WHAT BORDISM-THEORETIC ANOMALY CANCELLATION CAN DO FOR U

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Abstract. We perform a bordism computation to show that the $E_{7(7)}(\mathbb{R})$ U-duality symmetry of 4d $\mathcal{N} = 8$ supergravity could have an anomaly invisible to perturbative methods; then we show that this anomaly is trivial. We compute the relevant bordism group using the Adams and Atiyah-Hirzebruch spectral sequences, and we show the anomaly vanishes by computing η-invariants on the Wu manifold, which generates the bordism group.

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1. Introduction

One of the most surprising discoveries in the field of string theory is the existence of duality symmetries. These symmetries show that the same theory can be described in superficially different ways. In some cases, this can be seen via a transformation of the parameters of the theory, or even the spacetime itself. One such symmetry is U-duality, given by the group $E_n(\mathbb{Z})$. By starting with an 11-dimensional theory which encompasses the type IIA string theory, and compactifying on an n-torus, we gain an $\mathrm{SL}_n(\mathbb{Z})$ symmetry from the modular group on n-torus. We arrive at the same theory by compactifying 10d type IIB on a $n-1$-torus, and obtain an $\mathrm{O}(n-1, n-1, \mathbb{Z})$ symmetry related to T-duality. The group $E_n(\mathbb{Z})$ is then generated by the two aforementioned groups.

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In the low energy regime of the 11d theory, which is 11d supergravity, we have an embedding of $E_n(\mathbb{Z}) \hookrightarrow E_{n(n)}$ upon applying the torus compactification procedure. The latter group is the U-duality of supergravity. One finds a maximally noncompact form of E_n after the compactification, and this is denoted $E_{n(n)}(\mathbb{R})$. The maximally noncompact form of a Lie group of rank n contains n more noncompact generators than compact generators. For the purpose of this paper, we reduce 11-dimensional supergravity on a 7-dimenisonal torus. This gives a maximal supergravity theory, i.e. 4d $\mathcal{N} = 8$ supergravity, with an E_7 symmetry [1]. The noncompact form is $E_{7(7)}$ which has 63 compact generators and 70 noncompact generators.

Because this is a symmetry of the theory, one can ask if it is anomalous, and in particular if there are any global anomalies. Since 4d $\mathcal{N} = 8$ supergravity arises as the low energy effective theory of string theory, then a strong theorem of quantum gravity saying that there are no global symmetries implies that the U-duality symmetry must be gaugable. Therefore, the existence of any global anomaly would require a mechanism for its cancellation. It would therefore be an interesting question to consider if additional topological terms need to be added to cancel the nonperturbative anomaly as in [DDHM21], but we show that with the matter content of 4d maximal supergravity is sufficient to cancel the anomaly on the nose.

The main theorem in this paper evaluates the order of the global anomaly for $E_{7(7)}(\mathbb{R})$. This is equal to the order of a bordism group in degree 5 that can be computed from the Adams spectral sequence. We find that the global anomaly is $\mathbb{Z}/2$ valued, but nonetheless is trivial when we take into account the matter content of 4d $\mathcal{N} = 8$ supergravity. In order to see the cancellation we first find the manifold generator of the bordism group, which happens to be the Wu manifold, and compute η-invariants on it. This bordism computation is also mathematically intriguing because we find ourselves working over the entire Steenrod algebra, however the specific properties of the problem we are interested in make this tractable.

This work only focuses on U-duality as a continuous group, because the cohomology of the discrete group that arises in string theory is not known, and a strategy we employ of taking the maximal compact subgroup will not work. But one could imagine running a similar Adams computation for the group $E_7(\mathbb{Z})$ and checking that the anomaly vanishes. There are also a plethora of dualities that arise from compactifying 11d supergravity that one can also compute anomalies of, among them are the U-dualities that arise from compactifying on lower dimensional tori. In upcoming work [DY23] we study the anomalies of T-duality in a setup where the group is small enough to be computable, but big enough to yield interesting anomalies.

The structure of the paper is as follows: in §2 we present the symmetries and tangential structure for the maximal 4d supergravity theory with U-duality symmetry and turn it into a bordism computation. We also give details on the field content of the theory and how it is compatible with the type of manifold we are considering. In §3 we review the possibility of global anomalies, and invertible field theories. In §4 we perform the spectral sequence computation and give the manifold generator for the bordism group in question. In §5 we show that the anomaly vanishes by considering the field content on the manifold generator.

2. Placing the U-duality symmetry on manifolds

In this section, we review how the $E_{7(7)}$ U-duality symmetry acts on the fields of 4d $\mathcal{N} = 8$ supergravity; then we discuss what kinds of manifolds are valid backgrounds in the presence of this

[1] Dimensional reduction of IIB supergravity on an 6-dimensional torus also yields the same symmetry.

symmetry. We assume that we have already Wick-rotated into Euclidean signature. We determine a Lie group H_4 with a map $\rho_4\colon H_4 \to \mathrm{O}_4$ such that 4d $\mathcal{N}=8$ supergravity can be formulated on 4-manifolds M equipped with a metric and an H_4-connection $P, \Theta \to M$, such that $\rho_4(\Theta)$ is the Levi-Civita connection. As we review in §3, anomalies are classified in terms of bordism; once we know H_4 and ρ_4, Freed-Hopkins' work [FH21b] tells us what bordism groups to compute.

The field content of 4d $\mathcal{N}=8$ supergravity coincides with the spectrum of type IIB closed string theory compactified on T^6 and consists of the following fields:

- 70 scalar fields,
- 56 gauginos (spin 1/2),
- 28 vector bosons (spin 1),
- 8 gravitinos (spin 3/2), and
- 1 graviton (spin 2).

Cremmer-Julia [CJ79] exhibited an $\mathfrak{e}_{7(7)}$ symmetry of this theory, meaning an action on the fields for which the Lagrangian is invariant. Here, $\mathfrak{e}_{7(7)}$ is the Lie algebra of the real, noncompact Lie group $E_{7(7)}$, which is the split form of the complex Lie group $E_7(\mathbb{C})$. Cartan constructed $E_{7(7)}$ explicitly as follows: the 56-dimensional vector space

$$V \coloneqq \Lambda^2(\mathbb{R}^8) \oplus \Lambda^2((\mathbb{R}^8)^*) \tag{2.1}$$

has a canonical symplectic form coming from the duality pairing. $E_{7(7)}$ is defined to be the subgroup of $\mathrm{Sp}(V)$ preserving the quartic form

$$q(x^{ab}, y_{cd}) = x^{ad}y_{bc}x^{cd}y_{da} - \frac{1}{4}x^{ab}y_{ab}x^{cd}y_{cd} + \frac{1}{96}\left(\epsilon_{abc\cdots h}x^{ab}x^{cd}x^{ef}x^{gh} + \epsilon^{abc\cdots h}y_{ab}y_{cd}y_{ef}y_{gh}\right). \tag{2.2}$$

Thus, by construction, $E_{7(7)}$ comes with a 56-dimensional representation, which we denote **56**.

$E_{7(7)}$ is noncompact; its maximal compact is $\mathrm{SU}_8/\{\pm 1\}$, giving us an embedding $\mathfrak{su}_8 \subset \mathfrak{e}_{7(7)}$. Thus $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$; let $\widetilde{E}_{7(7)}$ denote the universal cover, which is a double cover.

Now we can review the $E_{7(7)}$-action on the fields of 4d $\mathcal{N}=8$ supergravity:

1. The 70 scalar fields can be repackaged into a single field valued in $E_{7(7)}/(\mathrm{SU}_8/\{\pm 1\})$ with its usual $E_{7(7)}$-action.
2. The gauginos transform in the representation **56** above.
3. The vector bosons transform in a 28-dimensional representation of $\mathfrak{e}_{7(7)}$ which we call **28**. Restricted to $\mathfrak{su}_8$, this representation is $\Lambda^2\mathbb{R}^8$.
4. The gravitinos transform in an 8-dimensional representation of $\mathfrak{e}_{7(7)}$, which we denote **8**; restricted to $\mathfrak{su}_8$, this is the defining representation.
5. The graviton transforms in the trivial representation.

See [FM13, §2] for a concise review. The $\mathfrak{e}_{7(7)}$-action exponentiates to an $\widetilde{E}_{7(7)}$-action on the fields.

The presence of fermions (the gauginos and gravitinos) means that we must have data of a spin structure, or something like it, to formulate this theory. In quantum physics, a strong form of G-symmetry is to couple to a G-connection, suggesting that we should formulate 4d $\mathcal{N}=8$ supergravity on spin 4-manifolds M together with an $\widetilde{E}_{7(7)}$-bundle $P \to M$ and a connection on P. The spin of each field tells us which representation of Spin_4 it transforms as, and we just learned how the fields transform under the $\widetilde{E}_{7(7)}$-symmetry, so we can place this theory on manifolds M with a *geometric* $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$*-structure*, i.e. a metric and a principal $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$-bundle $P \to M$ with connection whose induced O_4-connection is the Levi-Civita connection. The fields are sections of associated bundles to P and the representations they transform in. The Lagrangian is invariant under the $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$-symmetry, so defines a functional on the space of fields, and we can study this field theory as usual.

However, we can do better! We will see that the representations above factor through a quotient H_4 of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, so the same procedure above works with H_4 in place of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$. A lift of the structure group to H_4 is less data than a lift all the way to $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, so we expect to be able to define 4d $\mathcal{N} = 8$ supergravity on more manifolds. This is similar to the way that the $\mathrm{SL}_2(\mathbb{Z})$ duality symmetry in type IIB string theory can be placed not just on manifolds with a $\mathrm{Spin}_{10} \times \mathrm{Mp}_2(\mathbb{Z})$-structure [2], but on the larger class of manifolds with a $\mathrm{Spin}_{10} \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$-structure [PS16, §5], or how certain SU_2 gauge theories can be defined on manifolds with a $\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{SU}_2$ structure [WWW19].

