Bordism for the 2-group symmetries of the heterotic and CHL strings

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Abstract. In the presence of a nonzero B-field, the symmetries of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string form a 2-group, or a categorified group, as do the symmetries of the CHL string. We express the bordism groups of the corresponding tangential structures as twisted string bordism groups, then compute them through dimension 11 modulo a few unresolved ambiguities. Then, we use these bordism groups to study anomalies and defects for these two string theories.

Contents

0. Introduction

String theory has long been a place where higher-categorical structures in mathematics meet their applications. This is true for a few different reasons, but one crucial reason is that many fields in superstring and supergravity theories have mathematical incarnations that are higher-categorical objects, and so even precisely setting up mathematical questions coming out of string theory, let alone solving them, often requires engaging with or developing the foundations of various kinds of geometric objects with higher structure. This paper is concerned with the appearance of a higher structure called a *2-group* in two specific string theories, and how including this structure affects computations of bordism groups for the tangential structures of these theories. These bordism groups control anomalies and extended objects for these theories. The main results of this paper are computations of bordism groups and their generating manifolds through dimension 11, except for a few ambiguities we did not addres, for the tangential structures underlying these two string theories.

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For the higher structures we investigate in this paper, the story begins with the *Kalb-Ramond field*, or the *B-field*. This is an analogue of the field strength of an electromagnetic field, represented as a closed differential 2-form with a quantization condition. Locality of quantum field theory means expressing the field strength of the electromagnetic field as a section of a sheaf, specifically as a connection on a principal $\mathbb{T}$-bundle, where $\mathbb{T}$ is the circle group. For the B-field, everything is one degree higher: it comes to us as a closed differential 3-form with a quantization condition, which we would like to express as a geometric object that sheafifies. This cannot be a connection on a principal G-bundle for a finite-dimensional Lie group G; instead, one models the B-field as a connection on a $\mathbb{T}$-*gerbe*, which is a categorification of a principal $\mathbb{T}$-bundle. A $\mathbb{T}$-gerbe on a manifold M is, roughly speaking, a bundle of groupoids on M which is locally equivalent to $\mathrm{pt}/\mathbb{T}$. There are several ways to make this precise; we discuss one, Murray's *bundle gerbes* [**Mur96**], in Definition 1.1.

In this article, we consider higher structures in two string theories: the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string, and the Chaudhuri-Hockney-Lykken (CHL) string. The former is a ten-dimensional superstring theory whose low-energy limit is ten-dimensional $\mathcal{N}=1$ supergravity, and the latter is a nine-dimensional theory obtained from the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory by compactifying on a circle. Both of these theories have B-fields, but Green and Schwarz [**GS84**] showed that in order to cancel an anomaly, the B-field and the gauge field must satisfy a relation known as a *Bianchi identity*. Fiorenza-Schreiber-Stasheff [**FSS12**] and Sati-Schreiber-Stasheff [**SSS12**] describe how the Bianchi identity mixes the data of the B-field and the gauge field into data that can be interpreted as a connection on a principal bundle for a 2-group $\mathbb{G}$, specifically a *string 2-group* $\mathcal{S}tr(G,\mu)$ associated to the data of a compact Lie group G and a class $\mu\in H^4(BG;\mathbb{Z})$; typically, G is the gauge group and μ is determined by the anomaly polynomial.

2-groups have been used in the theoretical and mathematical physics literature for some time now. This program began in earnest with work of Baez, Crans, Lauda, Stevenson, and Schreiber [**Bae02, BC04, BL04, BSCS07, BS07**]; more recently, 2-groups, their symmetries, and their anomalies have made a resurgence in quantum field theory following work of Córdova-Dumitriescu-Intrilligator [**CDI19**] and Benini-Córdova-Hsin [**BCH19**] identifying many examples of 2-group symmetries in commonly studied QFTs. See also Sharpe [**Sha15**] and the references therein.

In the first part of this article, we introduce the Bianchi identity and 2-groups, then review work of Fiorenza-Schreiber-Stasheff [**FSS12**] and Sati-Schreiber-Stasheff [**SSS12**] mentioned above. These authors work in the setting of stacks on the site $\mathcal{M}an$ of smooth manifolds; the data of the B-field (Q,Θ_Q) and the principal G-bundle with connection (P,Θ_P) on a manifold M refine to maps from M to classifying stacks of these data. The data of an identification of two differential characteristic classes associated to Θ_P and Θ_Q gives rise to

(1) a principal $\mathcal{S}tr(G,\mu)$-bundle lifting P for a specified choice of μ (Proposition 1.35), and
(2) local data of solutions to the Bianchi identity (Proposition 1.37, [**FSS12**, §6.3]).

