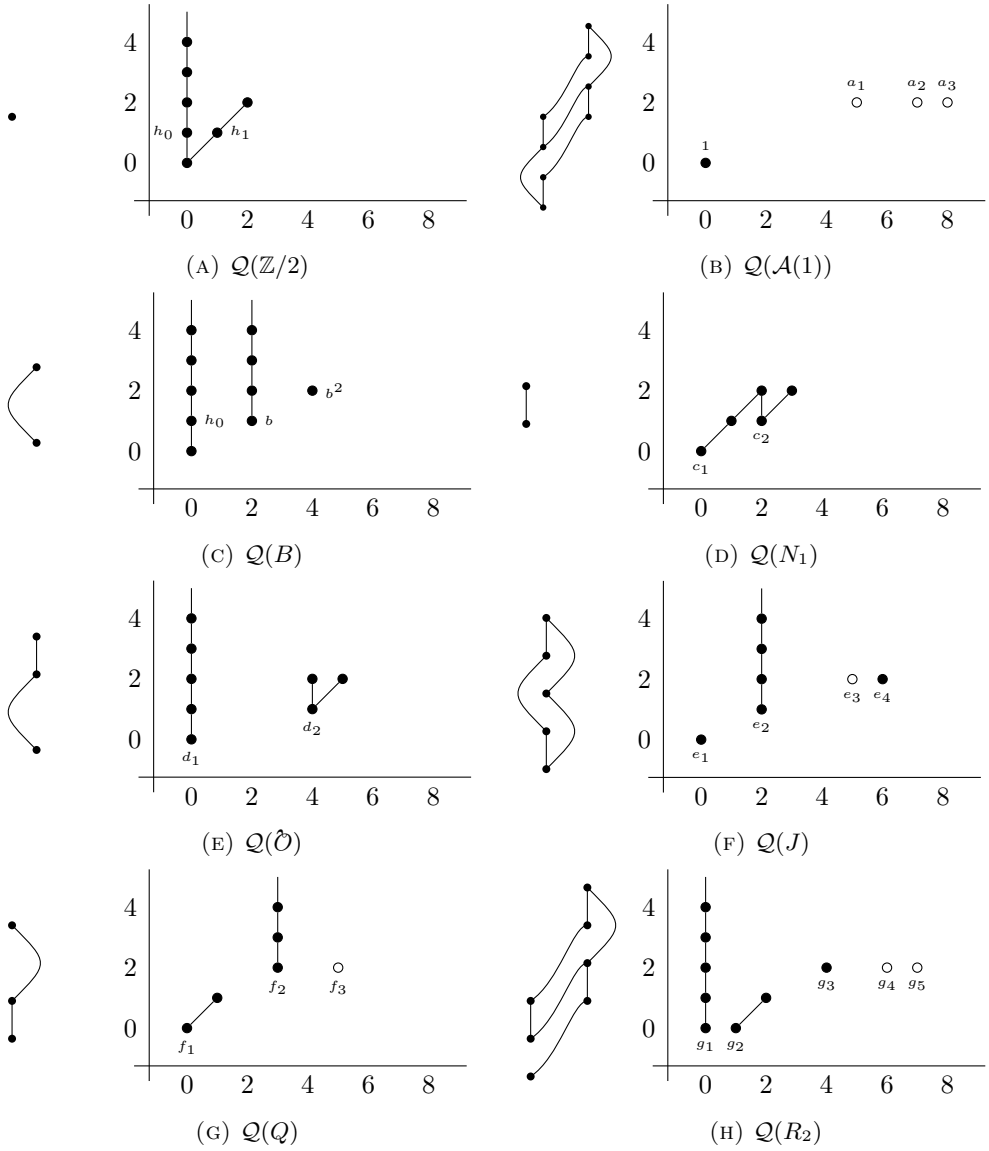


FIGURE 4. Ext charts $\mathcal{Q}(M)$ for various $\mathcal{A}(1)$-modules M, with M displayed on the left for reference. The filled-in classes are the image of the map $\partial_2 \colon \mathrm{Ext}(M) \to \mathcal{Q}(M)$. These charts are calculated in Theorem B.12.

Proof of Proposition 3.14. As described in §3.1, we will compute the $\tau_{\leq 2}ko$-homology of $B\mathbb{Z}/n$, as well as of the $(0, y)$ twist thereof. The (twisted) supercohomology groups are then $SH^k(X, a, b) \cong \mathrm{Hom}(\tau_{\leq 2}ko_k(X, a, b), \mathbb{C}^\times)$. Thus in the rest of this proof we focus only on (twisted) $\tau_{\leq 2}ko$-homology. Lemma 3.6 takes care of the calculation away from the prime 2, so in the rest of the proof, we work at 2, where we use the HASS.

In the untwisted case, split $\tau_{\leq 2}ko_*(X)$ into the known summand $\tau_{\leq 2}ko_*(\mathrm{pt})$ and the reduced $\tau_{\leq 2}ko$-homology of X. We compute the latter using the reduced HASS, which differs only in taking the $\mathcal{A}(1)$-module structure on reduced cohomology as input.

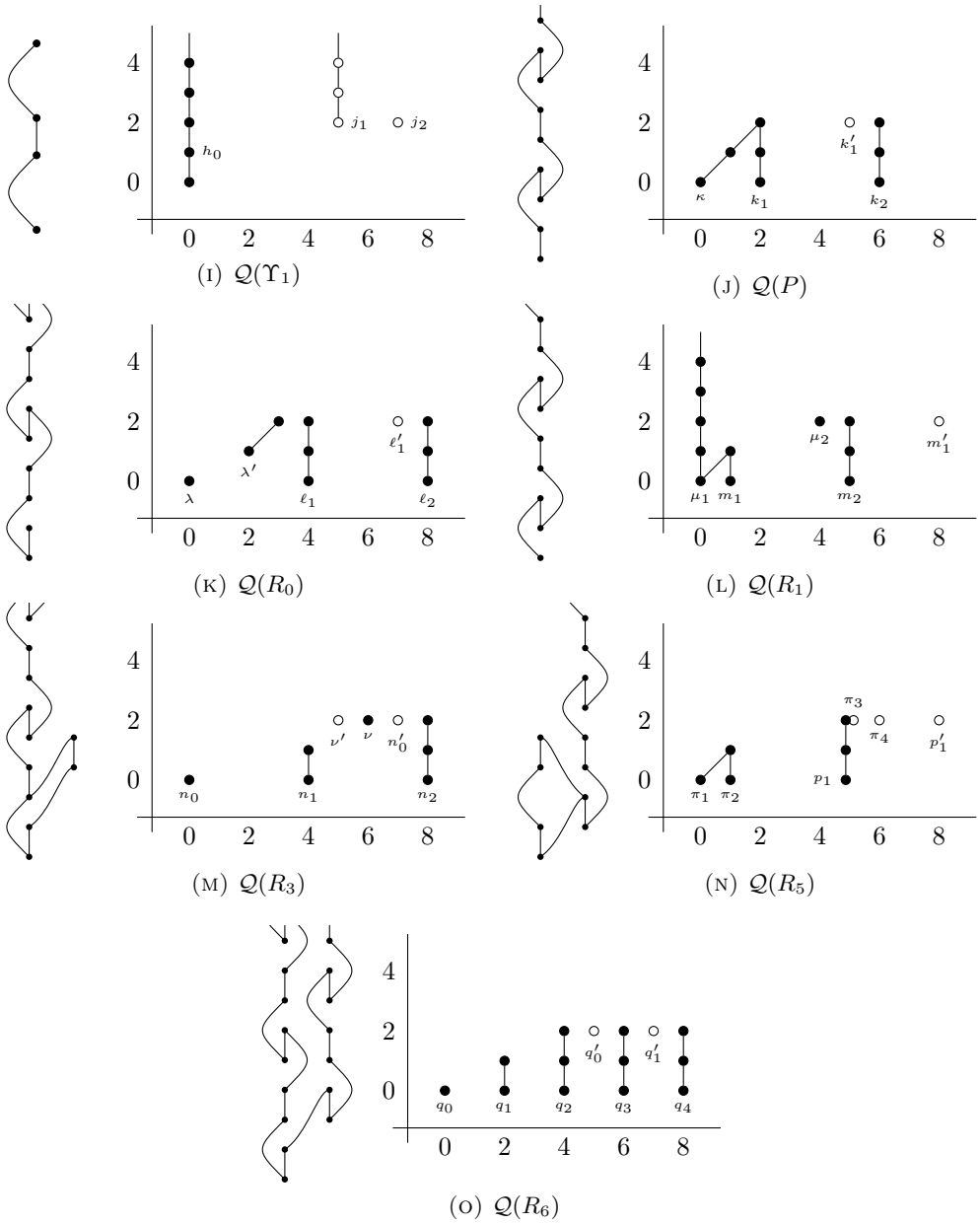


FIGURE 4. Ext charts $\mathcal{Q}(M)$ for various $\mathcal{A}(1)$-modules M, with M displayed on the left for reference. The filled-in classes are the image of the map $\partial_2 \colon \mathrm{Ext}(M) \to \mathcal{Q}(M)$. These charts are calculated in Theorem B.12.

Let k be the largest number such that 2^k divides n. The inclusion $\mathbb{Z}/2^k \to \mathbb{Z}/n$ induces a 2-local stable equivalence $B\mathbb{Z}/2^k \to B\mathbb{Z}/n$, so we may without loss of generality assume $n = 2^k$.

As described in the user's guide, our first step in these computations is to determine the $\mathcal{A}(1)$-modules $\widetilde{H}^*(B\mathbb{Z}/2^k; \mathbb{Z}/2)$ and $H^*_{0,y}(B\mathbb{Z}/2^k; \mathbb{Z}/2)$ for all $k \geq 1$. These can be looked up: there are

$\mathcal{A}(1)$-module isomorphisms

$$\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \Sigma P, \tag{3.15a}$$

$$H^*_{0,y}(B\mathbb{Z}/2;\mathbb{Z}/2) \cong R_1, \tag{3.15b}$$

and for all $k \geq 1$,

$$\widetilde{H}^*(B\mathbb{Z}/2^k;\mathbb{Z}/2) \cong \Sigma\mathbb{Z}/2 \oplus \bigoplus_{i\geq 1}^{\infty} \Sigma^{4i+2}B \oplus \Sigma^{4i+3}B \tag{3.15c}$$

$$H^*_{0,y}(B\mathbb{Z}/2^k;\mathbb{Z}/2) \cong \bigoplus_{i\geq 0}^{\infty} \Sigma^{4i}B \oplus \Sigma^{4i+1}B. \tag{3.15d}$$

For (3.15a) and (3.15b), this is essentially Definition 3.11; see [Cam17, §7.8] and [BC18, Figure 4]. For (3.15c) and (3.15d), see [Cam17, Figure 7.5], [DDHM24, Figure 17], [BGHR24, Lemma 3.8], [BDD+25, Proposition 13.20], or [Dha26, Corollary 2.10]. We draw P and R_1 in Figure 4j, left, resp. Figure 4l, left, and we draw the decompositions (3.15c) and (3.15d) in Figure 5, top left resp. bottom left.

Next, using the $\mathcal{Q}^{s,t}(-)$ charts for $\mathbb{Z}/2$, B, P, and R_1 we computed in Figures 4a, 4c, 4j and 4l, respectively, we obtain the E_2-page of the HASS for the $\tau_{\leq 2}ko$-homology, twisted by $(0,0)$ and $(0,y)$, of $B\mathbb{Z}/2^k$. For $B\mathbb{Z}/2$, untwisted, this is Figure 4j shifted right by 1; for $B\mathbb{Z}/2$, twisted, the E_2-page is exactly Figure 4l.

For $k > 1$ we give the E_2-pages for the untwisted and twisted computations in Figure 5, top center resp. bottom center, as well as a brief generators-and-relations summary here.

- First $(B\mathbb{Z}/2^k, 0, 0)$. By Theorem B.12, part (2), the map $\partial_2 \colon \mathrm{Ext}(\widetilde{H}^*(B\mathbb{Z}/2^k;\mathbb{Z}/2),\mathbb{Z}/2) \to \mathcal{Q}(\widetilde{H}^*(B\mathbb{Z}/2^k;\mathbb{Z}/2))$ is surjective, so we can reuse notation from the presentation of $\mathrm{Ext}(\widetilde{H}^*(B\mathbb{Z}/2^k;\mathbb{Z}/2),\mathbb{Z}/2)$ given in [BDD+25, Theorem 13.36].
  - $\theta \in \mathcal{Q}^{0,1}$, no relations.
  - For each $i \geq 1$, classes $\kappa_i \in \mathcal{Q}^{0,4i-2}$ and $\theta\kappa_i \in \mathcal{Q}^{0,4i-1}$, with the relations $h_1\kappa_i = 0$ and $h_1\theta\kappa_i = 0$.[13]
  - For each $i \geq 1$, classes $\lambda_i \in \mathcal{Q}^{1,4i+1}$ and $\theta\lambda_i \in \mathcal{Q}^{1,4i+2}$, with $h_1\lambda_i = 0$ and $h_1\theta\lambda_i = 0$.
  - For each $i \geq 1$, classes $\mu_i \in \mathcal{Q}^{2,4i+4}$ and $\theta\mu_i \in \mathcal{Q}^{2,4i+5}$, each annihilated by both h_0 and h_1.
- For $(B\mathbb{Z}/2^k, 0, y)$:[14]
  - A class $1 \in \mathcal{Q}^{0,0}$, with no relations.
  - A class $\vartheta \in \mathcal{Q}^{0,1}$, with $h_1\vartheta = 0$.
  - Classes $k_i \in \mathcal{Q}^{0,4i}$ and $\vartheta k_i \in \mathcal{Q}^{0,4i+1}$, $i \geq 1$, killed by h_1.
  - Classes $\ell_i \in \mathcal{Q}^{1,4i-1}$ and $\vartheta\ell_i \in \mathcal{Q}^{1,4i}$ for all $i \geq 0$, killed by h_1.
  - Classes $m_i \in \mathcal{Q}^{2,4i+2}$ and $\vartheta m_i \in \mathcal{Q}^{2,4i+3}$ for all $i \geq 0$, killed by both h_0 and h_1.

[13] There is an algebra structure on $\mathrm{Ext}_{\mathcal{A}(1)}(H^*(B\mathbb{Z}/2^k;\mathbb{Z}/2),\mathbb{Z}/2)$ coming from the group multiplication $\mathbb{Z}/2^k \times \mathbb{Z}/2^k \to \mathbb{Z}/2^k$. With respect to this algebra structure, $\theta\kappa_i$ is indeed the product of θ and κ_i. Since we have not proven that the HASS for $\tau_{\leq 2}ko$-homology is multiplicative, we cannot say that $\theta\kappa_i \in \mathcal{Q}^{0,4i-1}$ is necessarily a product in the same way, and the notation is optimistic. Nevertheless, we expect that the HASS *is* multiplicative (see Conjecture 4.4), which would make $\theta\kappa_i = \theta \cdot \kappa_i$ in $\mathcal{Q}(H^*(B\mathbb{Z}/2^k;\mathbb{Z}/2))$ as well. Similar considerations apply to λ_i and $\theta\lambda_i$, and to μ_i and $\theta\mu_i$.

[14] An analogue of Footnote 13 applies here, except that to our knowledge the algebra structure on $\mathrm{Ext}(H^*_{0,y}(B\mathbb{Z}/2^k;\mathbb{Z}/2))$ has not been worked out. Our notation is based on partial computations of this algebra structure, and we do not claim it implies products in $\mathcal{Q}$.

When $k = 1$, in both the twisted and untwisted cases, almost all differentials vanish for degree reasons or because this is forced by commutativity with the $s\mathbb{E}$-action. The exceptions are:

- For the untwisted computation, for all $i \geq 2$, $d_2 \colon E_2^{0,4i-2} \to E_2^{2,4i-1}$ could a priori send $k_i \mapsto k'_{i-1}$. For all $i \geq 1$, k_i is in the image of the map from $\partial_2 \colon \mathrm{Ext}(\Sigma P) \to \mathcal{Q}(\Sigma P)$, as we saw in Table 2. By Theorem 2.26, part (3), ∂_2 is the map on E_2-pages from the ordinary Adams spectral sequence for $\widetilde{ko}_*(B\mathbb{Z}/2)$ to the HASS for $\tau_{\leq 2}\widetilde{ko}_*(B\mathbb{Z}/2)$. Therefore we may compute this differential in the ordinary Adams spectral sequence and push it forward using f_2. But in the ordinary Adams spectral sequence, this differential vanishes for degree reasons. Thus this d_2 vanishes.
- In the twisted computation, for $i \geq 1$, $d_2 \colon E_2^{0,4i} \to E_2^{2,4i-1}$ may a priori send ℓ_i to $h_1\lambda'$ $(i = 1)$ or ℓ'_{i-1} $(i > 1)$. Again, Table 2 tells us that ℓ_i is in the image of ∂_2, so to show the differential vanishes in the HASS it suffices to show it vanishes in the usual ko-Adams spectral sequence, which was shown in [Cam17, §7.8].

Thus in both cases the Adams spectral sequence collapses. On the E_∞-page, all extension questions are detected by the action of h_0, finishing the computation and giving the groups in Table 1. Thus in the rest of the proof we restrict to $k > 1$.

Inspecting Figure 5, top and bottom center, there are many differentials which could be nonzero. As in the $k = 1$ case, Theorem B.12 shows their sources and targets are in $\mathrm{Im}(\partial_2)$, so we may compute these differentials in the corresponding ko-homology Adams spectral sequences, then push them forward. Though these ko-homology differentials have been studied in several different references, including [Sie13, Cam17, DL21a, BGHR24, DDHM24, BDD+25, Dha26], a combination of complicated arguments and some errors in the literature (see Remark 3.17) make it hard to apply their arguments in this proof. As such, we give a direct account.

As with all of the cited calculations, we use a technique inspired by the May–Milgram theorem [MM81] to compute all of the HASS differentials. Specifically, consider the Postnikov truncation map $\tau_{\leq 0} \colon ko \to H\mathbb{Z}$, a map of E_∞-ring spectra. For any ko-module M, we get an induced map from the ko-module Baker–Lazarev Adams spectral sequence of M to the corresponding $H\mathbb{Z}$-module spectral sequence for $H\mathbb{Z} \wedge_{ko} M$. On E_2-pages, this map is precisely the map $(\tau_{\leq 0})_* \colon \mathrm{Ext}_{\mathcal{A}(1)} \to \mathrm{Ext}_{\mathcal{A}(0)}$ induced by the inclusion $\mathcal{A}(0) \coloneqq \langle \mathrm{Sq}^1 \rangle \hookrightarrow \mathcal{A}(1)$ [BDD+25, Proposition 13.1].[15] Thus we have a diagram

$$\begin{array}{ccc} & \mathrm{Ext}_{\mathcal{A}(1)}(M) & \\ {\scriptstyle (\tau_{\leq 0})_*} \swarrow & & \searrow {\scriptstyle \partial_2} \\ \mathrm{Ext}_{\mathcal{A}(0)}(M) & & \mathcal{Q}(M). \end{array} \tag{3.16}$$

We will pull differentials back along $(\tau_{\leq 0})_*$, then push them forward along ∂_2.

Remark 3.17. Let $M = ko \wedge (B\mathbb{Z}/2^k)$. In this case, [BDD+25, Corollary 13.30] calculates the map $(\tau_{\leq 0})_*$, and shows it is far from injective. This means that the image of $d_r(x)$ in the ko-module spectral sequence under $(\tau_{\leq 0})_*$ does not uniquely determine $d_r(x)$. As such, the statement and proof of [BDD+25, Proposition 13.38] are not correct: see [Dha26, Remark 4.7]. For example, when $k = 3$, this method of proof cannot tell whether $d_3(\kappa_2) = h_0^2\theta\lambda_1$ or $d_3(\kappa_2) = h_0^2\theta\lambda_1 + v\theta$. This error

[15] In *loc. cit.* M is assumed to be of the form $ko \wedge X$ for a spectrum X, but the more general version we gave here is true, and follows from the naturality of the Baker–Lazarev Adams spectral sequence in the ring spectrum used as a base.

does not affect any results in that paper other than Proposition 13.38. We thank Prerna Dhankhar for pointing this error out to us.

Dhankhar [Dha26, Proposition 4.6] determines the correct values of these differentials in degrees 14 and below, as well as in all degrees if $k = 2$.

Fortunately, this ambiguity in the values of the differentials disappears in the HASS.

Definition 3.18. Let $\mathbb{M}$ be an $\mathbb{E}$- or $s\mathbb{E}$-module. A class $x \in \mathbb{M}$ is part of an *h_0-tower* if for all $j \geq 0$, $h_0^k x \neq 0$.

Lemma 3.19. *Let $M \in \{ko \wedge (B\mathbb{Z}/2^k), M^{ko}h_{0,y}\}$ and let $x \in \mathrm{Ext}_{\mathcal{A}(1)}(H^*_{ko}(M))$ be part of an h_0-tower. If $x \in \ker((\tau_{\leq 0})_*)$, then for $j \gg 0$, $h_0^j x \in \ker(\partial_2)$.*

Proof. For $M = ko \wedge (B\mathbb{Z}/2^k)$, $\ker((\tau_{\leq 0})_*)$ is calculated in [BDD+25, Corollary 13.30], and $\ker(\partial_2)$ can be calculated using the $\mathcal{A}(1)$-module decomposition (3.15c) and the computation of ∂_2 on each summand given in Theorem B.12. Comparing the two finishes the proof for this choice of M. The proof for the other choice of M is analogous. □

Corollary 3.20. *The differentials in the HASSes computing $SH^*(B\mathbb{Z}/2^k)$ and $SH^*(B\mathbb{Z}/2^k, 0, y)$ vanish except on the E_k-page, where they are uniquely determined by h_0- and h_1-linearity and the following values on generators.*

$$
\begin{aligned}
d_k(\kappa_1) &= h_0^k\theta & d_k(\ell_0) &= h_0^{k+1}\vartheta \\
d_k(\kappa_i) &= h_0^{k-1}\theta\lambda_{i-1},\ i > 1 & d_k(k_i) &= h_0^{k-1}\vartheta\ell_{i-1} \\
d_k(\lambda_i) &= h_0^{k+1}\theta\kappa_i & d_k(\ell_i) &= h_0^{k+1}\vartheta k_i\ i \geq 1,
\end{aligned} \tag{3.21}
$$

and d_k vanishes on the classes 1, θ, ϑ, μ_i, m_i, $\theta\kappa_i$, ϑk_i, $\theta\lambda_i$, $\vartheta\ell_i$, $\theta\mu_i$, and ϑm_i.

Proof. Differentials vanish on μ_i, m_i, $\theta\mu_i$, and ϑm_i because that is the only possibility compatible with h_0-linearity of differentials. For the remaining claimed differentials, use May–Milgram's identification of Adams differentials over $\mathcal{A}(0)$ with differentials in the $\mathbb{Z}/2$-Bockstein spectral sequence for homology with $\mathbb{Z}_{(2)}$ coefficients [MM81]. The latter differentials are uniquely determined by the values of $H_*(B\mathbb{Z}/2^k; \mathbb{Z}_{(2)})$, and therefore so are the differentials in the $H\mathbb{Z}$-module Adams spectral sequence for $H\mathbb{Z} \wedge_{ko} M$. Lemma 3.19 ports these differentials over to the HASS, determining all differentials on classes $h_0^j x$ for $j \gg 0$. This determines all differentials modulo the kernel of h_0. For degree reasons, the only differentials that could be in the kernel of h_0 are $d_2(\theta\kappa_i) \in \{0, \mu_{i-1}\}$ for $i \geq 2$ and $d_2(\vartheta k_i) \in \{0, m_0\}$ for $i \geq 1$. These differentials can be shown to vanish because the corresponding differentials in the ko-module Adams spectral sequence for M vanish, which follows from the Leibniz rule. □

Thus we can draw the E_∞-page in Figure 5, top right (untwisted case), resp. bottom right (twisted case). Despite the presence of tricky extension questions in the Adams spectral sequence for $\widetilde{ko}_*(B\mathbb{Z}/2^k)$ and $ko_*(B\mathbb{Z}/2^k, 0, y)$ (see, for example, [DDHM24, §13.2] and [BDD+25, §13.2]), in the HASSes for $\tau_{\leq 2}ko$-homology, the corresponding extension questions for multiplication by 2 are completely solved by the h_0-action on the E_∞-page. Therefore we obtain the supercohomology groups claimed in Table 1. □

Remark 3.22 (Eventual periodicity). Looking at Table 1, the twisted supercohomology groups of $B\mathbb{Z}/n$ are periodic in degrees 3 and above. This is an instance of a quite general fact: if E is a ring spectrum with finitely many nonzero homotopy groups and X is a space with periodic cohomology,

one can use the Atiyah–Hirzebruch spectral sequence to show that $E^*(X)$ is periodic in degrees N and above for some $N \gg 0$. Thus this applies to $\tau_{\leq n} ko_*(B\mathbb{Z}/n)$, $\tau_{\leq n} ku_*(B\mathbb{Z}/n)$, and twists thereof, even though the (ordinary and twisted) ko- and ku-(co)homology groups of $B\mathbb{Z}/n$ are not periodic. See for example [BG03, §2.2 and §3.4] and [BG10, §6.3 and §7.3].

Remark 3.23 (Gysin sequences). Table 1 has another pattern: for k big enough, $SH^k(B\mathbb{Z}/n) \cong SH^{k+2}(B\mathbb{Z}/n, 0, y)$ and $SH^k(B\mathbb{Z}/n, 0, y) \cong SH^{k+2}(B\mathbb{Z}/n)$. This is also not a coincidence. Following the discussion in [DDK+26, Example 7.35], but with SH in place of $MTSpin$, we have Gysin sequences for the complex line bundle $L \to B\mathbb{Z}/n$ associated to the representation as the n^{th} roots of unity:

$$\cdots \to SH^k(B\mathbb{Z}/n) \xrightarrow{e^{SH}(L)} SH^{k+2}(B\mathbb{Z}/n, 0, y) \longrightarrow SH^{k+2}(S^1) \longrightarrow SH^{k+1}(B\mathbb{Z}/n) \to \cdots \tag{3.24a}$$

$$\cdots \to SH^k(B\mathbb{Z}/n, 0, y) \xrightarrow{e^{SH}(L)} SH^{k+2}(B\mathbb{Z}/n) \longrightarrow SH^{k+2}(S^1) \longrightarrow SH^{k+1}(B\mathbb{Z}/n) \to \cdots \tag{3.24b}$$

Here $e^{SH}(V)$ denotes the product with the Euler class in (twisted) $\tau_{\leq 2} ko$-cohomology. See [DDK+26, §4.1] for more information.

A quick Atiyah–Hirzebruch spectral sequence argument shows $SH^k(S^1) = 0$ for $k > 3$, implying that the Euler class maps in (3.24) are isomorphisms for those k.

	$n \equiv 2 \bmod 4$		$n \equiv 0 \bmod 4$	
k	$SH^k(B\mathbb{Z}/n)$	$SH^k(B\mathbb{Z}/n, 0, y)$	$SH^k(B\mathbb{Z}/n)$	$SH^k(B\mathbb{Z}/n, 0, y)$
0	$\mathbb{C}^\times$	$\mathbb{C}^\times$	$\mathbb{C}^\times$	$\mathbb{C}^\times$
1	$\mathbb{Z}/n \oplus \mathbb{Z}/2$	$\mathbb{Z}/2n$	$\mathbb{Z}/2n$	$\mathbb{Z}/2n$
2	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
$4j$, $j > 0$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$4j+1$, $j > 0$	$\mathbb{Z}/(n/2)$	$\mathbb{Z}/4n$	$\mathbb{Z}/(n/2)$	$\mathbb{Z}/2n \oplus \mathbb{Z}/2$
$4j+2$, $j > 0$	0	0	$\mathbb{Z}/2$	0
$4j+3$, $j > 0$	$\mathbb{Z}/4n$	$\mathbb{Z}/(n/2)$	$\mathbb{Z}/2n \oplus \mathbb{Z}/2$	$\mathbb{Z}/(n/2)$

TABLE 1. Twisted supercohomology groups of cyclic groups. We prove these isomorphisms in Proposition 3.14.