Let $-1 \in \mathrm{Spin}_4$ be the nonidentity element of the kernel of $\mathrm{Spin}_4 \to \mathrm{SO}_4$ and let x be the nonidentity element of the kernel of $\widetilde{E}_{7(7)} \to E_{7(7)}$. The key fact allowing us to descend to a quotient is that -1 acts nontrivially on the representations of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$ above, and x acts nontrivially, but on a given representation, they both act by 1 or they both act by -1. Therefore the $\mathbb{Z}/2$ subgroup of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$ generated by $(-1, x)$ acts trivially, and we can form the quotient

$$H_4 \coloneqq \mathrm{Spin}_4 \times_{\{\pm 1\}} \widetilde{E}_{7(7)} = (\mathrm{Spin}_4 \times \widetilde{E}_{7(7)})/\langle(-1, x)\rangle. \tag{2.3}$$

The representations that the fields transform in all descend to representations of H_4, so following the procedure above, we can define 4d $\mathcal{N} = 8$ supergravity on manifolds M with a *geometric* H_4-*structure*, i.e. a metric, an H_4-bundle $P \to M$, and a connection on P whose induced O_4-connection is the Levi-Civita connection.

Remark 2.4. As a check to determine that we have the correct symmetry group, we can compare with other string dualities. The U-duality group contains the S-duality group for type IIB string theory, which comes geometrically from the fact that 4d $\mathcal{N} = 8$ supergravity can be constructed by compactifying type IIB string theory on T^6. Therefore the ways in which the duality groups mix with the spin structure must be compatible. As explained by Pantev-Sharpe [PS16, §5], the $\mathrm{SL}_2(\mathbb{Z})$ duality symmetry of type IIB string theory mixes with the spin structure to form the group $\mathrm{Spin}_{10} \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$.

Therefore the way in which the U-duality group mixes with $\{\pm 1\} \subset \mathrm{Spin}_4$ must also be nontrivial. Extensions of a group G by $\{\pm 1\}$ are classified by $H^2(BG; \{\pm 1\})$. If G is connected, BG is simply connected and the Hurewicz and universal coefficient theorems together provide a natural identification

$$H^2(BG; \{\pm 1\}) \xrightarrow{\cong} \mathrm{Hom}(\pi_2(BG), \{\pm 1\}) = \mathrm{Hom}(\pi_1(G), \{\pm 1\}). \tag{2.5}$$

As $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$, there is only one nontrivial extension of $E_{7(7)}$ by $\{\pm 1\}$, namely the universal cover $\widetilde{E}_{7(7)} \to E_{7(7)}$. That is, by comparing with S-duality, we again obtain the group H_4, providing a useful double-check on our calculation above.

3. Anomalies, invertible field theories, and bordism

3.1. **Generalities on anomalies and invertible field theories.** It is sometimes said that in mathematical physics, if you ask four people what an anomaly is, you will get five answers. The

[2] the actual symmetry group is an extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow 1.$$

goal of this section is to explain our perspective on anomalies, due to Freed-Teleman [FT14], and how to reduce the determination of the anomaly to a question in algebraic topology, an approach due to Freed-Hopkins-Teleman [FHT10] and Freed-Hopkins [FH21b].

Whatever an anomaly is, it signals a mild inconsistency in the definition of a quantum field theory. For example, if a quantum field theory Z is n-dimensional, one ought to be able to evaluate it on a closed n-manifold M, possibly equipped with some geometric structure, to obtain a complex number $Z(M)$, called the *partition function* of M. If Z has an anomaly, $Z(M)$ might only be defined after some additional choices, and in the absence of those choices $Z(M)$ is merely an element of a one-dimensional complex vector space $\alpha(M)$.

The theory Z is local in M, so $\alpha(M)$ should also be local in M. One way to express this locality is to ask that $\alpha(M)$ is the state space of M for some $(n+1)$-dimensional quantum field theory α, called the *anomaly field theory* α of Z. The condition that the state spaces of α are one-dimensional follows from the fact that α is an *invertible field theory* [FM06, Definition 5.7], meaning that there is some other field theory α^{-1} such that $\alpha\otimes\alpha^{-1}$ is isomorphic to the trivial field theory $\mathbf{1}$.[3,4] This approach to anomalies is due to Freed-Teleman [FT14]; see also Freed [Fre14, Fre19].

We can therefore understand the possible anomalies associated to a given n-dimensional quantum field theory Z by classifying the $(n+1)$-dimensional invertible field theories with the same symmetry type as Z. The classification of invertible *topological* field theories is due to Freed-Hopkins-Teleman [FHT10], who lift the question into stable homotopy theory; see Grady-Pavlov [GP21, §5] for a recent generalization to the nontopological setting.

Supergravity with its U-duality symmetry is a unitary quantum field theory, and therefore its anomaly theory satisfies the Wick-rotated analogue of unitarity: *reflection positivity*. Freed-Hopkins [FH21b] classify reflection-positive invertible field theories, again using stable homotopy theory. Let $\mathrm{O}\coloneqq\lim_{n\to\infty}\mathrm{O}_n$ be the infinite orthogonal group.

Theorem 3.1 (Freed-Hopkins [FH21b, Theorem 2.19])**.** *Let $n\ge 3$, H_n be a compact Lie group, and $\rho_n\colon H_n\to\mathrm{O}_n$ be a homomorphism whose image contains SO_n. Then there is canonical data of a topological group H and a continuous homomorphism $\rho\colon H\to\mathrm{O}$ such that the pullback of ρ along $\mathrm{O}_n\hookrightarrow\mathrm{O}$ is ρ_n.*

In other words, when the hypotheses of this theorem hold, we have more than just H_n-structures on n-manifolds; we can define H-structures on manifolds of any dimension, by asking for a lift of the classifying map of the stable tangent bundle $M\to B\mathrm{O}$ to BH; a manifold equipped with such a lift is called an *H-manifold*. Following Lashof [Las63], this allows us to define bordism groups Ω_k^H and a homotopy-theoretic object called the *Thom spectrum MTH*, whose homotopy groups are the H-bordism groups.

Theorem 3.2 (Freed-Hopkins [FH21b])**.** *With H_n as in Theorem 3.1, the abelian group of deformation classes of n-dimensional reflection-positive invertible topological field theories on H_n-manifolds is naturally isomorphic to the torsion subgroup of $[MTH,\Sigma^{n+1}I_{\mathbb{Z}}]$.*

Freed-Hopkins then conjecture (*ibid.*, Conjecture 8.37) that the whole group $[MTH,\Sigma^{n+1}I_{\mathbb{Z}}]$ classifies all reflection-positive invertible field theories, topological or not.

[3]The relationship between invertibility and one-dimensional state spaces is that $\alpha\otimes\alpha^{-1}\simeq\mathbf{1}$ means that on any closed, n-manifold M, there is an isomorphism of complex vector spaces $\alpha(M)\otimes\alpha^{-1}(M)\cong\mathbf{1}(M)=\mathbb{C}$. This forces $\alpha(M)$ and $\alpha^{-1}(M)$ to be one-dimensional. Often the converse is also true: see Schommer-Pries [SP18].

[4]In some cases, we do not want to assume α extends to closed n-manifolds; see Freed-Teleman [FT14] for more information. But the U-duality anomaly we investigate in this paper does extend.

The notation $[MTH, \Sigma^{n+1} I_{\mathbb{Z}}]$ means the abelian group of homotopy classes of maps between MTH and an object $\Sigma^{n+1} I_{\mathbb{Z}}$ belonging to stable homotopy theory; see [FH21b, §6.1] for a brief introduction in a mathematical physics context. We mentioned MTH above; $I_{\mathbb{Z}}$ is the *Anderson dual of the sphere spectrum* [And69, Yos75], characterized up to homotopy equivalence by its universal property, which says that there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{n-1}(E), \mathbb{Z}) \longrightarrow [E, \Sigma^n I_{\mathbb{Z}}] \longrightarrow \mathrm{Hom}(\pi_n(E), \mathbb{Z}) \longrightarrow 0. \tag{3.3}$$

Applying this when $E = MT\xi$, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\Omega_{n+1}^H, \mathbb{Z}) \xrightarrow{\varphi} [MTH, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\Omega_{n+2}^H, \mathbb{Z}) \longrightarrow 0 \tag{3.4}$$

decomposing the group of possible anomalies of unitary QFTs on H_n-manifolds. These two factors admit interpretations in terms of anomalies.

(1) The quotient $\mathrm{Hom}(\Omega_{n+2}^H, \mathbb{Z})$ is a free abelian group of degree-$(n+2)$ characteristic classes of H-manifolds. The map ψ sends an anomaly field theory to its anomaly polynomial. This is the part of the anomaly visible to perturbative methods, and sometimes is called the *local anomaly.*
(2) The subgroup $\mathrm{Ext}(\Omega_{n+1}^H, \mathbb{Z})$ is isomorphic to the abelian group of torsion bordism invariants $f\colon \Omega_{n+1}^H \to \mathbb{C}^\times$. These classify the reflection-positive invertible *topological* field theories α_f: the correspondence is that the bordism invariant f is the partition function of α_f. This part of an anomaly field theory is generally invisible to perturbative methods and is called the *global anomaly.*

Work of Yamashita-Yonekura [YY21] and Yamashita [Yam21] relates this short exact sequence to a differential generalized cohomology theory extending $\mathrm{Map}(MTH, \Sigma^{n+1} I_{\mathbb{Z}})$.

3.2. **Specializing to the U-duality symmetry type.** For us, $n = 4$ and the symmetry type is $H_4 = \mathrm{Spin} \times_{\{\pm 1\}} \widetilde{E}_{7(7)}$. This group is not compact, so Theorems 3.1 and 3.2 above do not apply. However, we can work around this obstacle: Marcus [Mar85] proved that the anomaly polynomial of the $E_{7(7)}$ symmetry vanishes,[5] meaning that the anomaly field theory is a *topological* field theory. Thinking of topological field theories as symmetric monoidal functors $\mathcal{B}ord_n^{H_n} \to \mathcal{C}$, we can freely adjust the structure we put on manifolds in these theories as long as the induced map on bordism categories is an equivalence. We make two adjustments.