Inspired by this, we introduce the tangential structures ξ^{het} and ξ^{CHL}, which are special cases of a general construction of Sati-Schreiber-Stasheff [**SSS12**, Definition

2.8]: a ξ_n^{het}-structure on a spin manifold M is data of a principal $\mathbb{G}_n^{\mathrm{het}}$-bundle, where $\mathbb{G}_n^{\mathrm{het}} := \mathcal{S}tr(\mathrm{Spin}_n \times (\mathrm{E}_8 \times \mathrm{E}_8 \rtimes \mathbb{Z}/2), c_1 + c_2 - \lambda)$ (1.42), whose associated Spin_n-bundle via the quotient $\mathbb{G}_n^{\mathrm{het}} \to \mathrm{Spin}_n$ is the principal Spin_n-bundle of spin frames (Definition 1.41). This is compatible as n varies, allowing us to stabilize and define a ξ^{het}-structure as usual. The definition of ξ^{CHL} in Definition 1.52, which coincides with $B\mathrm{String}^{2a}$ in [**SSS12**, (2.18), §2.2.3], is analogous. Related tangential structures appear in [**Sat11b, FSS15a, FSS15b, FSS21**].

Given a tangential structure, we can compute bordism groups, and indeed the point of this paper is to compute ξ^{het} and ξ^{CHL} bordism groups in low dimensions. These bordism groups can then be used to learn more about the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic and CHL strings. We have two primary applications in mind.

(1) The *cobordism conjecture* of McNamara-Vafa [**MV19**] is an application to the question of what kinds of spacetime backgrounds are summed over in quantum gravity. Such backgrounds are often taken to be manifolds or something closely related equipped with data of a tangential structure ξ. The cobordism conjecture says that if ξ is the most general tangential structure which can appear in this way in any particular d-dimensional theory of quantum gravity, then $\Omega_k^\xi = 0$ for $3 \le k \le d-1$. We will see that $\Omega_k^{\xi^{\mathrm{het}}}$ and $\Omega_k^{\xi^{\mathrm{CHL}}}$ are often nonzero in that range. This is consistent with the cobordism conjecture: it suggests that ξ^{het} and ξ^{CHL} are not the most general tangential structures that can be summed over. Typically these bordism groups are killed by allowing singular manifolds corresponding to considering the theory with branes or other defects, so one can use bordism computations to predict new defects in string theories.

(2) A broad class of n-dimensional quantum field theories come with data of an *anomaly*, which in many cases can roughly be described an $(n+1)$-dimensional invertible field theory α. In some cases one wants to trivialize α, meaning exhibiting an isomorphism from α to the trivial field theory. By work of Freed-Hopkins-Teleman [**FHT10**] and Freed-Hopkins [**FH21b**], invertible field theories can be classified using bordism group computations. For both the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string and the CHL string, the bordism groups indicating a potential anomaly are nonzero, and it would be interesting to check whether the corresponding anomalies are nontrivial.

See §3, as well as Questions 0.1 to 0.3 below, for more on these applications and what we can learn from our bordism computations.

Our main theorems are the following two computations of the ξ^{het} and ξ^{CHL} bordism groups in low dimensions.

Theorem A. *For $k \le 10$, the ξ^{het}-bordism groups are:*

$$\begin{aligned}
\Omega_0^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}\\
\Omega_1^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\\
\Omega_2^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\\
\Omega_3^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/8\\
\Omega_4^{\xi^{\mathrm{het}}} &\cong \mathbb{Z} \oplus \mathbb{Z}/2
\end{aligned}$$

$$\begin{aligned}
\Omega_5^{\xi^{\mathrm{het}}} &\cong 0\\
\Omega_6^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2\\
\Omega_7^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/16\\
\Omega_8^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^{\oplus i}\\
\Omega_9^{\xi^{\mathrm{het}}} &\cong (\mathbb{Z}/2)^{\oplus j}\\
\Omega_{10}^{\xi^{\mathrm{het}}} &\cong (\mathbb{Z}/2)^{\oplus k}.
\end{aligned}$$

Here, either $i = 1$, $j = 4$, and $k = 4$, or $i = 2$, $j = 6$, and $k = 5$.

