# BOSONIZATION AND ANOMALY INDICATORS OF (2+1)-D FERMIONIC TOPOLOGICAL ORDERS 

ARUN DEBRAY, WEICHENG YE, AND MATTHEW YU


#### Abstract

We provide a mathematical proposal for the anomaly indicators of symmetries of $(2+1)$-d fermionic topological orders, and work out the consequences of our proposal in several nontrivial examples. Our proposal is an invariant of a super modular tensor category with a fermionic group action, which gives a (3+1)-d topological field theory (TFT) that we conjecture to be invertible; the anomaly indicators are partition functions of this TFT on 4manifolds generating the corresponding twisted spin bordism group. Our construction relies on a bosonization construction due to Gaiotto-Kapustin and Tata-Kobayashi-Bulmash-Barkeshli, together with a "bosonization conjecture" which we explain in detail. In the second half of the paper, we discuss several examples of our invariants relevant to condensed-matter physics. The most important example we consider is $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ time-reversal symmetry with symmetry algebra $\mathcal{T}^{2}=(-1)^{F} C$, which many fermionic topological orders enjoy, including the $\mathrm{U}(1)_{5}$ spin Chern-Simons theory. Using newly developed tools involving the Smith long exact sequence, we calculate the cobordism group that classifies its anomaly, present the generating manifold, and calculate the partition function on the generating manifold which serves as our anomaly indicator. Our approach allows us to reproduce anomaly indicators known in the literature with simpler proofs, including $\mathbb{Z} / 4^{T f}$ time-reversal symmetry with symmetry algebra $\mathcal{T}^{2}=(-1)^{F}$, and other symmetry groups in the 10 -fold way involving Lie group symmetries.


## Contents

1. Introduction ..... 2
1.1. Summary of Main Results ..... 4
2. Preliminaries: Fermionic Symmetry, Anomaly and (2+1)D Fermionic Topological Orders ..... 6
2.1. Fermionic Symmetry and Anomaly ..... 6
2.2. Fermionic Topological Order with Symmetry Action ..... 11
3. Spin TFT: bosonization, conjectures and application to Fermionic Topological Orders ..... 15
3.1. Unpacking the Conjectures ..... 15
3.2. Partition Functions for super-MTC and Anomaly Indicators ..... 29
4. Warmup: $\mathbb{Z} / 4^{T f}$ ..... 32
5. $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ ..... 35

[^0]5.1. The Power of Smith: $\Omega_{4}^{\mathrm{EPin}}$ and $\Omega_{4}^{\mathrm{EPin}[k]} \quad 35$
5.2. Anomaly Indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ Symmetry 50
6. Conclusion and Discussion 55

Appendix A. Anomaly Indicators with Lie group Symmetry: 10-fold way 57
A.1. Class A and class C 58
A.2. Class AI, AII, AIII 62
A.3. Class CI, CII 64

Appendix B. Data of Fermionic Topological Orders 65
B.1. $\mathrm{U}(1)_{5} \quad 65$
B.2. $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1} \quad 65$
B.3. $\mathrm{SO}(3)_{3} \quad 66$

Appendix C. Anomaly Cascade for Fermionic Topological Orders 67
References 70

## 1. Introduction

A topological order is a unique state of matter that emerges in certain gapped quantum systems [Wen04]. Unlike traditional phases of matter, such as solids, liquids, and gases, topological order is not defined by local order parameters or spontaneous symmetry breaking, but rather by the presence of long-range quantum entanglement. One of the most notable examples of these exotic phases is the fractional quantum Hall effect [STG99]. This is a fermionic topological order in $(2+1)$-dimensions and exists for two-dimensional electron systems subjected to strong magnetic fields. The electrons exhibit a collective behavior giving rise to particles known as anyons, quasiparticle excitations with nontrivial statistics that may be neither bosonic nor fermionic. For our set up in $(2+1)$-d, a fermionic topological order is mathematically axiomatized to be a super modular tensor category (MTC) [ $\left.\mathrm{BGH}^{+} 17\right]$, the definition of which we will further delve into in $\S 2$.

There is a rich interplay between topological orders and symmetry. Most notably, anyons may transform in a projective representation under the symmetry action, and we sometimes say that anyons carry "fractional" quantum numbers. This is known as the phenomenon of symmetry fractionalization. Symmetry action on a topological order gives a categorical group action on the corresponding tensor category [BBCW19, GV17, BB22b]. Moreover, the symmetry action on the topological order can have a 't Hooft anomaly [Hoo80]. In the condensed matter setting, this suggests that the symmetry action cannot be realized as an on-site symmetry. The topological order therefore has to be realized on the boundary of an invertible field theory or a symmetry-protected topological phase (SPT) in one dimension higher. Classifying the invertible phases would then give a classification of the anomalies. We will perform this classification by computing the relevant cobordism groups [FH21, WS14] for symmetries associated to fermionic theories.

One can use 't Hooft anomalies as a means of constraining the IR phase that a UV theory flows to, but doing so requires computing the specific value of the anomaly and not just the group that classifies it. A very common scenario in the high energy literature is when the UV is described by a free fermion theory with a global symmetry [GKS18, CDGK20, KS18, DGY23], and in the IR the theory flows to a strongly interacting field theory, in particular a topological order. Another scenario that appears in the condensed matter literature involves the UV lattice system having some Lieb-Schultz-Mattis-type anomaly $\left[\mathrm{CZB}^{+} 16, \mathrm{ET} 20, \mathrm{YGH}^{+} 22\right]$, and anomaly matching can help
us identify which IR theories can emerge from the UV lattice system [YGH+22, ZHW21, YZ23b]. However, computing the anomaly in the IR is a much more involved process than in the UV, see e.g. [TY17] where the authors compute the $\mathbb{Z} / 16$ valued anomaly for time-reversal in (2+1)-d topological theories by using a crosscap background. It is therefore desirable to understand how to compute anomalies for fermionic topological orders in a systematic way.

There has been significant progress in calculating the anomaly of (2+1)-d bosonic topological orders with symmetry, including [BBCW19, WL17, LL19, WLL16, BBC ${ }^{+}$19, BB20, KB21, YZ23a]. In particular, for a time reversal symmetry $\mathbb{Z} / 2^{T}$ that may permute anyons, [WL17] proposed a set of anomaly indicators to detect the anomaly of any bosonic topological order, and [ $\left.\mathrm{BBC}^{+} 19\right]$ derived the formula by calculating the partition function of a certain topological field theory (TFT) on the generating manifold of the corresponding unoriented bordism group $\Omega_{4}^{O}$. In order to extend the calculation to a general finite group symmetry, which may contain anti-unitary elements and/or permutes anyons, [BB20] gave a state-sum construction of the anomaly theory. Following these ideas, [YZ23a] revisited the construction in the language of extended TFTs and wrote down a general recipe to obtain the explicit anomaly indicators for any bosonic topological order with general symmetry groups. This recipe is also generalized to include Lie group symmetries.

Many of these strategies have been applied to tackle fermionic topological orders, including [WL17, LL19, NMLW21, TKBB23, BB22a, ABK21, KB21]. In particular, [WL17] also proposed an anomaly indicator to detect the anomaly of fermionic topological order with the $\mathbb{Z} / 4^{T f}$ symmmetry. Later, [TKBB23] generalized the construction and calculation in [BB20] to the fermionic setting and derived the conjectured anomaly indicator. Lie group symmetries were also studied in the fermionic setting in [LL19, NMLW21, KB21].

The purpose of this paper is to classify and compute the anomaly for symmetries of $(2+1)$-d fermionic topological orders. We provide a general mathematical approach and also study several example symmetries. In the body of our article, we focus on abelian time-reversal symmetry, with symmetry algebra $\mathcal{T}^{2}=(-1)^{F}$ for the $\mathbb{Z} / 4^{T f}$ symmetry and $\mathcal{T}^{2}=(-1)^{F} C$ for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, with $\mathcal{T}$ the time-reversal generator, $C$ charge conjugation and $(-1)^{F}$ fermion parity. The tools that we employ can be applied to general symmetry groups which may be discrete or continuous, abelian or non-abelian, contain anti-unitary elements and/or permute anyons. We follow the general procedure developed in $\left[\mathrm{BBC}^{+} 19, \mathrm{BB} 20\right.$, YZ23a] for bosonic topological orders and generalize it to the fermionic context in a spirit similar to [TKBB23, KB21]. We make a conjecture that the computation of the partition function for a spin TFT can be done via a bosonization procedure, which relates the spin theory to a bosonic one. This conjecture is vital in establishing if a particular spin TFT associated to an anomaly is invertible, and if its partition function is a cobordism invariant. Moreover, our approach utilizes a handle decomposition [GS99] of a manifold following [YZ23a]. Compared to [TKBB23, KB21] which uses a cellulation of a manifold, our calculation is much simpler and will produce closed-form expressions for partition functions and anomaly indicators.

The general procedure can be summarized by a sequence of steps. We first compute the group that classifies the anomaly of a fermionic symmetry and identify the generating 4-manifold for the dual bordism group. We then evaluate the partition function on the generating manifold of a certain spin TFT, which should be thought of as the anomaly theory whose boundary hosts the given (symmetry-enriched) topological order. The partition function is written in terms of the anyon content of the topological order as well as the specific symmetry action on the set of anyons. The result is the anomaly indicator for the fermionic symmetry. We showcase these steps
by deriving the anomaly indicators known in the literature for fermionic symmetries in the 10 -fold way classification [WS14, FH21]. As a brand new example with physical importance, we then focus on the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, which, as discussed in [DG21], is a symmetry of abelian spin Chern-Simons theories such as $\mathrm{U}(1)_{k}$ with $k=5,13, \ldots$ describing the $\nu=1 / k$ fractional quantum Hall effect. We use this example to showcase all the technical tools and the power of our general method. As a bonus, we use this example to demonstrate how to use the Smith homomorphism as a powerful method to perform the bordism calculations, in particular for resolving hard extension problems and constructing the generating manifold.
1.1. Summary of Main Results. We now give an overview of the organization of the paper, and present a summary of the main results.
(1) In $\S 2$, we give a brief summary of the background material and notation that will be used throughout the paper, including our definitions of fermionic symmetry, anomaly, and $(2+1)$-d fermionic topological order.
(2) In $\S 3$, we state the main conjecture serving as the backbone that makes our techniques implementable. The obvious steps to take is to construct a spin version of Crane-Yetter theory corresponding to the fermionic symmetry. However, this is a difficult open problem so we take a different approach by bosonization. We frame the problem of identifying the spin TFT and writing down its partition function into a bosonic problem, which serves as an easier method for making concrete computations of partition functions.

Heuristic Definition. Given a fermionic $G$-symmetry acting on a super MTC $\mathcal{C}$, we define a 4 d TFT $\alpha$ by the following recipe:
(a) Let $Z^{b}$ be the bosonic shadow of the 3 d spin TFT defined by $\mathcal{C}$. The $G$-action on $\mathcal{C}$ induces a $G$-symmetry of $Z^{b}$.
(b) Let $\alpha^{b}$ be the 4 d anomaly field theory of the $G$-symmetry of $Z^{b}$, as constructed in $\left[\mathrm{BBC}^{+} 19, \mathrm{BB} 22 \mathrm{~b}, \mathrm{YZ} 23 \mathrm{a}\right]$.
(c) Let $\alpha$ be the fermionization of $\alpha^{b}$.

Anomaly indicators refer to the value of $\alpha$ on closed 4-manifolds, especially generators of bordism groups of interest.

Most of $\S 3$ is devoted to building up the ingredients of the precise version of this definition; said precise definition is given in $\S 3.2$, with formulas for the partition function of $\alpha$ in (3.46) and (3.51). Ideas similar to the above heuristic definition appear in work of Tata-Kobayashi-Bulmash-Barkeshli [TKBB23].

A priori, the 4 d TFT $\alpha$ defined above has no relation to the anomaly of the $G$-symmetry on $\mathcal{C}$, but computations in examples suggest that the two are the same. A significant goal of this paper is to provide a conjectural explanation of this phenomenon.

In order to explain our conjecture, we need a few pieces of notation. Gaiotto-Kapustin [GK16] implement the bosonic shadow construction and its inverse as a kernel transform with an anomalous TFT called $z_{c}$ : one tensors with $z_{c}$, then sums over spin structures or (higher) $\mathbb{Z} / 2$ gauge fields. Tata-Kobayashi-Bulmash-Barkeshli [TKBB23] show how to incorporate the $G$-action into a more general definition of $z_{c}$. Because $z_{c}$ is defined on manifolds with data of a (possibly twisted) spin structure and a $\mathbb{Z} / 2$ higher gauge field, we re-express it in $\S 3.1 .5$ as a defect between two TFTs, specifically two Dijkgraaf-Witten type TFTs $F_{\text {Spin }}$ and $F_{B}$ defined by summing over spin structures, resp. $\mathbb{Z} / 2$ higher gauge fields.

Conjecture 3.33 (Bosonization Conjecture). Given a fermionic $G$-symmetry acting on a super MTC $\mathcal{C}$ with anomaly $\widetilde{\alpha}$, let $F_{\widetilde{\alpha}}$ denote the Dijkgraaf-Witten type theory obtained by summing $\widetilde{\alpha}$ over (twisted) spin structures, and let $F_{\widetilde{\beta}}$ be the same construction applied to the bosonic shadow of $\alpha$. Then, as theories of manifolds with (twisted) spin structures, $z_{c}$ extends from an $\left(F_{\text {Spin }}, F_{B}\right)$-defect to an $\left(F_{\widetilde{\alpha}}, F_{\widetilde{\beta}}\right)$-defect.

The restriction to (twisted) spin manifolds is to work around the appearance of a different anomaly; see §3.1.4. We explain in §3.1.5 how the bosonization conjecture implies the following key result.

Theorem. Assuming Conjecture 3.33, the theory $\alpha$ we define in §3.2 is the anomaly of the $G$-action on $\mathcal{C}$.

In particular, the bosonization conjecture applied to this context implies that $\alpha$ is invertible and the anomaly indicators we calculate are Reinhardt bordism invariants; we show in Corollary 3.54 that (again assuming Conjecture 3.33) they are bordism invariants in the usual sense. In other words:

Conjecture. The bosonic shadows $Z_{b}$ of the anomaly (3+1)-d fermionic theory assemble into a partition function $\mathcal{Z}^{f}$, which is a cobordism invariant in the (twisted) spin cobordism group that classifies anomalies of the (2+1)-d fermionic theory.
(3) We then turn to specific examples of fermionic symmetries. We start with reproducing the known anomaly indicators for the $\mathbb{Z} / 4^{T f}$ symmetry with symmetry algebra $\mathcal{T}^{2}=(-1)^{F}$ in $\S 4$. Next we undertake the task of understanding the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry with symmetry algebra $\mathcal{T}^{2}=(-1)^{F} C$.

We generalize the case of $\mathbb{Z} / 4^{T}$ to $\mathbb{Z} / k^{T}$ with $4 \mid k$ and denote the tangential structure for this symmetry by EPin $[k]$. When $k=4$ it reduces to the EPin structure which already appears in the literature. The Atiyah-Hirzeburch spectral sequence and the Adams spectral sequence can determine that $\Omega_{4}^{\mathrm{EPin}[k]}$ is of order 4 [BG97, WWZ20], but there is a highly nontrivial extension problem that meant the specific isomorphism type of this group was an open question. Utilizing the recently developed techniques involving the Smith homomorphism [HKT20, $\left.\mathrm{DDK}^{+} 23\right]$, we resolve the extension problem and the classification of anomalies as well as the generating manifolds are summarized as follows:

## Theorem.

(a) If $k \equiv 4 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 4$. Let $\mathcal{M}$ denote the manifold we construct in Theorem 5.43, which is the total space of a Klein bottle bundle over $S^{2}$, then $\mathcal{M}$ generates $\Omega_{4}^{\mathrm{EPin}[k]}$.
(b) If $k \equiv 0 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, with a basis given by the bordism classes of $\mathcal{M}$ and the K3 surface.

This is a combination of Theorems 5.28 and 5.43.
For $k=4$, this bordism group has been studied in the literature, but we are the first to compute it. Botvinnik-Gilkey [BG97] and Barrera-Yanez [BY99, Theorem 3.1] study this and other epin $[k]$ bordism groups using algebraic and analytic methods, respectively, but do not determine the isomorphism type of $\Omega_{4}^{\mathrm{EPin}[k]}$; Wan-Wang-Zheng [WWZ20, $\S$ B.1] erroneously reported that $\Omega_{4}^{\mathrm{EPin}[4]}$ is isomorphic to $\mathbb{Z} / 2 \oplus \mathbb{Z} / 2$; and Córdova-HsinZhang [CHZ23] show that a closely related group is isomorphic to $\mathbb{Z} / 4$ but do not study epin $[k]$ bordism.

The general formula for the anomaly indicator is then obtained by calculating the partition function on the generator $\mathcal{M}$ of $\Omega_{4}^{\mathrm{EPin}}$, which is given in Proposition 5.56. We use it to compute the anomalies for abelian Chern-Simons theories, as well as $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$, and we find:
Theorem 5.60. The fermionic topological order $\mathrm{U}(1)_{k}$ (for $k=5,13, \ldots$, as given in [DG21]), $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ realize a time-reversal symmetry, with algebra $\mathcal{T}^{2}=$ $(-1)^{F} C$. The anomaly for $\mathrm{U}(1)_{k}, \mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ evaluates to $\nu=0,2,3 \in \mathbb{Z} / 4$, respectively.
(4) In Appendix A, we extend our calculation to Lie group symmetries, including seven out of the ten symmetries in the 10 -fold way classification of fermionic symmetries. ${ }^{1}$ We write down the generating manifolds of the corresponding bordism groups and calculate the partition functions on the generating manifolds following our general recipe in $\S 3.2$. The calculation correctly reproduces the anomaly indicators for these symmetries known in the literature [LL19, NMLW21]. The results and anomaly indicators are summarized in Propositions A.6, A.18, A.23, and A.28.
(5) In Appendix B, we collect data of the fermionic topological orders and symmetry actions we explicitly discuss, including $\mathrm{U}(1)_{5}, \mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)$.
(6) In Appendix C, we provide an additional test of our conjectures and results by comparing our calculation in Theorem 5.60 with a different method of calculating anomalies: the "anomaly cascade conjecture" of Bulmash-Barkeshli [BB22a] (here appearing as Conjecture C.3) identifying the behavior of the anomaly field theory with respect to the filtration of $\mho_{\xi}^{4}$ coming from the Atiyah-Hirzebruch spectral sequence. In Propositions C. 8 and C. 14 and Lemma C.11, we calculate the anomalies of the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry on $U(1)_{5}$, $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$, and $\mathrm{SO}(3)_{3}$ assuming this conjecture; our results are consistent with Theorem 5.60 , providing support for our conjectures.

Bulmash-Barkeshli relate the image of an anomaly in the layers of the Atiyah-Hirzebruch filtration to tensor-category-theoretic data, meaning that to use their conjecture we must perform calculations with the tensor categories we study. To do so, we make use of a technique called zesting due to Bruillard-Galindo-Hagge-Ng-Plavnik-Rowell-Wang [ $\mathrm{BGH}^{+}$17] (see also $\left[\mathrm{DGP}^{+} 21\right]$ ) to produce modular extensions of super MTCs with prescribed fusion rules.

## 2. Preliminaries: Fermionic Symmetry, Anomaly and (2+1)D Fermionic Topological Orders

In this section, we collect all the background knowledge that will be used throughout the paper. This includes the basic setup of a fermionic symmetry and its anomaly in §2.1, as well as necessary information about $(2+1)$-d fermionic topological order with a symmetry action on it in $\S 2.2$. Readers familiar with these topics can skip the exposition in this section and use it as a reference.
2.1. Fermionic Symmetry and Anomaly. In this subsection, we give a more careful definition of what we mean by a fermionic symmetry, as well as the relevant bordism group involved in the anomaly classification for the symmetry.

[^1]In order to specify a symmetry in a fermionic system, we need to specify the symmetry group $G_{f}$, identify the $\mathbb{Z} / 2^{f}$ subgroup corresponding to the distinguished fermion parity symmetry, and all the anti-unitary elements in the symmetry group. It is helpful to summarize all the information of a fermionic symmetry into the following triple of data.

Definition 2.1. A fermionic symmetry is the data of a group $G_{b}$, a homomorphism $s: G_{b} \rightarrow\{ \pm 1\}$, and a central extension

$$
\begin{equation*}
1 \rightarrow \mathbb{Z} / 2 \rightarrow G_{f} \rightarrow G_{b} \rightarrow 1 \tag{2.2}
\end{equation*}
$$

We refer to $G_{b}$ as the bosonic symmetry group of the fermionic symmetry.
As is conventional, we will name a fermionic symmetry in terms of its full fermionic symmetry group $G_{f}$.

The homomorphism $s$ defines a class in $H^{1}(B G ; \mathbb{Z} / 2)$ that, with a slight abuse, we also call $s$, and the extension by $\mathbb{Z} / 2$ defines a class $\omega \in H^{2}(B G ; \mathbb{Z} / 2)$. The data of ( $\left.G_{b}, s, \omega\right)$ characterizes a fermionic symmetry up to isomorphism of $G_{b}, s$, and the extension. Hence, we will also identify a fermionic symmetry by the triple $\left(G_{b}, s, \omega\right)$.

In particular, given a fermionic symmetry, $s$ tells us whether elements of $G_{b}$ act unitarily or antiunitarily:

$$
s(\mathbf{g})= \begin{cases}0 & \text { if } \mathbf{g} \text { acts unitarily }  \tag{2.3}\\ 1 & \text { if } \mathbf{g} \text { acts anti-unitarily }\end{cases}
$$

$s$ also induces a map $B s: B G_{b} \rightarrow B \mathbb{Z} / 2$, and the pullback of the tautological bundle on $B \mathbb{Z} / 2$ across $s$ gives a 1-dimensional line bundle on $B G_{b}$. This line bundle is denoted by $\sigma$ in this paper and will play a crucial role. In particular, $w_{1}(\sigma)=s$.

The symmetry groups that we consider in the main text are listed as follows, given in terms of the triple data $\left(G_{b}, s, \omega\right)$ :
(1) $\mathbb{Z} / 4^{T f}$. Here $G_{b}=\mathbb{Z} / 2$, $s$ the nontrivial element in $H^{1}(B \mathbb{Z} / 2 ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2$, and $\omega$ is the nontrivial element in $H^{2}(B \mathbb{Z} / 2 ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2$.
(2) $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$. Here $G_{b}=\mathbb{Z} / 4$, $s$ the nontrivial element in $H^{1}(B \mathbb{Z} / 4 ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2$, and $\omega$ is the trivial element in $H^{2}(B \mathbb{Z} / 4 ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2$.
In this paper, we use ${ }^{T}$ to denote that some elements in the symmetry group are anti-unitary, and hence the $s$ for these groups are nontrivial. We also use ${ }^{f}$ to denote that some element in the symmetry group corresponds to fermion parity $(-1)^{F}$.

We will sometimes need maps between different fermionic symmetries which we define as follows.
Definition 2.4. A map $\phi$ between two different fermionic symmetries $\left(G_{1}, s_{1}, \omega_{1}\right)$ and $\left(G_{2}, s_{2}, \omega_{2}\right)$ is a map $\phi: G_{1} \rightarrow G_{2}$ such that $\phi^{*}\left(s_{2}\right)=s_{1}$ and $\phi^{*}\left(\omega_{2}\right)=\omega_{1}$.

This definition guarantees that the induced maps between, e.g., the relevant bordism groups and the relevant (twisted) cohomology are all well-defined. For simplicity, we will usually only state the $\operatorname{map} \phi: G_{1} \rightarrow G_{2}$ and let the reader check that the two requirements are satisfied.

Remark 2.5. There are a few other ways to package the data of a fermionic symmetry.

- A normal 1-type [Kre99, §2].
- Fermionic groups in the sense of Stehouwer [Ste22, Definition 1].
- A twist of spin bordism provided by the data of a map $B G_{b} \rightarrow B \mathrm{O} / B$ Spin [DY23a, §1.2.3].

All of these are equivalent to our Definition 2.1, in that they provide $G_{b}, s$, and $\omega$ in the sense above, and are equivalent to such data.

In the situations we consider in this paper, the anomaly of a fermionic symmetry acting on an $n$-dimensional field theory $Z$ is an $(n+1)$-dimensional invertible field theory $\alpha:^{2}$ there is some other $(n+1)$-dimensional theory $\alpha^{\prime}$ and an isomorphism $\alpha \otimes \alpha^{\prime} \stackrel{\simeq}{\leftrightharpoons} \mathbf{1}$, where $\mathbf{1}$ is the trivial theory whose value on objects is the monoidal unit, and whose value on morphisms is the identity. The notion of an invertible field theory is due to Freed-Moore [FM06, Definition 5.7]. The connection to anomalies is due to Freed-Teleman [FT14, Fre14]; see also [Wit00, FHT10] and see Freed [Fre23] for a nice overview.

To talk about examples of invertible field theories, we must specify what kinds of manifolds we place them on.

Definition 2.6. A tangential structure is a map $\xi: B \rightarrow B O$, which we, without loss of generality, take to be a fibration. Given $\xi$, a $\xi$-structure on a vector bundle $V \rightarrow X$ is a lift of the classifying $\operatorname{map} f_{V}: X \rightarrow B$ O of $V$ to $B$, i.e. it is a map $\widetilde{f}_{V}: X \rightarrow B$ such that $f_{V} \simeq \xi \circ \widetilde{f}_{V}$. A $\xi$-structure on a manifold $M$ means a $\xi$-structure on $T M$.

Two $\xi$-structures on a vector bundle $V \rightarrow X$ are equivalent if they belong to the same connected component of the space of $\xi$-structures on $V$. Whenever we count $\xi$-structures, we are referring to equivalence classes of $\xi$-structures.

Given a family of groups $H(n)$ with maps $H(n) \rightarrow H(n+1)$ and homomorphisms $\rho(n): H(n) \rightarrow$ $\mathrm{O}(n)$ commuting with the maps $H(n) \rightarrow H(n+1)$ and $\mathrm{O}(n) \rightarrow \mathrm{O}(n+1)$, one can take the classifying space of the colimit to obtain a $\xi$-structure $B \rho: B H \rightarrow B$ O. This is a good source of examples of $\xi$-structures: for example, when $H(n)=\mathrm{SO}(n)$, this notion of a $\xi$-structure is equivalent to an orientation, and when $H(n)=\operatorname{Spin}(n)$, it is equivalent to a spin structure. Equivalence of $\xi$-structures coincides with equality of orientations, resp. spin structures.

Following Lashof [Las63], one may define bordism groups $\Omega_{k}^{\xi}$ of closed $k$-manifolds with $\xi$ structures, recovering the usual notions of oriented bordism, spin bordism, etc. Likewise, one can define bordism categories of $\xi$-manifolds, and therefore $\xi$-structured topological field theories.

Definition 2.7. Let $V \rightarrow X$ be a vector bundle and $\xi: B \rightarrow B O$ be a tangential structure. An $(X, V)$-twisted $\xi$-structure on a vector bundle $E \rightarrow M$ is data of a map $f: M \rightarrow X$ and a $\xi$-structure on $E \oplus f^{*}(V)$.
Lemma 2.8 (Shearing). Let $V_{t} \rightarrow B$ denote the tautological virtual vector bundle. If $\eta: B \times X \rightarrow$ $B O$ is the map classified by the vector bundle $\xi^{*}\left(V_{t}\right) \boxplus V$, then $\eta$-structures are equivalent to $(X, V)$-twisted $\xi$-structures.

We do not know the origin of this result; see [DDHM23, Lemma 10.18] for a proof. ${ }^{3}$ Lemma 2.8 implies that we can consider bordism groups and topological field theories of $(X, V)$-twisted $\xi$-manifolds.

Ansatz 2.9. Let $\left(G_{b}, s, \omega\right)$ be a fermionic symmetry and assume that there is a vector bundle $V \rightarrow B G_{b}$ such that $w_{1}(V)=s$ and $w_{2}(V)=\omega$. The data of $\left(G_{b}, s, \omega\right)$ acting as a symmetry of a (spacetime dimension $n$ ) spin TFT $Z: \mathcal{B}$ ord ${ }_{n}^{\text {Spin }} \rightarrow \mathcal{C}$ induces an extension of $Z$ to a (possibly

[^2]anomalous) TFT $\hat{Z}: \mathcal{B}$ ord $n_{n}^{\xi} \rightarrow \mathcal{C}$ of manifolds with $a\left(B G_{b}, V\right)$-twisted spin structure $\xi . \hat{Z}$ lives on the boundary of an $(n+1)$-dimensional invertible field theory $\alpha$ of $\left(B G_{b}, V\right)$-twisted spin manifolds. We refer to $\alpha$ as the anomaly of $Z$ with its $\left(G_{b}, s, \omega\right)$-symmetry.

For a detailed discussion of this ansatz, see Freed-Hopkins [FH21, §2, §3] in general and Stehouwer [Ste22, Ste23] for fermionic symmetries specifically, as well as [KTTW15, Wen13, DGG21, Tho20, WG20] for further justification from a physically motivated point of view. The assumption in Ansatz 2.9 that a bundle $V \rightarrow B G_{b}$ with the required characteristic classes exists is not always true: see [GKT89, DY22, DY23a] for some counterexamples. In all examples relevant for this paper, though, such a bundle exists, so we will not worry about this nuance.

However, Ansatz 2.9 chooses $V$, and generally there is more than one possible choice. Fortunately, if $V$ and $W$ have the same first two Stiefel-Whitney classes, the notions of $(X, V)$-twisted spin structure and $(X, W)$-twisted spin structures are equivalent, which follows from [HJ20, Corollary 3.3.8] (see also [Deb21, Theorem 1.39]). In practice, one can forget about $V$ and just remember $w_{1}(V)$ and $w_{2}(V)$. This leads to a more expansive definition of a twisted spin structure than we gave in Definition 2.7.

Definition 2.10 (Wang [Wan08, Definition 8.2]). Let $X$ be a space and choose $s \in H^{1}(X ; \mathbb{Z} / 2)$ and $\omega \in H^{2}(X ; \mathbb{Z} / 2)$. A $(X, s, \omega)$-twisted spin structure on a vector bundle $V \rightarrow M$ is data of a map $f: M \rightarrow X$ and trivializations of $w_{1}(V)+f^{*}(s)$ and $w_{2}(V)+f^{*}\left(s^{2}+\omega\right)$.

The Whitney sum formula implies $(X, V)$-twisted spin structures are in natural bijection with $\left(X, w_{1}(V), w_{2}(V)\right)$-twisted spin structures. For us, $X$ will always be $B G_{b}$.
$(X, s, \omega)$-twisted spin structures are tangential structures in the sense of Definition 2.6.
Definition 2.11. Let

$$
\begin{equation*}
\xi_{X, s, \omega}: X\left\langle s, s^{2}+\omega\right\rangle \longrightarrow B \mathrm{O} \times X \tag{2.12a}
\end{equation*}
$$

denote the fiber of the map

$$
\begin{equation*}
\left(w_{1}+s, w_{2}+s^{2}+\omega\right): B \mathrm{O} \times X \longrightarrow K(\mathbb{Z} / 2,1) \times K(\mathbb{Z} / 2,2) \tag{2.12b}
\end{equation*}
$$

We will also use $\xi_{X, s, \omega}$ to refer to the tangential structure obtained by composing the map (2.12a) with the map $B O \times X \rightarrow B O$ given by projection onto the first factor.

Lemma 2.13. $(X, s, \omega)$-twisted spin structures are in natural bijection with $\xi_{X, s, \omega}$-structures.
Similarly, one can study twisted orientations.
Definition 2.14 (Olbermann [Olb07, §1.4]). Let $X$ be a space and $s \in H^{1}(X ; \mathbb{Z} / 2)$. An $(X, s)$ twisted orientation on a vector bundle $V \rightarrow M$ is data of a map $f: M \rightarrow X$ and a trivialization of $w_{1}(V)-f^{*}(s)$.

Just as in Definition 2.11 and Lemma 2.13, we let $\xi_{X, s}$ denote the tangetial structure given by composing the fiber of $w_{1}+s: B \mathrm{O} \times X \rightarrow K(\mathbb{Z} / 2,1)$ with the projection $B \mathrm{O} \times X \rightarrow B \mathrm{O}$; then $\xi_{X, s}$-structures are in natural bijection with $(X, s)$-twisted orientations.

Therefore we are in the business of classifying invertible field theories of $\left(B G_{b}, V\right)$-twisted spin manifolds. These were classified by Freed-Hopkins-Teleman [FHT10] in terms of Reinhardt bordism [Rei63], also called SKK bordism [KKNO73] or Madsen-Tillmann bordism [MT01, MW07]. However, because we are interested in anomalies of unitary field theories, we can make the simplifying assumption that the anomaly invertible field theory comes with data of reflection positivity, the

Wick-rotated analogue of unitarity. Freed-Hopkins [FH21] classify reflection positive invertible topological field theories in terms of (ordinary) bordism. ${ }^{4}$ Before we state Freed-Hopkins' precise result, we need a little more notation.

Definition 2.15. The Pontrjagin dual of the sphere spectrum is the spectrum $I_{U(1)}$ representing the generalized cohomology theory $I_{\mathrm{U}(1)}^{*}$ whose value on a space or spectrum $X$ is

$$
\begin{equation*}
I_{\mathrm{U}(1)}^{n}(X):=\operatorname{Hom}\left(\pi_{n}^{s}(X), \mathrm{U}(1)\right) . \tag{2.16}
\end{equation*}
$$

For any spectrum $E$, we let $I_{\mathrm{U}(1)} E:=\operatorname{Map}\left(E, I_{\mathrm{U}(1)}\right)$; the generalized cohomology theory defined by $I_{\mathrm{U}(1)} E$ is

$$
\begin{equation*}
\left(I_{\mathrm{U}(1)} E\right)^{n}(X) \cong \operatorname{Hom}\left(E_{n}(X), \mathrm{U}(1)\right) \tag{2.17}
\end{equation*}
$$

It is not trivial that (2.16) and (2.17) satisfy the Eilenberg-Steenrod axioms: the proof makes use of the fact that $\mathrm{U}(1)$ is an injective abelian group. In fact, one can define $I_{A}$ analogously for any injective abelian group $A$, and it is common to use $\mathbb{Q} / \mathbb{Z}, \mathbb{R} / \mathbb{Z}$, or $\mathbb{C}^{\times}$in place of $U(1)$. Brown-Comenetz [BC76] were the first to consider a spectrum of this sort, with $A=\mathbb{Q} / \mathbb{Z}$, and so $I_{\mathbb{Q} / \mathbb{Z}}$ is sometimes called the Brown-Comenetz dual of the sphere spectrum; the use of $A=\mathrm{U}(1)$ is more common in physics applications, beginning with Bunke-Schick [BS13, §4.2.3].
Definition 2.18. Recall that the bordism groups $\Omega_{*}^{\xi}(X)$ are the generalized homology theory associated to a spectrum $M T \xi$. We use the notation $\mho_{\xi}^{*}$ to denote the generalized cohomology theory represented by $I_{\mathrm{U}(1)} M T \xi .^{5}$
(2.17) thus implies $\mho_{\xi}^{n}$ is a group of bordism invariants of $n$-dimensional $\xi$-manifolds.

Theorem 2.19 (Freed-Hopkins [FH21, Theorem 8.29]). There is a natural isomorphism from the group of isomorphism classes of reflection positive, invertible, $n$-dimensional TFTs $\alpha$ on manifolds with $\xi$-structure such that $\alpha\left(S^{n}\right)=1$ to $\mho_{\xi}^{n}$.

Here we need to specify a $\xi$-structure on $S^{n}$; we use the one induced by the trivial $\xi$-structure on $\underline{\mathbb{R}}^{n+1} \rightarrow S^{n}$, together with the isomorphism $T S^{n} \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$. This condition on $\alpha\left(S^{n}\right)$ will not play an important role in our paper.

Theorem 2.19 and Ansatz 2.9 tell us that to study anomalies of fermionic symmetries, we should compute $\mho_{\xi}^{4}$ when $\xi$ is a $\left(B G_{b}, V\right)$-twisted spin structure. It is equivalent to compute the Pontrjagin-dual bordism groups. An element in the bordism group associated to the tangential structure in Ansatz 2.9 is given by the following triple of data:

- a manifold $M$;
- a $G_{b}$-bundle $P$ on $M$, specified by a map $f: M \rightarrow B G_{b}$;
- a spin-structure $\zeta$ on $f^{*}(V) \oplus T M$.

In particular, to choose a spin-structure on $f^{*}(V) \oplus T M$, we must have

$$
\begin{equation*}
w_{1}(T M)=f^{*}(s), \quad w_{2}(T M)=w_{1}(T M)^{2}+f^{*}(\omega) \tag{2.20}
\end{equation*}
$$

In many situations we will not explicitly state the last two data when talking about an element in the bordism group, as they can be easily recovered from context. As an explicit example, in $\S 4$ we focus

[^3]on the $\mathbb{Z} / 4^{T f}$ symmetry. From the point of view of Ansatz 2.9, the associated tangential structure is a $(B \mathbb{Z} / 2,3 \sigma)$-twisted spin-structure, where $\sigma$ is the tautological line bundle on $B \mathbb{Z} / 2 \cong B \mathrm{O}(1)$. This is simply the notion of a pin ${ }^{+}$structure, as shown by Stolz [Sto88, §8], and using Eq. (2.20), we obtain the usual characteristic class condition $w_{2}(T M)=0$ for a pin ${ }^{+}$manifold $M$.

To compute twisted spin bordism or cobordism groups, we can use Lemma 2.8 to deduce that the $(X, V)$-twisted spin bordism groups are naturally isomorphic to $\Omega_{*}^{\mathrm{Spin}}\left(X^{V-r_{V}}\right)$, where $r_{V}$ is the rank of $V$ and $X^{E}$ denote the Thom spectrum of a virtual vector bundle $E \rightarrow X .{ }^{6}$ Then we can start applying the machinery of spectral sequences to do the calculation. In $\S 5.1$, we will also do the calculation with the help of the Smith homomorphism, as developed in [HKT20, DDK ${ }^{+} 23$ ].

Remark 2.21. In this paper, we are interested in anomalies of $(2+1) \mathrm{d}$ theories. The corresponding bordism groups, which are in degree 4, can contain free summands, so their Pontrjagin duals can contain $U(1)$ summands. From the anomaly classification perspective, these summands are irrelevant, as the useful invariant is the deformation class of the anomaly. However, these $U(1)$ summands have a useful physical meaning: they describe Hall conductance, which will be discussed in Appendix A.1.
2.2. Fermionic Topological Order with Symmetry Action. In this subsection, we give a quick review of the mathematical setup of (2+1)-d fermionic topological order with symmetry action, and list all the relevant data involved in the calculation of anomaly indicators. For a more comprehensive review of these concepts and notations, see [BBCW19, EGNO16, ENO10, Tur94, Sel11, BK01], and [GV17, BB22b, ABK21] for a targeted review of fermionic topological order and super modular tensor category.

Definition 2.22. A (2+1)-d fermionic topological order is a super modular tensor category (superMTC). This is a braided fusion category with Müger center given by the category of super vector spaces. ${ }^{7}$

Remark 2.23. It is well-known that a $(2+1)$-d bosonic topological order is a unitary modular tensor category (unitary-MTC). Another way to interpret a super-MTC is that it is almost a unitary-MTC, except that the object which has trivial double braiding with all other objects has endomorphisms given by $s \mathcal{V}$ ect, rather than $\mathcal{V}$ ect.

We now describe all the relevant data of a super-MTC $\mathcal{C}$. First, there is a finite set of simple objects $a$. They are referred to as (simple) anyons in the context of topological orders. Moreover, there is a special anyon $\psi$ in the Müger center which represents the local fermion. The set of morphisms $\operatorname{Hom}(a, b)$ between two objects $a$ and $b$ forms a $\mathbb{C}$-linear vector space. In the context of bosonic topological order, $\operatorname{Hom}(a, b)$ can be viewed as the Hilbert space of states associated to a 2-sphere that hosts anyons $a$ and $\bar{b}$. For fermionic topological order, we can have a local fermion $\psi$ in the background. Hence, it is sometimes useful to consider the $\mathbb{Z} / 2$-graded Hilbert space $\operatorname{Hom}(a, b) \oplus \operatorname{Hom}(a \times \psi, b)$, with the grading denoting the background fermion number. This can be viewed as the Hilbert space of states associated to a 2 -sphere that hosts anyons $a$ and $\bar{b}$ in fermionic topological order.

[^4]$\mathcal{C}$ also has the structure of fusion and braiding. Fusion means that there is a bifunctor $\times$ such that acting it on anyons $a$ and $b$ gives
\[

$$
\begin{equation*}
a \times b \cong \bigoplus_{c} N_{a b}^{c} c \tag{2.24}
\end{equation*}
$$

\]

where $N_{a b}^{c}$ is interpreted as the dimension of the channels of how two anyons $a$ and $b$ fuse into a third anyon $c$.

There are two related vector spaces, $V_{a b}^{c}$ and $V_{c}^{a b}$, referred to as the fusion and splitting vector spaces, respectively. The two vector spaces are dual to each other, and depicted graphically as:

$$
\begin{equation*}
\left.\left(d_{c} / d_{a} d_{b}\right)^{1 / 4}{ }_{c}^{c} / d_{a} d_{b}\right)^{1 / 4} \underbrace{\mu}_{c}=\left\langle a, b ;\left.c\right|_{\mu} \in V_{a b}^{c},\right. \tag{2.25}
\end{equation*}
$$

where $\mu=1, \ldots, N_{a b}^{c}, d_{a}$ is the quantum dimension of $a$, and the factors $\left(\frac{d_{c}}{d_{a} d_{b}}\right)^{1 / 4}$ are a normalization convention for the diagrams.

More generally, for any integer $n$ and $m$ there are vector spaces $V_{b_{1}, b_{2}, \ldots, b_{m}}^{a_{1}, a_{2}, \ldots, a_{n}}$, which are referred to as the fusion space of $m$ anyons into $n$ anyons. These vector spaces have a natural basis in terms of tensor products of the elementary splitting spaces $V_{c}^{a b}$ and fusion spaces $V_{a b}^{c}$. For instance, we have

$$
\begin{equation*}
V_{d}^{a b c} \cong \sum_{e} V_{e}^{a b} \otimes V_{d}^{e c} \cong \sum_{f} V_{d}^{a f} \otimes V_{f}^{b c} \tag{2.27}
\end{equation*}
$$

The two vector spaces are related to each other by a basis transformation referred to as $F$-symbols, which is diagrammatically shown as follows


There is a trivial anyon denoted by 1 such that $1 \times a=a \times 1=a$. We denote $\bar{a}$ as the anyon conjugate to $a$, for which $N_{a \bar{a}}^{1}=1$, i.e.

$$
\begin{equation*}
a \times \bar{a}=1 \oplus \cdots \tag{2.29}
\end{equation*}
$$

Note that $\bar{a}$ is unique for a given $a$.
The $R$-symbols define the braiding properties of the anyons, and can be defined via the the following diagram:

$$
\begin{equation*}
\left.\oint_{c}^{a}=\sum_{\nu}\left[R_{c}^{a b}\right]_{\mu \nu}^{a}\right\}_{c \uparrow}{ }^{a} . \tag{2.30}
\end{equation*}
$$

Under a basis transformation, $\Gamma_{c}^{a b}: V_{c}^{a b} \rightarrow V_{c}^{a b}$, the $F$ and $R$ symbols change according to:

$$
\begin{align*}
& F_{d e f}^{a b c} \rightarrow \tilde{F}_{d}^{a b c}=\Gamma_{e}^{a b} \Gamma_{d}^{e c} F_{d e}^{a b c}\left[\Gamma_{f}^{b c}\right]^{\dagger}\left[\Gamma_{d}^{a f}\right]^{\dagger} \\
& R_{c}^{a b} \rightarrow \tilde{R}_{c}^{a b}=\Gamma_{c}^{b a} R_{c}^{a b}\left[\Gamma_{c}^{a b}\right]^{\dagger} . \tag{2.31}
\end{align*}
$$

where we have suppressed splitting space indices and dropped brackets on the $F$-symbol for clarity of notation. In this paper, we refer to this basis transformation as a vertex basis transformation.

On the other hand, physical quantities, like the topological twist $\theta_{a}$ and the modular $S$-matrix $S_{a b}$, should always be basis-independent combinations of the data. The topological twist $\theta_{a}$ is defined via the diagram:

$$
\begin{equation*}
\theta_{a}=\theta_{\bar{a}}=\sum_{c, \mu} \frac{d_{c}}{d_{a}}\left[R_{c}^{a a}\right]_{\mu \mu}=\frac{1}{d_{a}} \tag{2.32}
\end{equation*}
$$

Finally, the modular $S$-matrix $S_{a b}$, is defined as

$$
\begin{equation*}
S_{a b}=D^{-1} \sum_{c} N_{\bar{a} b}^{c} \frac{\theta_{c}}{\theta_{a} \theta_{b}} d_{c}=\frac{1}{D} \tag{2.33}
\end{equation*}
$$

where $D=\sqrt{\sum_{a} d_{a}^{2}}$ is the total dimension.
In a super-MTC $\mathcal{C}$, we have a special anyon $\psi$ which physically corresponds to the local fermion. 1 and $\psi$ form the Müger center $s \mathcal{V}$ ect, meaning that we must have $\theta_{\psi}=-1, \psi \times \psi=1$ and $\psi$ braids trivially with all anyons in $\mathcal{C}$. This also implies that the set of anyon labels of a super-MTC decomposes as $\mathcal{C}=\mathcal{C}_{0} \times\{1, \psi\}$. However, it does not mean that $\mathcal{C}$ is simply the Deligne tensor product of some unitary-MTC $\mathcal{C}_{0}$ with the Müger center $\{1, \psi\}$. Still, the $S$-matrix of $\mathcal{C}$ does have a decomposition:

$$
S=\tilde{S} \otimes \frac{1}{\sqrt{2}}\left(\begin{array}{ll}
1 & 1  \tag{2.34}\\
1 & 1
\end{array}\right)
$$

where we demand that $\tilde{S}$ is unitary, hence we have "modular" in the terminology "super modular tensor category".

Now we want to equip $\mathcal{C}$ with a group action.
Definition 2.35 (Galindo-Venegas-Ramírez [GV17, Definition 3.11]). A fermionic action of a fermionic symmetry $\left(G_{b}, s, \omega\right)$ on a super-MTC $\mathcal{C}$ is a categorical $G_{b}$ action on $\mathcal{C}$, such that $\psi$ is preserved under $G_{b}$ and the action of $G_{b}$ on $\psi$ canonically corresponds to $\omega \in H^{2}\left(B G_{b} ; \mathbb{Z} / 2\right)$.

See also Bulmash-Barkeshli [BB22b] for a related but inequivalent definition,
We break down this definition into pieces and give concrete formulas to represent these conditions. First of all, given an element $\mathbf{g} \in G_{b}$, we assign a functor $\rho_{\mathbf{g}}$ to it. And we say that the data of the functors $\rho_{\mathbf{g}}$ defines a homomorphism $\rho: \operatorname{Aut}(\mathcal{C})$.
$\mathbf{g}$ can permute the anyons and we use $\mathbf{g}_{a}$ to denote the (simple) anyon we get after the $\mathbf{g}$ action on the (simple) anyon labeled by $a$. According to the value of $s(\mathbf{g})$ as in Eq. (2.3), $\mathbf{g}$ is either a unitary or an anti-unitary element, and is mapped to either a unitary or anti-unitary monoidal functor, respectively. We also introduce the related notation $\varsigma(\mathbf{g})$

$$
\varsigma(\mathbf{g})= \begin{cases}1 & \text { if } \mathbf{g} \text { is unitary }  \tag{2.36}\\ * & \text { if } \mathbf{g} \text { is anti-unitary }\end{cases}
$$

where $*$ denotes complex conjugation. Moreover, as requested, the local fermion $\psi$ is preserved under every $\rho_{\mathbf{g}}$.