3.3.2. *The* 16*-fold way and "the* 2*-fold way".* Let $B \in H^2(B^2\mathbb{Z}/2; \mathbb{Z}/2)$ denote the tautological class. In this subsubsection, we will prove $SH^4(B^2\mathbb{Z}/2, 0, B) \cong \mathbb{Z}/16$, then identify this with the "16-fold way" classification of [Kit06, JFR24]. Similarly, we will prove $SH^5(B^2\mathbb{Z}/2, 0, B) \cong \mathbb{Z}/2$, then identify this with the two different "fermionic $(3+1)$D topological orders" S theory and T theory in [JF25].

Proposition 3.25 (Johnson-Freyd [JF25, Remark 2.2.5])**.** *There are isomorphisms*

$$SH^k(B^2\mathbb{Z}/2, 0, B) = \begin{cases} \mathbb{Z}/2, & k = 5 \\ \mathbb{Z}/16, & k = 4 \\ 0, & k = 1, 2, 3 \\ \mathbb{C}^\times, & k = 0. \end{cases} \tag{3.26}$$

Johnson-Freyd uses the Atiyah–Hirzebruch spectral sequence to make this computation; he only discusses SH^4 and SH^5 explicitly, but the lower-degree groups follow clearly from his proof. This

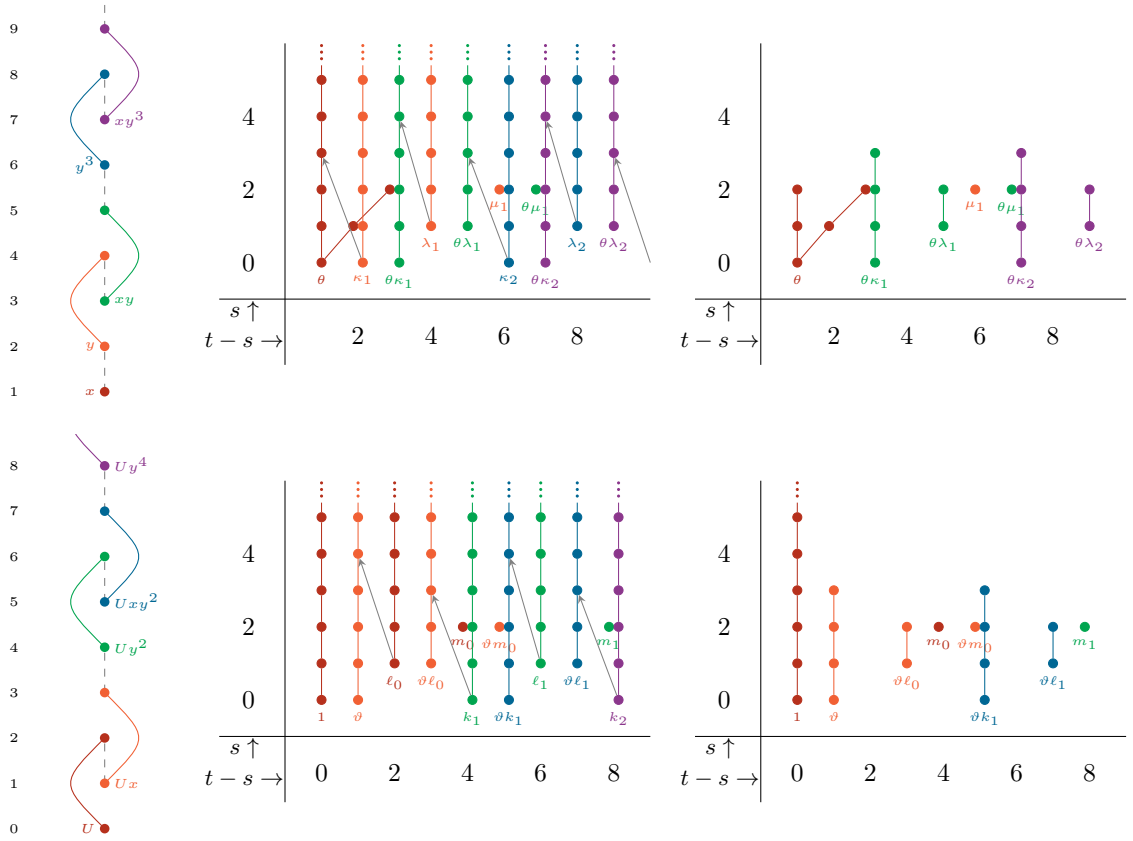


FIGURE 5. Upper left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(B\mathbb{Z}/n;\mathbb{Z}/2)$, when $n \equiv 0 \bmod 4$. Let 2^k be the largest power of 2 dividing n. The dashed lines represent nonzero 2^k-Bockstein actions. Figure adapted from [Cam17, Figure 7.5]. Upper center: the $E_2 = E_k$-page of the HASS computing $\tau_{\leq 2}ko_*(B\mathbb{Z}/n)$ (here $k = 3$). Notation follows [BDD+25, Theorem 13.36]. We discuss the pictured differentials in Corollary 3.20. Upper right: the $E_{k+1} = E_\infty$-page. Lower left: the $\mathcal{A}(1)$-module structure on $H^*((B\mathbb{Z}/n)^{L-2};\mathbb{Z}/2)$, where L is the complex line bundle induced from the rotation representation of $\mathbb{Z}/n$. The dashed lines again represent nonzero 2^k-Bockstein operations. Lower center: the $E_2 = E_k$-page of the HASS computing $\tau_{\leq 2}ko_*(B\mathbb{Z}/n, 0, y)$, including the differentials we compute in the proof of Corollary 3.20 (here $k = 3$). Lower right: the $E_{k+1} = E_\infty$-page.

proof requires addressing a tricky extension question in SH^4. We will see that the corresponding extension question in the HASS is much easier to solve.

Proof. Since $B^2\mathbb{Z}/2$ is trivial at odd primes, we focus on $p = 2$, where we use the HASS. Recall Serre's computation of $H^*(B^k\mathbb{Z}/2;\mathbb{Z}/2)$ [Ser53, Théorème 2]; specialized to $k = 2$, we have

$$H^*(B^2\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[B, C, E, \cdots], \tag{3.27}$$

where $|B| = 2$, $|C| = 3$, $|E| = 5$, $\mathrm{Sq}(B) = B + C + B^2$, $\mathrm{Sq}(C) = C + E + C^2$, and $\mathrm{Sq}(E) = E + C^2 + \mathrm{Sq}^4(E) + E^2$.

Using Definition 3.7 we have an $\mathcal{A}(1)$-module isomorphism

$$H^*_{0,B}(B^2\mathbb{Z}/2;\mathbb{Z}/2) \cong \Upsilon_1 \oplus \Sigma^4\Upsilon_1 \oplus \Sigma^5\mathcal{A}(1) \oplus L, \tag{3.28}$$

where L is 5-connected (and therefore cannot affect the computation of SH^4). We draw the isomorphism (3.28) in Figure 6, left. Then, using the $\mathcal{Q}(-)$ charts we computed in Figures 4b and 4i, we draw the E_2-page of the HASS for $\tau_{\leq 2}ko_*(B^2\mathbb{Z}/2,0,B)$ in Figure 6, right. Already we see that $SH^0 \cong \mathbb{C}^\times$ and that SH^1, SH^2 and SH^3 vanish, so for the rest of the proof we focus on SH^4 and SH^5.

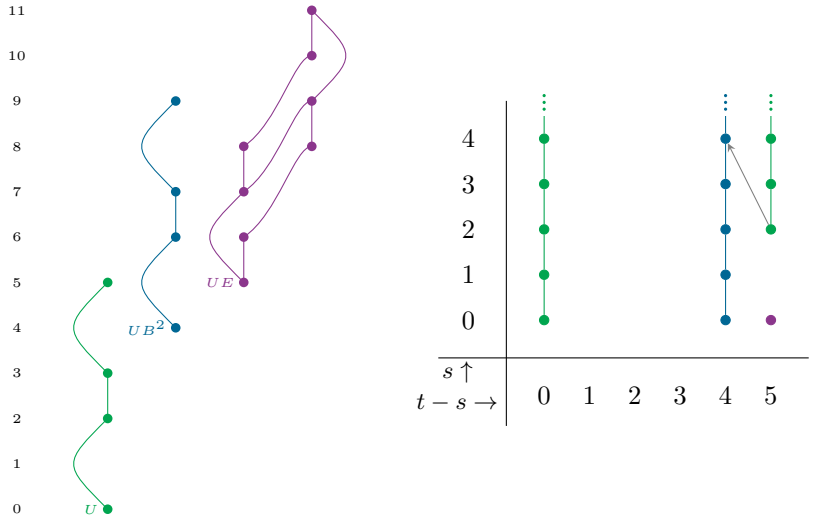


FIGURE 6. Left: the $\mathcal{A}(1)$-module structure on $H^*_{0,B}(B^2\mathbb{Z}/2;\mathbb{Z}/2)$ in low degrees, as in (3.28). Right: the E_2-page of the HASS computing $(\tau_{\leq 2}ko)_*(B^2\mathbb{Z}/2,0,B)$. We use this in the proof of Proposition 3.25, where we show that the d_2 depicted is nonzero.

In topological degrees 5 and below, all differentials vanish for degree reasons, except for the differentials d_r from the h_0-tower in topological degree 5 to the h_0-tower in topological degree 4. Since the h_0-action on E_∞ lifts to multiplication by 2, by inspection of Figure 6, right, one sees that the proposition statement is equivalent to $d_r \neq 0$ for $r = 2$, and that is what we will prove.

First, it is not too hard to show that for *some* $r \geq 2$, $d_r \neq 0$: if not, $(\tau_{\leq 2}ko)_4(B^2\mathbb{Z}/2,0,B) \otimes \mathbb{Q}$ would be nonzero, but $\tau_{\leq 2}ko$ is rationally equivalent to $H\mathbb{Q}$ (since it has $\pi_0 \cong \mathbb{Z}$ and all other homotopy groups torsion), and $H_4(B^2\mathbb{Z}/2;\mathbb{Q}) \cong 0$. If $d_r \neq 0$, then by inspection $(\tau_{\leq 2}ko)_4(B^2\mathbb{Z}/2,0,B) \cong \mathbb{Z}/2^{r+2}$ (since h_0 lifts to multiplication by 2), with a minimum order of 16 if $r = 2$. Then, from the $(r+1)$-page, there is only a single $\mathbb{Z}/2$ in topological degree 5, implying $SH^5 \cong \mathbb{Z}/2$. Thus it remains only to compute SH^4.

The E^2-page of the Atiyah–Hirzebruch spectral sequence for $(\tau_{\leq 2}ko)_*(B^2\mathbb{Z}/2,0,B)$ provides an upper bound for the orders of these twisted $\tau_{\leq 2}ko$-homology groups, as differentials lower the order. In total degree 4, we have a $\mathbb{Z}/4$ and two copies of $\mathbb{Z}/2$, giving an upper bound of order 16. Putting these two together, we conclude that $(\tau_{\leq 2}ko)_4(B^2\mathbb{Z}/2,0,B) \cong \mathbb{Z}/16$. □

The $\mathbb{Z}/16$ in $SH^4(B^2\mathbb{Z}/2,0,B)$ is the same 16 in Kitaev's 16-fold way [Kit06, §9] (see also [BGH+17, LKW17, JFR24]), as well as in the fermionic 16-fold way of [KPY26b]. In the rest

of this subsubsection, we describe this connection, as well as the meaning of $SH^k(B^2\mathbb{Z}/2, 0, B)$ for $0 \leq k \leq 5$.

Definition 3.29.

(1) A *Picard n-groupoid* is a symmetric monoidal n-category which is an n-groupoid (all k-morphisms are invertible, for $1 \leq k \leq n$), such that all objects are invertible with respect to the tensor product.

(2) If $\mathcal{G}$ is a Picard n-groupoid, the geometric realization of the nerve of $\mathcal{G}$ is a grouplike E_∞-space (grouplike because all objects of $\mathcal{G}$ have inverses, E_∞ via the monoidal product), and therefore Ω^∞ of a spectrum $|\mathcal{G}|$, which we call the *classifying spectrum* of $\mathcal{G}$.

By construction, $\pi_0(|\mathcal{G}|)$ is isomorphic to the abelian group of isomorphism classes of objects of $\mathcal{G}$, $\pi_1(|\mathcal{G}|) \cong \mathrm{Aut}_{\mathcal{G}}(\mathbf{1})$, $\pi_2(|\mathcal{G}|) \cong \mathrm{Aut}_{\mathrm{Aut}_{\mathcal{G}}(\mathbf{1})}(\mathrm{id}_{\mathbf{1}})$, and so on; in particular, $\pi_k(|\mathcal{G}|) = 0$ if $k \notin [0, n]$.

Theorem 3.30 (Stable homotopy hypothesis [MOP+22])**.** *The assignment from a Picard n-groupoid to its classifying spectrum defines an equivalence of ∞-categories from Picard n-groupoids to spectra whose homotopy groups are concentrated in degrees $[0, n]$.*

Remark 3.31. For $n \leq 2$, Theorem 3.30 was proven earlier. The case $n = 1$ is a folk theorem, with proofs or sketches given in [BCC93, HS05, Dri06, Pat12, JO12, GK14, Sha23]. The case $n = 2$ is a theorem of Gurski–Johnson–Osorno [GJO19].

A priori, the 16-fold way is about tensor (1-)categories. In fact, as explained in [JF25, JFR24], it lives in the world of fusion 2-categories, and the explicit way it is relevant is discussed below. Since braided fusion 2-categories are in particular E_2-monoidal 2-categories, they belong to a Morita 5-category by work of Johnson-Freyd–Scheimbauer [JFS17, Theorem/Definition 1.12].

The connection to supercohomology passes through the following fact.

Proposition 3.32 (Johnson-Freyd–Reutter [JFR24], Johnson-Freyd [JF25])**.** *Let $\mathcal{F}$ be the Picard 5-groupoid of nondegenerate, braided fermionic fusion 2-categories with $s\mathcal{V}ect$ as 1-morphisms. Then for $0 \leq k \leq 5$ there are injective homomorphisms*

$$SH^k(B^2\mathbb{Z}/2, 0, B) \hookrightarrow \pi_{5-k}(|\mathcal{F}|). \tag{3.33}$$

For $k = 0, 5$ this map is an isomorphism.

For context, the Picard spectrum $\mathcal{F}$ is constructed out of the core of the (unpointed) Morita 5-category of nondegenerate, braided fermionic fusion 2-categories with $s\mathcal{V}ect$ as 1-morphisms. The monoidal structure on objects is the Deligne tensor product of fusion 2-categories [Déc24a], and the monoidal structure on higher morphisms is given by composition of bimodules.

Combining Propositions 3.25 and 3.32, we recover some known facts in and around the 16-fold way.

- On π_0, we recover the discussion in [DY26, Remark 3.16]. $\pi_0(|\mathcal{F}|) \cong SH^5(B^2\mathbb{Z}/2, 0, B) \cong \mathbb{Z}/2$, meaning that the abelian group of Morita classes of nondegenerate braided fermionic strongly fusion 2-categories is $\mathbb{Z}/2$. The two Morita classes $0, 1 \in \mathbb{Z}/2$ correspond to the theories $\mathcal{S} \coloneqq Z(2s\mathcal{V}ect)$, resp. $\mathcal{T}$, defined by Johnson-Freyd [JF25, §2.2]; see also Johnson-Freyd–Reutter [JFR24, §3.1].
- On π_1, we recover Johnson-Freyd's interpretation of the 16-fold way [JF25, §2]:

$$SH^4(B^2\mathbb{Z}/2, 0, B) \cong \mathrm{Aut}(\mathcal{S}) \cong \mathbb{Z}/16$$

(see *ibid.*, Remarks 2.2.5 and 2.3.9). In more detail, the 16-fold way is the group parameterizing the minimal nondegenerate extension of $s\mathcal{V}ect$. In [JFR24], Johnson-Freyd–Reutter proved the existence of minimal nondegenerate extensions of $s\mathcal{V}ect$ using the theory $\mathcal{S}$, and found that choices of automorphisms for $\mathcal{S}$ corresponded to choices of minimal nondegenerate extension for $s\mathcal{V}ect$. Since nondegenerate braided fermionic strongly fusion 2-categories are classified by $SH^5(B^2\mathbb{Z}/2, 0, B)$, their automorphisms are classified by $SH^4(B^2\mathbb{Z}/2, 0, B)$ which is the group we compute.

- $\pi_5(|\mathcal{F}|) \cong SH^0(B^2\mathbb{Z}/2, 0, B) \cong \mathbb{C}^\times$ is a standard fact: $\Omega^4\mathcal{F} \simeq s\mathcal{L}ine_{\mathbb{C}}$, the Picard groupoid of invertible super vector spaces, so this $\mathbb{C}^\times$ is the nonzero scalars.

Remark 3.34. Barkeshli–Hsin–Kobayashi [BHK24] also studied the automorphisms of $\mathcal{S}$, in the context of quantum codes on the boundary. For their applications, they use the subgroup $\mathbb{Z}/8$ which exchanges the two "magnetic" objects in $\mathcal{S}$, and this group corresponds to the 8 Ising type braided fusion categories among the 16 minimal nondegenerate extensions of $s\mathcal{V}ect$.

It is natural to guess that (3.33) lifts to a map of spectra.

Conjecture 3.35. There is a map of spectra $\tau_{\geq 0}(\Sigma^5 I_{\mathbb{C}^\times} M^{\tau_{\leq 2} ko}(f_B)) \to |\mathcal{F}|$ inducing the maps (3.33) on homotopy groups.

Here $f_B \colon B^2\mathbb{Z}/2 \to B\mathrm{GL}_1(\tau_{\leq 2} ko)$ is the map $B^2\mathbb{Z}/2 \to B\mathrm{O}/B\mathrm{Spin}$ from Definition A.14, part (2) followed by the map $(\tau_{\leq 2})_* \circ T \colon B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{GL}_1(\tau_{\leq 2} ko)$ from Definition A.25.

One can show that the Postnikov invariants of $\tau_{\geq 0}(\Sigma^5 I_{\mathbb{C}^\times} M^{\tau_{\leq 2} ko}(f_B))$ vanish, so that it is a sum of shifts of Eilenberg–Mac Lane spectra. Thus an affirmative answer to Conjecture 3.35 would be progress towards computing the k-invariants of $|\mathcal{F}|$, which have category-theoretic meaning (see [JO12, GJOS17] for lower-categorical analogues).

3.3.3. *A fourfold way over* $\mathbb{R}$. Recently, Sanford [San26] proved the following analogue of the sixteen-fold way over $\mathbb{R}$.

Theorem 3.36 (Sanford [San26, Theorem 5.10])**.** *The kernel of the map from the Witt group of* $\mathcal{V}ect_{\mathbb{R}}$ *to the Witt group of* $s\mathcal{V}ect_{\mathbb{R}}$ *is isomorphic to* $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Analogously to the reformulation in [JFR24, §4.4], Theorem 3.36 can be rephrased as follows.

Corollary 3.37. *Let* $s\mathcal{A}lg_{\mathbb{R}}$ *denote the Morita category of real superalgebras and* $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ *denote the classifying spectrum of its Picard* 2*-groupoid of units (see Definition 3.29). Then there is an isomorphism*

$$(\Sigma^{-2}|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|)^5(B^2\mathbb{Z}/2, 0, B) \cong \mathbb{Z}/2 \times \mathbb{Z}/2. \tag{3.38}$$

We will prove this directly, using our calculations from the proof of Proposition 3.25. Just like in that proof, the Atiyah–Hirzebruch spectral sequence approach to Corollary 3.37 is conceptually simpler but involves some more difficult computations.

Remark 3.39. To make sense of $(0, B)$-twisted supercohomology, we require $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ to be a ko-module. We define its ko-module structure as follows: since Postnikov truncation is lax symmetric monoidal (see [HNP25, Example/Proposition 4.15]) and the homotopy groups of $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ are concentrated between degrees 0 and 2 by construction, $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ acquires a canonical $\tau_{\leq 2}\mathbb{S}$-module structure. Since the unit map $\mathbb{S} \to ko$ is 2-connected, it induces an equivalence $\tau_{\leq 2}\mathbb{S} \xrightarrow{\simeq} \tau_{\leq 2} ko$, giving $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ the structure of a $\tau_{\leq 2} ko$-module. Finally, precomposing with the algebra map $ko \to \tau_{\leq 2} ko$, we get a ko-module structure on $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$.

In order to prove Corollary 3.37, we need a simpler description of $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$. We will give it in Proposition 3.50 after a few lemmas.

The following lemma is well-known.

Lemma 3.40. *Up to ko-module equivalence, there is a unique ko-module M such that*

1. $\pi_0(M) \cong \pi_1(M) \cong \mathbb{Z}/2$,
2. $\pi_n(M) = 0$ *for* $n \neq 0, 1$, *and*
3. $\eta\colon \pi_0(M) \to \pi_1(M)$ *is nonzero.*

Proof. Existence is straightforward: $\Sigma^{-1}\tau_{1:2}ko$ meets the conditions in the lemma statement. On to uniqueness.

Consider the Postnikov fiber sequence

$$\tau_{\geq 1}M \longrightarrow M \xrightarrow{\tau_{\leq 0}} \tau_{\leq 0}M, \tag{3.41a}$$

which is a fiber sequence of ko-modules. $\tau_{\geq 1}M$ has a single nonzero homotopy group, which is $\pi_1 \cong \mathbb{Z}/2$, so $\tau_{\geq 1}M \simeq H\mathbb{Z}/2$. By a similar argument, $\tau_{\leq 0}M \simeq H\mathbb{Z}/2$. If one takes the cofiber of $\tau_{\leq 0}$ in (3.41a), the result is a cofiber sequence

$$M \xrightarrow{\tau_{\leq 0}} H\mathbb{Z}/2 \xrightarrow{k} \Sigma^2 H\mathbb{Z}/2, \tag{3.41b}$$

for some ko-module map k. In particular, the homotopy class of k in $\pi_0\mathrm{Map}_{ko}(H\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}/2) \cong H^2_{ko}(H\mathbb{Z}/2)$ determines M up to ko-module equivalence, as the fiber of k. By Lemma 1.31, part (1), $H^*_{ko}(H\mathbb{Z}/2) \cong \mathcal{A}(1)$, whose subgroup of homogeneous, degree-2 elements is isomorphic to $\mathbb{Z}/2$, so there are exactly two homotopy classes of ko-modules belonging to a fiber sequence of the form (3.41b).

If $k \simeq 0$, (3.41b) splits and $M \simeq H\mathbb{Z}/2 \vee \Sigma H\mathbb{Z}/2$. Since η acts trivially on $\pi_*(H\mathbb{Z}/2)$, η acts trivially on this module; therefore there is at most ko-module up to homotopy equivalence meeting the three conditions in the lemma statement. □

Lemma 3.42. *There are $\mathcal{A}(1)$-module isomorphisms*

$$H^*_{ko}(\tau_{1:2}ko) \cong \Sigma Q \oplus \Sigma^5 \mathring{\mathcal{O}} \tag{3.43a}$$

$$H^*_{ko}(H\mathbb{Z}/8) \cong \Upsilon_1 \oplus \Sigma\Upsilon_1. \tag{3.43b}$$

The ko-module structure on $H\mathbb{Z}/8$ is induced from the $H\mathbb{Z}$-module structure on $H\mathbb{Z}/8$ by the Postnikov quotient $\tau_{\leq 0}\colon ko \to H\mathbb{Z}$. Names of $\mathcal{A}(1)$-modules are as in Definition 3.11.

Proof sketch. Apply H^*_{ko} to the cofiber sequences of ko-modules

$$\tau_{\geq 4}ko \xrightarrow{\tau_{\geq 3}} \tau_{\geq 1}ko \xrightarrow{\tau_{\leq 2}} \tau_{1:2}ko \tag{3.44a}$$

$$H\mathbb{Z} \xrightarrow{8} H\mathbb{Z} \longrightarrow H\mathbb{Z}/8 \tag{3.44b}$$

to obtain long exact sequences of $\mathcal{A}(1)$-modules. By Lemma 1.51, $H^*_{ko}(\tau_{\geq 4}ko) \cong \Sigma^4 \mathring{\mathcal{O}}$ and $H^*_{ko}(\tau_{\geq 1}ko) \cong \Sigma Q$, and by [Bak20, §5], $H^*_{ko}(H\mathbb{Z}) \cong \Upsilon_1$. The lemma statement then follows directly from exactness of these two long exact sequences. □

Corollary 3.45. $\pi_0\mathrm{Map}_{ko}(\Sigma^{-1}\tau_{1:2}ko, \Sigma^3 H\mathbb{Z}/8)$ *is isomorphic to either* $\mathbb{Z}/4$ *or* $\mathbb{Z}/8$.

Proof. Since $H\mathbb{Z}/8$ is 2-local, we apply the Baker–Lazarev Adams spectral sequence. The E_2-page is

$$\begin{aligned} E_2^{s,t} &\cong \operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_{ko}(\Sigma^3 H\mathbb{Z}/8), H^*_{ko}(\Sigma^{-1}\tau_{1:2}ko)) \\ &\cong \operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(\Sigma^3\Upsilon_1 \oplus \Sigma^4\Upsilon_1, Q \oplus \Sigma^4\mathring{\mathcal{O}}) \end{aligned} \tag{3.46a}$$

by Lemma 3.42. Since $\Upsilon_1 \cong \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ (Definition 2.34), the change-of-rings theorem (see for example [BC18, §4.5]) simplifies the E_2-page to

$$\operatorname{Ext}^{s,t}_{\mathcal{A}(0)}(\Sigma^3\mathbb{Z}/2 \oplus \Sigma^4\mathbb{Z}/2, Q \oplus \Sigma^4\mathring{\mathcal{O}}). \tag{3.46b}$$

One can directly verify that there are $\mathcal{A}(0)$-module isomorphisms $Q \cong \mathcal{A}(0) \oplus \Sigma^3\mathbb{Z}/2$ and $\mathring{\mathcal{O}} \cong \mathbb{Z}/2 \oplus \Sigma^2\mathcal{A}(0)$ (see, e.g., [DDHM24, Figure 31]). Ext commutes with direct sums in both arguments, reducing the computation of (3.46b) to that of $\operatorname{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathcal{A}(0))$ and $\operatorname{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2)$. The former consists of a single $\mathbb{Z}/2$ in bidegree $(s,t) = (0,-1)$, which follows from graded self-injectivity of $\mathcal{A}(0)$ (see [Mar83, Proposition 12.8, Theorem 12.9]) or can be worked out directly, and the latter is $\mathbb{Z}/2[h_0]$ with h_0 in bidegree $(1,1)$ [BC18, Remark 4.5.4].

Putting it all together, the E_2-page contains four $\mathbb{Z}/2[h_0]$ summands ("h_0-towers") in topological degrees -1, 0, 0, and 1, together with $\mathbb{Z}/2$ summands in filtration 0 and topological degrees -4, -3, 2, and 3. Since $H\mathbb{Z}/8$ vanishes rationally, $\pi_*\mathrm{Map}_{ko}(\Sigma^{-1}\tau_{1:2}ko, \Sigma^3 H\mathbb{Z}/8) \otimes \mathbb{Q} \cong 0$. Therefore differentials must kill all but finitely many of the elements of the four h_0-towers. This means that one of the h_0-towers in degree 0 supports a d_{r_1} differential to the h_0-tower in degree -1, so this entire tower vanishes after the E_{r_1}-page, and the other h_0-tower in degree 0 must receive a d_{r_2} differential from the h_0-tower in degree 1, for some $r_2 \geq 2$. Thus, on the E_∞-page, in topological degree 0 we have $\mathbb{Z}/2[h_0]/(h_0^{r_2+1})$; since h_0 lifts to multiplication by 2, this implies $\pi_0\mathrm{Map}_{ko}(\Sigma^{-1}\tau_{1:2}ko, \Sigma^3 H\mathbb{Z}/8) \cong \mathbb{Z}/2^{r_2}$.

Finally, we need to show $r_2 \leq 3$. If it were larger, a group of homotopy classes of maps into $H\mathbb{Z}/8$ would contain elements on which $8 \neq 0$, but this cannot occur. □

In what follows, we will make use of the *Anderson dual of the sphere*, $I_{\mathbb{Z}}$ [And69]; see [FMS07, Appendix B] or [FH21, §5.3] for a review.