(1) First, forget the metric and connection in the definition of a geometric H_4-structure. The space of such data is contractible and therefore can be ignored for topological field theories.
(2) We can then replace H_4 with its maximal compact subgroup: for any Lie group G with $\pi_0(G)$ finite, inclusion of the maximal compact subgroup $K \hookrightarrow G$ is a homotopy equivalence [Mal45, Iwa49] and defines a natural equivalence of groupoids $\mathcal{B}un_K(X) \xrightarrow{\simeq} \mathcal{B}un_G(X)$ on spaces X, hence a symmetric monoidal equivalence of bordism categories of manifolds with these kinds of bundles.

Spin_4 is compact, and the maximal compact of $\widetilde{E}_{7(7)}$ is SU_8, so the maximal compact of H_4 is the group $\mathrm{Spin}_4 \times_{\{\pm 1\}} \mathrm{SU}_8$. Now Theorems 3.1 and 3.2 apply: the stabilization of $\mathrm{Spin}_4 \times_{\{\pm 1\}} \mathrm{SU}_8$ is

[5] Marcus' analysis does not discuss the question of H_4 versus $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, but this does not matter: in many cases including the one we study, the anomaly polynomial for a d-dimensional field theory on G-manifolds is an element of $H^{d+2}(BG; \mathbb{Q})$, and rational cohomology is insensitive to finite covers such as $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)} \to H_4$. Thus Marcus' computation applies in our case too.

$\mathrm{Spin}\text{-}\mathrm{SU}_8 \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_8$, and the anomaly field theory is classified by the torsion subgroup of $[MT(\mathrm{Spin}\text{-}\mathrm{SU}_8), \Sigma^6 I_{\mathbb{Z}}]$, which is determined by $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

In Theorem 4.21, we prove $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \cong \mathbb{Z}/2$, so there is potential for the anomaly field theory to be nontrivial.

Concretely, a manifold with a spin-SU_8 structure is an oriented manifold M with a principal $\mathrm{SU}_8/\{\pm 1\}$-bundle $P \to M$ and a trivialization of $w_2(M) + a(P)$, where a is the unique nonzero element of $H^2(B(\mathrm{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$.

Remark 3.5. Computing bordism groups to determine whether an anomaly is trivial is a well-established technique in the mathematical physics literature: other papers taking this approach include [Wit86, Kil88, Mon15, Mon17, GPW18, Hsi18, STY18, DGL20, GEM19b, MM19, TY19, WW19b, WW19a, WWZ20, WY19, BLT20, DL21b, DL20, DL21a, GOP+20, HKT20, HTY20, JF20, KPMT20, Tho20, WW20b, WW20a, WW20c, FH21a, FH21b, DDHM21, DGG21, Koi21, LOT21b, LOT21a, LT21a, TY21, Yu21, WNC21, DGL22, Deb22, LY22, Tac22, Yon22].

4. Spectral sequence computation

The E_2 page for U-duality in the Adams spectral sequence is [Ada58, Theorem 2.1, 2.2]

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow \pi_{s-t}(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8))_2^\wedge \cong (\Omega_{s-t}^{\mathrm{Spin}\text{-}\mathrm{SU}_8})_2^\wedge\,,$$

which converges to the 2-completion of the desired bordism group via the Pontrjagin-Thom construction. It is important to remark that the standard techniques of phrasing the bordism question as one involving spin bordism of a Thom spectrum over $B(\mathrm{SU}_8/\{\pm 1\})$ does not work in this case. In order to do so, one would need to find a real, finite-dimensional representation of $\mathrm{SU}_8/\{\pm 1\}$ which is oriented, but not spin. By this we mean, a representation $\rho : \mathrm{SU}_8/\{\pm 1\} \to \mathrm{SO}_n$ that does not lift to Spin_n. If such a representation exists then a Spin-SU_8 structure is naturally equivalent to $(B(\mathrm{SU}_8/\{\pm\}), \rho)$-twisted spin structure. However, one can show that all representations of $\mathrm{SU}_8/\{\pm 1\}$ are spin. Without being able to use the change of rings theorem and work over $\mathcal{A}(1)$, had spin bordism been available, we thus need to work over $\mathcal{A}$, the entire mod-2 Steenrod algebra. It would be interesting to find more problems where similar complications occur when trying to work with twisted spin bordism.

In order to set up the Adams computation, a necessary step is to establish the two theorem in §4.2 with the goal to give the Steenrod actions on $H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2)$. Applying the Thom isomorphism takes care of the rest. We also detail the simplifications that make working over the entire Steenrod algebra accessible. We refer the reader to [BC18] which highlights many of the computational details of the Adams spectral sequence, but mainly employs a change of rings to work over $\mathcal{A}(1)$. We start by showing that computing the 2-completion is sufficient for the tangential structure we are considering.

4.1. Nothing interesting at odd primes.

We will show that the Adams spectral sequence computation that we run which only gives the two torsion part of the anomaly is sufficient for our purposes.

Proposition 4.1. *$\Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ has no p-torsion when p is an odd prime.*

Proof. The quotient $\mathrm{Spin} \times \mathrm{SU}_8 \to \mathrm{Spin}\text{-}\mathrm{SU}_8$ is a double cover, hence on classifying spaces is a fiber bundle with fiber $B\mathbb{Z}/2$. $H^*(B\mathbb{Z}/2; \mathbb{Z}/p) = \mathbb{Z}/p$ concentrated in degree 0, so $B(\mathrm{Spin} \times \mathrm{SU}_8) \to B(\mathrm{Spin}\text{-}\mathrm{SU}_8)$ is an isomorphism on $\mathbb{Z}/p$ cohomology (e.g. look at the Serre spectral sequence for this fiber bundle). The Thom isomorphism lifts this to an isomorphism of cohomology of the relevant Thom spectra, and then the stable Whitehead theorem implies that the forgetful map $\Omega_*^{\mathrm{Spin}}(B\mathrm{SU}_8) \to \Omega_*^{\text{Spin-SU}_8}$ is an isomorphism on p-torsion.

The same argument applies to the double cover $\mathrm{Spin} \times \mathrm{SU}_8 \to \mathrm{SO} \times \mathrm{SU}_8$, so the p-torsion in $\Omega_*^{\text{Spin-SU}_8}$ is the same as the p-torsion in $\Omega_*^{\mathrm{SO}}(B\mathrm{SU}_8)$. Now apply the Atiyah-Hirzebruch spectral sequence. Averbuh [Ave59] and Milnor [Mil60, Theorem 5] prove there is no p-torsion in Ω_*^{SO}, and Borel [Bor51, Proposition 29.2] shows there is no p-torsion in $H_*(B\mathrm{SU}_8;\mathbb{Z})$ and $H_*(B\mathrm{SU}_8;\mathbb{Z}/2)$. Therefore the only way to obtain p-torsion in $\Omega_*^{\mathrm{SO}}(B\mathrm{SU}_8)$ would be from a differential between free summands, but all free summands in Ω_*^{SO} and $H_*(B\mathrm{SU}_8;\mathbb{Z})$ are contained in even degrees, so there are no differentials between free summands, and therefore no p-torsion. □

4.2. **Computing the cohomology of $B(\text{Spin-SU}_8)$.** We first start with computing $H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}/2)$ and finding the low degree classes, as well as Steenrod actions. Then by borrowing these results and further applying a Serre spectral sequence, we are able to run the Adams spectral sequence in §4.3. For more on the Serre spectral sequence and its application to physical problems see [GEM19a, Yu21, LY22, LT21b, DL21b, DGL22].

Theorem 4.2. *$H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b,c,d,e,\dots]$ with $|a| = 2$, $|b| = 3$, $|c| = 4$, $|d| = 5$, and $|e| = 6$, and there are no other generators or relations below degree 7. The Steenrod squares are*

$$\begin{aligned}
\mathrm{Sq}(a) &= a + b + a^2\\
\mathrm{Sq}(b) &= d + b^2\\
\mathrm{Sq}(c) &= e + \mathrm{Sq}^3(c) + c^2\\
\mathrm{Sq}(d) &= d + b^2 + \mathrm{Sq}^3(d) + \mathrm{Sq}^4(d) + d^2.
\end{aligned} \tag{4.3}$$

Proof. We first give the cohomology of $B(\mathrm{SU}_8/\{\pm 1\})$ by using the Serre spectral sequence for the fibration $\mathrm{SU}_8/\{\pm 1\} \to \mathrm{pt} \to B\mathrm{SU}_8/\{\pm 1\}$. The cohomology $H^*(\mathrm{SU}_8/\{\pm 1\};\mathbb{Z}/2)$ is given in [BB65, Theorem 7.2] which we reproduce here:

$$\mathbb{Z}/2[z_1]/z_1^8 \otimes \bigwedge[z_2,\dots,z_7]\,, \quad \dim z_i = 2i-1\,. \tag{4.4}$$

The E_2 page of

$$E_2^{p,q} = H^p(B(\mathrm{SU}_8/\{\pm 1\}); H^q(\mathrm{SU}_8/\{\pm 1\};\mathbb{Z})) \Longrightarrow H^{p+q}(\mathrm{pt};\mathbb{Z}) \tag{4.5}$$

for q degree up to 8 and p degree up to 6 is given by:

$$\begin{array}{c|cccccc}
8 & z_1^8, z_1^5 z_2, z_1^4 z_2, z_1^5 z_3, z_1 z_4, z_2 z_3 & 0 & & & & \\
7 & z_1^7, z_1^4 z_2, z_1^2 z_3, z_4 & 0 & & & & \\
6 & z_1^6, z_1^3 z_2, z_1 z_3 & 0 & & & & \\
5 & z_1^5, z_1^2 z_2, z_3 & 0 & & & & \\
4 & z_1^4, z_1 z_2 & 0 & & & & \\
3 & z_1^3, z_2 & 0 & & & & \\
2 & y = z_1^2 & 0 & & & & \\
1 & z_1 & 0 & & & & \\
0 & 1 & 0 & a & b & (a^2, c) & (ab, d) & (a^3, b^2, e) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6\,.
\end{array} \tag{4.6}$$