The action of $\rho_{\mathbf{g}}$ on the fusion space $V_{a b}^{c}$ can be written in terms of the following matrix form

$$
\begin{equation*}
\rho_{\mathbf{g}}|a, b ; c\rangle_{\mu}=\sum_{\nu} U_{\mathbf{g}}\left(\mathbf{g}_{a},{ }^{\mathbf{g}} b ;{ }^{\mathbf{g}} c\right)_{\mu \nu} K^{s(\mathbf{g})}\left|\mathbf{g}_{a}, \mathbf{g}_{b} ; \mathbf{g}_{c}\right\rangle_{\nu} \tag{2.37}
\end{equation*}
$$

where $U_{\mathbf{g}}\left(\mathbf{g}_{a}, \mathbf{g}_{b} ; \mathbf{g}_{c}\right)$ is an $N_{a b}^{c} \times N_{a b}^{c}$ matrix, and $K$ denotes complex conjugation which appears when $s(\mathbf{g})=1$ to account for the fact that the action $\rho_{\mathbf{g}}$ is now $\mathbb{C}$-anti-linear. Thus, under a vertex basis transformation, $\Gamma_{c}^{a b}: V_{c}^{a b} \rightarrow V_{c}^{a b}, U_{\mathbf{g}}(a, b ; c)_{\mu \nu}$ also transforms according to

$$
\begin{equation*}
\tilde{U}_{\mathbf{g}}(a, b, c)=\left[\Gamma_{\overline{\mathbf{g}}_{c}}^{\overline{\mathrm{g}}_{a} \overline{\mathbf{g}}_{b}}\right]^{\varsigma(\mathbf{g})} U_{\mathbf{g}}(a, b, c)\left[\left(\Gamma_{c}^{a b}\right)^{-1}\right] \tag{2.38}
\end{equation*}
$$

Here we introduce the shorthand notation $\overline{\mathbf{g}}=\mathbf{g}^{-1}$. We will call this set of data $U$-symbols.
To preserve the monoidal structure and braiding, the $F$ and $R$ symbols should transform according to:

$$
\begin{align*}
& U_{\mathbf{g}}\left({ }^{\mathbf{g}} a,{ }^{\mathbf{g}} b ;{ }^{\mathbf{g}} e\right) U_{\mathbf{g}}\left(\mathbf{g}_{e},{ }^{\mathbf{g}} c ;{ }^{\mathbf{g}} d\right) F_{\mathbf{g}_{d} \mathbf{g}_{e} \mathbf{g}_{f}{ }^{\mathbf{g}} U_{\mathbf{g}}}{ }^{-1}\left(\mathbf{g}_{b,}{ }^{\mathbf{g}} c ;{ }^{\mathbf{g}} f\right) U_{\mathbf{g}}^{-1}\left(\mathbf{g}_{a,}{ }^{\mathbf{g}} f ;{ }^{\mathbf{g}} d\right)=K^{s(\mathbf{g})} F_{d e f}^{a b c} K^{s(\mathbf{g})} \\
& U_{\mathbf{g}}\left({ }^{\mathbf{g}} b,{ }^{\mathbf{g}} a ;{ }^{\mathbf{g}} c\right) R_{\mathbf{g}_{c}}^{\mathbf{g}_{a}{ }_{b}} U_{\mathbf{g}}\left({ }^{\mathbf{g}} a,{ }^{\mathbf{g}} b ;{ }^{\mathbf{g}} c\right)^{-1}=K^{s(\mathbf{g})} R_{c}^{a b} K^{s(\mathbf{g})} . \tag{2.39}
\end{align*}
$$

where we have suppressed the additional indices that appear when $N_{a b}^{c}>1$. Likewise, the modular $S$-matrix $S_{a b}$ and $\theta_{a}$ should transform according to:

$$
\begin{align*}
S_{\mathbf{g}_{a} \mathbf{g}_{b}} & =S_{a b}^{\varsigma(\mathbf{g})}  \tag{2.40}\\
\theta_{\mathbf{g}_{a}} & =\theta_{a}^{\varsigma(\mathbf{g})} \tag{2.41}
\end{align*}
$$

Moreover, we choose $\rho_{\mathbf{1}}$ to always be the identity functor. When $G$ is continuous, we further choose $\rho_{\mathbf{g}}$ such that $\rho_{\mathbf{g}}$ 's for different $\mathbf{g}$ 's in the same connected component are the same functor.

Secondly, to account for the multiplication rule of $G_{b}$, there should be a natural isomorphism $\eta(\mathbf{g}, \mathbf{h})$ connecting $\rho_{\mathbf{g}} \circ \rho_{\mathbf{h}}$ with $\rho_{\mathbf{g h}}$ :

$$
\begin{equation*}
\eta(\mathbf{g}, \mathbf{h}): \quad \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}} \Longrightarrow \rho_{\mathbf{g h}} \tag{2.42}
\end{equation*}
$$

By the definition of natural isomorphism, first of all, for every anyon $a, \eta(\mathbf{g}, \mathbf{h})$ assigns an isomorphism $\eta_{\mathbf{g h}_{a}}(\mathbf{g}, \mathbf{h}) \in \operatorname{Hom}\left(\mathbf{g}\left({ }^{\mathbf{h}} \mathbf{a}\right),{ }^{\mathbf{g h}} \mathbf{a}\right)$ to ${ }^{\mathbf{g h}} a$. Therefore, for every simple anyon $a$, we must have ${ }^{\mathbf{g}}\left(\mathbf{h}_{a}\right)=\mathbf{g h}_{a}$ and $\eta_{a}(\mathbf{g}, \mathbf{h})$ is simply a $\mathrm{U}(1)$ phase factor. We will call this set of data $\eta$-symbols. Acting on the fusion space $V_{a b}^{c}$, we have the following consistency condition between $U$ and $\eta$-symbols

$$
\begin{equation*}
\frac{\eta_{a}(\mathbf{g}, \mathbf{h}) \eta_{b}(\mathbf{g}, \mathbf{h})}{\eta_{c}(\mathbf{g}, \mathbf{h})}=U_{\mathbf{g}}(a, b ; c)^{-1} K^{s(\mathbf{g})} U_{\mathbf{h}}\left(\overline{\mathbf{g}}_{a}, \overline{\mathbf{g}}_{b ;} \overline{\mathbf{g}}_{c}\right)^{-1} K^{s(\mathbf{g})} U_{\mathbf{g h}}(a, b ; c) \tag{2.43}
\end{equation*}
$$

To satisfy the group associativity on the nose, we impose the following constraint on $\eta$-symbols

$$
\begin{equation*}
\eta_{a}(\mathbf{g}, \mathbf{h}) \eta_{a}(\mathbf{g h}, \mathbf{k})=\eta_{a}(\mathbf{g}, \mathbf{h} \mathbf{k}) \eta_{\overline{\mathrm{s}}_{a}}(\mathbf{h}, \mathbf{k})^{\varsigma(\mathbf{g})} \tag{2.44}
\end{equation*}
$$

Considering the local fermion $\psi$, we further require that the cocycle $\eta_{\psi}(\mathbf{g}, \mathbf{h})$, which in fact represents an element $\left[\eta_{\psi}\right]$ in $H^{2}\left(B G_{b} ; \mathbb{Z} / 2\right)$, coincides with $\omega \in H^{2}\left(B G_{b} ; \mathbb{Z} / 2\right)$ in the definition of the fermionic symmetry, i.e.

$$
\begin{equation*}
\left[\eta_{\psi}\right]=\omega \tag{2.45}
\end{equation*}
$$

Given a set of functors $\left\{\rho_{\mathbf{g}}\right\}$ connected by some natural isomorphism $\eta_{0}(\mathbf{g}, \mathbf{h})$ as in Eq. (2.43), we may have different choices of $\eta(\mathbf{g}, \mathbf{h})$.

Definition 2.46. For the same set of functors $\left\{\rho_{\mathbf{g}}\right\}$, different solutions $\eta_{a}(\mathbf{g}, \mathbf{h})$ of Eq. (2.43), (2.44), and (2.45) are referred to as different symmetry fractionalization classes.

In fact, Eq. (2.44) and Eq. (2.45) may never be satisfied by any choice of $\eta(\mathbf{g}, \mathbf{h})$. Such requirements define two obstructions that take values in certain cohomology, which are referred to as the obstruction to symmetry fractionalization. When the two obstructions vanish, different
symmetry fractionalization classes form a torsor over $H_{[\rho]}^{2}\left(B G_{b} ; \mathcal{A} /\{1, \psi\}\right)$. Here $\mathcal{A}$ is the set of Abelian anyons in $\mathcal{C}$, which is thought of as a module of $G_{b}$ under the action $\rho$. These are discussed in detail in [GV17, BB22b, ABK21].

Finally, two different sets of functors $\rho_{\mathbf{g}}$ and $\tilde{\rho}_{\mathbf{g}}$ can be identified if they are connected by some natural isomorphism $\gamma(\mathbf{g})$

$$
\begin{equation*}
\gamma(\mathbf{g}): \rho_{\mathbf{g}} \Longrightarrow \tilde{\rho}_{\mathbf{g}} \tag{2.47}
\end{equation*}
$$

which we refer to as the symmetry action gauge transformation. This changes $U_{\mathbf{g}}(a, b ; c)$ and $\eta_{a}(\mathbf{g}, \mathbf{h})$ in the following way:

$$
\begin{align*}
U_{\mathbf{g}}(a, b ; c) & \rightarrow \frac{\gamma_{a}(\mathbf{g}) \gamma_{b}(\mathbf{g})}{\gamma_{c}(\mathbf{g})} U_{\mathbf{g}}(a, b ; c) \\
\eta_{a}(\mathbf{g}, \mathbf{h}) & \rightarrow \frac{\gamma_{a}(\mathbf{g h})}{\gamma_{a}(\mathbf{g})\left(\gamma_{\overline{\mathbf{g}}_{a}}(\mathbf{h})\right)^{\varsigma(\mathbf{g})}} \eta_{a}(\mathbf{g}, \mathbf{h}) . \tag{2.48}
\end{align*}
$$

Different gauge inequivalent choices of $\{\eta\}$ and $\{U\}$ characterize distinct symmetry fractionalization classes. In this paper we will always fix the gauge

$$
\begin{array}{r}
\eta_{1}(\mathbf{g}, \mathbf{h})=\eta_{a}(\mathbf{1}, \mathbf{g})=\eta_{a}(\mathbf{g}, \mathbf{1})=1 \\
U_{\mathbf{g}}(1, b ; c)=U_{\mathbf{g}}(a, 1 ; c)=1 \tag{2.49}
\end{array}
$$

In summary, a $(2+1)$-d fermionic topological order is described by a super-MTC $\mathcal{C}$, and a fermionic symmetry with data $\left(G_{b}, s, \omega\right)$ acting on $\mathcal{C}$ requires the data $\left\{\rho_{\mathbf{g}} ; U_{\mathbf{g}}(a, b ; c), \eta_{a}(\mathbf{g}, \mathbf{h})\right\}$ associated to $G_{b}$, satisfying various consistency conditions as in Eqs. (2.39), (2.43), (2.44) and (2.45).

## 3. Spin TFT: bosonization, conjectures and application to Fermionic Topological Orders

In this section, we build up the background to state the bosonization conjecture in Conjecture 3.33. After stating this we will discuss how to use the conjecture to arrive at Conjecture 3.45 which states that the anomaly of the fermionic topological order should be an invertible field theory. The former two points are the main focus of $\S 3.1$. With these formal statements in place, we will have established the groundwork to calculate the partition function for the anomaly of fermionic topological order. In $\S 3.2$, we present details of how to build the partition function from the contents of the super modular tensor category and the manifold generator of the bordism group.
3.1. Unpacking the Conjectures. Bosonization is a general correspondence between bosonic systems (whose fundamental degrees of freedom are bosonic) and fermionic systems (whose fundamental degrees of freedom are fermionic). After some background of higher category theory in §3.1.1 and boundary theories from relative field theory point of view in $\S 3.1 .2$, we formalise our notion of bosonization/fermionization in $\S 3.1 .3$ and $\S 3.1 .4$. Much of the details is aimed to incorporate the data of the global symmetry, the tangential structure, and the anomaly, in a convenient formalism for establishing anomaly indicators as the partition functions for invertible fermionic theories.
3.1.1. A little more category theory. Throughout this subsection, we work with $n$-dimensional TFTs valued in a symmetric monoidal $k$-category $\mathcal{C}_{k}$, where $1 \leq k \leq n$. In this section, by the dimension $n$ of a TFT we refer to the spacetime dimension; in other sections, this is typically written as $d+1$. We need $\mathcal{C}_{k}$ to satisfy the following two properties.
(1) $\mathcal{C}_{k}$ is additive (see Gaiotto-Johnson-Freyd [GJF19, §4.3] for the definition of additive higher categories), and there is a finite path integral construction for TFTs valued in $\mathcal{C}_{k}$ in the sense of Freed-Quinn [FQ93] for $k=1$ and Freed [Fre94] and Freed-Hopkins-LurieTeleman [FHLT09] (see also [Fre93, Fre95, Fre99]) for $k>1 .{ }^{8}$ The finite path integral is the mathematical instantiation of the procedure of gauging a finite symmetry of a topological field theory; this symmetry could include fermion parity.
(2) Let $\mathcal{C}_{k}^{\times}$denote the sub- $k$-category of invertible objects and invertible (higher) morphisms; $\mathcal{C}_{k}^{\times}$is a Picard $k$-groupoid, meaning its classifying space is canonically $\Omega^{\infty}$ of a connective spectrum $\left|\mathcal{C}_{k}^{\times}\right|$; see, e.g., $\left[\right.$Fre19, §6.5] for more information. We need $\left|\mathcal{C}_{k}^{\times}\right|$to be homotopy equivalent to the connective cover of $I_{\mathrm{U}(1)}$, the Pontrjagin dual of the sphere spectrum from Definition 2.15. This assumption on $\mathcal{C}_{k}$ allows us to lift $\mathrm{U}(1)$-valued bordism invariants into invertible TFTs valued in $\mathcal{C}_{k}$; see [FH21, Fre19] for more information.

If 1 denotes the tensor unit in $\mathcal{C}_{k}, \Omega \mathcal{C}_{k}:=\operatorname{End}_{\mathfrak{C}_{k}}(\mathbf{1})$ is a symmetric monoidal $(k-1)$-category satisfying the above two conditions, so if we have chosen $\mathcal{C}_{k}$, we define $\mathcal{C}_{k-1}:=\Omega \mathfrak{C}_{k}$.

Categories $\mathcal{C}_{k}$ satisfying the above two properties are known for $k \leq 2$ : for $k=1$, one may use $s \mathcal{V e c t} \mathbb{C}_{\mathbb{C}}$, the category of complex super vector spaces, and for $k=2$, one may use $s \mathcal{A l g} g_{\mathbb{C}}$, the Morita bicategory of complex superalgebras [Fre12, DG18]. For $k>2$ our choice of $\mathcal{C}_{k}$ is therefore an ansatz: it is a conjecture of Freed-Hopkins [FH21, §5.3] (see also Freed [Fre19, Remark 5.9, §6.8]), that such $\mathcal{C}_{k}$ exist. We will assume $\Omega^{k-2} \mathcal{C}_{k} \simeq s \mathcal{A} l g_{\mathbb{C}}$, which implies $\Omega^{k-1} \mathcal{C}_{k} \simeq s \mathcal{V}$ ect $t_{\mathbb{C}}$. We will often heuristically think of the objects of $\mathcal{C}_{k}$ as algebras, monoidal categories, etc., by analogy with the common use of monoidal $\mathbb{C}$-linear categories, resp. braided monoidal $\mathbb{C}$-linear categories to discuss 3d, resp. 4d TFTs.
3.1.2. G-symmetric theories as boundaries. Understanding the anomaly requires us to put the theory on the boundary of some (invertible) TFT. In this subsubsection we formalize this notion from the point of view of Atiyah-Segal-style TFT [Ati88, Seg88].

Let $F_{\text {triv }}: \mathcal{B o r d}{ }_{n+1}^{\xi} \rightarrow \mathcal{C}_{k}$ be the trivial $(n+1)$-dimensional TFT, i.e. the symmetric monoidal functor sending all objects to the tensor unit 1 and all (higher) morphisms to the identity. The following definition is a formalization of the notion of an $n$-dimensional boundary.

Definition 3.1 (Freed-Teleman [FT14, Definition 2.1]). Let $\alpha: \mathcal{B o r d}{\underset{n+1}{ }}_{\xi} \rightarrow \mathcal{C}_{k}$ be a topological field theory, and for the $(n+1)$-d trivial TFT $F_{\text {triv }}: \mathcal{B}$ ord $n_{n+1}^{\xi} \rightarrow \mathcal{C}_{k}$, let $\tau_{\leq n} F_{\text {triv }}$ be the restriction of $F_{\text {triv }}$ to the sub- $k$-category of $\mathcal{B}$ ord $d_{n+1}^{\xi}$ consisting of all objects and $j$-morphisms for $1 \leq j<k$, and only the identity $k$-morphisms. A topological field theory relative to $\alpha$ is a homomorphism

$$
\begin{equation*}
\mathfrak{Z}: \tau_{\leq n} F_{\text {triv }} \longrightarrow \tau_{\leq n} \alpha \tag{3.2}
\end{equation*}
$$

If we did not impose $\tau_{\leq n}$, all relative TFTs would be equivalences between $F_{\text {triv }}$ and $\alpha$, as the category $\mathcal{F} u n^{\otimes}\left(\mathcal{B} \operatorname{ord}_{n+1}, \mathcal{C}_{k}\right)$ is a $k$-groupoid (see, e.g., [EGNO16, Exercise 2.10.15] or [CR18, Lemma 2.13] for the case $k=1$, [Lur09b, Remark 2.4.7(a), Theorem 2.4.18] for the case $k=n$, and $[$ Fre13, $\S 6]$ for general $k$ ).

[^5]We may equivalently characterize a TFT relative to $\alpha$ as a homomorphism in the other direction:

$$
\begin{equation*}
\mathfrak{Z}^{\prime}: \tau_{\leq n} \alpha \longrightarrow \tau_{\leq n} F_{\text {triv }} \tag{3.3}
\end{equation*}
$$

For an absolute TFT $Z$ and an $n$-manifold $M$, the partition function of $Z$ on $M$ is a complex number; but if $\mathfrak{Z}$ is a TFT relative to $\alpha, \mathfrak{Z}(M)$ is an element of the state space $\alpha(M) \in s \mathcal{V}$ ect $\mathbb{C}_{\mathbb{C}}$.

Definition 3.4. Fix two tangential structures $\xi: B_{1} \rightarrow B O$ and $\eta: B_{2} \rightarrow B O$, and a map $\Phi: \eta \rightarrow \xi$, i.e. a map $B_{2} \rightarrow B_{1}$ commuting with the maps to $B O$. This induces a forgetful functor $f_{\Phi}: \mathcal{B}$ ord $m_{m}^{\eta} \rightarrow \mathcal{B}$ ord $d_{m}^{\xi}$ for all $m$, which is symmetric monoidal.

Let $\alpha: \mathcal{B}$ ord ${ }_{n+1}^{\xi} \rightarrow \mathcal{C}_{k}$ be a topological field theory. Then an $\eta$-structured TFT relative to $\alpha$ is a TFT relative to $f_{\Phi} \circ \alpha: \mathcal{B}$ ord ${ }_{n+1}^{\eta} \rightarrow \mathcal{C}_{k}$.

This allows us to give additional structure to boundary theories, e.g. spin boundary theories of an oriented TFT.

Let $G$ be a finite group, and given a tangential structure $\xi: B \rightarrow B O$, let $\xi(G)$ denote the tangential structure $\xi \circ \pi_{1}: B \times B G \rightarrow B \mathrm{O}$, where $\pi_{1}$ is projection onto the first factor. Thus a $\xi(G)$-structure is data of a $\xi$-structure and a principal $G$-bundle.

There is an $(n+1)$-dimensional theory $F_{G}$ of unoriented manifolds, called finite gauge theory, obtained by performing the finite path integral to sum $F_{\text {triv }}: \mathcal{B}$ ord $d_{n+1}^{\text {id }(G)} \rightarrow \mathcal{C}_{k}$ over $G$-bundles [DW90, FQ93]. The following fact is a consequence of the cobordism hypothesis:
Lemma 3.5. There is an equivalence of categories between TFTs $Z: \mathcal{B o r d}_{n}^{\xi(G)} \rightarrow \mathcal{C}_{k-1}$ and $\xi$-structured TFTs $\mathfrak{Z}$ relative to $F_{G}$.

See, e.g., [FMT22, Remark 3.10], as well as [FRS02, FPSV15, FSV13]. Lemma 3.5 has the concrete interpretation that being a field theory with global $G$-symmetry is equivalent to being a boundary theory to $F_{G}$.

Remark 3.6 (Partition functions relative to $F_{G}$ ). Let $\mathfrak{Z}$ be an $n$-dimensional TFT relative to $F_{G}$ and $Z$ be the corresponding TFT of manifolds with a principal $G$-bundle under the equivalence of Lemma 3.5. In this remark, we spell out the way in which the partition functions of $\mathfrak{Z}$ and $Z$ are "the same," a perspective we learned from Gaiotto-Kulp [GK21, §2.1].

Let $M$ be a closed $n$-manifold. Then the natural transformation (3.2) implies the partition function $\mathfrak{Z}(M)$ is, rather than a complex number, a morphism $\varphi_{M}: \mathbf{1}(M) \rightarrow F_{G}(M)$. Here both $\mathbf{1}$ and $F_{G}$ are $(n+1)$-dimensional theories, $\mathbf{1}(M)$ and $F_{G}(M)$ are state spaces: specifically, the complex vector spaces $\mathbb{C}$ and $\mathbb{C}\left[\mathcal{B} u n_{G}(M)\right],{ }^{9}$ respectively. Therefore the morphism is determined by $\varphi_{M}(1)$, and we identify $\mathfrak{Z}(M)$ with the "partition vector" $\varphi_{M}(1) \in \mathbb{C}\left[\mathcal{B} u n_{G}(M)\right]$.

Meanwhile, $Z$ is an absolute TFT that requires the additional data of a principal $G$-bundle $P \rightarrow M$, such that isomorphic principal $G$-bundles have equal partition functions. Therefore $Z$ defines a function

$$
\begin{align*}
\psi_{M}: \pi_{0}\left(\mathcal{B} u_{G}(M)\right) & \longrightarrow \mathbb{C} \\
P & \longmapsto Z(M, P) . \tag{3.7}
\end{align*}
$$

In other words, $\psi_{M}$ is an element of $\mathbb{C}\left[\mathcal{B} u n_{G}(M)\right]$.
The relationship between $Z$ and $\mathfrak{Z}$ espoused by Lemma 3.5 is that $\varphi_{M}(1)=\psi_{M}$.

[^6]If objects of $\mathcal{C}_{k}$ are algebras of some kind, $F_{G}(\mathrm{pt})$ is the group algebra $\mathbf{1}[G]$, and Lemma 3.5 tells us that theories with a (nonanomalous) $G$-symmetry are equivalent data to (either left or right) $\mathbf{1}[G]$-modules.

Example 3.8. When $n+1=3$ and the target category is $\mathcal{C} a t_{\mathbb{C}}\left[E_{1}\right]$, the tricategory of monoidal $\mathbb{C}$-linear categories, the tensor unit is $\mathcal{V e c t} \mathbb{C}_{\mathbb{C}}$ and $F_{G}(\mathrm{pt})=\mathcal{V e c t}[G]$. In this case, the correspondence between $(2+1)$-d theories with a $G$-symmetry and $\mathcal{V e c t}[G]$-modules is spelled out by FreedTeleman [FT22, §3.1].

There is an analogue of $F_{G}$ called $F_{\text {Spin }}$, an oriented TFT obtained by performing the finite path integral to sum $F_{\text {triv }}: \mathcal{B}$ ord ${ }_{n+1}^{\text {Spin }} \rightarrow \mathcal{C}_{k}$ over all spin structures inducing a fixed orientation. This is a well-studied idea, often studied under the name "gauging fermion parity" and going back to work of Seiberg-Witten [SW86] and Álvarez-Gaumé-Ginsparg-Moore-Vafa [ÁGGMV86] interpreting the Gliozzi-Sherck-Olive projection on a superstring worldsheet as a sum over spin structures. We are specifically interested in dimension $n=3$, where $F_{\text {Spin }}$ is the theory $\mathcal{S}$ discussed by Johnson-Freyd [JF20, §2.2]; we discuss this in more detail in Definition 3.34. Lemma 3.5 generalizes to imply that $n$-dimensional spin TFTs are equivalent data to TFTs relative to $F_{\text {Spin }}$.
3.1.3. The Jordan-Wigner transform for $2 d$ TFTs. As a simplified case to the contents later in this paper, in this subsubsection, we consider $n=2$, and discuss the contents of bosonization and Jordan-Wigner transformation from our point of view. The contents here have been studied elsewhere in the literature from the field theoretic point of view and on the lattice [KT17, KTT19, CKR18a, GK21, Ina23, JSW20]. When $n=2$, a special feature is that we can avoid discussing nontrivial higher form symmetries after bosonization, making the discussion relatively clean. We will come back to this nontrivial point in §3.1.4.

Given a closed 2-manifold $\Sigma$ with spin structure $\mathfrak{s}$, let $a(\Sigma, \mathfrak{s}) \in\{ \pm 1\}$ denote its Arf invariant; this is a spin bordism invariant and defines an isomorphism $a: \Omega_{2}^{\text {Spin }} \rightarrow\{ \pm 1\}$. Spin structures on any manifold $M$ are a torsor over $H^{1}(M ; \mathbb{Z} / 2)$; thus, given a principal $\mathbb{Z} / 2$-bundle $P \rightarrow M$ and a spin structure $\mathfrak{s}$ on $M$, let $\mathfrak{s}+P$ denote the spin structure given by acting on $\mathfrak{s}$ by the class $w_{1}(P) \in H^{1}(M ; \mathbb{Z} / 2)$. Then the function taking a closed surface $\Sigma$, a spin structure $\mathfrak{s}$ on $\Sigma$, and a principal $\mathbb{Z} / 2$-bundle $P \rightarrow \Sigma$ to

$$
\begin{equation*}
a_{J W}:(\Sigma, \mathfrak{s}, P) \longmapsto a(\Sigma, \mathfrak{s}+P) \tag{3.9}
\end{equation*}
$$

is a bordism invariant $a_{J W}: \Omega_{2}^{\text {Spin }}(B \mathbb{Z} / 2) \rightarrow\{ \pm 1\}$, and therefore by property 2 of $\mathcal{C}_{k}$ lifts to define a 2-dimensional invertible TFT

$$
\begin{equation*}
z_{c}: \mathcal{B} \operatorname{ord}_{2}^{\operatorname{Spin} \times \mathbb{Z} / 2} \longrightarrow s \mathcal{A} l g_{\mathbb{C}} \tag{3.10}
\end{equation*}
$$

Because this theory is defined for manifolds with a spin structure and a principal $\mathbb{Z} / 2$-bundle, it exists relative to both $F_{\mathbb{Z} / 2}$ and $F_{\text {Spin }}$. That is, if we fix the spin structure, $z_{c}$ is a theory on manifolds with a $\mathbb{Z} / 2$-bundle, hence by Lemma 3.5 is equivalent data to a spin TFT relative to $F_{\mathbb{Z} / 2}$ (i.e. in the sense of Definition 3.4), meaning a natural transformation from (the truncations of) the trivial spin TFT to $F_{\mathbb{Z} / 2}$ regarded as a spin TFT. Since spin TFTs are equivalent to theories relevant to $F_{\text {Spin }}, z_{c}$ is also equivalent to data of a homomorphism

$$
\begin{equation*}
z_{c}^{\prime}: \tau_{\leq 2} F_{\text {Spin }} \longrightarrow \tau_{\leq 2} F_{\mathbb{Z} / 2 \times \text { Spin }} \tag{3.11a}
\end{equation*}
$$

Likewise, holding the $\mathbb{Z} / 2$-bundle fixed and letting the spin structure vary, $z_{c}$ defines a homomorphism

$$
\begin{equation*}
z_{c}^{\prime \prime}: \tau_{\leq 2} F_{\mathbb{Z} / 2 \times \text { Spin }} \longrightarrow \tau_{\leq 2} F_{\mathbb{Z} / 2} \tag{3.11b}
\end{equation*}
$$

Composing $z_{c}^{\prime \prime}$ and $z_{c}^{\prime}$, we obtain a homomorphism $\tau_{\leq 2} F_{\text {Spin }} \rightarrow \tau_{\leq 2} F_{\mathbb{Z} / 2}$, i.e. a defect between these two TFTs, akin to a bimodule between two algebras. In general, given three 3 -d TFTs $A, B$, and $C$, a $(B, A)$-defect $Z^{\prime}$, and a $(C, B)$-defect $Z^{\prime \prime}$, one can compose the homomorphisms defining $Z^{\prime}$ and $Z^{\prime \prime}$ as in Eq. (3.11) above, to form a $(C, A)$-defect which we denote $Z^{\prime \prime} \otimes_{B} Z^{\prime} .{ }^{10}$ If $A=F_{\text {triv }}$, a $(B, A)$-defect is the same thing as a TFT relative to $B$.

The point of all this is that tensoring with $\alpha$ exchanges theories relative to $F_{\mathbb{Z} / 2}$ (i.e. $\left(F_{\text {triv }}, F_{\mathbb{Z} / 2}\right)$ defects) with theories relative to $F_{\text {Spin }}$ (i.e. $\left(F_{\text {Spin }}, F_{\text {triv }}\right)$-defects):

Definition 3.12. Let $\mathcal{Z}_{f}: \mathcal{B o r d}{ }_{2}^{\text {Spin }} \rightarrow s \mathcal{A} l g_{\mathbb{C}}$ be a 2 d spin TFT. The bosonization of $\mathcal{Z}_{f}$ is the TFT

$$
\begin{equation*}
Z_{b}:=\mathcal{Z}_{f} \otimes_{F_{\mathrm{Spin}}} z_{c}: \mathcal{B} \operatorname{ord}_{2}^{\mathrm{SO} \times \mathbb{Z} / 2} \rightarrow s \mathcal{A} l g_{\mathbb{C}} \tag{3.13}
\end{equation*}
$$

Likewise, given a TFT $W_{b}: \mathcal{B}$ ord ${ }^{\mathrm{SO} \times \mathbb{Z} / 2} \rightarrow s \mathcal{A l} g_{\mathbb{C}}$, its fermionization is $\mathcal{W}_{f}:=W^{b} \otimes_{F_{\mathbb{Z} / 2}} z_{c}$. The result of the tensor product for bosonization is summarized in a sandwich construction in Figure 1.

Taking the tensor product of two theories $Z_{1}$ and $Z_{2}$ over $F_{\text {Spin }}$, resp. over $F_{\mathbb{Z} / 2}$ amounts to first forming the usual tensor product $Z_{1} \otimes Z_{2}$ of TFTs, then summing over spin structures, resp. principal $\mathbb{Z} / 2$-bundles. Freed-Quinn [FQ93, (2.9)] give a formula for the partition functions of finite path integral TFTs, allowing us to write down formulas for the partition functions of the bosonization or fermionization of a theory. First, the bosonization; let $\Sigma$ be a closed, oriented surface, $P \rightarrow \Sigma$ be a principal $\mathbb{Z} / 2$-bundle, $\mathcal{S} \operatorname{pin}(\Sigma)$ be the groupoid of spin structures on $\Sigma$, and $\mathcal{Z}_{f}: \mathcal{B o r d}_{2}^{\mathrm{Spin}} \rightarrow s \mathcal{A} l g_{\mathbb{C}}$ be a TFT. Then [Tho20, (2.3)]

$$
\begin{equation*}
Z_{b}(\Sigma, P)=\frac{1}{2^{\# \pi_{0}(\Sigma)}} \sum_{\mathfrak{s} \in \pi_{0} \operatorname{spin}(\Sigma)} a_{J W}(\Sigma, P, \mathfrak{s}) \mathcal{Z}_{f}(\Sigma, \mathfrak{s}) \tag{3.14a}
\end{equation*}
$$

Likewise, with $\Sigma$ as above, choose a spin structure $\mathfrak{s}$ on $\Sigma$ and a TFT $W_{b}: \mathcal{B o r d}{ }_{2}^{\mathrm{SO} \times \mathbb{Z} / 2} \rightarrow s \mathcal{A l} g_{\mathbb{C}}$; then [Tho20, (2.7)]

$$
\begin{equation*}
\mathcal{W}_{f}(\Sigma, \mathfrak{s})=\frac{1}{2^{\# \pi_{0}(\Sigma)}} \sum_{P \in \pi_{0} \mathcal{B} u n_{\mathbb{Z} / 2}(\Sigma)} a_{J W}(\Sigma, P, \mathfrak{s}) W_{b}(\Sigma, P) \tag{3.14b}
\end{equation*}
$$

The resemblance to the Fourier transform is no coincidence; we will return to this point in §3.1.6.
Definition 3.15 (Freed-Moore [FM06, §5.5]). For any $\lambda \in \mathbb{C}^{\times}$and $n \geq 0$, define the Euler TFT $e_{\lambda}$ to be the invertible, $n$-dimensional topological field theory whose partition function on a closed manifold $M$ is $\lambda^{\chi(M)}$, where $\chi$ denotes the Euler characteristic.

Remark 3.16. Freed-Moore's definition is equivalent to Definition 3.15, but more explicit; that Definition 3.15 suffices to define an invertible TFT follows from Freed-Hopkins-Teleman's classification of invertible field theories in terms of Reinhardt bordism invariants [FHT10] and the fact that the Euler number is a Reinhardt bordism invariant [Rei63, Theorem 1].

Lemma 3.17. The TFTs $z_{c} \otimes_{F_{\mathbb{Z} / 2}} z_{c}$ and $z_{c} \otimes_{F_{\text {Spin }}} z_{c}$ are both isomorphic to $e_{1 / 2}$.

[^7]

Figure 1. The figure depicts the procedure of bosonization, where $z_{c}$ is a right $F_{\text {Spin-module. The opposite procedure of fermionization starts with } Z_{b} \text { and inserts }}$ $z_{c}$ as a right $F_{\mathbb{Z} / 2}$-module.

Proof sketch. First, show that $z_{c} \otimes_{F_{z / 2}} z_{c}$ and $z_{c} \otimes_{F_{\text {spin }}} z_{c}$ are invertible by using a theorem of Schommer-Pries [SP18, Theorem 11.1] that reduces checking invertibility to checking the values of these TFTs on $S^{1}$ with either the bounding spin structure or the trivial $\mathbb{Z} / 2$-bundle. ${ }^{11}$ To do so, use the description of the state spaces of a finite path integral theory given by Freed-Quinn in [FQ93, (2.10)].

Recall from the discussion around Definition 2.15 that the definition of $I_{\mathrm{U}(1)}$ makes sense with an arbitrary injective abelian group in place of $\mathrm{U}(1)$. There is a homotopy equivalence from $\left|s \mathcal{A l g} g_{\mathbb{C}}^{\times}\right|$to the connective cover of $\Sigma^{2} I_{\mathbb{C} \times}$ [DG18, Proposition 4.20], ${ }^{12}$ so invertible field theories $Z: \mathcal{B o r d}{ }_{2}^{\xi} \rightarrow s \mathcal{A l} g_{\mathbb{C}}$ are equivalent data to homotopy classes of maps

$$
\begin{equation*}
\left|\overline{\mathcal{B} \text { ord } d_{2}^{\xi}}\right| \rightarrow \Sigma^{2} I_{\mathbb{C}^{\times}}, \tag{3.18}
\end{equation*}
$$

where $\overline{\mathcal{D}}$ denotes the Picard 2-groupoid completion of $\mathcal{D}$. The universal property of $I_{\mathbb{C}^{\times}}$is a natural isomorphism $\left[E, \Sigma^{m} I_{\mathbb{C}}\right] \cong \operatorname{Hom}\left(\pi_{m}(E), \mathbb{C}^{\times}\right)$just as in Definition 2.15, so homotopy classes of maps of the form in Eq. (3.18) (i.e. isomorphism classes of 2d invertible TFTs) are equivalent data to their partition functions. ${ }^{13}$ Therefore it suffices to check that the partition functions of $z_{c} \otimes_{F_{\mathrm{Z} / 2}} z_{c}$ and $z_{c} \otimes_{F_{\text {Spin }}} z_{c}$ on a closed, connected, oriented surface $\Sigma_{g}$ of genus $g$ are both equal to $e_{1 / 2}\left(\Sigma_{g}\right)=2^{2 g-2}$, which can be done using Eq. (3.14).
Corollary 3.19. The bosonization of the fermionization of a TFT $Z_{b}$ : $\mathcal{B o r d}_{2}^{\mathrm{SO} \times \mathbb{Z} / 2} \rightarrow \operatorname{sAlg}_{\mathbb{C}}$ is isomorphic to $Z_{b} \otimes e_{1 / 2}$, and likewise the fermionization of the bosonization of $\mathcal{Z}_{f}: \mathcal{B}$ ord ${ }_{2}^{\mathrm{Spin}} \rightarrow s \mathcal{A} l g_{\mathbb{C}}$ is isomorphic to $\mathcal{Z}_{f} \otimes e_{1 / 2}$.

[^8]The factor of $e_{1 / 2}$ is the analogue of the factor of $1 /(2 \pi)$ in the Fourier inversion formula.
Tensoring with $z_{c}$ is an equivalence of categories - however, this equivalence is not monoidal. One way to see this is to compare the groups of invertible objects on the two sides, which are not isomorphic. Instead, bosonization and fermionization behave like the Fourier transform: the Fourier transform is not a ring homomorphism; rather, it exchanges multiplication on one side with convolution on the other. There is a symmetric monoidal convolution product defined on 2 d $\mathrm{SO} \times \mathbb{Z} / 2 \mathrm{TFTs}$, and bosonization sends tensor product to convolution [Tho20, §2.2].

Remark 3.20 (More general tangential structures). The Arf invariant of spin surfaces generalizes to the Arf-Brown-Kervaire invariant of pin ${ }^{-}$surfaces [Bro71, KT90], so the theory $z_{c}$ extends to a theory on pin $^{-}$surfaces with a $\mathbb{Z} / 2$-bundle. One can therefore make the same definitions to define a bosonization-fermionization correspondence between $2 \mathrm{~d} \mathrm{pin}^{-} \mathrm{TFTs}$ and $2 \mathrm{~d} \mathrm{O} \times \mathbb{Z} / 2 \mathrm{TFTs}$, and the analogue of Corollary 3.19 holds. This is due to Thorngren [Tho20, $\S 2.2$ ]; see Kobayashi [Kob19, $\S 4]$ for an application and [Ste16, Tur20, MS23] for classification results of 2d pin ${ }^{-}$TFTs.

Likewise, an $r$-spin structure on a surface $\Sigma$ is the tangential structure described by the $r$ fold cover $\mathrm{SO}(2) \rightarrow \mathrm{SO}(2)$ (so 1-spin structures are orientations and 2-spin structures are spin structures in the usual sense). The Arf invariant extends to $r$-spin surfaces [GGP12, RW14], so since $r$-spin structures on $\Sigma$ extending a given orientation are a torsor over $H^{1}(\Sigma ; \mathbb{Z} / r)$ [RW14, §2.3], one can follow a similar line of argument to define a correspondence between $2 \mathrm{~d} \mathrm{SO} \times \mathbb{Z} / r$ TFTs and 2d $r$-spin TFTs, albeit with some subtleties because $r$-spin structures do not make sense above dimension 2. The state spaces of $F_{r-\text { Spin }}$ were constructed by Runkel [Run20, §6]. See [Nov15, Ste16, CS23, RS21, SS22, CMS23, Sze23] for more work on 2d $r$-spin TFT. From the point of view of physics, spin structures are to fermions as $r$-spin structures are to parafermions, and the parafermionic version of the Jordan-Wigner transform is due to Fradkin-Kadanoff [FK80]; see also recent work of Radičević [Rad18, §4.2], Chen-Haghighat-Wang [CHW23], and Duan-JiaLee [DJL23, §3].
3.1.4. Higher dimensions: bosonic shadows and bosonization conjectures. There has been a great deal of recent research generalizing the 2 d bosonization/fermionization correspondence of the previous subsubsection to higher dimensions. Different generalizations adopt different perspectives; we will use a construction of Gaiotto-Kapustin [GK16] in all dimensions, generalized by Tata-Kobayashi-Bulmash-Barkeshli [TKBB23] to general twisted spin structures in spacetime dimension 4. We encourage the reader to check out the related but different approaches of [Tho20, Kob22a, CKR18b].

We start with a fermionic symmetry presented by data $\left(G_{b}, s, \omega\right)$ as in Definition 2.1, and recall from Section 2.1 the definitions of $\left(B G_{b}, s, \omega\right)$-twisted spin structures $\xi_{B G_{b}, s, \omega}$ and $\left(B G_{b}, s\right)$-twisted orientations $\xi_{B G_{b}, s}$. We will eventually focus on $n=3,4$. The basic story is pretty similar to before: there is a 3 -dimensional kernel theory $z_{c}$, which is a defect between two 4 -dimensional theories, and the bosonization/fermionization correspondence is implemented by tensoring with $z_{c}$. However, there are three key differences.
(1) On the fermionic side, we use $\left(B G_{b}, s, \omega\right)$-twisted spin structures, and the bosonic side must also take this generalization into account.
(2) Instead of using principal $\mathbb{Z} / 2$-bundles to build $F_{\mathbb{Z} / 2}$, one has to use $\mathbb{Z} / 2$ (higher) gerbes. ${ }^{14}$ Another way to say this is that the ordinary $\mathbb{Z} / 2$ symmetry on the bosonic side of the correspondence is replaced with a $B^{n-2} \mathbb{Z} / 2$ symmetry, or an $(n-2)$-form $\mathbb{Z} / 2$ symmetry.
(3) Moreover, $z_{c}$ carries an anomaly with respect to this $B^{n-2} \mathbb{Z} / 2$ symmetry, as do the theories on the bosonic side of the correspondence. This means that rather than building $F_{\mathbb{Z} / 2}$ by summing the trivial theory over $\mathbb{Z} / 2$ higher gerbes, we must sum a nontrivial invertible theory, much like in the finite path integral construction of Dijkgraaf-Witten theory first constructed by Freed-Quinn [FQ93] and then generalized and extended in [Fre94, FHLT09, Mor15, Tro16, CRS19, SW19, SW20, Har20].
Recall that by property 2 , bordism invariants $a: \Omega_{n}^{\xi} \rightarrow \mathrm{U}(1)$ categorify to invertible TFTs $\alpha_{a}: \mathcal{B}$ ord $d_{n}^{\xi} \rightarrow \mathcal{C}$ such that $a$ is the partition function of $\alpha_{a}$. Also recall $\left(B G_{b}, s\right)$-twisted orientations (Definition 2.14) and the tangential structure $\xi_{B G_{b}, s}$ characterizing them. Then, let $\xi_{B G_{b}, s} \times$ $K(\mathbb{Z} / 2, n-1)$ denote the tangential structure which consists of a $\xi_{B G_{b}, s}$-structure and a map to $K(\mathbb{Z} / 2, n-1)$.
Definition 3.21. Given a fermionic symmetry written as $\left(G_{b}, s, \omega\right)$, let $\alpha_{0}: \mathcal{B}$ ord $d_{n+1}^{\xi_{B G, s} \times K(\mathbb{Z} / 2, n-1)} \rightarrow$ $\mathcal{C}_{k}$ be the invertible TFT characterized by the property that its partition function is

$$
\begin{equation*}
(M, f: M \rightarrow B G) \longmapsto \exp \left(\pi i \int_{M}\left(\mathrm{Sq}^{2}([B])+f^{*}(\omega) B\right)\right) \tag{3.22}
\end{equation*}
$$

Here $M$ is a closed $(n+1)$-manifold and $B$ is a $\mathbb{Z} / 2(n-2)$-gerbe, or equivalently a map $M \rightarrow B^{n-2} \mathbb{Z} / 2$. Isomorphism classes of this data are in natural bijection with classess $[B] \in$ $H^{n-2}(M ; \mathbb{Z} / 2)$.

The theory $\alpha_{0}$ can be thought of as a "classical higher Dijkgraaf-Witten theory," where instead of using a finite group, we use a finite higher group. ${ }^{15,16}$
Definition 3.23. Let $F_{B}: \mathcal{B}$ ord $d_{n+1}^{S O} \rightarrow \mathcal{C}_{k}$ denote the quantum Dijkgraaf-Witten theory obtained from $\alpha_{0}$ by using the finite path integral to sum over $\mathbb{Z} / 2(n-2)$-gerbes.

Like in Footnote 16, for $\mathcal{C}_{k}$ the existence of the finite path integral is a hypothesis, but one can construct $F_{B}$ as a TFT with target $n \mathcal{V}$ ect independently of our hypothesis.

In Gaiotto-Kapustin's version of bosonization/fermionization, both the kernel theory $z_{c}$ and the bosonic theories $Z_{b}$ are anomalous with anomaly $\alpha_{0}$; that is, they are boundary theories for $F_{B}$. Our next step is to define $z_{c}$.

[^9]For the rest of this section we fix a choice of specific spaces in the homotopy types $B \mathrm{O}$ and $B G_{b}$. For $H \in\left\{\mathrm{O}, G_{b}\right\}$, use the geometric realization of the nerve of the topological category with a single object $*$ and $\operatorname{Hom}(*, *) \cong H$. Fix cocycles $W_{1} \in Z^{1}(B \mathrm{SO} ; \mathbb{Z} / 2)$ and $W_{2} \in Z^{2}(B \mathrm{O} ; \mathbb{Z} / 2)$ representing the cohomology classes $w_{1}$, resp. $w_{2}$, and cocycles $S \in Z^{1}\left(B G_{b} ; \mathbb{Z} / 2\right)$ and $W \in Z^{2}\left(B G_{b} ; \mathbb{Z} / 2\right)$ representing $s$, resp. $\omega$.

In the rest of this section, given a closed $n$-manifold $M$ and a cocycle $\widetilde{B} \in Z^{n-2}(M ; \mathbb{Z} / 2)$, we let $\sigma(M, \widetilde{B}) \in\{ \pm 1\}$ denote the $G u$-Wen Grassmann integral of $\widetilde{B}$ on $M$, as defined by Tata [Tat20, $\S 7]$; see also [CR07, Cim09, GW14, GK16, Kob19] for other works giving special cases of this definition and [TKBB23, §IV] for a comparison of different definitions.

Definition 3.24 (Gaiotto-Kapustin [GK16]). Let $M$ be an $n$-manifold, possibly with boundary, together with data of
(1) a class $\widetilde{B} \in Z^{n-1}(M ; \mathbb{Z} / 2)$,
(2) a map $f: M \rightarrow B G_{b}$.
(3) a map $\gamma: M \rightarrow B$ O representing the classifying map of $T M$,
(4) a cochain $\chi \in C^{0}(M ; \mathbb{Z} / 2)$ such that $\delta(\chi)=\gamma^{*}\left(W_{1}\right)+f^{*}(S)$, and
(5) a cochain $\zeta \in C^{1}(M ; \mathbb{Z} / 2)$ such that $\delta(\zeta)=\gamma^{*}\left(W_{2}\right)+\gamma^{*}\left(W_{1}\right) \cup \gamma^{*}\left(W_{1}\right)+f^{*}(W)$.

Given this data, define

$$
\begin{equation*}
z_{c}(M, \widetilde{B}, \gamma, \zeta):=\sigma(M, \widetilde{B})(-1)^{\int_{M} \zeta \smile \widetilde{B}} \in \mathrm{U}(1) \tag{3.25}
\end{equation*}
$$

Remark 3.26. If one replaces $W_{2}$ with a different representative $W_{2}^{\prime}$, then there is some cochain $A \in$ $C^{1}(B S O ; \mathbb{Z} / 2)$ with $\delta(A)=W_{2}-W_{2}^{\prime}$. One can then replace the data $(\widetilde{B}, \gamma, \zeta)$ with $\left(\widetilde{B}, f, \zeta+\gamma^{*}(A)\right)$ to obtain the same value of $z_{c}$. Thus the choice of $W_{2}$, while important in order to have a definition, does not affect what follows; likewise for the chain-level choices of $W_{1}, W$, and $S$.