Lemma 3.47. *Up to ko-module equivalence, there is a unique ko-module N such that*

1. $\pi_0(N) \cong \mathbb{Z}/8$, $\pi_1(N) \cong \mathbb{Z}/2$, *and* $\pi_2(N) \cong \mathbb{Z}/2$,
2. $\pi_n(N) = 0$ *for* $n \notin \{0,1,2\}$, *and*
3. $\eta^2 \colon \pi_0(N) \to \pi_2(N)$ *is nonzero.*

Proof. The universal property of Anderson duality implies that it is equivalent to show that there is a unique ko-module N' up to ko-module equivalence satisfying conditions (2) and (3), but with (1) replaced with $\pi_0(N') \cong \mathbb{Z}/2$, $\pi_1(N') \cong \mathbb{Z}/2$, and $\pi_2(N') \cong \mathbb{Z}/8$. This is because, if N' satisfies these desiderata, then $N \coloneqq \Sigma^3 I_{\mathbb{Z}} N'$ satisfies the conditions in the lemma statement, and vice versa, so uniqueness of N' implies uniqueness of N. Thus in the rest of the proof we will focus on N'.

For existence, take $N' \coloneqq \tau_{\leq 2}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})$. The facts we need about its homotopy groups and η^2-action are standard, and follow from Mahowald's [Mah82, Lemma 7.3], resp. Bruner–Mira–Stanley–Snaith's [BMSS15, §3.1] computation of $ko_*(B\mathbb{Z}/2)$ as a graded abelian group, resp. ko_*-module, after using the Smith isomorphism $\Sigma^{-1}\Sigma^\infty B\mathbb{Z}/2 \simeq (B\mathbb{Z}/2)^{\sigma-1}$ to pass from $\Sigma^{-1}ko \wedge B\mathbb{Z}/2$ to $ko \wedge (B\mathbb{Z}/2)^{\sigma-1}$: see, e.g., [Koc96, Lemma 2.6.5].

Thus we turn to uniqueness. By a similar line of reasoning as we used to build (3.41b) during the proof of Lemma 3.40, there is a fiber sequence of ko-modules

$$N' \xrightarrow{\tau_{\leq 1}} M \xrightarrow{k} \Sigma^3 H\mathbb{Z}/8, \tag{3.48}$$

where M is a ko-module satisfying the conditions of Lemma 3.40. Thus, as we proved in Lemma 3.40, $M \simeq \Sigma^{-1}\tau_{1:2}ko$ as ko-modules, so by Corollary 3.45, the group of homotopy classes of possible maps k in (3.48) is isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/8$.

Precomposing with $\tau_{\geq 1}\colon \Sigma H\mathbb{Z}/2 \simeq \tau_{\geq 1}M \to M$ defines a homomorphism

$$\tau_{\geq 1}^*\colon \pi_0\mathrm{Map}_{ko}(M, \Sigma^3 H\mathbb{Z}/8) \longrightarrow \pi_0\mathrm{Map}_{ko}(\Sigma H\mathbb{Z}/2, \Sigma^3 H\mathbb{Z}/8) \tag{3.49}$$

sending k to the Postnikov invariant of $\tau_{\geq 1}N'$. Since $2 = 0$ in $\pi_0\mathrm{Map}_{ko}(\Sigma H\mathbb{Z}/2, \Sigma^3 H\mathbb{Z}/8)$, then if k is even, then $\tau_{\geq 1}^*(k) = 0$, implying $\tau_{\geq 1}N' \simeq \Sigma H\mathbb{Z}/2 \vee \Sigma^2 H\mathbb{Z}/8$. This is incompatible with our requirement that $\eta^2\colon \pi_0(N') \to \pi_2(N')$ is nonzero.

Thus k is an odd element of $\mathbb{Z}/4$ or $\mathbb{Z}/8$. In this case, there is an automorphism of $\Sigma^3 H\mathbb{Z}/8$ sending $k \mapsto 1$. Thus there is at most one homotopy class of ko-modules meeting the conditions in the lemma statement. □

Proposition 3.50. *There is a ko-module equivalence*

$$|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})| \simeq \Sigma^2 I_{\mathbb{Z}}(\tau_{\leq 2}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})). \tag{3.51}$$

We believe Proposition 3.50 is known to some experts in the field, but are unaware of an explicit reference in the literature. See [GJF19, §6.4] for a closely related result.

Proof sketch. We will show both of these ko-modules meet the hypotheses of Lemma 3.47, so that the conclusion of that lemma implies they are ko-module equivalent. In fact, for $\Sigma^2 I_{\mathbb{Z}}\tau_{\geq 0}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})$, this follows from our analysis of $\tau_{\leq 2}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})$ during the proof of Lemma 3.47 together with the universal property of Anderson duality.

Thus we focus on $|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$. Freed [Fre12, (1.40)] showed that $\pi_*(|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|)$ satisfies conditions (1) and (2) of Lemma 3.47. That $\eta^2 \neq 0$ as maps $\pi_0(|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|) \to \pi_2(|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|)$ can be shown in the same way as is shown for $s\mathcal{A}lg_{\mathbb{C}}$ in [DG18, §4.3.5]. □

Corollary 3.52. *Let $h_{0,B}\colon B^2\mathbb{Z}/2 \to B\mathrm{GL}_1(ko)$ be the twist induced by $(0, B)$ as defined in Remark A.27. There is an isomorphism*

$$(\Sigma^{-2}|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|)^5(B^2\mathbb{Z}/2, 0, B) \cong \pi_4(ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \wedge_{ko} M^{ko}h_{0,B}). \tag{3.53}$$

Proof sketch. Apply Proposition 3.50 to replace $\Sigma^{-2}|\mathrm{Pic}(s\mathcal{A}lg_{\mathbb{R}})|$ with $\Sigma^4 I_{\mathbb{Z}}(\tau_{\leq 2}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1}))$. But since $ko_n((B\mathbb{Z}/2)^{\sigma-1}) = 0$ for $n = 3, 4, 5$ (see for example [Cam17, Theorem 6.4] or [DKS26, Example 5.31]), truncation defines a ko-module equivalence

$$\tau_{\leq 2}^*\colon \Sigma^4 I_{\mathbb{Z}}(\tau_{\leq 2}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})) \xrightarrow{\simeq} \Sigma^4 I_{\mathbb{Z}}(\tau_{\leq 5}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})). \tag{3.54}$$

Next apply the universal property of Anderson duality to identify the degree-5 twisted $\Sigma^4 I_{\mathbb{Z}}(\tau_{\leq 5}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1}))$-cohomology of $B^2\mathbb{Z}/2$ with the degree-4 twisted $\tau_{\leq 5}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1})$-homology of $B^2\mathbb{Z}/2$. Finally, since $\tau_{\leq 5}$ is 5-connected, it induces an isomorphism on (twisted) homology in degrees 5 and below, so we may remove $\tau_{\leq 4}$, giving the right-hand side of (3.53). □

Finally we can return to our reformulation of Sanford's result [San26, Theorem 5.10].

Proof of Corollary 3.37. By Corollary 3.52, it suffices to show

$$\pi_4(ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \wedge_{ko} M^{ko}h_{0,B}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{3.55}$$

We will do this with the Baker–Lazarev Adams spectral sequence. The E_2-page is the Ext of the $\mathcal{A}(1)$-module

$$H^*_{ko}(ko \wedge (B\mathbb{Z}/2)^{\sigma-1} \wedge_{ko} M^{ko}h_{0,B}) \cong H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2) \otimes H^*_{ko}(M^{ko}h_{0,B}) \tag{3.56a}$$

by Lemma 1.55 (and the identification $H^*_{ko}(ko \wedge X) \cong H^*(X; \mathbb{Z}/2)$ [DKS26, Lemma 5.16]). By (3.28) and the identification $B\mathbb{Z}/2 \simeq \Sigma(B\mathbb{Z}/2)^{\sigma-1}$ mentioned above,

$$H^*_{ko}(M^{ko}h_{0,B}) \otimes H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2) \cong (\Upsilon_1 \oplus \Sigma^4\Upsilon_1 \oplus \Sigma^5\mathcal{A}(1) \oplus L) \otimes P, \tag{3.56b}$$

where L is 5-connected, and therefore can be ignored for our π_4 computation.

Since $\Upsilon_1 \cong \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ (Definition 2.34), the change-of-rings theorem [BC18, §4.5] implies

$$\mathrm{Ext}_{\mathcal{A}(1)}(\Upsilon_1 \otimes P, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}(0)}(P, \mathbb{Z}/2). \tag{3.57}$$

One can check directly that, as an $\mathcal{A}(0)$-module, P is free on a homogeneous basis with one element in each nonnegative even degree. Since $\mathrm{Ext}_{\mathcal{A}(0)}(\mathcal{A}(0), \mathbb{Z}/2)$ consists of a $\mathbb{Z}/2$ in bidegree $(0,0)$, we conclude that $\mathrm{Ext}_{\mathcal{A}(0)}(P, \mathbb{Z}/2)$ consists of a $\mathbb{Z}/2$ in each bidegree $(s,t) = (0, 2k)$ for each $k \geq 0$.

By a similar argument, $\mathrm{Ext}_{\mathcal{A}(1)}(\mathcal{A}(1) \otimes P; \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathbb{Z}/2}(P, \mathbb{Z}/2)$. Since $\mathbb{Z}/2$ is a field and P is connective, these Ext groups vanish except when $s = 0$ and $t \geq 0$.

Combining all of these facts, we see that in topological degree 4, the E_2-page of the Baker–Lazarev Adams spectral sequence has $E_2^{0,4} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and for all $t - s \leq 5$, the E_2-page vanishes for $s > 0$. Thus the spectral sequence collapses in this range to imply (3.55). □

3.3.4. *The quaternion group of order 8.* When performing the calculations of either ko-homology or supercohomology, it is sometimes helpful to be able to compute the Atiyah–Hirzebruch filtration of a given class x. In the mathematical physics literature, this is often called the *layer* of x [WG20]. This controls where the image of x on the E^∞-page of the AHSS is.

Specializing to supercohomology, for any space X and twisting data (a, b), there is a map $(-)_{\mathrm{Maj}} \colon SH^k(X, a, b) \to H^{k-2}(X; \mathbb{Z}/2)$, called the *Majorana layer* [WG20, DYY25], dual to the 1-connected cover

$$\tau_{\geq 2} \colon \Sigma^2 H\mathbb{Z}/2 \simeq \tau_{\geq 2}\tau_{\leq 2}ko \longrightarrow \tau_{\leq 2}ko. \tag{3.58}$$

Johnson-Freyd–Yu [JFY21, §1] asked whether there is a finite group G such that the Majorana layer homomorphism $SH^4(BG, 0, 0) \to H^2(BG; \mathbb{Z}/2)$ is nonzero, and Décoppet–Johnson-Freyd [Déc24b, Appendix A] showed that this happens for $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. This information helps determine whether the corresponding topological orders are bosonic gauge theories in the usual sense. In [DYY25, Appendix C] (see also *ibid.*, "Definition" 3.15), we defined the *$p + ip$ layer* associated to a ko-homology or spin bordism class, which is the part of the Atiyah–Hirzebruch filtration corresponding to $\pi_4 ko \cong \mathbb{Z}$. The $p + ip$ layer acts as an obstruction for an invertible field theory to be the anomaly of a unitary TFT [DYY25, DYY26b].

Using the map Φ^{ko} from Theorem 1.35, which maps from the Atiyah–Hirzebruch to the Adams filtration, we can use the Adams filtration of a class to constrain which layer it lies in. We will apply this for $G = Q_8$ and show that the Majorana layer homomorphism $SH^4(BG, 0, 0) \to H^2(BG; \mathbb{Z}/2)$ is in fact zero, using the HASS to calculate $SH^*(BQ_8)$ and Φ^{ko} to compute the Majorana layer homomorphism.

First, the supercohomology calculation.

Proposition 3.59. *Let $j \geq 1$. Then there are isomorphisms*

$$\begin{aligned} SH^0(BQ_8) &\cong \mathbb{C}^\times & SH^{4j}(BQ_8) &\cong \mathbb{Z}/2 \\ SH^1(BQ_8) &\cong (\mathbb{Z}/2)^{\oplus 3} & SH^{4j+1}(BQ_8) &\cong (\mathbb{Z}/2)^{\oplus 2} \\ SH^2(BQ_8) &\cong (\mathbb{Z}/2)^{\oplus 3} & SH^{4j+2}(BQ_8) &\cong (\mathbb{Z}/2)^{\oplus 3} \\ & & SH^{4j-1}(BQ_8) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/4)^{\oplus 2}. \end{aligned} \tag{3.60}$$

The unique nonzero element of $(\tau_{\leq 2} ko)_{4j}(BQ_8) \cong \mathbb{Z}/2$ has Adams filtration 1.

Proof. We will compute reduced $\tau_{\leq 2}ko$-homology and trust the reader to dualize and add back in $SH^*(\mathrm{pt})$ at the end. Mitchell–Priddy [MP84, Theorem D] produce a 2-local stable splitting

$$\Sigma^\infty BQ_8 \simeq_{(2)} \Sigma^\infty B\mathrm{SL}_2(\mathbb{F}_3) \vee C \vee C, \tag{3.61}$$

where C is the fiber of the map $\Sigma^\infty B\mathrm{Pin}_2^- \to \Sigma^\infty B\mathrm{Spin}_3$ induced by the usual inclusion $\mathrm{Pin}_2^- \hookrightarrow \mathrm{Spin}_3$. Thus we may study $\widetilde{SH}^*(B\mathrm{SL}_2(\mathbb{F}_3)))$ and $SH^*(C)$ separately, then 2-localize and direct-sum them together.

Bayen–Bruner [BB96, §3] produce $\mathcal{A}(1)$-module isomorphisms

$$\widetilde{H}^*(B\mathrm{SL}_2(\mathbb{F}_3); \mathbb{Z}/2) \cong \bigoplus_{j \geq 1}^{\infty} \Sigma^{4j-1}\mathbb{Z}/2 \oplus \Sigma^{4j}\mathbb{Z}/2 \tag{3.62a}$$

$$H^*(C; \mathbb{Z}/2) \cong \bigoplus_{j \geq 0}^{\infty} \Sigma^{4j+1} N_1, \tag{3.62b}$$

using Quillen's calculation [Qui72, §8] of $H^*(B\mathrm{SL}_2(\mathbb{F}_3); \mathbb{Z}/2)$ (see [MP84, Theorem 2.4] for the ring structure), resp. Mitchell–Priddy's calculation [MP84, §6] of $H^*(C; \mathbb{Z}/2)$.

Thus, using the charts for $\mathcal{Q}(\mathbb{Z}/2)$ and $\mathcal{Q}(N_1)$ that we computed in Figures 4a and 4d, we draw the E_2-pages of the HASSes for $\tau_{\leq 2}ko$ of $B\mathrm{SL}_2(\mathbb{F}_3)$ and C in Figure 7, upper left and lower left, respectively. Theorem B.12 shows that $\partial_2 \colon \mathrm{Ext}(M) \to \mathcal{Q}(M)$ is surjective for $M \in \{\mathbb{Z}/2, N_1\}$, so for both $B\mathrm{SL}_2(\mathbb{F}_3)$ and C, the map on E_2-pages from the usual Adams spectral sequences for ko-homology to the HASSes for $\tau_{\leq 2}ko$-homology are surjective. Thus the same is true on E_3 if there are no differentials in the ko-homology Adams spectral sequence on E_2.

Bayen–Bruner [BB96, §3] compute all differentials in the Adams spectral sequences for $\widetilde{ko}_*(B\mathrm{SL}_2(\mathbb{F}_3))$ and $ko_*(C)$ (see *ibid.*, Figures 1, 2, 3, and 4). As described in the previous paragraph, these imply the following differentials in the HASSes for $B\mathrm{SL}_2(\mathbb{F}_3)$ and C.

- For $B\mathrm{SL}_2(\mathbb{F}_3)$, Bayen–Bruner's computations together with our calculation of ∂_2 in Theorem B.12 imply that in the HASS, $d_2 = 0$ and d_3 is uniquely determined by the fact that it vanishes except in topological degrees $t - s = 4j$, $j \geq 1$. On all later pages, differentials vanish for degree reasons, so $E_4 = E_\infty$.
- For C, Bayen–Bruner's computations together with Theorem B.12 imply that in the HASS, d_2 is nonzero precisely in topological degrees $t - s = 4j + 1$, $j \geq 1$, which determines it uniquely. From E_3 onward, the HASS collapses for degree reasons.

We draw the E_∞-pages of these spectral sequences in Figure 7, top right and bottom right, respectively. For C, degree reasons imply there are no extension problems and we are done. For $B\mathrm{SL}_2(\mathbb{F}_3)$, there a priori could be a hidden extension in degrees $4j + 1$, $j \geq 1$. These classes are in the image of the map from ko, and Bayen–Bruner [BB96, Theorem 5] show that the corresponding extension in ko-homology splits, implying the extension in $\tau_{\leq 2}ko_{4j+1}(BQ_8)$ also splits.

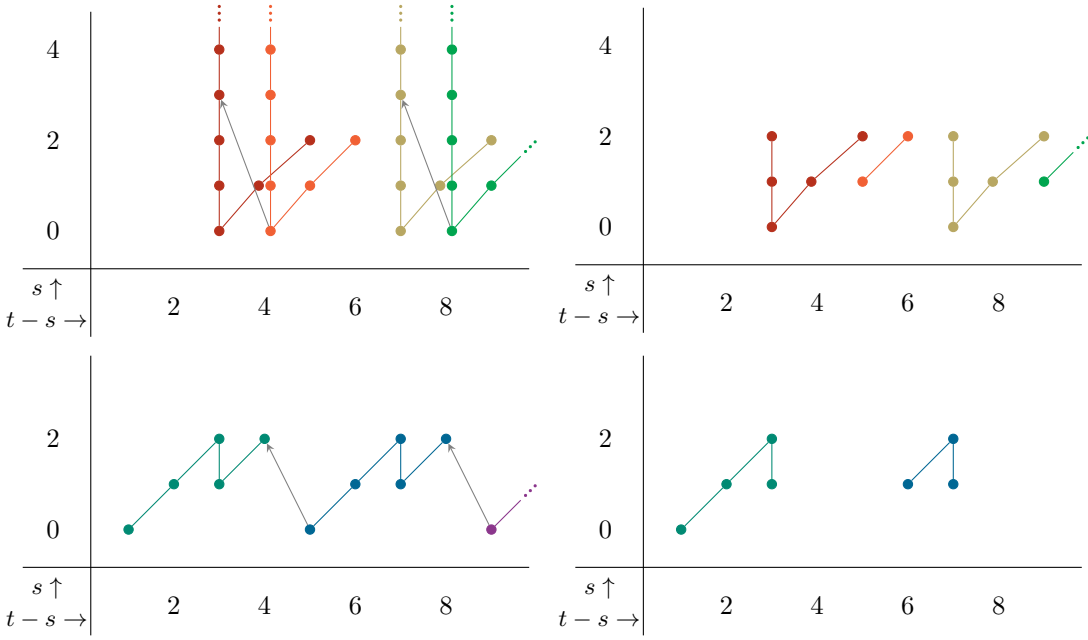


FIGURE 7. The $\tau_{\leq 2}ko$ HASSes for the direct summands of BQ_8. Upper left: the $E_2 = E_3$-page of the HASS computing $\widetilde{(\tau_{\leq 2}ko)}_*(B\mathrm{SL}_2(\mathbb{F}_3))^\wedge_2$. In the proof of Proposition 3.59, we deduce the pictured differentials from the corresponding differentials in the ordinary Adams spectral sequence for $\widetilde{ko}_*(B\mathrm{SL}_2(\mathbb{F}_3))$, which Bayen–Bruner [BB96, §3] calculated. Upper right: the $E_4 = E_\infty$-page. Lower left: the E_2-page of the corresponding HASS for the spectrum $C \coloneqq \mathrm{fib}(\Sigma^\infty B\mathrm{Pin}_2^- \to \Sigma^\infty B\mathrm{Spin})$, whose differentials follow in the same way. Lower right: the $E_3 = E_\infty$-page of this HASS. We use these computations in Proposition 3.59 as key steps in the computation of $SH^*(BQ_8)$.

The last claim to verify is the Adams filtration of the unique nonzero class x_{4j} in degrees $4j$, $j \geq 1$. This class is in the $B\mathrm{SL}_2(\mathbb{F}_3)$ direct summand, since we have just shown $\tau_{\leq 2}ko_{4j}(C) = 0$. Inspecting Figure 7, upper right, we see that the image of x_{4j} in the E_∞-page has filtration 1, so we are done. □

Corollary 3.63. *For all j, the Majorana layer homomorphism*

$$(-)_{\mathrm{Maj}} \colon SH^{4j}(BQ_8) \to H^{4j-2}(BQ_8; \mathbb{Z}/2)$$

vanishes.

Proof. By the universal property of Anderson duality, it suffices to show that the Anderson dual map $\tau_{\geq 2} \colon H_{4j-2}(BQ_8; \mathbb{Z}/2) \to (\tau_{\leq 2}ko)_{4j}(BQ_8)$ is zero. Equivalently, we want to show that $\tau_{\leq 2}ko_{4j}(BQ_8)$ has no elements x of Atiyah–Hirzebruch filtration 2 or higher. The map Φ^{ko} between the AHSS and Adams filtrations implies that any such element x would then have Adams filtration at least 2, which we proved does not happen in Proposition 3.59. □

4. Further directions

In this section, we list some computations and conjectures regarding the HASSes we studied in this paper and potential generalizations.

4.1. **The HASS for restricted supercohomology.** Though we focused on $\tau_{\leq 2}ko$ and applications to supercohomology in this paper, our main theorems hold for all Postnikov truncations of ko and ku, and the calculation of the corresponding HASSes proceeds similarly. It would be interesting to study the HASS for other truncations of ko, or of any truncation of ku.

In the specific example of $\tau_{\leq 1}ko$, Pontrjagin dual to *restricted supercohomology* rSH [Fre08, GW14], one could apply HASS computations towards several problems.

Example 4.1. Restricted supercohomology classifies levels of spin Chern–Simons theories [Jen05, FN23] as well as a class of twists of string bordism and topological modular forms: see [FHT10, JF20, DY23a, TY23a, TY23b, TY25a, BLM26].

Example 4.2. Let $J_{\mathrm{PL}}\colon \mathrm{PL} \to \mathrm{GL}_1(\mathbb{S})$ be the piecewise linear J-homomorphism. By work of Sullivan [Sul66], maps from a space M to the cofiber $\mathrm{GL}_1(\mathbb{S})/\mathrm{PL}$ of J_{PL} are closely related to the classification of piecewise linear manifold structures on M. Sullivan [Sul96, Theorem 4(i)] (see also Madsen–Milgram [MM79, Theorem 4.34(a)]) showed that, 2-locally,[16] $\mathrm{GL}_1(\mathbb{S})/\mathrm{PL}$ splits as a space into a product of Eilenberg–Mac Lane spaces and one factor of $\Omega^{\infty-4}rSH$. Thus to classify the PL structures on M, one needs to compute $rSH^4(M)$, which by Anderson duality is determined by $\tau_{\leq 1}ko_i(M)$ for $i = 3, 4$.

Prior work computing $[-, \mathrm{GL}_1(\mathbb{S})/\mathrm{PL}]$ for applications to classifying PL structures includes [Wal68, LdM71, Rud74, Rub79, Muk03, Muk04, Muk06, Wan22, Bak26]. These works used a variety of ad hoc methods to resolve extension questions in rSH-cohomology of various manifolds of interest, such as projective spaces and Dold manifolds. The $\tau_{\leq 1}ko$-HASS may enable a systematic approach or an extension to more classes of manifolds.

Example 4.3. Johnson-Freyd–Treumann [JFT20, §6] describe how understanding $rSH^*(BQ_{16})$ can be used to calculate fractional Pontrjagin classes of representations of the Conway group Co_0, with applications to conformal field theory. See also the discussion in [JF20, §2.3].

We predict that, like for (extended) supercohomology, the HASS will solve extension problems better than the AHSS (which in this case simplifies to the long exact sequence of [Fre08, (1.3)]).

4.2. **Generalizations.** In this section we gather some conjectures on the HASSes we constructed and some possible generalizations.

First, both the R-module Adams and Atiyah–Hirzebruch spectral sequences are *multiplicative*: applied to an R-algebra spectrum, one obtains a natural product on each page of these spectral sequences, and differentials satisfy a Leibniz rule with respect to this product. More generally, there is an analogue of the cross product relating the spectral sequences of two R-modules M and N with that of $M \wedge_R N$. The analogous statement at the level of filtered spectra is the claim that the Atiyah–Hirzebruch and Adams filtrations, as functors from R-modules to filtered R-modules have lax symmetric monoidal structures.[17] Surprisingly, this does not appear to be in the literature in general.

[16] The homotopy type of $\mathrm{GL}_1(\mathbb{S})/\mathrm{PL}$ away from 2 is closely related to that of KO [Sul96, Theorem 4(ii)].

[17] See Hedenlund [Hed20, Theorem II.3.5], Antieau [Ant24, Theorem 8.9], and Wu [Wu26, Theorem E] for the details of the connection between multiplicative spectral sequences as defined above and lax symmetric monoidal structures on filtrations.

- The symmetric monoidal structure on $\mathrm{AH}^{\mathbb{S}}_\bullet$ is discussed in [vN25, Remark 2.25]. The same proof also applies to $\mathrm{AH}^R_\bullet$, because the result in *loc. cit.* follows directly from the results in [Hed20, §II.1.4], which are proven in the generality of a stable ∞-category with t-structure, such as $\mathcal{M}od_R$.
- For $\mathrm{Ad}^{\mathbb{S}}_\bullet$, the lax symmetric monoidal structure is due to Pstrągowski [Pst23b]: see Patchkoria–Pstrągowski [PP26, Theorem 5.64 and Remark 5.65], who also removed an Adams type assumption. For more general E_∞-ring spectra R, this does not seem to be written down, though it would presumably follow from a good theory of synthetic R-module spectra (see Remark 2.23).

Conjecture 4.4. There is a lax symmetric monoidal structure on $\mathrm{Ad}^R_\bullet$ refining the multiplicative structure on the Baker–Lazarev Adams spectral sequence, such that the natural transformations $\Phi^{ko}\colon \mathrm{AH}^{ko}_\bullet \Rightarrow \mathrm{Ad}^{ko}_{g(\bullet)}$ and $\Phi^{\ell}\colon \mathrm{AH}^{\ell}_\bullet \Rightarrow \mathrm{Ad}^{\ell}_{g^c(\bullet)}$ we constructed in Theorem 1.35 are lax monoidal.

Carrick [Car25] studies a map between the Adams–Novikov and slice filtrations in genuine equivariant homotopy theory, and proves that this map sends E_∞-algebras to E_∞-algebras, so Conjecture 4.4 suggests a parallel to his result.

Conjecture 4.4 would imply that Φ^{ko} and Φ^{ℓ} commute with products, and should also lead to a proof that the HASSes we constructed for $\tau_{\leq n}ko$ and $\tau_{\leq n}\ell$ are also multiplicative. This would be helpful for computations.

Remark 4.5. We gave two quite different proofs of Theorem 1.35 in two cases of ko and ℓ. The proof for ℓ began with an identification of the ℓ-module Atiyah–Hirzebruch filtration with the $H\mathbb{Z}_{(p)}$-based, ℓ-module Adams filtration (Theorem 1.41). Showing this is a monoidal natural isomorphism would immediately imply Conjecture 4.4 in the case of ℓ.

Unfortunately, this approach does not seem to work for ko. The approach we found in this case seems much harder to make monoidal.

Our other dream is that there are surely more examples of this Atiyah–Hirzebruch-to-Adams phenomenon. We list a few that we are especially optimistic about.

Conjecture 4.6. Recall (e.g. from [Hys26, §6]) that the *connective, self-conjugate K-theory spectrum*, $ksc \coloneqq ko/\eta^2$, is an E_∞-ko-algebra.