$\Omega_{11}^{\xi^{\mathrm{het}}}$ *is an abelian group of order* 64 *isomorphic to one of* $\mathbb{Z}/8\oplus\mathbb{Z}/8$, $\mathbb{Z}/16\oplus\mathbb{Z}/4$, $\mathbb{Z}/32 \oplus \mathbb{Z}/2$, *or* $\mathbb{Z}/64$.

This is a combination of Theorems 2.62 and 2.74. In §2.2.1, we find manifold representatives for all classes in $\Omega_k^{\xi^{\mathrm{het}}}$ for $k \le 10$ except potentially for two missing classes X_8 and X_9 of dimensions 8, resp. 9 and their products with S^1_{nb}. These classes may or may not be zero depending on the fate of an Adams differential. In §2.2.2, we find a manifold representing X_8: if the unaddressed Adams differential vanishes, X_8 should be added to the list of generators in §2.2.1, and if the differential does not vanish, then X_8 bounds as a ξ^{het}-manifold.

Our calculation of ξ^{CHL}-bordism builds on work of Hill [**Hil09**, Theorem 1.1], who computes $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$ in dimensions 14 and below.

Theorem B. *For $k \le 11$, there is an abstract isomorphism from $\Omega_*^{\xi^{\mathrm{CHL}}}$ to the free and 2-torsion summands of $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$. Therefore, by Hill's computation* [**Hil09**]*, there are isomorphisms*

$$\begin{aligned}
\Omega_0^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z} & \Omega_6^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/2\\
\Omega_1^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/2 & \Omega_7^{\xi^{\mathrm{CHL}}} &\cong 0\\
\Omega_2^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/2 & \Omega_8^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\\
\Omega_3^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/8 & \Omega_9^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\\
\Omega_4^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z} & \Omega_{10}^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2\\
\Omega_5^{\xi^{\mathrm{CHL}}} &\cong 0 & \Omega_{11}^{\xi^{\mathrm{CHL}}} &\cong \mathbb{Z}/8.
\end{aligned}$$

This is a combination of Theorems 2.90 and 2.92. We also obtain some information about manifold representatives of generators of these groups.

The computational tool we use to prove Theorems A and B is standard: the Adams spectral sequence. This spectral sequence has seen plenty of applications in the mathematical physics literature, and there is a standard procedure reviewed by Beaudry-Campbell [**BC18**] for simplifying the E_2-page for a wide class of tangential structures, namely those which can be described as oriented, spin^c, spin, or string bordism twisted by a virtual vector bundle. For example, the twisted string bordism computations of [**FK96, Fan99, FW10**] make use of this simplifying technique. Unfortunately, this procedure is unavailable to us: in Lemma 2.2, we prove that ξ^{het} and ξ^{CHL} cannot be described as twists of this sort. However, we are still able to describe them as twists in a more general sense due to Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG$^+$14a, ABG$^+$14b**]: adapting an argument of Hebestreit-Joachim [**HJ20**], one learns that the Thom spectra for ξ^{het} and ξ^{CHL} can be produced as the *MTString*-module Thom spectra associated to certain maps to $B\mathrm{GL}_1(\mathit{MTString})$. Using this structure, in joint work with Matthew Yu, we are able to prove a theorem simplifying the calculation of the E_2-page:

Theorem C (Debray-Yu [**DY23**]). *In topological degrees* 15 *and below, the* E_2*-pages of the Adams spectral sequences computing* 2*-completed twisted string bordism for a class of twists including those for* ξ^{het} *and* ξ^{CHL} *can be computed as* Ext *over the subalgebra* $\mathcal{A}(2)$ *of the Steenrod algebra.*

What we prove is more precise and holds in more generality; see Theorem 2.20 and Corollary 2.22 for that version of the result.[1]

The $\mathcal{A}(2)$-module Ext groups we have to compute are simpler than what one a priori has to work with over the entire Steenrod algebra $\mathcal{A}$. We do not need this simplification at odd primes; there the full Adams spectral sequence is easier to work with, and the absence of a simplification does not hinder us (though see also [**DY23**, §3.2]).

The reason we computed these bordism groups in this paper is with applications to physics, specifically to anomalies and the cobordism conjecture, in mind. We discuss some implications of our calculations in §3; for example, one of the $\mathbb{Z}/2$ summands of $\Omega_1^{\xi^{\mathrm{het}}}$ corresponds to the non-supersymmetric 7-brane recently discovered by Kaidi-Ohmori-Tachikawa-Yonekura [**KOTY23**]. We end this section of the introduction with some questions related to these physics predictions.