The data $(\gamma, f, \chi, \zeta)$ in Definition 3.24 induce a $\left(B G_{b}, s, \omega\right)$-twisted spin structure on $M$ refining the $\left(B G_{b}, s\right)$-twisted orientation picked out by $\gamma$ and $\chi$, by providing a trivialization of the cohomology classes $w_{1}(M)+f^{*}(s)$ and $w_{2}(M)+w_{1}(M)^{2}+f^{*}(\omega)$. Moreover, if $M$ admits a $\left(B G_{b}, s, w\right)$-twisted spin structure, then assigning to $(f, \zeta)$ the $\left(B G_{b}, s, \omega\right)$-twisted spin structure it induces defines a homotopy equivalence from the space of data $(f, \zeta)$ to the space of $\left(B G_{b}, s, \omega\right)$ twisted spin structures on $M$ refining the $\left(B G_{b}, s\right)$-twisted orientation defined by $\gamma$ and $\chi$. Similarly, there is a canonical homotopy equivalence between the space of data $(\gamma, f, \chi, \zeta, \widetilde{B})$ on $M$ and the space of pairs of $\left(B G_{b}, s, \omega\right)$-twisted spin structures refining the $\left(B G_{b}, s\right)$-twisted orientation defined by $(\gamma, \chi)$ and a $\mathbb{Z} / 2$ (higher) gerbe $B$. These equivalences are compatible with taking boundaries, meaning we have data of a homotopy equivalence from the $n$-dimensional bordism category of manifolds with data of $(\widetilde{B}, f, \gamma, \chi, \zeta)$ to $\mathcal{B}$ ord $n_{n}^{\xi_{B G_{b}, s, \omega} \times K(\mathbb{Z} / 2, n-1)}$, and we can ask whether the function $z_{c}$ is the partition function of a TFT.

Unfortunately, $z_{c}$ is not invariant enough to be a partition function. Given two equivalent data $\left(\widetilde{B}_{0}, f_{0}, \gamma_{0}, \chi_{0}, \zeta_{0}\right)$ and ( $\left.\widetilde{B}_{1}, f_{1}, \gamma_{1}, \chi_{1}, \zeta_{1}\right)$ we think of a path between these data as the data $(\bar{B}, \bar{f}, \bar{\gamma}, \bar{\chi}, \bar{\zeta})$ such that the pullback of this data to $M \times\{i\}$ is $\left(\widetilde{B}_{i}, f_{i}, \gamma_{i}, \chi_{i}, \zeta_{i}\right)$. Using this extension to $M \times[0,1]$, which we think of as a bordism between our two choices of data, Tata-Kobayashi-Bulmash-Barkeshli [TKBB23], show that going from the data on $M \times\{0\}$ to the data on $M \times\{1\}$, $z_{c}$ is multiplied by

$$
\begin{equation*}
(-1)^{\int_{M \times[0,1]}\left(\bar{f}^{*}(\omega) \bar{B}+\mathrm{Sq}^{2}(\bar{B})\right)} . \tag{3.27}
\end{equation*}
$$

When $G_{b}=1$, this was previously shown by Gaiotto-Kapustin [GK16].

This bordism is pictured in Figure 2.


Figure 2. The red and green faces show the two partition functions are off by a phase when traversing the path that connects the two equivalent sets of data $\left(\widetilde{B}_{0}, \gamma_{0}, \zeta_{0}\right)$ and $\left(\widetilde{B}_{1}, \gamma_{1}, \zeta_{1}\right)$.

So $z_{c}$ is, like many irregular verbs, inconsistent in a consistent way. Recall Freed-Quinn's construction [FQ93] of the state spaces of the Dijkgraaf-Witten theory $F_{B}$, in which one introduces a groupoid $\mathcal{G}(M)$ of data of cocycles representing a cohomology class, cycles representing the fundamental class, etc., and shows that the state spaces of the classical theory $\alpha_{0}$ are a line bundle over $\mathcal{G}(M)$, and that the state spaces of the quantum theory $F_{B}$, where we have summed over the classes $B$, are the sections of the line bundle.

The conclusion is that a complex number that is an invariant of ( $\widetilde{B}, f, \gamma, \chi, \zeta)$ but transforms as (3.27) is actually an element of the state space $F_{B}(M)$. That is:

Proposition 3.28 (Gaiotto-Kapustin [GK16], Tata-Kobayashi-Bulmash-Barkeshli [TKBB23]). $z_{c}$, regarded as a nonextended ${ }^{17}$ spin theory, has the structure of a boundary theory for $F_{B}$.

Gaiotto-Kapustin and Tata-Kobayashi-Bulmash-Barkeshli do not phrase their results in this way, but Proposition 3.28 follows from what they prove. In particular, their arguments apply to the case when $M$ has nonempty boundary.

Let $F_{C}$ be the result of applying the finite path integral to sum over $\left(B G_{b}, s, \omega\right)$-twisted spin structures with fixed $\left(B G_{b}, s\right)$-twisted orientation. Just like in the 2d case, we can encode the dependence of $z_{c}$ on the $\left(B G_{b}, s, \omega\right)$-twisted spin structure into the statement that $z_{c}$ is, as an oriented theory, an $\left(F_{C}, F_{B}\right)$-defect, and then bosonization and fermionization are hardly different from Definition 3.12.

Definition 3.29 (Tata-Kobayashi-Bulmash-Barkeshli [TKBB23, §VI, §VII]).
(1) Let $\mathcal{Z}_{f}$ be an $n$-dimensional TFT on manifolds with a $\left(B G_{b}, s, \omega\right)$-twisted spin structure. The bosonic shadow or bosonization of $\mathcal{Z}_{f}$ is $Z_{b}:=\mathcal{Z}_{f} \otimes_{F_{C}} z_{c}$, which is a boundary theory for $F_{B}$.
(2) Let $Z_{b}$ be a boundary theory for $F_{B}$; then its fermionization is $\mathcal{Z}_{f}:=Z_{b} \otimes_{F_{B}} z_{c}$, which is an $n$-dimensional $\left(B G_{b}, s, \omega\right)$-twisted spin TFT.

So bosonization and fermionization exchange (nonanomalous) ( $B G_{b}, s, w$ )-twisted spin TFTs with TFTs with a $\left(B G_{b}, s\right)$-twisted orientation, a $B^{n-2} \mathbb{Z} / 2$ symmetry, and the specific anomaly theory $\alpha_{0}$.

[^10]Remark 3.30. In dimension $2, z_{c}$ is an invertible TFT, and one naturally wonders whether this is true in all dimensions. Because $z_{c}$ is not defined absolutely, but only relative to $\alpha_{0},{ }^{18}$ it is less clear how to define invertibility, because if $M$ and $N$ are two boundary theories to the same theory $Z$, $M \otimes N$ is a $(Z \otimes Z)$-boundary condition, and we need extra data to obtain an absolute theory.

However, because the bordism invariant used to define $\alpha_{0}$ has order 2 , there is an equivalence $\alpha_{0} \otimes \alpha_{0} \cong F_{\text {triv }}$; after choosing such an equivalence, the tensor product of two $\alpha_{0}$ boundary theories $M$ and $N$ becomes an absolute TFT (i.e. it is a boundary theory of the trivial theory), and therefore we may ask whether $M \otimes N$ is trivial. For $M=N=z_{c}$, ultimately because $z_{c}$ is built from $\mathbb{Z} / 2$-valued cocycles, and so in this sense $z_{c}$ is an invertible (anomalous) field theory.

Some other works have studied invertible boundary theories, including [Ina21, Que21]; see also [ENO10, §4.1], [DY23b, Definition 1.3.1], and [Déc23, §4.1].

Remark 3.31 (More general tangential structures). In addition to the generalization in Tata-Kobayashi-Bulmash-Barkeshli that we have just surveyed, several other works have studied generalizations of Gaiotto-Kapustin's construction to other tangential structures. Bhardwaj [Bha17, $\S 3.3$ ] studies bosonization of 3 d pin ${ }^{+}$TFTs, and Kobayashi [Kob19, Kob22a] generalizes to both $\mathrm{pin}^{+}$and pin ${ }^{-}$TFTs in all dimensions. Gukov-Hsin-Pei [GHP21, §6], Hsin-Ji-Jian [HJJ22, §5], and Kobayashi [Kob22b] study analogues of bosonization and fermionization for field theories on manifolds with "Wu structure," i.e. a trivialization of a Wu class.
3.1.5. Adding symmetries and the conjecture. We are interested in using the bosonic shadow procedure to compute anomaly field theories, essentially by reducing the more complicated fermionic case to the better-understood bosonic case. In this subsubsection, we will often say "category" when referring to $k$-categories; whenever we do this, the value of $k$ will either be clear or can be understood from context.

Let $\widetilde{\alpha}: \mathcal{B o r d}{ }_{4}^{\xi_{B G_{b}, s, \omega}} \rightarrow \mathcal{C}_{k}$ be an invertible TFT, and suppose that we have data of a trivialization $\tau$ of the restriction

$$
\begin{equation*}
\left.\widetilde{\alpha}\right|_{\text {Spin }}: \mathcal{B o r d}_{4}^{\text {Spin }} \longrightarrow \mathcal{B o r d}_{4}^{\xi_{B G_{b}, s, \omega}} \xrightarrow{\widetilde{\alpha}} \mathcal{C}_{k}, \tag{3.32}
\end{equation*}
$$

where the first map is induced by regarding a spin structure as a $B G_{b}, s, \omega$ )-twisted spin structure with trivial $G_{b}$-bundle. Change of tangential structure induces a forgetful functor $\Phi_{\widetilde{\alpha}}$ from the category of boundary theories to $\widetilde{\alpha}$ to the category of boundary theories to $\left.\widetilde{\alpha}\right|_{\text {Spin }}$ - which, thanks to $\tau$, is the category of 3 -dimensional spin TFTs.

Let $\mathcal{Z}_{f}$ be an 3-dimensional spin TFT, and suppose that $\mathcal{Z}_{f}$ is in the image of $\Phi_{\tilde{\alpha}}$, i.e. that $\mathcal{Z}_{f}$ can be extended to an anomalous TFT on $\left(B G_{b}, s, \omega\right)$-twisted spin manifolds with anomaly $\widetilde{\alpha}$. Choose such an extension $\widetilde{\mathcal{Z}}_{f}$ of $\mathcal{Z}_{f}$. If $F_{\widetilde{\alpha}}: \mathcal{B}$ ord ${ }_{n}^{\xi_{B G_{b}, s} \times K(\mathbb{Z} / 2, n-1)} \rightarrow \mathcal{C}_{k}$ denotes the theory obtained from $\widetilde{\alpha}$ by summing over twisted spin structures with fixed principal $G$-bundle, then the $F_{\text {Spin }}$-boundary theory $\mathcal{Z}_{f}$ extends to the $F_{\widetilde{\alpha}}$-boundary theory $\widetilde{\mathcal{Z}}_{f}$.

Let $\widetilde{\beta}$ be the bosonization of $\widetilde{\alpha}$ in the sense of Definition 3.29 , and let $F_{\widetilde{\beta}}$ be the theory produced by summing $\widetilde{\beta}$ over $\mathbb{Z} / 22$-gerbes. We would like to say "the anomaly of the bosonization is the bosonization of the anomaly." One might hope to make that precise by asking that $z_{c}$ extends from an $\left(F_{\text {Spin }}, F_{B}\right)$-defect to an $\left(F_{\widetilde{\alpha}}, F_{\widetilde{\beta}}\right)$-defect. However, $F_{\widetilde{\beta}}$ is anomalous: it is a bosonization,

[^11]so carries the anomaly theory $\alpha_{0}$ from Definition 3.21. $F_{\widetilde{\alpha}}$ does not carry this anomaly. This means that without some kind of additional data, it does not make sense to refer to $\left(F_{\widetilde{\alpha}}, F_{\widetilde{\beta}}\right)$ defects: see Figure 3. By treating the combined module $\tilde{z}_{c}\left(F_{\widetilde{\beta}}\right)$ as a single unit, one can couple to it $F_{\widetilde{\alpha}}$ in such a way that the anomaly only residing on $F_{\widetilde{\beta}}$ is trivialized by $\tilde{z}_{c}$. In fact, $\widetilde{z}_{c}$ must contain an anomaly because the entire system of $F_{\widetilde{\alpha}} \boxtimes_{\widetilde{z}_{c}} F_{\widetilde{\beta}}$ must in the end be nonanomalous.


Figure 3. We want to state Conjecture 3.33 implements the slogan "the bosonization of the anomaly is the anomaly of the bosonization," but since bosonizations typically have nontrivial anomalies, this cannot be done naïvely: one needs extra data to reconcile the bulk theories 1 and $\alpha_{0}$, respectively the (trivial) anomaly of $F_{\widetilde{\alpha}}$ and the anomaly of $F_{\widetilde{\beta}}$.

To solve this, we do something which may look odd: we introduce a $\left(B G_{b}, s, \omega\right)$-twisted spin structure as an additional background field. The partition function (3.22) vanishes on $\left(B G_{b}, s, \omega\right)$ twisted spin manifolds - in fact, the $\left(B G_{b}, s, \omega\right)$-twisted spin structure provides a trivialization and therefore trivializes the anomaly. Though it may seem strange to introduce a twisted spin structure after bosonizing and getting rid of twisted spin structures, this is fine from the point of view of calculating an anomaly indicator on a 4 -manifold $X: X$ already has a twisted spin structure, so $\alpha_{0}$ is trivial on $X$ and therefore using this procedure works.

Conjecture 3.33 (Bosonization conjecture). Let $z_{c}$ denote Gaiotto-Kapustin's 3-dimensional $\left(F_{\text {Spin }}, F_{B}\right)$ defect (i.e. Tata-Kobayashi-Barkeshli-Bulmash's construction for $G_{b}=1$ ). Then, as theories of $\left(B G_{b}, s, \omega\right)$-twisted spin manifolds, $z_{c}$ canonically extends to an $\left(F_{\widetilde{\alpha}}, F_{\widetilde{\beta}}\right)$ defect $\widetilde{z}_{c}$.

The importance of $\tilde{z}_{c}$ will be the main focus of the next subsection. Its existence is crucial for realizing the anomaly of a fermionic topological order as a cobordism invariant.
3.1.6. Invertibility of the fermionic anomaly from bosonization. We will use the language of fusion 2 -categories for the purpose of discussing invertibility of the anomaly of the ( $2+1$ )-d fermionic TFT. The fusion 2-category has the right property to serve as a bulk theory for the fermionic TFT in consideration, and it cures the slight non-degeneracy on the boundary. When $\widetilde{\alpha}$ is the associated anomaly theory of a fermionic topological order $\mathcal{T}_{f}$, invertibility under the tensor product given by stacking is not obvious at first glance because $\widetilde{\alpha}$ is the anomaly for a slightly degenerate braided fusion category. A fully nondegenerate braided fusion category is an invertible element in the Morita 4-category of braided fusion categories [BJSS21] denoted $\mathcal{M}_{\text {or }}^{2}$ (2Vect). Then invertibility of the anomaly follows from the fact that the Crane-Yetter theory [CY93] is invertible [SP18].
would then guarantee that the anomaly theory is invertible. In principle, one would want to generalize the result in the nondegenerate case to understand if slightly degenerate braided fusion categories are invertible in $\mathcal{M} \operatorname{Mor}_{2}(2 s \mathcal{V}$ ect $) .{ }^{19}$ Doing this would imply that fermionic topological orders, or super MTCs (with a choice of central charge), define fully extended invertible framed 4d TFTs and would be a partial description of a "spin Crane-Yetter" construction. One can then use this to construct $\widetilde{\alpha}$ as an invertible fermionic TFT. Instead of taking this approach, we will elaborate on how Conjecture 3.33 can be used as credence for the fact that $\widetilde{\alpha}$ is invertible, but we do not prove this.

We begin at the level of $z_{c}$ which maps between $\mathcal{T}_{f}$ and its bosonized theory $\mathcal{T}_{b}$; this is the bottom map in Figure 4 . While $z_{c}$ is a $\left(F_{\text {Spin }}, F_{\mathbb{Z} / 2}\right)$ bimodule, it can be described by a spin- $\mathbb{Z} / 2^{f}$ gauge theory in (3+1)-d. Its dynamical field is a $\mathbb{Z} / 2$-valued 1-cocycle $\eta$ that solves $\delta \eta=w_{2}$, aka a spin structure.

In the following definition, we define the fusion 2 -category corresponding to a particular 4d TFT. The notion of fusion 2-categories is due to Douglas-Reutter [DR18, Definition 2.1.6]; see there for the definition.

Definition 3.34 ([JF20, §2.2]). The $(3+1)$-d spin- $\mathbb{Z} / 2^{f}$ gauge theory $\mathcal{S}$ is a nondegenerate braided fusion 2 -category with two components: the component of the identity is given by $2 s \mathcal{V} e c t$, and a magnetic component that contains two objects.

- The surface operators 1 and $c$ form the identity component. $c$ is called the Cheshire string or Kitaev chain as it is the condensation of the fermion $\psi$ in $s \mathcal{V}$ ect along a surface, and satisfies fusion rule $c^{2} \cong \mathbf{1} . \mathbf{1}$ is the condensation of the vacuum 1 in $s \mathcal{V}$ ect along a surface.
- The non-identity component has a magnetic object $m$, which is required for detecting $c$, and another object $m^{\prime}=m \otimes c$. Under fusion the magnetic object obeys $m^{2} \cong \mathbf{1}$.

Remark 3.35. The (3+1)-d spin- $\mathbb{Z} / 2^{f}$ gauge theory $\mathcal{S}$ has been explored in great detail in physics literature, which goes under the name of "(3+1)-d fermionic $\mathbb{Z} / 2$ gauge theory" or " $3+1$ )-d fermionic toric code" [BHK23, HOS04, FHH22, BCFV14, CH23].

The potential degeneracy of $\mathcal{T}_{f}$ is cured when coupled to $z_{c}$ because the line $\psi$ condenses in the bulk to $c$, and $m$ only existed in the bulk.

There is a choice of isomorphism between $Z_{(1)}\left(\mathcal{M} \operatorname{od}\left(\mathcal{T}_{f}\right)\right)$, the Drinfel'd center of $\mathcal{M} \operatorname{od}\left(\mathcal{T}_{f}\right)$, and $\mathcal{S}$ which corresponds to a choice of minimal modular extension. For each choice of isomorphism of $Z_{(1)}\left(\mathcal{M} \operatorname{lod}\left(\mathcal{T}_{f}\right)\right) \cong \mathcal{S}$ one can assign to $\mathcal{T}_{f} \otimes_{F_{\text {Spin }}} z_{c}=\mathcal{T}_{b}$ its anomaly theory $\widetilde{\beta}$; this is moving into the bulk in Figure 4. From the point of view of the tensor product on the side of $\widetilde{\alpha}$, the theory $\widetilde{\beta}$ is not invertible. As we have been thinking of bosonization as an analogue of the Fourier transform, invertibility with respect to the tensor product ought to be exchanged with invertibility with respect to some sort of convolution.

Definition 3.36 (Convolution kernel). Suppose $n=2 m$ and define $\kappa_{\text {conv }}$ : $\mathcal{B o r d}{ }_{n}^{\mathrm{O} \times B^{m} \mathbb{Z} / 2 \times B^{m} \mathbb{Z} / 2} \rightarrow$ $\mathcal{C}_{k}$ to be the invertible TFT characterized by the property that its partition function on a closed $n$-manifold $M$ with $\mathbb{Z} / 2(m-2)$-gerbes $P$ and $Q$ is

$$
\begin{equation*}
\kappa_{\mathrm{conv}}(M, P, Q):=\exp \left(\pi i \int_{M}[P]^{2}+[P][Q]\right) \tag{3.37}
\end{equation*}
$$

[^12]where $[P] \in H^{m}(M ; \mathbb{Z} / 2)$ denotes the cohomology class classifying the higher gerbe $P .^{20}$
Definition 3.38 (Thorngren [Tho20, (2.27)]). With $m$ and $n$ as above, let $Z_{1}$ and $Z_{2}$ be TFTs of manifolds with a $\mathbb{Z} / 2(m-2)$-gerbe. The convolution of $Z_{1}$ and $Z_{2}$ is the TFT
\[

$$
\begin{equation*}
Z_{1} \star Z_{2}: \mathcal{B} \text { ord }{ }_{n}^{\mathrm{O} \times B^{m} \mathbb{Z} / 2} \rightarrow \mathcal{C}_{k} \tag{3.39}
\end{equation*}
$$

\]

defined by summing the TFT

$$
\begin{equation*}
W(M, P, Q):=Z_{1}(M, P) \otimes Z_{2}(M, P \odot Q) \otimes \kappa_{\mathrm{conv}}(M, P, Q) \tag{3.40}
\end{equation*}
$$

over the first $(m-2)$-gerbe $P$, where $\odot$ is the tensor product of gerbes, which adds their cohomology classes.

Example 3.41. The partition function of $Z_{1} \star Z_{2}$ on an $n$-manifold $M$ with $(m-2)$-gerbe $Q$ is

$$
\begin{equation*}
\left(Z_{1} \star Z_{2}\right)(M, Q):=\chi_{\infty}\left(\mathcal{G e r b e}_{\mathbb{Z} / 2}^{m-2}(M)\right) \sum_{P \in \mathcal{G} \operatorname{erbe}_{\mathbb{Z} / 2}^{m-2}(M)} Z_{1}(M, P) Z_{2}(M, P \odot Q) \exp \left(\pi i \int_{M}[P]^{2}+[P][Q]\right) \tag{3.42}
\end{equation*}
$$

where $\mathcal{G e r b e} e_{\mathbb{Z} / 2}^{m-2}(M)$ denotes the $m$-groupoid of $(m-2)$-gerbes, generalizing the $m=1$ case of groupoid of principal $\mathbb{Z} / 2$-bundles. The function $\chi_{\infty}$ is the (higher) groupoid cardinality of a higher groupoid; see, for example, [Qui95, Lei08, BHW10].

Remark 3.43 (Symmetric monoidality of the convolution product). We predict, but do not attempt to prove, that the convolution product extends to a symmetric monoidal structure on the category of TFTs on $n$-manifolds with $\mathbb{Z} / 2(m-2)$-gerbes, and that the bosonization functor should admit a symmetric monoidal structure with respect to the usual tensor product on the fermionic side and the convolution product on the bosonic side. (For the latter conjecture, one must address somehow the anomaly on the bosonic side (Definition 3.21).)

Remark 3.44 (Generalizations of Definition 3.38). One can straightforwardly generalize Definition 3.38 to other finite cyclic groups $A$, albeit restricted to oriented manifolds with higher $A$-gerbes to ensure the integral in (3.37) is defined. It would be interesting to explore generalizations to (finite) higher abelian groups, similarly to the perspectives taken by Freed-Teleman [FT22, §9], Liu [Liu23], and Freed-Moore-Teleman [FMT22, §3.5] on electric-magnetic duality. It would also be interesting to generalize Definition 3.38 to odd-dimensional TFTs.

In coupling to a bulk $z_{c}$ we have gained nondegeneracy at the price of the product structure on $\widetilde{\beta}$ changing. This leads us to conjecturing invertibility of $\widetilde{\alpha}$ as a step in the bosonization conjecture:

Conjecture 3.45. The data of $\widetilde{\beta}$ being invertible under the convolution product is ported through the conjectured existence of $\widetilde{z}_{c}$ to $\widetilde{\alpha}$ which is invertible with respect to the regular tensor product.

The heart of Conjecture 3.33 is that composing the maps
should be equivalent to going into the bulk for $\mathcal{T}_{f}$ immediately. As a slogan, one could say that "the bosonization of the anomaly is the anomaly of the bosonization." This means that traversing between $\mathcal{T}_{f}$ and $\mathcal{T}_{b}$ in Figure 4 automatically maps the corresponding content of bulks, and vice

[^13]

Figure 4. The figure shows the bulk boundary systems on the fermionic and bosonic side. The bottom of the figure displays two (2+1)-d topological orders related by $z_{c}$, while the top of the figure shows the corresponding $(3+1) \mathrm{d}$ bulk theories. By Conjecture $3.33, z_{c}$ extends to $\widetilde{z}_{c}$ which relates a noninvertible theory with an invertible one.
versa when mapping between $\widetilde{\alpha}$ and $\widetilde{\beta}$ for the boundaries. Therefore, to find the anomaly field theory of a $\left(B G_{b}, s, w\right)$-twisted spin extension of a super MTC, we bosonize, and compute the bosonic anomaly indicator of the corresponding ( $B G_{b}, s$ )-twisted oriented theory with its $B \mathbb{Z} / 2$ symmetry. Once one has calculated the bulk theory for the bosonic theory, upon fermionizing one obtains the anomaly field theory of the original super MTC. This is the strategy we will use to find anomaly indicators of spin TFTs.
3.2. Partition Functions for super-MTC and Anomaly Indicators. In the context of bosonic topological orders, $\left[\mathrm{BBC}^{+} 19, \mathrm{BB} 22 \mathrm{~b}\right.$, YZ23a] presented the bulk TFT in terms of a generalized version of the Crane-Yetter model [CY93]. The bulk TFT can be identified by calculating the partition function on the generating manifolds for the relevant bordism group. These partition functions on the complete list of generating manifolds give a list of anomaly indicators. This is a gadget that characterizes the anomaly in terms of a specific element inside the relevant cobordism group, and is presented in terms of the data of the topological order and symmetry action at hand.

We wish to directly adapt these methods in the bosonic case to compute the anomaly indicators for fermionic topological orders. Specifically, we need to calculate the partition function of the anomaly theory on a complete list of generators of ( $B G_{b}, s, \omega$ )-twisted spin bordism groups, and the anomaly can be accordingly identified as an element in the (Pontryagin dual) cobordism group, $\mho_{\xi}^{4}$. According to Conjecture 3.33, we can use the same Crane-Yetter model to obtain the partition function of the bosonized theory, denoted as $\widetilde{\beta}$ in $\S 3.1$ and $\S 3.1 .6$. We are now tasked with calculating the partition function of $\widetilde{\alpha}$ that hosts the fermionic topological order on its boundary, as first proposed in [TKBB23].

As discussed around Eq. (2.20), for a fermionic symmetry given by data $\left(G_{b}, s, \omega\right)$, the data of a generating manifold contains three pieces of data: a manifold $M$, a $G_{b}$-bundle $P$ on $M$ specified by a map $f: M \rightarrow B G_{b}$, and a spin-structure $\zeta$ on $f^{*}(V) \oplus T M$ where $V$ satisfies Eq. (2.20). In this subsection, we will present the full data $M, P, \zeta$ as the argument of $\mathcal{Z}_{f}$, but later when dealing with specific examples we will usually omit $P$ and $\zeta$ to avoid clutter. For such a generating
manifold, using the bosonic shadows of Definition 3.29, the partition function for the spin TFT can be decomposed as

$$
\begin{equation*}
\mathcal{Z}_{f}(M, P, \zeta)=\frac{1}{\sqrt{\left|H^{2}(M ; \mathbb{Z} / 2)\right|}} \sum_{[\mathfrak{L}] \in H_{1}(M ; \mathbb{Z} / 2)} Z_{b}(M, P, \mathfrak{L}) z_{c}(M, \mathfrak{L}, \zeta) \tag{3.46}
\end{equation*}
$$

Here we use a slightly different notation for the arguments of $z_{c}$ that were introduced in Definition 3.24: instead of using $\widetilde{B}$ we use $\mathfrak{L}$ which represents the Poincaré dual. ${ }^{21}$ The partition function is written as the summation running over $[\mathfrak{L}] \in H_{1}(M ; \mathbb{Z} / 2)$.

Each summand is the multiplication of the bosonic shadow $Z_{b}$ and $z_{c}$. The bosonic shadows $Z_{b}(M, P, \mathfrak{L})$ are insensitive to $\zeta$, and are constructed in terms of the super-MTC data and the symmetry action. Moreover, we need to insert an extra fermion loop into the cycles of $M$ represented by $\mathfrak{L}$ when calculating $Z_{b}(M, P, \mathfrak{L})$. In contrast, $z_{c}(M, \mathfrak{L}, \zeta)$ is explicitly dependent on $\zeta$ and on neither the super-MTC data nor the symmetry action data.

Remark 3.47. A physical interpretation of $z_{c}(M, \mathfrak{L}, \zeta)$ can be given as follows. The vector space of a spin TFT associated with some $d$-dimensional manifold are $\mathbb{Z} / 2$-graded, with the grading denoting the fermion number. More precisely, $\mathfrak{L}$ represents the grading of the vector spaces associated with every 3 -cycle in the following way: if a 3 -cycle intersects $\mathfrak{L}$ an even (odd) number of times, the vector space associated with the 3 -cycle should have grading 0 (or 1 ). Hence the summation runs over all possible assignment of grading. When working with spin TFTs we must specify a spin-structure $\zeta$, which originates from the sign coming from the exchange of two fermionic operators. Given a specific $\zeta$, identical assignment of grading should give identical phase in front of the bosonic shadows $Z_{b}$, denoted by $z_{c}(M, \mathfrak{L}, \zeta)$.

In the presence of time-reversal symmetry, there is an explicit definition in terms of triangulation stated in e.g. [Kob19, Tat20, TKBB23]. For practicality, given a set of generators $\left[\mathfrak{L}_{i}\right] \in H_{1}(M ; \mathbb{Z} / 2)$, we can assign a certain phase $z_{c}\left(M, \mathfrak{L}_{i}, \zeta\right)$ to each $f_{i}$ according to $\zeta$ in the following way,

$$
z_{c}\left(M, \mathfrak{L}_{i}, \zeta\right)=\left\{\begin{array}{l} 
\pm 1 \text { if } \mathfrak{L}_{i} \text { is orientable, } \quad \text { i.e., } w_{1}\left(\mathfrak{L}_{i}\right)=0 \quad \bmod 2  \tag{3.48}\\
\pm i \text { if } \mathfrak{L}_{i} \text { is unorientable, i.e., } w_{1}\left(\mathfrak{L}_{i}\right)=1 \quad \bmod 2
\end{array}\right.
$$

In particular, for orientable $\mathfrak{L}_{i}$, the + or $-\operatorname{sign}$ tracks whether the cycle $\mathfrak{L}_{i}$ has a bounding or non-bounding spin-structure, respectively. According to [GK16, Kob19, TKBB23], $z_{c}$ is a quadratic refinement of a higher cup product pairing:

$$
\begin{equation*}
z_{c}\left(M, \mathfrak{L}_{i}+\mathfrak{L}_{j}, \zeta\right)=z_{c}\left(M, \mathfrak{L}_{i}, \zeta\right) z_{c}\left(M, \mathfrak{L}_{j}, \zeta\right)(-1)^{\int_{M} \hat{\mathfrak{L}}_{i} \cup_{2} \hat{\mathfrak{L}}_{j}} \tag{3.49}
\end{equation*}
$$

where $\hat{\mathfrak{L}}_{i, j} \in H^{3}(M, \mathbb{Z} / 2)$ are Pontrjagin dual to $\mathfrak{L}_{i, j}$. In this way, for a given $\zeta$, the extra phase $z_{c}(M, \mathfrak{L}, \zeta)$ can be identified for every $[\mathfrak{L}] \in H_{1}(M, \mathbb{Z} / 2)$,

To calculate $Z_{b}(M, P, \mathfrak{L})$, we can just directly port the rules developed in [YZ23a] for a given a handle decomposition of $M$. We emphasize that even though the input data for [YZ23a] is a bosonic theory described by a unitary-MTC, most aspects carries over to super-MTC. The only complication here is that we need to add an extra fermion loop on the cycle represented by $\mathfrak{L}$. For the reader's convenience, we copy the recipe in [YZ23a] so that we may subsequently use it to calculate the bosonic shadow $Z_{b}(M, P, \mathfrak{L})$, given the data of a super-MTC and some symmetry action on it.

[^14](1) Identify a handle decomposition of the manifold $M$. On each 1-handle add appropriate defects according to the $G_{b}$-bundle structure $P$. This step is identical for every $\mathfrak{L}$.
(2) The $S^{1}$ boundary of each 2 -handle is separated by the defects into segments. Associate an anyon $a$ to an arbitrary segment on the $S^{1}$ boundary of each 2-handle. The anyons on the other segments are related to $a$ by the $G_{b}$-actions given by the defects. Write down the $\eta$-factor coming from the natural isomorphism for $a$ that connects the functor of successive $G_{b}$-actions to the identity functor.
(3) Associate a dual vector $\left\langle a_{1}, \ldots ; b_{1},\left.\ldots\right|_{\mu \ldots} K^{s(\mathbf{g})} \text { and a vector }\left.\right|^{\mathbf{g}} a_{1}, \ldots ;{ }^{\mathrm{g}} b_{1}, \ldots\right\rangle_{\tilde{\mu} \ldots}$ to the two $D^{3}$ planes of the attaching region $S^{0} \times D^{3}$ of every 1-handle, where $a_{1}, \ldots$ and $b_{1}, \ldots$ are labels of anyons running out of and into the lower $D^{3}$ plane of the attaching region of the 1-handle. In the particular case of a nontrivial $\mathfrak{L}$, we should include a local fermion $\psi$ loop according to $\mathfrak{L}$. Write down the $U$-factor from
$$
\left\langle a_{1}, \ldots ; b_{1},\left.\left.\ldots\right|_{\mu \ldots} K^{s(\mathbf{g})} \rho_{\mathbf{g}}^{-1}\right|_{\mathbf{g}} ^{a_{1}}, \ldots ;{ }^{\mathbf{g}} b_{1}, \ldots\right\rangle_{\tilde{\mu} \ldots}=U_{\mathbf{g}}^{-1}\left(\mathbf{g}_{\left.a_{1}, \ldots ;{ }^{\mathbf{g}} b_{1}, \ldots\right)_{\tilde{\mu} \ldots, \mu \ldots} .}\right.
$$
(4) Given the prescribed anyon labels associated to the $S^{1}$ lines corresponding to 2-handles and vectors associated to the $D^{3}$ balls corresponding to 1-handles, possibly with some extra fermion loop in the presence of nontrivial $\mathfrak{L}$, evaluate the anyon diagram from the Kirby diagram $\langle K\rangle$ of $M$.
(5) Assemble the result for the bosonic shadow as follows:
\[

$$
\begin{align*}
Z_{b}(M, P, \mathfrak{L})=D^{-\chi+2\left(N_{4}-N_{3}\right)} \times \sum_{\text {labels }} & \left(\frac{\prod_{2 \text { handle } i} d_{a_{i}}}{\prod_{1 \text {-handle } x}\left(\prod_{2 \text {-handle } j \text { across } x} d_{a_{j}}\right)^{1 / 2}}\right.  \tag{3.51}\\
& \left.\times\left(\prod_{i}(\eta \text {-factors })_{i}\right) \times\left(\prod_{x}(U \text {-factors })_{x}\right) \times\langle K\rangle\right) .
\end{align*}
$$
\]

Here $N_{k}$ is the number of $k$-handles in the handle decomposition, and $\chi \equiv N_{0}-N_{1}+N_{2}-$ $N_{3}+N_{4}$ is the Euler number of $M$.

Lemma 3.52. The expression Eq. (3.51) is independent of the exact form of the handle decomposition, position of defects, and various gauge transformations.

Proof sketch. The proof is the same as in [YZ23a, §C.1, §C.2, §C.3], because the proof there simply uses the fact that the category under consideration is a pre-modular tensor category and does not reference the modularity property.

Moreover, the proof in [YZ23a, $\S \mathrm{C} .4]$ also shows that $Z_{1}(M,[0])$ (when there is no insertion of the fermion loop) is an invariant for oriented bordism.

The invariants we have just defined are the partition functions of a topological field theory, because we built them by fermionization. From Conjecture 3.33, we think this TFT is an anomaly theory, meaning it should be invertible. We do not have a general proof, though it is true in all examples we computed.

Conjecture 3.53 (Invertibility conjecture). The fermionic anomaly indicators we defined above are the partition functions of an invertible TFT.

Corollary 3.54. Assuming Conjecture 3.53, the fermionic anomaly indicators are $\left(B G_{b}, s, \omega\right)$ twisted spin bordism invariants.
Proof. Recall that $\xi_{B G_{b}, s, \omega}$ denotes the tangential structure corresponding to a $\left(B G_{b}, s, \omega\right)$-twisted spin structure, and let $\xi_{B G_{b}, s, \omega}(n)$ denote the pullback of $\xi_{B G_{b}, s, \omega}$ along $B \mathrm{O}(n) \rightarrow B \mathrm{O}$. Freed-Hopkins-Teleman [FHT10] showed that the partition functions of invertible TFTs are Reinhart bordism invariants, meaning the anomaly indicators define a homomorphism

$$
\begin{equation*}
\phi: \pi_{4}\left(\Sigma^{4} M T \xi_{B G, s, \omega}(4)\right) \longrightarrow \mathbb{C}^{\times} \tag{3.55}
\end{equation*}
$$

We would like to obtain ordinary bordism invariants. The obstruction to lifting a Reinhart bordism invariant $\phi$ to an ordinary bordism invariant is $\phi\left(S^{4}\right)$, where $S^{4}$ carries the tangential structure arising from the boundary of $D^{5}$ (in particular, it has a trivial $G_{b}$-bundle). Anomaly indicators on this manifold are equal to 1 , so we obtain an actual bordism invariant.

## 4. Warmup: $\mathbb{Z} / 4^{T f}$

As a warmup, in this section we consider the $\mathbb{Z} / 4^{T f}$ symmetry and rederive the anomaly indicator for the $\mathbb{Z} / 4^{T f}$ symmetry, which was first proposed in [WL17]. In light of Definition 2.1, the triple which defines what we call the $\mathbb{Z} / 4^{T f}$ symmetry is $G_{b}=\mathbb{Z} / 2$ with both $s$ and $\omega$ nontrivial. This symmetry can also be represented by the algebra $\mathcal{T}^{2}=(-1)^{F}$, where $\mathcal{T}$ is the generator of $\mathbb{Z} / 2$ time-reversal and $(-1)^{F}$ denotes fermion parity. In the 10 -fold way classification, this symmetry is in "class DIII".

It is well-known that the relevant bordism group for the $\mathbb{Z} / 4^{T f}$ symmetry is $\Omega_{4}^{\text {Pin }+} \cong \mathbb{Z} / 16$, generated by $\mathbb{R P}^{4}$ with nontrivial $\mathbb{Z} / 2$-bundle and either of its two pin ${ }^{+}$structures [Gia73, $\S 2$, Theorem 3.4(a)].


Figure 5. The Kirby diagram of $\mathbb{R P}^{4}$. The two blue spheres illustrate the attaching region of the 1-handle and the red lines illustrate the attaching region of the 2 -handle. The 1-handle is unorientable.

We can proceed to derive the anomaly indicator for the $\mathbb{Z} / 4^{T f}$ symmetry by calculating the partition function of the generating manifold $\mathbb{R P}^{4}$ according to the procedure outlined in $\S 3.2$. The result is summarized in the following proposition.

Proposition 4.1. The anomaly indicator of fermionic topological orders with the $\mathbb{Z} / 4^{T f}$ symmetry is given by

$$
\begin{equation*}
\mathcal{I}=\frac{1}{\sqrt{2} D} \sum_{a} d_{a} \theta_{a} \eta_{a} \tag{4.2}
\end{equation*}
$$

where

$$
\eta_{a}=\left\{\begin{array}{lr}
\eta_{a}(\mathcal{T}, \mathcal{T}), & \mathcal{T}_{a}=a  \tag{4.3}\\
i \eta_{a}(\mathcal{T}, \mathcal{T}) U_{\mathcal{T}}(a, \psi ; a \times \psi) F^{a, \psi, \psi}, & \mathcal{T}_{a=a \times \psi}, \\
0, & \text { otherwise }
\end{array}\right.
$$

This expression is first proposed in [WL17] and derived in [TKBB23, Appendix J], although the derivation there involves cell decomposition instead of handle decomposition and is hence rather involved.

Proof sketch of Proposition 4.1. We derive the formula for the anomaly indicator by calculating the partition function of $\mathbb{R} \mathbb{P}^{4}$. According to Eq. (3.46), $\mathbb{R} \mathbb{P}^{4}$ has $H_{1}\left(\mathbb{R} \mathbb{P}^{4} ; \mathbb{Z} / 2\right) \cong \mathbb{Z} / 2$, hence the partition function can be decomposed as the sum of two bosonic shadows. corresponding to whether or not we insert a fermion loop into the noncontractible cycle. $H^{2}\left(\mathbb{R} \mathbb{P}^{4} ; \mathbb{Z} / 2\right) \cong \mathbb{Z} / 2$ and thus there is a $\frac{1}{\sqrt{2}}$ factor in Eq. (3.46). Moreover, the minimal handle-decomposition of $\mathbb{R} \mathbb{P}^{4}$ contains 1 0 -handle, 1 1-handle, 12 -handle, 13 -handle, and 14 -handle, and its Kirby diagram is given in Figure 5.

(A)

(B)

Figure 6. Anyon diagrams from the Kirby diagram in Fig. 5, with no extra fermion loop (left) or one extra sanddune-colored fermion loop in the blue 1-handle (right). The red line illustrates the 2-handle, the blue circles illustrate the 1-handle, and the dark red lines illustrate morphisms. Note that in comparison with Fig. 5 where both segments flow upward, here one segment flows upward and another segment flows downward due to the nonorientable cycle.

When translating this Kirby diagram into anyon diagrams, we start with inserting no fermion loop into the diagram. We then need to label the 2-handle and the 1-handle by anyons and morphisms in a proper way. First, we label the 2-handle by anyons. Because of the nontrivial $\mathbb{Z} / 2$-bundle on $\mathbb{R P}^{4}$, anyons and morphisms are acted upon by $\mathcal{T}$ when crossing the 1 -handle, hence we label two red segments in Figure 5 by $a$ and ${ }^{\mathcal{T}} a$, respectively. Moreover, because the cycle is unorientable, in comparison to Figure 5 we need to flip the flow of one red segment when drawing the anyon diagram, as shown in Figure 6a. Next, we label the 1-handle by morphisms. On the

1-handle we need to associate a morphism in $\operatorname{Hom}\left(a,{ }^{\mathcal{T}} a\right)$, which is nonempty only when $a={ }^{\mathcal{T}} a$. In this way, the Kirby diagram can be translated to the anyon diagram in Figure 6a.

The $\eta$-factor associated to this diagram comes from $\rho_{\mathcal{T}}^{-1} \circ \rho_{\mathcal{T}}^{-1}$ acting on $\mathcal{T} a$, which gives $\eta \tau_{a}(\mathcal{T}, \mathcal{T})^{*}=\eta_{a}(\mathcal{T}, \mathcal{T})$. The $U$-factor associated to this diagram is simply 1. Finally, the anyon diagram in Figure 6a evaluates to $d_{a} \theta_{a}$. After carefully counting all the other factors involving quantum dimensions as in Eq. (3.51), we arrive at the expression of the first bosonic shadow $Z_{1}$,

$$
\begin{equation*}
Z_{1}=\frac{1}{D} \sum_{\left\{{ }^{\mathcal{T}_{a=a\}}^{a}}\right.} d_{a} \theta_{a} \times \eta_{a}(\mathcal{T}, \mathcal{T}) \tag{4.4}
\end{equation*}
$$

The bracket denotes the condition on the anyon $a$ that goes into the sum.
Then we insert a fermion loop into the 1-handle/noncontractible cycle. We label the 2-handle and the 1 -handle in a similar fashion, and we obtain the anyon diagram in Figure 6 b. From the 1-handle we have the constraint $\mathcal{T}^{\mathcal{T}} a=a \times \psi$, such that the morphism associated to the 1-handle is nonempty. The $\eta$-factor associated to this diagram is also $\eta_{a}(\mathcal{T}, \mathcal{T})$, while the $U$-factor associated to this diagram is $U_{\mathcal{T}}(a, \psi ; a \times \psi)$. Finally, the anyon diagram in Figure 6 b evaluates to $d_{a} \theta_{a} F^{a, \psi, \psi}$. After carefully counting all the other factors involving quantum dimensions as in Eq. (3.51), we arrive at the expression of the second bosonic shadow $Z_{2}$,

$$
\begin{equation*}
Z_{2}=\frac{1}{D} \sum_{\substack{a \\\left\{\mathcal{T}^{\mathcal{T}} a=a \times \psi\right\}}} d_{a} \theta_{a} \times \eta_{a}(\mathcal{T}, \mathcal{T}) U_{\mathcal{T}}(a, \psi ; a \times \psi) F^{a, \psi, \psi} \tag{4.5}
\end{equation*}
$$

At the very end, we need to sum over the two bosonic shadows, weighted by the phase factor $z_{c}$. According to Eq. (3.48), we can choose the phase factor in front of $Z_{2}$ to be $+i$, which amounts to choosing a pin ${ }^{+}$-structure on $\mathbb{R P}^{4}$ among the two choices. The partition function resulted from the other choice will be related to this one by complex conjugation. Therefore, we have

$$
\begin{equation*}
\mathcal{I}=\mathcal{Z}_{f}\left(\mathbb{R P}^{4}\right)=\frac{1}{\sqrt{2}}\left(Z_{1}+i Z_{2}\right) \tag{4.6}
\end{equation*}
$$

Plugging into Eqs. (4.4) and (4.5), we arrive at the partition function of $\mathbb{R P}^{4}$ taking the form of Eq. (4.2). This is our desired anomaly indicator for the $\mathbb{Z} / 4^{T f}$ symmetry. It is straightforward to check that this expression is invariant under the vertex basis transformation, Eq. (2.31) and Eq. (2.38), as well as the symmetry action gauge transformation, Eq. (2.48).

As a straightforward application, by directly plugging into Eq. (4.2) the data of fermionic topological orders $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ with the $\mathbb{Z} / 4^{T f}$ symmetry (collected in Appendix B), we have

Proposition 4.7. The anomaly of fermionic topological orders $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ with the $\mathbb{Z} / 4^{T f}$ symmetry has anomaly $\nu=2,3 \in \mho_{\mathrm{Pin}^{+}}^{4} \cong \mathbb{Z} / 16$, respectively.

Remark 4.8. There is also the $\mathbb{Z} / 2^{T} \times \mathbb{Z} / 2^{f}$ symmetry defined by the triple $\left(G_{b}, s, \omega\right)$ such that $G_{b}=\mathbb{Z} / 2$ with $s$ nontrivial and $\omega$ trivial, with symmetry algebra $\mathcal{T}^{2}=1$. It is in "class BDI" of the 10 -fold way classification of fermionic symmetries. The relevant bordism group is $\Omega_{4}^{\mathrm{Pin}-} \cong 0$ [ABP69, Theorem 5.1], hence the partition function on any pin ${ }^{-}$manifold is 1 and there is no associated anomaly indicator.

$$
\text { 5. } \mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}
$$

In this section, we go to the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry and derive the anomaly indicator of any fermionic topological order with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry. In light of Definition 2.1, the triple $\left(G_{b}, s, \omega\right)$ is given by $G_{b}=\mathbb{Z} / 4$ with $s$ nontrivial and $\omega$ trivial, The symmetry algebra is $\mathcal{T}^{2}=(-1)^{F} C$, with $\mathcal{T}$ the time-reversal generator, $C$ charge conjugation and $(-1)^{F}$ fermion parity. Such $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry shows up in many interesting fermionic topological orders, especially $\mathrm{U}(1)_{k}$, with $k=5,13,17,25, \ldots$, as discussed in [DG21].

In order to obtain the anomaly indicator and eventually the anomaly of these fermionic topological orders with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, we first need to identify the relevant tangential structure and calculate the relevant bordism and cobordism group. Despite the simplicity of the symmetry group, this bordism group and its generator have not been calculated before (though see [BG97, WWZ20] for some partial progress), thanks to tricky extension problems in both the Atiyah-Hirzebruch and Adams spectral sequences. So we undertake this calculation in $\S 5.1$, where we see that the bordism group is isomorphic to $\mathbb{Z} / 4$, implying that anomalies of $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetries in (2+1)-d are also classified by $\mathbb{Z} / 4$. Then in $\S 5.2$ we calculate the partition function of a manifold representative of a generator of the bordism group, following the recipe outlined in $\S 3.2$, and obtain the anomaly indicator with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry.

The additional information we use to resolve the extension question and compute the bordism group comes from a long exact sequence built from the Smith homomorphism. The Smith homomorphism was first studied by Conner-Floyd [CF64, Theorem 26.1], then later generalized to many situations by many authors; see [COSY20, HKT20, DDK $\left.{ }^{+} 23\right]$ for discussions aimed at a mathematical physics audience. The use of the long exact sequence associated to the Smith homomorphism and its cofiber, identified explicitly in [ $\left.\mathrm{DDK}^{+} 23\right]$, is a newer technique, but has already proven helpful to resolve differentials and extension questions in several bordism computations in [DDHM23, Deb23, DL223].
5.1. The Power of Smith: $\Omega_{4}^{\mathrm{EPin}}$ and $\Omega_{4}^{\mathrm{EPin}[k]}$. In this subsection, we calculate the bordism group involved in the anomaly of the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry in ( $2+1$ )-d fermionic systems and identify a generating manifold. Interestingly, we can easily generalize the calculation to the $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry, ${ }^{22}$ with $k$ a multiple of 4. In $\S 5.1 .1$, we will present results for general $k$ as well as their derivation. In §5.1.2, we also present the generating manifold of the bordism group, which sets up the stage of later calculation of the anomaly indicator.