(1) Let $\mathrm{HAd}^{ksc}_\bullet$ denote the (2-primary) hastened Adams filtration associated to the fiber sequence $\Sigma^2 ko \xrightarrow{\eta^2} ko \to ksc$. Then, there is a natural transformation of functors $\mathcal{M}od_{ksc} \to \mathcal{F}i\ell_{ksc}$)

$$\Phi^{ksc}\colon \mathrm{AH}^{ksc}_\bullet \Longrightarrow \mathrm{HAd}^{ksc}_{g^{ksc}(\bullet)} \tag{4.7}$$

which, when composed with $(-)_{-\infty}$, is 2-completion. (The function g^{ksc} is defined in Theorem 1.49.)

(2) There is a hastened Adams spectral sequence for $\tau_{\leq k}ksc$-homology analogous to the ones we constructed for ko and ℓ.

The hastened Adams filtration mentioned in part (1) of Conjecture 4.6, as well as the induced spectral sequence, do not appear in the literature to our knowledge, but are known to some experts in the field. They can be constructed by adapting the argument we gave for $\tau_{\leq 2}ko$, using the fact that $\eta^2\colon \Sigma^2 ko \to ko$ factors through $\tau_{\leq 2}ko \to ko$.

Another direction we speculate in is motivic and equivariant analogues of our main results. Motivic and G-equivariant stable homotopy theory are generalizations of stable homotopy theory in

which, approximately speaking, topological spaces are replaced with smooth projective varieties over a base scheme, resp. topological spaces with a G-action, where G is typically a compact Lie group. In these settings, there are analogues of ko and ku whose Adams and Atiyah–Hirzebruch spectral sequences behave somewhat similarly to the classical case (at least for $G = \mathbb{Z}/2$ in equivariant stable homotopy theory).

- In motivic stable homotopy theory, the analogue of ku is $kg\ell$, the effective cover of K-theory, constructed by Voevodsky (see [Voe98, §6.2] and [Voe02, §2]). The analogue of ko, denoted kq, is the very effective cover of Hermitian K-theory.[18]
- In $\mathbb{Z}/2$-equivariant stable homotopy theory, the analogues of ku, resp. ko are kr, resp. $ko_{\mathbb{Z}/2}$, the equivariant connective covers (see [Lew95, Proposition 3.3]) of Atiyah's Real K-theory [Ati66] resp. $\mathbb{Z}/2$-equivariant KO-theory.[19]

In all of these cases, there is a version of the Adams spectral sequence whose E_2-page is isomorphic to Ext over an algebra analogous to $\mathcal{E}(1)$ or $\mathcal{A}(1)$. Examples of such motivic and $\mathbb{Z}/2$-equivariant Adams spectral sequences appear in [Hil11, IS11, Orm11, OØ13, Ghe18, Qui19, GHIR20, CKQ21, CQ21, DFN+25, LPT25, Mor25, MPT25, Dha26, Mor26].

Conjecture 4.8. For R an E_∞-$\mathbb{Z}/2$-ring spectrum, let $\mathcal{M}od_R$ denote the ∞-category of R-modules in genuine $\mathbb{Z}/2$-spectra. Then:

- Let $\mathrm{AH}^R_\bullet\colon \mathcal{M}od_R \to \mathcal{F}i\ell_R$ be the $\mathbb{Z}/2$-equivariant Atiyah–Hirzebruch filtration, defined analogously to Definition 1.5.
- For $R = kr$, $ko_{\mathbb{Z}/2}$, let $\mathrm{Ad}^R_\bullet\colon \mathcal{M}od_R \to \mathcal{F}i\ell_R$ be the $\mathbb{Z}/2$-equivariant, $H\underline{\mathbb{Z}/2}$-based, R-module Adams filtration, where $\underline{\mathbb{Z}/2}$ denotes the constant Mackey functor on $\mathbb{Z}/2$.[20]

We conjecture that, for these choices of R, there are natural transformations of functors $\Phi^R_{\mathrm{mot}}\colon \mathrm{AH}^R_\bullet \Rightarrow \mathrm{Ad}^R_{\widetilde{g^R}(\bullet)}$, where $\widetilde{g}^R$ is g applied to the underlying spectrum of R, and such that, upon taking underlying spectra, one recovers the natural transformations of Theorem 1.35.

For groups G larger than $\mathbb{Z}/2$, very little is known about G-equivariant Adams spectral sequences for ko_G or G-equivariant analogues of kr. For this reason, we restrict to $G = \mathbb{Z}/2$ in Conjecture 4.8.

Conjecture 4.9. Let F be either $\mathbb{C}$ or $\mathbb{R}$, and let $kg\ell$ and kq denote the effective cover of K-theory, resp. the very effective cover of Hermitian K-theory, over the field F. When R is one of these E_∞-rings, let $\mathcal{M}od_R$ be the ∞-category of R-modules in F-motivic spectra. Then:

- Let $\mathrm{AH}^R_\bullet\colon \mathcal{M}od_R \to \mathcal{F}i\ell_R$ be the motivic Atiyah–Hirzebruch filtration, defined similarly to Definition 1.5 (see also [DFN+25] for some examples).
- Let $\mathrm{Ad}^R_\bullet\colon \mathcal{M}od_R \to \mathcal{F}i\ell_R$ be the $H\underline{\mathbb{Z}/2}$-based, R-module Adams filtration.

We conjecture that, for these choices of F and R, there are natural transformations of functors $\Phi^R_{\mathrm{eq}}\colon \mathrm{AH}^R_\bullet \to \mathrm{Ad}^R_{\widehat{g}(\bullet)}$, where $\widehat{g}$ is g after Betti realization. We conjecture that after Betti realization r_F to spectra ($F = \mathbb{C}$) or $\mathbb{Z}/2$-spectra ($F = \mathbb{R}$), these natural transformations recover $\Phi^{r_{\mathbb{C}}(R)}$ (Theorem 1.35), resp. $\Phi^{r_{\mathbb{R}}(R)}_{\mathrm{mot}}$ (Conjecture 4.8).

[18] Spitzweck–Østvær [SØ12] constructed the very effective cover functor and Bachmann [Bac17] first applied it to KQ to define kq. Ananyevskiy–Röndigs–Østvær [ARØ20] argued that kq is a good motivic analogue of ko, answering questions of Hill [Hil11, §5.2] and Isaksen–Shkembi [IS11].

[19] We also recommend Bruner–Greenlees [BG10] and Greenlees [Gre18] for more on kr and its relatives.

[20] To construct this we also need maps $ko_{\mathbb{Z}/2} \to H\underline{\mathbb{Z}/2}$ and $kr \to H\underline{\mathbb{Z}/2}$. For kr, Dugger [Dug05, §6] constructs a map $kr \to H\underline{\mathbb{Z}}$, and the map we want is that map followed by mod $\overline{2}$ reduction. For $ko_{\mathbb{Z}/2}$, precompose with the map $c\colon ko_{\mathbb{Z}/2} \to kr$ defined in [GHIR20, Definition 10.11].

Balderrama–Culver–Quigley's motivic Lefschetz principle [BCQ25, Proposition 5.2.1] and Dhankhar–Field–Nigam–Quigley–Yang's real motivic Lefschetz principle [DFN+25, Proposition 5.1] strongly suggest that Conjecture 4.9 generalizes at least to other characteristic 0 algebraically closed and real closed fields. It may generalize further.

Appendix A. Two definitions of twisted supercohomology are equivalent

In this appendix, we will characterize twisted supercohomology in two different ways.

(1) The first way, which we give in Definition A.16, uses the language of 2-categories. From this point of view, supercohomology has been used to study fusion 2-categories in the work of Johnson-Freyd–Yu [JFY21, JFY22], Décoppet–Yu [DY23b, DY26], Décoppet [Déc24b, §4, Appendix], Décoppet–Huston–Johnson-Freyd–Nikshych–Penneys–Plavnik–Reutter–Yu [DHJF+24], Johnson-Freyd [JF25], Teixeira–Yu [TY25b, §2], Gu–Luo–Wang–Zhang [GLWZ26], and Kobayashi–Prem–Yu [KPY26a, §A.4].

(2) The second way we characterize twisted supercohomology, in Definition A.25, is homotopical, using the Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14a, ABG+14b] approach to twisted generalized cohomology. From this point of view, it is easier to compute twisted supercohomology groups.

We will prove these two approaches are equivalent in Theorem A.35. In particular, these two definitions give rise to two different twists labeled by $a \in H^1(X; \mathbb{Z}/2)$ and $b \in H^2(X; \mathbb{Z}/2)$ for a space X, and we prove that these two different formalisms for twisted supercohomology are the same.

A.1. The categorical twist.

Throughout this section, when we write "2-category" we mean a bicategory, i.e. we do not require composition to be strictly associative, only up to coherent isomorphism.

Definition A.1. Let $s\mathcal{A}lg$ denote the symmetric monoidal 2-category whose objects are complex superalgebras, 1-morphisms are super bimodules, and 2-morphisms are grading-preserving bimodule homomorphisms. The symmetric monoidal structure comes from the tensor product with the Koszul sign rule.

Given a symmetric monoidal 2-category $\mathcal{C}$, let $\mathrm{Pic}(\mathcal{C})$ denote the sub-2-category of $\mathcal{C}$ of $\otimes$-invertible objects and composition-invertible 1- and 2-morphisms. This is a *Picard 2-groupoid*, meaning a symmetric monoidal 2-groupoid whose objects are all invertible with respect to the monoidal product.

The following theorem is a consequence of Baez–Dolan's cobordism hypothesis [BD95], proven by Lurie [Lur09].

Theorem A.2. *There is a canonical* O_2*-action on* $\mathrm{Pic}(s\mathcal{A}lg)$.

Let $\mathbb{G} \coloneqq \mathrm{Aut}(\mathrm{Pic}(s\mathcal{A}lg))$; since $\mathrm{Pic}(s\mathcal{A}lg)$ is a Picard 2-groupoid, $\mathbb{G}$ is a symmetric 3-group, i.e. a Picard 3-groupoid with a single isomorphism class of objects. Thus the classifying space $B\mathbb{G}$ is an E_∞-algebra whose homotopy groups vanish outside of degrees $[1, 3]$ (see Theorem 3.30).

Theorem A.2 in particular means that there is a functor ρ from the fundamental ∞-groupoid of $B\mathrm{O}_2$ to $B\mathbb{G}$, which one can think of as a "principal $\mathbb{G}$-bundle," and thus obtain an associated bundle $\mathcal{X} \to B\mathrm{O}_2$ with "fiber" $\mathrm{Pic}(s\mathcal{A}lg)$. This has a few more concrete consequences.

Corollary A.3.

(1) *The isomorphism* $\pi_1(B\mathrm{O}_2) \cong \mathbb{Z}/2$ *means that* ρ *defines an involution on* $\mathrm{Pic}(s\mathcal{A}lg)$. *By* [Lur09, Example 2.4.12]*, this is the duality involution.*

From the $\mathrm{Pic}(s\mathcal{A}lg)$*-bundle point of view, this says that the monodromy of the bundle* $\mathcal{X} \to B\mathrm{O}_2$ *around the nontrivial loop is the duality involution.*

(2) ρ *sends* $\pi_2(B\mathrm{O}_2) \cong \mathbb{Z}$ *to the* Serre automorphism S *of* $\mathrm{Pic}(s\mathcal{A}lg)$ [Lur09, Example 2.4.14].

(3) *Since* $\mathrm{Pic}(s\mathcal{A}lg)$ *is a* 2*-category, the action* ρ *is trivial on homotopy groups in degrees* 4 *and above. Since* $\pi_3(B\mathrm{O}_2) \cong 0$*, the action is also trivial in degree* 3.

Proposition A.4 (Gunningham [Gun16, §2.5, §2.6])**.** *The restriction of* ρ *across the map* $r\colon \mathrm{Spin}_2 \to \mathrm{O}_2$*, defined to be the double cover of the identity component, is canonically trivializable.*

Heuristically, the pullback of $\mathcal{X}$ to $B\mathrm{Spin}_2$ is canonically equivalent to the trivial bundle.

Proof. Since $\mathrm{Pic}(s\mathcal{A}lg)$ is a 2-category, this question factors through the 3-skeleton of $B\mathrm{Spin}_2$, which is S^2. Thus it suffices to find a trivialization for the generator of $\pi_2(B\mathrm{Spin}_2) \cong \mathbb{Z}$. The map $r\colon B\mathrm{Spin}_2 \to B\mathrm{SO}_2$ induces multiplication by 2 on π_2, since $\mathrm{Spin}_2 \to \mathrm{SO}_2$ is a double cover; the map $B\mathrm{SO}_2 \to B\mathrm{O}_2$ is the universal cover, hence is an isomorphism on π_2. Thus the generator of $\pi_2(B\mathrm{Spin}_2)$ acts via ρ as the *square* of the Serre automorphism, and it suffices to show this acts trivially.

Let A be a superalgebra. Gunningham [Gun16, §2.6] shows that trivializing the square of the Serre automorphism acting on A is naturally equivalent to specifying an isomorphism of bimodules from $A^! \coloneqq \mathrm{Hom}_{A^e}(A, A^e)$ to $A^* \coloneqq \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$, where $A^e \coloneqq A \otimes A^{\mathrm{op}}$ (here all tensor products and Homs are in the super sense: see (*ibid.*), Example 2.5). If A is invertible, its inverse is its dual up to a contractible space of choices, and the dual in the Morita 2-category of superalgebras is the opposite superalgebra (*ibid.*). Thus for A an invertible superalgebra, $A \otimes A^{\mathrm{op}}$ is canonically equivalent to $\mathbb{C}$ as superalgebras, and therefore $A^!$ is canonically equivalent to A^*. □

Proposition A.5 (Lurie [Lur09, Example 2.4.15])**.** *The map* ρ *factors as a composition*

$$\rho\colon B\mathrm{O}_2 \xrightarrow{i} B\mathrm{O} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{k} B\mathbb{G}, \tag{A.6}$$

where i *is induced from the usual inclusion* $\mathrm{O}_2 \hookrightarrow \mathrm{O}$*,* J *is the* J*-homomorphism, and both* J *and* k *are maps of* E_∞*-spaces.*

Lurie does not discuss $B\mathbb{G}$ or ρ specifically; the point is that, for any Picard n-groupoid $\mathcal{G}$, the O_n-action on $\mathcal{G}$ guaranteed by the cobordism hypothesis factors in an analogous way through the map J. See [DS26] for a detailed discussion of this point,

Lemma A.7. *Let* $u_*\colon B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(\tau_{\leq 2}ko)$ *be the map induced by the unit map* $\mathbb{S} \to \tau_{\leq 2}ko$*. Then the map* k *from Proposition A.5 factors through* u_**.*

Proof. Because $\mathrm{Pic}(s\mathcal{A}lg)$ is a 2-category, the ∞-group $\mathbb{G}$ is in fact a 3-group: it has no non-identity n-morphisms for $n > 3$. Thus $\pi_n(B\mathbb{G}) = 0$ for $n > 3$, so the map $k\colon B\mathrm{GL}_1(\mathbb{S}) \to B\mathbb{G}$ factors through $\tau_{\leq 3}B\mathrm{GL}_1(\mathbb{S})$.

There is a natural identification of $\pi_n(B\mathrm{GL}_1(R))$ with 1 (if $n = 0$), $\pi_0(R)^\times$ (if $n = 1$), or $\pi_{n-1}(R)$ (if $n > 1$); applying this, the map u_* is an isomorphism on homotopy groups in degrees 3 and below, and kills all higher-degree homotopy groups of $B\mathrm{GL}_1(\mathbb{S})$; thus u_* is 3-truncation, finishing the proof. □

Definition A.8. Throughout this appendix, let $q\colon B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}$ denote the cofiber, in the ∞-category of E_∞-spaces, of the map $\iota\colon B\mathrm{Spin} \to B\mathrm{O}$ induced from the map $\mathrm{Spin}_n \to \mathrm{O}_n$, $n \geq 3$, which is the universal cover of the connected component of the identity.

Lemma A.9. *The space of null-homotopies of the composition*

$$B\mathrm{Spin} \xrightarrow{\iota} B\mathrm{O} \xrightarrow{k\circ J} B\mathbb{G}, \tag{A.10}$$

as a map of E_∞-spaces, is contractible (hence in particular nonempty).

Corollary A.11. *Descending $k \circ J$ across q, we obtain a map of E_∞-spaces*

$$\widetilde{\rho}\colon B\mathrm{O}/B\mathrm{Spin} \longrightarrow B\mathbb{G}. \tag{A.12}$$

The unique nonzero element of $\pi_1(B\mathrm{O}/B\mathrm{Spin}) \cong \mathbb{Z}/2$ acts on $\mathrm{Pic}(s\mathcal{A}lg)$ by duality, and the unique nonzero element of $\pi_2(B\mathrm{O}/B\mathrm{Spin}) \cong \mathbb{Z}/2$ acts by the Serre automorphism, which squares to 0 by Proposition A.4.

The actions of the generators follow from the description of the actions of π_1 and π_2 of $B\mathrm{O}_2$ in Corollary A.3, together with the fact that $\pi_i(B\mathrm{O}_2) \to \pi_i(B\mathrm{O}) \to \pi_i(B\mathrm{O}/B\mathrm{Spin})$ is surjective for $i = 1, 2$.

Proof of Lemma A.9. Since $B\mathrm{Spin}$, $B\mathrm{O}$, and $B\mathbb{G}$ are grouplike E_∞-spaces, it is equivalent to show that the space of null-homotopies of the corresponding map of spectra

$$bspin \longrightarrow bo \longrightarrow b\mathbb{G} \tag{A.13}$$

is contractible. Here, under the equivalence between connective spectra and grouplike E_∞-spaces, *bspin*, *bo*, and $b\mathbb{G}$ are the connective spectra corresponding to $B\mathrm{Spin}$, $B\mathrm{O}$, and $B\mathbb{G}$ respectively.[21]

Since $B\mathrm{Spin}$ is 3-connected, *bspin* is also 3-connected. Since $B\mathbb{G}$ is 3-truncated, so is $b\mathbb{G}$. Thus $[bspin, b\mathbb{G}] = 0$: all maps of the form (A.13) are null-homotopic. The space of null-homotopies of any such map is homotopy equivalent to the connective cover of $\mathrm{Map}_{\mathrm{Sp}}(bspin, \Sigma^{-1}b\mathbb{G})$, which is contractible by an analogous connectivity argument. □

Definition A.14.

(1) Let $A\colon B\mathbb{Z}/2 \to B\mathrm{O}/B\mathrm{Spin}$ be the composition of the map $B\mathbb{Z}/2 \to B\mathrm{O}$ given by the tautological line bundle followed by $q\colon B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}$.
(2) Let $B\colon B^2\mathbb{Z}/2 \simeq B\mathrm{SO}/B\mathrm{Spin} \to B\mathrm{O}/B\mathrm{Spin}$ be the map induced from the usual map $B\mathrm{SO} \to B\mathrm{O}$.

Proposition A.15 ([DY23a, Proposition 1.37])**.** *The map $S \coloneqq (A, B)\colon B\mathbb{Z}/2 \times B^2\mathbb{Z}/2 \to B\mathrm{O}/B\mathrm{Spin}$ is a homotopy equivalence of spaces.*

Both $B\mathrm{O}/B\mathrm{Spin}$ and $B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ are E_∞-spaces; however, they are not equivalent even as E_1-spaces (*ibid.*, Lemma 1.41).

We are now ready for our first definition of twisted supercohomology.

Definition A.16. Let X be a space, $a \in H^1(X; \mathbb{Z}/2)$, and $b \in H^2(X; \mathbb{Z}/2)$. The pair (a, b) defines a map

$$f_{a,b}\colon X \xrightarrow{(a,b)} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \xrightarrow[(A.15)]{S} B\mathrm{O}/B\mathrm{Spin} \xrightarrow[(A.11)]{\widetilde{\rho}} B\mathbb{G}. \tag{A.17a}$$

[21] We warn the reader that some authors use *bo* to denote the spectrum *ko*. In our notation, *bo* is the 0-connected cover of *ko*. We apologize for this notational inconsistency and hope that, since we do not use the notation *bo* outside of the proof of Lemma A.9, it does not cause any confusion.

Let $P_k \to B\mathbb{G}$ be the associated bundle

$$E\mathbb{G} \times_{\mathbb{G}} B^{k-2}|\mathrm{N}(\mathrm{Pic}(s\mathcal{A}lg))| \longrightarrow B\mathbb{G}, \tag{A.17b}$$

where $|\mathrm{N}(-)|$ denotes the geometric realization of the nerve. The (a,b)-*twisted categorical supercohomology of* X in degree k, denoted $SH_c^k(X;a,b)$, is the abelian group of homotopy classes of sections of $f_{a,b}^* P_k \to X$.

Since $\mathrm{Pic}(s\mathcal{A}lg)$ is symmetric monoidal, its nerve is an E_∞-space, hence may be delooped arbitrarily many times as we do in (A.17b). In particular, for $0 \le k < 2$ we interpret B^{k-2} as Ω^{2-k}. The mixing construction over the higher group $\mathbb{G}$ refers to the (geometric realization of the nerve of the) action (higher) groupoid associated to the $\mathbb{G}$-action.

Lemma A.18. *Let X be a space and $b \in H^2(X;\mathbb{Z}/2)$. There is a natural isomorphism from $SH_c^*(X;0,b)$ to the b-twisted supercohomology of X as defined by Décoppet* [Déc24b, Definition 4.3].

This is essentially true by definition: indeed, we built Definition A.16 to generalize Décoppet's definition to the case $a \neq 0$.

Let $x \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ be the unique nonzero element.

Example A.19. We will show $SH_c^1(B\mathbb{Z}/2, x, 0) \cong \mathbb{Z}/2$. This is by definition the space of sections of the pullback $f_{x,0}^* P_1 \to B\mathbb{Z}/2$, whose fiber is $\Omega\mathrm{Pic}(s\mathcal{A}lg) \simeq \mathrm{Pic}(s\mathcal{V}ect)$. We will use the standard CW structure on $B\mathbb{Z}/2$ to make computations; the results are independent of the choice of CW structure. The map $(x,0)\colon B\mathbb{Z}/2 \to K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2) \cong B\mathrm{O}/B\mathrm{Spin}$ is an equivalence on the 1-skeleton, and is trivial on the 2-cell of $B\mathbb{Z}/2$. The monodromy around the cells of $B\mathbb{Z}/2$ can be described as follows.

- Around the 1-cell e^1 of $B\mathbb{Z}/2$, the monodromy of $f_{x,0}^* P_1$ is the same as the monodromy of P_1 around the 1-cell of $B\mathrm{O}/B\mathrm{Spin}$, which by Corollary A.3 sends a morphism ϕ in $\mathrm{Pic}(s\mathcal{V}ect)$ to $\phi^\vee$.
- Around the 2-cell e^2 of $B\mathbb{Z}/2$, the monodromy of $f_{x,0}^* P_1$ is the identity.
- Since $f_{x,0}^* P_1$ is a bundle of Picard groupoids, which have no higher morphisms, all higher-dimension cells of $B\mathbb{Z}/2$ necessarily carry trivial monodromy.

We will build a section s of $f_{x,0}^* P_1 \to B\mathbb{Z}/2$ one cell at a time. Since this is a bundle whose fibers are Picard groupoids, our section is data over the 0- and 1-cells and a condition on the 2-cell, then automatically extends to higher degrees.

(0) $B\mathbb{Z}/2$ has a single 0-cell e^0, so over this cell, our section chooses a single object s_0 of $\mathrm{Pic}(s\mathcal{V}ect)$, isomorphic to the even line $\mathbb{C}$ or the odd line $\Pi\mathbb{C}$. We may without loss of generality assume that s_0 *is* $\mathbb{C}$ or $\Pi\mathbb{C}$: if not, we can choose an isomorphism $\alpha\colon s_0 \xrightarrow{\cong} \mathbb{C}$ (or $\Pi\mathbb{C}$) and replace our section s with $\alpha^{-1}\circ s\circ\alpha$, which is in the same homotopy class of sections. Thus we assume s_0 is $\mathbb{C}$ or $\Pi\mathbb{C}$ in the rest of this argument.

(1) To extend s over the 1-skeleton, we must choose an isomorphism $s_1\colon s_0 \xrightarrow{\cong} s_0^\vee$ (since the monodromy around e^1 is the duality involution on $\mathrm{Pic}(s\mathcal{V}ect)$). As of yet we can choose s_1 freely, giving a $\mathbb{C}^\times$ of choices for either $s_0 \cong \mathbb{C}$ or $s_0 \cong \Pi\mathbb{C}$ a priori. However, like for s_0, different choices can lead to homotopy-equivalent sections. Specifically, if one acts on s_0 by $\lambda \in \mathbb{C}^\times$, one necessarily obtains a homotopy-equivalent section, yet s_1 is multiplied by λ^2 (on the domain s_0 we act by λ, and on the codomain $s_0^\vee$ we act by $\lambda^\vee = \lambda^{-1}$; moving the latter action to the domain we get λ^2). Thus, if $s_1 = \lambda^2 \widetilde{s}_1$ for another choice $\widetilde{s}_1$, the data

of s_1 and $\widetilde{s}_1$ define homotopy-equivalent sections over the 1-skeleton. So for both $\mathbb{C}$ and $\Pi\mathbb{C}$, we have *one* choice of homotopy class of section.

(2) The 2-cell e^2 is attached by $\partial e^2 = 2e_1$, which gives the condition that "squaring" s_1 must equal the monodromy around the 2-cell. By "squaring" we mean that the composition

$$s_0 \xrightarrow{s_1} s_0^\vee \xrightarrow{(s_1^\vee)^{-1}} s_0^{\vee\vee} \xrightarrow{\psi^{-1}} s_0 \tag{A.20}$$

So the composition is the number $s_1^{-1}s_1 = 1$. We saw that the monodromy of $f_{x,0}^* P_1$ is also equal to 1, so the condition on the 2-cell is met by both $s_0 = \mathbb{C}$ and $s_0 = \Pi\mathbb{C}$.

To summarize, there are two choices of section up to homotopy, so $SH_c^1(B\mathbb{Z}/2, x, 0)$ is an abelian group of order 2, hence is isomorphic to $\mathbb{Z}/2$.

Example A.21. We will show $SH_c^1(B\mathbb{Z}/2, x, x^2) \cong 0$. Most of the story is the same, except that since the degree-2 part of the twist is nonzero, around the unique 2-cell e^2 of $B\mathbb{Z}/2$, the monodromy of $f_{x,x^2}^* P_1$ equals the monodromy of the 2-cell in $K(\mathbb{Z}/2, 2)$, which by Corollary A.3 is the grading operator $(-1)^F$.

Thus when we attempt to build a section s, the argument is exactly the same for s_0 and s_1: s_0 is isomorphic to $\mathbb{C}$ or $\Pi\mathbb{C}$, and given s_0 there is a $\mathbb{C}^\times$ of choices for s_1, all of which are homotopy equivalent. Now, though, the condition on the 2-cell is that $s_1^{-1}s_1$ equals the monodromy around the 2-cell. For s_0 an even line this is fine, as $(-1)^F$ acts by 1 on even lines, so we have a unique homotopy class of section with $s_0 \cong \mathbb{C}$. But if s_0 is an odd line, $(-1)^F$ acts as -1, and there is no choice of s_1 such that $s_1^{-1}s_1 = -1$. Thus there is a unique homotopy class of section, so $SH_c^1(B\mathbb{Z}/2, x, x^2) \cong 0$.

Remark A.22. The reader may enjoy calculating $SH_c^1(B\mathbb{Z}/2, 0, x^2) \cong \mathbb{Z}/4$ and $SH_c^1(B\mathbb{Z}/2, 0, 0) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ using arguments similar to those in Examples A.19 and A.21.

A.2. **The homotopical twist.** Now we introduce the other twist, using Ando–Blumberg–Gepner–Hopkins–Rezk's definition [ABG+14a, ABG+14b] of twisted generalized cohomology. Given an E_∞-ring spectrum R, May [May77, §III.2] defines a grouplike E_∞-space $\mathrm{GL}_1(R)$, called the *∞-group of units* of R, whose definition is manifestly natural in R; see [ABG+14b, §1.2] for a review. Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14a, Definition 2.10, Corollary 2.14] construct a natural transformation M from the fundamental ∞-groupoid of $B\mathrm{GL}_1(R)$ to the category of R-module spectra.