Since we are converging to a point, there must be a d_2 differential from z_1 to a, and a d_3 differential will be from $y = z_1^2$ to b. Repeating the transgression, we see that d_4 maps z_2 to c, d_5 maps z_1^4 to d, and d_6 maps z_3 to e. With the generators in low degree at our disposal, we now give the Steenrod action on these generators. For this we consider the fibration $B\mathrm{SU}_8 \to B(\mathrm{SU}_8/\{\pm 1\}) \to B^2\mathbb{Z}/2$. Using the fact that $H^*(B^2\mathbb{Z}/2;\mathbb{Z}/2) = \mathbb{Z}/2\{T,\, y = \mathrm{Sq}^1 T,\, z = \mathrm{Sq}^2\mathrm{Sq}^1 T,\, \ldots\}$, we have

$$E_2^{p,q} = H^p(B^2\mathbb{Z}/2; H^q(B\mathrm{SU}_8;\mathbb{Z}/2)) \Longrightarrow H^{p+q}(B(\mathrm{SU}_8/\{\pm 1\};\mathbb{Z}/2) \tag{4.7}$$

in q degree below 11 given by:

$$\begin{array}{c|ccccccccc}
10 & c_2c_3 \\
9 & 0 \\
8 & c_2^2, c_4 \\
7 & 0 & 0 & 0 & 0 \\
6 & c_3 & 0 & c_3T & c_3y \\
5 & 0 & 0 & 0 & 0 \\
4 & c_2 & 0 & c_2T & c_2y & c_2T^2 & (c_2z, c_2Ty) \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & T & y & T^2 & (z, Ty) & (T^3, y^2) & (T^2y, Tz) & (T^4, Ty^2, yz) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\,,
\end{array} \tag{4.8}$$

where c_i are the mod 2 reduction of Chern classes for the cohomology of $B\mathrm{SU}_8$. We immediately see that the classes a and b pull back to T and $\mathrm{Sq}^1 T$ respectively, since there are no differential that hit these two generators. Furthermore c pulls back to c_2 while d pulls back to z. Thus, we find that:

$$\begin{aligned}
\mathrm{Sq}^1 a &= b, & \mathrm{Sq}^2 a &= a^2, \\
\mathrm{Sq}^1 b &= 0, & \mathrm{Sq}^2 b &= d, \\
\mathrm{Sq}^1 d &= b^2, & \mathrm{Sq}^2 d &= 0\,.
\end{aligned} \tag{4.9}$$

Lastly, we need to determine the action of the Steenrod operators on c and e. We now present a Lemma, whose corollary completes the proof.

Lemma 4.10. *The classes c and e are in the image of the mod 2 reduction map*

$$r\colon H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}) \to H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}/2).$$

Corollary 4.11. $\mathrm{Sq}^1(c) = 0$ *and* $\mathrm{Sq}^1(e) = 0$.

Proof. Sq^1 is the Bockstein for the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. Therefore if x is in the image of $r_4\colon H^*(-;\mathbb{Z}/4) \to H^*(-;\mathbb{Z}/2)$, then $\mathrm{Sq}^1(x) = 0$. And the mod 2 reduction map $\mathbb{Z} \to \mathbb{Z}/2$ factors through $\mathbb{Z}/4$. ◻

Proof of Lemma 4.10. The map r induces a map of Serre spectral sequences for the fibration $B\mathbb{Z}/2 \to B\mathrm{SU}_8 \to B(\mathrm{SU}_8/\{\pm 1\})$. Let's run the Serre spectral sequence with $\mathbb{Z}$ coefficients. It has signature

$$E_2^{*,*} = H^*(B(\mathrm{SU}_8/\{\pm 1\}); H^*(B\mathbb{Z}/2;\mathbb{Z})) \Longrightarrow H^*(B\mathrm{SU}_8;\mathbb{Z}). \tag{4.12}$$

Since $B(\mathrm{SU}_8/\{\pm 1\})$ is simply connected, we do not need to worry about local coefficients. We know that $H^*(B\mathbb{Z}/2;\mathbb{Z}) \cong \mathbb{Z}[z]/2z$, where $|z| = 2$, and $H^*(B\mathrm{SU}_8;\mathbb{Z}) \cong \mathbb{Z}[c_2, \ldots, c_8]$, with $|c_i| = 2i$, so we may run the spectral sequence in reverse. The E_2 page for (4.12) is:

$$\begin{array}{c|ccccccc} 6 & z^3 & 0 & 0 & \alpha z^3 & c_2 z^3 & \beta z^3 & (c_3 z^3, \alpha z^3) \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & z^2 & 0 & 0 & \alpha z^2 & c_2 z^2 & \beta z^2 & (c_3 z^2, \alpha^2 z^2) \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & z & 0 & 0 & \alpha z & c_2 z & \beta z & (c_3 z, \alpha^2 z) \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \alpha & c_2 & \beta & (c_3, \alpha^2) \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6\,. \end{array} \tag{4.13}$$

As $H^2(B\mathrm{SU}_8;\mathbb{Z}) = 0$, $z \in E_2^{0,2} = H^2(B\mathbb{Z}/2;\mathbb{Z})$ admits a differential. The only option is a transgressing d_3; let $\alpha := d_3(z)$. Since $2z = 0$, $2\alpha = 0$. The Leibniz rule (now with signs) tells us

$$d_2(z^2) = z d_2(z) + d_2(z) z = 2\alpha z = 0. \tag{4.14}$$

Therefore if z^2 admits a differential, the differential must be the transgressing $d_5\colon E_4^{0,4} \to E_4^{5,0}$, see (4.13). But z^2 does admit a differential. One way to see this is to compute the pullback $H^4(B\mathrm{SU}_8;\mathbb{Z}) \to H^4(B\mathbb{Z}/2;\mathbb{Z})$. Since $H^4(B\mathrm{SU}_8;\mathbb{Z})$ is generated by c_2 of the defining representation $\mathbb{C}^8$, we can restrict that representation to $\mathbb{Z}/2$ and compute its second Chern class to compute the pullback map. As a representation of $\mathbb{Z}/2$, $\mathbb{C}^8$ is a direct sum of 8 copies of the sign representation, so its total Chern class is $c(8\sigma) = (1+z)^8$ by the Whitney sum rule, and the z^2 term is $\binom{8}{2} z^2$, which is even. Since $2z^2 = 0$, this implies c_2 pulls back to 0. If z^2 did not support a differential, then it would be in the image of this pullback map, so we have discovered that z^2 admits a differential, specifically d_5. Let $\beta := d_5(z^2)$. From the spectral sequence we see that $H^4(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z})$ is isomorphic to $H^4(B\mathrm{SU}_8;\mathbb{Z})$, for which c_2 is an element of. By using the mod 2 reduction map from $H^4(B\mathrm{SU}_8;\mathbb{Z}) \to H^4(B\mathrm{SU}_8;\mathbb{Z}/2)$, and the pullback map induced from $\mathrm{SU}_8 \xrightarrow{f} \mathrm{SU}_8/\{\pm 1\}$ we see that c is the mod 2 reduction of c_2 in $H^4(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z})$. This is summarized in the following diagram:

$$\begin{tikzcd} c_2 \in H^4(B\mathrm{SU}_8/\{\pm 1\};\mathbb{Z}) \arrow[r, "\text{mod } 2"] \arrow[d, equal] & H^4(B\mathrm{SU}_8/\{\pm 1\};\mathbb{Z}/2) \arrow[d, "f^*"] \\ H^4(B\mathrm{SU}_8;\mathbb{Z}) \arrow[r] & H^4(B\mathrm{SU}_8;\mathbb{Z}/2)\,. \end{tikzcd}$$

□

With the fact that c is the mod 2 reduction of c_2, we still consider defining $e = \mathrm{Sq}^2(c)$, rather than the mod 2 reduction of c_3. This choice of definition presents ambiguities in the action of the Steenrod squares on e, and the relations in the cohomology ring, but the ambiguities are in too high of a degree to affect the computation at hand. □

Remark 4.15. Toda [Tod87] uses another approach to compute $H^*(BG;\mathbb{Z}/2)$ when G is compact, simple, and not simply connected: the Eilenberg-Moore spectral sequence

$$E_2^{p,q} = \mathrm{coTor}^{p,q}_{H^*(B\pi_1(G);\mathbb{Z}/2)}(H^*(B\widetilde{G};\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow H^{p+q}(BG;\mathbb{Z}/2), \tag{4.16}$$

where $\widetilde{G} \to G$ is the universal cover, the coalgebra structure on $H^*(B\pi_1(G);\mathbb{Z}/2)$ comes from multiplication on $\pi_1(G)$, and the comodule structure on $H^*(B\widetilde{G};\mathbb{Z}/2)$ comes from the inclusion $\pi_1(G) \hookrightarrow \widetilde{G}$ and multiplication in $\widetilde{G}$. If you apply this to $G = \mathrm{SU}_8/\{\pm 1\}$, however, the E_2-page of the Eilenberg-Moore spectral sequence is identical to the E_2-page of the Serre spectral sequence (4.7) in the range relevant to us.

We now compute the cohomology of the tangential stucture that we actually need for U-duality via a Serre spectral sequence using the fibration

$$B\mathbb{Z}/2 \longrightarrow B(\text{Spin-SU}_8) \longrightarrow B(\text{SO} \times \text{SU}_8/\{\pm\})\,, \tag{4.17}$$

and knowledge of the cohomology of $B(\text{SU}_8/\{\pm\})$.