Definition 5.1. Let $k$ be divisible by 4. An epin $[k]$ structure is a ( $B \mathbb{Z} / k, \sigma$ )-twisted spin structure. When $k=4$ we will also refer to an epin[4] structure as an epin structure.

As discussed after Eq. (2.3), here $\sigma$ is a line bundle on $B \mathbb{Z} / k$, defined as the pullback of the tautological bundle on $B \mathbb{Z} / 2 \cong B O(1)$ across the nontrivial classifying map $B s: B \mathbb{Z} / k \rightarrow B \mathbb{Z} / 2$. The name "epin" is due to Wan-Wang-Zheng [WWZ20], who studied epin[4] structures. ${ }^{23}$

According to Ansatz 2.9, the tangential structure involved in the classification of anomaly is an epin $[k]$ structure. It is straightforward to see that $w_{1}(\sigma)$ is nontrivial while $w_{2}(\sigma)$ is trivial,

[^15]and that this uniquely characterizes them in $H^{*}(B \mathbb{Z} / k ; \mathbb{Z} / 2)$, so that the requirement in Eq. (2.20) about realizing $s$ and $\omega$ as Stiefel-Whitney classes is indeed satisfied.

The anomaly of the $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry in (2+1)-d is hence classified by the Pontrjagin dual of $\Omega_{4}^{\mathrm{EPin}[k]}$. Remarkably, we will see in Theorem 5.28 that for all $k$ the associated Atiyah-Hirzeburch spectral sequences (and even Adams spectral sequences) have identical entries on all pages, yet the extension problems on the $E_{\infty}$-pages differ for different $k$.

To solve the extension problem, the Smith homomorphism serves a crucial role, and here we give a brief review of the Smith homomorphism together with the long exact sequence associated to it. We start with a simple lemma; recall the definition of $(X, V)$-twisted $\xi$-structures from Definition 2.7.

Lemma 5.2. Let $V, W \rightarrow X$ be vector bundles of ranks $r_{V}, r_{W}$, respectively, and suppose $M$ is a closed $n$-manifold with an $(X, V)$-twisted $\xi$-structure. If $i: N \hookrightarrow M$ is a closed $\left(n-r_{W}\right)$-submanifold of $M$ such that the $\bmod 2$ fundamental class $i_{*}(N) \in H_{n-r_{W}}(M ; \mathbb{Z} / 2)$ is Poincaré dual to the $\bmod 2$ Euler class $e(W)$, then the $(X, V)$-twisted $\xi$-structure on $M$ induces an $(X, V \oplus W)$-twisted $\xi$-structure on $N$.

This follows directly from the fact that, since $[N]$ is Poincare dual to the Euler class of $W$, the normal bundle $\nu \rightarrow N$ of $N \hookrightarrow M$ is isomorphic to $W$; and $\left.T M\right|_{N} \cong T N \oplus \nu$.

The conditions in Lemma 5.2 typically do not uniquely determine the diffeomorphism class of $N$. However, with a little care, the assignment from $M$ to $N$ can be made compatible with bordism.

Let $\Omega_{\xi}^{*}$ denote " $\xi$-cobordism" [Ati61], the generalized cohomology theory defined by the spectrum $M T \xi$ whose generalized homology theory is $\xi$-bordism. $\Omega_{\xi}^{*}$ is different from $\mho_{\xi}^{*}$, as the latter was built using Pontrjagin duality; the values of these two theories are very different even when evaluated on the point.
Definition 5.3. Let $V, W \rightarrow X$ be vector bundles of ranks $r_{V}$, resp. $r_{W}$ and let $e^{\xi}(W) \in$ $\Omega_{\xi}^{r_{W}}\left(X^{W-r_{W}}\right)$ be the $\xi$-cobordism Euler class of $W$. Taking the cap product with $e^{\xi}(W)$ defines a homomorphism

$$
\begin{equation*}
S_{W}: \Omega_{n}^{\xi}\left(X^{V-r_{V}}\right) \longrightarrow \Omega_{n-r_{W}}^{\xi}\left(X^{V \oplus W-\left(r_{V}+r_{W}\right)}\right) \tag{5.4}
\end{equation*}
$$

This is called a Smith homomorphism.
For the details of the definition of twisted $\xi$-cobordism Euler classes and the bordism-invariance of $S_{W}$, see [ $\left.\mathrm{DDK}^{+} 23, \S I I I . \mathrm{C}\right]$. In particular, one needs stronger results than Lemma 5.2 to get the theory of the Smith homomorphism off of the ground; we included Lemma 5.2 to provide intuition for the more general construction in Definition 5.3.

Remark 5.5. It would be nice to have a simpler description of $S_{W}$, due to the abstruseness of Euler classes in twisted generalized cohomology. This is often possible.
(1) Euler classes are natural in maps of spectra, so given a map of spectra $\phi: M T \xi \rightarrow E$, the image of the Smith homomorphism under $\phi$ is the cap product with the $E$-cohomology Euler class. Usually one chooses $E$ to be $H \mathbb{Z}$ or $H \mathbb{Z} / 2$, sending $e^{\xi}(W)$ to the usual Euler class, resp. top Stiefel-Whitney class, of $W$, in order to only worry about cap products in ordinary homology.
(2) Both bordism classes and ordinary homology classes of a manifold $M$ can often be represented by maps of manifolds $N \rightarrow M$. The Smith homomorphism can then be recast as asking, given $M$ and $W \rightarrow M$, find a manifold $M$ and a map $f: N \rightarrow M$ whose bordism or
homology class is Poincaré dual to the Euler class of $W$. Then the Smith homomorphism sends the $V$-twisted $\xi$-bordism class of $M$ to the $(V \oplus W)$-twisted $\xi$-bordism class of $N$.

These two facts lead to the usual interpretation of the Smith homomorphism as "taking the Poincaré dual of the Euler class/of the top Stiefel-Whitney class."

In most cases, using Euler classes in $\mathbb{Z}$ or $\mathbb{Z} / 2$ cohomology, rather than in cobordism, suffices; this includes all Smith homomorphisms studied in this paper. But there are examples where one must use a better approximation to $\xi$-cobordism to correctly define the Smith homomorphism. One such example appears in $\left[\mathrm{DDK}^{+} 23\right.$, Appendix B].

Theorem 5.6 ([DDK $\left.\left.{ }^{+} 23\right]\right)$. Let $V, W \rightarrow X$ be vector bundles of ranks $r_{V}$, $r_{W}$, respectively, and $p: S(W) \rightarrow X$ be the sphere bundle of $W$. Then there is a long exact sequence
$\cdots \rightarrow \Omega_{k}^{\xi}\left(S(W)^{p^{*} V-r_{V}}\right) \xrightarrow{p_{*}} \Omega_{k}^{\xi}\left(X^{V-r_{V}}\right) \xrightarrow{S_{W}} \Omega_{k-r_{W}}^{\xi}\left(X^{V \oplus W-\left(r_{V}+r_{W}\right)}\right) \rightarrow \Omega_{k-1}^{\xi}\left(S(W)^{p^{*} V-r_{V}}\right) \rightarrow \cdots$
This long exact sequence connects bordism groups with different twists and in different dimensions, hence if one case is easier to determine, the long exact sequence can be very helpful to deriving results in other cases. We will apply Theorem 5.6 a few times in this paper with $X=B \mathbb{Z} / k$, and we need the following lemma regarding its sphere bundle.

Lemma 5.8. Given a short exact sequence, $1 \rightarrow \hat{G} \rightarrow G \rightarrow \mathbb{Z} / 2 \rightarrow 1$, let $\sigma$ be the 1-dimensional line bundle on $G$ defined as the pullback of the tautological bundle on $B \mathbb{Z} / 2 \cong B \mathrm{O}(1)$. The map $S(\sigma) \rightarrow B \mathbb{Z} / k$ is homotopy equivalent to the map $B \hat{G} \rightarrow B G$ induced by the inclusion $\hat{G} \hookrightarrow G$.

Compare [DL223, Lemma C.2].
Remark 5.9. If the projection $G \rightarrow \mathbb{Z} / 2$ corresponds to an element $s \in H^{1}(B G ; \mathbb{Z} / 2)$ as in Eq. (2.3), then $\hat{G}$ is the subgroup of unitary symmetries.

Proof of Lemma 5.8. The sphere bundle of $\sigma$ pulls back from the sphere bundle of the universal line bundle $L \rightarrow B \mathbb{Z} / 2$ across the classifying map $f: B G \rightarrow B \mathbb{Z} / 2$ for $\sigma$. But $S(L)=E \mathbb{Z} / 2$, which is contractible. The homotopy pullback of a diagram $B \xrightarrow{f} D \stackrel{g}{\leftarrow} C$ such that $C$ is contractible is the homotopy fiber of $f$, so $S(\sigma) \rightarrow B \mathbb{Z} / k$ is the homotopy fiber of the map $B G \rightarrow B \mathbb{Z} / 2$ induced by the quotient $G \rightarrow \mathbb{Z} / 2$. The classifying space functor turns short exact sequences into fiber sequences, and applying this to $1 \rightarrow \hat{G} \rightarrow G \rightarrow \mathbb{Z} / 2 \rightarrow 1$, we can conclude.
5.1.1. Computing $\Omega_{4}^{\mathrm{EPin}[k]}$ : Spectral sequences and the Smith homomorphism. First of all, we collect some results of group cohomology of $\mathbb{Z} / k$, which will be needed when writing down entries of the spectral sequence.

We recall that if $4 \mid k$,

$$
\begin{array}{cl}
H^{*}(B \mathbb{Z} / k ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2[x, y] /\left(x^{2}\right), & |x|=1,|y|=2 \\
H^{*}(B \mathbb{Z} / k ; \mathbb{Z}) \cong \mathbb{Z}[\bar{y}] /(k \bar{y}), & |\bar{y}|=2 \tag{5.11}
\end{array}
$$

and $\bar{y} \bmod 2=y$. Now we deal with the twisted integral cohomology groups.
Lemma 5.12. As a module over $A_{k}:=H^{*}(B \mathbb{Z} / k ; \mathbb{Z}) \cong \mathbb{Z}[\bar{y}] /(k \bar{y})$ with $|\bar{y}|=2$,

$$
\begin{equation*}
H^{*}\left(B \mathbb{Z} / k ; \mathbb{Z}_{w_{1}(\sigma)}\right) \cong\left(\Sigma^{-1} \tau_{>0} A_{k} \cdot \bar{x}\right) / 2 \bar{x} \tag{5.13}
\end{equation*}
$$

The class $\bar{x} \in H^{1}\left(B \mathbb{Z} / k ; \mathbb{Z}_{w_{1}(\sigma)}\right)$ is the twisted Euler class of $\sigma \rightarrow B \mathbb{Z} / k$, and $\bar{x} \bmod 2=x$.

To unpack the notation in Eq. (5.13) a bit: $\tau_{>0} A_{k}$ means the submodule of $A_{k}$ in positive degrees. $\Sigma^{-1} \tau_{>0} A_{k}$ means to take a copy of $\tau_{>0} A_{k}$ and lower the grading by 1 ; just as $\tau_{>0} A_{k}$ is generated as an $A_{k}$-module by $y$ in degree $2, H^{*}\left(B \mathbb{Z} / k ; \mathbb{Z}_{w_{1}(\sigma)}\right)$ is generated as an $A_{k}$-module by $\bar{x}$ in degree 1. Thus the twisted cohomology groups of $B \mathbb{Z} / k$ begin $0, \mathbb{Z} / 2,0, \mathbb{Z} / 2,0, \ldots$, with the copies of $\mathbb{Z} / 2$ generated by $\bar{x}, \overline{y x}, \bar{y}^{2} \bar{x}$, and so on.

Proof sketch of Lemma 5.12. Following Čadek [Čad99, Lemma 1], consider the Gysin sequence for $\sigma \rightarrow B \mathbb{Z} / k$. The sphere bundle of $\sigma$ is homotopy equivalent to $B \mathbb{Z} /(k / 2)$, so the Gysin sequence is a long exact sequence of the form

$$
\begin{equation*}
\cdots \rightarrow H^{n-1}\left(B \mathbb{Z} / k ; \mathbb{Z}_{w_{1}(\sigma)}\right) \stackrel{\cdot \bar{x}}{\rightarrow} H^{n}(B \mathbb{Z} / k ; \mathbb{Z}) \rightarrow H^{n}(B \mathbb{Z} /(k / 2) ; \mathbb{Z}) \rightarrow H^{n}\left(B \mathbb{Z} / k ; \mathbb{Z}_{w_{1}(\sigma)}\right) \rightarrow \cdots \tag{5.14}
\end{equation*}
$$

Studying the effect of the map $B \mathbb{Z} /(k / 2) \rightarrow B \mathbb{Z} / k$ on cohomology, we see that Eq. (5.14) breaks into a bunch of short exact sequences, from which the lemma follows.

Remark 5.15. This Gysin sequence is closely analogous to the Smith long exact sequence in Theorem 5.6, just with bordism replaced with homology. Hence this problem can be thought of as our first example solved by the Smith homomorphism.

Remark 5.16. Though the twisted cohomology groups in Lemma 5.12 do not form a ring, the product of two classes in $\mathbb{Z}_{w_{1}(\sigma)}$-cohomology lands in untwisted $\mathbb{Z}$-cohomology, inducing a $(\mathbb{Z} \times \mathbb{Z} / 2)$ graded ring structure on $H^{*}\left(B \mathbb{Z} / k ; \mathbb{Z} \oplus \mathbb{Z}_{w_{1}(\sigma)}\right)$, as observed by Čadek [Čad99, §1] (see also Costenoble-Waner [CW92, CW16]). It is possible to extend Lemma 5.12 to show

$$
\begin{equation*}
H^{*}\left(B \mathbb{Z} / k ; \mathbb{Z} \oplus \mathbb{Z}_{w_{1}(\sigma)}\right) \cong \mathbb{Z}[\bar{x}, \bar{y}] /\left(2 \bar{x}, k \bar{y},(k / 2) \bar{y}-\bar{x}^{2}\right) \tag{5.17}
\end{equation*}
$$

with $|\bar{x}|=(1,1)$ and $|\bar{y}|=(2,0)$, for example by using the local coefficients Serre spectral sequence [Sie67, Theorem 2.19] in a manner similar to [Deb21, Theorem 5.49] and [MCB23, Appendices A.4, E.5.a.b, E.5.b.b]; we do not need this extra structure, so do not prove it.

Corollary 5.18. Let $\beta_{\mathrm{U}(1)}: H^{k}(-; \mathrm{U}(1)) \rightarrow H^{k+1}(-; \mathbb{Z})$ denote the Bockstein associated to the short exact sequence

$$
\begin{equation*}
1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{e^{2 \pi i(-)}} \mathrm{U}(1) \longrightarrow 1 \tag{5.19}
\end{equation*}
$$

Then, as an $A_{k}$-module,

$$
\begin{equation*}
H^{*}\left(B \mathbb{Z} / k ;(\mathrm{U}(1))_{w_{1}(\sigma)}\right) \cong\left(\Sigma^{-2} \tau_{>0} A_{k} \cdot \bar{x}\right) / 2 \bar{x} \tag{5.20}
\end{equation*}
$$

meaning the twisted $\mathrm{U}(1)$-valued cohomology groups of $B \mathbb{Z} / k$ begin $\mathbb{Z} / 2,0, \mathbb{Z} / 2,0, \ldots$, with the copies of $\mathbb{Z} / 2$ generated by classes whose $\beta_{\mathrm{U}(1)}$-images are $\bar{x}, \overline{y x}, \bar{y}^{2} \bar{x}$, etc.

We will use the notation $\bar{x}$ and $\bar{y}$ for both the classes in $H^{*}(B \mathbb{Z} / k ; \mathbb{Z})$ and their $\beta_{\mathrm{U}(1)}$-preimages in $H^{*-1}(B \mathbb{Z} / k ; \mathrm{U}(1))$; the meaning will always be clear from context.
Proof. Plug Lemma 5.12 into the long exact sequence in cohomology associated to the exponential exact sequence (5.19).The result follows as soon as we know $H^{*}\left(B \mathbb{Z} / k ; \mathbb{R}_{w_{1}(\sigma)}\right)$ vanishes in all degrees, which follows from Lemma 5.12 and the universal coefficient theorem.
Proposition 5.21 (Botvinnik-Gilkey [BG97, §5]). For all $k, \Omega_{4}^{\operatorname{EPin}[k]}$ has order 4.
We obtain this result through the Atiyah-Hirzeburch spectral sequence, which is a different technique from Botvinnik-Gilkey. We include this proof because we will use the details of our
argument later, both in Theorem 5.28 to finish the computation of $\Omega_{4}^{\operatorname{EPin}[k]}$ and in Appendix C to provide an interpretation of the layers of the Atiyah-Hirzebruch spectral sequence in the context of anomalies of fermionic topological order.

We actually show that $\mho_{\mathrm{EPin}[k]}^{4}$ has order 4 , which by Definition 2.18 is equivalent to Proposition 5.21. We have two reasons for our change to Pontrjagin-dualized bordism: to simplify the differentials and to make contact with a physically motivated interpretation of this spectral sequence due to [BB22a, WG20]. This technique is spiritually similar to a strategy of Campbell [Cam17, $\S 7.4]$, also used in [FH20, §5.1] and [Deb21, §5.3.1]; heuristically, the difference is whether U(1) carries the discrete topology, as it does for us, or the usual topology, as it does for Campbell.

By Lemma 2.8 and Definition 5.1, there is an isomorphism $\mho_{\mathrm{EPin}[k]}^{4} \cong \mho_{\mathrm{Spin}}^{4}\left((B \mathbb{Z} / k)^{\sigma-1}\right)$ natural in $k$. Thus we will study the Atiyah-Hirzebruch spectral sequence

$$
\begin{equation*}
E_{2}^{p, q}=H^{p}\left(X ; \mho_{\mathrm{Spin}}^{q}\right) \Longrightarrow \mho_{\mathrm{Spin}}^{p+q}(X) \tag{5.22}
\end{equation*}
$$

Using (2.17), the coefficient groups $\mho_{\text {Spin }}^{*}$ begin $U(1), \mathbb{Z} / 2, \mathbb{Z} / 2,0, U(1), 0,0,0$ in degrees 0 through 7.

Lemma 5.23. Let $i: \mathbb{Z} / 2 \rightarrow \mathrm{U}(1)$ be the unique injective homomorphism, $r_{2}: \mathbb{Z} \rightarrow \mathbb{Z} / 2$ be reduction $\bmod 2$, and $\beta_{\mathrm{U}(1)}$ be the Bockstein from Corollary 5.18. Then, in (5.22),
(1) $d_{2}: E_{2}^{p, 1} \rightarrow E_{2}^{p+2,0}$ is identified with the map $i \circ \mathrm{Sq}^{2}: H^{p}(X ; \mathbb{Z} / 2) \rightarrow H^{p+2}(X ; \mathrm{U}(1))$,
(2) $d_{2}: E_{2}^{p, 2} \rightarrow E_{2}^{p+2,1}$ is identified with $\mathrm{Sq}^{2}: H^{p}(X ; \mathbb{Z} / 2) \rightarrow H^{p+2}(X ; 2)$, and
(3) $d_{3}: E_{3}^{p, 4} \rightarrow E_{3}^{p+3,2}$ is identified with the map $H^{p}(X ; \mathrm{U}(1)) \rightarrow \operatorname{ker}\left(\mathrm{Sq}^{2}\right) \subset H^{p+3}(X ; \mathbb{Z} / 2)$ given by $\mathrm{Sq}^{2} \circ r_{2} \circ \beta_{\mathrm{U}(1)}$.

Remark 5.24. The statement of part (3) of Lemma 5.23 relies on the proof of part (2): because $E_{2}^{*, 3}=0$ and $d_{2}$ out of $E_{2}^{*, 2} \cong H^{*}(X ; \mathbb{Z} / 2)$ is identified with $\mathrm{Sq}^{2}, E_{3}^{p, 2} \cong \operatorname{ker}\left(\mathrm{Sq}^{2}\right) \subset H^{p}(X ; \mathbb{Z} / 2)$ as promised in (3). We will prove part (2) without reference to part (3), so there is no circular logic.
Proof of Lemma 5.23. Let $I_{\mathbb{Z}}: \mathcal{S} p^{\mathrm{op}} \rightarrow \mathcal{S} p$ denote the Anderson duality functor [And69, Yos75]; then, there is a map $\alpha: I_{\mathrm{U}(1)} X \rightarrow \Sigma I_{\mathbb{Z}} X$ whose fiber is $\operatorname{Map}(X, H \mathbb{C})$ [And69, Definition 4.11]. ${ }^{24,25}$ Consider the following zigzag of maps of spectra:

$$
\begin{equation*}
I_{\mathrm{U}(1)} M T \operatorname{Spin} \stackrel{\phi}{\leftarrow} I_{\mathrm{U}(1)} k o \xrightarrow{\alpha} \Sigma I_{\mathbb{Z}} k o \underset{\longleftarrow}{\varsigma} \Sigma I_{\mathbb{Z}} K O \underset{\simeq}{\underset{\simeq}{\varkappa}} \Sigma^{5} K O \tag{5.25}
\end{equation*}
$$

where $\alpha$ is as above and

- $\phi$ is the Pontrjagin dual of the Atiyah-Bott-Shapiro map MTSpin $\rightarrow k o$ [ABS64],
- $\zeta$ is the Anderson dual of the connective cover map $k o \rightarrow K O$, and
- $\chi$ is the map implementing the shifted Anderson self-duality of $K O$ [And69, Theorem $4.16] .{ }^{26}$
The differentials corresponding to ours in the $K O$-cohomology Atiyah-Hirzebruch spectral sequence were computed by Bott [Bot69] (see Anderson-Brown-Peterson [ABP67, Proof of Lemma 5.6] for

[^16]an explicit description), and we will chase them through (5.25) to determine the differentials in the theorem statement. Specifically, in the $K O$-cohomology Atiyah-Hirzebruch spectral sequence,
(0) $d_{2}: E_{2}^{p, 0} \rightarrow E_{2}^{p+2,-1}$ is $\mathrm{Sq}^{2} \circ r: H^{p}(X ; \mathbb{Z}) \rightarrow H^{p+2}(X ; \mathbb{Z} / 2)$, where $r$ is the reduction mod 2 map;
$(-1) d_{2}: E_{2}^{p,-1} \rightarrow E_{2}^{p+2,-2}$ is $\mathrm{Sq}^{2}: H^{p}(X ; \mathbb{Z} / 2) \rightarrow H^{p+2}(X ; \mathbb{Z} / 2)$;
$(-2) d_{3}: E_{3}^{p,-2} \rightarrow E_{3}^{p+3,-4}$ is $\beta \circ \mathrm{Sq}^{2}: H^{p}(X ; \mathbb{Z} / 2) \rightarrow H^{p+3}(X ; \mathbb{Z})$, where $\beta$ is the Bockstein $H^{*}(-; \mathbb{Z} / 2) \rightarrow H^{*+1}(-; \mathbb{Z})$.

We need $\Sigma^{5} K O$, which amounts to adding 5 to $q$ in the above formulas.
Anderson-Brown-Peterson [ABP67] showed that the Atiyah-Bott-Shapiro map induces an isomorphism on homology groups in degrees 7 and below, so by the universal property of $I_{\mathrm{U}(1)}, \phi$ is an isomorphism on homotopy groups in degrees -7 and above, i.e. is an isomorphism on generalized cohomology of a point in cohomological degrees 7 and below. This implies that in the map of cohomological Atiyah-Hirzebruch spectral sequences induced by $\phi$, the $E_{2}$-pages coincide for $q \leq 7$ and this identification commutes with all differentials out of $E_{r}^{p, q}$ for $q \leq 7$. Therefore in the degrees we care about, the effect of $\alpha$ on differentials may as well be the identity map.

Likewise, by definition $k o \rightarrow K O$ is an isomorphism on homotopy groups in degrees 0 and above. By invoking the universal property of Anderson duality [And69, Lemma 4.13] or by explicitly tracing through the long exact sequence induced by $H \mathbb{C} \wedge X \rightarrow I_{\mathrm{U}(1)} X \rightarrow \Sigma I_{\mathbb{Z}} X$, we learn $\zeta$ is an isomorphism on homotopy groups in degrees 1 and below, i.e. is an isomorphism on generalized cohomology of a point in degrees -1 and above. Arguing as we did for $\phi$, we learn $\zeta$ is an isomorphism on $E_{2}$-pages of Atiyah-Hirzebruch spectral sequences in degrees -1 and above, and commutes with all differentials emerging from $E_{r}^{p, q}$ with $q \geq-1$. Once again in the range we care about we can treat this as the identity.

That leaves $\chi$ and $\alpha$, and $\chi$ is a homotopy equivalence, so up to isomorphism will not change Atiyah-Hirzebruch differentials; we focus on $\alpha$. We saw above that the fiber of $\alpha$ is $F:=\operatorname{Map}(k o, H \mathbb{C})$; by definition $F^{*}(\mathrm{pt}) \cong H^{*}(k o ; \mathbb{C})$, which is isomorphic to $\mathbb{C}$ in nonnegative degrees $0 \bmod 4$ and otherwise vanishes. Using this and the long exact sequence in cohomology of a point associated to the cofiber sequence

$$
\begin{equation*}
\operatorname{Map}(k o, H \mathbb{C}) \longrightarrow I_{\mathrm{U}(1)} k o \longrightarrow \Sigma\left(I_{\mathbb{Z}} k o\right) \tag{5.26}
\end{equation*}
$$

we see that $\alpha$ is an isomorphism in cohomological degrees 1 and 2 ; comparing with the $K O$ cohomology differential that we described above, we have proven part (2) of the theorem.

For the other two differentials in the theorem statement, we use the long exact sequence in cohomology associated to (5.26) to see that, when passing from $\Sigma I_{\mathbb{Z}} k o$ to $I_{\mathrm{U}(1)} k o$ in degrees -1 and 0 and in degrees 3 and 4 , we must precompose with $\beta_{\mathrm{U}(1)}$, finishing the proof.

Proof of Proposition 5.21. Now we can directly write the $E_{2}$-page of our Atiyah-Hirzebruch spectral sequence Eq. (5.22). We do so in Figure 7, where $x, y, \bar{x}$, and $\bar{y}$ are as in Eq. (5.10) and Corollary 5.18.

Since we are running the Atiyah-Hirzebruch spectral sequence for $(B \mathbb{Z} / k)^{\sigma-1}$ as a Thom spectrum, we need to pass through the Thom isomorphism. Let $U \in H^{0}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathbb{Z} / 2\right)$ denote the mod 2 Thom class; because $\sigma$ is not orientable, we do not have an integral Thom class, so given $\gamma \in H^{*}\left(B \mathbb{Z} / k ;(\mathrm{U}(1))_{w_{1}(\sigma)}\right)$, we let $\bar{U} \gamma \in H^{*}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathrm{U}(1)\right)$ denote the image of $\gamma$ under the Thom isomorphism. We can now write down the entries in Eq. (5.22).

| 4 | $\bar{U} \bar{x}$ | $\bar{U} \bar{y} \bar{x}$ |  | $\bar{U} \bar{y}^{2} \bar{x}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 |  |  |  |  |  |  |
| 2 |  |  |  | $U$ | $U x$ | Uy | Uxy | $U y^{2}$ | $U x y^{2}$ |
| 1 | U | $U x$ | Uy | Uxy | $U y^{2}$ | $U x y^{2}$ |
| 0 | $\bar{U} \bar{x}$ |  | $\bar{U} \overline{y x}$ |  | $\bar{U} \bar{y}^{2} \bar{x}$ |  |
|  | 0 | 1 | 2 | 3 | 4 | 5 |
| $\begin{aligned} & \text { FIGURE } \\ & V_{\text {Spin }}^{*}(1) \end{aligned}$ | $\begin{aligned} & \text { 7. The } \\ & \left.\mathbb{Z}(k)^{\sigma-1}\right) \text {. } \end{aligned}$ | Atiyah-Hirzebruch |  | spectral | sequence | computing |

We have formulas for the differentials thanks to Lemma 5.23, and now we evaluate them. As described above, $\beta_{\mathrm{U}(1)}$ is an isomorphism in positive degrees in the twisted cohomology of a finite group, so we will suppress it in the arguments below.

Recall that because $\sigma-1$ has rank 0 , the mod 2 Thom isomorphism $a \mapsto U a$ is a degreepreserving isomorphism of graded abelian groups $H^{*}(B \mathbb{Z} / k ; \mathbb{Z} / 2) \rightarrow H^{*}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathbb{Z} / 2\right)$, but it does not commute with the Steenrod squares. Instead, $\mathrm{Sq}^{1}(U)=U w_{1}(\sigma-1)$ and $\mathrm{Sq}^{2}(U)=$ $U w_{2}(\sigma-1)$ [Tho52, Théorème II.2]; the values of Steenrod squares on classes of the form $\bar{U} a$ for $a \in H^{*}(B \mathbb{Z} / k ; \mathbb{Z} / 2)$ then follow from the Cartan formula. We have $w_{1}(\sigma-1)=x$ and $w_{2}(\sigma-1)=0$. Using this formula, we obtain the following differentials.
(1) For the $d_{3}$ beginning in degree $q=4$ : the Thom isomorphism commutes with $r$ and $\beta$, so $r(\bar{U} \bar{x})=U w_{1}(\sigma)=U x$ and $d_{2}(\bar{U} \bar{x})=\mathrm{Sq}^{2}(U x)=0$.
(2) For the $d_{2}$ beginning in degree $q=2, d_{2}(U)=0, d_{2}(U x)=0, d_{2}(U y)=U y^{2}$, and $d_{2}(U x y)=U x y^{2}$.
(3) For the $d_{2}$ beginning in degree $q=2, d_{2}(U y) \neq 0$, and is the class whose $\beta_{\mathrm{U}(1)}$ Bockstein is $\bar{U} \bar{y}^{2} \bar{x}$. All other $d_{2}$ differentials to or from classes in total degree 4 vanish for degree reasons.
Thus the piece of the $E_{4}$-page in total degree 4 consists of $E_{4}^{0,4}=\mathbb{Z} / 2 \cdot \bar{U} \bar{x}$ and $E_{4}^{2,2}=\mathbb{Z} / 2 \cdot U y$. All higher differentials vanish on these two summands for degree reasons, so $E_{\infty}^{p, 4-p}$ consists of two $\mathbb{Z} / 2$ summands, implying $\mho_{\text {Spin }}^{4}\left((B \mathbb{Z} / k)^{\sigma-1}\right)$ is an abelian group of order 4.

Remark 5.27. The extension question raised by Proposition 5.21 is a common feature of AtiyahHirzebruch spectral sequence computations of twisted spin bordism groups in low degrees; for example, in the analogous computation of $\mho_{\text {Pin }^{-}}^{2} \cong \mathbb{Z} / 8$, the Atiyah-Hirzebruch spectral sequence reports this $\mathbb{Z} / 8$ as three $\mathbb{Z} / 2$ summands on the $E_{\infty}$-page, and likewise $\mho_{\text {Pin }}^{4} \cong \mathbb{Z} / 16$ is broken into four $\mathbb{Z} / 2$ summands on the $E_{\infty}$-page of its Atiyah-Hirzebruch spectral sequence. The same is true for the homologically graded Atiyah-Hirzebruch spectral sequences computing bordism groups.

Typically one can resolve these extension questions using the Adams spectral sequence, whose extra structure can often be used to disambiguate these extension problems; see Beaudry-Campbell [BC18, §4.8, Figure 30]. For example, Campbell [Cam17, Theorems 6.4 and 6.7] runs the Adams
spectral sequences computing pin ${ }^{+}$and pin ${ }^{-}$bordism in low degree: the $\mathbb{Z} / 8$ and $\mathbb{Z} / 16$ of interest are visible on the $E_{\infty}$-page. ${ }^{27}$

For EPin $[k]$, Botvinnik-Gilkey [BG97, §5] and (for $k=4$ ) Wan-Wang-Zheng [WWZ20, §B.1] run the Adams spectral sequence to compute $\Omega_{4}^{\mathrm{EPin}[k]}$. However, the Adams spectral sequence does not resolve the ambiguity on its $E_{\infty}$-page: there is the potential for a "hidden extension," and without addressing it, one cannot distinguish between $\mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ and $\mathbb{Z} / 4$. So we have to try something different.

It turns out that the answer depends on $k$.

## Theorem 5.28.

(1) If $k \equiv 4 \bmod 8, \Omega_{4}^{\operatorname{EPin}[k]} \cong \mathbb{Z} / 4$.
(2) If $k \equiv 0 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$.

For $k=4$, Theorem 5.28 corrects a mistake in the literature (e.g. [WWZ20, §B.1]), where this group was claimed to be isomorphic to $\mathbb{Z} / 2 \oplus \mathbb{Z} / 2$. We will discuss manifold representatives for generators of $\Omega_{4}^{\mathrm{EPin}[k]}$ in §5.1.2.

Proof of Theorem 5.28. We will prove part (1) first, then use it to prove part (2).
We will fit $\Omega_{4}^{\mathrm{EPin}[k]}$ into a long exact sequence from Theorem 5.6, whose other terms are already known. In Theorem 5.6 , choose $\xi$ to be spin-structures, $X=B \mathbb{Z} / k, V=0$, and $W=\sigma$ that was introduced as the pullback of the tautological line bundle from $B \mathbb{Z} / 2$. By directly plugging this data into Eq. (5.7) we get a long exact sequence

$$
\begin{equation*}
\cdots \longrightarrow \Omega_{m}^{\mathrm{Spin}}(B \mathbb{Z} /(k / 2)) \longrightarrow \Omega_{m}^{\mathrm{Spin}}(B \mathbb{Z} / k) \longrightarrow \Omega_{m-1}^{\mathrm{EPin}[k]} \longrightarrow \Omega_{m-1}^{\mathrm{Spin}}(B \mathbb{Z} /(k / 2)) \longrightarrow \cdots \tag{5.29}
\end{equation*}
$$

For $k=4$, we can directly write down many entries in the long exact sequence: work of MahowaldMilgram [MM76] implies $\Omega_{4}^{\text {Spin }}(B \mathbb{Z} / 2) \cong \mathbb{Z}$ and $\Omega_{5}^{\text {Spin }}(B \mathbb{Z} / 2)=0,{ }^{28}$ and Bruner-Greenlees [BG10, Example 7.3.3] show $\Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} / 4) \cong \mathbb{Z} / 4 .{ }^{29}$ Therefore, the long exact sequence (5.29) has the form

$$
\begin{equation*}
0 \longrightarrow \mathbb{Z} / 4 \xrightarrow{S_{\sigma}} \Omega_{4}^{\operatorname{EPin}[k]} \xrightarrow{\varphi} \mathbb{Z} \tag{5.30}
\end{equation*}
$$

We already know $\Omega_{4}^{\mathrm{EPin}}$ is finite, so $\varphi=0$ and $S_{\sigma}$ is an isomorphism. Therefore, we immediately establish that $\Omega_{4}^{\mathrm{EPin}}=\Omega_{4}^{\mathrm{EPin}[4]} \cong \mathbb{Z} / 4$.

For other values of $k \equiv 4 \bmod 8$, the Smith homomorphism $S_{\sigma}: \Omega_{m}^{\mathrm{Spin}}(B \mathbb{Z} / k) \rightarrow \Omega_{m-1}^{\mathrm{EPin}[k]}$ is not an isomorphism, but we can still follow the calculation of $k=4$ after applying the trick of localizing at $2 .{ }^{30}$ If $k \equiv 4 \bmod 8$, then $k / 2$ is two times an odd integer, so the inclusion $B \mathbb{Z} / 2 \hookrightarrow B \mathbb{Z} /(k / 2)$ induces an isomorphism on $\mathbb{Z}_{(2)}$-cohomology, hence also on 2-localized generalized homology (e.g. using the Atiyah-Hirzebruch spectral sequence). Therefore for $k \equiv 4 \bmod 8$, we can replace $B \mathbb{Z} /(k / 2)$ with $B \mathbb{Z} / 2$ in Eq. (5.29). And the long exact sequence (5.29) has the form

$$
\begin{equation*}
0 \longrightarrow \mathbb{Z} / 4 \xrightarrow{S_{\sigma}} \Omega_{4}^{\operatorname{EPin}[k]} \otimes \mathbb{Z}_{(2)} \xrightarrow{\varphi} \mathbb{Z}_{(2)} \tag{5.31}
\end{equation*}
$$

[^17]Thus, the localization of $\Omega_{4}^{\operatorname{EPin}[k]}$ at 2 is isomorphic to $\mathbb{Z} / 4$; since we know from Proposition 5.21 that $\Omega_{4}^{\mathrm{EPin}[k]}$ is either $\mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ or $\mathbb{Z} / 4$, we conclude that for all $k \equiv 4 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 4$.

When $8 \mid k$, the algebraic information of the bordism groups in the Smith long exact sequence is not enough to clarify whether $\Omega_{4}^{\operatorname{EPin}[k]}$ is isomorphic to $\mathbb{Z} / 4$ or to $\mathbb{Z} / 2 \oplus \mathbb{Z} / 2$. Thus we have to do something different: study the map of Atiyah-Hirzebruch spectral sequences induced by the map $p: B \mathbb{Z} / k \rightarrow B \mathbb{Z} / 4$. This map is the pullback map on cohomology on the $E_{2}$-page and commutes with all differentials, giving us an induced map on the $E_{\infty}$-page which is compatible with the filtration on $\Omega_{p+q}^{\operatorname{EPin}[k]}$. Like in the proof of Proposition 5.21, we will study the $\mathcal{U}_{\text {Spin }}^{*}$ Atiyah-Hirzebruch spectral sequence; see that proof for the strategy and notation. We will let $p^{*}$ denote the pullback map associated to $p$ in both cohomology and Pontrjagin-dualized bordism; the specific map will always be clear from context.

For any $\ell$ divisible by 4 , let $E_{*}^{*, *}(\ell)$ denote the spectral sequence for $\mho_{\text {Spin }}^{*}\left((B \mathbb{Z} / \ell)^{\sigma-1}\right)$. In the proof of Proposition 5.21, we saw that for both $\ell=4$ and $\ell=k, E_{\infty}^{\bullet, 4-\bullet}$ has two nonzero summands: $E_{\infty}^{0,4} \cong \mathbb{Z} / 2$ and $E_{\infty}^{2,2} \cong \mathbb{Z} / 2$. Thus in both spectral sequences, the group we want to compute is an extension of $E_{\infty}^{0,4}$ by $E_{\infty}^{2,2}$, and compatibility of these filtrations with the map $(B \mathbb{Z} / k)^{\sigma-1} \rightarrow(B \mathbb{Z} / 4)^{\sigma-1}$ implies that there is a commutative diagram of short exact sequences


We know $\mho_{\text {Spin }}^{4}\left((B \mathbb{Z} / 4)^{\sigma-1}\right) \cong \mathbb{Z} / 4$, so $f_{0}: \mathbb{Z} / 2 \rightarrow \mathbb{Z} / 4$ sends $1 \mapsto 2$ and $g_{0}: \mathbb{Z} / 4 \rightarrow \mathbb{Z} / 2$ is reduction $\bmod 2$. Next we want to understand the vertical arrows.

- Since $E_{\infty}^{2,2}(4)$ is generated by the class $U y, p_{1}^{*}$ is determined by the image of $U y$ under the pullback $H^{2}\left((B \mathbb{Z} / 4)^{\sigma-1} ; \mathbb{Z} / 2\right) \rightarrow H^{2}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathbb{Z} / 2\right)$. Naturality of the Thom isomorphism implies that it suffices to compute the pullback of $y$ under $p^{*}: H^{2}(B \mathbb{Z} / 4 ; \mathbb{Z} / 2) \rightarrow$ $H^{2}(B \mathbb{Z} / k ; \mathbb{Z} / 2)$. It turns out $p^{*}(y)=0,{ }^{31}$ so $U y \mapsto 0$ as well and $p_{1}^{*}=0$.
- By contrast, the generator $\bar{U} \bar{x} \in H^{0}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathrm{U}(1)\right)$ pulls back from $(B \mathbb{Z} / 2)^{\sigma-1}$ : naturality of $\beta_{\mathrm{U}(1)}$ implies we may as well check on the corresponding class in $H^{1}\left((B \mathbb{Z} / k)^{\sigma-1} ; \mathbb{Z}\right)$, which is the Euler class of $\sigma)$. The map $\mathbb{Z} / k \rightarrow \mathbb{Z} / 2$ factors through $\mathbb{Z} / 4$, so $p_{2}^{*}$ is an isomorphism.

Suppose $\mathcal{U}_{\text {Spin }}^{4}\left((B \mathbb{Z} / k)^{\sigma-1}\right) \cong \mathbb{Z} / 4$. Then we still have $f: \mathbb{Z} / 2 \rightarrow \mathbb{Z} / 4$ sends $1 \mapsto 2$ and $g: \mathbb{Z} / 4 \rightarrow$ $\mathbb{Z} / 2$ is reduction mod 2 . The commutativity of the right-hand square in (5.32) implies $p_{2}^{*} \circ g_{0}=g \circ p^{*}$, so $p^{*}: \mathbb{Z} / 4 \rightarrow \mathbb{Z} / 4$ must map 1 to either 1 or -1 . However, using the commutativity of the left square, $p^{*} \circ f_{0}=f \circ p_{1}^{*}$, so $p(1)= \pm 1$ cannot be satisfied. Therefore $\mho_{\text {Spin }}^{4}\left((B \mathbb{Z} / k)^{\sigma-1}\right) \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$.

Remark 5.33. There are a few other ways to show that the extension in the Atiyah-Hirzebruch spectral sequence splits when $8 \mid k$. One can write down a very similar proof by studying the analogous comparison map on Adams spectral sequences, for example, using Botvinnik-Gilkey's description [BG97, §5] of the $E_{\infty}$-page of the Adams spectral sequences for $(B \mathbb{Z} / k)^{\sigma-1}$ for all

[^18]$k \equiv 0 \bmod 4$. Alternatively, studying the Smith long exact sequence (5.29) produces an isomorphism
\[

$$
\begin{equation*}
\Omega_{4}^{\mathrm{EPin}[k]} \cong \operatorname{coker}\left(\Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} /(k / 2)) \rightarrow \Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} / k)\right) \tag{5.34}
\end{equation*}
$$

\]

$\Omega_{5}^{\text {Spin }}(B \mathbb{Z} /(k / 2))$ and $\Omega_{5}^{\text {Spin }}(B \mathbb{Z} / k)$ are known to be generated by lens space bundles $Q_{\ell}^{5}(1, j)$ over $S^{2}$, with a complete invariant given by a collection of $\eta$-invariants [BGS97, §5], and Barrera-Yanez and Gilkey [BYG99, Theorem 1.3(2)], using work of Donnelly [Don78], found a formula for the values of these $\eta$-invariants on $Q_{\ell}^{5}(1, j) .{ }^{32}$ Using this formula, one can completely understand the $\operatorname{map} \Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} /(k / 2)) \rightarrow \Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} / k)$ and therefore compute $\Omega_{4}^{\mathrm{EPin}[k]}$ using Eq. (5.34). The Smith long exact sequence is crucial here: Barrera-Yanez [BY99, §3] discovered that a similar $\eta$-invariant argument without the extra information of the Smith homomorphism is insufficient to resolve the extension question.

Having determined the bordism groups corresponding to the $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry, it is natural to ask what are the maps between different bordism groups induced by maps between these fermionic symmetry groups. We can also read off this information by following the maps between Atiyah-Hirzeburch spectral sequences, similar to the proof of $\Omega_{4}^{\operatorname{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ when $8 \mid k$. We collect some of these results in the following proposition.

## Proposition 5.35.

(1) When $k \equiv 4 \bmod 8$, the projection $\mathbb{Z} / k^{T} \rightarrow \mathbb{Z} / 2^{T}$ induces the map $\Omega_{4}^{\operatorname{EPin}[k]} \rightarrow \Omega_{4}^{\operatorname{Pin}^{+}}$such that $1 \in \Omega_{4}^{\operatorname{EPin}[k]} \cong \mathbb{Z} / 4$ is mapped to $4 \in \Omega_{4}^{\operatorname{Pin}^{+}} \cong \mathbb{Z} / 16$.
(2) When $k \equiv 4 \bmod 8$, the projection $\mathbb{Z} /(2 k)^{T} \rightarrow \mathbb{Z} / k^{T}$ induces the map $\Omega_{4}^{\mathrm{EPin}[2 k]} \rightarrow \Omega_{4}^{\mathrm{EPin}[k]}$ such that $(1,0),(0,1) \in \Omega_{4}^{\mathrm{EPin}[2 k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ are mapped to $2,0 \in \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 4$, respectively.
(3) When $8 \mid k$, the projection $\mathbb{Z} /(2 k)^{T} \rightarrow \mathbb{Z} / k^{T}$ induces the map $\Omega_{4}^{\mathrm{EPin}[2 k]} \rightarrow \Omega_{4}^{\mathrm{EPin}[k]}$ such that $(1,0),(0,1) \in \Omega_{4}^{\operatorname{EPin}[2 k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ are mapped to $(1,0),(0,0) \in \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, respectively.

Remark 5.36. In physics, the UV symmetry $G_{\mathrm{UV}}$ and IR symmetry $G_{\mathrm{IR}}$ are in general different but related by a homomorphism $\phi: G_{\mathrm{UV}} \rightarrow G_{\mathrm{IR}}$. Anomaly matching in this context means that the UV anomaly is in fact the pullback of the IR anomaly induced by the homomorphism $\phi$. Therefore, such maps between different bordism groups induced by maps between different symmetry groups are potentially important in this context. Concrete examples involving such interplay include the so-called "emergibility" problem [ $\mathrm{YGH}^{+} 22$, ZHW21] and the "intrinsically gapless SPT" phase [TVV21, WP23].
5.1.2. Manifold Generator for $\Omega_{4}^{\operatorname{EPin}[k]}$. In this sub-subsection, we define an epin $[k]$ manifold $\mathcal{M}$ and show in Theorem 5.43 that for $k \equiv 4 \bmod 8$, the bordism class of $\mathcal{M}$ generates $\Omega_{4}^{\mathrm{EPin}[k]}$; for $k \equiv 0 \bmod 8$, we show $\mathcal{M}$ and the K3 surface generate. We again use the Smith long exact sequence in our proof. A lot of topological information about $\mathcal{M}$ is given in our proof; notably, we present the Kirby diagram of $\mathcal{M}$ in Figure 8, which will be explicitly needed in the derivation of the anomaly indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry.

In order to describe the generator $\mathcal{M}$ of $\Omega_{4}^{\operatorname{EPin}[k]}$, we need to collect some necessary information about the Klein bottle $K$, which will be an important piece in the construction. According to [EE69], the Klein bottle can be realized as $\mathbb{C} / \Gamma$, where $\mathbb{C}$ is the complex plane and $\Gamma$ is the group

[^19]generated by automorphism $A, B$ of the plane such that
\[

$$
\begin{equation*}
A z=\bar{z}+1 / 2, \quad B z=z+i \tag{5.37}
\end{equation*}
$$

\]

The fundamental group of the Klein bottle is generated by an orientable loop $a$, shown on the complex plane as a straight line from $z=0$ to $z=i$, and an unorientable loop $b$, shown as a straight line from $z=0$ to $z=1 / 2$. They satisfy the relation $b a b^{-1}=a^{-1}$, i.e.,

$$
\begin{equation*}
\pi_{1}(K) \cong\left\langle a, b \mid b a b^{-1}=a^{-1}\right\rangle \tag{5.38}
\end{equation*}
$$

An element $a^{m} b^{n}, m, n \in \mathbb{Z}$ of the group will be denoted by $(m, n)$ in the paper. $\pi_{1}(K)$ can also be written as $\mathbb{Z} \rtimes \mathbb{Z}$, where the normal $\mathbb{Z}$ summand is generated by $a$, the other $\mathbb{Z}$ summand is generated by $b$ and the semidirect product displays the nontrivial $b$ action on $a$. Moreover, because the universal cover of $K$ is contractible, $K \simeq B(\mathbb{Z} \rtimes \mathbb{Z})$. The $\mathbb{Z} / 2$-cohomology ring of $K$ is given by

$$
\begin{equation*}
H^{*}(K ; \mathbb{Z} / 2)=\mathbb{Z} / 2\left[A_{a}, A_{b}\right] /\left(A_{a}^{2}+A_{b} A_{a}, A_{b}^{2}\right), \quad\left|A_{a}\right|=\left|A_{b}\right|=1 \tag{5.39}
\end{equation*}
$$

Here $A_{a}, A_{b}$ can be identified as the cohomology classes of the following cochains:

$$
\begin{equation*}
A_{a}\left(a^{m} b^{n}\right)=m \bmod 2, \quad A_{b}\left(a^{m} b^{n}\right)=n \bmod 2 \tag{5.40}
\end{equation*}
$$

Definition 5.41. Let $\alpha: \mathbb{R} / \mathbb{Z} \rightarrow \operatorname{Diff}(K)$ be the map sending $t \in \mathbb{R}$ to the automorphism of $\mathbb{C}$ defined by $z \mapsto z+t$, which descends to a diffeomorphism of $K=\mathbb{C} / \Gamma$ that only depends on the value of $\mathbb{R} / \mathbb{Z}$. We will also think of this as a map out of $U(1)$ via the exponential isomorphism $\mathbb{R} / \mathbb{Z} \xrightarrow{\cong} \mathrm{U}(1)$.