Definition A.23 ([ABG+14a, Definition 2.20])**.** For R an E_∞-ring spectrum, the *R-module Thom spectrum* of a map $f\colon X \to B\mathrm{GL}_1(R)$, denoted $M^R f$, is the colimit of the X-shaped diagram

$$X \xrightarrow{f} B\mathrm{GL}_1(R) \xrightarrow{M} \mathcal{M}od_R. \tag{A.24}$$

Here we have identified X and $B\mathrm{GL}_1(R)$ with their respective fundamental ∞-groupoids.

Using Definition A.23, Ando–Blumberg–Gepner [ABG18, Definition 1.2] define f-twisted R-homology to be $\pi_*(M^R f)$ and f-twisted R-cohomology to be $\pi_{-*}\mathrm{Map}_R(M^R f, R)$. This is close to how we define twisted supercohomology, but because supercohomology is not a ring spectrum we modify their approach slightly.

We will work with the Postnikov 2-truncation $\tau_{\leq 2} ko$ of the connective real K-theory spectrum ko. $\tau_{\leq 2} ko$ inherits an E_∞-ring structure from the standard E_∞-ring structure on ko.

Definition A.25. Let $T\colon BO/B\mathrm{Spin} \to B\mathrm{GL}_1(ko)$ be the map defined in [DY23a, (1.36)]. Let X be a space, $a \in H^1(X;\mathbb{Z}/2)$, and $b \in H^2(X;\mathbb{Z}/2)$. Let $g_{a,b}$ be the composition

(A.26a)
$$g_{a,b}\colon X \xrightarrow{(a,b)} K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2) \xrightarrow[\text{(A.15)}]{S} BO/B\mathrm{Spin} \xrightarrow[\text{[DY23a]}]{T} B\mathrm{GL}_1(ko) \xrightarrow{(\tau_{\le 2})_*} B\mathrm{GL}_1(\tau_{\le 2}ko).$$

Let $M^{\tau_{\le 2}ko}g_{a,b}$ denote the $\tau_{\le 2}ko$-module Thom spectrum of $g_{a,b}$ in the sense of Definition A.23. The (a,b)-*twisted homotopical supercohomology of* X in degree k is

$$SH^k_h(X;a,b) \coloneqq \pi_{-k}(I_{\mathbb{C}^\times} M^{\tau_{\le 2}ko}g_{a,b}). \tag{A.26b}$$

Remark A.27. With notation as in Definition A.25, let

$$h_{a,b} \coloneqq T\circ S\circ (a,b)\colon X \longrightarrow B\mathrm{GL}_1(ko). \tag{A.28}$$

In other words, we use the twisting data (a,b) to make a ko-module Thom spectrum $M^{ko}h_{a,b}$ analogously to how we made a $\tau_{\le 2}ko$-module Thom spectrum in Definition A.25. By [DY23a, Lemma 1.8], there is a $\tau_{\le 2}ko$-module equivalence $M^{ko}h_{a,b} \wedge_{ko} \tau_{\le 2}ko \simeq M^{\tau_{\le 2}ko}g_{a,b}$.

Example A.29. We will show $SH^1_h(B\mathbb{Z}/2,x,0) \cong \mathbb{Z}/2$. By definition and the universal property of $I_{\mathbb{C}^\times}$,

$$SH^1_h(B\mathbb{Z}/2,x,0) \cong \mathrm{Hom}(\pi_1(M^{\tau_{\le 2}ko}g_{x,0}),\mathbb{C}^\times). \tag{A.30}$$

Thus it suffices to show that $\pi_1(M^{\tau_{\le 2}ko}g_{x,0}) \cong \mathbb{Z}/2$. Because the map T from Definition A.25 factors through the map $B\mathrm{GL}_1(MTSpin) \to B\mathrm{GL}_1(ko)$ (see [DY23a, (1.36)]), $g_{a,b}$ factors through the map on $B\mathrm{GL}_1$ induced by the composition of maps of E_∞-ring spectra $MTSpin \to ko \to \tau_{\le 2}ko$. Thus, by [DY23a, Lemma 1.8],

$$M^{\tau_{\le 2}ko}g_{x,0} \simeq M^{MTSpin}g_{x,0} \wedge_{MTSpin} \tau_{\le 2}ko, \tag{A.31}$$

where $M^{MTSpin}g_{x,0}$ is the $MTSpin$-module Thom spectrum associated to the lift of the map $g_{a,b}$ to $B\mathrm{GL}_1(MTSpin)$. The map $MTSpin \to \tau_{\le 2}ko$ is 2-connected, because it is the composition of the Atiyah–Bott–Shapiro map $MTSpin \to ko$, which Anderson–Brown–Peterson [ABP67] showed is 7-connected, with $\tau_{\le 2}$, which is 2-connected by definition. Thus the map

$$M^{MTSpin}g_{x,0} \to M^{MTSpin}g_{x,0} \wedge_{MTSpin} \tau_{\le 2}ko \simeq M^{\tau_{\le 2}ko}g_{a,b} \tag{A.32}$$

is also 2-connected (this can be proven, e.g., with the Künneth spectral sequence). Thus (A.32) is an isomorphism on π_1. There is an $MTSpin$-module equivalence $M^{MTSpin}g_{x,0} \simeq MTPin^-$ (in this language, this is [DDK+26, Example 6.23], but the essence of the proof dates to [Pet68, §7]), so

$$\pi_1(M^{\tau_{\le 2}ko}g_{x,0}) \cong \pi_1(MTPin^-) \cong \Omega_1^{\mathrm{Pin}^-}, \tag{A.33}$$

which is isomorphic to $\mathbb{Z}/2$ [ABP69, Theorem 5.1].

Example A.34. We will show $SH^1_h(B\mathbb{Z}/2,x,x^2) \cong 0$. The proof is completely analogous to that of Example A.29, but with $M^{MTSpin}g_{x,x^2}$ in place of $M^{MTSpin}g_{x,0}$. There is an $MTSpin$-module equivalence $M^{MTSpin}g_{x,x^2} \simeq MTPin^+$ (in this language, this is [DDK+26, Example 6.23], but the real content of the proof is in [Sto88, §8]), so $\pi_1(M^{\tau_{\le 2}ko}g_{x,x^2}) \cong \Omega_1^{\mathrm{Pin}^+}$, which vanishes [Gia73b, §2].

Compare Examples A.19 and A.21.

A.3. **The two twists are the same.** Our main result in this appendix is:

Theorem A.35. *There is a natural isomorphism $\kappa\colon SH^*_c(X,a,b) \xrightarrow{\cong} SH^*_h(X,a,b)$.*

We will thus just say "supercohomology" and "$SH^*(X,a,b)$" to refer to this object.

Before we prove Theorem A.35, we will need a few lemmas.

Definition A.36. Let $pic(s\mathcal{A}lg)$ denote the classifying spectrum of $s\mathcal{A}lg$ (see Definition 3.29), i.e. a connective spectrum whose infinite loop space is equivalent as E_∞-spaces to $|\mathrm{N}(\mathrm{Pic}(s\mathcal{A}lg))|$.

By Theorem 3.30, this definition characterizes $pic(s\mathcal{A}lg)$ up to a contractible space of choices.

Lemma A.37. *There is a canonical homotopy equivalence of spectra $pic(s\mathcal{A}lg) \xrightarrow{\simeq} \Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko)$.*

Proof. Debray–Gunningham [DG18, §4.3.5] constructed a map $t\colon pic(s\mathcal{A}lg) \to \Sigma^2 I_{\mathbb{C}^\times}$ and showed that it is the connective cover, giving a canonical equivalence $pic(s\mathcal{A}lg) \to \tau_{\geq 0}\Sigma^2 I_{\mathbb{C}^\times}$. The Hurewicz map $u\colon \mathbb{S} \to \tau_{\leq 2} ko$ is an isomorphism on homotopy groups in degrees 2 and below (this is standard for $\mathbb{S} \to ko$, and to get $\tau_{\leq 2} ko$, use the universal property of Postnikov truncation). Thus by the universal property of $I_{\mathbb{C}^\times}$, the map $\Sigma^2 I_{\mathbb{C}^\times}(u)\colon \Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko) \to \Sigma^2 I_{\mathbb{C}^\times}(\mathbb{S}) = \Sigma^2 I_{\mathbb{C}^\times}$ is an isomorphism on nonnegative-degree homotopy groups, hence a homotopy equivalence on connective covers. Let $\widetilde{u}$ denote this equivalence on connective covers. Finally, since $\pi_i(\tau_{\leq 2} ko)$ vanishes for $i \geq 3$, $\pi_j(\Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko))$ vanishes for $j < 0$, so $\Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko)$ is canonically equivalent to its connective cover. Combining these facts, we have that $\widetilde{u}^{-1} \circ t$ defines a homotopy equivalence $pic(s\mathcal{A}lg) \xrightarrow{\simeq} \Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko)$. □

Remark A.38. Passing the $\tau_{\leq 2} ko$-module structure on $\Sigma^2 I_{\mathbb{C}^\times}(\tau_{\leq 2} ko)$ through Lemma A.37, we get a canonical $\tau_{\leq 2} ko$-module structure on $pic(s\mathcal{A}lg)$. This gives rise to an ∞-group morphism $\xi\colon \mathrm{GL}_1(\tau_{\leq 2} ko) \to \mathbb{G}$ by restricting the action map $\tau_{\leq 2} ko \to \mathrm{End}(\mathrm{pic}(s\mathcal{A}lg))$ to the ∞-group of units and observing that the units act by automorphisms, rather than just endomorphisms.

Corollary A.39. *The following diagram commutes up to homotopy.*

$$\begin{array}{ccc} B\mathrm{O}/B\mathrm{Spin} & \xrightarrow{\widetilde{\rho}\ (\text{A.11})} & B\mathbb{G}. \\ & {\scriptstyle S\circ(a,b)}\searrow \quad \nearrow{\scriptstyle \xi}\ (\text{A.25}),\ (\text{A.38}) & \\ & B\mathrm{GL}_1(\tau_{\leq 2} ko) & \end{array} \tag{A.40}$$

Heuristically, Corollary A.39 tells us that if $P \to X$ is a principal $\mathbb{G}$-bundle whose classifying map factors through $B\mathrm{O}/B\mathrm{Spin}$, P has a canonical reduction of structure group to $\mathrm{GL}_1(\tau_{\leq 2} ko)$. We have not provided definitions to make this heuristic parse, but as an idea, it helped us plot the proof of Theorem A.35.

Proof. Since we constructed $\widetilde{\rho}$ in Lemma A.9 and Corollary A.11 by canonically descending the map $k \circ J\colon B\mathrm{O} \to B\mathbb{G}$ across $B\mathrm{O}/B\mathrm{Spin}$, it suffices to prove the corollary with $B\mathrm{O}$ in place of $B\mathrm{Spin}$, $k \circ J$ in place of $\widetilde{\rho}$, and $S \circ (a,b)$ precomposed with the quotient $q\colon B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}$. By [DY23a, Theorem 1.13] (see also (*ibid.*, Remark 1.14) and [May77, Lemma IV.2.6]) we may replace $S \circ (a,b) \circ q$ with

$$B\mathrm{O} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{u_*} B\mathrm{GL}_1(\tau_{\leq 2} ko), \tag{A.41}$$

where u_* is the map induced by the unit map $u\colon \mathbb{S} \to \tau_{\leq 2} ko$ of E_∞-ring spectra. That is, we want to show that the diagram

$$\begin{array}{ccccc} BO & \xrightarrow{J} & B\mathrm{GL}_1(\mathbb{S}) & \xrightarrow{u_*} & B\mathrm{GL}_1(\tau_{\leq 2} ko) \\ & & & \searrow^{k} & \downarrow{\scriptstyle \xi} \\ & & & & B\mathbb{G} \end{array} \tag{A.42}$$

commutes, meaning k factors through $B\mathrm{GL}_1(\tau_{\leq 2} ko)$, which we proved in Lemma A.7: one can trace through the definition of ξ to see that $k \simeq \xi \circ u_*$. □

To prove Theorem A.35, we use May–Sigurdsson's [MS06] and Ando–Blumberg–Gepner's [ABG18] formalisms for parametrized stable homotopy theory. In these settings, the analogues of unbased, resp. based spaces are spaces over B (i.e. spaces X with maps $X \to B$), resp. *ex-spaces*, diagrams $B \to X \to B$ whose composition is the identity (see [MS06, §1.3] for conventions). As with ordinary spectra, which are built from based spaces, May–Sigurdsson's parametrized spectra are built from ex-spaces.

For a space X, let $\mathcal{T}op_{/X}$ denote the category of CGWH spaces over X, $\mathcal{K}_X$ denote the category of ex-spaces over X, and $\mathcal{S}_X$ denote May–Sigurdsson's category of parametrized orthogonal spectra over X, defined in [MS06, §11.2]. We will let $\mathcal{S}p_{/X}$ denote Ando–Blumberg–Gepner's ∞-category of parametrized spectra over X [ABG18, Definition 3.3], which is the ∞-category of functors $X^{\mathrm{op}} \to \mathcal{S}p$.[22]

Definition A.43. Given a space over B $f\colon X \to B$, let X_+ denote the ex-space $X \amalg B$, where the map $B \to X \amalg B$ is the inclusion of the second factor and the map $X \amalg B \to B$ is $f \amalg \mathrm{id}$.

Remark A.44. In particular, if $E \in \mathcal{S}_B$ and X is a space over B, $E^*(X)$ denotes the *unreduced* E-cohomology of X, which is naturally isomorphic to the *reduced* E-cohomology of X_+ (see [MS06, §20.1]).[23]

Let Π be a locally compact, well-based topological group, $P \to X$ be a principal Π-bundle, and E be a spectrum with Π-action. In (*ibid.*, §15.4), May–Sigurdsson construct an associated parametrized spectrum $P \times_\Pi E$ to a principal Π-bundle $P \to X$ and a spectrum E with a Π-action (i.e. a Π-spectrum indexed on a trivial Π-universe in the sense of Lewis–May [LMSM86, Definition I.2.1][24]). Then, in (*ibid.*, §22.1), May–Sigurdsson remark that it follows from the definitions that if E_P denotes the parametrized cohomology theory over X associated to $P \times_\Pi E$ ([MS06, Definition 20.2.4]), then the (unreduced) cohomology $E_P^0(X)$ is naturally isomorphic to the abelian group of homotopy classes of sections of $P \times_\Pi \Omega^\infty E$.

Our $\mathbb{G}$ plays the role of Π, but we do not know whether it is a topological group, as it is merely a grouplike E_1-space. We fix this as follows.[25] Let $\Omega\colon s\mathcal{S}et \to s\mathcal{G}rp$ denote *Kan's loop group*

[22] Here we implicitly identify X with its fundamental ∞-groupoid. In particular, $X^{\mathrm{op}} \simeq X$ (see [ABG18, Remark 3.2]); we use X^{op} here following Ando–Blumberg–Gepner [ABG18], who parametrize over more general things than spaces.

[23] The notation X_+ in (*ibid.*, Proof of Theorem 20.1.5) should agree with Definition A.43. This is in conflict with (*ibid.*, Notation 5.1.3), but this appears to be a typo: using Definition A.43 is necessary for the statements to type-check.

[24] Lewis–May assume that the group G that acts is a compact Lie group; as May–Sigurdsson [MS06, §15.4] note, since we are using the trivial G-universe, this assumption can be removed.

[25] An alternate approach would be to work with the grouplike E_1-space $\Omega(BO/B\mathrm{Spin})$, since the map (A.17a) factors through $BO/B\mathrm{Spin}$. Following Maycock Parker [May88], $\Omega(BO/B\mathrm{Spin})$ is modeled by the projective orthogonal

functor [Kan58a, Kan58b], where $s\mathcal{S}et$ and $s\mathcal{G}rp$ are the categories of simplicial sets and simplicial groups, respectively. If $|–|$ denotes geometric realization, then for any simplicial set X, there is a natural homotopy equivalence of E_1-spaces $|\Omega X| \xrightarrow{\sim} \Omega|X|$. Moreover, $|\Omega X|$, being the geometric realization of a simplicial group, is a compactly generated, Hausdorff, paracompact topological group. Thus we may replace $\mathbb{G}$ with $|\Omega(\mathrm{Sing}(B\mathbb{G}))|$, where $\mathrm{Sing}\colon \mathcal{T}op \to s\mathcal{S}et$ denotes the singular complex functor, to obtain a topological group equivalent to $\mathbb{G}$ as E_1-spaces. In the rest of this section, we will implicitly use this construction to make $\mathbb{G}$ into a topological group, which one can check is well-based.

However, the topological group $\mathbb{G}$ we constructed in this way is not locally compact. To surmount this, we check that for the applications we have in mind, May–Sigurdsson's local compactness hypothesis is unnecessary.

Lemma A.45. *Let Π be a compactly generated, Hausdorff, paracompact topological group (e.g. the geometric realization of a simplicial group). Let $P \to X$ be a principal Π-bundle of CGWH spaces and E be an orthogonal Ω-spectrum with Π-action. Then:*

1. *applying the associated bundle construction $P \times_\Pi -$ levelwise to E produces a spectrum over X in the sense of* [MS06, §11.2]*, generalizing the construction in (ibid., §15.4), and*
2. *if E_P^* denotes the parametrized cohomology theory over X associated to $P \times_\Pi E$ in the sense of* [MS06, Definition 20.2.4]*, then the unreduced cohomology $E_P^0(X)$ is naturally isomorphic to the abelian group of homotopy classes of sections of $P \times_\Pi \Omega^\infty E$.*

Remark A.46. May–Sigurdsson [MS06, §3.3, §15.4] consider a much more general setting of a group Γ containing Π as a normal subgroup, $G \coloneqq \Gamma/\Pi$, and a fiber bundle of G-spaces $F \to E \to B$. For us, $\Pi = \Gamma$, $G = 1$, and $E \to B$ is $\mathrm{id}\colon B \to B$.

Proof of Lemma A.45. For the most part, the proof amounts to tracing through May–Sigurdsson's definitions and theorems and observing that the ones we need do not actually require Π to be locally compact. We highlight the key parts.

1. In [MS06, Proposition 1.6.1] and the surrounding discussion, May–Sigurdsson explain that they restrict to locally compact Π so as to avoid confusion as to differing definitions of a Π-space. For us, the assumptions of the lemma hand us all of the Π-spaces we need as CW complexes, so this ambiguity does not apply.
2. Paralleling [MS06, §3.3], we want to show that $P \times_\Pi –$ defines a functor from based Π-spaces to ex-spaces over X, both within $\mathcal{CGWH}$. This follows from standard properties of quotients of CGWH spaces, along with the fact that Π acts freely on $P \times F$ to get Hausdorffness of the quotient.
3. The spectrum-level associated bundle construction [MS06, Notation 15.4.1] goes through in exactly the same way.

Once we have our parametrized spectrum $P \times_\Pi E$ over X, the rest is not any different from the case May–Sigurdsson considered, but we will be explicit. Let $[–,–]_{\mathcal{C}}$ denote the homotopy classes of maps in a model category $\mathcal{C}$,[26] and consider X as a space over itself by $\mathrm{id}\colon X \to X$. Then by [MS06,

group of the super real Hilbert space $\ell^2 \oplus \Pi\ell^2$, where orthogonal transformations send homogeneous elements to homogeneous elements, yet can change their gradings. This group is also not locally compact.

[26] In all cases we use, May–Sigurdsson construct the model structure on $\mathcal{C}$. Specifically, see [MS06, §6.2] for $\mathcal{T}op_{/X}$ and $\mathcal{K}_X$, and (*ibid.*, §12.3) for $\mathcal{S}_X$.

Definition 20.2.4] and Remark A.44,

$$E_P^0(X) \cong [\Sigma_X^\infty X_+, P \times_\Pi E]_{\mathcal{S}_X}, \tag{A.47a}$$

where Σ_X^∞ and Ω_X^∞ are defined in (*ibid.*, Definition 11.24) and $(-)_+$ is Definition A.43. Σ_X^∞ and Ω_X^∞ are shown to partake in a Quillen adjunction in [MS06, Proposition 12.6.2], so we can rewrite $E_P^0(X)$ as

$$\cong [X_+, \Omega_X^\infty(P \times_\Pi E)]_{\mathcal{K}_X} \tag{A.47b}$$

$$\cong [X, P \times_\Pi \Omega^\infty E]_{\mathcal{T}op_{/X}}, \tag{A.47c}$$

by the definition of Ω_B^∞ as the 0^{th} space of a parametrized Ω-spectrum and the fact that May–Sigurdsson [MS06, Notation 15.41] define $P \times_\Pi -$ levelwise. To finish the proof, observe that (A.47c) is by definition the abelian group of homotopy classes of sections of the bundle of spaces $P \times_\Pi \Omega^\infty E \to X$. □

Lemma A.48. *Let X be a CGWH space and $r\colon X \to \mathrm{pt}$ be the crush map.*

1. *The parametrized spectrum $\Sigma_X^\infty X_+$ is the unit with respect to May–Sigurdsson's symmetric monoidal structure* [MS06, Theorem 11.2.5] *on $\mathcal{S}_X$.*
2. *$r^*\mathbb{S}$ is the unit with respect to Ando–Blumberg–Gepner's symmetric monoidal structure* [ABG18, §6.1] *on $\mathcal{S}p_{/X}$.*

Given a map $f\colon Y \to X$ and $E \in \mathcal{S}p_{/X}$, i.e. a functor $X \xrightarrow{E} \mathcal{S}p$, the pullback f^*E is by definition the composition $Y \xrightarrow{f} X \xrightarrow{E} \mathcal{S}p$. Thus in Lemma A.48, $r^*\mathbb{S}$ is the functor $X \to \mathcal{S}p$ sending $x \mapsto \Sigma_+^\infty x$.

Proof. Part (2) follows directly from the fact that r^* is symmetric monoidal, which is proven in [ABG18, Proposition 6.8]. Thus in the rest of the proof we focus on part (1).

May–Sigurdsson describe the monoidal unit S_X of $\mathcal{S}_X$ in [MS06, Definition 11.2.2] (see also (*ibid.*, Theorem 11.2.5)).[27] So we need to identify $\Sigma_X^\infty X_+$ and S_X. Let V be a finite-dimensional inner product space and S^V be the one-point compactification of V. Unwinding [MS06, (11.2.1), Definition 11.2.4], it suffices to identify the ex-spaces $S_X^V \wedge_X X_+$ and S_X^V, where S_X^V is the ex-space $S^V \times X$ with the map $X \to S^V \times X$ given by $x \mapsto (x, \infty)$ and $S^V \times X \to X$ is projection onto the second factor. At this point, one can directly check that there is a natural homeomorphism of ex-spaces $S_X^V \wedge_X X_+ \simeq S_X^V$ using the definition of $\wedge_X$ [MS06, Definition 1.3.4]. □

Proof of Theorem A.35. Let $E_k \coloneqq \Sigma^{k-2} I_{\mathbb{C}^\times}(\tau_{\leq 2} ko)$. By definition, $\mathbb{G}$ acts on the Picard 2-groupoid $\mathrm{Pic}(s\mathcal{A}lg)$, hence also on $pic(s\mathcal{A}lg)$, and we showed $pic(s\mathcal{A}lg) \simeq \Sigma^{2-k} E_k$ in Lemma A.37. Thus we obtain the structure of a spectrum with $\mathbb{G}$-action on E_k. Let $\pi\colon f_{a,b}^* E\mathbb{G} \to X$ be pullback of the universal principal $\mathbb{G}$-bundle to X by the map $f_{a,b}$ from (A.17a) and $E_{k,\pi} \coloneqq (f_{a,b}^* E\mathbb{G}) \times_{\mathbb{G}} E_k$ be the associated bundle we constructed in Lemma A.45 following May–Sigurdsson [MS06, §11.2].

Let $E_{k,\pi}^*$ be the parametrized cohomology theory for spaces over X associated to $E_{k,\pi}$ as in [MS06, §22.1]. Then, by Lemma A.45, $E_{k,\pi}^0(X)$ is naturally isomorphic to the abelian group of homotopy classes of sections of $f_{a,b}^* E\mathbb{G} \times_{\mathbb{G}} \Omega^\infty E_k$, which by Definition A.16 is $SH_c^k(X, a, b)$.

By [MS06, Definition 20.2.4], $E_{k,\pi}^0(X) \cong \pi_0 r_* F_X(\Sigma_X^\infty X_+, P \times_{\mathbb{G}} E_k)$ where $r\colon X \to \mathrm{pt}$ is the crush map, and F_X and Σ_X^∞ are as defined in [MS06, Proof of Theorem 11.2.5, resp. Definition 11.2.4] (set

[27] In *loc. cit.*, May–Sigurdsson actually describe S_X as an $\mathcal{I}$-*space*, i.e. a functor from the category $\mathcal{I}$ defined in [MMSS01, Example 4.4] or [MM02, Definition II.2.1] (set $G = \{1\}$) to $\mathcal{T}op$. This is okay: May–Sigurdsson [MS06, Definition 11.2.3] define their parametrized spectra to be $\mathcal{I}$-spaces together with slightly more data. Here the extra data form an evident homotopy equivalence compatible with our identifications.

$G = \{1\}$). Ando–Blumberg–Gepner [ABG18, Theorem B.2] prove an equivalence of *Wirthmüller contexts* ([FHM03, Definition 2.12(iii)]) between $\mathcal{S}p_{/X}$ and $\mathcal{S}_X$ – specifically, from Ando–Blumberg–Gepner's Wirthmuller context on $\mathcal{S}p_{/X}$ in [ABG18, Corollary 6.9] to May–Sigurdsson's on $\mathcal{S}_X$ (see [MS06, §2.2]). This equivalence amounts to the data of a natural equivalence of symmetric monoidal ∞-categories $W_X \colon \mathcal{S}_X \xrightarrow{\simeq} \mathcal{S}p_{/X}$ commuting with natural isomorphisms between the respective base-change functors f^*, f_*, and $f_!$ for a map of spaces $f \colon X \to Y$, as well as the parametrized mapping spectrum F_X. Invoking this equivalence, we now have a natural isomorphism

$$SH^k_c(X, a, b) \xrightarrow{\cong} \pi_0 r_* F_X(W_X(\Sigma^\infty_X X_+), P \times_{\mathbb{G}} E_k), \tag{A.49}$$

where on the right-hand side, r_*, F_X, and the associated bundle construction are in the Ando–Blumberg–Gepner sense. We have not discussed the associated bundle in this sense, and we also need to compute $W_X(\Sigma^\infty_X X_+)$; let us do both now. Recall that $\mathcal{S}p_{/X}$ is the ∞-category of functors $X \to \mathcal{S}p$ (see Footnote 22).

- By part (1) of Lemma A.48, $\Sigma^\infty_X X_+$ is the monoidal unit of $\mathcal{S}_X$. Since W_X is symmetric monoidal, $W_X(\Sigma^\infty_X X_+)$ is the monoidal unit of $\mathcal{S}p_{/X}$, which is $r^*\mathbb{S}$ (Lemma A.48, part (2)).
- The associated bundle $P \times_{\mathbb{G}} E_k$ is defined as follows: the $\mathbb{G}$-action on E_k defines a functor $B\mathbb{G} \to \mathcal{S}p$ whose value at the basepoint of $B\mathbb{G}$ is E_k. The associated bundle of spectra to P is the composition $X \to B\mathbb{G} \to \mathcal{S}p$, where the first map is the classifying map for P. To compare with May–Sigurdsson's associated bundle, which was defined levelwise, consider the monodromy of $W_{B\mathbb{G}}(E\mathbb{G} \times_{\mathbb{G}} E_k)$. This is by definition a map $B\mathbb{G} \to \mathcal{S}p$ which agrees with the above description of the associated bundle.

By Corollary A.39, we can "change structure group" to $\mathrm{GL}_1(\tau_{\leq 2} ko)$. More precisely, we can replace the associated bundle $P \times_{\mathbb{G}} E_k$ with the functor

$$\Phi \colon X \xrightarrow{S \circ (a,b)} B\mathrm{GL}_1(\tau_{\leq 2} ko) \to \mathcal{M}od_{\tau_{\leq 2} ko} \xrightarrow{(-)^\vee} \mathcal{M}od_{\tau_{\leq 2} ko}, \tag{A.50}$$

where $(-)^\vee$ is $\mathrm{Map}_{\mathbb{S}}(-, \Sigma^k I_{\mathbb{C}^\times})$. (That is, the composition of Φ with the forgetful functor from $\tau_{\leq 2} ko$-modules to spectra is naturally equivalent to $P \times_{\mathbb{G}} E_k$.)