Theorem 4.18. *$H^*(B(\text{Spin-SU}_8);\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b,c,w_4,d,e,\ldots]$ with $|a|=2$, $|b|=3$, $|c|=4$, $|w_4|=4$,$|d|=5$, and $|e|=6$. The map $\text{Spin-SU}_8 \to \text{SO}\times\text{SU}_8/\{\pm\}$ induces a quotient map on cohomology, and the Steenrod squares are giving in (4.3) along with*

$$\begin{aligned}\text{Sq}^1 w_4 &= ab + d\,,\\ \text{Sq}^2 w_4 &= aw_4 + \ldots\,.\end{aligned} \tag{4.19}$$

Proof. We run the spectral sequence with signature

$$E_2^{*,*} = H^*(B(\text{SO}\times\text{SU}_8/\{\pm\}); H^*(B\mathbb{Z}/2;\mathbb{Z}/2)) \Rightarrow H^*(B(\text{Spin-SU}_8);\mathbb{Z}/2)\,,$$

where the E_2 page is given by:

$$\begin{array}{c|cccccc}
5 & t^5 & 0 & & & & \\
4 & t^4 & 0 & & & & \\
3 & t^3 & 0 & & & & \\
2 & t^2 & 0 & (t^2a, t^2a+t^2w_2) & \ldots & & \\
1 & t & 0 & (ta, ta+tw_2) & (tb, tb+tw_3) & \ldots & \\
0 & 1 & 0 & (a, a+w_2) & (b, b+w_3) & \begin{pmatrix} a^2, c, w_4, a^2+w_2^2, \\ a(a+w_2)\end{pmatrix} & \begin{pmatrix} ab+d+w_5, (a+w_2)(b+w_3) \\ a(b+w_3), b(a+w_2)\end{pmatrix} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5\,.
\end{array} \tag{4.20}$$

The w_i are the Stiefel-Whitney classes of $B\text{SO}$, and t is the generator of the cohomology $H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$. The differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ hits the class for the extension that gives Spin-SU$_8$, which is $a+w_2$, and identifies $a = w_2$. Applying the Leibniz gives the differential on t^{2n+1}. We then use Kudo's transgression theorem [Kud56], which says that Steenrod squares commute with transgression in the Serre spectral sequence. There must therefore be a $d^3 : E_3^{0,2} \to E_3^{3,0}$ which hits $b+w_3$, since $\text{Sq}^1 t = t^2$ kills $\text{Sq}^1(a+w_1)$. In total degree 4, there is a d_4 differential that takes t^4 to $ab+d+w_5$, i.e. this differential takes $\text{Sq}^2 t^2$ to $\text{Sq}^2(b+w_3)$ [6]. We see that there is a new class w_4 which pulled back from BSO. Applying the Wu-formula then establishes (4.19). □

4.3. **The Adams Computation.**

In this section we give the details of the $\mathcal{A}$ modules and the E_∞ page of the spectral sequence computation in order to show that

[6]We slightly change the basis for the degree 5 generators here so that the d_4 differential identifies w_5 with $ab+d$ and therefore $\text{Sq}^1(w_4U) = w_5U = (ab+d)U$ agrees with $\text{Sq}^2(bU)$. This is necessary in order to have a valid $\mathcal{A}$ module. We point to [Ada21] as a reference for the fact that M_n does not lift from an $\mathcal{A}(1)$ module to an $\mathcal{A}$ module for any finite n. This means in the degree we are considering, there must be a node in degree 4 that is joined with $(ab+d)U$ upon acting by Sq^1.

Theorem 4.21. *Up to degree 5, the first few groups of* Spin-SU_8 *bordism are*

$$\begin{aligned}\Omega_0^{\text{Spin-SU}_8} &\cong \mathbb{Z}\\ \Omega_1^{\text{Spin-SU}_8} &\cong 0\\ \Omega_2^{\text{Spin-SU}_8} &\cong 0\\ \Omega_3^{\text{Spin-SU}_8} &\cong 0\\ \Omega_4^{\text{Spin-SU}_8} &\cong \mathbb{Z}^2\\ \Omega_5^{\text{Spin-SU}_8} &\cong \mathbb{Z}/2.\end{aligned} \tag{4.22}$$

Treating $d \in H^5(B(\mathrm{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$ as a characteristic class, the bordism invariant $(M, P) \mapsto \int_M d(P) \in \mathbb{Z}/2$ realizes the isomorphism $\Omega_5^{\text{Spin-SU}_8} \to \mathbb{Z}/2$.

Proof. The first simplification to working with the entire Steenrod alebra is that the only higher Steenrod operator beyond Sq^2 in $\mathcal{A}$ that we must incorporate for the purpose of working up to degree 5 is Sq^4. As input, we need the $\mathcal{A}$-module structure on $H^*(MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_8); \mathbb{Z}/2)$, which by the Thom isomorphism is given by $\mathbb{Z}/2[a, b, c, w_4, d, e, \dots]\{U\}$, where $U : H^*(B\mathrm{SO}; \mathbb{Z}/2) \to H^*(MT\mathrm{SO}; \mathbb{Z}/2)$ is the Thom class coming from the tautological bundle over $B\mathrm{SO}$. For any cohomology class x coming from $B\mathrm{SO}$, we can get the Steenrod squares of Ux from the $\mathcal{A}$-module structure on $MT\mathrm{SO}$. We have also previously determined the action of Steenrod squares on elements of the cohomology of $B\mathrm{SU}_8/\{\pm\}$, and therefore we know the Steenrod action on all elements in $H^*(MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_8); \mathbb{Z}/2)$. We thus have [BC18, Remark 3.3.5]

$$\mathrm{Sq}^k(Ux) = \sum_{i=0}^{k} \mathrm{Sq}^i(U)\mathrm{Sq}^{k-i}(x) = \sum_{i=0}^{k} Uw_i\, \mathrm{Sq}^{k-i}(x), \tag{4.23}$$

where $w_1 = 0$ when pulled back from $MT\mathrm{SO}$ and $w_2 = a, w_3 = b, w_5 = ab + d$ by the proof of Theorem 4.18. After localizing at $p = 2, MT\mathrm{SO}$ is a direct sum of Eilenberg-MacLane spectra, which in low degree is

$$H^*(MT\mathrm{SO}; \mathbb{Z}/2) \cong H^*(H\mathbb{Z}) \oplus \Sigma^4 H^*(H\mathbb{Z}) \oplus \Sigma^5 H^*(H\mathbb{Z}/2) \oplus \dots. \tag{4.24}$$

Under the quotient map in cohomology

$$H^*(MT\mathrm{SO} \wedge B(\mathrm{SU}_8/\{\pm\}); \mathbb{Z}/2) \to H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2); \mathbb{Z}/2)$$

the three elements in (4.24) survive, and in addition we pick up a new $\mathcal{A}$ module coming from Uc which is purely associated to $B(\mathrm{SU}_8/\{\pm\})$. In degree six and below, this module is denoted by $C\eta$. We let $\Sigma^k C\eta$ denote the shift of $C\eta$ in which the grading of every element is increased by k. Then, there is an isomorphism of $\mathcal{A}$-modules

$$H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2 \oplus \Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2) \oplus \Sigma^4 C\eta \oplus \Sigma^5 \mathcal{A} \oplus P, \tag{4.25}$$

where P contains no nonzero elements in degrees 5 and below, and we used the fact that $H^*(H\mathbb{Z}) = \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ and $H^*(H\mathbb{Z}/2) = \mathcal{A}$. The red summand is generated by U, and is worked out in Figure 1 by using (4.23). The green summand is generated by Ua^2, and the purple summand is generated by Ud. To compute the E_2-page of the Adams spectral sequence we need to know Ext of each summand in (4.25) ((Ext(–) means $\mathrm{Ext}_{\mathcal{A}}^{*,*}(-; \mathbb{Z}/2)$)) By using the change of rings [BC18, Section 4.5], we get $\mathrm{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2)$, and since $\mathcal{A}(0)$ only includes Sq^1,

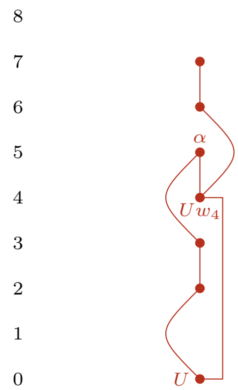

FIGURE 1. The only relevant higher Steenrod operation in this degree is Sq^4, which acts on U to give Uw_4. This is connected to $\alpha = (ab+d)U$ by Sq^1.

this just gives $\mathbb{Z}/2[h_0]$, where $h_0 \in \mathrm{Ext}^{1,1}$. The same logic applies for the Ext of the green summand, and the Ext of the purple summand contributes a $\mathbb{Z}/2$ in degree 5.

We need to compute $\mathrm{Ext}_{\mathcal{A}}(C\eta)$, at least in low degrees. We can do this with a standard technique: $C\eta$ is part of a short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \Sigma^2\mathbb{Z}/2 \longrightarrow C\eta \longrightarrow \mathbb{Z}/2 \longrightarrow 0, \tag{4.26}$$

and a short exact sequence of $\mathcal{A}$-modules induces a long exact sequence of Ext groups. It is conventional to draw this as if on the E_1-page of an Adams-graded spectral sequence [TODO: see [BC18] for more. . .]. We draw the short exact sequence (4.26) in Figure 2, left, and we draw the induced long exact sequence in Ext in Figure 2, right. Looking at this long exact sequence, there are three boundary maps that could be nonzero in the range displayed; because boundary maps commute with the $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2)$-action, these boundary maps are all determined by

$$\partial\colon \mathrm{Ext}^{0,2}_{\mathcal{A}}(\Sigma^2\mathbb{Z}/2) \to \mathrm{Ext}^{1,2}_{\mathcal{A}}(\mathbb{Z}/2). \tag{4.27}$$

This boundary map is either 0 or an isomorphism, and it must be an isomorphism, because

$$\mathrm{Ext}^{0,2}_{\mathcal{A}}(C\eta) = \mathrm{Hom}_{\mathcal{A}}(C\eta, \Sigma^2\mathbb{Z}/2) = 0, \tag{4.28}$$

and if the boundary map vanished, we would obtain $\mathbb{Z}/2$ for this Ext group. Thus we know $\mathrm{Ext}_{\mathcal{A}}(C\eta)$ in the range we need.