Since $\mathbb{R} / \mathbb{Z}$ is connected, the image of $\alpha$ is contained in the subgroup $\operatorname{Diff}_{0}(K) \subset \operatorname{Diff}(K)$ of diffeomorphisms in the connected component of the identity.

Lemma 5.42 (Earle-Eels [EE69, §11]). The map $\alpha: \mathbb{R} / \mathbb{Z} \rightarrow \operatorname{Diff}_{0}(K)$ is a homotopy equivalence; in particular, $B \alpha: B(\mathbb{R} / \mathbb{Z}) \rightarrow B \operatorname{Diff}(K)$ is an isomorphism on $\pi_{k}$ for $k>1$. ${ }^{33}$

Therefore to specify a fiber bundle whose fiber is the Klein bottle, we can write down a principal $\mathbb{R} / \mathbb{Z}$-bundle, or equivalently a $\mathrm{U}(1)$-bundle, then take the associated Klein bottle bundle. Moreover, the map $\pi_{2}\left(B \operatorname{Diff}_{0}(K)\right) \rightarrow \pi_{2}(B \operatorname{Diff}(K))$ induced by the inclusion Diff ${ }_{0}(K) \hookrightarrow \operatorname{Diff}(K)$ is an isomorphism, so we conclude $\pi_{2}(B \operatorname{Diff}(K)) \cong \mathbb{Z}$. For the universal Klein bottle bundle $K \rightarrow E \rightarrow B \operatorname{Diff}(K)$, the boundary map $\partial_{0}: \pi_{2}(B \operatorname{Diff}(K)) \rightarrow \pi_{1}(K)$ in the long exact sequence of homotopy groups maps the generator of $\pi_{2}(B \operatorname{Diff}(K)) \cong \mathbb{Z}$ to $b^{2} \in \pi_{1}(K)$.

Now we can write down a generator $\mathcal{M}$ of $\Omega_{4}^{\mathrm{EPin}[k]}$. The generator we choose is a Klein bottle bundle over $S^{2}$.

Theorem 5.43. The following data defines a closed 4-manifold with epin $[k]$ structure.
(1) The manifold $\mathcal{M}$ itself is a nontrivial Klein bottle bundle over $S^{2}$ such that the classifying map of the Klein bottle bundle $f_{\mathcal{M}}: S^{2} \rightarrow B \operatorname{Diff}(K)$ is $k / 2$ times the generator of $\pi_{2}(B \operatorname{Diff}(K)) \cong$ $\mathbb{Z}$ picked out by the isomorphism $\pi_{2}(B \mathrm{U}(1)) \stackrel{\cong}{\rightrightarrows} \pi_{2}(B \operatorname{Diff}(K))$ described above and the standard isomorphism $\pi_{2}(B \mathrm{U}(1)) \cong \mathbb{Z}$.
(2) The principal $\mathbb{Z} / k$-bundle on $\mathcal{M}$ is defined by a map $f: \mathcal{M} \rightarrow B \mathbb{Z} / k$, which is determined by the induced map $\pi_{1}(\mathcal{M}) \rightarrow \pi_{1}(B \mathbb{Z} / k) \cong \mathbb{Z} / k$, such that $a \rightarrow 0$ and $b \rightarrow 1$.

[^20](3) The spin structure on $T \mathcal{M} \oplus f^{*}(\sigma)$ is chosen such that the orientable cycle a has a nonbounding spin-structure. ${ }^{34}$
If $k \equiv 4 \bmod 8$, the bordism class of $\mathcal{M}$ generates $\Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 4$. If $k \equiv 0 \bmod 8$, the bordism classes of $\mathcal{M}$ and the K3 surface generate $\Omega_{4}^{\operatorname{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$.

For $k \equiv 4 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 4$, so this lemma detects the unique generator outright. For $k \equiv 0 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, so this lemma provides one of two generators, while the other generator may be taken to be the K 3 surface with trivial $\mathbb{Z} / k$-bundle.

Remark 5.44. To find manifold representatives of the generators of $\Omega_{4}^{\mathrm{EPin}[k]}$, in principle one could coax the generators out of the Smith homomorphism $\Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} / k) \stackrel{\cong}{\leftrightharpoons} \Omega_{4}^{\operatorname{EPin}[k]}$ we studied in §5.1. Assume $k=4$ for now, so that $\Omega_{4}^{\operatorname{EPin}[4]} \cong \mathbb{Z} / 4$ by Theorem 5.28 . Let $M$ be a closed spin 5 -manifold with principal $\mathbb{Z} / 4$-bundle $P \rightarrow M$ such that $(M, P)$ is a generator for $\Omega_{5}^{\mathrm{Spin}}(B \mathbb{Z} / 4) \cong \mathbb{Z} / 4$. Let $f: M \rightarrow B \mathbb{Z} / 4$ be the classifying map for $P$. Because the Smith homomorphism $\Omega_{5}^{\text {Spin }}(B \mathbb{Z} / 4) \rightarrow$ $\Omega_{4}^{\mathrm{EPin}[4]}$ is an isomorphism, if $N \subset M$ is a smooth representative of the Poincaré dual of $f^{*}(x)$, then $\left(N,\left.P\right|_{N}\right)$ generates $\Omega_{4}^{\mathrm{EPin}[4]} .{ }^{35} \Omega_{5}^{\text {Spin }}(B \mathbb{Z} / 4)$ is well-understood: one choice of the generating manifold, called $Q_{4}^{5}$, is a fiber bundle over $S^{2}$ with fiber the lens space $L_{4}^{3}=S^{3} /(\mathbb{Z} / 4)$ [BGS97, §5]. If $P \rightarrow L_{4}^{3}$ denotes the quotient $\mathbb{Z} / 4$-bundle $S^{3} \rightarrow S^{3} /(\mathbb{Z} / 4)=L_{4}^{3}$, with classifying map $g: L_{4}^{3} \rightarrow B \mathbb{Z} / 4$, the Poincaré dual of $g^{*}(x) \in H^{1}\left(L_{4}^{3} ; \mathbb{Z} / 2\right)$ can be represented by a Klein bottle [BW69], suggesting that the Poincaré dual of $f^{*}(x) \in H^{1}\left(Q_{4}^{5} ; \mathbb{Z} / 2\right)$ can be represented by a Klein bottle bundle over $S^{2}$. However, rather than construct the generator of $\Omega_{4}^{\mathrm{EPin}[4]}$ as a submanifold of $Q_{4}^{5}$, in our presentation we only used the Smith homomorphism as inspiration that we should look for a Klein bottle bundle over $S^{2}$. Then the task is to write one down with the right properties such that it indeed generates $\Omega_{4}^{\mathrm{EPin}[4]}$.

For other values of $k$ not much changes; when $k \equiv 4 \bmod 8$ the story is almost exactly the same, except that the Smith homomorphism is merely surjective, not an isomorphism, before localizing at 2. For $k \equiv 0 \bmod 8, \Omega_{4}^{\mathrm{EPin}[k]} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$; the above argument works for a generator of one of the two $\mathbb{Z} / 2$ summands, and the second summand is generated by the K3 surface, as follows from the generator of $\Omega_{4}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right)$ identified below, together with Eq. (5.47).

We start the proof of Theorem 5.43 by proving the following detection lemma for the generator of $\Omega_{4}^{\mathrm{EPin}[k]}$, which is built from Smith homomorphism as well; then we check that $\mathcal{M}$ indeed satisfies the requirement in the detection lemma.

Lemma 5.45. Suppose $\mathcal{M}$ is a closed 4-dimensional epin $[k]$ manifold, with the associated principal $\mathbb{Z} / k$-bundle $\mathcal{P} \rightarrow \mathcal{M}$ classified by a map $f: \mathcal{M} \rightarrow B \mathbb{Z} / k$. Let $i: \mathcal{N} \hookrightarrow \mathcal{M}$ be a smooth representative of the Poincaré dual of $f^{*}(y) \in H^{2}(\mathcal{M} ; \mathbb{Z} / 2)$, and $\mathcal{S} \subset \mathcal{N}$ be the Poincaré dual of $(i \circ f)^{*}(x) \in$ $H^{1}(\mathcal{N} ; \mathbb{Z} / 2)$. Then $\mathcal{S}$ has a spin- $\mathbb{Z} / k$ structure induced from the epin $[k]$ structure on $\mathcal{M}$, and if $\mathcal{S} \cong S_{n b}^{1}$ as a spin- $\mathbb{Z} / k$ manifold, then $[\mathcal{M}] \in \Omega_{4}^{\operatorname{EPin}[k]}$ is neither zero nor the class of the K3 surface.

Here, a spin- $\mathbb{Z} / k$ structure is a $\operatorname{Spin} \times\{ \pm 1\} \mathbb{Z} /(2 k)$ structure where the diagonal $\{ \pm 1\}$ subgroup is quotiented out.

[^21]Proof. Let $\rho: \mathbb{Z} / k \rightarrow \mathrm{U}(1)$ be the standard one-(complex-)-dimensional rotation representation and $V_{\rho} \rightarrow B \mathbb{Z} / k$ be the associated complex line bundle arising as the pullback of the tautological line bundle on $B \mathrm{U}(1) \cong \mathbb{C P}^{\infty}$.

Recall the long exact sequence from Theorem 5.6, built around the Smith homomorphism. We need two instances of this long exact sequence, both with $\xi=$ Spin and $X=B \mathbb{Z} / k$ :
(1) $V=\sigma$ and $W=V_{\rho}$, and
(2) $V=V_{\rho} \oplus \sigma$ and $W=\sigma$.

Lemma 5.8 identifies $S(\sigma) \simeq B \mathbb{Z} /(k / 2)$. Moreover, by using the same procedure as in Lemma 5.8, since the homotopy fiber of the map $B \mathbb{Z} / k \rightarrow B \mathrm{U}(1)$ is $\mathrm{U}(1) / \mathbb{Z} / k=\mathrm{U}(1)$ as a topological space, we find that $S\left(V_{\rho}\right)=B \mathbb{Z}$. Then, we have long exact sequences

$$
\begin{align*}
& \cdots \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right) \rightarrow \Omega_{k}^{\mathrm{EPin}[k]} \xrightarrow{S_{V_{\rho}}} \Omega_{k-2}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right) \rightarrow \Omega_{k-1}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right) \rightarrow \cdots  \tag{5.46a}\\
& \cdots \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathbb{Z} /(k / 2))^{V_{\rho}-2}\right) \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right) \xrightarrow{S_{\sigma}} \Omega_{k-1}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho}-2}\right) \rightarrow \cdots \tag{5.46b}
\end{align*}
$$

Many of these bordism groups are known.

- Campbell [Cam17, $\S 7.8, \S 7.9]$ identifies $\Omega_{*}^{\text {Spin }}\left(\left(B \mathbb{Z} / 2^{\ell}\right)^{V_{\rho}-1}\right)$ with spin- $\mathbb{Z} / 2^{\ell+1}$-bordism. The case of $\ell=1$ was computed in [Gia76], and [Cam17] calculates the other bordism groups in degrees 4 and below. Specifically, we need $\Omega_{2}^{\text {Spin-Z/k }}=0$ and $\Omega_{1}^{\text {Spin- } \mathbb{Z} /(2 k)} \cong \mathbb{Z} /(2 k)$ [Cam17, Theorems 7.9 and 7.10].
- Botvinnik-Gilkey [BG97] study the Adams spectral sequence for $k o_{*}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)$, and Barrera-Yanez [BY99, Theorem 3.1] resolves some extension questions. In dimensions 7 and below this spectral sequence is isomorphic to the Adams spectral sequence for $\Omega_{*}^{\text {Spin }}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)$. Though Botvinnik-Gilkey do not explicitly identify their twisted kohomology groups, from their computations [BG97, §5] it follows that $\Omega_{m}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)$ vanishes for $m=1,3$ and is isomorphic to $\mathbb{Z} / 2$ for $m=2$.
- In $\left[\mathrm{DDK}^{+} 23\right.$, Footnote 27$]$ it is shown that $\Omega_{4}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right) \cong \mathbb{Z} / 2$, generated by the K3 surface with trivial map to $\mathbb{Z}$. Thus, the map $\Omega_{4}^{\text {Spin }} \rightarrow \Omega_{4}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right)$ is surjective.
Put these computations into (5.46a), we obtain a long exact sequence

$$
\begin{equation*}
\underbrace{\Omega_{3}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)}_{=0} \stackrel{\iota}{\longrightarrow} \underbrace{\Omega_{4}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right)}_{\cong \mathbb{Z} / 2} \longrightarrow \Omega_{4}^{\mathrm{EPin}[k]} \stackrel{S_{V_{\rho}}}{\longrightarrow} \underbrace{\Omega_{2}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)}_{\cong} . \tag{5.47}
\end{equation*}
$$

First we see that $\iota$ is injective; since we know the K3 surface with trivial map to $B \mathbb{Z}$ generates $\Omega_{4}^{\mathrm{Spin}}\left((B \mathbb{Z})^{\sigma-1}\right) \cong \mathbb{Z} / 2$, a $\mathbb{Z} / 2$ subgroup of $\Omega_{4}^{\mathrm{EPin}[k]}$ is generated by the K3 surface with trivial $\mathbb{Z} / k$-bundle. The complement of this subgroup maps under $S_{V_{\rho}}$ to a nonzero element of $\Omega_{2}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)$, and since $\Omega_{4}^{\mathrm{EPin}[k]}$ has four elements from Proposition 5.21, $S_{V_{\rho}}$ is surjective. Thus, if $\mathcal{M}$ is an epin $[k] 4$-manifold whose bordism class is distinct from 0 and [K3], and $i: \mathcal{N} \hookrightarrow \mathcal{M}$ is a smooth representative of the Poincaré dual of $f^{*}(y) \in H^{2}(\mathcal{M} ; \mathbb{Z} / 2)$, then $[\mathcal{N}]=S_{V_{\rho}}([\mathcal{M}])$ in $\Omega_{2}^{\text {Spin }}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right) \cdot{ }^{36}$

Now (5.46b) gives:

$$
\begin{equation*}
\underbrace{\Omega_{2}^{\mathrm{Spin}-\mathbb{Z} / k}}_{=0} \longrightarrow \underbrace{\Omega_{2}^{\mathrm{Spin}}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right)}_{\cong \mathbb{Z} / 2} \xrightarrow[\cong \mathbb{Z} /(2 k)]{S_{\sigma}} \underbrace{\Omega_{1}^{\mathrm{Spin}-\mathbb{Z} /(2 k)}}_{1}, \tag{5.48}
\end{equation*}
$$

[^22]which identifies the map $S_{\sigma}: \Omega_{2}^{\text {Spin }}\left((B \mathbb{Z} / k)^{V_{\rho} \oplus \sigma-3}\right) \rightarrow \Omega_{1}^{\text {Spin- } /(2 k)}$ with the map $\mathbb{Z} / 2 \rightarrow \mathbb{Z} /(2 k)$ sending $1 \mapsto k$. Thus with $\mathcal{M}$ and $\mathcal{N}$ above, if $\mathcal{S} \subset \mathcal{N}$ is a smooth representative of the Poincaré dual of $(i \circ f)^{*}(x) \in H^{1}(\mathcal{N} ; \mathbb{Z} / 2)$, then $\mathcal{S}$ has a spin- $\mathbb{Z} /(2 k)$ structure and $[\mathcal{S}]$ corresponds to $k \in \Omega_{1}^{\text {Spin-Z/(2k) }} \cong \mathbb{Z} /(2 k)$. To finish off the theorem, all we need to know is that $S_{n b}^{1}$ with trivial $\mathbb{Z} /(2 k)$-bundle also represents $k \in \Omega_{1}^{\text {Spin- } \mathbb{Z} /(2 k)} \cong \mathbb{Z} /(2 k)$. For $k=4$, [DDHM23, Footnote 52] explains how to coax this out of the Atiyah-Hirzebruch spectral sequence computing $\Omega_{*}^{\text {Spin- } \mathbb{Z} /(2 k)}$; the proof for arbitrary $k$ is the same.

Proof of Theorem 5.43. We start the proof by deriving a few topological properties of $\mathcal{M}$, and we show that $\mathcal{M}$ is indeed an epin manifold. Finally we check that $\mathcal{M}$ satisfies the desired properties of Lemma 5.45, hence indeed a generator of $\Omega_{4}^{\mathrm{EPin}[k]}$ that is not the class of the K3 surface.

First we derive a few topological properties of $\mathcal{M}$ and give its $\mathbb{Z} / 2$ cohomology ring. Consider the long exact sequence of homotopy groups

$$
\begin{equation*}
\underset{\cong \mathbb{Z}}{\pi_{2}\left(S^{2}\right)} \xrightarrow[\cong]{\cong \mathcal{M}} \underset{\substack{ \\\pi_{1}(K)}}{\pi_{1}(\mathcal{M})} \longrightarrow \underset{\cong 0}{\pi_{1}\left(S^{2}\right)} \tag{5.49}
\end{equation*}
$$

The map $\partial_{\mathcal{M}}$ is $\partial_{0} \circ\left(f_{\mathcal{M}}\right)_{*}$, where $f_{\mathcal{M}}: \mathcal{M} \rightarrow B \operatorname{Diff}(K)$ is the classifying map of the Klein bottle bundle and $\partial_{0}: \pi_{2}(B \operatorname{Diff}(K)) \rightarrow \pi_{1}(K)$ is the boundary map for the universal bundle of $K$. Hence, $\partial_{\mathcal{M}}$ maps the generator of $\pi_{2}\left(S^{2}\right) \cong \mathbb{Z}$ to $k / 2$ times the generator of $\pi_{2}(B \operatorname{Diff}(K)) \cong \mathbb{Z}$ and further to $(0, k) \in \pi_{1}(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$. From the exactness of Eq. (5.49) we see that $\pi_{1}(\mathcal{M}) \cong \mathbb{Z} \rtimes \mathbb{Z} / k$, i.e.,

$$
\begin{equation*}
\pi_{1}(\mathcal{M}) \cong\left\langle a, b \mid b a b^{-1}=a^{-1}, b^{k}=1\right\rangle \tag{5.50}
\end{equation*}
$$

By taking the classifying space of Eq. (5.49), we have the following fiber sequence:

$$
\begin{equation*}
\underset{\cong K}{B(\mathbb{Z} \rtimes \mathbb{Z})} \longrightarrow B(\mathbb{Z} \rtimes \mathbb{Z} / k) \longrightarrow \underset{\cong \mathbb{C P}^{\infty}}{B^{2} \mathbb{Z}} \tag{5.51}
\end{equation*}
$$

Then we observe that we have the following commutative diagram

where $p$ is the projection to the base $S^{2}, \rho$ is induced from the standard rotation representation of $\mathbb{Z} / k$, and $i$ is the natural inclusion of $S^{2}$ into $\mathbb{C P}{ }^{\infty}$. The cohomology rings of all of these spaces except $\mathcal{M}$ are known, and we can use them to build the cohomology of $\mathcal{M}$. Specifically, we have

- $H^{*}\left(\mathbb{C P}^{\infty}, \mathbb{Z} / 2\right) \cong \mathbb{Z} / 2\left[c_{1}\right], \quad\left|c_{1}\right|=2$
- $H^{*}\left(S^{2}, \mathbb{Z} / 2\right) \cong \mathbb{Z} / 2[B] /\left(B^{2}\right), \quad|B|=2$
- $H^{*}(B \mathbb{Z} / k ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2[x, y] /\left(x^{2}\right), \quad|x|=1,|y|=2$
- $H^{*}(K ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2\left[A_{a}, A_{b}\right] /\left\{A_{a}^{2}+A_{b} A_{a}, A_{b}^{2}\right\}, \quad\left|A_{a}\right|=\left|A_{b}\right|=1$
- $H^{*}(B(\mathbb{Z} \rtimes \mathbb{Z} / k) ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2\left[\hat{\mathcal{A}}_{a}, \hat{\mathcal{A}}_{b}, \hat{\mathcal{B}}\right] /\left\{\hat{\mathcal{A}}_{a}^{2}+\hat{\mathcal{A}}_{a} \hat{\mathcal{A}}_{b}, \quad \hat{\mathcal{A}}_{b}^{2}\right\}, \quad\left|\hat{\mathcal{A}}_{a}\right|=\left|\hat{\mathcal{A}}_{b}\right|=1,|\hat{\mathcal{B}}|=2$

In particular, $\hat{\mathcal{A}}_{a}, \hat{\mathcal{A}}_{b}$ and $\hat{\mathcal{B}}$ in $H^{*}(B(\mathbb{Z} \rtimes \mathbb{Z} / k) ; \mathbb{Z} / 2)$ can be defined by the following cochains:

$$
\begin{align*}
\hat{\mathcal{A}}_{a}\left(a^{m} b^{n}\right) & =m \bmod 2  \tag{5.53a}\\
\hat{\mathcal{A}}_{b}\left(a^{m} b^{n}\right) & =n \bmod 2  \tag{5.53b}\\
\hat{\mathcal{B}}\left(a^{m_{1}} b^{n_{1}}, a^{m_{2}} b^{n_{2}}\right) & =\frac{\left(n_{1} \bmod k\right)+\left(n_{2} \bmod k\right)-\left(n_{1}+n_{2} \bmod k\right)}{k} . \tag{5.53c}
\end{align*}
$$

Then from the Serre spectral sequence with respect to $K \rightarrow \mathcal{M} \rightarrow S^{2}$, the $\mathbb{Z} / 2$ cohomology ring of $\mathcal{M}$ is given by

$$
\begin{equation*}
H^{*}(\mathcal{M} ; \mathbb{Z} / 2)=\mathbb{Z} / 2\left[\mathcal{A}_{a}, \mathcal{A}_{b}, \mathcal{B}\right] /\left\{\mathcal{A}_{a}^{2}+\mathcal{A}_{a} \mathcal{A}_{b}, \mathcal{A}_{b}^{2}, \mathcal{B}^{2}\right\} \tag{5.54}
\end{equation*}
$$

In particular, the conditions $\mathcal{A}_{a}^{2}=\mathcal{A}_{a} \mathcal{A}_{b}$ and $\mathcal{A}_{b}^{2}=0$ come from the pullback of conditions $\hat{\mathcal{A}}_{a}^{2}=\hat{\mathcal{A}}_{a} \hat{\mathcal{A}}_{b}, \hat{\mathcal{A}}_{b}^{2}=0$ in $H^{*}(B(\mathbb{Z} \rtimes \mathbb{Z} / k) ; \mathbb{Z} / 2)$, and the condition $\mathcal{B}^{2}=0$ comes from the pullback of the condition $B^{2}=0$ in $H^{*}\left(S^{2} ; \mathbb{Z} / 2\right)$. Maps between these cohomology groups induced by maps between different spaces can be obtained by inspecting the explicit cochain representatives or the Serre spectral sequence. We have


Now we are ready to show that $\mathcal{M}$ is an epin $[k]$ manifold. According to the requirement in Eq. (2.20), we just need to show that

- $w_{1}(T \mathcal{M})=f^{*}(x)$ : Since the induced map $\pi_{1}(\mathcal{M}) \rightarrow \mathbb{Z} / 4$ maps $a \mapsto 0$ and $b \mapsto 1$ where $a$ is orientable and $b$ is unorientable, from the explicit cochain representatives we immediately have $w_{1}(T \mathcal{M})=f^{*}(x)=\mathcal{A}_{b}$.
- $w_{2}(T \mathcal{M})=0$ : For any $u \in H^{2}(\mathcal{M} ; \mathbb{Z} / 2)$, consider $u^{2}=\operatorname{Sq}^{2}(u)=\nu_{2}(T \mathcal{M}) u$ where $\nu_{2}$ is the second Wu class given by $\nu_{2}(T \mathcal{M})=w_{1}(T \mathcal{M})^{2}+w_{2}(T \mathcal{M})$. Because $u$ can be written as a combination of $\mathcal{A}_{a}^{2}$ and $\mathcal{B}, u^{2}=0$ and hence $\nu_{2}(T \mathcal{M})=0$. Therefore, we must have $w_{2}(T \mathcal{M})=w_{1}^{2}(T \mathcal{M})=\mathcal{A}_{b}^{2}=0$.
We conclude that $\mathcal{M}$ is indeed an epin $[k]$ manifold.
Now we can directly check that the constructed $\mathcal{M}$ satisfies the condition stated in Lemma 5.45. In particular, from Eq. (5.55) $f^{*}(y)=\mathcal{B}$, and $\mathcal{N}$ as a smooth representative of the Poincaré dual of $\mathcal{B}=p^{*}(B)$ can be chosen to be the Klein bottle $K$ with induced $\mathbb{Z} / k$-bundle structure and spin-structure from $\mathcal{M}$. Finally, $(i \circ f)^{*}(x)=A_{b} \in H^{1}(K ; \mathbb{Z} / 2)$ and $\mathcal{S}$ as a smooth representative of $A_{b}$ can be chosen to be $a$, which by construction is indeed $S_{n b}^{1}$. Therefore, we establish that $\mathcal{M}$ with the stated $\mathbb{Z} / k$-bundle structure and spin-structure is indeed a nonzero element of $\Omega_{4}^{\operatorname{EPin}[k]}$ that is not the class of K3 surface.

Finally, after obtaining the generating manifold $\mathcal{M}$, we will proceed with the goal of writing down the specific anomaly indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry in the next section. The final topological ingredient that we need is the Kirby diagram of $\mathcal{M}$ for $k=4$. The minimal handle-decomposition of $\mathcal{M}$ contains 1 0-handle, 2 1-handles, 2 2-handles, 23 -handles, and 1 4-handle, and its Kirby diagram is given in Figure 8.

We end this subsection by briefly explaining how to obtain the Kirby diagram. Recall that the fundamental groups of both the Klein bottle and $\mathcal{M}$ are generated by an orientable cycle $a$ and an unorientable cycle $b$, such that they satisfy the condition $b a b^{-1} a=1$. The dark blue and blue 1-handle, shown in Figure 8, correspond to cycle $a$ and $b$, respectively. The red line which spans the blue and dark blue balls represents the 2-handle coming from the Klein bottle, and can be drawn by following the path $b a b^{-1} a$. The orange line which spans only the blue balls represent the


Figure 8. The Kirby diagram of the generator $M$. The blue balls and dark blue balls illustrate two 1-handles, and the red lines and orange lines illustrate two 2 -handles. The 1-handle denoted by blue balls is unorientable while the 1-handle denoted by dark blue balls is orientable.

2-handle coming from the base $S^{2}$. Since the fibration is given by $\pi_{2}\left(S^{2}\right) \rightarrow \pi_{1}(K)$ that sends the generator 1 to $(0,4)$, the equator of $S^{2}$ is wrapped along the unorientable loop $b$ of the Klein bottle four times, and this is reflected by the orange line traveling along the blue 1-handle four times. The red line crosses the orange line under and over exactly once, reflecting the +1 intersection number of the two 2 -handles. We also need the orange 2 -handle to have self-intersection zero, hence we add an extra loop on the upper part of the diagram to cancel the self-crossing on the right of the rightmost blue ball. The Kirby diagram of $\mathcal{M}$ in Figure 8 is thus obtained.
5.2. Anomaly Indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ Symmetry. We have now determined that the anomaly of the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry in (2+1)-dimensions is $\mathbb{Z} / 4$ classified by the first case of Theorem 5.28, and also given the necessary topological data of the generating manifold $\mathcal{M}$ in Theorem 5.43. In this subsection, we present the anomaly indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry in Proposition 5.56, and plug into the data of some simplest fermionic topological orders with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ action to obtain the actual value of their anomaly summarized in Theorem 5.60. By applying the rules of calculating anomaly indicators for general fermionic symmetries in $\S 3.2$, computing the anomaly indicators for $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ is now straightforward.

Proposition 5.56. The anomaly indicator of fermionic topological order with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry is given by

$$
\begin{align*}
& \mathcal{I}=\frac{1}{2 D^{2}} \sum_{\substack{a, b, y, z, u \\
\mu \nu \rho \sigma \tilde{\mu} \tilde{\nu} \tilde{\rho} \tilde{\sigma} \alpha \beta \mu^{\prime}}} d_{a} \frac{\theta_{u}}{\theta_{b}}\left(R_{\mathcal{T}_{y}{ }^{\mathcal{T}}{ }^{a}, \mathcal{T}^{3}{ }_{a}}\right)_{\tilde{\rho} \sigma}^{*}\left(F_{\mathcal{T}_{z}, a, \tau^{2}{ }_{a}}^{*}\right)_{(u, \alpha, \beta)\left(y, \tilde{\sigma}, \mu^{\prime}\right)}^{*}\left(F_{\mathcal{T}_{z}, a, \mathcal{T}^{2}{ }_{a}}\right)_{(u, \alpha, \beta)\left(\mathcal{T}^{2} y, \rho, \mu\right)}  \tag{5.57}\\
& \times U_{\mathcal{T}}^{-1}\left({ }^{\mathcal{T}} a,{ }^{\mathcal{T}^{3}} a ;{ }^{\mathcal{T}} y\right)_{\sigma \tilde{\sigma}}^{*} U_{\mathcal{T}}^{-1}\left(b,{ }^{\mathcal{T}^{2}} y ;{ }^{\mathcal{T}} z\right)_{\mu \tilde{\mu}} U_{\mathcal{T}}^{-1}\left(a, \mathcal{T}^{2} a ; \mathcal{T}^{2} y\right)_{\rho \tilde{\rho}} \\
& \times \eta_{a}(\mathcal{T}, \mathcal{T})^{*} \eta_{\mathcal{T}^{2}{ }_{a}}(\mathcal{T}, \mathcal{T})^{*} \eta_{\mathcal{T}^{2}{ }_{a}}\left(\mathcal{T}^{2}, \mathcal{T}^{2}\right)^{*} \times \mathcal{U}_{b y z},
\end{align*}
$$

where

Roman letters in the formulas denote anyons, and Greek letters denote bases of fusion spaces. The use of a dash in the subscript of F-symbols is when the fusion involves the fermion and therefore only has a single (unique) channel.

Proof sketch of Proposition 5.56. Even though the expression of the anomaly indicator is relatively complicated, the derivation still follows closely the procedure outlined in $\S 3.2$. We just need to calculate the partition function of the generating manifold $\mathcal{M}$ of $\Omega_{4}^{\text {EPin }}$, and its important properties are presented in §5.1.2. In particular $\mathcal{M}$ has $H_{1}(\mathcal{M}, \mathbb{Z} / 2) \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, hence the partition function can be decomposed as the sum of four bosonic shadows; and $H^{2}(\mathcal{M} ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, thus there is a $\frac{1}{2}$ factor in Eq. (3.46). The minimal handle-decomposition of $\mathcal{M}$ contains 10 -handle, 2 1-handles, 2 2-handles, 2 3-handles, and 14 -handle, and its Kirby diagram is given in Figure 8.

To translate this Kirby diagram into anyon diagrams, we start with inserting no fermion loop into the diagram. We need to label the 2-handles and 1-handles by anyons and morphisms according to the recipe. First we need to label segments of the orange and red loop by anyons. We follow the ordering of the two anyon loops in Figure 9a and 9b, and label the anyons at position 1 by $a$ and $b$, respectively. According to the $\mathbb{Z} / 4$ bundle structure of $\mathcal{M}$ given in Appendix 5.1.2, the blue 1-handle has a nontrivial $\mathbb{Z} / 4$ bundle put on it. In particular, we take the convention that when an anyon crosses from the top blue circle to the bottom one, it receives an action by $\mathcal{T}^{-1}=\mathcal{T}^{3}$; crossing from the bottom blue circle to the top results in an action by $\mathcal{T}$. This produces the labels in Figure 10a. To draw the anyon diagram Figure 10a, we also need to reverse the flow of anyons on several segments because of the presence of the unorientable blue 1-handle. Finally we label the morphisms by $x, y, z$ (together with $\mu, \nu$ labels that we omit in the figures for clarity), and the morphisms are also acted upon by $\mathcal{T}$ when crossing the blue 1-handle. The morphisms are nonempty only when $\mathcal{T}^{\mathcal{T}} b=b$ and ${ }^{\mathcal{T}} z=z$. This gives Figure 10a.

Now we start inserting some fermion loops into the diagram. Since $H_{1}(\mathcal{M} ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, there are three inequivalent possibilities: adding one extra fermion loop to the dark blue handle,


Figure 9. Illustration of how anyons travel in the orange (left) and red (right) lines. Anyons will travel from 1 to 8 and back to 1 . We remark that some arrows on the diagrams are reversed with respect to the ordering of vertices. The anyons at position 1 on the orange and red lines are labeled by $a$ and $b$, respectively.
adding one extra fermion loop to the blue handle, and adding one extra fermion loop crossing both the blue and dark blue handle. The labels can then be obtained in a similar fashion, and we obtain Figures 10b, 11a and 11b.

Then we can directly translate these anyon diagrams into compact expressions in terms of the data for a super-MTC according to the standard rules of computing anyon diagrams [BK01, Sel11, BBCW19, YZ23a]. After adding the correct factors of $U$-symbols, $\eta$-symbols and quantum dimensions according to Eq. (3.51), we obtain the compact form of individual bosonic shadows


Figure 10. Anyon diagrams from the Kirby diagram in Fig. 8, with no extra fermion loop (left) or one extra fermion loop (colored brown and labeled with $\psi$ at its starting and ending points) in the dark blue 1-handle (right). The red and orange lines illustrate two 2-handles, the blue and dark blue circles illustrate two 1-handles, and the dark red lines illustrate morphisms. We omit the $\mu \nu$ labels for clarity of the diagram.
$Z_{i}, i=1, \ldots, 4$. For example, we have

$$
\begin{align*}
& Z_{1}=\frac{1}{D^{2}} \sum_{\substack{a, b, y, z, u \\
\mu \nu \rho \sigma \tilde{\mu} \tilde{\rho} \tilde{\sigma} \alpha \beta \\
\mathcal{T}^{\mathcal{T}} b=b \\
\mathcal{T}_{z=z\}}}} d_{a} \frac{\theta_{u}}{\theta_{b}}\left(R_{\mathcal{\mathcal { T }} y}^{\mathcal{T}_{a}, \mathcal{T}^{3}{ }_{a}}\right)_{\tilde{\rho} \sigma}^{*}\left(F_{z}^{b, a,,^{\mathcal{T}^{2}} a}\right)_{(u, \alpha, \beta)(y, \tilde{\sigma}, \tilde{\nu})}^{*}\left(F_{z}^{b, a, \mathcal{T}^{2} a}\right)_{(u, \alpha, \beta)\left(\mathcal{T}^{2} y, \rho, \mu\right)} \delta_{\tilde{\mu} \nu}  \tag{5.58}\\
& \times U_{\mathcal{T}}^{-1}\left({ }^{\mathcal{T}} a,{ }^{\mathcal{T}^{3}} a ;{ }^{\mathcal{T}} y\right)_{\sigma \tilde{\sigma}}^{*} U_{\mathcal{T}}^{-1}\left(b,{ }^{\mathcal{T}} y ; z\right)_{\nu \tilde{\nu}}^{*} U_{\mathcal{T}}^{-1}\left(b,{ }^{\mathcal{T}^{2}} y ; z\right)_{\mu \tilde{\mu}} U_{\mathcal{T}}{ }^{-1}\left(a,{ }^{\mathcal{T}^{2}} a ;{ }^{\mathcal{T}^{2}} y\right)_{\rho \tilde{\rho}} \\
& \times \eta_{a}(\mathcal{T}, \mathcal{T})^{*} \eta_{\mathcal{T}^{2}{ }_{a}}(\mathcal{T}, \mathcal{T})^{*} \eta_{\mathcal{T}^{2}}{ }_{a}\left(\mathcal{T}^{2}, \mathcal{T}^{2}\right)^{*},
\end{align*}
$$

which is the first case for $\mathcal{U}_{b y z}$ in Proposition 5.56. The brackets denote conditions on the anyon $b$ and $z$ that go into the sum. The remaining three bosonic shadows correspond to the other three nontrivial cases for $\mathcal{U}_{\text {byz }}$.


Figure 11. Anyon diagrams from the Kirby diagram in Fig. 8 continued, with one extra fermion loop (colored brown and labeled with $\psi$ at its starting and ending points) in either the blue 1-handle (left) or crossing both blue and dark blue 1-handles (right). Again, the red and orange lines illustrate two 2-handles, the blue and dark blue circles illustrate two 1-handles, and the dark red lines illustrate morphisms. We omit the $\mu \nu$ labels for clarity of the diagram.

Finally, we explain the phase, denoted by $z_{c}$ in Eq. (3.46), in front of each bosonic shadow $Z_{i}, i=1, \ldots, 4$. This directly comes from the spin-structure we choose for $\mathcal{M}$ in Theorem 5.43. In particular, the phase in front of $Z_{2}$ is -1 and reflects the fact that the orientable cycle of $\mathcal{M}$ corresponds to $S^{1}$ with non-bounding spin-structure. Fermions therefore pick up a minus sign when traversing this $S^{1}$. We also choose the spin-structure of $\mathcal{M}$ such that the phase in front of $Z_{3}$ and $Z_{4}$ is $+i .{ }^{37}$ Summing the expressions of bosonic shadows $Z_{i}$ up with the correct phase in the front according to Eq. (3.46), we have

$$
\begin{equation*}
\mathcal{I}=\mathcal{Z}_{f}(\mathcal{M})=\frac{1}{2}\left(Z_{1}-Z_{2}+i Z_{3}+i Z_{4}\right) \tag{5.59}
\end{equation*}
$$

[^23]Plugging into the specific expression of $Z_{i}, i=1, \ldots, 4$, we finally arrive at the compact expression of the anomaly indicator $\mathcal{I}$ for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, given in Proposition 5.56.

Theorem 5.60. The anomaly of fermionic topological orders $\mathrm{U}(1)_{5}, \mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ with the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry has anomaly $\nu=0,2,3 \in \mho_{\mathrm{EPin}}^{4} \cong \mathbb{Z} / 4$, respectively.

Proof. After directly plugging the data of $\mathrm{U}(1)_{5}, \mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ (collected in Appendix B) into the formula in Eq. (5.57), the result is

$$
\begin{equation*}
\mathcal{I}=1,-1,-i \tag{5.61}
\end{equation*}
$$

Thus, we immediately see that the anomalies of these fermionic topological orders correspond to $\nu=0,2,3$, respectively, in $\mho_{\text {EPin }}^{4} \cong \mathbb{Z} / 4$.

To give more credence to our computation, in Appendix C, we use the anomaly cascade developed in [BB22a] to rederive the anomaly of these fermionic topological orders. ${ }^{38}$ The calculation of $\mathrm{U}(1)_{5}$ can be easily generalized to $\mathrm{U}(1)_{k}, k=5,13, \ldots$, which all have the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry as discussed in [DG21]. It turns out that just like $\mathrm{U}(1)_{5}$ they all have anomaly $\nu=0$.

Remark 5.62. From the partition function of $\mathrm{SO}(3)_{3}$, we see that the anomaly indicator does take values in $\left\{i^{k}, k=0, \ldots, 3\right\}$, as dictated by $\mho_{\text {EPin }}^{4} \cong \mathbb{Z} / 4$ we obtain in Section 5.1 from Smith homomorphism. This is not at all obvious from the explicit formula in Eq. (5.57). This fact gives yet another way to solve the extension problem (assuming Conjecture 3.53). The manifolds in $\Omega_{4}^{\text {EPin }}$ include the K3 surface and $\mathcal{M}$. Evaluating the partition function for $\mathrm{SO}(3)_{3}$ on $\mathcal{M}$ and obtaining the value $-i$ indicate that indeed the extension exists and $\mathcal{M}$ is indeed the generating manifold. This method is in similar spirit to the method of using $\eta$-invariants to solve extension problems, such as in [Gil84, Gil85, BG87a, Gil87, Gil88a, Gil88b, Sto88, Gil89, BG95, GB96, BGS97, BY99, BYG99, BY06, Mal11, MR15, Hsi18, KPMT20, DGL22, HTY22, DDHM23].

## 6. Conclusion and Discussion

The main focus of our work was on detecting anomalies for fermionic topological orders, especially time-reversal anomalies of $\mathbb{Z} / 4^{T f}$ symmetry and $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry. The $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry is realized already in $\mathrm{U}(1)_{k}$ theories, and we hope that for low values of $k$ our result can be interesting for applications to the fractional quantum Hall effect. Going beyond just discrete symmetries, in Appendix A we showcase even more examples of our techniques by computing the anomaly indicators for all symmetries in Freed-Hopkins' tenfold way involving Lie group symmetries. The mathematical underpinning that makes our results sensible arises from how we built up the bosonization conjecture, and the invertibility conjecture. We then spelled out how to make the formal mathematics explicit by calculating the partition function of the anomaly theory using bosonic shadows and techniques from geometric topology.

We wrap up by giving a quick summary of interesting future directions.
(1) One of the most important manifolds we need to consider is the K3 surface. The bordism class of the K3 surface generates $\Omega_{4}^{\text {Spin }} \cong \mathbb{Z}$, and K3 equipped with a trivial $G_{b}$-bundle often appears as a generator of bordism groups associated to many different fermionic symmetry groups. These include fermionic symmetry $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ when $8 \mid k$ with the associated bordism group $\Omega_{4}^{\mathrm{EPin}[k]}$, according to Theorem 5.43. Hence, the K3 surface is relevant to
${ }^{38}$ For $\mathrm{U}(1)_{5}$ and $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$, the calculation in the appendix reproduces the result $\nu=0,2$ obtained by the anomaly indicator. For $\mathrm{SO}(3)_{3}$, naïvely the anomaly cascade can only tell us that the anomaly $\nu$ is odd.
the calculation of anomaly indicators for many fermionic symmetries. Furthermore, the partition function of K3 gives the formula for the chiral central charge of a super-MTC, similar to how the partition function of $\mathbb{C P}^{2}$ gives the formula for the chiral central charge of a unitary-MTC [CKY93], which is called the Gauss-Milgram formula in the literature. Such a formula is very important in understanding the properties of the corresponding fermionic/bosonic topological order. For example, it gives (the fractional part of) the thermal Hall conductance as discussed in $\S$ B.1, and it is relevant to understanding the boundary properties of the topological order [KZ20, You23].

We can anticipate that the formula for the partition function on the K3 surface we get by directly reading the Kirby diagram is very complicated, akin to the complication of the formula in Eq. (5.57). Aasen-Jones-Walker approach this problem from the point of view of characteristic bordism [KT90]; see [Wal21].
(2) It would be interesting to study anomaly indicators for other symmetries, e.g., dihedral group symmetries. Dihedral groups appear as the point groups of many 2d wallpaper groups, hence anomaly indicators of dihedral group symmetries can have potential application in understanding the "emergibility" of topological orders in these lattice systems. Furthermore, dihedral group symmetries have also been found in abelian Chern-Simons theories [DG21, Tables 1-4]. Therefore, it would be interesting to explore the anomalies of these symmetries. The papers [Gia76, Ped17, GOP ${ }^{+}$20, KPMT20, WWZ20, Deb21, DDHM23] have collectively calculated most of the degree- 4 twisted spin bordism groups of $B D_{2 n}$ controlling these anomalies. Moreover, it would be desirable to extend the calculation to general space group symmetries as well, including the 17 wallpaper group symmetries.
(3) Having gained a comprehensive understanding of the anomaly of fermionic topological orders, it becomes evident that a thorough understanding of the gapped boundaries of invertible/SPT states is essential. This is termed "Clay's problem" by Freed-Teleman, following Córdova-Ohmori's work [CO19, CO20]. We hope that the insights obtained from our paper can serve as a building block for this problem.

For example, when $k \equiv 0 \bmod 8$, we have $\mathcal{U}_{\mathrm{EPin}[k]}^{4} \cong \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$. From Proposition 5.35 and the anomaly of $\mathrm{SO}(3)_{3}$ for $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, we see that $\mathrm{SO}(3)_{3}$ with $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry action has an anomaly with value $(1,0)$. It will be interesting to identify if there is any fermionic topological order that has $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry and an anomaly with value $(0,1)$, or that there is no such fermionic topological order and we have another example of symmetry-enforced gaplessness, i.e., certain element of 't Hooft anomaly being not realized by any symmetry-enriched topological order.

Moreover, we also hope that our result may help generate a necessary and sufficient condition of symmetry-enforced gaplessness. It is worth mentioning that there have already been a lot of attempts in this direction, including [CO19, CO20, NMLW21, Bre23, BS23, YC23].
(4) The bosonization conjecture used to argue invertibility of $\widetilde{\alpha}$ in $\S 3.1$ and $\S 3.1 .6$ is reasonable from a physical perspective, but could be improved if one can offer a rigorous mathematical proof using a spin Crane-Yetter construction. There are some discussions of CraneYetter construction which generates a fully-extended framed or oriented TFT [CY93, CKY94, BJS18, BJSS21, Tha21], and we wish to extend the construction to generate a fully-extended spin TFT, which will eventually prove the bosonization conjecture and invertibility conjecture we have.
(5) Our method of calculating bordism groups and generating manifolds using the Smith homomorphism can be very helpful in calculating bordism groups associated to other symmetries. Specifically, there are fourfold reflection symmetries in many 3-dimensional space groups, whose point groups contain the fourfold rotoreflection symmetry group $S_{4}$ (in Schönflies notation $[\mathrm{Ae} 16]^{39}$ ) as a subgroup. This symmetry corresponds to an order 4 anti-unitary symmetry according to the crystalline equivalence principle [TE18, Deb21]. Our result of $\Omega_{4}^{\mathrm{EPin}} \cong \mathbb{Z} / 4$ will certainly help in the classification and construction of topological crystalline states protected by these point group symmetries. For example, in [ZQG22, Table II] the classifications of $S_{4}$-symmetric topological superconductor for systems with spinless/spin- $1 / 2$ fermions is claimed to be $\mathbb{Z} / 2$, while according to our result it should be $\mathbb{Z} / 4$. It will be interesting to see how this will modify other results in the tables and eventually obtain a full classification of topological crystalline states protected by space group symmetries.
(6) In [BB22a] the anomaly has been interpreted as the obstruction to extending certain data associated to the symmetry action on the super-MTC to a unitary-MTC. We adopt this interpretation in Appendix C and see that the anomalies obtained via this approach agree with the anomaly indicator computations. It would be desirable to have a mathematically rigorous connection between the two approaches.

## Appendix A. Anomaly Indicators with Lie group Symmetry: 10-fold way

In this section, we give examples of anomaly indicators for fermionic symmetries involving Lie groups. These include seven out of ten symmetries in the 10 -fold way classification of fermionic symmetries [WS14, FH21], as listed in Table 1. The anomaly indicators for class AII and class AIII involving $\mathrm{U}(1)$ symmetry were first presented in [LL19], and a derivation for abelian topological orders was given in [KB21]. [NMLW21] proposed anomaly indicators for all these symmetries, with the help of replacing the time-reversal symmetry with the mirror symmetry under the crystalline equivalence principle. We will obtain the same expressions in the most general setting by following the recipe in §3.2. In doing so we showcase how to apply our general recipe to Lie group symmetries, and write down anomaly indicators of them without resorting to any additional assumptions. Moreover, for symmetries in class A or class C, even though there is no 't Hooft anomaly associated to them in $(2+1)$-d, we calculate the partition functions on some generating manifolds as well, which correspond to an element in the cobordism group, and interpret the result as the formula for thermal and $\mathrm{U}(1)$ Hall conductance. For class CI and class CII symmetry, we also demonstrate that certain elements in the cobordism group which classifies the anomaly can never be realized by any fermionic topological order, demonstrating the phenomenon of "symmetry-enforced gaplessness" [WS14, NMLW21].