Let $\widetilde{\Phi}$ be the composition of only the first two arrows in (A.50), so $\Phi = F_X(\widetilde{\Phi}, r^*\Sigma^k I_{\mathbb{C}^\times})$, where $r \colon X \to \mathrm{pt}$ is the crush map. Therefore we can invoke the tensor-hom adjunction:

$$\begin{aligned} \pi_0 r_* F_X(r^*\mathbb{S}, P \times_{\mathbb{G}} E_k) &\cong \pi_0 r_* F_X(r^*\mathbb{S}, F_X(\widetilde{\Phi}, r^*\Sigma^k I_{\mathbb{C}^\times})) \\ &\cong \pi_0 r_* F_X(r^*\mathbb{S} \wedge_X \widetilde{\Phi}, r^*\Sigma^k I_{\mathbb{C}^\times}) \\ &\cong \pi_0 r_* F_X(\widetilde{\Phi}, r^*\Sigma^k I_{\mathbb{C}^\times}), \end{aligned} \tag{A.51}$$

since $r^*\mathbb{S}$ is the monoidal unit (Lemma A.48, part (2)). Now we can invoke the $(r_!, r^*)$ adjunction [ABG18, §5.4]:

$$\pi_0 r_* F_X(\widetilde{\Phi}, r^*\Sigma^k I_{\mathbb{C}^\times}) \cong \pi_0 \mathrm{Map}(r_!\widetilde{\Phi}, \Sigma^k I_{\mathbb{C}^\times}). \tag{A.52}$$

By [ABG18, Proposition 4.23] (where f is the trivial family with fiber pt), $r_!\widetilde{\Phi}$ is the $\tau_{\leq 2} ko$-module Thom spectrum of the map $S \circ (a, b) \colon X \to B\mathrm{GL}_1(\tau_{\leq 2} ko)$, so (A.52) is naturally isomorphic to $SH^k_h(X, a, b)$ by Definition A.25. □

Remark A.53. Many applications of twisted supercohomology only need twists that factor through $K(\mathbb{Z}/2, 2) \simeq B\mathrm{SO}/B\mathrm{Spin} \to B\mathrm{O}/B\mathrm{Spin}$, i.e. for which the degree-1 part of the twist is zero. This includes the applications to fusion 2-categories discussed in [JFY21, JFY22, DY23b, Déc24b, DHJF+24, JF25, TY25b, DYY25, DY26, DYY26b, GLWZ26, KPY26a], though see [DYY25, §5].

The theorems in this appendix are quite a bit easier restricted to this class of twists because one can take advantage of uniqueness theorems similar to Antieau–Gepner–Gómez's [AGG14, Theorem 1.1], showing there is a unique nontrivial map $K(\mathbb{Z}/2,2) \to B\mathrm{GL}_1(KO)$ up to homotopy. By contrast, there is a $\mathbb{Z}/2$-torsor of ways to extend to twists over $B\mathrm{O}/B\mathrm{Spin}$. This torsor manifests in several different ways, including but not limited to:

(1) The choice of whether to put a (BG_b, a, b)-twisted spin structure on the tangent bundle or stable normal bundle of a manifold, hence whether to consider tangential or normal Thom spectra MTH, resp. MH.
(2) The choice of whether a (BG_b, a, b)-twisted spin structure on a vector bundle V includes a trivialization of $w_2(V) + b$ or of $w_2(V) + a^2 + b$.
(3) The choice of Pin^+ or Pin^-, and relatedly one's sign conventions for Clifford algebras.
(4) The choice of homotopy equivalence $B\mathrm{O}/B\mathrm{Spin} \simeq B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ (see [DY23a, Remark 1.40]).
(5) The set of homotopy classes of E_1-equivalences $\mathrm{PO}(\mathcal{H}) \simeq \Omega(B\mathrm{O}/B\mathrm{Spin})$ (see Footnote 25).

See also [DS26, §1] for some related conventions.

So why did we work at this level of generality in this paper? Though we did not need degree-one twists for the main results of [DYY25], we conjectured there that our results should extend to this more general class of twists, corresponding to not-necessarily-unitary symmetries of fermionic QFTs. The tools we built in this paper will hopefully be useful in studying that conjecture.

Appendix B. Ext and $\mathcal{Q}$ computations

In this appendix, we compute $\mathcal{Q}(M)$ for each $\mathcal{A}(1)$-module M in Definition 3.11. Our main theorem in this section is Theorem B.12, which we stated heuristically in the main text as Theorem 3.13.

Before we prove that theorem, we prove some lemmas computing Ext groups of some $\mathcal{A}(1)$-modules that we will need. Neither the statements nor the proofs of these lemmas are necessary to understand the statements of the Ext charts, just to give a complete account of the calculations.

The following lemma is well-known, and in any case can be checked directly.

Lemma B.1. *There is a unique isomorphism $f\colon B \to \mathcal{A}(1) \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$ of graded vector spaces. It is also an $\mathcal{A}(1)$-module isomorphism and a $\mathbb{Z}/2$-coalgebra isomorphism.*

The following lemma is also standard; for example, it follows by combining the $\mathbb{E}$-module and $\mathbb{Z}/2$-algebra structures on $\mathrm{Ext}_{\mathcal{A}(1)}(B)$, described in [Bay94, §4.5] and [BC18, Example 4.5.6], respectively.

Lemma B.2. *There is an $\mathbb{E}$-algebra isomorphism $\mathrm{Ext}_{\mathcal{A}(1)}(B) \cong \mathbb{E}[b]/(h_1, v + h_0 b^2, w + b^4)$, where $b \in \mathrm{Ext}^{1,3}$.*

By Lemma B.1, the algebra structure can be thought of as arising either from the coalgebra structure on B or from $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$ by the change-of-rings theorem (see, e.g., [BC18, §4.5]).

Remark B.3. We can use Lemmas B.1 and B.2 to compute the $\mathbb{E}$-module structure on $\mathrm{Ext}_{\mathcal{A}(1)}(B \otimes M)$ for an $\mathcal{A}(1)$-module M in terms of the $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, b]$-module structure on $\mathrm{Ext}_{\mathcal{E}(1)}(M)$. Specifically, $\mathrm{Ext}_{\mathcal{A}(1)}(B \otimes M)$ is naturally isomorphic to the $\mathbb{E}$-module with underlying bigraded vector space $\mathrm{Ext}_{\mathcal{E}(1)}(M)$ and $\mathbb{E}$-action induced from the $\mathbb{E}$-algebra structure on $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$ from Lemma B.2.

We apply this to the following examples.

- $R_6 \coloneqq P \otimes B$, so the $\mathbb{E}$-module structure on $\mathrm{Ext}_{\mathcal{A}(1)}(R_6)$ follows from the $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$-module structure on $\mathrm{Ext}_{\mathcal{E}(1)}(P)$ calculated in [Deb21, Proposition 4.47].
- $B \otimes \mathring{O}$, using the $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$-module structure on $\mathrm{Ext}_{\mathcal{E}(1)}(\mathring{O})$ calculated by Adams–Priddy [AP76, §3].

Next we calculate $\mathrm{Ext}(R_1)$. Since R_1 is a coalgebra (Remark 3.12), $\mathrm{Ext}(R_1)$ is an $\mathbb{E}$-algebra. The bigraded vector space structure, together with the actions of h_0 and h_1, appears in Gitler–Mahowald–Milgram [GMM68, §2]; the rest of the module structure is implicit in their computation.

Lemma B.4. *There is an $\mathbb{E}$-algebra isomorphism*

(B.5)

$$\mathrm{Ext}_{\mathcal{A}(1)}(R_1) \xrightarrow{\cong} \mathbb{E}[q, p_i : i \geq 0]/(h_0 p_0 + h_1, h_0 q + v, q^2 + w, h_1 p_i, q p_i + h_0^2 p_{i+1}, p_i p_j \text{ for all } i, j \geq 0),$$

where $q \in \mathrm{Ext}^{2,6}$ and $p_i \in \mathrm{Ext}^{0,4i+1}$.

Proof. The key observation is that the claimed algebra structure is the unique $\mathbb{Z}^2$-graded $\mathbb{E}$-algebra structure compatible with the underlying $\mathbb{Z}^2$-graded $\mathbb{E}$-module structure, so it suffices to compute the $\mathbb{E}$-module structure. Gitler–Mahowald–Milgram [GMM68, §2] compute $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(R_1)$ as an $\mathbb{Z}/2[h_0, h_1]/(h_0 h_1, h_1^3)$-module; to determine the actions of v and w, one may follow the arguments of Campbell [Cam17, §7.8] and Beaudry–Campbell [BC18, Figures 25 and 26], who study $\mathrm{Ext}_{\mathcal{A}(1)}(R_1)$ using the long exact sequence associated to a short exact sequence of $\mathcal{A}(1)$-modules. This long exact sequence commutes with the action of elements of $\mathbb{E}$, and this suffices to prove the claimed v- and w-actions. □

Lemma B.6.

(1) *Up to isomorphism, there is a unique nonsplit $\mathcal{A}(1)$-module extension of $\mathring{O}$ by ΣR_3. Call this $\mathcal{A}(1)$-module R_7.*
(2) *There is an $\mathcal{A}(1)$-module isomorphism*

$$\mathring{O} \otimes R_1 \cong R_7 \oplus \bigoplus_{i \geq 0} \Sigma^{4i+3} \mathcal{A}(1). \tag{B.7}$$

See Figure 8 for a picture of R_7 and of the isomorphism (B.7).

Proof sketch. Part (1) can be checked directly using the Adem relations. For part (2), we need the technology of *Margolis homology* [AM71, Mar83], which we briefly review. Let $Q_0 \coloneqq \mathrm{Sq}^1$ and $Q_1 \coloneqq \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$; then for $i = 0, 1$, Q_i squares to 0 and acts naturally on any $\mathcal{A}(1)$-module, so we may regard an $\mathcal{A}(1)$-module M as a chain complex with differential Q_i. The resulting homology groups are called the *Margolis homology* of M.

Back to the show. To prove (2), one can embed R_7 as a submodule of $\mathring{O} \otimes R_1$, then verify that this embedding is an isomorphism on Q_0- and Q_1-Margolis homology. We draw this embedding and Margolis homology computation in Figure 8. By a theorem of Wall [Wal62, Theorem 2] (see also [AM71, Theorem 4.1]), this implies the embedding $R_7 \hookrightarrow \mathring{O} \otimes R_1$ is an $\mathcal{A}(1)$-module isomorphism after quotienting out all free summands. To finish the proof of (B.7), we need to find the locations of these free summands. This is a standard Poincaré polynomial argument similar to those in Freed–Hopkins [FH21, §§D.4–D.6]. □

Recall from Definition 3.11 that R_1 was defined to be the cohomology of the Thom spectrum $(B\mathbb{Z}/2)^{2\sigma-2}$. Thus, by the Thom isomorphism theorem, R_1 is one-dimensional over $\mathbb{Z}/2$ in each nonnegative degree n, generated by the class Ux^n, where $U \in H^0$ is the Thom class and $x \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the unique nonzero class.

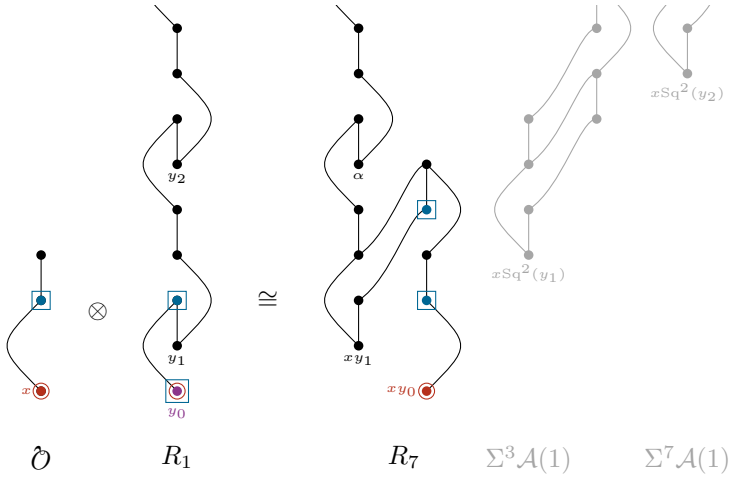


FIGURE 8. A picture proof of Lemma B.6, part (2): that $\mathcal{O} \otimes R_1 \simeq R_7$. Here $\alpha \coloneqq \mathrm{Sq}^2(x)\mathrm{Sq}^2(y_1) + xy_2$. The circled classes generate the Q_0-Margolis homology of each $\mathcal{A}(1)$-module, and the squared classes generate the Q_1-Margolis homology.

Lemma B.8. *Let $\Theta\colon \mathcal{A}(1) \oplus \bigoplus_{i\geq 0} \Sigma^{4i+1}\mathcal{A}(1) \to R_1$ be the $\mathcal{A}(1)$-module map sending the generator of the $\Sigma^k\mathcal{A}(1)$ summand to Ux^k. Then Θ is surjective with kernel isomorphic to ΣR_7.*

This is proven by direct calculation. See Figure 9 for a picture illustrating the proof.

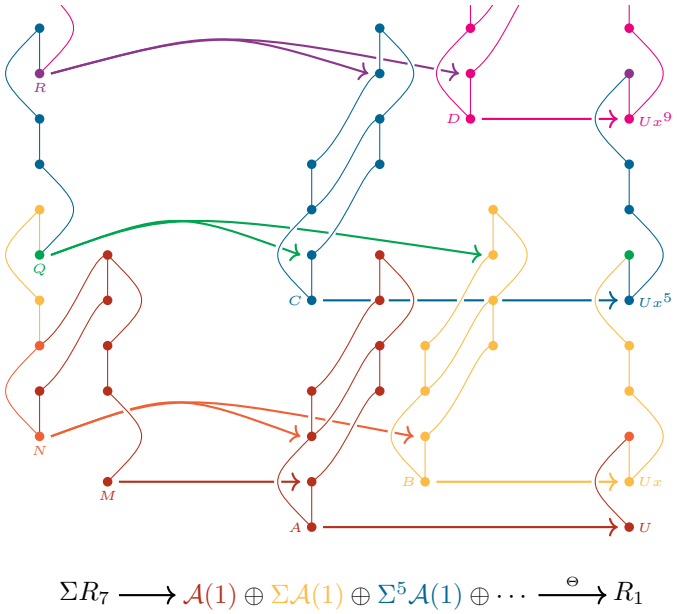


FIGURE 9. A picture proof of Lemma B.8: that the kernel of the map $\Theta\colon \mathcal{A}(1) \oplus \Sigma\mathcal{A}(1) \oplus \Sigma^5\mathcal{A}(1) \oplus \cdots \to R_1$ is isomorphic to R_7. The inclusion of the kernel sends $M \mapsto \mathrm{Sq}^1(A)$, $N \mapsto \mathrm{Sq}^2(A) + \mathrm{Sq}^1(B)$, $Q \mapsto \mathrm{Sq}^2\mathrm{Sq}^3(B) + \mathrm{Sq}^1(C)$, $R \mapsto \mathrm{Sq}^2\mathrm{Sq}^3(C) + \mathrm{Sq}^1(D)$, etc.

Corollary B.9. *For $s, t \geq 0$, there is an isomorphism $\operatorname{Ext}^{s,t}_{\mathcal{A}(1)}(R_7) \cong \operatorname{Ext}^{s+1,t+1}_{\mathcal{A}(1)}(R_1)$ compatible with the $\mathbb{E}$-actions on both sides.*

Proof. The map Θ from Lemma B.8 is the first step in a minimal resolution $P_\bullet$ of R_1, so $P_{\bullet-1}$ (for $\bullet \geq 1$) forms a minimal resolution for $\ker(\Theta) \cong \Sigma R_7$, which implies the corollary statement on Ext groups. □

We have just one more auxiliary lemma before we can give the Adams charts: we have to calculate the $\mathcal{A}(1)$-modules $M \otimes \mathring{\mathcal{O}}$ for the $\mathcal{A}(1)$-modules M we are interested in.

Lemma B.10. *There are $\mathcal{A}(1)$-module isomorphisms*

$$\mathbb{Z}/2 \otimes \mathring{\mathcal{O}} \cong \mathring{\mathcal{O}} \qquad \mathcal{A}(1) \otimes \mathring{\mathcal{O}} \cong \mathcal{A}(1) \oplus \Sigma^2\mathcal{A}(1) \oplus \Sigma^3\mathcal{A}(1) \tag{B.11a}$$

$$\mathring{\mathcal{O}} \otimes \mathring{\mathcal{O}} \cong Q_{2,1} \qquad J \otimes \mathring{\mathcal{O}} \cong \mathcal{A}(1) \oplus \Sigma^2 R_2 \tag{B.11b}$$

$$Q \otimes \mathring{\mathcal{O}} \cong \Sigma^3\mathbb{Z}/2 \oplus \mathcal{A}(1) \qquad R_2 \otimes \mathring{\mathcal{O}} \cong Q_{0,1} \oplus \Sigma\mathcal{A}(1) \oplus \Sigma^2\mathcal{A}(1) \tag{B.11c}$$

$$\Upsilon_1 \otimes \mathring{\mathcal{O}} \cong \Upsilon_1 \oplus \Sigma^2\mathcal{A}(1) \qquad P \otimes \mathring{\mathcal{O}} \cong \Sigma^2 R_0 \oplus \bigoplus_{i \geq 0} \Sigma^{4i}\mathcal{A}(1) \tag{B.11d}$$

$$R_0 \otimes \mathring{\mathcal{O}} \cong R_3 \oplus \bigoplus_{i \geq 0} \Sigma^{4i+2}\mathcal{A}(1) \qquad R_3 \otimes \mathring{\mathcal{O}} \cong \Sigma^3 R_5 \oplus \mathcal{A}(1) \oplus \bigoplus_{i \geq 0} \Sigma^{4i+2}\mathcal{A}(1) \tag{B.11e}$$

$$R_5 \otimes \mathring{\mathcal{O}} \cong \Sigma^3 P \oplus \mathcal{A}(1) \oplus \Sigma\mathcal{A}(1) \oplus \bigoplus_{i \geq 0} \Sigma^{4i+3}\mathcal{A}(1). \tag{B.11f}$$

Proof. For $\mathbb{Z}/2$ this is tautological. For $\mathcal{A}(1)$, $\mathcal{A}(1) \otimes M$ is a free $\mathcal{A}(1)$-module on the graded vector space M. For $\mathring{\mathcal{O}}$, Q, and R_2, use Proposition 2.39 with $(j_2, n_2) = (1, 0)$ and (j_1, n_1) equal to $(1, 0)$, $(3, 0)$, and $(3, 1)$ respectively; the isomorphism type of the free summand of the tensor product then follows from a dimension-counting argument. For J, this is [Mil24, Lemma 3.4(2)]. For Υ_1, use $\Upsilon_1 \cong \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ and the fact that, as an $\mathcal{A}(0)$-module, $\mathring{\mathcal{O}} \cong \mathbb{Z}/2 \oplus \Sigma^2\mathcal{A}(0)$.

For M equal to P, R_0, R_3, and R_5, use the fact that $\mathring{\mathcal{O}}$ is *stably invertible* [AP76, §3] and M is Q_1*-locally stably invertible* [Yu95] to deduce that $M \otimes \mathring{\mathcal{O}}$ is also Q_1-locally stably invertible and is isomorphic to $M \otimes \Sigma^{-1} R_0 \otimes \mathring{\mathcal{O}}$ modulo free summands [Bru14, Theorem 5.2]. $R_0 \otimes \mathring{\mathcal{O}} \cong R_3$ modulo free summands [DKS26, (6.45)], and Yu [Yu95, Lemma 2.5] calculates the tensor product of any two of P, R_0, R_3, and R_5 modulo free summands. Finally, put the free summands back with a Poincaré polynomial argument like the one in the proof of Lemma B.6. □

Below, bidegrees are of the form $(t - s, s)$.

Theorem B.12. *Recall from Lemma 2.55 that $s\mathbb{E} \cong \mathbb{Z}/2[h_0, h_1]/(h_0 h_1, h_1^3)$, with $|h_0| = (0, 1)$ and $|h_1| = (1, 1)$.*

(1) *We have the following $s\mathbb{E}$-module isomorphisms.*

$$\mathcal{Q}(\mathbb{Z}/2) \cong s\mathbb{E} \tag{B.13a}$$

$$\mathcal{Q}(\mathcal{A}(1)) \cong s\mathbb{E}[a_1, a_2, a_3]/ \qquad (h_0, h_1, a_i^2, a_i a_j \text{ for all } i, j), \tag{B.13b}$$

where $|a_1| = (5, 2)$, $|a_2| = (7, 2)$, and $|a_3| = (8, 2)$.

$$\mathcal{Q}(B) \cong s\mathbb{E}[b]/(h_1, b^3, h_0b^2), \qquad |b| = (2,1). \tag{B.13c}$$

$$\mathcal{Q}(N_1) \cong s\mathbb{E}\{c_1, c_2\}/(h_0c_1, h_1^2c_1 + h_0c_2, h_1^2c_2),$$
$$|c_1| = (0,0), \quad |c_2| = (2,1). \tag{B.13d}$$

$$\mathcal{Q}(\mathring{\mathcal{O}}) \cong s\mathbb{E}\{d_1, d_2\}/(h_1d_1, h_0^2d_2, h_1^2d_2),$$
$$|d_1| = (0,0), \quad |d_2| = (4,1). \tag{B.13e}$$

$$\mathcal{Q}(J) \cong s\mathbb{E}\{e_1, e_2, e_3, e_4\}\big/$$
$$(h_0e_1, h_1e_1, h_1e_2, h_0e_3, h_1e_3, h_0e_4, h_1e_4), \tag{B.13f}$$

where $|e_1| = (0,0)$, $|e_2| = (2,1)$, $|e_3| = (5,2)$, *and* $|e_4| = (6,2)$.

$$\mathcal{Q}(Q) \cong s\mathbb{E}\{f_1, f_2, f_3\}\big/$$
$$(h_0f_1, h_1^2f_1, h_1f_2, h_0f_3, h_1f_3), \tag{B.13g}$$

where $|f_1| = (0,0)$, $|f_2| = (3,2)$, *and* $|f_3| = (5,2)$.

$$\mathcal{Q}(R_2) \cong s\mathbb{E}\{g_1, g_2, g_3, g_4, g_5\}\big/$$
$$(h_1g_1, h_0g_2, h_1^2g_2, h_0g_3, h_1g_3,$$
$$h_0g_4, h_1g_4, h_0g_5, h_1g_5), \tag{B.13h}$$

where $|g_1| = (0,0)$, $|g_2| = (1,0)$, $|g_3| = (4,2)$, $|g_4| = (6,2)$, *and* $|g_5| = (7,2)$.

$$\mathcal{Q}(\Upsilon_1) \cong s\mathbb{E}[j_1, j_2]/(h_1, j_1^2, j_2^2, j_1j_2, h_0j_2),$$
$$|j_1| = (5,2), \quad |j_2| = (7,2). \tag{B.13i}$$

$$\mathcal{Q}(P) \cong s\mathbb{E}\{\kappa, k_i, k_i' : i \geq 1\}\big/$$
$$(h_0\kappa, h_0^3k_i, h_1k_i, h_0k_i', h_1k_i', h_1^2\kappa + h_0^2k_1), \tag{B.13j}$$

where $|\kappa| = (0,0)$, $|k_i| = (4i-2, 0)$, *and* $|k_i'| = (4i+1, 2)$.

$$\mathcal{Q}(R_0) \cong s\mathbb{E}\{\lambda, \lambda', \ell_i, \ell_i' : i \geq 1\}\big/$$
$$(h_0\lambda, h_1\lambda, h_0\lambda', h_1^2\lambda',$$
$$h_0^3\ell_i, h_1\ell_i, h_0\ell_i', h_1\ell_i'), \tag{B.13k}$$

where $|\lambda| = (0,0)$, $|\lambda'| = (2,1)$, $|\ell_i| = (4i, 0)$, *and* $|\ell_i'| = (4i+3, 2)$.

$$\mathcal{Q}(R_1) \cong s\mathbb{E}\{\mu_1, \mu_2, m_i, m_i' : i \geq 1\}\big/$$
$$(h_1\mu_1 + h_0m_1, h_0\mu_2, h_1\mu_2,$$
$$h_0^3m_i, h_0m_i', h_1m_i, h_1m_i'), \tag{B.13l}$$

where $|\mu_1| = (0,0)$, $|\mu_2| = (4,2)$, $|m_i| = (4i+1, 0)$, *and* $|m_i'| = (4i+8, 2)$.

$$\mathcal{Q}(R_3) \cong (s\mathbb{E}/h_1)\{\nu, \nu', n_i, n_i' : i \geq 0\}\big/$$
$$(h_0\nu, h_0\nu', h_0n_0, h_0n_i', h_0^2n_1,$$
$$h_0^3n_i \text{ for } i \geq 2), \tag{B.13m}$$

where $|n_i| = (4i, 0)$, $|n_i'| = (4i+7, 2)$, $|\nu'| = (5,2)$, *and* $|\nu| = (6,2)$.

$$\begin{aligned}\mathcal{Q}(R_5) \cong s\mathbb{E}\{\pi_1, \pi_2, \pi_3, \pi_4, p_i, p_i' : i \geq 1\}\big/ \\ (h_0\pi_1, h_1\pi_1 + h_0\pi_2, h_1\pi_2, \\ h_0\pi_3, h_1\pi_3, h_0\pi_4, h_1\pi_4, \\ h_0^3 p_i, h_1 p_i, h_0 p_i', h_1 p_i'),\end{aligned} \tag{B.13n}$$

where $|\pi_1| = (0,0)$, $|\pi_2| = (1,0)$, $|\pi_3| = (5,2)$, $|\pi_4| = (6,2)$, $|p_i| = (4i+1, 0)$, *and* $|p_i'| = (4i+4, 2)$.

$$\begin{aligned}\mathcal{Q}(R_6) \cong (s\mathbb{E}/h_1)\{q_i, q_i' : i \geq 0\}\big/ \\ (h_0^3 q_i, h_0 q_i' \text{ for } i \geq 0, \\ h_0 q_0, h_0^2 q_1)\end{aligned} \tag{B.13o}$$

where $|q_i| = (2i, 0)$ *and* $|q_i'| = (2i+5, 2)$.

In the cases of $\mathbb{Z}/2$, B, *and* Υ_1, *these are also equivalences of* $s\mathbb{E}$*-algebras.*

(2) *For* $M \in \{\mathbb{Z}/2, B, N_1, \mathring{O}\}$, *the map* $\partial_2 \colon \operatorname{Ext}_{\mathcal{A}(1)}(M) \to \mathcal{Q}(M)$ *is surjective. For the remaining* $\mathcal{A}(1)$*-modules we consider,* $\operatorname{coker}(\partial_2)$ *is spanned by the classes listed in Table 2.*

For all $\mathcal{A}(1)$-modules M appearing in this theorem, we draw M and $\mathcal{Q}(M)$ in Figure 4.

M	generators of $\operatorname{coker}(\partial_2)$
$\mathcal{A}(1)$	a_1, a_2, a_3
J	e_3
Q	f_3
R_2	g_4, g_5
Υ_1	j_1, j_2
P	k_i', $i \geq 1$
R_0	ℓ_i', $i \geq 1$
R_1	m_i', $i \geq 1$
R_3	ν', n_i', $i \geq 0$
R_5	π_3, π_4, p_i', $i \geq 1$
R_6	q_i', $i \geq 0$

TABLE 2. For each $\mathcal{A}(1)$-module M in the table, we give an $s\mathbb{E}$-module generating set for the cokernel of the map $\partial_2 \colon \operatorname{Ext}(M) \to \mathcal{Q}(M)$ from (2.28). For $M = \mathbb{Z}/2$, B, N_1, and $\mathring{O}$, $\operatorname{coker}(\partial_2) = \{0\}$. In all cases, the image of ∂_2 is spanned as an $s\mathbb{E}$-module by the generators we gave in (B.13) that are not present in this table. This table is part of Theorem B.12.

In the proof of Theorem B.12, we will need to know $\operatorname{Ext}_{\mathcal{A}(1)}(M)$ as an $\mathbb{E}$-module for lots of $\mathcal{A}(1)$-modules M. For the most part, the Ext groups we need have been worked out in the literature, together with the actions of h_0 and h_1, giving the module structure over $\mathbb{Z}/2[h_0, h_1]/(h_0h_1, h_1^3)$. Most of these references leave the rest of the $\mathbb{E}$-module structure implicit. Thus in Table 3 we gather what is to our knowledge the original references for these computations.