Question 0.1. What does the Kaidi-Ohmori-Tachikawa-Yonekura 7-brane correspond to in Hořava-Witten theory, and what does this look like in bordism? Hořava-Witten [**HW96a, HW96b, Wit96**] proposed that the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string can be identified with a certain limit of M-theory compactified on an interval; thus this ought to correspond to a notion of bordism of manifolds with boundary. Conner-Floyd [**CF66**, §16] define a notion bordism of compact manifolds with boundary — is this the correct kind of bordism for applications to McNamara-Vafa's conjecture?

We discuss some additional extended objects predicted by our bordism computations to exist in the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic and CHL strings in §3.1.

Question 0.2. Is the $\mathbb{Z}/2$ symmetry exchanging the two E_8-bundles in $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory anomalous? Because $\Omega_{11}^{\xi^{\mathrm{het}}}$ is nonzero, we were unable to rule out this anomaly.

Witten [**Wit86**, §4] and Tachikawa-Yonekura [**TY21**] show that the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string is anomaly-free in certain cases, but they do not address the $\mathbb{Z}/2$ symmetry.

Question 0.3. Does the CHL string have an anomaly? This anomaly could be nontrivial, because $\Omega_{10}^{\xi^{\mathrm{CHL}}}\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$.

There is another application of twisted string bordism to physics that we did not address in this paper: studying elliptic genera, the Witten genus and related invariants, along the lines of, e.g., Bunke-Naumann [**BN14**], McTague [**McT14**], Han-Huang-Duan [**HHD21**], and Berwick-Evans [**BE23**]. It would be interesting to study whether the calculations in this paper could be applied in similar contexts.

[1] We also provide a proof sketch of the case we need in Remark 2.26.

Outline. We begin in §1.1 by introducing the fields present in 10d $\mathcal{N} = 1$ supergravity, the low-energy limit of heterotic string theory. We discuss how the Green-Schwarz anomaly cancellation condition imposes an equation called the *Bianchi identity* (1.10) on the fields in this theory. We then generalize this to a *twisted Bianchi identity* (1.12) associated to data of a Lie group G and a class $\mu \in H^4(BG;\mathbb{Z})$. In §1.2, we relate these Bianchi identities to the presence of a 2-group symmetry in this field theory. We begin by reviewing 2-groups, their principal bundles, and their connections, and in Example 1.22 define the string cover $\mathcal{S}(G,\mu)$ corresponding to a group G and a class $\mu \in H^4(BG;\mathbb{Z})$. Then we review work of Fiorenza-Schreiber-Stasheff [**FSS12**] and Sati-Schreiber-Stasheff [**SSS12**] relating the Bianchi identity to twisted string structures. Using this, we define the heterotic tangential structure in Definition 1.41, which is the topological part of the structure necessary for defining $\mathcal{N} = 1$ supergravity. Then, in §1.3, we introduce the CHL string and define the CHL tangential structure using what we learned in §1.2.

In §2, we compute the bordism groups $\Omega_*^{\xi^{\mathrm{het}}}$ and $\Omega_*^{\xi^{\mathrm{CHL}}}$ in low degrees. For the latter we are able to completely compute them in dimensions 11 and below, but for the former, we have only partial information above dimension 7, occluded by Adams differentials and an extension problem we could not solve. We begin in §2.1 by discussing how to simplify the Thom spectra $MT\xi^{\mathrm{het}}$ and $MT\xi^{\mathrm{CHL}}$; we prove in Lemma 2.2 that a standard approach does not work, and so we use a different idea: construct $MT\xi^{\mathrm{het}}$ and $MT\xi^{\mathrm{CHL}}$ as $MTString$-module Thom spectra using machinery developed by Ando-Blumberg-Gepner-Hopkins-Rezk. We review this machinery and discuss how it leads to Corollary 2.22, a special case of the main theorem of our work [**DY23**] joint with Matthew Yu, simplifying the calculation of the E_2-page of the Adams spectral sequence at 2 for a wide class of twisted string bordism groups. Next, in §2.2, we undertake this computation for ξ^{het}. We do not have such a simplification at odd primes, so in §2.3 we press ahead directly with the Adams spectral sequence for ξ^{het}, proving in Theorem 2.74 that $\Omega_*^{\xi^{\mathrm{het}}}$ lacks odd-primary torsion in degrees 11 and below. Finally, in §2.4 we run the analogous calculations for the CHL string, again using Corollary 2.22 at $p = 2$ and arguing more directly at odd primes.