In particular, we should pay special attention to how to write down the $\eta$-symbols in the presence of Lie group symmetries. This was discussed in detail in [YZ23a] in the context of bosonic topological order and we repeat it here. Consider a manifold $M$ with a $G_{b}$-bundle on it defined by the map $f: M \rightarrow B G_{b}$. We want to obtain the $\eta$-factor for some 2 -handle $h$ if we label $h$ by an anyon $a$. We write down this $\eta$-factor in terms of the (fractional) charge of $a$.

[^24]| Class | $G_{f}$ | $G_{b}$ | $s$ | $\omega$ | Tangential | Bordism Group | Generator |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| D | $\mathbb{Z} / 2^{f}$ | 1 |  |  | Spin $^{c}$ | $\mathbb{Z}$ | K3 |
| DIII | $\mathbb{Z} / 4^{T f}$ | $\mathbb{Z} / 2$ | $x$ | $x^{2}$ | $\operatorname{Pin}^{+}$ | $\mathbb{Z} / 16$ | $\mathbb{R P}^{4}$ |
| BDI | $\mathbb{Z} / 2^{T} \times \mathbb{Z} / 2^{f}$ | $\mathbb{Z} / 2$ | $x$ | 0 | Pin $^{-}$ | 0 | - |
| A | $\mathrm{U}_{f}(1)$ | $\mathrm{U}(1)$ | 0 | $w_{2}$ | $\operatorname{Spin}^{c}$ | $(\mathbb{Z})^{2}$ | $\mathbb{C P}^{2}, S^{2} \times S^{2}$ |
| AI | $\mathrm{U}_{f}(1) \rtimes \mathbb{Z} / 2^{T}$ | $\mathrm{O}(2)$ | $w_{1}$ | $w_{2}$ | $\operatorname{Pin}^{\tilde{c}-}$ | $\mathbb{Z} / 2$ | $\mathbb{C P}^{2}$ |
| AII | $\mathrm{U}_{f}(1) \rtimes_{\mathbb{Z} / 2} \mathbb{Z} / 4^{T f}$ | $\mathrm{O}(2)$ | $w_{1}$ | $w_{2}+w_{1}^{2}$ | $\operatorname{Pin}^{\tilde{c}+}$ | $(\mathbb{Z} / 2)^{3}$ | $\mathbb{R P P}^{4}, \mathbb{C P}^{2}, S^{2} \times S^{2}$ |
| AIII | $\mathrm{U}_{f}(1) \times_{\mathbb{Z} / 2} \mathbb{Z} / 4^{T f}$ | $\mathrm{U}(1) \times \mathbb{Z} / 2$ | $x$ | $w_{2}+x^{2}$ | $\operatorname{Pin}^{c}$ | $\mathbb{Z} / 8 \oplus \mathbb{Z} / 2$ | $\mathbb{R P}^{4}, \mathbb{C P}^{2}$ |
| C | $\mathrm{SU}_{f}(2)$ | $\mathrm{SO}(3)$ | 0 | $w_{2}$ | $\operatorname{Spin}^{h}$ | $(\mathbb{Z})^{2}$ | $\mathbb{C P}^{2}, S^{4}$ |
| CI | $\mathrm{SU}_{f}(2) \times \mathbb{Z} / 2 \mathbb{Z} / 4^{T f}$ | $\mathrm{O}(3)$ | $w_{1}$ | $w_{2}$ | $\operatorname{Pin}^{h+}$ | $\mathbb{Z} / 4 \oplus \mathbb{Z} / 2$ | $\mathbb{R P}^{4}, \mathbb{C P}^{2}$ |
| CII | $\mathrm{SU}_{f}(2) \times \mathbb{Z} / 2^{T}$ | $\mathrm{O}(3)$ | $w_{1}$ | $w_{2}+w_{1}^{2}$ | $\operatorname{Pin}^{h-}$ | $(\mathbb{Z} / 2)^{3}$ | $\mathbb{R P}^{4}, \mathbb{C P}^{2}, S^{4}$ |

Table 1. We list the fermionic symmetries of the 10 -fold way classification and present them in terms of both $G_{f}$ and $\left(G_{b}, s, \omega\right)$ as in Definition 2.1. Here, $x$ is the generator of $H^{1}(\mathbb{Z} / 2 ; \mathbb{Z} / 2)$ and $w_{1,2,3}$ are the Stiefel-Whitney classes of (special) orthogonal groups $\mathrm{U}(1)=\mathrm{SO}(2), \mathrm{O}(2), \mathrm{SO}(3)$ or $\mathrm{O}(3)$. The table also gives the corresponding tangential structures, corresponding bordism groups in (3+1)-d and the generating manifolds. Class DIII and class BDI with $G_{b}=\mathbb{Z} / 2$ are discussed in $\S 4$ and see Item 1 for some comments about class D. We discuss the last seven cases involving $\mathrm{U}(1)$ and $\mathrm{SO}(3)$ symmetry in this appendix.

Definition A.1. Suppose $G_{b}$ is a connected Lie group. Denote $e^{2 \pi i q_{a}} \in \mathrm{U}(1)$ as the phase factor obtained from pairing $\left[\eta_{a}\right] \in H^{2}\left(B G_{b}, \mathrm{U}(1)\right)$ with the generator of $H_{2}\left(B G_{b} ; \mathbb{Z}\right) \cong \mathbb{Z}$. Then $q_{a} \in[0,1)$ is defined as the (fractional) charge of the anyon $a$ under the symmetry $G_{b}$.

Note that in our convention, for the symmetries in the 10 -fold way classification involving $\mathrm{U}(1)$, the local fermion $\psi$ carries charge $\frac{1}{2}$. This is with respect to the subgroup $\mathrm{U}_{b}(1)$ of the bosonic symmetry group $G_{b}$. Compared with the subgroup $\mathrm{U}_{f}(1)$ of $G_{f}$, which is the double cover of $\mathrm{U}_{b}(1)$, the charge differs by a factor of 2 , and the local fermion $\psi$ carries charge $1 .{ }^{40}$ This is a convention adopted by many physics papers. We use our convention so that all symmetry groups are discussed on equal footing. We also mention that for the symmetries in the 10 -fold way classification involving $\mathrm{SO}(3)$, the local fermion $\psi$ carries a projective representation, or is a spinor, under $\mathrm{SO}(3)$.

The $\eta$-factor can be expressed in terms of the charge $q_{a}$ as follows [YZ23a]. In the presence of a connected Lie group symmetry $G_{b}, f$ maps a 2 -chain $[h]$ in $M$, which represents the 2 -handle $h$, to a 2-chain $f_{*}[h]$ in $B G_{b}$, which represents $n \in H_{2}\left(B G_{b} ; \mathbb{Z}\right) \cong \mathbb{Z}$. Then the desired $\eta$-factor is simply $e^{2 n \pi i q_{a}}$. Intuitively, such a phase factor can be viewed as the phase the anyon $a$ experiences when traveling along the $S^{1}$ boundary given the nontrivial background $G_{b}$-bundle structure, hence the expression is written in terms of the charge of $a$.
A.1. Class A and class C. In this subsection, we start by considering the fermionic symmetries corresponding to "class A" and "class C." The necessary information of the two symmetries is collected in Table 1. A special feature of "class A" and "class C" is that there is no 't Hooft-like anomaly for the two symmetries, but there can still be a nontrivial partition function that gives various Hall conductance. We will also see later in this appendix that the calculation of anomaly

[^25]indicators for some other 10 -fold way symmetries reduce to these two cases by restricting to a $\mathrm{U}(1)$ or $\mathrm{SO}(3)$ subgroup.

Let us start with class A. The associated tangential structure is well-known to be spin ${ }^{c}$. We have $\Omega_{4}^{\text {Spin }^{c}} \cong \mathbb{Z} \oplus \mathbb{Z}$ [Sto68, Chapter XI $]$, generated by, ${ }^{41}$

- $\mathbb{C P}^{2}$, with the tautological $\mathrm{U}(1)$ bundle,
- $S^{2} \times S^{2}$, with a U(1)-bundle on it whose classifying map is identified with $(2,2)$ in the abelian group $\left[S^{2}, B \mathrm{U}(1)\right] \times\left[S^{2}, B \mathrm{U}(1)\right]=\left(\pi_{2}(B \mathrm{U}(1))\right)^{2} \cong \mathbb{Z}^{2}$.
The Pontrjagin dual of the bordism group is $\mho_{\mathrm{Spin}^{c}}^{4} \cong \mathrm{U}(1) \oplus \mathrm{U}(1)$. Therefore, there is no $(2+1)$-d anomaly associated to class A. Still, in this case, the partition function defined in §3.2 identifies an element in $\mho_{\mathrm{Spin}^{c}}^{4}$ as well. In the physics literature, the two $U(1)$ pieces are interpreted as two theta terms in $(3+1)$-d, which give the thermal Hall conductance and $U(1)$ Hall conductance.

Given an element $\left(\Theta_{1}, \Theta_{2}\right) \in \mho_{\mathrm{Spin}^{c}}^{4} \cong \mathrm{U}(1) \oplus \mathrm{U}(1)$, the partition function on a manifold $M$ with chosen $\mathrm{U}(1)$-bundle structure and spin-structure can be written as follows [LM89, SW16, WS14]

$$
\begin{equation*}
\mathcal{Z}_{f}(M)=\exp \left(i\left(\Theta_{1} I_{1}+\Theta_{2} I_{2}\right)\right) \tag{A.2}
\end{equation*}
$$

where

$$
\begin{gather*}
I_{1}=\frac{1}{8}\left(-\operatorname{Sign}(M)+\int_{M}\left(c_{1}\right)^{2}\right)  \tag{A.3}\\
I_{2}=\int_{M}\left(c_{1}\right)^{2} \tag{A.4}
\end{gather*}
$$

Here $\operatorname{Sign}(M)$ is the signature of $M$, and $c_{1}$ is the first Chern class of the bosonic $\mathrm{U}(1)$ bundle. ${ }^{42}$ The thermal Hall conductance $\kappa$ and $\mathrm{U}_{f}(1)$ Hall conductance $\sigma_{H}$ are related to $\left(\Theta_{1}, \Theta_{2}\right)$ by ${ }^{43}$

$$
\begin{equation*}
\kappa=\frac{\Theta_{1}}{2 \pi} \quad(\bmod 1), \quad \sigma_{H}=\frac{8 \Theta_{2}+\Theta_{1}}{2 \pi} \quad(\bmod 1) \tag{A.5}
\end{equation*}
$$

Given a fermionic topological order with class A symmetry, by calculating the partition function on the two manifold representatives, we have

## Proposition A.6.

$$
\begin{equation*}
e^{i \Theta_{1}}=\frac{\mathcal{Z}_{f}\left(S^{2} \times S^{2}\right)}{\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)^{8}}, \quad \exp \left(i \Theta_{2}\right)=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right) \tag{A.7}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)=\frac{1}{\sqrt{2} D} \sum_{a} d_{a}^{2} \theta_{a} e^{2 \pi i q_{a}}  \tag{A.8}\\
\mathcal{Z}_{f}\left(S^{2} \times S^{2}\right)=\frac{1}{2 D} \sum_{a, b} d_{a} d_{b} S_{a b} e^{4 \pi i q_{a}} e^{4 \pi i q_{b}} . \tag{A.9}
\end{gather*}
$$

Here $q_{a}$ is the fractional charge of anyon a defined in Definition A.1.

[^26]Proof sketch of Proposition A.6. Since both manifolds are simply connected, in both calculation there is only one bosonic shadow to sum over, and hence the calculation and the final expressions are greatly simplified.

The partition function of $\mathbb{C P}^{2}$ can be calculated as follows. The minimum handle decomposition of $\mathbb{C P}^{2}$ contains 10 -handle, 12 -handle and 14 -handle. The Kirby diagram can be found in [GS99]; we draw it in Eq. (A.10). The topological twist reflects the +1 intersection number of $\mathbb{C P}^{2}$. Now we label the 2 -handle by anyon $a$. From the $\mathrm{U}(1)$ bundle structure on $\mathbb{C P}^{2}$, the $\eta$-factor is simply $e^{2 \pi i q_{a}}$, where $q_{a} \in[0,1)$ is the fractional charge of anyon $a$ as in Definition A.1. The anyon diagram associated to the Kirby diagram is evaluated as

$$
\begin{equation*}
\left\rangle=d_{a} \theta_{a}\right. \tag{A.10}
\end{equation*}
$$

Assembling all factors as in Eq. (3.51) and Eq. (3.46), we have

$$
\begin{equation*}
\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)=\frac{1}{\sqrt{2} D} \sum_{a} d_{a}^{2} \theta_{a} e^{2 \pi i q_{a}} \tag{A.11}
\end{equation*}
$$

The partition function of $S^{2} \times S^{2}$ can be calculated in a very similar fashion. The minimum handle decomposition of $S^{2} \times S^{2}$ contains 10 -handle, 2 2-handles and 14 -handle, and the Kirby diagram is given in [GS99] and drawn in Eq. (A.12). In particular, the two circles correspond to the equators of the two $S^{2}$ pieces. Now we label the red and orange 2-handle by anyon $a$ and $b$, respectively. From the $\mathrm{U}(1)$ bundle structure on $S^{2} \times S^{2}$, the $\eta$-factors are $e^{4 \pi i q_{a}}$ and $e^{4 \pi i q_{b}}$, respectively, where again $q_{a, b}$ is the fractional charge of anyon $a$ and $b$, respectively. The anyon diagram associated to the Kirby diagram is evaluated as


Assembling all factors as in Eq. (3.51) and Eq. (3.46), we have

$$
\begin{equation*}
\mathcal{Z}_{f}\left(S^{2} \times S^{2}\right)=\frac{1}{2 D} \sum_{a, b} d_{a} d_{b} S_{a b} e^{4 \pi i q_{a}} e^{4 \pi i q_{b}} \tag{A.13}
\end{equation*}
$$

The discussion of class C is very similar to the discussion of class A. The associated tangential structure is Spin ${ }^{h}$. Freed-Hopkins [FH21, Theorem 9.97] showed $\Omega_{4}^{\text {Spin }^{h}} \cong \mathbb{Z} \oplus \mathbb{Z}$ (see also [BM23, Mil23]), and Hu [Hu23, Appendix A] found the following set of generators: ${ }^{44}$

- $\mathbb{C P}^{2}$, with the tautological $\mathrm{U}(1) \subset \mathrm{SO}(3)$ bundle,
- $S^{4}$, with an $\mathrm{SO}(3)$-bundle over it, whose classifying map is identified with $f: S^{4} \cong \mathbb{H} \mathbb{P}^{1} \subset$ $\mathbb{H P}^{\infty} \cong B \mathrm{SU}(2) \xrightarrow{p_{*}} B \mathrm{SO}(3)$, where $p: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ is the natural projection. ${ }^{45}$

[^27]The Pontrjagin dual of the bordism group is $\mho_{\text {Spin }^{h}}^{4} \cong \mathrm{U}(1) \oplus \mathrm{U}(1)$. Therefore, again there is no $(2+1)$-d anomaly associated to class C, but we can obtain (the fractional part of) the thermal Hall conductance and $\mathrm{SO}(3)$ Hall conductance from the partition functions.

Given an element $\left(\Theta_{1}, \Theta_{2}\right) \in \mho_{\text {Spin }^{h}}^{4} \cong \mathrm{U}(1) \oplus \mathrm{U}(1)$, the partition function on a manifold $M$ with chosen $\mathrm{SO}(3)$-bundle structure and spin-structure can be written as follows [Hu23, WS14]

$$
\begin{equation*}
\mathcal{Z}_{f}(M)=\exp \left(i\left(\Theta_{1} I_{1}+\Theta_{2} I_{2}\right)\right), \tag{A.14}
\end{equation*}
$$

where

$$
\begin{gather*}
I_{1}=\frac{1}{4}\left(-\operatorname{Sign}(M)+\int_{M} p_{1}\right)  \tag{A.15}\\
I_{2}=\int_{M} p_{1} \tag{A.16}
\end{gather*}
$$

Here $\operatorname{Sign}(M)$ is the signature of $M$, and $p_{1}$ is the first Pontrjagin class of the bosonic $\mathrm{SO}(3)$ bundle. The thermal Hall conductance $\kappa$ and $\mathrm{U}_{f}(1)$ Hall conductance $\sigma_{H}$ are related to $\left(\Theta_{1}, \Theta_{2}\right)$ by ${ }^{46}$

$$
\begin{equation*}
\kappa=\frac{\Theta_{1}}{\pi} \quad(\bmod 2), \quad \sigma_{H}=\frac{4 \Theta_{2}+\Theta_{1}}{\pi} \quad(\bmod 2) . \tag{A.17}
\end{equation*}
$$

Given a fermionic topological order with $\mathrm{SO}(3)$ action, by calculating the partition function on the two manifold representatives, we have

## Proposition A.18.

$$
\begin{equation*}
e^{i \Theta_{1}}=\frac{1}{\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)^{4}}, \quad \exp \left(i \Theta_{2}\right)=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right) \tag{A.19}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)=\frac{1}{\sqrt{2} D} \sum_{a} d_{a}^{2} \theta_{a} e^{2 \pi i q_{a}} \tag{A.20}
\end{equation*}
$$

Here $q_{a} \in\left\{0, \frac{1}{2}\right\}$ is the fractional charge of anyon a defined in Definition A.1, and labels whether a carries integer ( $q_{a}=0$ ) or spinor ( $q_{a}=\frac{1}{2}$ ) representation under $\mathrm{SO}(3)$.

Proof sketch of Proposition A.18. The calculation of the partition function of $\mathbb{C P}^{2}$ completely parallels the calculation in the proof of Proposition A.6, and we immediately obtain the result in Eq. (A.20). The only subtlety is that here $q_{a}$ only takes value in $\left\{0, \frac{1}{2}\right\}$ because $H^{2}(B \mathrm{SO}(3) ; \mathrm{U}(1)) \cong$ $\mathbb{Z} / 2 .^{47}$

We just need to focus on the partition function of $S^{4}$. Again, since $S^{4}$ is simply connected, we just have one bosonic shadow to sum over. Moreover, the handle decomposition of $S^{4}$ is extremely simple, i.e., it just contains 10 -handle and 14 -handle glued together along the boundary $S^{3}$. Hence following the formula in Eq. (3.51), we immediately have

$$
\begin{equation*}
\mathcal{Z}_{f}\left(S^{4}\right)=1 \tag{A.21}
\end{equation*}
$$

By directly evaluating $I_{1}$ and $I_{2}$ for the two generating manifolds, we obtain Eq. (A.19).

[^28]Here we see an interesting phenomenon: the partition function on some manifold is always 1 for any fermionic topological order with given symmetry. From this we can derive some interesting physical consequences. In the current example of symmetry in class C, by inspecting Eq. (A.19) and Eq. (A.17), we have

Corollary A.22. Any fermionic topological order with class $C$ symmetry action must have $\mathrm{SO}(3)$ Hall conductance given by an even integer.
A.2. Class AI, AII, AIII. Now we go to class AI, AII and AIII, whose fermionic symmetry groups all contain $\mathrm{U}(1)$ as a subgroup. The definitions of these fermionic symmetries are in Table 1. It turns out that the anomaly indicators for these symmetries can all be obtained from the anomaly indicators of class A and class DIII ( $\mathbb{Z} / 4^{T f}$ symmetry), whose anomaly indicators have been obtained in Proposition A. 6 and Proposition 4.1. See [YZ23a, Section VI] for a similar calculation in the context of bosonic topological order. We list the result below.

Proposition A.23. The classification of anomaly and anomaly indicators of fermionic topological orders with symmetries in class AI, AII and AIII are given by

- Class AI. The anomaly is classified by $\mathbb{Z} / 2$, with the anomaly indicator $\mathcal{I}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$.
- Class AII. The anomaly is classified by $(\mathbb{Z} / 2)^{3}$, with the anomaly indicator $\mathcal{I}_{1}=\mathcal{Z}_{f}\left(\mathbb{R} \mathbb{P}^{4}\right)$, $\mathcal{I}_{2}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$ and $\mathcal{I}_{3}=\mathcal{Z}_{f}\left(S^{2} \times S^{2}\right)$.
- Class AIII. The anomaly is classified by $\mathbb{Z} / 8 \oplus \mathbb{Z} / 2$, with the anomaly indicator of the $\mathbb{Z} / 8$ piece $\mathcal{I}_{1}=\mathcal{Z}_{f}\left(\mathbb{R} \mathbb{P}^{4}\right)$, and the $\mathbb{Z} / 2$ piece $\mathcal{I}_{2}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$.
The partition functions of $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$ are calculated in Proposition $A .6$ and the partition function of $\mathbb{R}^{4}{ }^{4}$ is calculated in Proposition 4.1.

Remark A.24. Even though these anomaly indicators have the same expressions as the expressions for class A or class DIII symmetries, because the classification of anomaly is different, they actually take values in different sets. For example, for class $\mathrm{AI}, \mathcal{I}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$ takes values only in $\{ \pm 1\}$.

These results are straightforward if we know the generating manifolds of the corresponding bordism groups. Hence, we end this subsection by commenting on the generating manifolds for these symmetries.

First consider class AI. The corresponding tangential structure is [FH21, Ste22] $\operatorname{Pin}^{\tilde{c}-}:=$ $\left(\operatorname{Pin}^{-} \ltimes \operatorname{Spin}_{2}\right) /\{ \pm 1\}$. Here $\mathrm{Pin}^{-}$acts on $\operatorname{Spin}_{2}$ by $\operatorname{Pin}^{-} \rightarrow \mathrm{O} \xrightarrow{\text { det }}\{ \pm 1\}$ and $\{ \pm 1\}$ acts on the circle group $\operatorname{Spin}_{2} \cong \mathrm{U}(1)$ by complex conjugation; then, to obtain $\operatorname{Pin}^{\tilde{c}-}$, quotient by the diagonal $\{ \pm 1\}$ subgroup. Similarly for $\operatorname{Pin}^{\tilde{c}+}$ below. $\Omega_{4}^{\mathrm{Pin}^{\tilde{c}-}} \cong \mathbb{Z} / 2[\mathrm{FH} 21]$, generated by $\mathbb{C P}^{2}$ with tautological $\mathrm{U}(1)$ bundle. ${ }^{48}$

Next we consider class AII. Freed-Hopkins showed $\Omega_{4}^{\text {Pin }}{ }^{\tilde{c}+} \cong(\mathbb{Z} / 2)^{\oplus 3}$ [FH21, Theorem 9.87]. Because both $\mathrm{U}(1)$ (class AI) and $\mathbb{Z} / 2^{T}$ (class DIII) are subgroups of $\mathrm{O}(2)^{T}$, spin ${ }^{c}$ manifolds and $\operatorname{pin}^{+}$manifolds all have canonically induced $\operatorname{pin}^{\tilde{c}+}$ structures. Therefore, a natural candidate set of manifold representatives consists of $\mathbb{C P}^{2}, S^{2} \times S^{2}$, and $\mathbb{R P}^{4}$, with induced pin ${ }^{\tilde{c}+}$ structures. However, it is not explicitly proven in the literature that these three manifolds are linearly independent in $\Omega_{4}^{\text {Pin }^{\tilde{c}+}}$. Here we explicitly present the proof, which again falls quickly to the power of the Smith long exact sequence.

[^29]Proposition A.25. The classes of $\mathbb{C P}^{2}, S^{2} \times S^{2}$, and $\mathbb{R} \mathbb{P}^{4}$ are linearly independent in $\Omega_{4}^{\text {Pin }}{ }^{\bar{c}+}$, hence form a generating set. Here $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$ have pin ${ }^{\tilde{c}+}$ structures induced from their spin ${ }^{c}$ structures, and $\mathbb{R P}^{4}$ has its pin ${ }^{\tilde{c}+}$ structure induced from either of its two pin ${ }^{+}$structures.

Proof. Observe that the bordism invariant $\int w_{1}^{4}: \Omega_{4}^{\mathrm{Pin}^{\tilde{c}+}} \rightarrow \mathbb{Z} / 2$ vanishes on $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$, but does not vanish on $\mathbb{R P}^{4}$. Hence we mainly need to prove that $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$ are linearly independent in $\Omega_{4}^{\operatorname{Pin}^{\tilde{c}+}}$, which amounts to proving that the map $\Omega_{4}^{\mathrm{Spin}^{c}} \rightarrow \Omega_{4}^{\mathrm{Pin}^{\tilde{c}+}}$ induced by the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{O}(2)$ maps two generators in $\Omega_{4}^{\text {Spin }^{c}}$ to two generators in $\Omega_{4}^{\mathrm{Pin}^{\tilde{c}+}}$.

Let $V_{t} \rightarrow B \mathrm{O}(2)$ be the tautological rank- 2 vector bundle and $\sigma:=\operatorname{Det}\left(V_{t}\right)$. A pin ${ }^{\tilde{c}+}$ structure is equivalent to a $\left(B O(2), 3 V_{t}\right)$-twisted spin-structure ${ }^{49}$ [FH21, (10.2)] (see [SSGR18, Lemma D.8] for a related but different characterization). Consider the Smith long exact sequence from Theorem 5.6 with $X=B \mathrm{O}(2), V=3 V_{t}$, and $W=\sigma$; by Lemma $5.8, S(W) \rightarrow B \mathrm{O}(2)$ is homotopy equivalent to the map $B \mathrm{U}(1) \rightarrow B \mathrm{O}(2)$. Therefore we have a long exact sequence

$$
\begin{equation*}
\cdots \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{U}(1))^{3 V_{t}-6}\right) \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{O}(2))^{3 V_{t}-6}\right) \xrightarrow{S_{⿱}} \Omega_{k-1}^{\mathrm{Spin}}\left((B \mathrm{O}(2))^{3 V_{t}+\sigma-7}\right) \rightarrow \cdots \tag{A.26}
\end{equation*}
$$

Using this, we interpret the pieces of (A.26) as follows:

- Because $2 V_{t} \rightarrow B \mathrm{U}(1)$ is spin, $\Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{U}(1))^{3 V_{t}-6}\right) \cong \Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{U}(1))^{V_{t}-2}\right)$, which is identified with spin ${ }^{c}$ bordism [BG87a, BG87b].
- The map $\Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{U}(1))^{3 V_{t}-6}\right) \rightarrow \Omega_{k}^{\mathrm{Spin}}\left((B \mathrm{O}(2))^{3 V_{t}-6}\right)$ can be identified with the map $\Omega_{k}^{\text {Spin }^{c}} \rightarrow \Omega_{k}^{\operatorname{Pin}^{\tilde{c}+}}$ given by the induced $\operatorname{pin}^{\tilde{c}+}$ structure described above, because both are induced by the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{O}(2)$.
- The characteristic-class data for a $\left(B \mathrm{O}(2), 3 V_{t}+\sigma-7\right)$-twisted spin structure is $w_{1}\left(3 V_{t}+\sigma\right)=$ 0 and $w_{2}\left(3 V_{t}+\sigma\right)=w_{2}$. This tangential structure corresponds to the fermionic symmetry defined by the triple $\left(G_{b}=\mathrm{O}(2), s=0, \omega=w_{2}\right)$, which was first studied by GuillouMarin [GM80] and is often called a spin-O(2) structure. Spin-O(2) structures are also studied in [KT90, DDHM22, HHLZ22, LS22, Ste22, DDHM23]. ${ }^{50}$

Thus (A.26) becomes

$$
\begin{equation*}
\cdots \longrightarrow \Omega_{5}^{\mathrm{Pin}^{\tilde{c}+}} \longrightarrow \Omega_{4}^{\mathrm{Spin}-\mathrm{O}(2)} \longrightarrow \Omega_{4}^{\mathrm{Spin}^{c}} \longrightarrow \Omega_{4}^{\mathrm{Pin}^{\tilde{c}+}} \longrightarrow \Omega_{3}^{\mathrm{Spin-O}(2)} \longrightarrow \cdots \tag{A.27a}
\end{equation*}
$$

so plugging in $\Omega_{4}^{\operatorname{Spin}^{c}} \cong \mathbb{Z}^{2}$ [Sto68, Chapter XI], $\Omega_{4}^{\text {Pin }^{\tilde{c}+}} \cong(\mathbb{Z} / 2)^{\oplus 3}$ and $\Omega_{5}^{\text {Pin }^{\tilde{c}+}}=0$ [FH21, Theorem $9.87]$, and $\Omega_{3}^{\text {Spin-O(2) }} \cong \mathbb{Z} / 2[S t e 22, \S 4.1]$ and $\Omega_{4}^{\text {Spin-O(2) }} \cong \mathbb{Z}^{2}[G M 80, \S V]$, (A.27a) simplifies to

$$
\begin{equation*}
0 \longrightarrow \mathbb{Z}^{2} \longrightarrow \mathbb{Z}^{2} \longrightarrow \underset{\Omega_{4}^{\text {Spin }^{c}}}{ } \longrightarrow \underset{\Omega_{4}^{\text {Pin }^{\bar{c}+}}}{(\mathbb{Z} / 2)^{\oplus 3} \longrightarrow \mathbb{Z} / 2 .} \tag{A.27b}
\end{equation*}
$$

Exactness implies that any generating set of $\Omega_{4}^{\mathrm{Spin}^{c}}$ is still linearly independent (over $\left.\mathbb{Z} / 2\right)$ in $\Omega_{4}^{\mathrm{Pin}}{ }^{\tilde{c}+}$. We conclude.

Finally we consider class AIII. The relevant tangential structure is called $\operatorname{pin}^{c}$ in the literature, and Bahri-Gilkey [BG87a, Theorem $0.2(\mathrm{~b})$ ] show that there is an isomorphism $\varphi: \Omega_{4}^{\operatorname{Pin}^{c}} \xlongequal{\cong} \mathbb{Z} / 8 \oplus \mathbb{Z} / 2$,

[^30]such that the two pin ${ }^{c}$ structures ${ }^{51}$ on $\mathbb{R P}^{4}$ are sent by $\varphi$ to $( \pm 1,0) \in \mathbb{Z} / 8 \oplus \mathbb{Z} / 2$ and the pin ${ }^{c}$ structure on $\mathbb{C P}^{2}$ induced by its $\operatorname{spin}^{c}$ structure from $\S A .1$ is sent to $(0,1)$. Thus we may take $\mathbb{R} \mathbb{P}^{4}$ and $\mathbb{C P}^{2}$ as our generators.
A.3. Class CI, CII. In this last subsection, we consider class CI and class CII, both of whose fermionic symmetry groups contain $\mathrm{SO}(3)$ as a subgroup. The necessary information of these symmetries is listed in Table 1. The anomaly indicators for these symmetries can also be obtained from the anomaly indicators of class C and class DIII, obtained in Proposition A. 18 and Proposition 4.1. For these two symmetries, it turns out that certain element in the group that classifies the anomaly can never be realized by any fermionic topological order. Hence, any system that saturates this anomaly can only be gapless. This is the phenomenon of "symmetry-enforced gaplessness," as discussed in e.g. [WS14, NMLW21, YZ23a]. We list the results below.

Proposition A.28. The classification of anomaly and anomaly indicators of fermionic topological orders with symmetries in class CI and CII are given by

- Class CI. The anomaly is classified by $\mathbb{Z} / 4 \oplus \mathbb{Z} / 2$, with the anomaly indicator of the $\mathbb{Z} / 4$ piece $\mathcal{I}_{1}=\mathcal{Z}_{f}\left(\mathbb{R} \mathbb{P}^{4}\right)$, and the $\mathbb{Z} / 2$ piece $\mathcal{I}_{2}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$. However, despite the $\mathbb{Z} / 4$ classification, $\mathcal{I}_{1}$ can only take values in $\{ \pm 1\}$.
- Class CII. The anomaly is classified by $(\mathbb{Z} / 2)^{3}$, with the anomaly indicator $\mathcal{I}_{1}=\mathcal{Z}_{f}\left(\mathbb{R P}^{4}\right)$, $\mathcal{I}_{2}=\mathcal{Z}_{f}\left(\mathbb{C P}^{2}\right)$ and $\tilde{\mathcal{I}}=\mathcal{Z}_{f}\left(S^{4}\right)$. However, $\tilde{\mathcal{I}}$ is identically 1 from Proposition A. 18.
The partition function of $\mathbb{C P}^{2}$ is calculated in Proposition $A .18$ and the partition function of $\mathbb{R} \mathbb{P}^{4}$ is calculated in Proposition 4.1.

Again, these results are straightforward if we determine the generating manifolds of the corresponding bordism groups, $\Omega_{4}^{\text {Pin }^{h \pm}}$. The bordism groups were computed by Freed-Hopkins [FH21, Theorem 9.97] to be $\mathbb{Z} / 4 \oplus \mathbb{Z} / 2$ for $\operatorname{pin}^{h+}$ and $(\mathbb{Z} / 2)^{\oplus 3}$ for $\operatorname{pin}^{h-}$. To describe the generators, we use the fact that $\operatorname{spin}^{h}$ and $\operatorname{pin}^{+}$structures naturally induce $\mathrm{pin}^{h \pm}$ structures. The standard inclusion of $\mathrm{SO}(3)$ into $\mathrm{O}(3)$ suggests that $\operatorname{spin}^{h}$ structures can define either kind of pin ${ }^{h \pm}$ structure. We can embed the nonzero element of $\mathbb{Z}_{2}^{T}$ to $\operatorname{diag}(-1,-1,-1)$ in $\mathrm{O}(3)$ such that a pin ${ }^{+}$structure induces a pin $^{h+}$ structure. Via the canonical embedding $\mathrm{O}(1) \cong \mathbb{Z}_{2} \hookrightarrow \mathrm{O}(3)$, a pin ${ }^{+}$structure also induces a $\operatorname{pin}^{h-}$ structure. We let the reader check that the two maps pull back the classes $s$ and $\omega$ of class CI and class CII correctly into corresponding elements in class DIII, as in Definition 2.4 to define a map of fermionic symmetry groups.

Lemma A. 29 (Guo-Putrov-Wang [GPW18, Claims 3 and 6]).
(1) There is an isomorphism $\phi: \Omega_{4}^{\operatorname{Pin}^{h+}} \xlongequal{\cong} \mathbb{Z} / 4 \oplus \mathbb{Z} / 2$, such that
(a) $\phi\left(\left[\mathbb{R}^{4}\right]\right)=(1,0)$, where $\mathbb{R}^{4} \mathbb{P}^{4}$ has the pin ${ }^{h+}$ structure induced from its pin ${ }^{+}$structure.
(b) $\phi\left(\left[\mathbb{C P}^{2}\right]\right)=(0,1)$, where $\mathbb{C P}^{2}$ has the pin ${ }^{h+}$ structure induced from its spin ${ }^{h}$ structure from above.
(2) There is an isomorphism $\psi: \Omega_{4}^{\mathrm{Pin}^{h-}} \xlongequal{\cong}(\mathbb{Z} / 2)^{\oplus 3}$, such that the bordism classes of $\mathbb{R} \mathbb{P}^{4}, \mathbb{C P}^{2}$, and $S^{4}$ are a set of $\mathbb{Z} / 2$-basis for $\Omega_{4}^{\text {Pin }^{h-}}$. Here $\mathbb{C P}^{2}$ and $S^{4}$ have pin ${ }^{h-}$ structures induced from their spin ${ }^{h}$ structures and $\mathbb{R P}^{4}$ has its pin ${ }^{h-}$ structure induced from either of its pin ${ }^{+}$ structures.

[^31]Finally, we observe that for both class CI and class CII symmetries, certain element in the classification of anomaly can never be realized by any fermionic topological order with given symmetry action. For class CI, these are elements corresponding to $\mathcal{I}_{1}= \pm i$. For class CII, these are elements corresponding to $\tilde{\mathcal{I}}=-1$. Therefore, any state with these anomalies can never be a fermionic topological order, and hence must be gapless. This is exactly the phenomenon of "symmetry-enforced gapless".

For the sake of completeness, here we repeat the argument in [NMLW21] that, for class CI, $\mathcal{I}_{1}$ takes values only in $\{ \pm 1\}$ instead of $\{ \pm 1, \pm i\}$ from the classification of anomaly. We need to show that $\mathcal{I}_{1}$ for any fermionic topological order can never take the values $\pm i$. To see this, note that the summation in $\mathcal{I}_{1}=\mathcal{Z}_{f}\left(\mathbb{R P}^{4}\right)$ involves two types of anyons, one type satisfying $\mathcal{T}^{\mathcal{T}} a=a$ and another type satisfying $\mathcal{T}^{\mathcal{T}} a=a \times \psi$. However, the second type of anyons actually does not exist for any topological order with class CI symmetry. This is because by inspecting Eq. (2.44), we see that $a$ and ${ }^{\mathcal{T}} a$ must have the same $\mathrm{SO}(3)$ charge, i.e., either they both have $q=0$ (integer spin) or both have $q=\frac{1}{2}$ (half-integer spin), while $\psi$ must have $q=\frac{1}{2}$.

## Appendix B. Data of Fermionic Topological Orders

For the reader's convenience, in this section we explicitly write down the data of the fermionic topological orders considered in this paper, i.e., $\mathrm{U}(1)_{5}, \mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ (semion-fermion theory), and $\mathrm{SO}(3)_{3}$, together with the data of the $\mathbb{Z} / 4^{T f}$ or the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry action. After directly plugging in the data into Eq. (3.51), we see that the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ anomalies of the three fermionic topological orders correspond to $\nu=0,2,3$ in $\mho_{\mathrm{EPin}}^{4} \cong \mathbb{Z} / 4$, respectively.
B.1. $\mathrm{U}(1)_{5}$. Anyons in $\mathrm{U}(1)_{5}$ can be labeled by integers $a=0, \ldots, 9$. $F$-symbols can be chosen to be all 1 while $R$-symbols can be chosen to be

$$
\begin{equation*}
R^{a, b}=\exp \left(\frac{\pi i}{5} a b\right) \tag{B.1}
\end{equation*}
$$

Here we omit the subscript of the $R$-symbol since the outcome of the fusion rules is unique.
The $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry permutes anyons in $\mathrm{U}(1)_{5}$ by $a \mapsto 3 a(\bmod 10)$. This symmetry is a "genuine" $\mathbb{Z} / 4^{T}$ action in the sense that all nontrivial elements in $\mathbb{Z} / 4^{T}$ permute anyons. All the $U$-symbols and $\eta$-symbols can be chosen to be 1 .
B.2. $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$. Anyons of $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$, or the semion-fermion theory, can be labeled by $1, s, \tilde{s}, \psi$. This theory is a direct product of the free-fermion theory $\{1, \psi\}$ and the semion theory $\mathrm{U}(1)_{2}$. The fusion rules can be given as follows:

$$
\begin{array}{r}
s \times s=\tilde{s} \times \tilde{s}=\psi \times \psi=1 \\
s \times \psi=\tilde{s}, \tag{B.3}
\end{array}
$$

The nontrivial $F$-symbols are $F^{a b c}=-1$ when $(a, b, c)$ is any combination of only $s$ and $\tilde{s}$. The $R$-symbols, with the anyons ordered by $(1, s, \psi, \tilde{s})$, are

$$
R^{a b}=\left(\begin{array}{cccc}
1 & 1 & 1 & 1  \tag{B.4}\\
1 & i & 1 & i \\
1 & 1 & -1 & -1 \\
1 & i & -1 & -i
\end{array}\right)
$$

The theory has the $\mathbb{Z} / 4^{T f}$ time-reversal symmetry, which exchanges $s$ and $\tilde{s}$. Denote the generator of bosonic $\mathbb{Z} / 2^{T}$ group as $\mathcal{T}$. $U$-symbols can be written as a matrix, with the row and
the column denoting $(a, b) \in\{1, s, \psi, \tilde{s}\}$ of $U_{\mathcal{T}}(a, b ; a \times b)$

$$
U_{\mathcal{T}}(a, b ; a \times b)=\left(\begin{array}{cccc}
1 & 1 & 1 & 1  \tag{B.5}\\
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1
\end{array}\right)
$$

We can choose $\eta$-symbols such that $\eta_{\psi}(\mathcal{T}, \mathcal{T})=-1, \eta_{s}(\mathcal{T}, \mathcal{T})=-i$ and $\eta_{\tilde{s}}(\mathcal{T}, \mathcal{T})=i$.
The $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry we consider acts on the theory through the natural projection $p: \mathbb{Z} / 4^{T} \rightarrow \mathbb{Z} / 2^{T}$. The $U$-symbols and $\eta$-symbols of $\mathbb{Z} / 4^{T}$ can be simply obtained from the pullback of $\mathbb{Z} / 2^{T}$.
B.3. $\mathrm{SO}(3)_{3} . \mathrm{SO}(3)_{3}$ can be thought of as a subcategory of $\mathrm{SU}(2)_{6}$. Anyons in $\mathrm{SU}(2)_{6}$ can be labeled by integers $a=0, \ldots, 6$, and anyons in $\mathrm{SO}(3)_{3}$ can be labeled by $1, s, \tilde{s}, \psi$, which are identified as $0,2,4,6$ in $\mathrm{SU}(2)$, respectively. With these identifications, the fusion rules can be given as follows:

$$
\begin{align*}
\psi \times \psi & =1  \tag{B.6}\\
\psi \times s & =\tilde{s},  \tag{B.7}\\
\psi \times \tilde{s} & =s  \tag{B.8}\\
s \times s=\tilde{s} \times \tilde{s} & =1+s+\tilde{s}  \tag{B.9}\\
s \times \tilde{s} & =\psi+s+\tilde{s} . \tag{B.10}
\end{align*}
$$

Let $q=e^{\pi i / 4}$. The $R$-symbols of $\mathrm{SU}(2)_{6}$ are relatively easy to display:

$$
\begin{equation*}
R_{c}^{a, b}=(-1)^{(a+b-c) / 2} q^{\frac{1}{8}(c(c+2)-a(a+2)-b(b+2))} \tag{B.11}
\end{equation*}
$$

The $F$-symbols require a set of auxiliary functions

$$
\begin{align*}
\lfloor n\rfloor & =\sum_{m=1}^{n} q^{(n+1) / 2-m}  \tag{B.12}\\
\lfloor n\rfloor! & =\lfloor n\rfloor\lfloor n-1\rfloor \cdots\lfloor 1\rfloor  \tag{B.13}\\
\Delta(a, b, c) & =\sqrt{\frac{\lfloor(a+b-c) / 2\rfloor!\lfloor(a-b+c) / 2\rfloor!\lfloor(-a+b+c) / 2\rfloor!}{\lfloor(a+b+c+2) / 2\rfloor!}} \tag{B.14}
\end{align*}
$$

for $n \geq 1$, and with $\Delta$ defined only when $a, b, c$ satisfy the triangle inequality. We also define $\lfloor 0\rfloor!=1$. With these definitions, we can define the $F$-symbols by the following formula:

$$
\begin{align*}
& F_{d e f}^{a b c}=(-1)^{(a+b+c+d) / 2} \Delta(a, b, e) \Delta(c, d, e) \Delta(b, c, f) \Delta(a, d, f) \sqrt{\lfloor e+1\rfloor\lfloor f+1\rfloor} \times \\
& \times \sum_{n}^{\prime} \frac{(-1)^{n / 2}\lfloor(n+2) / 2\rfloor!}{\lfloor(a+b+c+d-n) / 2\rfloor!\lfloor(a+c+e+f-n) / 2\rfloor!\lfloor b+d+e+f-n\rfloor!} \times \\
& \times \frac{1}{\lfloor(n-a-b-e) / 2\rfloor!\lfloor(n-c-d-e) / 2\rfloor!\lfloor(n-b-c-f) / 2\rfloor!\lfloor(n-a-d-f) / 2\rfloor!} \tag{B.15}
\end{align*}
$$

where the summation runs over even integers such that $\max (a+b+e, c+d+e, b+c+f, a+d+f) \leq$ $n \leq \min (a+b+c+d, a+c+e+f, b+d+e+f)$. The quantum dimensions are given by $d_{1}=d_{\psi}=1$ and $d_{s}=d_{\tilde{s}}=1+\sqrt{2}$, with total quantum dimension $\mathcal{D}^{2}=8+4 \sqrt{2}$. The topological spins are $\theta_{1}=1, \theta_{\psi}=-1, \theta_{s}=i, \theta_{\tilde{s}}=-i$.

The theory has the $\mathbb{Z} / 4^{T f}$ time-reversal symmetry, which exchanges $s$ and $\tilde{s}$. Denote the generator of bosonic $\mathbb{Z} / 2^{T}$ group as $\mathcal{T}$. The non-trivial $U$ symbols are

$$
\begin{align*}
U_{\mathcal{T}}(s, \tilde{s} ; \psi)=U_{\mathcal{T}}(\tilde{s}, \psi ; s)= & U_{\mathcal{T}}(\psi, s ; \tilde{s})=U_{\mathcal{T}}(s, s ; s)=U_{\mathcal{T}}(\tilde{s}, \tilde{s} ; \tilde{s}) \tag{B.16}
\end{align*}=i
$$

and $U_{\mathcal{T}}(a, b ; c)=-i$ when two of $(a, b, c)$ are $s$ and the third is $\tilde{s}$ or vice-versa. Finally, the $\eta$ symbols are all trivial except

$$
\begin{equation*}
\eta_{\psi}(\mathcal{T}, \mathcal{T})=-1 \tag{B.18}
\end{equation*}
$$

One can check exhaustively by a computer that these data satisfy all of the consistency conditions.
The $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry we consider acts on the theory through the natural projection $p: \mathbb{Z} / 4^{T} \rightarrow \mathbb{Z} / 2^{T}$. The $U$-symbols and $\eta$-symbols of $\mathbb{Z} / 4^{T}$ can be simply obtained from the pullback of $\mathbb{Z} / 2^{T}$.

## Appendix C. Anomaly Cascade for Fermionic Topological Orders

In this section, we give another argument that the anomaly vanishes for the fermionic topological order $\mathrm{U}(1)_{5}$ with given $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry action, and also show that the two other fermionic topological orders $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ and $\mathrm{SO}(3)_{3}$ appearing in the paper have nontrivial anomaly. We use a conjecture of Bulmash-Barkeshli [BB22a]: the conjecture proceeds by unpacking the anomaly in terms of its layers in the Atiyah-Hirzebruch spectral sequence, and each layer then has an interpretation as an obstruction to extending some data when gauging the given symmetry. Following the conjecture, we examine whether the obstruction in each layer is trivial or not for the fermionic topological orders considered in this paper, hence obtaining the anomalies of them. Unpacking the anomaly in this way has the benefit that one can explicitly see the implementation of gauging and the obstruction of it at the level of the skeletalization data of super MTCs. A technique coming from tensor categories called zesting also finds a nice application here and this is an opportunity to showcase it.

Because Bulmash-Barkeshli's interpretation of the anomaly for a fermionic symmetry acting on a super-MTC is expressed in terms of the layers of the Atiyah-Hirzebruch filtration, we will quickly walk through the explicit data that appears in this filtration on $\mho_{\mathrm{Spin}}^{4}\left(\left(B G_{b}\right)^{V-r_{V}}\right)$. This filtration is induced from the Postnikov filtration on $I_{\mathrm{U}(1)} M T S$ in [Mau63, Theorem 3.3]. Specifically, from the construction of the Atiyah-Hirzebruch spectral sequence, we learn the following.

Lemma C.1. Given a fermionic symmetry with data $\left(G_{b}, s, \omega\right)$, let $V \rightarrow B G_{b}$ be a vector bundle on $B G_{b}$ of rank $r$ such that $w_{1}(V)=s$ and $w_{2}(V)=\omega$. The data of the filtration in total degree 4 of the $E_{\infty}$-page of the Atiyah-Hirzebruch spectral sequence for $\mho_{\text {Spin }}^{*}\left(\left(B G_{b}\right)^{V-r_{V}}\right)$ mplies that there are abelian groups $F^{1}$ and $F^{2}$ and short exact sequences

$$
\begin{gather*}
0 \longrightarrow E_{\infty}^{2,2} \longrightarrow F^{1} \longrightarrow E_{\infty}^{0,4} \longrightarrow 0  \tag{C.2a}\\
0 \longrightarrow E_{\infty}^{3,1} \longrightarrow F^{2} \longrightarrow F^{1} \longrightarrow 0  \tag{C.2b}\\
0 \longrightarrow E_{\infty}^{4,0} \longrightarrow \delta_{\text {Spin }}^{4}\left(X^{V-r}\right) \longrightarrow F^{2} \longrightarrow 0 \tag{C.2c}
\end{gather*}
$$

In addition:

- $E_{\infty}^{0,4}$ is a subgroup of $H^{0}\left(X ; \mathrm{U}(1)_{s}\right)$.
- $E_{\infty}^{2,2}$ is a subgroup of $H^{2}(X ; \mathbb{Z} / 2)$.
- $E_{\infty}^{3,1}$ is a subquotient of $H^{3}(X ; \mathbb{Z} / 2)$.
- $E_{\infty}^{4,0}$ is a quotient of $H^{4}\left(X ; \mathrm{U}(1)_{s}\right)$.