Proof sketch. The strategy is exactly the same in all sixteen cases; therefore we walk through the case $M = N_1$ in detail, then discuss the few differences that occur in the remaining fifteen cases.

M	$\mathrm{Ext}(M)$ (partial $\mathbb{E}$-mod. str.)	$\mathrm{Ext}(M)$ (full $\mathbb{E}$-mod. str.)
$\mathbb{Z}/2$	—	[Liu62, Theorem 3]
B	—	[Bay94, §4.5]
N_1	—	[Per78, §VII.2]
$\mathring{\mathcal{O}}$	[AP76, §3]	[Per78, §VII.1]
J	[AP76, §3]	[Per78, §VII.1]
Q	[AP76, §3]	[Per78, §VII.1]
R_2	[AP76, §3]	[Per78, §VII.1]
Υ_1	—	[Mil75, Lemma 5.8]
P	[GMM68, §2]	[BG10, Figure A.9.17]
R_0	[GMM68, §2]	[DKS26, Example 5.40]
R_1	[GMM68, §2]	Lemma B.4
R_3	[Yu95, Theorem 3.1]	[DKS26, Example 5.37]
R_5	[Yu95, Theorem 3.1]	[DKS26, Example 5.34]
R_6	[Sie13, Theorem A.0.19]	Remark B.3
$B \otimes \mathring{\mathcal{O}}$	[DL20, Appendix A]	Remark B.3
$N_1 \otimes \mathring{\mathcal{O}}$	[Pea14, Figure 6]	[DKS26, Proposition 5.99]
$Q_{2,1}$	[AP76, §3]	[Per78, §VII.1]
$Q_{0,1}$	[AP76, §3]	[Per78, §VII.1]
R_7	—	Corollary B.9

TABLE 3. In the proof of Theorem B.12, we will need to know the $\mathbb{E}$-module structures on $\mathrm{Ext}(M)$ for various $\mathcal{A}(1)$-modules M. In this table, we list the first reference we know of explicitly giving each $\mathbb{E}$-module structure. In many cases, an earlier reference described $\mathrm{Ext}(M)$ either as merely a bigraded vector space or as a module over $\mathbb{Z}/2[h_0, h_1]/(h_0h_1, h_1^3)$, leaving the rest of the $\mathbb{E}$-module structure implicit. In those cases, we give both references for completeness. The reader may also find the Ext calculations in [BG10, Appendix A], [BC18, §§4.4–4.6, 5.2, 5.3, 5.5], [BR21, §6], and [BDD+25, Proposition 14.11] helpful.

We will compute the long exact sequence from Theorem 2.26 in order to find $\mathcal{Q}(N_1)$. We draw this computation in Figure 10. Consulting Table 3, we have $\mathbb{E}$-module isomorphisms

$$\mathrm{Ext}(N_1) \cong \mathbb{E}\{c_1, c_2\}/(h_0c_1, h_1^2c_1 + h_0c_2, vc_1 + h_1^2c_2, vc_2) \tag{B.14a}$$

$$\mathrm{Ext}(\mathring{\mathcal{O}} \otimes N_1) \cong \mathbb{E}\{p, q, r\}/(h_0p, h_1p, vp, wp = h_1^2r, h_0q, h_1^2q + h_0r, vq + h_1^2r, vr), \tag{B.14b}$$

where the $(t-s, s)$ bidegrees of the generators are $|c_1| = (0,0)$, $|c_2| = (2,1)$, $|p| = (0,0)$, $|q| = (4,1)$, and $|r| = (6,2)$. The map $f_2 \colon \mathrm{Ext}^{s,t}(\mathring{\mathcal{O}} \otimes N_1) \to \mathrm{Ext}^{s+3,t+7}(N_1)$ in the long exact sequence (2.28) is $\mathbb{E}$-linear, hence is determined by its values on p, q, and r. For degree reasons, $f_2(p)$ is either 0 or vc_1, $f_2(q)$ is 0 or wc_1, and $f_2(r)$ is 0 or wc_2. Moreover, $\mathbb{E}$-linearity implies all three of these boundary maps are zero or all three are nonzero: w, h_1^2, and v respectively act injectively on the domain and codomain of ∂ for p, q, and r respectively, so all three are detected by whether $f_2(wp)$ is 0 or vwc_1.

Thus we need to find $f_2(p)$. For degree reasons, the Künneth map $\kappa \colon \mathrm{Ext}(\mathring{\mathcal{O}}) \otimes \mathrm{Ext}(N_1) \to \mathrm{Ext}(\mathring{\mathcal{O}} \otimes N_1)$ must send $\delta \otimes c_1 \mapsto p$. Recall from Lemma 2.55 that, if N_1 is replaced with $\mathbb{Z}/2$, $f_2(\delta) = v$. Thus

$$f_2(p) = f_2(\kappa(\delta \otimes c_1)) = f_2(\delta)f_2(c_1) = vc_1, \tag{B.15}$$

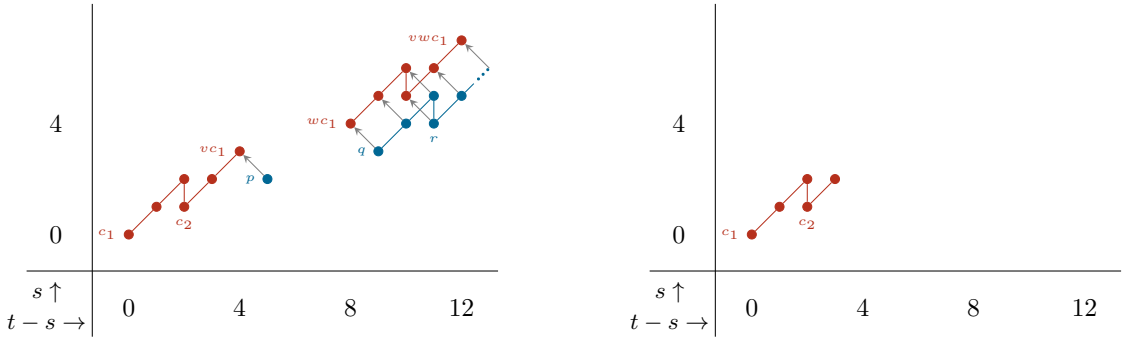


FIGURE 10. The long exact sequence from Theorem 2.26, used to compute $\mathcal{Q}(N_1)$ in the proof of Theorem B.12. Left: the map $f_2\colon \mathrm{Ext}^{s,t}(N_1 \otimes \mathring{\mathcal{O}}) \to \mathrm{Ext}^{s+3,t+7}(N_1)$. This map is injective in all degrees, which determines the entire long exact sequence; in particular, $\mathcal{Q}(N_1) \cong \mathrm{coker}(f_2)$ and $\partial_2\colon \mathrm{Ext}(N_1) \to \mathcal{Q}(N_1)$ is surjective. Right: $\mathcal{Q}(N_1)$.

using that $\mathbb{E}$ acts naturally on the entire long exact sequence (2.28). As stated above, this forces $\partial(q) = wc_1$ and $\partial(r) = wc_2$.

After computing the boundary map, we see that all classes in topological degrees 4 and above vanish, leaving only the vector space spanned by c_1, h_1c_1, $h_1^2c_1 = h_0c_2$, and c_2, which is the $s\mathbb{E}$-module claimed in the theorem statement.

For the remaining fifteen $\mathcal{A}(1)$-modules M appearing in the theorem statement, one calculates $\mathcal{Q}(M)$ in the same way, using the long exact sequence in Theorem 2.26. As input, we need to compute the $\mathcal{A}(1)$-module structure on $\mathring{\mathcal{O}} \otimes M$ and the $\mathbb{E}$-module structures on $\mathrm{Ext}(M)$ and $\mathrm{Ext}(\mathring{\mathcal{O}} \otimes M)$. We calculated $\mathring{\mathcal{O}} \otimes M$ in most cases in Lemma B.10; see Table 3 for all Ext calculations needed in this proof. This leaves open $\mathring{\mathcal{O}} \otimes M$ for $M \in \{B, R_1, R_6\}$, which we did not discuss in Lemma B.10. We computed $\mathring{\mathcal{O}} \otimes R_1$ in Lemma B.6, part (2). We calculate $\mathrm{Ext}(\mathring{\mathcal{O}} \otimes B)$ in Remark B.3. Finally, since $R_6 \coloneqq P \otimes B$, $\mathring{\mathcal{O}} \otimes R_6 \cong B \otimes (\mathring{\mathcal{O}} \otimes P)$. Remark B.3 thus reduces the computation of $\mathrm{Ext}_{\mathcal{A}(1)}(\mathring{\mathcal{O}} \otimes R_6)$ to that of $\mathrm{Ext}_{\mathcal{E}(1)}(\mathring{\mathcal{O}} \otimes P)$. Lemma B.10 reduces to finding $\mathrm{Ext}_{\mathcal{E}(1)}$ of R_0 and $\mathcal{A}(1)$. There is an $\mathcal{E}(1)$-module isomorphism $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$ ([Bay94, §4.7.2]); $\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{E}(1))$ is tautologically $\mathbb{Z}/2$ concentrated in bidegree $(0,0)$, and $\mathrm{Ext}_{\mathcal{E}(1)}(R_0)$ is given in [Deb21, Proposition 4.47]. □

Remark B.16. As we discussed in Remark 3.12, R_1 is a coalgebra, so $\mathcal{Q}(R_1)$ is an $s\mathbb{E}$-algebra. In Theorem B.12, we only computed the $s\mathbb{E}$-module structure on this algebra. This is because, unlike for $\mathbb{E}$ acting on $\mathrm{Ext}(R_1)$ (c.f. Lemma B.4), our method of proof does not uniquely determine the $s\mathbb{E}$-algebra structure.

REFERENCES

[ABG+14a] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology. *J. Topol.*, 7(3):869–893, 2014. arXiv:1403.4325. 5, 50, 54

[ABG+14b] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. Units of ring spectra, orientations and Thom spectra via rigid infinite loop space theory. *J. Topol.*, 7(4):1077–1117, 2014. arXiv:1403.4320. 5, 50, 54

[ABG18] Matthew Ando, Andrew J. Blumberg, and David Gepner. Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map. *Geom. Topol.*, 22(7):3761–3825, 2018. arXiv:1112.2203. 54, 57, 59, 60

[ABP67] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. *Ann. of Math. (2)*, 86:271–298, 1967. 55

[ABP69] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. Pin cobordism and related topics. *Comment. Math. Helv.*, 44:462–468, 1969. 17, 55

[Ada69] J. F. Adams. Lectures on generalised cohomology. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, Lecture Notes in Math., No. 99, pages 1–138. Springer, Berlin-New York, 1969. 11, 17

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974. 10, 14, 21

[Ada23] Katharine Adamyk. Classifying and extending Q_0 local $\mathcal{A}(1)$-modules. *New York J. Math.*, 29:939–980, 2023. arXiv:2107.02837. 23, 30

[AGG14] Benjamin Antieau, David Gepner, and José Manuel Gómez. Actions of $K(\pi, n)$ spaces on K-theory and uniqueness of twisted K-theory. *Trans. Amer. Math. Soc.*, 366(7):3631–3648, 2014. arXiv:1106.5099. 61

[AH61] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. Amer. Math. Soc., Providence, RI, 1961. 8

[Ait24] Colin Aitken. *Model Structures on Infinity-Categories of Filtrations*. PhD thesis, The University of Chicago, 2024. arXiv:2402.17921. 2, 9

[AM71] J. F. Adams and H. R. Margolis. Modules over the Steenrod algebra. *Topology*, 10:271–282, 1971. 62

[And69] D.W. Anderson. Universal coefficient theorems for K-theory, 1969. `https://faculty.tcu.edu/gfriedman/notes/Anderson-UCT.pdf`. 42

[Ant24] Benjamin Antieau. Spectral sequences, décalage, and the Beilinson t-structure, 2024. arXiv:2411.09115. 2, 8, 9, 47

[Ant25a] Benjamin Antieau. Filtrations and cohomology II: the Gauss-Manin connection, 2025. arXiv:2511.14731. 2

[Ant25b] Benjamin Antieau. Filtrations and cohomology III: cohomology of E_∞ rings, 2025. arXiv:2512.15509. 2

[AP76] J. F. Adams and S. B. Priddy. Uniqueness of BSO. *Math. Proc. Cambridge Philos. Soc.*, 80(3):475–509, 1976. 18, 62, 64, 67

[AP25a] Keita Allen and Lucas Piessevaux. Synthetic equivariant spectra for finite abelian groups and motivic homotopy theory, 2025. arXiv:2510.20197. 2

[AP25b] Toni Annala and Piotr Pstrągowski. A note on weight filtrations at the characteristic, 2025. arXiv:2502.19626. 2

[AR24] Benjamin Antieau and Noah Riggenbach. Cyclotomic synthetic spectra, 2024. arXiv:2411.19929. 2

[Ari21] Stefano Ariotta. Coherent cochain complexes and Beilinson t-structures, 2021. With an appendix by Achim Krause. arXiv:2109.01017. 2

[ARØ20] Alexey Ananyevskiy, Oliver Röndigs, and Paul Arne Østvær. On very effective hermitian K-theory. *Math. Z.*, 294(3–4):1021–1034, 2020. arXiv:1712.01349. 49

[AS06] Michael Atiyah and Graeme Segal. Twisted K-theory and cohomology. In *Inspired by S. S. Chern*, volume 11 of *Nankai Tracts Math.*, pages 5–43. World Sci. Publ., Hackensack, NJ, 2006. arXiv:abs/math/0510674. 12

[Ati66] M. F. Atiyah. K-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966. 49

[Bac17] Tom Bachmann. The generalized slices of Hermitian K-theory. *J. Topol.*, 10(4):1124–1144, 2017. arXiv:1610.01346. 49

[Bak20] Andrew Baker. Homotopy theory of modules over a commutative S-algebra: some tools and examples, 2020. arXiv:2003.12003. 11, 14, 15, 30, 41

[Bak26] Andrew Baker. Poincaré duality spaces related to the Joker, 2026. arXiv:2603.29425. 47

[Bay94] Dilip Kumar Bayen. *The connective real K-homology of finite groups*. PhD thesis, Wayne State University, 1994. 17, 18, 61, 67, 68

[BB96] Dilip Bayen and Robert R. Bruner. Real connective K-theory and the quaternion group. *Trans. Amer. Math. Soc.*, 348(6):2201–2216, 1996. 17, 18, 45, 46

[BC76] Edgar H. Brown and Michael Comenetz. Pontrjagin duality for generalized homology and cohomology theories. *American Journal of Mathematics*, 98(1):1–27, 1976. 27, 28

[BC18] Agnès Beaudry and Jonathan A. Campbell. A guide for computing stable homotopy groups. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 89–136. Amer. Math. Soc., [Providence], RI, 2018. arXiv:1801.07530. 2, 3, 17, 27, 30, 33, 42, 44, 61, 62, 67

[BCC93] M. Bullejos, P. Carrasco, and A. M. Cegarra. Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-MacLane's classification theorem. *Math. Proc. Cambridge Philos. Soc.*, 114(1):163–189, 1993. 39

[BCQ25] William Balderrama, Dominic Leon Culver, and J. D. Quigley. The motivic lambda algebra and motivic Hopf invariant one problem. *Geom. Topol.*, 29(3):1489–1570, 2025. arXiv:2112.07479. 50

[BD95] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995. arXiv:q-alg/9503002. 50

[BDD+25] Noah Braeger, Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. Cobordism utopia: U-dualities, bordisms, and the Swampland, 2025. arXiv:2505.15885. 17, 18, 33, 34, 35, 37, 67

[BG03] R. R. Bruner and J. P. C. Greenlees. The connective K-theory of finite groups. *Mem. Amer. Math. Soc.*, 165(785):viii+127, 2003. 36

[BG10] Robert R. Bruner and J. P. C. Greenlees. *Connective real K-theory of finite groups*, volume 169 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. 36, 49, 67

[BGH+17] Paul Bruillard, César Galindo, Tobias Hagge, Siu-Hung Ng, Julia Yael Plavnik, Eric C. Rowell, and Zhenghan Wang. Fermionic modular categories and the 16-fold way. *J. Math. Phys.*, 58(4):041704, 2017. arXiv:1603.09294. 3, 5, 38

[BGHR24] Noé Bárcenas, Luis Eduardo García-Hernández, and Raphael Reinauer. The Gromov-Lawson-Rosenberg conjecture for $\mathbb{Z}/4 \times \mathbb{Z}/4$, 2024. arXiv:2408.07895. 17, 18, 30, 33, 34

[BHHM08] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald. On the existence of a v_2^{32}-self map on $M(1,4)$ at the prime 2. *Homology Homotopy Appl.*, 10(3):45–84, 2008. arXiv:0710.5426. 3, 4, 6, 19, 20

[BHK24] Maissam Barkeshli, Po-Shen Hsin, and Ryohei Kobayashi. Higher-group symmetry of (3+1)D fermionic $\mathbb{Z}_2$ gauge theory: logical CCZ, CS, and T gates from higher symmetry. *SciPost Phys.*, 16(5):122, 2024. arXiv:2311.05674. 40

[BHS26] Robert Burklund, Jeremy Hahn, and Andrew Senger. Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra. *Geom. Topol.*, 30(5):1625–1717, 2026. arXiv:2010.10325. 2

[BJ04] Hans Joachim Baues and Mamuka Jibladze. Computation of the E_3-term of the Adams spectral sequence, 2004. arXiv:math/0407045. 12

[BJM24] J. Francis Baer, Maxwell Johnson, and Peter Marek. Stable comodule deformations and the synthetic Adams-Novikov spectral sequence, 2024. arXiv:2402.14274. 2

[BK72] A. K. Bousfield and D. M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. *Topology*, 11:79–106, 1972. 8

[BK73a] A. K. Bousfield and D. M. Kan. Pairings and products in the homotopy spectral sequence. *Trans. Amer. Math. Soc.*, 177:319–343, 1973. 8

[BK73b] A. K. Bousfield and D. M. Kan. A second quadrant homotopy spectral sequence. *Trans. Amer. Math. Soc.*, 177:305–318, 1973. 8

[BL01] Andrew Baker and Andrey Lazarev. On the Adams spectral sequence for R-modules. *Algebr. Geom. Topol.*, 1:173–199, 2001. arXiv:math/0105079. 2, 8, 10, 13, 14, 15

[BLM26] Jonathan Beardsley, Kiran Luecke, and Jack Morava. Brauer–Wall groups and truncated Picard spectra of K-theory. *Algebr. Geom. Topol.*, 26(5):1749–1780, 2026. arXiv:2306.10112. 47

[BM26] Jonathan Buchanan and Stephen McKean. KSp-characteristic classes determine Spin^h cobordism. *Algebr. Geom. Topol.*, 26(2):485–551, 2026. arXiv:2312.08209. 30

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019. arXiv:1802.03261. 2

[BMSS15] Robert Bruner, Khairia Mira, Laura Stanley, and Victor Snaith. Ossa's theorem via the Kunneth formula. *Mathematics and Statistics*, 3(3):58–64, 2015. arXiv:1008.0166. 42

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979. 15

[Bou03] A. K. Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geom. Topol.*, 7:1001–1053, 2003. arXiv:math/0312531. 8

[BP25] Robert Burklund and Piotr Pstrągowski. Quivers and the Adams spectral sequence. *Adv. Math.*, 471:110270, 2025. arXiv:2305.08231. 14

[BR08] Andrew Baker and Birgit Richter. Uniqueness of E_∞ structures for connective covers. *Proc. Amer. Math. Soc.*, 136(2):707–714, 2008. arXiv:math/0506422. 11

[BR21] Robert R. Bruner and John Rognes. *The Adams spectral sequence for topological modular forms*, volume 253 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2021. 2, 3, 4, 6, 17, 18, 19, 20, 67

[Bro24] James A. Brotherston. Monoidal model structures on filtered chain complexes relating to spectral sequences, 2024. arXiv:2402.09207. 9

[Bro25] James A. Brotherston. A distributive lattice of model structures relating to spectral sequences. *Homology Homotopy Appl.*, 27(2):91–122, 2025. arXiv:2402.09893. 9

[Bru14] Robert R. Bruner. Idempotents, localizations and Picard groups of $A(1)$-modules. In *An alpine expedition through algebraic topology*, volume 617 of *Contemp. Math.*, pages 81–108. Amer. Math. Soc., Providence, RI, 2014. arXiv:1211.0213. 64

[Cam17] Jonathan A. Campbell. Homotopy theoretic classification of symmetry protected phases, 2017. arXiv:1708.04264. 2, 6, 17, 18, 33, 34, 37, 43, 62

[Car25] Christian Carrick. Slice spectral sequences through synthetic spectra, 2025. arXiv:2510.19501. 2, 48

[CD26] Christian Carrick and Jack Morgan Davies. A synthetic approach to detecting v_1-periodic families. *Trans. Amer. Math. Soc.*, 379(3):1985–2031, 2026. arXiv:2401.16508. 2, 21

[CDDH26] Vivek Chakrabhavi, Arun Debray, Markus Dierigl, and Jonathan J. Heckman. Reflection branes, bordisms, and U dualities. *Phys. Rev. D*, 113(6):066015, 2026. arXiv:2509.03573. 17

[CDvN24] Christian Carrick, Jack Morgan Davies, and Sven van Nigtevecht. The descent spectral sequence for topological modular forms, 2024. arXiv:2412.01640. 2

[CDvN25] Christian Carrick, Jack Morgan Davies, and Sven van Nigtevecht. Descent spectral sequences through synthetic spectra. *Int. Math. Res. Not. IMRN*, 15:Paper No. rnaf216, 26, 2025. arXiv:2407.01507. 2

[CESLW20] Joana Cirici, Daniela Egas Santander, Muriel Livernet, and Sarah Whitehouse. Model category structures and spectral sequences. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(6):2815–2848, 2020. arXiv:1805.00374. 9

[CKQ21] Dominic Leon Culver, Hana Jia Kong, and J. D. Quigley. Algebraic slice spectral sequences. *Doc. Math.*, 26:1085–1119, 2021. arXiv:2007.08682. 49

[CLW25] Joana Cirici, Muriel Livernet, and Sarah Whitehouse. Model category structures on truncated multicomplexes for complex geometry. *Bull. Lond. Math. Soc.*, 57(12):4163–4177, 2025. arXiv:2501.13509. 9

[CQ21] Dominic Leon Culver and J. D. Quigley. kq-resolutions I. *Trans. Amer. Math. Soc.*, 374(7):4655–4710, 2021. arXiv:1905.11952. 49

[Cre25] Julie Creemers. Spectral sequences obtained from towers in the ∞-category of spectra: décalage, exact couples and Massey products. Master's thesis, Utrecht University, 2025. https://studenttheses.uu.nl/items/6a22ab2e-906a-4e19-b079-caf0dc3fbd8b. 2

[Cul21] D. Culver. The Adams spectral sequence for 3-local tmf. *J. Homotopy Relat. Struct.*, 16(1):1–40, 2021. arXiv:1902.04230. 21

[Dav26] Donald M. Davis. *The connective KO theory of the Eilenberg-MacLane space* $K(\mathbb{Z}/2, 2)$, volume 7 of *New York Journal of Mathematics. NYJM Monographs*. State University of New York, University at Albany, Albany, NY, 2026. arXiv:2502.14982. 17

[DDHM24] Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. The chronicles of IIBordia: dualities, bordisms, and the Swampland. *Adv. Theor. Math. Phys.*, 28(3):805–1025, 2024. arXiv:2302.00007. 17, 18, 28, 33, 34, 35, 42

[DDK+26] Arun Debray, Sanath K. Devalapurkar, Cameron Krulewski, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren. The Smith fiber sequence and invertible field theories. *Commun. Math. Phys.*, 407(2):25, 2026. arXiv:2405.04649. 36, 55

[Deb21] Arun Debray. Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle, 2021. arXiv:2102.02941. 12, 17, 18, 30, 62, 68

[Déc24a] Thibault D. Décoppet. The 2-Deligne tensor product. *Kyoto J. Math.*, 64(1):1–29, 2024. arXiv:abs/2103.16880. 39

[Déc24b] Thibault Didier Décoppet. Extension theory and fermionic strongly fusion 2-categories. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 20:092, 2024. With an appendix by Décoppet and Theo Johnson-Freyd. arXiv:2403.03211. 1, 30, 44, 50, 53, 60

[DFN+25] Prerna Dhankhar, Rebecca Field, Arjun Nigam, J. D. Quigley, and Albert Jinghui Yang. Very effective algebraic and hermitian K-theory of the cyclic group of order two, 2025. arXiv:2509.25006. 49, 50

[DG18] Arun Debray and Sam Gunningham. The Arf-Brown TQFT of pin$^-$ surfaces. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 49–87. Amer. Math. Soc., Providence, RI, 2018. arXiv:1803.11183. 43, 56

[DGL22] Joe Davighi, Ben Gripaios, and Nakarin Lohitsiri. Anomalies of non-Abelian finite groups via cobordism. *J. High Energy Phys.*, 2022(9):147, 2022. arXiv:2207.10700. 18

[DGM81] Donald M. Davis, Sam Gitler, and Mark Mahowald. The stable geometric dimension of vector bundles over real projective spaces. *Trans. Amer. Math. Soc.*, 268(1):39–61, 1981. 24

[Dha26] Prerna Dhankhar. On the motivic K-theories of classifying spaces of cyclic 2-groups, 2026. arXiv:2610.27016. 18, 33, 34, 35, 49

[DHJF+24] Thibault D. Décoppet, Peter Huston, Theo Johnson-Freyd, Dmitri Nikshych, David Penneys, Julia Plavnik, David Reutter, and Matthew Yu. The classification of fusion 2-categories, 2024. arXiv:2411.05907. 1, 50, 60

[DKS26] Arun Debray, Cameron Krulewski, and Luuk Stehouwer. Unraveling the Bott spiral, 2026. arXiv:2605.00316. 17, 24, 43, 44, 64, 67

[DL20] Joe Davighi and Nakarin Lohitsiri. Anomaly interplay in U(2) gauge theories. *J. High Energy Phys.*, 2020(5):098, 2020. arXiv:2001.07731. 17, 67

[DL21a] Joe Davighi and Nakarin Lohitsiri. The algebra of anomaly interplay. *SciPost Phys.*, 10(3):074, 2021. arXiv:2011.10102. 17, 18, 34

[DL21b] Joe Davighi and Nakarin Lohitsiri. Omega *vs.* pi, and 6d anomaly cancellation. *J. High Energy Phys.*, 2021(5):267, 2021. arXiv:2012.11693. 17

[DL23] Joe Davighi and Nakarin Lohitsiri. Toric 2-group anomalies via cobordism. *J. High Energy Phys.*, 2023(7):019, 2023. With an appendix by Arun Debray. arXiv:2302.12853. 17

[Dri06] Vladimir Drinfeld. Infinite-dimensional vector bundles in algebraic geometry. In Pavel Etingof, Vladimir Retakh, and I. M. Singer, editors, *The Unity of Mathematics: In Honor of the Ninetieth Birthday of I.M. Gelfand*, pages 263–304. Birkhäuser Boston, Boston, MA, 2006. arXiv:math/0309155. 39

[DS26] Amartya Shekhar Dubey and Luuk Stehouwer. The J-homomorphism from localising the cobordism hypothesis, 2026. arXiv:2611.19615. 51, 61

[Dug05] Daniel Dugger. An Atiyah-Hirzebruch spectral sequence for KR-theory. *K-Theory*, 35(3–4):213–256, 2005. Published in 2006. arXiv:math/0304099. 49

[DW25] Donald M. Davis and W. Stephen Wilson. The connective KO-theory of the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$, I: the E_2 page. *J. Homotopy Relat. Struct.*, 20(4):511–521, 2025. arXiv:2410.23094. 17

[DY23a] Arun Debray and Matthew Yu. Adams spectral sequences for non-vector-bundle Thom spectra, 2023. arXiv:2305.01678. 3, 10, 17, 29, 47, 52, 55, 56, 61