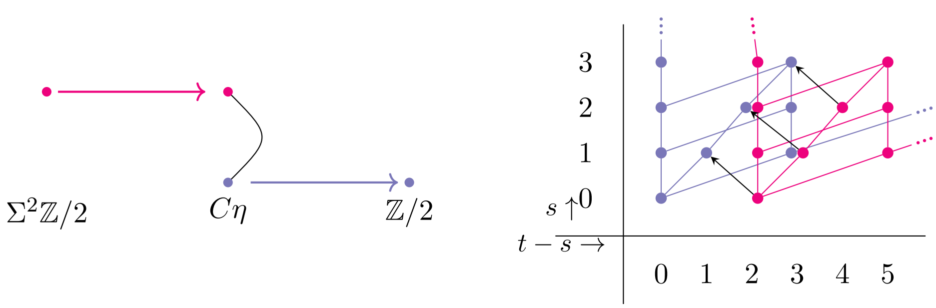

FIGURE 2. Left: the short exact sequence (4.26). Right: the induced long exact sequence in Ext groups.

Compiling the information of Ext on (4.25) we draw the E_2-page of the Adams spectral sequence through topological degree 5 in Figure 3 [7].

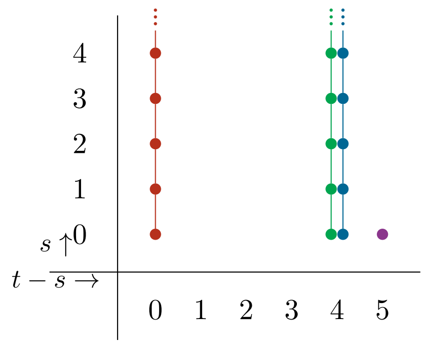

FIGURE 3. The E_2-page of the Adams spectral sequence computing $\Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

In this range, the only differentials that could be nonzero go from the 5-line to the 4-line. Usually we would need to know the 6-line in order to determine if there are any differentials from the 6-line to the 5-line, so that we could evaluate $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$, but the 5-line is concentrated in filtration zero, and all Adams differentials land in filtration 2 or higher, so what we have computed is good enough.

Returning to the differentials from the 5-line to the 4-line: Adams differentials must commute with the action of h_0 on the E_r-page, and h_0 acts by 0 on the 5-line but injectively on the 4-line, so these differentials must also vanish. Thus the spectral sequence collapses giving the bordism groups in the theorem statement. The fact that $\Omega_5^{\mathrm{Spin}\times_{\{\pm\}}\mathrm{SU}_8} \cong \mathbb{Z}/2$ is detected by $\int d$ follows from the fact that its image in the E_∞-page is in Adams filtration zero, corresponding to Ext of the free $\Sigma^5\mathcal{A}$ summand generated by Ud; see [FH21a, §8.4]. [8]

□

4.4. **Determining the Manifold Generator.** We now determine the generator of $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \cong \mathbb{Z}/2$. We start by considering a map $\widetilde{\Phi}\colon \mathrm{SU}_2 \to \mathrm{SU}_8$ sending a matrix A to its fourfold block sum $A\oplus A\oplus A\oplus A$. This sends $-1\mapsto -1$, so $\widetilde{\Phi}$ descends to a map

$$\Phi\colon \mathrm{SO}_3 = \mathrm{SU}_2/\{\pm 1\} \longrightarrow \mathrm{SU}_8/\{\pm 1\}. \tag{4.29}$$

Recall that $H^*(B\mathrm{SO}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]$ and that there are three classes a, b, and d in $H^*(B(\mathrm{SU}_8/\{\pm 1\});\mathbb{Z}/2)$.

Lemma 4.30. *$\Phi^*(a) = w_2$, $\Phi^*(b) = w_3$, and $\Phi^*(d) = w_2w_3$.*

This will imply that to find a generator, all we have to do is find a closed, oriented 5-manifold M with a principal SO_3-bundle $P\to M$ with $w_2(M) = w_2(P)$ and $w_2(P)w_3(P)\neq 0$. This is easier than directly working with $\mathrm{SU}_8/\{\pm 1\}$!

Proof of Lemma 4.30. Once we show $\Phi^*(a) = w_2$, we're done:

$$\Phi^*(b) = \Phi^*(\mathrm{Sq}^1(a)) = \mathrm{Sq}^1(\Phi^*(a)) = \mathrm{Sq}^1(w_2) = w_3, \tag{4.31a}$$

[7] On the right side of the figure, the modules in red, blue, and purple are pulled back from $MT\mathrm{SO}$.

[8] While we do not draw the $\mathcal{A}(1)$ modules up to degree 6, there is a way to access information in this degree. We know that if we replace the spin bordism of $B\mathrm{SU}_8/\{\pm\}$ with the oriented bordism of $B\mathrm{SU}_8$, then the Atiyah-Hirzebruch spectral sequence for oriented bordism tensored with $\mathbb{Q}$ tells us in degree 6, there should be one $\mathbb{Q}$ summand that is detected by c_3 of the SU_8-bundle.

where the last equal sign follows by the Wu formula. In a similar way

$$\Phi^*(d) = \Phi^*(\mathrm{Sq}^2(b)) = \mathrm{Sq}^2(\Phi^*(b)) = \mathrm{Sq}^2(w_3) = w_2w_3, \tag{4.31b}$$

again using the Wu formula. So all we have to do is pull back a.

Consider the commutative diagram of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathrm{SU}_2 \arrow[r] \arrow[d, "\widetilde{\Phi}"] & \mathrm{SO}_3 \arrow[r] \arrow[d, "\Phi"] & 1 \\ 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathrm{SU}_8 \arrow[r] & \mathrm{SU}_8/\{\pm 1\} \arrow[r] & 1. \end{tikzcd} \tag{4.32}$$

Taking classifying spaces, this shows that the pullback of the fiber bundle $B\mathbb{Z}/2 \to B\mathrm{SU}_8 \to B(\mathrm{SU}_8/\{\pm 1\})$ along the map $\Phi\colon B\mathrm{SO}_3 \to B(\mathrm{SU}_8/\{\pm 1\})$ is the fiber bundle $B\mathbb{Z}/2 \to B\mathrm{SU}_2 \to B\mathrm{SO}_3$. We therefore obtain a map between the Serre spectral sequences computing the mod 2 cohomology rings of $B\mathrm{SU}_2$ and $B\mathrm{SU}_8$, and it is an isomorphism on $E_2^{0,*}$, i.e. on the cohomology of the fiber.

Both $B\mathrm{SU}_2$ and $B\mathrm{SU}_8$ are simply connected, so $H^1(–;\mathbb{Z}/2)$ vanishes for both spaces. Therefore in both of these Serre spectral sequences, the generator x of $E_2^{0,1} = H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ must admit a differential. The only differential that can be nonzero is the transgressing $d_2\colon E_2^{0,1} \to E_2^{2,0}$; in $E_2(\mathrm{SU}_8)$, we saw in (4.6) that $d_2(x) = a$, and in $E_2(\mathrm{SU}_2)$, $d_2(x) = w_2$, because w_2 is the only nonzero element of $E_2^{2,0} = H^2(B\mathrm{SO}_3;\mathbb{Z}/2)$. Since the pullback map of spectral sequences commutes with differentials, this means $\Phi^*(a) = w_2$ as desired. □

Now let $W \coloneqq \mathrm{SU}_3/\mathrm{SO}_3$, which is a closed, oriented 5-manifold called the *Wu manifold*, and let $P \to W$ be the quotient $\mathrm{SU}_3 \to \mathrm{SU}_3/\mathrm{SO}_3$. For completness we prove the following proposition about the cohomology of the Wu manifold.

Proposition 4.33. *$H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$ with $|z_2| = 2$ and $|z_3| = 3$. The Steenrod squares are*

$$\begin{aligned} \mathrm{Sq}(z_2) &= z_2 + z_3 \\ \mathrm{Sq}(z_3) &= z_3 + z_2z_3, \end{aligned} \tag{4.34}$$

and the Stiefel-Whitney class is $w(W) = 1 + z_2 + z_3$. Moreover, $w(P) = 1 + z_2 + z_3$. Thus $w_2(P)w_3(P) \neq 0$, meaning (W, P) is our sought-after generator of $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

Proof. Once we know the cohomology ring and the Steenrod squares are as claimed, the total Stiefel-Whitney class of W follows from Wu's theorem as follows. The second Wu class v_2 is defined to be the Poincaré dual of the map

$$x \mapsto \int_W \mathrm{Sq}^2(x)\colon H^3(W;\mathbb{Z}/2) \to H^5(W;\mathbb{Z}/2) \to \mathbb{Z}/2 \tag{4.35}$$

via the Poincaré duality identification $H^2(W;\mathbb{Z}/2) \cong (H^3(W;\mathbb{Z}/2))^\vee$. Wu's theorem shows that $v_2 = w_2 + w_1^2$, so since $H^1(W;\mathbb{Z}/2) = 0$, $w_1 = 0$ and $w_2 = v_2$. Since $\mathrm{Sq}^2(z_3) = z_2z_3$, $w_2 \neq 0$, so it must be z_2. Then $w_3 = \mathrm{Sq}^1(w_2) = z_3$; w_4 is trivial for degree reasons; and $w_5 = 0$ follows from the Wu formula calculating $\mathrm{Sq}^1(w_4)$.