The subgroups and quotient groups follow from analyzing the differentials in this spectral sequence and their formulas in Lemma 5.23.

According to [BB22a], the four layers in the Atiyah-Hirzebruch spectral sequence capture the anomaly for a super-MTC as four layers of obstructions to gauging the fermionic symmetry, which they dub the anomaly cascade. We need the concept of minimal modular extension of the superMTC $\mathcal{C}$, which is some unitary-MTC which contains the original super-MTC as a sub-category [Müg03, DN20]. [ $\left.\mathrm{BGH}^{+} 17, \mathrm{GV} 17\right]$ state that given a super-MTC $\mathcal{C}$ if there is one minimal extension, then there are exactly 16 up to Witt equivalence, and [JR23] prove that minimal extension always exists. This minimal modular extension is interpreted as the bosonic theory obtained from gauging fermion parity in the fermionic theory.
Conjecture C. 3 (The anomaly cascade, Bulmash-Barkeshli [BB22a]). The anomaly for a superMTC constitutes four layers, which have the following interpretation in terms of extending certain data from the super-MTC $\mathcal{C}$ to some unitary-MTC $\mathcal{B}$ as the minimal modular extension of the super-MTC:

- The first layer: This is valued in $E_{\infty}^{0,4}$, and is the obstruction for the modular extension to be able to have time-reversal symmetry.
- The second layer: This is valued in $E_{\infty}^{2,2}$, and is the obstruction of extending the data of the homomorphism $\rho: G_{b} \rightarrow \operatorname{Aut}(\mathcal{C})$ to a homomorphism $\check{\rho}: G_{b} \rightarrow \operatorname{Aut}(\mathcal{B})$.
- The third layer: This is valued in $E_{\infty}^{3,1}$, and is the obstruction of extending the data of symmetry fractionalization, or the data of $\eta_{a}(\mathbf{g}, \mathbf{h})$ as defined in Definition 2.46.
- The fourth layer: This is valued in $E_{\infty}^{4,0}$, and is the anomaly of the extended $G_{b}$ action on the modular extension $\mathcal{B}$.

Remark C.4. Our presentation here is slightly different from the presentation in [BB22a] in terms of the value in each layer. We follow closely the data of the Atiyah-Hirzebruch spectral sequence on the infinity page, as in Lemma C.1. In particular, we demand that the first layer to be $E_{\infty}^{0,4}$, which is a quotient of $H^{0}\left(X ; \mathrm{U}(1)_{s}\right)$ rather than $H^{1}\left(X ; \mathbb{Z}_{s}\right)$ as in Bulmash-Barkeshli. When $s$ is nontrivial and there are anti-unitary symmetries present, $H^{0}\left(X ; \mathrm{U}(1)_{s}\right)$ is canonically isomorphic to $H^{1}\left(X ; \mathbb{Z}_{s}\right)$ : both are $\mathbb{Z} / 2$-valued and are connected to each other by the Bockstein induced by $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathrm{U}(1)$. When $s$ is trivial and there is no anti-unitary symmetry, comparing with [BB22a], we simply say that the obstruction in the first layer always vanishes.

In the rest of this appendix, we assume Conjecture C.3.
Going along the lines of [BB22a] we can regard the information of the first three layers with the fourth as giving a mixed anomaly between fermion parity and $G_{b}$. Since each subsequent layer carries more refined data about the interplay between the symmetry and the MTC, the first three layers must be trivialized in sequential order. If the first three layers are completely trivialized, then the only part of the anomaly is in the bosonic sector controlled by the fourth layer.

Conjecture C. 3 and the computation in Eq. (7) together imply:
Corollary C.5. The anomaly of a super-MTC with $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry has nontrivial contributions from the first layer in $E_{\infty}^{0,4} \cong \mathbb{Z} / 2$, and the third layer in $E_{\infty}^{3,1} \cong \mathbb{Z} / 2$.

As an added bonus for this appendix, we will review and apply a technical trick called zesting explained in $\left[\mathrm{BGH}^{+} 17\right]$, with further applications given in $\left[\mathrm{DGP}^{+} 21\right]$. Zesting is a procedure
to obtain one modular extension from another modular extension of a super-MTC. The zesting procedure may preserve many features of a fusion category such as its rank, Frobenius-Perron dimension, and grading, but can also alter certain data such as the central charge. Our strategy will be to use zesting to find a modular extension with the property that the central charge is 0 so that the unitary-MTC indeed has time-reversal symmetry [WL17, BB22a]; this will trivialize the first layer in Corollary C.5. One particularly useful fact about zesting that we use is that zesting an abelian MTC gives another abelian MTC.

Here we list some data of the new unitary-MTC obtained from zesting. Beginning from the fusion rules $\otimes$ of the old unitary-MTC, the fusion rules $\boxtimes$ of the new unitary-MTC are given by $\left[\mathrm{BGH}^{+} 17\right.$, Section 4.1]:

$$
a_{1} \boxtimes a_{2}=\left\{\begin{array}{lr}
\left(a_{1} \otimes \psi\right) \otimes a_{2} & \quad \text { if both } a_{1} \text { and } a_{2} \text { have odd grading, },  \tag{C.6}\\
a_{2} \otimes a_{2} & \text { if at least one of } a_{1} \text { or } a_{2} \text { has even grading. } .
\end{array}\right.
$$

Here, $a_{1}$ and $a_{2}$ are anyons in the original unzested theory, $\psi$ is the fermion, and an anyon $a$ in the old unitary-MTC has even (odd) grading if its braiding with $\psi$ is trivial (nontrivial). Zesting also gives a new set of braidings given by $R_{\boxtimes}^{a_{1}, a_{2}}\left[\mathrm{BGH}^{+} 17\right.$, Section 4.6]:
(C.7)

$$
R_{\boxtimes}^{a_{1}, a_{2}}=\left\{\begin{array}{lr}
b\left(R^{a_{1}, \psi} \otimes \operatorname{Id}_{a_{2}}\right) \circ R^{\psi \otimes a_{1}, a_{2}} \circ\left(F^{a_{2}, \psi, a_{1}}\right)^{-1} & \text { if both } a_{1} \text { and } a_{2} \text { have odd grading, }, \\
R^{a_{1}, a_{2}} \text { otherwise } .
\end{array}\right.
$$

Here $R$ is the braiding and $F$ is the $F$-symbol in the old unitary-MTC. There is a constant $b$ which one has the freedom to choose so that the resulting theory has certain properties, e.g. vanishing central charge.

Proposition C.8. The anomaly of $\mathrm{U}(1)_{5}$ for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry vanishes.
Proof. By Corollary C.5, we need to show that the obstruction in the first layer, valued in $E_{\infty}^{0,4}$, and the obstruction in the third layer, valued in $E_{\infty}^{3,1}$, each vanish.

- $\mathrm{U}(1)_{5}$ can be extended to an abelian bosonic unitary-MTC with $\mathbb{Z} / 4^{T}$ action.

The most natural candidate of a unitary-MTC that is a modular extension of $\mathrm{U}(1)_{5}$ is $\mathrm{U}(1)_{20}\left[\mathrm{BGH}^{+} 17\right.$, Section 2.6]. Anyons in $\mathrm{U}(1)_{20}$ can be labeled by integers $a=0, \ldots, 19$, and the original anyons $a=0, \ldots, 9$ of $\mathrm{U}(1)_{5}$ embed into $\mathrm{U}(1)_{20}$ as $a \mapsto 2 a$. However, $\mathrm{U}(1)_{20}$ has central charge 1 instead of 0 , hence it has no time-reversal symmetry. Fortunately, the central charge is an integer, and hence we can find another modular extension of $\mathrm{U}(1)_{5}$ that is related to $\mathrm{U}(1)_{20}$ by zesting such that this new modular extension has the desired time-reversal symmetry.

From Eq. (C.6) and Eq. (C.7), we can immediately write down the desired unitary-MTC $\mathcal{B}$, which is a modular extension of $\mathrm{U}(1)_{5}$ with zero central charge. The fusion rules of the objects in $\mathcal{B}$ have group structure $\mathbb{Z} / 2 \times \mathbb{Z} / 10$, with anyons labeled by ( $a, b$ ) with $a=0,1$ and $b=0, \ldots, 9$. The theory $\mathcal{B}$ describes as a $\mathbb{Z} / 2 \times \mathbb{Z} / 10$ gauge theory with an extra Dijkgraaf-Witten twist. $\mathcal{B}$ has trivial $F$-symbols and the $R$-symbols are given by

$$
\begin{equation*}
R_{\left(a_{1}+a_{2}, b_{1}+b_{2}\right)}^{\left(a_{1}, b_{1}\right)\left(,\left(2, b_{2}\right)\right.}=\exp \left(\pi i\left(a_{1} a_{2}+a_{1} b_{2}+\frac{1}{5} b_{1} b_{2}\right)\right) . \tag{C.9}
\end{equation*}
$$

The original anyon labeled by $a=1$ embeds into $\mathcal{B}$ as $(1,1) . \mathcal{B}$ has $\mathbb{Z} / 4^{T}$ time-reversal symmetry generated by the following action

$$
\begin{equation*}
(a, b) \longmapsto(a, 3 \times b(\bmod 10)), \tag{C.10}
\end{equation*}
$$

which is compatible with the original $\mathbb{Z} / 4^{T}$ time-reversal action on $U(1)_{5}$.

- The symmetry fractionalization data of the $\mathbb{Z} / 4^{T}$ symmetry in $U(1)_{5}$ can be extended to the new unitary-MTC $\mathcal{B}$.

All the $U$-symbols and $\eta$-symbols of $\mathrm{U}(1)_{5}$ can be set equal to 1 . It is straightforward to check that the $\mathrm{U}(1)$-symbols and $\eta$-symbols of $\mathcal{B}$ can be set equal to 1 as well.
Hence we conclude that all obstructions vanish and the full anomaly of $\mathrm{U}(1)_{5}$ indeed vanishes.
Lemma C.11. There is no modular extension of $\mathrm{SO}(3)_{3}$ with time-reversal symmetry.
Proof. A modular extension of $\mathrm{SO}(3)_{3}$ is $\mathrm{SU}(2)_{6}$. This theory has central charge $\frac{9}{4}$, an odd multiple of $\frac{1}{4}$. From the 16 fold way classification $\left[\mathrm{BGH}^{+} 17\right]$, all modular extensions of $\mathrm{SO}(3)_{3}$ must have central charge an odd multiple of $\frac{1}{4}$. Therefore, $\mathrm{SO}(3)_{3}$ has no modular extension that has time-reversal symmetry, and the obstruction in $E_{\infty}^{0,4}$ is thus nontrivial.

Remark C.12. The anomaly cascade cannot determine whether $\mathrm{SO}(3)_{3}$, for a particular $\mathbb{Z} / 4^{T}$ action (a particular set of choice of $U$ - and $\eta$-symbols), has anomaly 1 or 3 in $\Omega_{4}^{\text {EPin }} \cong \mathbb{Z} / 4$ in a straightforward manner, because both 1 and 3 lead to the same obstruction at the same level of the anomaly cascade. It is easiest to determine the explicit value by the anomaly indicator for the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry in Proposition 5.56.

The semion-fermion theory $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-1}$ has a simple modular extension $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-4}$, which also has $\mathbb{Z} / 4^{T}$ symmetry. The anyons in $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-4}$ are labeled by $(a, b), a=0,1, b=0, \ldots, 4$, and the anyons $1, s, \tilde{s}, \psi$ in the original semion-fermion theory correspond to $(0,0),(1,0),(1,2)$, $(0,2)$, respectively. The $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry is generated by

$$
\begin{equation*}
(a, b) \longmapsto(a+b \bmod 2,2 a+b \bmod 4) . \tag{C.13}
\end{equation*}
$$

Proposition C.14. The semion-fermion theory realizes the anomaly $\nu=2 \in \mathcal{U}_{\mathrm{EPin}}^{4}$.
Proof sketch. The symmetry fractionalization data of the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ action in $\mathrm{U}(1)_{2} \times \mathrm{U}(1)_{-4}$ cannot be made compatible with the $\eta$-symbols of the original semion-fermion theory. One can check that, in order to be compatible with the $\eta$-symbols of the original semion-fermion theory, Eq. (2.44) cannot be satisfied: the quotient between the left and right hand side is not 1 , but results in the double-braid between two anyons $a$ and $\mathcal{T}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in\{1, \psi\}=\mathbb{Z} / 2$. The phase one picks up from the double braiding is given by $\frac{\theta_{a \times \mathcal{T}}}{\theta_{a} \theta_{\mathcal{T}}}$, and a nontrivial $\mathcal{T}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in\{1, \psi\}$ defines a nontrivial element in $H^{3}\left(B \mathbb{Z} / 4^{T},\{1, \psi\}\right) \cong \mathbb{Z} / 2$. Therefore, we establish that the semion-fermion theory realizes the anomaly $\nu=2 \in \mathcal{V}_{\text {EPin }}^{4}$.

## References

[ABK21] David Aasen, Parsa Bonderson, and Christina Knapp. Characterization and Classification of Fermionic Symmetry Enriched Topological Phases. arXiv e-prints, page arXiv:2109.10911, September 2021. https://arxiv.org/abs/2109.10911. 3, 11, 15
[ABP67] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. Ann. of Math. (2), 86:271-298, 1967. 39, 40
[ABP69] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. Pin cobordism and related topics. Comment. Math. Helv., 44:462-468, 1969. 34
[ABS64] M.F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. Topology, 3(Supplement 1):3-38, 1964. 39
[Ae16] M. I. Aroyo and ed. International Tables for Crystallography. In a Nutshell. Princeton University Press, 2016. 57
[ÁGGMV86] L. Álvarez-Gaumé, P. Ginsparg, G. Moore, and C. Vafa. An $\mathrm{O}(16) \times \mathrm{O}(16)$ heterotic string. Phys. Lett. B, 171(2-3):155-162, 1986. 18
[AM94] M.F. Atiyah and I.G. MacDonald. Introduction To Commutative Algebra. Addison-Wesley series in mathematics. Avalon Publishing, 1994. 42
[And69] D.W. Anderson. Universal coefficient theorems for K-theory. 1969. https://faculty.tcu.edu/ gfriedman/notes/Anderson-UCT.pdf. 39, 40
[Ati61] M. F. Atiyah. Bordism and cobordism. Proc. Cambridge Philos. Soc., 57:200-208, 1961. https: //doi.org/10.1017/s0305004100035064. 36
[Ati88] Michael F. Atiyah. Topological quantum field theory. Publications Mathématiques de l'IHÉS, 68:175186, 1988. 16
[BB20] Daniel Bulmash and Maissam Barkeshli. Absolute anomalies in (2+1)D symmetry-enriched topological states and exact (3+1)D constructions. Physical Review Research, 2(4):043033, October 2020. https: //arxiv.org/abs/2003.11553. 3
[BB22a] Daniel Bulmash and Maissam Barkeshli. Anomaly cascade in (2+1)-dimensional fermionic topological phases. Phys. Rev. B, 105(15):155126, 2022. https://arxiv.org/abs/2109.10922. 3, 6, 39, 55, 57, 67, 68, 69
[BB22b] Daniel Bulmash and Maissam Barkeshli. Fermionic symmetry fractionalization in (2 +1 ) dimensions. Physical Review B, 105(12):125114, March 2022. https://arxiv.org/abs/2109.10913. 2, 4, 11, 13, 15, 29
$\left[\mathrm{BBC}^{+} 19\right]$ Maissam Barkeshli, Parsa Bonderson, Meng Cheng, Chao-Ming Jian, and Kevin Walker. Reflection and Time Reversal Symmetry Enriched Topological Phases of Matter: Path Integrals, Non-orientable Manifolds, and Anomalies. Communications in Mathematical Physics, 374(2):1021-1124, June 2019. https://arxiv.org/abs/1612.07792. 3, 4, 29
[BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. Symmetry Fractionalization, Defects, and Gauging of Topological Phases. Physical Review B, 100(11):115147, September 2019. https://arxiv.org/abs/1410.4540. 2, 3, 11, 52
[BC76] Edgar H. Brown, Jr. and Michael Comenetz. Pontrjagin duality for generalized homology and cohomology theories. Amer. J. Math., 98(1):1-27, 1976. 10
[BC18] Agnès Beaudry and Jonathan A. Campbell. A guide for computing stable homotopy groups. In Topology and quantum theory in interaction, volume 718 of Contemp. Math., pages 89-136. Amer. Math. Soc., [Providence], RI, [2018] (C)2018. https://arxiv.org/abs/1801.07530. 41
[BCFV14] F. J. Burnell, Xie Chen, Lukasz Fidkowski, and Ashvin Vishwanath. Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order. Physical Review B, 90(24):245122, December 2014. https://arxiv.org/abs/1302.7072. 27
[BCP14a] Ilka Brunner, Nils Carqueville, and Daniel Plencner. Orbifolds and topological defects. Comm. Math. Phys., 332(2):669-712, 2014. https://arxiv.org/abs/1307.3141. 16
[BCP14b] Ilka Brunner, Nils Carqueville, and Daniel Plencner. A quick guide to defect orbifolds. In String-Math 2013, volume 88 of Proc. Sympos. Pure Math., pages 231-241. Amer. Math. Soc., Providence, RI, 2014. https://arxiv.org/abs/1310.0062. 16
[BG87a] Anthony Bahri and Peter Gilkey. The eta invariant, $\mathrm{Pin}^{c}$ bordism, and equivariant $\mathrm{Spin}^{c}$ bordism for cyclic 2-groups. Pacific J. Math., 128(1):1-24, 1987. 55, 63
[BG87b] Anthony Bahri and Peter Gilkey. Pin $^{c}$ cobordism and equivariant Spin ${ }^{c}$ cobordism of cyclic 2-groups. Proceedings of the American Mathematical Society, 99(2):380-382, 1987. 63
[BG95] Boris Botvinnik and Peter Gilkey. An analytic computation of $\mathrm{k}_{4 \nu-1}\left(B Q_{8}\right)$. Topol. Methods Nonlinear Anal., 6(1):127-135, 1995. 55
[BG97] Boris Botvinnik and Peter Gilkey. The Gromov-Lawson-Rosenberg conjecture: the twisted case. Houston J. Math., 23(1):143-160, 1997. 5, 35, 38, 42, 43, 47
[BG10] Robert R. Bruner and J. P. C. Greenlees. Connective real K-theory of finite groups, volume 169 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010. 42
$\left[\mathrm{BGH}^{+} 17\right] \quad$ Paul Bruillard, César Galindo, Tobias Hagge, Siu-Hung Ng, Julia Yael Plavnik, Eric C Rowell, and Zhenghan Wang. Fermionic modular categories and the 16 -fold way. Journal of Mathematical Physics, 58(4):041704, 2017. https://arxiv.org/abs/1603.09294. 2, 6, 68, 69, 70
[BGS97] Boris Botvinnik, Peter Gilkey, and Stephan Stolz. The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology. J. Differential Geom., 46(3):374-405, 1997. 44, 46, 55
[Bha17] Lakshya Bhardwaj. Unoriented 3d TFTs. J. High Energy Phys., (5):048, front matter+33, 2017. https://arxiv.org/abs/1611.02728. 25
[BHK23] Maissam Barkeshli, Po-Shen Hsin, and Ryohei Kobayashi. Higher-group symmetry of (3+1)D fermionic $\mathbb{Z}_{2}$ gauge theory: logical CCZ, CS, and T gates from higher symmetry. 11 2023. https://arxiv.org/ abs/2311.05674. 27
[BHW10] John C. Baez, Alexander E. Hoffnung, and Christopher D. Walker. Higher dimensional algebra VII: Groupoidification. Theory Appl. Categ., 24:No. 18, 489-553, 2010. https://arxiv.org/abs/0908. 4305. 28
[BJS18] Adrien Brochier, David Jordan, and Noah Snyder. On dualizability of braided tensor categories. arXiv e-prints, page arXiv:1804.07538, April 2018. https://arxiv.org/abs/1804.07538. 56
[BJSS21] Adrien Brochier, David Jordan, Pavel Safronov, and Noah Snyder. Invertible braided tensor categories. Algebraic \& Geometric Topology, 21(4):2107-2140, 2021. https://arxiv.org/abs/2003.13812. 26, 56
[BK01] B. Bakalov and A.A. Kirillov. Lectures on Tensor Categories and Modular Functors. Translations of Mathematical Monographs. American Mathematical Society, 2001. 11, 52
[BM23] Jonathan Buchanan and Stephen McKean. KSp-characteristic classes determine Spin ${ }^{h}$ cobordism. 2023. https://arxiv.org/abs/2312.08209. 60
[Bot69] Raoul Bott. Lectures on $K(X)$. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New YorkAmsterdam, 1969. 39
[Bre94] Lawrence Breen. On the classification of 2-gerbes and 2-stacks. Astérisque, (225):160, 1994. 22
[Bre10] Lawrence Breen. Notes on 1- and 2-gerbes. 152:193-235, 2010. https://arxiv.org/abs/math/0611317. 22
[Bre23] T. Daniel Brennan. Anomaly Enforced Gaplessness and Symmetry Fractionalization for Spin Sym- $_{G}$ metries. arXiv e-prints, page arXiv:2308.12999, August 2023. https://arxiv.org/abs/2308. 12999. 56
[Bro71] Edgar H. Brown, Jr. The Kervaire invariant of a manifold. In Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), pages 65-71. Amer. Math. Soc., Providence, R.I., 1971. 21
[Bro82] Edgar H. Brown, Jr. The cohomology of $\mathrm{BSO}_{n}$ and $\mathrm{BO}_{n}$ with integer coefficients. Proc. Amer. Math. Soc., 85(2):283-288, 1982. 61
[BS13] Ulrich Bunke and Thomas Schick. Differential orbifold K-theory. J. Noncommut. Geom., 7(4):1027-1104, 2013. https://arxiv.org/abs/0905.4181. 10
[BS23] T. Daniel Brennan and Aiden Sheckler. Anomaly Enforced Gaplessness for Background Flux Anomalies and Symmetry Fractionalization. arXiv e-prints, page arXiv:2311.00093, October 2023. https://arxiv. org/abs/2311.00093. 56
[BW69] Glen E. Bredon and John W. Wood. Non-orientable surfaces in orientable 3-manifolds. Inventiones mathematicae, 7:83-110, 1969. 46
[BY99] Egidio Barrera-Yanez. The eta invariant of twisted products of even-dimensional manifolds whose fundamental group is a cyclic 2 group. Differential Geom. Appl., 11(3):221-235, 1999. 5, 44, 47, 55
[BY06] Egidio Barrera-Yanez. The eta invariant and the "twisted" connective $K$-theory of the classifying space for cyclic 2-groups. Homology Homotopy Appl., 8(2):105-114, 2006. 55
[BYG99] Egidio Barrera-Yanez and Peter B. Gilkey. The eta invariant and the connective $K$-theory of the classifying space for cyclic 2 groups. Ann. Global Anal. Geom., 17(3):289-299, 1999. 44, 55
[Čad99] Martin Čadek. The cohomology of $\mathrm{BO}(n)$ with twisted integer coefficients. J. Math. Kyoto Univ., 39(2):277-286, 1999. 38
[Cam17] Jonathan A. Campbell. Homotopy theoretic classification of symmetry protected phases. 2017. https: //arxiv.org/abs/1708.04264. 39, 41, 47
[Car23] Nils Carqueville. Orbifolds of topological quantum field theories. 2023. https://arxiv.org/abs/2307. 16674. 16
[CDGK20] Changha Choi, Diego Delmastro, Jaume Gomis, and Zohar Komargodski. Dynamics of QCD 3 with Rank-Two Quarks And Duality. JHEP, 03:078, 2020. https://arxiv.org/abs/1810.07720. 2
[CF64] P. E. Conner and E. E. Floyd. Differentiable periodic maps. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Band 33. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964. 35
[CH23] Yu-An Chen and Po-Shen Hsin. Exactly solvable lattice Hamiltonians and gravitational anomalies. SciPost Phys., 14(5):089, 2023. https://arxiv.org/abs/2110.14644. 27
[CHW23] Jin Chen, Babak Haghighat, and Qing-Rui Wang. Para-fusion category and topological defect lines in $\mathbb{Z}_{N}$-parafermionic CFTs. 2023. https://arxiv.org/abs/2309.01914. 21
[CHZ23] Clay Cordova, Po-Shen Hsin, and Carolyn Zhang. Anomalies of Non-Invertible Symmetries in (3+1)d. 8 2023. https://arxiv.org/abs/2308.11706. 5
[Cim09] David Cimasoni. Dimers on graphs in non-orientable surfaces. Lett. Math. Phys., 87(1-2):149-179, 2009. https://arxiv.org/abs/0804.4772. 23
[CKR18a] Yu-An Chen, Anton Kapustin, and Đorđe Radičević. Exact bosonization in two spatial dimensions and a new class of lattice gauge theories. Annals Phys., 393:234-253, 2018. https://arxiv.org/abs/ 1711.00515. 18
[CKR18b] Yu-An Chen, Anton Kapustin, and Đorđe Radičević. Exact bosonization in two spatial dimensions and a new class of lattice gauge theories. Annals Phys., 393:234-253, 2018. https://arxiv.org/abs/ 1711.00515. 21
[CKY93] Louis Crane, Louis H. Kauffman, and David N. Yetter. Evaluating the Crane-Yetter Invariant. 1993. https://arxiv.org/abs/hep-th/9309063. 56
[CKY94] Louis Crane, Louis H. Kauffman, and David N. Yetter. State-Sum Invariants of 4-Manifolds I. arXiv e-prints, pages hep-th/9409167, September 1994. https://arxiv.org/abs/hep-th/9409167. 56
[CM23] Nils Carqueville and Lukas Müller. Orbifold completion of 3-categories. 2023. https://arxiv.org/abs/ 2307.06485. 16
$\left[\mathrm{CMR}^{+} 21\right]$ Nils Carqueville, Vincentas Mulevicius, Ingo Runkel, Gregor Schaumann, and Daniel Scherl. Orbifold graph TQFTs. 2021. https://arxiv.org/abs/2101.02482. 16
[CMS23] Nils Carqueville, Ehud Meir, and Lorant Szegedy. Invariants of r-spin TQFTs and non-semisimplicity. 2023. https://arxiv.org/abs/2306.08608. 21
[CO19] Clay Córdova and Kantaro Ohmori. Anomaly obstructions to symmetry preserving gapped phases. 2019. https://arxiv.org/abs/1910.04962. 56
[CO20] Clay Córdova and Kantaro Ohmori. Anomaly constraints on gapped phases with discrete chiral symmetry. Physical Review D, 102(2):025011, July 2020. https://arxiv.org/abs/1912.13069. 56
[COSY20] Clay Córdova, Kantaro Ohmori, Shu-Heng Shao, and Fei Yan. Decorated $\mathbb{Z}_{2}$ symmetry defects and their time-reversal anomalies. Phys. Rev. D, 102(4):045019, 15, 2020. https://arxiv.org/abs/1910. 14046. 35
[CR07] David Cimasoni and Nicolai Reshetikhin. Dimers on surface graphs and spin structures. I. Comm. Math. Phys., 275(1):187-208, 2007. https://arxiv.org/abs/math-ph/0608070. 23
[CR16] Nils Carqueville and Ingo Runkel. Orbifold completion of defect bicategories. Quantum Topol., 7(2):203279, 2016. https://arxiv.org/abs/1210.6363. 16
[CR18] Nils Carqueville and Ingo Runkel. Introductory lectures on topological quantum field theory. In Advanced school on topological quantum field theory, volume 114 of Banach Center Publ., pages 9-47. Polish Acad. Sci. Inst. Math., Warsaw, 2018. https://arxiv.org/abs/1705.05734. 16
[CRS19] Nils Carqueville, Ingo Runkel, and Gregor Schaumann. Orbifolds of $n$-dimensional defect TQFTs. Geom. Topol., 23(2):781-864, 2019. https://arxiv.org/abs/1705.06085. 16, 22
[CRS20] Nils Carqueville, Ingo Runkel, and Gregor Schaumann. Orbifolds of Reshetikhin-Turaev TQFTs. Theory Appl. Categ., 35:Paper No. 15, 513-561, 2020. https://arxiv.org/abs/1809.01483. 16
[CS23] Nils Carqueville and Lóránt Szegedy. Fully extended r-spin TQFTs. Quantum Topol., 14(3):467-532, 2023. https://arxiv.org/abs/2107.02046. 21
[CW92] S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J., 39(2):325-351, 1992. 38
[CW16] Steven R. Costenoble and Stefan Waner. Equivariant ordinary homology and cohomology, volume 2178 of Lecture Notes in Mathematics. Springer, Cham, 2016. https://arxiv.org/abs/math/0310237. 38
[CY93] Louis Crane and David Yetter. A categorical construction of 4D topological quantum field theories. In Quantum topology, volume 3 of Ser. Knots Everything, pages 120-130. World Sci. Publ., River Edge, NJ, 1993. https://arxiv.org/abs/hep-th/9301062. 26, 29, 56
$\left[\mathrm{CZB}^{+} 16\right]$ Meng Cheng, Michael Zaletel, Maissam Barkeshli, Ashvin Vishwanath, and Parsa Bonderson. Translational Symmetry and Microscopic Constraints on Symmetry-Enriched Topological Phases: A View from the Surface. Physical Review X, 6(4):041068, December 2016. https://arxiv.org/abs/1511.02263. 2
[DDHM22] Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. The anomaly that was not meant IIB. Fortschr. Phys., 70(1):Paper No. 2100168, 31, 2022. https://arxiv.org/abs/2107.14227. 63
[DDHM23] Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. The chronicles of IIBordia: Dualities, bordisms, and the Swampland. 2023. https://arxiv.org/abs/2302.00007. 8, 35, 44, 48, 55, 56, 63
$\left[\mathrm{DDK}^{+} 23\right]$ Arun Debray, Sanath K. Devalapurkar, Cameron Krulewski, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren. A long exact sequence in symmetry breaking: order parameter constraints, defect anomaly-matching, and higher Berry phases. 2023. https://arxiv.org/abs/2309.16749. 5, 11, 35, 36, 37, 47, 60
[Deb21] Arun Debray. Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle. 2021. https://arxiv.org/abs/2102.02941. 9, 38, 39, 56, 57
[Deb23] Arun Debray. Bordism for the 2-group symmetries of the heterotic and CHL strings. 2023. https: //arxiv.org/abs/2304.14764. 35
[Déc23] Thibault D. Décoppet. On the dualizability of fusion 2-categories, and the relative 2-Deligne tensor product. 2023. https://arxiv.org/abs/2311.16827. 25
[DG18] Arun Debray and Sam Gunningham. The Arf-Brown TQFT of pin ${ }^{-}$surfaces. In Topology and quantum theory in interaction, volume 718 of Contemp. Math., pages 49-87. Amer. Math. Soc., Providence, RI, 2018. https://arxiv.org/abs/1803.11183. 16, 20
[DG21] Diego Delmastro and Jaume Gomis. Symmetries of Abelian Chern-Simons Theories and Arithmetic. JHEP, 03:006, 2021. https://arxiv.org/abs/1904.12884. 4, 6, 35, 55, 56
[DGG21] Diego Delmastro, Davide Gaiotto, and Jaume Gomis. Global anomalies on the Hilbert space. Journal of High Energy Physics, 2021(11):142, November 2021. https://arxiv.org/abs/2101.02218. 9
[DGL22] Joe Davighi, Ben Gripaios, and Nakarin Lohitsiri. Anomalies of non-Abelian finite groups via cobordism. J. High Energy Phys., (9):Paper No. 147, 44, 2022. https://arxiv.org/abs/2207.10700. 55
$\left[\mathrm{DGP}^{+} 21\right]$ Colleen Delaney, César Galindo, Julia Plavnik, Eric C Rowell, and Qing Zhang. Braided zesting and its applications. Communications in Mathematical Physics, 386:1-55, 2021. https://arxiv.org/abs/ 2005.05544. 6, 68
[DGY23] Diego Delmastro, Jaume Gomis, and Matthew Yu. Infrared phases of 2d QCD. JHEP, 02:157, 2023. https://arxiv.org/abs/2108.02202. 2
[DJL23] Zhihao Duan, Qiang Jia, and Sungjay Lee. $\mathbb{Z}_{N}$ duality and parafermions revisited. Journal of High Energy Physics, 2023(206), 2023. https://arxiv.org/abs/2309.01913. 21
[DL21] Joe Davighi and Nakarin Lohitsiri. The algebra of anomaly interplay. SciPost Phys., 10(3):074, 2021. https://arxiv.org/abs/2011.10102. 42
[DL223] Toric 2-group anomalies via cobordism. J. High Energy Phys., (7):Paper No. 19, 52, 2023. With a mathematical appendix by Arun Debray. https://arxiv.org/abs/2302.12853. 35, 37
[DN20] A. Davydov and D. Nikshych. On minimal non-degenerate extensions of braided tensor categories, March 2020. Tensor categories and topological quantum field theories, Mathematical Science Reserach Institute. 68
[Don78] Harold Donnelly. Eta invariants for G-spaces. Indiana Univ. Math. J., 27(6):889-918, 1978. 44
[DR18] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds. 2018. https://arxiv.org/abs/1812.11933. 27
[DW90] Robbert Dijkgraaf and Edward Witten. Topological gauge theories and group cohomology. Comm. Math. Phys., 129(2):393-429, 1990. 17
[DY22] Arun Debray and Matthew Yu. What bordism-theoretic anomaly cancellation can do for U. 2022. https://arxiv.org/abs/2210.04911.9
[DY23a] Arun Debray and Matthew Yu. Adams spectral sequences for non-vector-bundle Thom spectra. 2023. https://arxiv.org/abs/2305.01678. 7, 9
[DY23b] Thibault D. Décoppet and Matthew Yu. Fiber 2-functors and Tambara-Yamagami fusion 2-categories. 2023. https://arxiv.org/abs/2306.08117. 25
[EE69] Clifford J. Earle and James Eells. A fibre bundle description of Teichmüller theory. Journal of Differential Geometry, 3(1-2):19 - 43, 1969. 44, 45
[EGNO16] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor Categories. Mathematical Surveys and Monographs. American Mathematical Society, 2016. https://math.mit.edu/~etingof/egnobookfinal. pdf. 11, 16
[ENO10] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Fusion categories and homotopy theory. Quantum topology, 1(3):209-273, 2010. With an appendix by Ehud Meir. https://arxiv.org/abs/0909.3140. 11, 25
[ET20] Dominic V. Else and Ryan Thorngren. Topological theory of Lieb-Schultz-Mattis theorems in quantum spin systems. Physical Review B, 101(22):224437, 2020. https://arxiv.org/abs/1907.08204. 2
[FH20] Daniel S. Freed and Michael J. Hopkins. Invertible phases of matter with spatial symmetry. Adv. Theor. Math. Phys., 24(7):1773-1788, 2020. https://arxiv.org/abs/1901.06419. 39
[FH21] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. Geom. Topol., 25(3):1165-1330, 2021. https://arxiv.org/abs/1604.06527. 2, 4, 9, 10, 16, 39, 57, 60, 62, 63, 64
[FHH22] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings. Gravitational anomaly of (3+1)dimensional Z2 toric code with fermionic charges and fermionic loop self-statistics. Phys. Rev. B, 106(16):165135, 2022. https://arxiv.org/abs/2110.14654. 27
[FHLT09] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman. Topological quantum field theories from compact Lie groups. In A Celebration of Raoul Bott's Legacy in Mathematics, 52009. https://arxiv.org/abs/0905.0731. 16, 22
[FHT10] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Consistent orientation of moduli spaces. In The many facets of geometry, pages 395-419. Oxford Univ. Press, Oxford, 2010. https: //arxiv.org/abs/0711.1909. 8, 9, 19, 32
[FK80] Eduardo Fradkin and Leo P. Kadanoff. Disorder variables and para-fermions in two-dimensional statistical mechanics. Nuclear Physics B, 170(1):1-15, 1980. 21
[FM06] Daniel S. Freed and Gregory W. Moore. Setting the quantum integrand of M-theory. Comm. Math. Phys., 263(1):89-132, 2006. https://arxiv.org/abs/hep-th/0409135. 8, 19
[FMP07] João Faria Martins and Timothy Porter. On Yetter's invariant and an extension of the Dijkgraaf-Witten invariant to categorical groups. Theory Appl. Categ., 18:No. 4, 118-150, 2007. https://arxiv.org/ abs/math/0608484. 22
[FMP23] João Faria Martins and Timothy Porter. A categorification of Quinn's finite total homotopy TQFT with application to TQFTs and once-extended TQFTs derived from strict omega-groupoids. 2023. https://arxiv.org/abs/2301.02491. 22
[FMS07] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. The uncertainty of fluxes. Communications in Mathematical Physics, 271(1):247-274, Apr 2007. https://arxiv.org/abs/hep-th/0605198. 39
[FMT22] Daniel S. Freed, Gregory W. Moore, and Constantin Teleman. Topological symmetry in quantum field theory. 2022. https://arxiv.org/abs/2209.07471. 17, 28
[FPSV15] Jürgen Fuchs, Jan Priel, Christoph Schweigert, and Alessandro Valentino. On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories. Commun. Math. Phys., 339(2):385-405, 2015. https://arxiv.org/abs/1404.6646. 17
[FQ93] Daniel S. Freed and Frank Quinn. Chern-Simons theory with finite gauge group. Commun. Math. Phys., 156:435-472, 1993. https://arxiv.org/abs/hep-th/9111004. 16, 17, 19, 20, 22, 24
[Fre93] Daniel S. Freed. Extended structures in topological quantum field theory. In Quantum topology, volume 3 of Ser. Knots Everything, pages 162-173. World Sci. Publ., River Edge, NJ, 1993. https: //arxiv.org/abs/hep-th/9306045. 16
[Fre94] Daniel S. Freed. Higher algebraic structures and quantization. Comm. Math. Phys., 159(2):343-398, 1994. https://arxiv.org/abs/hep-th/9212115. 16, 22
[Fre95] Daniel S. Freed. Characteristic numbers and generalized path integrals. In Geometry, topology, $\mathcal{E}$ physics, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 126-138. Int. Press, Cambridge, MA, 1995. https://arxiv.org/abs/dg-ga/9406002. 16
[Fre99] Daniel S. Freed. Quantum groups from path integrals. In Particles and fields (Banff, AB, 1994), CRM Ser. Math. Phys., pages 63-107. Springer, New York, 1999. https://arxiv.org/abs/q-alg/9501025. 16
[Fre12] Dan Freed. Lectures on twisted $K$-theory and orientifolds. 2012. https://people.math.harvard.edu/ ~dafr/vienna.pdf. 16, 20
[Fre13] Daniel S. Freed. The cobordism hypothesis. Bull. Amer. Math. Soc. (N.S.), 50(1):57-92, 2013. https: //arxiv.org/abs/1210.5100. 16
[Fre14] Daniel S. Freed. Anomalies and invertible field theories. In String-Math 2013, volume 88 of Proc. Sympos. Pure Math., pages 25-45. Amer. Math. Soc., Providence, RI, 2014. https://arxiv.org/abs/1404.7224. 8
[Fre19] D.S. Freed. Lectures on Field Theory and Topology. CBMS Regional Conference Series in Mathematics. Conference Board of the Mathematical Sciences, 2019. 16
[Fre23] Daniel S. Freed. What is an anomaly? 2023. https://arxiv.org/abs/2307.08147. 8
[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators 1. Partition functions. Nucl. Phys. B, 646:353-497, 2002. https://arxiv.org/abs/hep-th/0204148. 17
[FSV13] Jürgen Fuchs, Christoph Schweigert, and Alessandro Valentino. Bicategories for boundary conditions and for surface defects in 3-d TFT. Commun. Math. Phys., 321:543-575, 2013. https://arxiv.org/ abs/1203.4568. 17
[FT14] Daniel S. Freed and Constantin Teleman. Relative quantum field theory. Comm. Math. Phys., 326(2):459476, 2014. https://arxiv.org/abs/1212.1692. 8, 16
[FT22] Daniel S. Freed and Constantin Teleman. Topological dualities in the Ising model. Geom. Topol., 26(5):1907-1984, 2022. https://arxiv.org/abs/1806.00008. 18, 22, 28
[GB96] Peter B. Gilkey and Boris Botvinnik. The eta invariant and the equivariant spin bordism of spherical space form 2 groups. In New developments in differential geometry (Debrecen, 1994), volume 350 of Math. Appl., pages 213-223. Kluwer Acad. Publ., Dordrecht, 1996. 55
[GEM19] Inaki García-Etxebarria and Miguel Montero. Dai-Freed anomalies in particle physics. JHEP, 08:003, 2019. https://arxiv.org/abs/1808.00009. 42
[GGP12] Hansjörg Geiges and Jesús Gonzalo Pérez. Generalised spin structures on 2-dimensional orbifolds. Osaka J. Math., 49(2):449-470, 2012. https://arxiv.org/abs/1004.1979. 21
[GHP21] Sergei Gukov, Po-Shen Hsin, and Du Pei. Generalized global symmetries of $T[M]$ theories. Part I. $J$. High Energy Phys., (4):Paper No. 232, 109, 2021. https://arxiv.org/abs/2010.15890. 25
[Gia73] V. Giambalvo. Pin and Pin' cobordism. Proc. Amer. Math. Soc., 39:395-401, 1973. 32
[Gia76] V. Giambalvo. Cobordism of spin manifolds with involution. Quart. J. Math. Oxford Ser. (2), 27(106):241-252, 1976. 47, 56
[Gil84] P. B. Gilkey. The eta invariant and the $K$-theory of odd-dimensional spherical space forms. Invent. Math., 76(3):421-453, 1984. 55
[Gil85] Peter B. Gilkey. The eta invariant for even-dimensional PIN ${ }_{c}$ manifolds. Adv. in Math., 58(3):243-284, 1985. 55
[Gil87] Peter B. Gilkey. The eta invariant and $\tilde{K} O$ of lens spaces. Math. Z., 194(3):309-320, 1987. 55
[Gil88a] Peter B. Gilkey. The eta invariant and equivariant Spin ${ }^{c}$ bordism for spherical space form groups. Canad. J. Math., 40(2):392-428, 1988. 55
[Gil88b] Peter B. Gilkey. The eta invariant and the equivariant unitary bordism of spherical space form groups. Compositio Math., 65(1):33-50, 1988. 55
[Gil89] Peter B. Gilkey. An analytic computation of the additive structure of $M \mathrm{U}^{*}\left(B Z_{4}\right)$. In Geometry and topology, pages 152-158. World Sci. Publishing, Singapore, 1989. 55
[GJF19] Davide Gaiotto and Theo Johnson-Freyd. Condensations in higher categories. 2019. https://arxiv. org/abs/1905.09566. 16, 27
[GK16] Davide Gaiotto and Anton Kapustin. Spin TQFTs and fermionic phases of matter. International Journal of Modern Physics A, 31:1645044-184, October 2016. https://arxiv.org/abs/1505.05856. 4, 21, 23, 24, 30
[GK21] Davide Gaiotto and Justin Kulp. Orbifold groupoids. JHEP, 02:132, 2021. https://arxiv.org/abs/ 2008.05960. 17, 18
[GKS18] Jaume Gomis, Zohar Komargodski, and Nathan Seiberg. Phases Of Adjoint QCD 3 And Dualities. SciPost Phys., 5(1):007, 2018. https://arxiv.org/abs/1710.03258. 2
[GKT89] J. Gunarwardena, B. Kahn, and C. Thomas. Stiefel-Whitney classes of real representations of finite groups. J. Algebra, 126(2):327-347, 1989. 9
[GM80] Lucien Guillou and Alexis Marin. Une extension d'un théorème de Rohlin sur la signature. In Seminar on Real Algebraic Geometry (Paris, 1977/1978 and Paris, 1978/1979), volume 9 of Publ. Math. Univ. Paris VII, pages 69-80. Univ. Paris VII, Paris, 1980. 63
[GMTW09] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. The homotopy type of the cobordism category. Acta Mathematica, 202(2):195-239, 2009. https://arxiv.org/abs/math/0605249. 20
$\left[\mathrm{GOP}^{+} 20\right]$ Meng Guo, Kantaro Ohmori, Pavel Putrov, Zheyan Wan, and Juven Wang. Fermionic finite-group gauge theories and interacting symmetric/crystalline orders via cobordisms. Comm. Math. Phys., 376(2):1073-1154, 2020. https://arxiv.org/abs/1812.11959. 56
[GPW18] Meng Guo, Pavel Putrov, and Juven Wang. Time reversal, SU( $N$ ) Yang-Mills and cobordisms: Interacting topological superconductors/insulators and quantum spin liquids in 3+1d. Annals of Physics, 394:244-293, 2018. https://arxiv.org/abs/1711.11587. 64
[Gra23] Daniel Grady. Deformation classes of invertible field theories and the Freed-Hopkins conjecture. 2023. https://arxiv.org/abs/2310.15866. 10
[GS99] R.E. Gompf and A. Stipsicz. 4-Manifolds and Kirby Calculus. Graduate studies in mathematics. American Mathematical Society, 1999. 3, 60
[GV17] César Galindo and César F. Venegas-Ramírez. Categorical fermionic actions and minimal modular extensions. 2017. https://arxiv.org/abs/1712.07097. 2, 11, 13, 15, 68
[GW14] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear $\sigma$ models and a special group supercohomology theory. Phys. Rev. B, 90:115141, Sep 2014. https://arxiv.org/abs/1201.2648. 23
[Har20] Yonatan Harpaz. Ambidexterity and the universality of finite spans. Proc. Lond. Math. Soc. (3), 121(5):1121-1170, 2020. https://arxiv.org/abs/1703.09764. 16, 22
[HHLZ22] Fei Han, Ruizhi Huang, Kefeng Liu, and Weiping Zhang. Cubic forms, anomaly cancellation and modularity. Adv. Math., 394:Paper No. 108023, 46, 2022. https://arxiv.org/abs/2005.02344. 63
[HJ20] Fabian Hebestreit and Michael Joachim. Twisted spin cobordism and positive scalar curvature. J. Topol., 13(1):1-58, 2020. https://arxiv.org/abs/1311.3164. 9
[HJJ22] Po-Shen Hsin, Wenjie Ji, and Chao-Ming Jian. Exotic invertible phases with higher-group symmetries. SciPost Phys., 12(2):Paper No. 052, 48, 2022. https://arxiv.org/abs/2105.09454. 25
[HKT20] Itamar Hason, Zohar Komargodski, and Ryan Thorngren. Anomaly Matching in the Symmetry Broken Phase: Domain Walls, CPT, and the Smith Isomorphism. SciPost Phys., 8:62, 2020. https: //arxiv.org/abs/1910.14039. 5, 11, 35
[HLN20] Fabian Hebestreit, Markus Land, and Thomas Nikolaus. On the homotopy type of L-spectra of the integers. Journal of Topology, 14(1):183-214, 2020. https://arxiv.org/abs/2004.06889. 39
[Hoo80] G. 't Hooft. Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking. In G. 't Hooft, C. Itzykson, A. Jaffe, H. Lehmann, P. K. Mitter, I. M. Singer, and R. Stora, editors, Recent Developments in Gauge Theories, pages 135-157. Springer US, Boston, MA, 1980. 2
[HOS04] T. H. Hansson, Vadim Oganesyan, and S. L. Sondhi. Superconductors are topologically ordered. Annals of Physics, 313(2):497-538, October 2004. https://arxiv.org/abs/cond-mat/0404327. 27
[HS14] Drew Heard and Vesna Stojanoska. K-theory, reality, and duality. Journal of K-theory, 14(3):526-555, 2014. https://arxiv.org/abs/1401.2581. 39
[Hsi18] Chang-Tse Hsieh. Discrete gauge anomalies revisited. 2018. https://arxiv.org/abs/1808.02881. 55
[HTY22] Chang-Tse Hsieh, Yuji Tachikawa, and Kazuya Yonekura. Anomaly inflow and p-form gauge theories. Comm. Math. Phys., 391(2):495-608, 2022. https://arxiv.org/abs/2003.11550. 55
[Hu23] Jiahao Hu. Invariants of real vector bundles. 2023. https://arxiv.org/abs/2310.05061. 60, 61
[Ina21] Kansei Inamura. Topological field theories and symmetry protected topological phases with fusion category symmetries. J. High Energy Phys., (5):Paper No. 204, 34, 2021. https://arxiv.org/abs/ 2103.15588. 25
[Ina23] Kansei Inamura. Fermionization of fusion category symmetries in $1+1$ dimensions. J. High Energy Phys., (10):Paper No. 101, 62, 2023. https://arxiv.org/abs/2206.13159. 18
[JF20] Theo Johnson-Freyd. (3+1)D topological orders with only a $\mathbb{Z}_{2}$-charged particle. 11 2020. https: //arxiv.org/abs/2011.11165. 18, 27
[JF22] Theo Johnson-Freyd. On the classification of topological orders. Commun. Math. Phys., 393(2):9891033, 2022. https://arxiv.org/abs/2003.06663. 11
[JFY22] Theo Johnson-Freyd and Matthew Yu. Topological orders in (4+1)-dimensions. SciPost Phys., 13(3):068, 2022. https://arxiv.org/abs/2104.04534. 11
[JR23] Theo Johnson-Freyd and David Reutter. Minimal nondegenerate extensions. Journal of the American Mathematical Society, 2023. https://arxiv.org/abs/2105.15167. 68
[JSW20] Wenjie Ji, Shu-Heng Shao, and Xiao-Gang Wen. Topological Transition on the Conformal Manifold. Phys. Rev. Res., 2(3):033317, 2020. https://arxiv.org/abs/1909.01425. 18
[KB21] Ryohei Kobayashi and Maissam Barkeshli. (3+1)D path integral state sums on curved U(1) bundles and $\mathrm{U}(1)$ anomalies of $(2+1) \mathrm{D}$ topological phases. November 2021. https://arxiv.org/abs/2111.14827. 3, 57
[KK12] Alexei Kitaev and Liang Kong. Models for gapped boundaries and domain walls. Communications in Mathematical Physics, 313(2):351-373, 2012. https://arxiv.org/abs/1104.5047. 11
[KKNO73] U. Karras, M. Kreck, W. D. Neumann, and E. Ossa. Cutting and pasting of manifolds; SK-groups. Publish or Perish, Inc., Boston, Mass., 1973. Mathematics Lecture Series, No. 1. 9
$\left[\mathrm{KLW}^{+} 20\right]$ Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng. Classification of topological phases with finite internal symmetries in all dimensions. JHEP, 09:093, 2020. https://arxiv.org/ abs/2003.08898. 11
[Kob19] Ryohei Kobayashi. Pin TQFT and Grassmann integral. Journal of High Energy Physics, 2019(12):14, December 2019. https://arxiv.org/abs/1905.05902. 21, 23, 25, 30
[Kob22a] Ryohei Kobayashi. Fermionic topological phases and bosonization in higher dimensions. PTEP, 2022(4):04A105, 2022. 21, 25
[Kob22b] Ryohei Kobayashi. Lattice construction of exotic invertible topological phases. Phys. Rev. B, 105:035153, Jan 2022. https://arxiv.org/abs/2106.10703. 25
[KPMT20] Justin Kaidi, Julio Parra-Martinez, and Yuji Tachikawa. Topological superconductors on superstring worldsheets. SciPost Phys., 9(1):Paper No. 010, 70, 2020. With a mathematical appendix by Arun Debray. https://arxiv.org/abs/1911.11780. 42, 55, 56
[Kre99] Matthias Kreck. Surgery and duality. Ann. of Math. (2), 149(3):707-754, 1999. https://arxiv.org/ abs/math/9905211. 7
[KS18] Zohar Komargodski and Nathan Seiberg. A symmetry breaking scenario for QCD 3 . JHEP, 01:109, 2018. https://arxiv.org/abs/1706.08775. 2
[KT90] R. C. Kirby and L. R. Taylor. Pin structures on low-dimensional manifolds. In Geometry of lowdimensional manifolds, 2 (Durham, 1989), volume 151 of London Math. Soc. Lecture Note Ser., pages 177-242. Cambridge Univ. Press, Cambridge, 1990. 21, 56, 63
[KT17] Anton Kapustin and Ryan Thorngren. Fermionic SPT phases in higher dimensions and bosonization. JHEP, 10:080, 2017. https://arxiv.org/abs/1701.08264. 18
[KTT19] Andreas Karch, David Tong, and Carl Turner. A web of 2d dualities: $\mathbf{Z}_{2}$ gauge fields and Arf invariants. SciPost Phys., 7:007, 2019. https://arxiv.org/abs/1902.05550. 18
[KTTW15] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. Fermionic symmetry protected topological phases and cobordisms. JHEP, 12:052, 2015. https://arxiv.org/abs/1406.7329. 9
[KWZ15] Liang Kong, Xiao-Gang Wen, and Hao Zheng. Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers. 2015. https://arxiv.org/abs/1502.01690. 11
[KWZ17] Liang Kong, Xiao-Gang Wen, and Hao Zheng. Boundary-bulk relation in topological orders. Nuclear Physics B, 922:62-76, 2017. https://arxiv.org/abs/1702.00673. 11
[KZ20] Liang Kong and Hao Zheng. A mathematical theory of gapless edges of 2d topological orders. Part I. Journal of High Energy Physics, 2020(2):150, February 2020. https://arxiv.org/abs/1905.04924. 56
[Las63] R. Lashof. Poincaré duality and cobordism. Trans. Amer. Math. Soc., 109:257-277, 1963. 8
[Lei08] Tom Leinster. The Euler characteristic of a category. Doc. Math., 13:21-49, 2008. https://arxiv.org/ abs/math/0610260. 28
[Lic65] W. B. R. Lickorish. On the homeomorphisms of a non-orientable surface. Proc. Cambridge Philos. Soc., 61:61-64, 1965. 45
[Liu23] Yu Leon Liu. Abelian duality in topological field theory. Comm. Math. Phys., 398(1):439-468, 2023. https://arxiv.org/abs/2112.02199. 22, 28
[LL19] Matthew F. Lapa and Michael Levin. Anomaly indicators for topological orders with U(1) and timereversal symmetry. Physical Review B, 100(16):165129, October 2019. https://arxiv.org/abs/1905. 00435. 3, 6, 57
[LM89] H.B. Lawson and M.L. Michelsohn. Spin Geometry (PMS-38), Volume 38. Princeton Mathematical Series. Princeton University Press, 1989. 59
[LS22] C. Lazaroiu and C. S. Shahbazi. Dirac operators on real spinor bundles of complex type. Differential Geom. Appl., 80:Paper No. 101849, 53, 2022. https://arxiv.org/abs/1809.09084. 63
[Lur09a] Jacob Lurie. Higher topos theory. Princeton University Press, 2009. https://www.math.ias.edu/ ~lurie/papers/HTT.pdf. 22
[Lur09b] Jacob Lurie. On the classification of topological field theories. In Current developments in mathematics, 2008, pages 129-280. Int. Press, Somerville, MA, 2009. https://arxiv.org/abs/0905.0465. 16
[LW19] Tian Lan and Xiao-Gang Wen. Classification of 3+1 d bosonic topological orders (ii): The case when some pointlike excitations are fermions. Physical Review X, 9(2):021005, 2019. https://arxiv.org/ abs/1801.08530. 11
[Mah82] Mark Mahowald. The image of $J$ in the EHP sequence. Ann. of Math. (2), 116(1):65-112, 1982. Correction in Annals of Mathematics, 120:399-400, 1984. 42
[Mal11] Arjun Malhotra. The Gromov-Lawson-Rosenberg conjecture for some finite groups. PhD thesis, University of Sheffield, 2011. https://arxiv.org/abs/1305.0455. 55
[Mau63] C. R. F. Maunder. The spectral sequence of an extraordinary cohomology theory. Proc. Cambridge Philos. Soc., 59:567-574, 1963. 67
[MCB23] Naren Manjunath, Vladimir Calvera, and Maissam Barkeshli. Nonperturbative constraints from symmetry and chirality on Majorana zero modes and defect quantum numbers in ( $2+1$ ) dimensions. Phys. Rev. B, 107:165126, Apr 2023. https://arxiv.org/abs/2210.02452. 38
[Mil23] Keith Mills. The structure of the spin^h bordism spectrum. 2023. https://arxiv.org/abs/2306. 17709. 60
[MM76] M. Mahowald and R. James Milgram. Operations which detect $S q^{4}$ in connective $K$-theory and their applications. Quart. J. Math. Oxford Ser. (2), 27(108):415-432, 1976. 42
[Mon15] Samuel Monnier. Higher abelian Dijkgraaf-Witten theory. Lett. Math. Phys., 105(9):1321-1331, 2015. https://arxiv.org/abs/1502.04706. 22
[Mor15] Jeffrey C. Morton. Cohomological twisting of 2-linearization and extended TQFT. J. Homotopy Relat. Struct., 10(2):127-187, 2015. https://arxiv.org/abs/1003.5603. 16, 22
[MR15] Arjun Malhotra and Kijti Rodtes. The Gromov-Lawson-Rosenberg conjecture for the semi-dihedral group of order 16. Glasg. Math. J., 57(2):365-386, 2015. https://arxiv.org/abs/1103.5817. 55
[MS23] Lukas Müller and Luuk Stehouwer. Reflection structures and spin statistics in low dimensions. 2023. https://arxiv.org/abs/2301.06664. 21
[MT01] Ib Madsen and Ulrike Tillmann. The stable mapping class group and $Q\left(\mathbb{C P}_{+}^{\infty}\right)$. Invent. Math., 145(3):509-544, 2001. 9
[Müg03] Michael Müger. On the structure of modular categories. Proc. London Math. Soc. (3), 87(2):291-308, 2003. https://arxiv.org/abs/math/0201017. 68
[MW07] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. Ann. of Math. (2), 165(3):843-941, 2007. https://arxiv.org/abs/math/0212321. 9
[MW20] Lukas Müller and Lukas Woike. Parallel transport of higher flat gerbes as an extended homotopy quantum field theory. J. Homotopy Relat. Struct., 15(1):113-142, 2020. https://arxiv.org/abs/1802. 10455. 22
[Ngu17] Hoang Kim Nguyen. On the infinite loop space structure of the cobordism category. Algebraic © Geometric Topology, 17(2):1021-1040, 2017. https://arxiv.org/abs/1505.03490. 20
[NMLW21] Shang-Qiang Ning, Bin-Bin Mao, Zhengqiao Li, and Chenjie Wang. Anomaly indicators and bulkboundary correspondences for three-dimensional interacting topological crystalline phases with mirror and continuous symmetries. Physical Review B, 104(7):075111, August 2021. https://arxiv.org/abs/ 2105.02682. 3, 6, 56, 57, 64, 65
[Nov15] Sebastian Novak. Lattice topological field theories in two dimensions. PhD thesis, Universität Hamburg, 2015. https://ediss.sub.uni-hamburg.de/bitstream/ediss/6467/1/Dissertation.pdf. 21
[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. Principal $\infty$-bundles: general theory. $J$. Homotopy Relat. Struct., 10(4):749-801, 2015. https://arxiv.org/abs/1207.0248. 22
[Olb07] Martin Olbermann. Conjugations on 6-Manifolds. PhD thesis, Heidelburg University, 2007. http: //archiv.ub.uni-heidelberg.de/volltextserver/7450/1/tmmain.pdf. 9
[Ped17] Riccardo Pedrotti. Stable classification of certain families of four-manifolds. Master's thesis, Max Planck Institute for Mathematics, 2017. 42, 56
[Por98] Tim Porter. Topological quantum field theories from homotopy $n$-types. J. London Math. Soc. (2), 58(3):723-732, 1998. https://doi.org/10.1112/S0024610798006838. 22
[Por07] Timothy Porter. Formal homotopy quantum field theories. II. Simplicial formal maps. In Categories in algebra, geometry and mathematical physics, volume 431 of Contemp. Math., pages 375-403. Amer. Math. Soc., Providence, RI, 2007. https://arxiv.org/abs/math/0512034. 22
[PT08] Timothy Porter and Vladimir Turaev. Formal homotopy quantum field theories. I. Formal maps and crossed C-algebras. J. Homotopy Relat. Struct., 3(1):113-159, 2008. https://arxiv.org/abs/math/ 0512032. 22
[Que21] Thomas Quella. Symmetry-protected topological phases beyond groups: The $q$-deformed bilinearbiquadratic spin chain. Phys. Rev. B, 103:054404, Feb 2021. https://arxiv.org/abs/2011. 12679. 25
[Qui95] Frank Quinn. Lectures on axiomatic topological quantum field theory. In Geometry and quantum field theory (Park City, UT, 1991), volume 1 of IAS/Park City Math. Ser., pages 323-453. Amer. Math. Soc., Providence, RI, 1995. 22, 28
[Rad18] Đorđe Radičević. Spin structures and exact dualities in low dimensions. 2018. https://arxiv.org/ abs/1809.07757. 21
[Rei63] Bruce L. Reinhart. Cobordism and the Euler number. Topology, 2(1):173-177, 1963. 9, 19
[Ric16] Nicolas Ricka. Equivariant Anderson duality and Mackey functor duality. Glasg. Math. J., 58(3):649-676, 2016. https://arxiv.org/abs/1408.1581. 39
[RS21] Ingo Runkel and Lóránt Szegedy. Topological field theory on $r$-spin surfaces and the Arf-invariant. $J$. Math. Phys., 62(10):Paper No. 102302, 34, 2021. https://arxiv.org/abs/1802.09978. 21
[Run20] Ingo Runkel. String-net models for nonspherical pivotal fusion categories. J. Knot Theory Ramifications, 29(6):2050035, 40, 2020. https://arxiv.org/abs/1907.12532. 21
[RW14] Oscar Randal-Williams. Homology of the moduli spaces and mapping class groups of framed, $r$-Spin and Pin surfaces. J. Topol., 7(1):155-186, 2014. https://arxiv.org/abs/1001.5366. 20, 21
[Seg88] G. B. Segal. The Definition of Conformal Field Theory, pages 165-171. Springer Netherlands, Dordrecht, 1988. 16
[Sel11] P. Selinger. A survey of graphical languages for monoidal categories. In Bob Coecke, editor, New Structures for Physics, pages 289-355. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. https: //arxiv.org/abs/0908.3347. 11, 52
[Sie67] Jerrold Siegel. Higher order cohomology operations in local coefficient theory. Amer. J. Math., 89:909931, 1967. 38
[Sie13] Christian Siegemeyer. On the Gromov-Lawson-Rosenberg Conjecture for Finite Abelian 2-Groups of Rank 2. PhD thesis, Westfälische Wilhelms-Universität Münster, 2013. https://repositorium. uni-muenster.de/document/miami/4882c034-0925-432c-b743-edf299b5192b/diss_siegemeyer.pdf. 42
[SP17] Christopher Schommer-Pries. Invertible topological field theories. 2017. https://arxiv.org/abs/1712. 08029. 20
[SP18] Christopher J. Schommer-Pries. Tori detect invertibility of topological field theories. Geom. Topol., 22(5):2713-2756, 2018. https://arxiv.org/abs/1511.01772. 20, 26
[SS22] Walker H. Stern and Lóránt Szegedy. Topological field theories on open-closed $r$-spin surfaces. Topology Appl., 312:Paper No. 108062, 40, 2022. https://arxiv.org/abs/2004.14181. 21
[SSGR18] Ken Shiozaki, Hassan Shapourian, Kiyonori Gomi, and Shinsei Ryu. Many-body topological invariants for fermionic short-range entangled topological phases protected by antiunitary symmetries. Phys. Rev. B, 98:035151, Jul 2018. https://arxiv.org/abs/1710.01886. 63
[ST10] Mihai D. Staic and Vladimir Turaev. Remarks on 2-dimensional HQFTs. Algebr. Geom. Topol., 10(3):1367-1393, 2010. https://arxiv.org/abs/0912.1380. 22
[Ste16] Walker H. Stern. Structured topological field theories via crossed simplicial groups. 2016. https: //arxiv.org/abs/1603.02614. 21
[Ste22] Luuk Stehouwer. Interacting SPT phases are not Morita invariant. Lett. Math. Phys., 112(3):Paper No. 64, 25, 2022. https://arxiv.org/abs/2110.07408. 7, 9, 62, 63
[Ste23] Luuk Stehouwer. Unitary fermionic topological field theory. PhD thesis, Rheinischen Friedrich-WilhelmsUniversität Bonn, 2023. 9
[STG99] Horst L Stormer, Daniel C Tsui, and Arthur C Gossard. The fractional quantum Hall effect. Reviews of Modern Physics, 71(2):S298, 1999. 2
[Sto68] Robert E. Stong. Notes on cobordism theory. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968. Mathematical notes. 59, 63
[Sto88] Stephan Stolz. Exotic structures on 4-manifolds detected by spectral invariants. Invent. Math., 94(1):147162, 1988. 11, 55
[SV23] Kürşat Sözer and Alexis Virelizier. Monoidal categories graded by crossed modules and 3-dimensional HQFTs. Adv. Math., 428:Paper No. 109155, 2023. https://arxiv.org/abs/2207.06534. 22
[SW86] N. Seiberg and E. Witten. Spin structures in string theory. Nuclear Phys. B, 276(2):272-290, 1986. 18
[SW16] Nathan Seiberg and Edward Witten. Gapped boundary phases of topological insulators via weak coupling. Progress of Theoretical and Experimental Physics, 2016(12):12C101, December 2016. https: //arxiv.org/abs/1602.04251. 59
[SW18] Christoph Schweigert and Lukas Woike. A parallel section functor for 2-vector bundles. Theory Appl. Categ., 33:Paper No. 23, 644-690, 2018. https://arxiv.org/abs/1711.08639. 16
[SW19] Christoph Schweigert and Lukas Woike. Orbifold construction for topological field theories. J. Pure Appl. Algebra, 223(3):1167-1192, 2019. https://arxiv.org/abs/1705.05171. 16, 22
[SW20] Christoph Schweigert and Lukas Woike. Extended homotopy quantum field theories and their orbifoldization. J. Pure Appl. Algebra, 224(4):106213, 42, 2020. https://arxiv.org/abs/1802.08512. 16, 22
[Sze23] Lóránt Szegedy. On invertible 2-dimensional framed and $r$-spin topological field theories. Homology Homotopy Appl., 25(1):105-126, 2023. https://arxiv.org/abs/1907.09428. 21
[Tat20] Sri Tata. Geometrically interpreting higher cup products, and application to combinatorial pin structures. 2020. https://arxiv.org/abs/2008.10170. 23, 30
[TE18] Ryan Thorngren and Dominic V. Else. Gauging spatial symmetries and the classification of topological crystalline phases. Phys. Rev. X, 8:011040, Mar 2018. https://arxiv.org/abs/1612.00846. 57
[Tha21] Ying Hong Tham. On the Category of Boundary Values in the Extended Crane-Yetter TQFT. PhD thesis, State University of New York at Stony Brook, 2021. https://arxiv.org/abs/2108.13467. 56
[Tho52] René Thom. Espaces fibrés en sphères et carrés de Steenrod. Ann. Sci. École Norm. Sup. (3), 69:109-182, 1952. 41
[Tho20] Ryan Thorngren. Anomalies and bosonization. Comm. Math. Phys., 378(3):1775-1816, 2020. https: //arxiv.org/abs/1810.04414. 9, 19, 21, 28
[TKBB23] Srivatsa Tata, Ryohei Kobayashi, Daniel Bulmash, and Maissam Barkeshli. Anomalies in (2+1)D fermionic topological phases and (3+1)D path integral state sums for fermionic SPTs. Communications in Mathematical Physics, 397(1):199-336, January 2023. https://arxiv.org/abs/2104.14567. 3, 4, 21, 23, 24, 29, 30, 33
[Tro16] Fabio Trova. Nakayama categories and groupoid quantization. 2016. https://arxiv.org/abs/1602. 01019. 16, 22
[Tur94] V.G. Turaev. Quantum Invariants of Knots and 3-manifolds. De Gruyter studies in mathematics. W. de Gruyter, 1994. 11
[Tur10] Vladimir Turaev. Homotopy quantum field theory, volume 10 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2010. Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier. 22
[Tur20] Alex Turzillo. Diagrammatic state sums for 2D pin-minus TQFTs. J. High Energy Phys., (3):019, 26, 2020. https://arxiv.org/abs/1811.12654. 21
[TVV21] Ryan Thorngren, Ashvin Vishwanath, and Ruben Verresen. Intrinsically gapless topological phases. Physical Review B, 104(7):075132, August 2021. https://arxiv.org/abs/2008.06638. 44
[TY17] Yuji Tachikawa and Kazuya Yonekura. On time-reversal anomaly of 2+1d topological phases. PTEP, 2017(3):033B04, 2017. https://arxiv.org/abs/1610.07010. 3
[VD23] Jackson Van Dyke. Projective symmetries of three-dimensional TQFTs. 2023. https://arxiv.org/ abs/2311.01637. 16
[Wal21] Kevin Walker. Formula for the anomaly of spin Chern-Simons theories, 2021. MathOverflow answer. https://mathoverflow.net/a/396453. 56
[Wan08] Bai-Ling Wang. Geometric cycles, index theory and twisted K-homology. J. Noncommut. Geom., 2(4):497-552, 2008. https://arxiv.org/abs/0710.1625. 9
[Wen04] X.G. Wen. Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons. Oxford Graduate Texts. OUP Oxford, 2004. 2
[Wen13] Xiao-Gang Wen. Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders. Physical Review D, 88(4):045013, August 2013. https://arxiv.org/abs/1303.1803. 9
[WG20] Qing-Rui Wang and Zheng-Cheng Gu. Construction and classification of symmetry-protected topological phases in interacting fermion systems. Phys. Rev. X, 10:031055, Sep 2020. https://arxiv.org/abs/ 1811.00536. 9, 39
[Win20] Bradley Windelborn. Classification of two-dimensional homotopy quantum field theories. Master's thesis, Australian National University, 2020. https://openresearch-repository.anu.edu.au/bitstream/ 1885/227146/1/Bradley_Windelborn_MPhil_thesis_2021.pdf. 22
[Wit00] Edward Witten. World-sheet corrections via D-instantons. J. High Energy Phys., (2):Paper 30, 18, 2000. https://arxiv.org/abs/hep-th/9907041. 8
[WL17] Chenjie Wang and Michael Levin. Anomaly indicators for time-reversal symmetric topological orders. Phys. Rev. Lett., 119(13):136801, 2017. https://arxiv.org/abs/1610.04624. 3, 32, 33, 69
[WLL16] Chenjie Wang, Chien-Hung Lin, and Michael Levin. Bulk-Boundary Correspondence for ThreeDimensional Symmetry-Protected Topological Phases. Physical Review X, 6(2):021015, April 2016. https://arxiv.org/abs/1512.09111. 3
[WP23] Rui Wen and Andrew C. Potter. Bulk-boundary correspondence for intrinsically gapless symmetryprotected topological phases from group cohomology. Phys. Rev. B, 107:245127, Jun 2023. https: //arxiv.org/abs/2208.09001. 44
[WS14] Chong Wang and T. Senthil. Interacting fermionic topological insulators/superconductors in three dimensions. Physical Review B, 89(19):195124, May 2014. https://arxiv.org/abs/1401.1142. 2, 4, 57, 59, 61, 64
[WWZ20] Zheyan Wan, Juven Wang, and Yunqin Zheng. Higher anomalies, higher symmetries, and cobordisms II: Lorentz symmetry extension and enriched bosonic/fermionic quantum gauge theory. Ann. Math. Sci. Appl., 5(2):171-257, 2020. https://arxiv.org/abs/1912.13504. 5, 35, 42, 56
[YC23] Xinping Yang and Meng Cheng. Gapped boundary of (4+1)d beyond-cohomology bosonic SPT phase. arXiv e-prints, page arXiv:2303.00719, March 2023. https://arxiv.org/abs/2303.00719. 56
[Yet93] David N. Yetter. TQFTs from homotopy 2-types. J. Knot Theory Ramifications, 2(1):113-123, 1993. https://doi.org/10.1142/S0218216593000076. 22
$\left[\mathrm{YGH}^{+} 22\right]$ Weicheng Ye, Meng Guo, Yin-Chen He, Chong Wang, and Liujun Zou. Topological characterization of Lieb-Schultz-Mattis constraints and applications to symmetry-enriched quantum criticality. SciPost Physics, 13(3):066, September 2022. https://arxiv.org/abs/2111.12097. 2, 3, 44
[Yos75] Zen-ichi Yosimura. Universal coefficient sequences for cohomology theories of CW-spectra. Osaka J. Math., 12(2):305-323, 1975. 39
[You23] Minyoung You. Gapped boundaries of fermionic topological orders and higher central charges. arXiv e-prints, page arXiv:2311.01096, November 2023. https://arxiv.org/abs/2311.01096. 56
[YZ23a] Weicheng Ye and Liujun Zou. Anomaly of $(2+1)$-dimensional symmetry-enriched topological order from $(3+1)$-dimensional topological quantum field theory. SciPost Phys., 15:004, 2023. https: //arxiv.org/abs/2210.02444. 3, 4, 29, 30, 31, 52, 57, 58, 62, 64
[YZ23b] Weicheng Ye and Liujun Zou. Classification of symmetry-enriched topological quantum spin liquids. 2023. https://arxiv.org/abs/2309.15118. 3
[ZHW21] Liujun Zou, Yin-Chen He, and Chong Wang. Stiefel liquids: Possible non-Lagrangian quantum criticality from intertwined orders. Physical Review X, 11(3):031043, July 2021. https://arxiv.org/abs/2101. 07805. 3, 44
[ZQG22] Jian-Hao Zhang, Yang Qi, and Zheng-Cheng Gu. Construction and classification of crystalline topological superconductor and insulators in three-dimensional interacting fermion systems. arXiv e-prints, page arXiv:2204.13558, April 2022. https://arxiv.org/abs/2204.13558. 57