[DY23b] Thibault D. Décoppet and Matthew Yu. Gauging noninvertible defects: a 2-categorical perspective. *Lett. Math. Phys.*, 113(2):36, 2023. arXiv:2211.08436. 50, 60

[DY26] Thibault D. Décoppet and Matthew Yu. The classification of 3+1D symmetry enriched topological order. *Comm. Math. Phys.*, 407(11):Paper No. 238, 2026. arXiv:2509.10603. 1, 39, 50, 60

[DYY25] Arun Debray, Weicheng Ye, and Matthew Yu. How to build anomalous (3+1)d topological quantum field theories, 2025. arXiv:2510.24834. 1, 5, 17, 18, 30, 44, 60, 61

[DYY26a] Arun Debray, Weicheng Ye, and Matthew Yu. Global structure in the presence of a topological defect. *Commun. Math. Phys.*, 407(8):159, 2026. arXiv:2501.18399. 17

[DYY26b] Arun Debray, Matthew Yu, and Weicheng Ye. Symmetric gapped states and symmetry-enforced gaplessness in 3 dimensions, 2026. arXiv:2602.12335. 17, 44, 60

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. 8, 9, 14, 15

[Emm26] Konstantin Emming. Filtrations in $\mathbb{C}$-motivic stable homotopy theory, 2026. arXiv:2608.04877. 2

[FGLW22] Xin Fu, Ai Guan, Muriel Livernet, and Sarah Whitehouse. Model category structures on multicomplexes. *Topology Appl.*, 316:108104, 2022. arXiv:2001.10873. 9

[FH21] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. *Geom. Topol.*, 25(3):1165–1330, 2021. arXiv:1604.06527. 17, 42, 62

[FHM03] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:107–131, 2003. arXiv:math/0206079. 60

[FHT10] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Consistent orientation of moduli spaces. In *The many facets of geometry*, pages 395–419. Oxford Univ. Press, Oxford, 2010. 47

[FMS07] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. The uncertainty of fluxes. *Communications in Mathematical Physics*, 271(1):247–274, Apr 2007. https://arxiv.org/abs/hep-th/0605198. 42

[FN23] Daniel S. Freed and Andrew Neitzke. The dilogarithm and abelian Chern-Simons. *J. Differential Geom.*, 123(2):241–266, 2023. arXiv:2006.12565. 47

[Fra11] John Francis. Integrals on spin manifolds and the K-theory of $\mathrm{K}(\mathbb{Z}, 4)$, 2011. https://sites.math.northwestern.edu/~jnkf/writ/bspin2011.pdf. 17, 18

[Fre08] Daniel S. Freed. Pions and generalized cohomology. *J. Differential Geom.*, 80(1):45–77, 2008. arXiv:hep-th/0607134. 1, 47

[Fre12] Daniel S. Freed. Lectures on twisted K-theory and orientifolds, 2012. https://people.math.harvard.edu/~dafr/vienna.pdf. 43

[Ghe18] Bogdan Gheorghe. The motivic cofiber of τ. *Doc. Math.*, 23:1077–1127, 2018. arXiv:1701.04877. 49

[GHIR20] Bertrand J. Guillou, Michael A. Hill, Daniel C. Isaksen, and Douglas Conner Ravenel. The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}. *Tunis. J. Math.*, 2(3):567–632, 2020. arXiv:1708.09568. 49

[Gia73a] V. Giambalvo. Cobordism of line bundles with a relation. *Illinois J. Math.*, 17:442–449, 1973. 17

[Gia73b] V. Giambalvo. Pin and Pin$'$ cobordism. *Proc. Amer. Math. Soc.*, 39:395–401, 1973. 17, 55

[Gia76] V. Giambalvo. Cobordism of spin manifolds with involution. *Quart. J. Math. Oxford Ser. (2)*, 27(106):241–252, 1976. 17

[GIKR22] Bogdan Gheorghe, Daniel C. Isaksen, Achim Krause, and Nicolas Ricka. $\mathbb{C}$-motivic modular forms. *J. Eur. Math. Soc. (JEMS)*, 24(10):3597–3628, 2022. arXiv:1810.11050. 2

[GJF19] Davide Gaiotto and Theo Johnson-Freyd. Symmetry-protected topological phases and generalized cohomology. *J. High Energy Phys.*, 2019(5):007, 2019. arXiv:1712.07950. 1, 43

[GJO19] Nick Gurski, Niles Johnson, and Angélica M. Osorno. The 2-dimensional stable homotopy hypothesis. *J. Pure Appl. Algebra*, 223(10):4348–4383, 2019. arXiv:1712.07218. 39

[GJOS17] Nick Gurski, Niles Johnson, Angélica M. Osorno, and Marc Stephan. Stable Postnikov data of Picard 2-categories. *Algebr. Geom. Topol.*, 17(5):2763–2806, 2017. arXiv:1606.07032. 40

[GK14] Nora Ganter and Mikhail Kapranov. Symmetric and exterior powers of categories. *Transform. Groups*, 19(1):57–103, 2014. arXiv:1110.4753. 39

[GLWZ26] Zhenbang Gu, Ran Luo, Yi-Nan Wang, and Yi Zhang. Understanding non-split 2-group symmetry: (3+1)D SymTFT, anomaly and bordism, 2026. arXiv:2608.08766. 17, 50, 60

[GMM68] S. Gitler, M. Mahowald, and R. James Milgram. The nonimmersion problem for RP^n and higher-order cohomology operations. *Proc. Nat. Acad. Sci. U.S.A.*, 60:432–437, 1968. 62, 67

[GOP+20] Meng Guo, Kantaro Ohmori, Pavel Putrov, Zheyan Wan, and Juven Wang. Fermionic finite-group gauge theories and interacting symmetric/crystalline orders via cobordisms. *Commun. Math. Phys.*, 376(2):1073–1154, 2020. arXiv:1812.11959. 17, 18

[Gre18] J. P. C. Greenlees. Four approaches to cohomology theories with reality. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 139–156. Amer. Math. Soc., [Providence], RI, 2018. arXiv:1705.09365. 49

[Gun16] Sam Gunningham. Spin Hurwitz numbers and topological quantum field theory. *Geom. Topol.*, 20(4):1859–1907, 2016. arXiv:1201.1273. 51

[Guo18] Meng Guo. *Some Calculations of Cobordism Groups and Their Applications in Physics*. PhD thesis, Harvard University, 2018. https://dash.harvard.edu/bitstream/handle/1/40050106/GUO-DISSERTATION-2018.pdf?sequence=4. 17

[GW14] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory. *Phys. Rev. B*, 90(11):115141, 2014. arXiv:1201.2648. 1, 47

[Hed20] Alice Hedenlund. *Multiplicative Tate spectral sequences*. PhD thesis, University of Oslo, 2020. https://www.mn.uio.no/math/personer/vit/rognes/theses/hedenlund-thesis.pdf. 2, 47, 48

[Hil07] Michael A. Hill. The 3-local tmf-homology of $B\Sigma_3$. *Proc. Amer. Math. Soc.*, 135(12):4075–4086, 2007. arXiv:math/0511649. 21

[Hil11] Michael A. Hill. Ext and the motivic Steenrod algebra over $\mathbb{R}$. *J. Pure Appl. Algebra*, 215(5):715–727, 2011. arXiv:0904.1998. 49

[HM25] Alice Hedenlund and Tasos Moulinos. The synthetic Hilbert additive group scheme. *Selecta Math. (N.S.)*, 31(5):Paper No. 89, 64, 2025. arXiv:2411.17441. 2

[HNP25] Yonatan Harpaz, Joost Nuiten, and Matan Prasma. On k-invariants for (∞, n)-categories. *Algebr. Geom. Topol.*, 25(2):721–790, 2025. arXiv:2011.12723. 40

[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra, 2022. arXiv:2206.11208. 2

[HS05] M.J. Hopkins and I.M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005. arXiv:math/0211216. 39

[Hys26] Logan Hyslop. Whitehead filtrations for computations in topological Hochschild homology. *Algebr. Geom. Topol.*, 26(7):2443–2474, 2026. arXiv:2311.06717. 48

[IS11] Daniel C. Isaksen and Armira Shkembi. Motivic connective K-theories and the cohomology of A(1). *J. K-Theory*, 7(3):619–661, 2011. arXiv:1002.2368. 49

[Jen05] Jerome A. Jenquin. Classical Chern-Simons on manifolds with spin structure, 2005. arXiv:math/0504524. 47

[JF20] Theo Johnson-Freyd. Topological Mathieu moonshine, 2020. arXiv:2006.02922. 47

[JF25] Theo Johnson-Freyd. $(3{+}1)D$ topological orders with only a $\mathbb{Z}_2$-charged particle. In *Quantum symmetries: tensor categories, TQFTs, and vertex algebras*, volume 813 of *Contemp. Math.*, pages 175–210. Amer. Math. Soc., [Providence], RI, 2025. arXiv:2011.11165. 1, 36, 39, 50, 60

[JFR24] Theo Johnson-Freyd and David Reutter. Minimal nondegenerate extensions. *J. Amer. Math. Soc.*, 37(1):81–150, 2024. arXiv:2105.15167. 3, 5, 36, 38, 39, 40

[JFS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. *Adv. Math.*, 307:147–223, 2017. arXiv:1502.06526. 39

[JFT20] Theo Johnson-Freyd and David Treumann. $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) = \mathbf{Z}/24$. *Int. Math. Res. Not. IMRN*, 2020(21):7873–7907, 2020. arXiv:1707.07587. 47

[JFY21] Theo Johnson-Freyd and Matthew Yu. Fusion 2-categories with no line operators are grouplike. *Bull. Aust. Math. Soc.*, 104(3):434–442, 2021. arXiv:2010.07950. 44, 50, 60

[JFY22] Theo Johnson-Freyd and Matthew Yu. Topological orders in $(4{+}1)$-dimensions. *SciPost Phys.*, 13(3):068, 2022. arXiv:2104.04534. 50, 60

[JO12] Niles Johnson and Angélica M. Osorno. Modeling stable one-types. *Theory Appl. Categ.*, 26:520–537, 2012. arXiv:1201.2686. 39, 40

[JW73] David Copeland Johnson and W. Stephen Wilson. Projective dimension and Brown-Peterson homology. *Topology*, 12:327–353, 1973. 11

[Kan58a] Daniel M. Kan. A combinatorial definition of homotopy groups. *Ann. of Math. (2)*, 67:282–312, 1958. 58

[Kan58b] Daniel M. Kan. On homotopy theory and c.s.s. groups. *Ann. of Math. (2)*, 68:38–53, 1958. 58

[Kit06] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Ann. Physics*, 321(1):2–111, 2006. arXiv:cond-mat/0506438. 3, 36, 38

[Kit09] Alexei Kitaev. Periodic table for topological insulators and superconductors. In *Advances in theoretical physics: Landau Memorial Conference*, volume 1134 of *AIP Conf. Proc.*, pages 22–30, 2009. arXiv:0901.2686. 3, 5

[KKSW26] Masashi Kawahira, Shotaro Kawanago, Shota Saito, and Hiroki Wada. Anomalies and discrete torsion in Type II and Type I worldsheet theories, 2026. arXiv:2608.30774. 17

[KL20] Nicholas Kuhn and Tyler Lawson. Are the AHSS and Adams spectral sequence the same when computing connective Morava K-theory of a space?, 2020. https://mathoverflow.net/q/359574. 14

[Kne25] Christian Kneißl. Spin cobordism and the gauge group of type I/heterotic string theory. *J. High Energy Phys.*, 2025(1):181, 2025. arXiv:2407.20333. 17, 18

[Koc96] S. O. Kochman. *Bordism, stable homotopy and Adams spectral sequences*, volume 7 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996. 42

[KPMT20] Justin Kaidi, Julio Parra-Martinez, and Yuji Tachikawa. Topological superconductors on superstring worldsheets. *SciPost Phys.*, 9(1):010, 2020. With a mathematical appendix by Arun Debray. arXiv:1911.11780. 17

[KPY26a] Ryohei Kobayashi, Abhinav Prem, and Matthew Yu. Realizing braided fusion 2-categories as (3+1)D mixed-state topological orders, 2026. arXiv:2610.03864. 50, 60

[KPY26b] Ryohei Kobayashi, Abhinav Prem, and Matthew Yu. Sixteen-fold way for fermionic topological orders, 2026. arXiv:2606.28682. 38

[KT17] Anton Kapustin and Ryan Thorngren. Fermionic SPT phases in higher dimensions and bosonization. *J. High Energy Phys.*, 2017(10):080, 2017. arXiv:1701.08264. 1, 28

[Kur25] Naoki Kuroda. Computations of Spin-Sp(4), Spin-SU(8), and Spin-Spin(16) bordism groups in dimensions up to 7, 2025. arXiv:2504.15014. 17

[LdM71] S. López de Medrano. *Involutions on manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], Band 59. Springer-Verlag, New York-Heidelberg, 1971. 47

[Lew95] L. Gaunce Lewis, Jr. Change of universe functors in equivariant stable homotopy theory. *Fund. Math.*, 148(2):117–158, 1995. 49

[Liu62] Arunas L. Liulevicius. The cohomology of a subalgebra of the Steenrod algebra. *Trans. Amer. Math. Soc.*, 104:443–449, 1962. 17, 67

[LKW17] Tian Lan, Liang Kong, and Xiao-Gang Wen. Modular extensions of unitary braided fusion categories and 2 + 1D topological/SPT orders with symmetries. *Commun. Math. Phys.*, 351(2):709–739, 2017. arXiv:1602.05936. 3, 5, 38

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. http://www.math.uchicago.edu/~may/BOOKS/equi.pdf. 57

[LOT21] Yasunori Lee, Kantaro Ohmori, and Yuji Tachikawa. Revisiting Wess-Zumino-Witten terms. *SciPost Phys.*, 10(3):61, 2021. arXiv:2009.00033. 17

[LPT25] Guchuan Li, Sarah Petersen, and Elizabeth Tatum. A spectrum-level splitting of the $ku_{\mathbb{R}}$-cooperations algebra, 2025. arXiv:2503.17149. 49

[LT21] Yasunori Lee and Yuji Tachikawa. Some comments on 6D global gauge anomalies. *Prog. Theor. Exp. Phys.*, 2021(8):08B103, 2021. arXiv:2012.11622. 17

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009. arXiv:0905.0465. 50, 51

[Lur17] Jacob Lurie. Higher algebra, 2017. https://www.math.ias.edu/~lurie/papers/HA.pdf. 7, 8, 10

[LW24a] Muriel Livernet and Sarah Whitehouse. Homotopy theory of spectral sequences. *Homology Homotopy Appl.*, 26(1):69–86, 2024. arXiv:2207.02156. 9

[LW24b] Muriel Livernet and Sarah Whitehouse. Spectral sequences via linear presheaves, 2024. arXiv:2406.02777. 9

[LWX24] Weinan Lin, Guozhen Wang, and Zhouli Xu. On the last Kervaire invariant problem, 2024. arXiv:2412.10879. 21

[LY22] Yasunori Lee and Kazuya Yonekura. Global anomalies in 8d supergravity. *J. High Energy Phys.*, 2022(7):125, 2022. arXiv:2203.12631. 17, 18

[Mah82] Mark Mahowald. The image of J in the EHP sequence. *Ann. of Math. (2)*, 116(1):65–112, 1982. Correction in *Annals of Mathematics*, 120:399–400, 1984. 42

[Mar74] H. R. Margolis. Eilenberg-Mac Lane spectra. *Proc. Amer. Math. Soc.*, 43:409–415, 1974. 24

[Mar83] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category. 42, 62

[Mau63] C. R. F. Maunder. The spectral sequence of an extraordinary cohomology theory. *Proc. Cambridge Philos. Soc.*, 59:567–574, 1963. 8, 9

[May77] J. Peter May. *E_∞ ring spaces and E_∞ ring spectra*. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. http://www.math.uchicago.edu/~may/BOOKS/e_infty.pdf. 54, 56

[May88] Ellen Maycock Parker. The Brauer group of graded continuous trace C^*-algebras. *Trans. Amer. Math. Soc.*, 308(1):115–132, 1988. 57

[Mil75] R. James Milgram. The Steenrod algebra and its dual for connective K-theory. In *Conference on homotopy theory (Evanston, Ill., 1974)*, volume 1 of *Notas Mat. Simpos.*, pages 127–158. Soc. Mat. Mexicana, México, 1975. 23, 24, 67

[Mil24] Keith Mills. The structure of the spin^h bordism spectrum. *Proc. Amer. Math. Soc.*, 152(8):3605–3616, 2024. arXiv:2306.17709. 64

[MM79] Ib Madsen and R. James Milgram. *The classifying spaces for surgery and cobordism of manifolds*. Annals of Mathematics Studies, No. 92. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1979. 47

[MM81] J. P. May and R. J. Milgram. The Bockstein and the Adams spectral sequences. *Proc. Amer. Math. Soc.*, 83(1):128–130, 1981. 13, 34, 35

[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002. http://www.math.uchicago.edu/~may/PAPERS/MMMFinal.pdf. 59

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. Lond. Math. Soc. (3)*, 82(2):441–512, 2001. http://www.math.uchicago.edu/~may/PAPERS/mmssLMSDec30.pdf. 59

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017. arXiv:1507.06869. 8

[MOP+22] Lyne Moser, Viktoriya Ozornova, Simona Paoli, Maru Sarazola, and Paula Verdugo. Stable homotopy hypothesis in the Tamsamani model. *Topology Appl.*, 316:108106, 2022. arXiv:2001.05577. 39

[Mor25] Jackson Morris. Rings of cooperations for hermitian K-theory over finite fields, 2025. arXiv:2509.02786. 49

[Mor26] Jackson Morris. On the ring of cooperations for real hermitian K-theory. *Ann. K-Theory*, 11(4):633–708, 2026. arXiv:2506.16672. 49

[MP84] Stephen A. Mitchell and Stewart B. Priddy. Symmetric product spectra and splittings of classifying spaces. *Amer. J. Math.*, 106(1):219–232, 1984. 45

[MPT25] Jackson Morris, Sarah Petersen, and Elizabeth Tatum. Splittings of truncated motivic Brown–Peterson cooperations algebras, 2025. arXiv:2509.19542. 49

[MR19] Fernando Muro and Constanze Roitzheim. Homotopy theory of bicomplexes. *J. Pure Appl. Algebra*, 223(5):1913–1939, 2019. arXiv:1802.07610. 9

[MS06] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006. http://www.math.uchicago.edu/~may/EXTHEORY/MaySig.pdf. 57, 58, 59, 60

[Muk03] Himadri Kumar Mukerjee. Classification of homotopy Dold manifolds. *New York J. Math.*, 9:271–293, 2003. 47

[Muk04] Himadri Kumar Mukerjee. Classification of homotopy real Milnor manifolds. *Topology Appl.*, 139(1–3):151–184, 2004. 47

[Muk06] Himadri Kumar Mukerjee. Classification of homotopy Wall's manifolds. *Topology Appl.*, 153(18):3467–3495, 2006. 47

[MZB+25] Ruochen Ma, Jian-Hao Zhang, Zhen Bi, Meng Cheng, and Chong Wang. Topological phases with average symmetries: the decohered, the disordered, and the intrinsic. *Phys. Rev. X*, 15(2):021062, 2025. arXiv:2305.16399. 1

[NRW+25] Shang-Qiang Ning, Xing-Yu Ren, Qing-Rui Wang, Yang Qi, and Zheng-Cheng Gu. Classification of interacting topological crystalline superconductors in three dimensions and beyond, 2025. arXiv:2512.25069. 13

[OØ13] Kyle M. Ormsby and Paul Arne Østvær. Motivic Brown-Peterson invariants of the rationals. *Geom. Topol.*, 17(3):1671–1706, 2013. arXiv:1208.5007. 49

[Orm11] Kyle M. Ormsby. Motivic invariants of p-adic fields. *J. K-Theory*, 7(3):597–618, 2011. arXiv:1002.5007. 49

[Pat12] Deepam Patel. De Rham $\mathcal{E}$-factors. *Invent. Math.*, 190(2):299–355, 2012. http://www.math.purdue.edu/~patel471/deRhamep.pdf. 39

[Pea14] Paul Thomas Pearson. The connective real K-theory of Brown-Gitler spectra. *Algebr. Geom. Topol.*, 14(1):597–625, 2014. 17, 67

[Ped17] Riccardo Pedrotti. Stable classification of certain families of four-manifolds. Master's thesis, Max Planck Institute for Mathematics, 2017. 12, 17, 18

[Per78] Michael Perloff. *PL-Spin Cobordism*. PhD thesis, Stanford University, 1978. 17, 67

[Pet68] F. P. Peterson. *Lectures on Cobordism Theory*. Lectures in Mathematics. Kinokuniya Book Store Co., Ltd., 1968. 55

[Pie26] Lucas Piessevaux. *Synthetic Equivariant Spectra and Complex Bordism*. PhD thesis, University of Bonn, 2026. https://bonndoc.ulb.uni-bonn.de/xmlui/handle/20.500.11811/14463. 2

[PP26] Irakli Patchkoria and Piotr Pstrągowski. Adams Spectral Sequences and Franke's Algebraicity Conjecture. *Mem. Amer. Math. Soc.*, 321(1636):v+140, 2026. arXiv:2110.03669. 2, 21, 48

[Pst23a] Piotr Pstrągowski. Perfect even modules and the even filtration, 2023. arXiv:2304.04685. 2

[Pst23b] Piotr Pstrągowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023. arXiv:1803.01804. 2, 21, 48

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972. 45

[Qui19] J. D. Quigley. The motivic Mahowald invariant. *Algebr. Geom. Topol.*, 19(5):2485–2534, 2019. arXiv:1801.06035. 49

[QZ25] Ryan Quinn and Qi Zhu. Multiplicative equivariant Thom spectra & structured Real orientations, 2025. arXiv:2512.15573. 2

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984. 10, 15

[Rub79] Ryszard L. Rubinsztein. Some remarks on the cohomotopy of $\mathbf{R}P^\infty$. *Quart. J. Math. Oxford Ser. (2)*, 30(117):119–128, 1979. 47

[Rud74] Yu. B. Rudyak. The normal invariants of certain manifolds. *Mat. Zametki*, 16:763–769, 1974. 47

[Sal26] Andrew Salch. Ext-groups of graded comodules, 2026. arXiv:2609.26465. 17

[San26] Sean Sanford. On the Mext groups of $\mathrm{sVect}_{\mathbb{R}}$ and $\mathrm{sVect}_{\mathbb{H}}$, 2026. arXiv:2609.20153. 5, 6, 40, 43

[Ser53] Jean-Pierre Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–232, 1953. 37

[Sha23] Amit Sharma. Picard groupoids and Γ-categories. *Cah. Topol. Géom. Différ. Catég.*, 64(3):287–324, 2023. arXiv:2002.05811. 39

[She25] Aiden Sheckler. Anomaly matching in 6d $\mathcal{N} = (2, 0)$ SCFTs from M5 cobordism. *J. High Energy Phys.*, 2025(12):116, 2025. arXiv:2501.04785. 17

[Sie13] Christian Siegemeyer. *On the Gromov-Lawson-Rosenberg Conjecture for Finite Abelian 2-Groups of Rank 2*. PhD thesis, Westfälische Wilhelms-Universität Münster, 2013. https://repositorium.uni-muenster.de/document/miami/4882c034-0925-432c-b743-edf299b5192b/diss_siegemeyer.pdf. 34, 67

[SØ12] Markus Spitzweck and Paul Arne Østvær. Motivic twisted K-theory. *Algebr. Geom. Topol.*, 12(1):565–599, 2012. arXiv:1008.4915. 49

[Sto88] Stephan Stolz. Exotic structures on 4-manifolds detected by spectral invariants. *Invent. Math.*, 94(1):147–162, 1988. 55

[Sto94] Stephan Stolz. Splitting certain M Spin-module spectra. *Topology*, 33(1):159–180, 1994. 17

[Sul66] Dennis Parnell Sullivan. *Triangulating homotopy equivalences*. PhD thesis, Princeton University, 1966. 47

[Sul96] D. P. Sullivan. Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric Topology Seminar Notes. In *The Hauptvermutung book*, volume 1 of *K-Monogr. Math.*, pages 69–103. Kluwer Acad. Publ., Dordrecht, 1996. 47

[TY23a] Yuji Tachikawa and Mayuko Yamashita. Anderson duality of topological modular forms and its differential-geometric manifestations, 2023. With an appendix by Sanath Devalapurkar. arXiv:2305.06196. 47

[TY23b] Yuji Tachikawa and Mayuko Yamashita. Topological modular forms and the absence of all heterotic global anomalies. *Commun. Math. Phys.*, 402(2):1585–1620, 2023. arXiv:2108.13542. 47

[TY25a] Yuji Tachikawa and Kazuya Yonekura. On invariants of two-dimensional minimally supersymmetric field theories, 2025. arXiv:2508.04916. 47

[TY25b] Daniel Teixeira and Matthew Yu. Mutual influence of symmetries and topological field theories, 2025. arXiv:2507.06304. 50, 60

[vN25] Sven van Nigtevecht. An introduction to filtered and synthetic spectra, 2025. arXiv:2509.21127. 2, 8, 21, 48

[Voe98] Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, Extra Vol. I, pages 579–604, 1998. https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/A1_homotopy_ICM_1998_Berlin_published.pdf. 49

[Voe02] Vladimir Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press,

Somerville, MA, 2002. https://www.math.ias.edu/Voevodsky/files/files-annotated/Dropbox/Published_papers/Motives/Open/Stage3/newopen.pdf. 49

[Wal62] C. T. C. Wall. A characterization of simple modules over the Steenrod algebra mod 2. *Topology*, 1:249–254, 1962. 62

[Wal68] C. T. C. Wall. Free piecewise linear involutions on spheres. *Bull. Amer. Math. Soc.*, 74:554–558, 1968. 47

[Wan22] Xueqi Wang. On the classification of certain 1-connected 7-manifolds and related problems. *Q. J. Math.*, 73(2):711–727, 2022. arXiv:1810.08474. 47

[WG18] Qing-Rui Wang and Zheng-Cheng Gu. Towards a complete classification of symmetry-protected topological phases for interacting fermions in three dimensions and a general group supercohomology theory. *Phys. Rev. X*, 8(1):011055, 2018. arXiv:1703.10937. 1, 13, 28, 30

[WG20] Qing-Rui Wang and Zheng-Cheng Gu. Construction and classification of symmetry-protected topological phases in interacting fermion systems. *Phys. Rev. X*, 10(3):031055, 2020. arXiv:1811.00536. 1, 28, 30, 44

[Wil73] W. Stephen Wilson. A new relation on the Stiefel-Whitney classes of spin manifolds. *Illinois J. Math.*, 17:115–127, 1973. 17

[Wu26] Yuchen Wu. Burklund-Lin-Wang-Xu methods in the cofiber-of-tau formalism and applications to equivariant slice differentials, 2026. arXiv:2606.02779. 47

[WW19] Zheyan Wan and Juven Wang. Higher anomalies, higher symmetries, and cobordisms I: classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory. *Ann. Math. Sci. Appl.*, 4(2):107–311, 2019. arXiv:1812.11967. 17

[WW25] Zheyan Wan and Juven Wang. Anomalous (3+1)d Fermionic Topological Quantum Field Theories via Symmetry Extension, 12 2025. arXiv:2512.25038. 1

[WWY26] Zheyan Wan, Juven Wang, and Shing-Tung Yau. Three families and Pontryagin class mod 3: Topological order via symmetry extension, 2026. arXiv:2605.26202. 1

[Yu95] Cherng-Yih Yu. *The connective real K-theory of elementary abelian 2-groups*. PhD thesis, The University of Notre Dame, 1995. 17, 64, 67

[ZWY+20] Jian-Hao Zhang, Qing-Rui Wang, Shuo Yang, Yang Qi, and Zheng-Cheng Gu. Construction and classification of point-group symmetry-protected topological phases in two-dimensional interacting fermionic systems. *Phys. Rev. B*, 101(10):100501, 2020. arXiv:1909.05519. 30

Department of Mathematics, University of Kentucky, 719 Patterson Office Tower, Lexington, KY 40506-0027

Email address: a.debray@uky.edu

Department of Physics and Astronomy, and Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, BC, Canada V6T 1Z1

Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong
Email address: victoryeofphysics@gmail.com

Mathematical Institute, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG
Email address: yumatthew70@gmail.com