So we need to compute the cohomology ring. Consider the Serre spectral sequence for the fiber bundle

$$\begin{tikzcd} \mathrm{SO}_3 \arrow[r] & \mathrm{SU}_3 \arrow[d] \\ & W, \end{tikzcd} \tag{4.36}$$

which has signature

$$E_2^{*,*} = H^*(W; H^*(\mathrm{SO}_3;\mathbb{Z}/2)) \Longrightarrow H^*(\mathrm{SU}_3;\mathbb{Z}/2). \tag{4.37}$$

A priori we must account for the action of $\pi_1(W)$ on $H^*(\mathrm{SO}_3;\mathbb{Z}/2)$, but using the long exact sequence in homotopy groups associated to a fiber bundle one deduces that W is simply connected because SU_3 is; therefore we do not have to worry about this. Moreover, because W is simply connected, the universal coefficient theorem tells us $H^1(W;\mathbb{Z}/2) = 0$.

As manifolds, $\mathrm{SO}_3 \cong \mathbb{RP}^3$, so $H^*(\mathrm{SO}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^4)$. Also, $H^*(\mathrm{SU}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[c_2, c_3]/(c_2^2, c_3^2)$, with $\lvert c_2\rvert = 3$ and $\lvert c_3\rvert = 5$ [Bor54, §8].

Lemma 4.38. *$H^2(W;\mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. The class $x \in E_2^{0,1} = H^1(\mathrm{SO}_3;\mathbb{Z}/2)$ supports a differential because $H^1(\mathrm{SU}_3;\mathbb{Z}/2) = 0$. Since the Serre spectral sequence is first-quadrant, the only option is a transgressing $d_2\colon E_2^{0,1} \to E_2^{2,0}$. Therefore $\dim H^2(W;\mathbb{Z}/2) \geq 1$. One can also see that this is an upper bound. Since $H^2(\mathrm{SU}_3;\mathbb{Z}/2) = 0$ as well, any additional classes in $E_2^{2,0} = H^2(W;\mathbb{Z}/2)$ have to be killed by a differential. But the only differential that could kill those classes is the transgressing d_2 we just mentioned, and x is the only nonzero element of $H^1(\mathrm{SO}_3;\mathbb{Z}/2)$, so there can't be anything else in $H^2(W;\mathbb{Z}/2)$. ☐

This is enough to get the cohomology ring: we already know H^0, H^1, and H^2 for the Wu manifold; Poincaré duality tells us $H^3(W;\mathbb{Z}/2) \cong \mathbb{Z}/2$, H^4 vanishes, and $H^5 \cong \mathbb{Z}/2$. Therefore there must be generators z_2 and z_3 for the cohomology ring in degrees 2 and 3, respectively, and their squares vanish by degree reasons. And by Poincare duality $z_2z_3 \neq 0$, so it is the generator of H^5. Therefore the cohomology ring is as we claimed.

Next we must determine the Steenrod squares. The fibration (4.36) pulls back from the universal SO_3-bundle $\mathrm{SO}_3 \to E\mathrm{SO}_3 \to B\mathrm{SO}_3$ via the classifying map f_P for P, inducing a map of Serre spectral sequences that commutes with the differentials. We draw this map in Figure 4. This map is an isomorphism on the line $E_2^{0,*}$, so $x \in E_2^{0,1}(\mathrm{SU}_3)$ pulls back from the generator $x \in E_2^{0,1}(E\mathrm{SO}_3)$ — and therefore $d_2(x) = z_2$ pulls back from a class in $E_2^{2,0} = H^2(B\mathrm{SO}_3;\mathbb{Z}/2)$. The only nonzero class in that degree is w_2, so $f_P^*(w_2) = z_2$, i.e. $w_2(P) = z_2$.

The Leibniz rule that in the Serre spectral sequence for SU_3, $d_2(x^2) = 2xd_2(x) = 0$. But because $H^2(\mathrm{SU}_2;\mathbb{Z}/2) = 0$, some differential must kill x^2. Apart from d_2, the only option is the transgressing $d_3\colon E_3^{0,2} \to E_3^{3,0}$, forcing $d_3(x^2) = z_3$. A similar argument in the Serre spectral sequence for $E\mathrm{SO}_3$ shows that in that spectral sequence, $d_3(x^2) = w_3$; therefore $f_P^*(w_3) = z_3$ and $w_3(P) = z_3$. Pullback commutes with Steenrod squares and $\mathrm{Sq}^1(w_2) = w_3$, so $\mathrm{Sq}^1(z_2) = z_3$. Finally, $f_P^*(w_2w_3) = z_2z_3$, and the Wu formula implies $\mathrm{Sq}^2(w_3) = w_2w_3$, so $\mathrm{Sq}^2(z_3) = z_2z_3$. We have computed all the Steenrod squares that could be nonzero for degree reasons, and along the way shown $w(P) = 1 + z_2 + z_3$: the higher-degree Stiefel-Whitney classes of a principal SO_3-bundle vanish. ☐

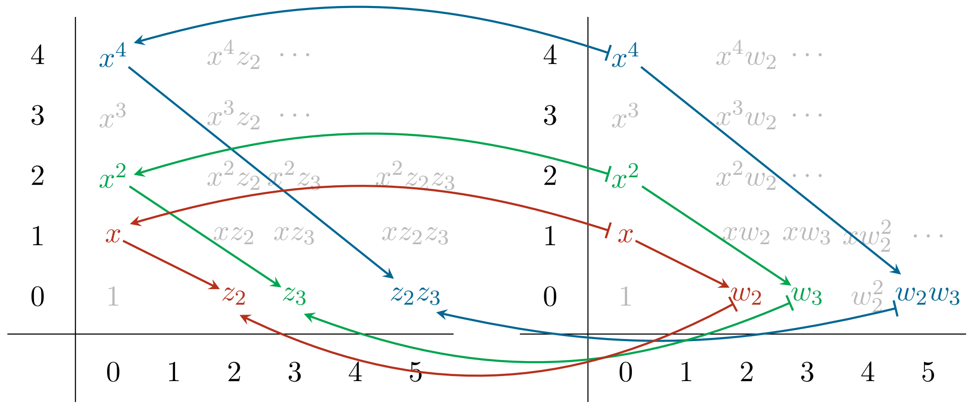

FIGURE 4. The fiber bundle $\mathrm{SO}_3 \to \mathrm{SU}_3 \to W$ pulls back from the universal SO_3-bundle $\mathrm{SO}_3 \to E\mathrm{SO}_3 \to B\mathrm{SO}_3$, inducing a map of Serre spectral sequences. This map commutes with differentials and is the identity on $E_2^{0,\bullet} = H^*(\mathrm{SO}_3;\mathbb{Z}/2)$, allowing us to conclude that w_2 pulls back to z_2, w_3 pulls back to z_3, and w_2w_3 pulls back to z_2z_3. This is a picture proof of part of Proposition 4.33.

5. Evaluating on the Anomaly

With the knowledge of the generating manifold for the $\mathbb{Z}/2$ in degree 5 as the Wu manifold, we can consider evaluating the anomaly of the theory with the field content given in §2. Since $\mathrm{SU}_8/\{\pm 1\}$ acts trivially on the scalars and the graviton only the remaining three fields could have anomalies. The next section is dedicated to showing:

Proposition 5.1. *The total anomaly of 4d $\mathcal{N} = 8$ supergravity arising from the gaugino, vector boson, and gravitino, vanish on the Wu manifold.*

5.1. **Evaluating on the Wu manifold.** The full anomaly denoted by $\mathcal{A}$ can be written schematically as

$$\text{“}\mathcal{A} = \mathcal{A}_{1/2}^{\mathrm{pert}} \otimes \mathcal{A}_{1}^{\mathrm{pert}} \otimes \mathcal{A}_{3/2}^{\mathrm{pert}} \otimes \mathcal{A}_{1/2}^{\mathrm{np}} \otimes \mathcal{A}_{1}^{\mathrm{np}} \otimes \mathcal{A}_{3/2}^{\mathrm{np}}\text{”} \tag{5.2}$$

where we have split up each part of the perturbative and nonperturbative anomaly coming from the gaugino, vector boson,and gravitino. Technically speaking, separating the anomaly in this way is not something that can be done canonically. By (3.4) the nontopological part arises as a quotient of the invertible theory by the topological theories. We write the anomaly in such a way in order to make it organizationally more clear. The Adams computation shows that the free part of $\Omega_6^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is nontrivial but it was shown in [Mar85, BHN10] that in fact the entire perturbative component of the anomaly vanishes.

The vector bosons can be defined without choosing a spin structure, and therefore the partition function of their anomaly field theory factors through the quotient by fermion parity. That is, the tangential structure is

$$\mathrm{SO} \times (\mathrm{SU}_8/\{\pm 1\}) = (\mathrm{Spin}\text{-}\mathrm{SU}_8)/\{\pm 1\}\,. \tag{5.3}$$

We will proceed in understanding the perturbative anomalies by isolating $\mathcal{A}_1^{\mathrm{pert}}$.