Purdue University, West Lafayette, Indiana
Email address: adebray@purdue.edu
Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong Email address: victoryeofphysics@gmail.com

Mathematical Institute, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG
Email address: yumatthew70@gmail.com


[^0]:    Date: December 22, 2023.
    It is a pleasure to thank Jaume Gomis for proposing this problem, Liujun Zou for collaboration in a related project and Yu-An Chen, Diego Delmastro, Dan Freed, and Juven Wang for helpful conversations. We thank Justin Kulp for providing us with tikz examples. The authors are listed in alphabetical order. Part of this research was conducted at the Perimeter Institute. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. The author MY is supported by the by the EPSRC Open Fellowship EP/X01276X/1.

[^1]:    ${ }^{1}$ Combining the result in Appendix A and the result in $\S 2$, we have discussed nine out of the ten symmetries in the 10 -fold way. The missing symmetry type is class D with $G_{f}=\mathbb{Z} / 2^{f}$, whose associated tangential structure is simply spin with no twist. We discuss this in $\S 6$, point 1.

[^2]:    ${ }^{2}$ Anomalies are a very general subject in physics; we are not claiming that everything called "anomaly" can be described in this way.
    ${ }^{3}$ The cited proof is given only for $\xi=$ Spin, but the same works for arbitrary $\xi$.

[^3]:    ${ }^{4}$ Though all invertible field theories we consider in this paper are topological, see also Freed-Hopkins' conjecture [FH21, Conjecture 8.37], recently proven by Grady [Gra23], on reflection-positive invertible field theories that are not necessarily topological.
    ${ }^{5}$ The notation $\mho_{\xi}^{*}$ for the dual of $\Omega_{*}^{\xi}$ is meant to parallel ordinary homology and cohomology: the dual of ordinary homology, which uses a right-side-up $H$, is ordinary cohomology, which is written with an upside-down $H$.

[^4]:    ${ }^{6}$ The term $-r_{V}$ shifts the Thom spectrum of $V$ so that the Thom class is in degree zero.
    ${ }^{7}$ Our definition of a fermionic topological order uses a specific presentation of a super-MTC, with simple objects given by the collection of anyons. We take this as the definition because it is most convenient for performing the main computations in this paper. Other methods of defining topological orders defines them up to gapped boundary, such as in [KK12, KWZ15, KWZ17, LW19, KLW ${ }^{+}$20, JF22, JFY22], and we will not be using this latter definition.

[^5]:    ${ }^{8}$ There is not yet a construction of the finite path integral for arbitrary additive $\mathcal{C}_{k}$, so its existence is part of our assumption. Compare [VD23, Hypothesis 1]. In many cases, though, the finite path integral has been constructed: in addition to the above articles, see Morton [Mor15], Trova [Tro16], Carqueville-Runkel-Schaumann [CRS19], Schweigert-Woike [SW19, SW20], and Harpaz [Har20], and see [CR16, BCP14a, BCP14b, SW18, CRS20, CMR ${ }^{+}$21, Car23, CM23] for some related constructions.

[^6]:    ${ }^{9}$ If $\mathcal{G}$ is a groupoid, $\mathbb{C}[\mathcal{G}]$ means the vector space of functions on $\pi_{0}(\mathcal{G})$, so $\mathbb{C}\left[\mathcal{B} u n_{G}(M)\right]$ is the vector space of functions on isomorphism classes of principal $G$-bundles on $M$. Since $M$ is closed and $G$ is finite, this is a finite-dimensional vector space.

[^7]:    ${ }^{10}$ Our notation is inspired by the fact that when the target category is $s \mathcal{A l} g_{\mathbb{C}}$ or $\mathcal{C} a t_{\mathbb{C}}\left[E_{1}\right]$, composition of defects corresponds to tensor product of bimodules, which is composition in the Morita category.

[^8]:    ${ }^{11}$ For 2d TFTs, Schommer-Pries' theorem requires an assumption on the tangential structure $\xi: B \rightarrow B \mathrm{O}(2)$ of the theory: specifically, if $B$ is connected we need that $S^{2}$ admits such a structure. For both theories appearing in this proof, the structure is $B \operatorname{Spin}_{2} \times B \mathbb{Z} / 2 \rightarrow B \mathrm{O}(2)$; the domain is connected and $S^{2}$ admits this structure, so we may use Schommer-Pries' theorem.
    ${ }^{12}$ To obtain $I_{\mathrm{U}(1)}$ instead of $I_{\mathbb{C} \times}$, one should use a Hermitian analogue of $s \mathcal{A l} g_{\mathbb{C}}[$ Fre12, (1.38)].
    ${ }^{13}$ The spectrum $\left|\overline{\mathcal{B o r d}}{ }_{2}^{\xi}\right|$ is known to be the Madsen-Tillmann spectrum $\Sigma^{2} M T \xi$ [GMTW09, Ngu17, SP17], so one could alternatively explicitly identify the abelian group of 2 d Spin $\times \mathbb{Z} / 2$ invertible TFTs $\operatorname{Hom}\left(\pi_{2}\left(\Sigma^{2} M T S p i n_{2} \wedge\right.\right.$ $\left.\left.(B \mathbb{Z} / 2)_{+}\right), \mathrm{U}(1)\right)$ and identify $z_{c} \otimes_{F_{\mathbb{Z} / 2}} z_{c}$ and $z_{c} \otimes_{F_{\mathrm{Spin}}} z_{c}$ in that group. Randal-Williams [RW14, Figure 5 , left] runs enough of the Adams spectral sequence for MTSpin ${ }_{2}$ to show that $\pi_{0}\left(\Sigma^{2} \operatorname{MTSpin}_{2}\right) \cong \mathbb{Z}, \pi_{1}\left(\Sigma^{2} M T S p i n_{2}\right) \cong \mathbb{Z} / 2$, and $\pi_{2}\left(\Sigma^{2}\right.$ MTSpin $\left._{2}\right) \cong \mathbb{Z} \oplus \mathbb{Z} / 2$, and that the map $\Sigma^{2} M T S p i n_{2} \rightarrow M T S p i n$ is an isomorphism on $\pi_{0}$ and $\pi_{1}$ and surjective on $\pi_{2}$. This is enough to set up the Atiyah-Hirzebruch spectral sequence for the $\Sigma^{2} M T S p i n_{2}$-homology of $B \mathbb{Z} / 2$, i.e. $\pi_{*}\left(\Sigma^{2} \operatorname{MTSpin}_{2} \wedge(B \mathbb{Z} / 2)_{+}\right)$, and solve it by comparing to the analogous spectral sequence for $\Omega_{*}^{\mathrm{Spin}}(B \mathbb{Z} / 2)$, with the conclusion that $\pi_{2}\left(\Sigma^{2} \operatorname{MTSpin}_{2} \wedge(B \mathbb{Z} / 2)_{+}\right) \cong \mathbb{Z} \oplus \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$, with generators $S^{2}, S_{n b}^{1} \times S_{n b}^{1}$ with trivial $\mathbb{Z} / 2$-bundle, and $S_{n b}^{1} \times S_{b}^{1}$ with $\mathbb{Z} / 2$-bundle pulled back from the nontrivial double cover on the second factor. Then one could check the isomorphism type of these two TFTs by checking only on these three generators.

[^9]:    ${ }^{14}$ We do not need a detailed understanding of higher gerbes in this paper; all we need is that for an abelian group $A, A \ell$-gerbes are objects that can be defined over a topological space $X$ whose isomorphism classes are in natural bijection with $H^{\ell+1}(X ; A)$ and which, like principal bundles, form a sheaf of $\infty$-groupoids, so that they are local objects in the sense of quantum field theory and can be background fields. Moreover, the addition on $H^{\ell+1}(X ; A)$ refines to a tensor product $\odot$ on higher gerbes. See Breen [Bre94, Bre10], Lurie [Lur09a, §7.2.2], and Nikolaus-Schreiber-Stevenson [NSS15] for precise definitions and more information.
    ${ }^{15}$ Analogues of Dijkgraaf-Witten theory using more general targets than $B G$ were first studied by Yetter [Yet93] and Quinn [Qui95], with additional work by Porter [Por98, Por07], Faria Martins-Porter [FMP07, FMP23], PorterTuraev [PT08], Staic-Turaev [ST10], Turaev [Tur10], Monnier [Mon15], Müller-Woike [MW20], Windelborn [Win20], Freed-Teleman [FT22], Liu [Liu23], and Sözer-Virelizier [SV23].
    ${ }^{16}$ Because the partition function of $\alpha_{0}$ is the integral of a cohomology class, rather than a more general bordism invariant, it is possible to construct $\alpha_{0}$ as valued in the 4 -category $\bigodot a t_{\mathbb{C}}\left[E_{2}\right]$ of braided monoidal, $\mathbb{C}$-linear 2-categories, the same target as the Crane-Yetter theory. This is because $\Sigma^{4} H \mathbb{C}^{\times}$is the 3-connective cover of $\left|\mathcal{C} a t_{\mathbb{C}}\left[E_{2}\right] \times\right|$, e.g. because $\Omega^{2}\left(\mathcal{C} a t_{\mathbb{C}}\left[E_{2}\right]\right) \simeq \mathcal{V} e c t_{\mathbb{C}}$ and $\left|\mathcal{V} e c t_{\mathbb{C}}^{\times}\right| \simeq \Sigma H \mathbb{C}^{\times}$, so precomposing the 3-connective covering $\operatorname{map} \tau_{\geq 3}: \Sigma^{4} H \mathbb{C}^{\times} \rightarrow\left|\mathcal{C} a t_{\mathbb{C}}\left[E_{2}\right]^{\times}\right|$with the map $\Sigma^{4} H \mathbb{Z} / 2 \rightarrow \Sigma^{4} H \mathbb{C}^{\times}$given by the unique injective group homomorphism $\mathbb{Z} / 2 \hookrightarrow \mathbb{C}^{\times}$, we obtain an invertible field theory valued in $\mathcal{C} a t_{\mathbb{C}}\left[E_{2}\right]$ from the degree- 4 mod 2 cohomology class defining $\alpha_{0}$.

[^10]:    ${ }^{17}$ The proposition is likely true with "nonextended" removed, but this is open: to our knowledge, $z_{c}$ has not been studied much in higher codimension.

[^11]:    ${ }^{18}$ We have considered $z_{c}$ both as a boundary theory of $F_{B}$ for manifolds with a $\left(B G_{b}, s\right)$-twisted orientation, and as a boundary theory of $\alpha_{0}$ for manifolds with a $\left(B G_{b}, s\right)$-twisted orientation and a map to $B^{n-1} \mathbb{Z} / 2$. To clarify, these two perspectives are equivalent, by summing over the maps to $B^{n-1} \mathbb{Z} / 2$; though we have mostly thought of $z_{c}$ as an $F_{B}$-boundary, in this remark we will think of it as an $\alpha_{0}$-boundary.

[^12]:    ${ }^{19} 2 \mathcal{V}$ ect and $2 s \mathcal{V}$ ect are the 2-categories $\Sigma(\mathcal{V e c t})$, resp. $\Sigma(s \mathcal{V e c t})$, the Karoubi completions of the 2-categories of $\mathcal{V}$ ect-, resp. sVect-module categories, as defined in [GJF19, §1.4].

[^13]:    ${ }^{20}$ Since this invertible TFT was defined as the integral of a cohomology class, an analogue of Footnote 16 applies to the construction of $\kappa_{\text {conv }}$ and the convolution product.

[^14]:    ${ }^{21}$ We also drop the orientation $\gamma$ from Eq. (3.24) to avoid clutter.

[^15]:    ${ }^{22}$ Similar to the $\mathbb{Z} / 4^{T} \times \mathbb{Z} / 2^{f}$ symmetry, in light of Definition 2.1, the $\mathbb{Z} / k^{T} \times \mathbb{Z} / 2^{f}$ symmetry discussed here can be defined in terms of the triple $\left(G_{b}, s, \omega\right)$ where $G_{b}=\mathbb{Z} / k$, with $s$ nontrivial and $\omega$ trivial. Since $H^{1}(B \mathbb{Z} / k ; \mathbb{Z} / 2) \cong \mathbb{Z} / 2$, this uniquely specifies a fermionic symmetry.
    ${ }^{23}$ Instead of defining epin structures as twisted spin structures, as we did in Definition 5.1, Wan-Wang-Zheng define them using a group EPin $\cong \mathbb{Z} / 4 \ltimes$ Spin and a map EPin $\rightarrow \mathrm{O}$ [WWZ20, (1.2), (1.6)]. It follows from their discussion in (ibid., §1) that the two definitions agree.

[^16]:    ${ }^{24}$ The map $\alpha$ and its fiber are sometimes stated differently, e.g. using $\mathbb{Q} / \mathbb{Z}$ and $H \mathbb{Q}$ instead of $\mathrm{U}(1)$ and $H \mathbb{C}$, respectively, or $\mathbb{R} / \mathbb{Z}$, resp. $H \mathbb{R}$, or $\mathbb{C}^{\times}$, resp. $H \mathbb{C}$. In all cases the arguments are essentially unchanged.
    ${ }^{25}$ There is a sense in which $I_{\mathbb{Z}}$ is the analogue of $I_{\mathrm{U}(1)}$, but with $\mathrm{U}(1)$ given the continuous topology, and the fiber sequence $\operatorname{Map}(X, H \mathbb{C}) \rightarrow I_{\mathrm{U}(1)} X \rightarrow \Sigma I_{\mathbb{Z}} X$ is a rotated counterpart of the exponential sequence. See, e.g., [FH21, §5.3].
    ${ }^{26}$ Anderson's lecture notes are unpublished; see Yosimura [Yos75, Theorem 4] for a published account of Anderson's proof. There are also at least four other proofs of the shifted Anderson self-duality of $K O$, each by very different methods, due to Freed-Moore-Segal [FMS07, Proposition B.11], Heard-Stojanoska [HS14, Theorem 8.1], Ricka [Ric16, Corollary 5.8], and Hebestreit-Land-Nikolaus [HLN20, Example 2.8].

[^17]:    ${ }^{27}$ For more examples contrasting the Atiyah-Hirzebruch and Adams spectral sequences when computing twisted spin bordism, see [KPMT20, Appendices E and F] and [Ped17].
    ${ }^{28}$ See also Mahowald [Mah82, Lemma 7.3], Bruner-Greenlees [BG10, Example 7.3.1], Siegemeyer [Sie13, Theorem 2.1.5], and García-Etxebarria-Montero [GEM19, (C.18)] for other works computing spin bordism or ko-homology of $B \mathbb{Z} / 2$ in these degrees.
    ${ }^{29}$ Siegemeyer [Sie13, §2.2] and Davighi-Lohitsiri [DL21, §A.3] provide additional computations of this group via other methods.
    ${ }^{30}$ This is a mathematical procedure whose effect on a finitely generated abelian group sends free summands to free summands, preserves all 2-power torsion, and sends all odd-power torsion to 0 . The reader is welcome to take this as a definition of localization in this paper; for a more general introduction to localization, see [AM94].

[^18]:    ${ }^{31}$ One way to see why $p^{*}(y)=0$ is as follows: if $V \rightarrow B \mathbb{Z} / 4$ is the complex line bundle associated to the rotation representation of $\mathbb{Z} / 4$ on $\mathbb{C}$, then $y=w_{2}(V)$, so to show $p^{*}(y)=0$, it suffices to show that $p^{*}(L)$ is spin, i.e. that the corresponding map $\mathbb{Z} / k \rightarrow \mathbb{Z} / 4 \rightarrow \mathrm{SO}_{2}$ lifts across the double cover $\operatorname{Spin}_{2} \rightarrow \mathrm{SO}_{2}$. This can be done, e.g. by sending $1 \in \mathbb{Z} / k$ to an eighth root of unity in $\mathrm{U}(1) \cong \operatorname{Spin}_{2}$.

[^19]:    32 See [DDHM23, §C.2] for a slight simplification of this formula.

[^20]:    ${ }^{33}$ In fact, $\pi_{0}(\operatorname{Diff}(K)) \neq 0$, so the map of classifying spaces is not an isomorphism on $\pi_{1}$. This follows from a theorem of Lickorish [Lic65].

[^21]:    ${ }^{34}$ Of the $\left|H_{1}(\mathcal{M} ; \mathbb{Z} / 2)\right|=4$ spin-structures, two satisfy this condition, and they correspond to a generator and the inverse of the generator of $\Omega_{4}^{\mathrm{EPin}[k]}$, respectively.
    ${ }^{35}$ According to Remark 5.5 , we should in principle be careful about using $f^{*}(x)$ versus the twisted spin cobordism Euler class of $f^{*}(\sigma)$, but by using $f^{*}(x)$ we simplify the calculations and do not lose any information in this example.

[^22]:    ${ }^{36}$ Again, because of Remark 5.5 one should be careful of which cohomology theory one takes duals in: in principle by using mod 2 cohomology instead of spin cobordism, one could lose information. But in the course of our calculation, we see that we learn enough to detect $\mathcal{M}$, so this simplification is OK. The same is true for $\mathcal{S}$ below.

[^23]:    ${ }^{37}$ If we choose a different spin-structure for $\mathcal{M}$ such that the phase in front of $Z_{3}$ and $Z_{4}$ is $-i$, then $\mathcal{M}$ with the new spin-structure is simply the inverse of $\mathcal{M}$ with the old spin-structure in the bordism group $\Omega_{4}^{\mathrm{EPin}}$. It is also straightforward to see that compared to $\mathcal{I}$, the partition function of $\mathcal{M}$ with the new spin-structure is simply complex-conjugated.

[^24]:    ${ }^{39}$ Note that $S_{4}$ here has nothing to do with the permutation group of four elements.

[^25]:    ${ }^{40}$ In this appendix, we use the subscript $f$ to emphasize that $\mathrm{U}_{f}(1)$ is a subgroup of the fermionic symmetry group $G_{f}$, while $\mathrm{U}(1)$ with no subscript is a subgroup of $G_{b}$. Similarly for $\mathrm{SU}_{f}(2)$.

[^26]:    ${ }^{41}$ We can also say that the two generating manifolds are $\mathbb{C P}^{2}$ and $S^{2} \times S^{2}$ with the $\mathrm{U}(1)$-bundle induced from their complex structure.
    ${ }^{42}$ In terms of the Riemann curvature tensor $R$ and the $\mathrm{U}(1)$ field strength $F$, i.e. from the point of view of Chern-Weil theory, $\operatorname{Sign}(M)=\frac{1}{192 \pi^{2}} \int_{M} \operatorname{tr}(R \wedge R)$ and $c_{1}=\frac{F}{2 \pi}$.
    ${ }^{43}$ Note that a $(2+1)$-d fermionic invertible state with class A symmetry has integer $\kappa$ and $\sigma_{H}$. Physically, we can stack invertible states with given fermionic topological order without changing the anyon content, and thus $\kappa$ and $\sigma_{H}$ for a fermionic topological order can be determined only up to contributions from invertible states. Therefore, for class A symmetry only the fractional part of $\kappa$ and $\sigma_{H}$ can be determined from the anyon content/super-MTC.

[^27]:    ${ }^{44}$ We can also say that the two generating manifolds are $\mathbb{C P}^{2}$ and $S^{4}$ with the $\mathrm{SO}(3)$-bundle induced from their almost quaternionic structure. Note that $S^{2} \times S^{2}$ with $\operatorname{spin}^{h}$ structure induced from its spin ${ }^{c}$ structure is not a generating manifold but is bordant to two copies of a generator of $\Omega_{4}^{\mathrm{Spin}}{ }^{h}$.
    ${ }^{45}$ This $\mathrm{SO}(3)$-bundle has an interesting property: its spin cobordism Euler class is nonzero, even though its $\mathbb{Z}$-cohomology Euler class vanishes. This means that the caveat raised in Remark 5.5 applies to the Smith homomorphism $\Omega_{4}^{\text {Spin }}{ }^{h} \rightarrow \Omega_{1}^{\text {Spin }}(B \mathrm{SO}(3))$ : using Euler classes in ordinary cohomology does not correctly compute the Smith homomorphism. See $\left[\mathrm{DDK}^{+} 23\right.$, Appendix B] for details of the computation of this spin cobordism Euler class and its consequences.

[^28]:    ${ }^{46}$ Note that a $(2+1)$-d fermionic invertible state with class C symmetry has even integer $\kappa$ and $\sigma_{H}$, hence here $\kappa$ and $\sigma_{H}$ can be determined $\bmod 2$.
    ${ }^{47}$ This follows from the Bockstein long exact sequence associated to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathrm{U}(1) \rightarrow 0$ and the facts that $H^{2}(B \mathrm{SO}(3) ; \mathbb{R})=0, H^{3}(B \mathrm{SO}(3) ; \mathbb{R})=0$, and $H^{3}(B \mathrm{SO}(3) ; \mathbb{Z}) \cong \mathbb{Z} / 2$ [Bro82, Theorem 1.5].

[^29]:    ${ }^{48}$ One way to see this is to observe that $\int w_{2}^{2}$ is a bordism invariant of pin ${ }^{\tilde{c}-}$ manifolds and is nonvanishing on $\mathbb{C P}^{2}$.

[^30]:    ${ }^{49}$ If the reader is comfortable with virtual vector bundles, pin $\tilde{c}+$ structures are also equivalent to $\left(B \mathrm{O}(2),-V_{t}\right)$-twisted spin structures. This is how pin ${ }^{\tilde{c}+}$ structures are presented as twisted spin structures in [FH21, (10.2)].
    ${ }^{50}$ Guillou-Marin do not describe their tangential structure as a twist of spin structure over $B O(2)$, but instead in terms of "characteristic submanifolds." See Kirby-Taylor [KT90, §2] for an explanation of how to pass from Guillou-Marin's description to one in terms of the Thom space $M O(2)$, from which the connection to spin- $\mathrm{O}(2)$ bordism is clearer.

[^31]:    ${ }^{51}$ If a manifold $M$ admits a pin ${ }^{c}$ structure, then its set of pin ${ }^{c}$ structures is a torsor over $H^{2}(M ; \mathbb{Z})$, analogous to spin ${ }^{c}$ structures. $\mathbb{R P}^{4}$ admits a pin ${ }^{c}$ structure, since the obstruction $\beta\left(w_{2}\right)$ lives in $H^{3}\left(\mathbb{R} \mathbb{P}^{4} ; \mathbb{Z}\right)=0$, so the set of $\operatorname{pin}^{c}$ structures is a torsor over $H^{2}\left(\mathbb{R P}^{2} ; \mathbb{Z}\right) \cong \mathbb{Z} / 2$.