Lemma 5.4. *The perturbative anomaly for the vector bosons independently vanishes*

Proof. With the knowledge that the manifold generator for the anomaly is the Wu manifold, we will further restrict to the SO_3 inside of $\mathrm{SU}_8/\{\pm 1\}$; we are left to computing $\Omega_6^{\mathrm{SO}}(B\mathrm{SO}_3)\otimes\mathbb{Q}$, which isolates the free summand. For the degree we are after, we can compute the bordism group via the AHSS. We take the E^2 page of

$$E^2_{p,q} = H_p(B\mathrm{SO}_3, \Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_6^{\mathrm{SO}}(B\mathrm{SO}_3)\,, \tag{5.5}$$

where

$$\Omega_*^{\mathrm{SO}}(\mathrm{pt}) = \{\,\mathbb{Z}\,,\,0\,,\,0\,,\,0\,,\mathbb{Z}\,,\,\mathbb{Z}/2\,,\,0\,,\dots\}\,, \tag{5.6}$$

and tensor with $\mathbb{Q}$. This is equivalent to the E_∞ page, as all differentials vanish, and is given by

$$\begin{array}{c|ccccccc}
6 & 0 \\
5 & 0 & 0 \\
4 & \mathbb{Q} & 0 & 0 & \mathbb{Q} \\
3 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Q} & 0 & 0 & 0 & \mathbb{Q} & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{5.7}$$

We see that the perturbative anomaly of the vector boson vanishes. $\square$

Corollary 5.8. *The perturbative anomalies from the fractional spin particles vanish on their own.*

Having established this corollary, we may now pullback the anomaly in (5.2) to the nonperturbative part, and the equation becomes literally true.

The η-invariant for the contributions in $\mathcal{A}^{\mathrm{np}}_{1/2}\otimes\mathcal{A}^{\mathrm{np}}_{3/2}$ is therefore a bordism invariant, and in particular the η-invariant is computed as two times some other representation and is twice another bordism invariant. In order to see this, we consider how **56**, **28**, and **8** split via our fourfold embedding of SU_2 into $\mathrm{SU}_8/\{\pm 1\}$ for the Wu manifold. We see that **56** gives the dimension of the alternating three forms in 8-dimensions, **28** the dimension of alternating two forms, and **8** is the defining representation. The branchings are given by

$$\mathbf{56} \to 2(10\times\mathbf{2} + 2\times\mathbf{4}), \tag{5.9}$$

$$\mathbf{28} \to 2(3\times\mathbf{3} + 5\times\mathbf{1}), \tag{5.10}$$

$$\mathbf{8} \to 4\times\mathbf{2}\,, \tag{5.11}$$

where the right hand side is in terms of $\mathfrak{su}_2$ representations. To see this, notice that the three forms can be split into

$$\wedge^2 V\otimes\wedge^1 V\otimes\wedge^0 V\otimes\wedge^0 V = \mathbb{C}\otimes V\otimes\mathbb{C}\otimes\mathbb{C} \tag{5.12}$$

in 6 ways, where V denotes the defining representation of $\mathfrak{su}_2$. It can also split into

$$\wedge^1 V\otimes\wedge^1 V\otimes\wedge^1 V\otimes\wedge^0 V = V\otimes V\otimes V\otimes\mathbb{C} \tag{5.13}$$

in 4 ways. Similarly, the two forms can be split into

$$\wedge^2 V\otimes\wedge^0 V\otimes\wedge^0 V\otimes\wedge^0 V \quad\text{and}\quad \wedge^1 V\otimes\wedge^1 V\otimes\wedge^0 V\otimes\wedge^0 V \tag{5.14}$$

in 4 ways and 6 ways, respectively.

To argue that the anomaly vanishes, we also want to show that $\eta_{\mathbf{R}}(\mathcal{D}_{\mathrm{Dirac}})$ is an integer. But since the local anomaly for the fermion vanished, the η-invariant is a bordism invariant. This can be seen from the Atiyah-Patodi-Singer (APS) index theorem, and the index for a Dirac operator makes sense on a 6-manifold. The anomaly in terms of the η-invariant on a manifold M is given by $\mathcal{A} = \exp(\pi i \eta_M(\mathcal{D})/2)$ [Wit16, FH21a], where

$$\eta_M(\mathcal{D}) = \left(\sum_{\lambda\neq 0} \mathrm{sign}(\lambda) + \dim\ker(i\mathcal{D}_M)\right)_{\mathrm{reg.}} . \tag{5.15}$$

But due to the special features of the Wu-manifold, we can instead just work with representations when evaluating the anomaly.

The gaugino was in the represenation **56**, and via the branching in (5.9), this is 4 times the η-invariant of some other representation; this implies $\mathcal{A}^{\mathrm{np}}_{1/2}$ is zero. As a spin 3/2 particle, the gravitino contains a spinor index as well as a Lorentz index, The η-invariant for the gravitino is therefore given by

$$\eta_{\mathrm{gravitino}} = \eta(\mathcal{D}_{\mathrm{Dirac}\times TW}) - 2\eta(\mathcal{D}_{\mathrm{Dirac}})\,, \tag{5.16}$$

where $\eta(\mathcal{D}_{\mathrm{Dirac}})$ is the anomaly of one Dirac fermion, and is $\eta(\mathcal{D}_{\mathrm{Dirac}\times TW})$ is the Dirac operator acting on the spinor bundle tensored with the tangent bundle. Thus, we need to use the fact that the tangent bundle of the Wu manifold is an associated bundle.

Lemma 5.17. *The tangent bundle of the Wu manifold W is given by*

$$TW = \mathrm{SU}(3)\times_{\mathrm{SO}(3)} \frac{\mathfrak{su}_3}{\mathfrak{so}_3}\,.$$

Proof. The fact that the Wu manifold is a homogeneous space allows us to use the following general procedure to construct its tangent bundle. For $H\subset G$ is a closed subgroup of a Lie group G, we have the following exact sequence of adjoint representations of H:

$$1 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{5.18}$$

The canonical principal H-bundle $H\to G/H$ gives an exact functor from representations of H to vector bundles over G/H. This gives a corresponding sequence of vector bundles:

$$1 \longrightarrow G\times_H \mathfrak{h} \longrightarrow G\times_H \mathfrak{g} \longrightarrow G\times_H \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{5.19}$$

There is an isomorphism $G\times_H \mathfrak{g}/\mathfrak{h}\to T(G/H)$ shown in [Cap19]. Let $p: G\to G/H$ and L_X be the left invariant vector field generated by $X\in\mathfrak{h}$. Then the mapping of $(g, X+\mathfrak{h})\in G\times(\mathfrak{g}/\mathfrak{h})$ to $T_g p\cdot L_X(g)\in T_{gH}(G/H)$ gives the isomorphism. Specifically for our problem, we have the SO_3-bundle $\mathrm{SU}_3\to W$, which by the present construction gives the desired result. □

Remark 5.20. This is an example of the "mixing construction": for a principal G-bundle $P\to M$ and a G-representation V, the space $P\times_G V$ is a vector bundle over M with rank equal to the dimension of V.

We are now left to understand $\frac{\mathfrak{su}_3}{\mathfrak{so}_3}$ as a representation of SO_3. The Lie algebra $\mathfrak{su}_3$ is an SU_3-representation, and restricting, it is also an SO_3 representation of dimension 8. But the **8** of $\mathfrak{su}_3$ branches as $\mathbf{8}\to\mathbf{1}+\mathbf{1}+\mathbf{3}+\mathbf{3}$ in $\mathfrak{so}_3$, so quotienting by $\mathfrak{so}_3$ then eliminates one of the **3** summands. Then $\eta(\mathcal{D}_{\mathrm{Dirac}\times TW}) = (\mathbf{1}+\mathbf{1}+\mathbf{3})\,\eta(\mathcal{D}_{\mathrm{Dirac}})$, which means the gravitino contributes $3\eta(\mathcal{D}_{\mathrm{Dirac}})$. By the branching in (5.11), $\eta(\mathcal{D}_{\mathrm{Dirac}})$ of **8** in $\mathfrak{su}_8$ is determined by **2** of $\mathfrak{su}_2$, and using the fact that

$\eta_{\sum_i \mathbf{R}_i}(\mathcal{D}_{\mathrm{Dirac}}) = \sum_i \eta_{\mathbf{R}_i}(\mathcal{D}_{\mathrm{Dirac}})$, we have a multiple of 4 worth of $\eta_{\mathbf{2}}(\mathcal{D}_{\mathrm{Dirac}})$ and that determines $\eta_{\mathrm{gravitino}}$. Then the anomaly $\mathcal{A}^{\mathrm{np}}_{3/2}$ associated to $\eta_{\mathrm{gravitino}}$ vanishes per the above discussion for the gauginos.

We now move onto the nonperturbative anomaly from the vector bosons, which is accessible from $\Omega_5^{\mathrm{SO}}(B\mathrm{SO}_3)$. By applying (5.6) to the AHSS , we only need to consider $H_5(B\mathrm{SO}_3, \mathbb{Z})$ as well as the $\mathbb{Z}/2$ element in bidegree $(0,5)$. One can evaluate the torsion part of $H_5(B\mathrm{SO}_3;\mathbb{Z})$ by the universal coefficient theorem, and looking at $H^6(B\mathrm{SO}_3;\mathbb{Z})$. We find that this is given by w_2w_3 of the SO_3 bundle and is nontrivial on the Wu manifold. Then the AHSS says $\Omega_5^{\mathrm{SO}}(B\mathrm{SO}_3) = \mathbb{Z}/2\times\mathbb{Z}/2$ detected by the bordism invariants $\int w_2(TM)w_3(TM)$ and $\int w_2(P)w_3(P)$; these are generated by W with trivial bundle, and W with the principal SO_3-bundle. We see that while the bosonic anomaly is in principle $\mathbb{Z}/2\times\mathbb{Z}/2$ valued, and coupling to spin structure eliminates one of the $\mathbb{Z}/2$. Using (5.10), for the representation of the vector boson, the anomaly is also twice of something as a bordism invariant. This is reasonable since the anomaly of multiple particles is the tensor product of their anomalies. [9]. The anomaly for the vector bosons is 2 times something as a bordism invariant, since the perturbative part vanished, and considering that we have argued that everything else in (5.2) vanishes aside from $\mathcal{A}^{\mathrm{np}}_1$, we have that $\mathcal{A} = \mathcal{A}^{\mathrm{np}}_1$. But $\mathcal{A}$ is $\mathbb{Z}/2$ valued, and with $\mathcal{A}^{\mathrm{np}}_1$ equating to 0 mod 2, the full anomaly vanishes, thus establishing proposition 5.1.

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