The final section, §3, is about applications to string theory. We first discuss the cobordism conjecture of McNamara-Vafa [**MV19**] in §3.1, and go over a few predictions that follow from the bordism group computations in §2. In §3.2, we briefly introduce anomalies of quantum field theories and their bordism-theoretic classification, and touch on questions raised by our bordism computations.

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1. Tangential structures for heterotic and CHL string theories

The goal of this section is to define the tangential structures ξ^{het} and ξ^{CHL} that are necessary to formulate the (low-energy limits of) the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string and the CHL string. By "tangential structure" we mean the topological part of the structure needed on a manifold to define a given field theory; see Definition 1.40 for the precise definition. The presence of a B-field in both theories means that these tangential structures arise as classifying spaces of higher groups. First, we introduce the heterotic string in §1.1, and see what data and conditions are told to us by Green-Schwarz anomaly cancellation; then in §1.2, we reinterpret that data as combining the gauge field and the B-field into a connection for a principal bundle for a higher group. Finally, in §1.3, we use the general theory from §1.2 to determine the tangential structure for the CHL string.

The material in this section is not new, though it was not always stated in this form before. The fact that a Bianchi identity/Green-Schwarz mechanism is expressing a lift to a connection for a higher-group principal bundle is well-known; see Fiorenza-Schreiber-Stasheff [**FSS12**] and Sati-Schreiber-Stasheff [**SSS12**].

1.1. The $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string. Heterotic string theories are ten-dimensional superstring theories whose low-energy limits are 10d $\mathcal{N} = 1$ supergravity theories. These supergravity theories can have Yang-Mills terms, and so are parametrized by the data of the gauge group G, a compact Lie group. However, not all choices of G yield valid supergravity theories; there is the potential for an anomaly that must be trivialized, and this is quite a strong constraint, implying that the connected component of the identity in G must be either $\mathrm{E}_8 \times \mathrm{E}_8$ or $G = \mathrm{SemiSpin}_{32}$[2] [**GS84, ATD10**]. The anomaly cancellation mechanism itself, due to Green-Schwarz [**GS84**], combines the different fields in the theory into a connection for a principal $\mathbb{G}$-bundle, where $\mathbb{G}$ is a higher group;[3] we use this subsection to discuss the fields and the Green-Schwarz condition, and the next subsection to discuss the role of higher group. In this paper, we will focus solely on the $\mathrm{E}_8 \times \mathrm{E}_8$ case; it would be interesting to study the analogues of the computations and applications in this paper in the $\mathrm{SemiSpin}_{32}$ case.

The group $\mathbb{Z}/2$ acts on $\mathrm{E}_8 \times \mathrm{E}_8$ by exchanging the two factors, and the setup of heterotic string theory, including the low-energy supergravity limit and Green-Schwarz' anomaly cancellation, is invariant under this symmetry, so we can expand the gauge group to $G \coloneqq (\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$.[4] This appears to have first been noticed by McInnes [**McI99**, §I]; see also [**dBDH$^+$00**, §2.2.1].

Enlarging the gauge group from $\mathrm{E}_8 \times \mathrm{E}_8$ to $(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$ is a choice, and requires justification — why this semidirect product, and not other or larger expansions? The answer is that $\mathbb{Z}/2$ is acting through the isomorphism $\mathbb{Z}/2 \cong$

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[2] The center of Spin_{4k} is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Quotienting by one copy of $\mathbb{Z}/2$ yields SO_{4k}; the quotients by the two other $\mathbb{Z}/2$ subgroups are isomorphic, and are called $\mathrm{SemiSpin}_{4k}$. See [**McI99**].

[3] Green-Schwarz' work only cancels the perturbative part of the anomaly; see §3.2 for more information.

[4] Though we often use the standard name "the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string" to refer to this theory, we will always consider the larger gauge group $(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$.

$\mathrm{Out}(\mathrm{E}_8 \times \mathrm{E}_8)$, meaning both that it acts nontrivially on the gauge group (and in turn, on the fields of the theory) and that no larger group acts faithfully in this way. Similar outer automorphism extensions have been considered in, for example, [**HLMM22, BG23**].

The fields of 10d $\mathcal{N} = 1$ supergravity on a manifold M include:

- a metric g,
- a spin structure on M,
- a principal G-bundle $P \to M$ with connection Θ_P,
- a *B-field* or *Kalb-Ramond field*, a gerbe $Q \to M$ with connection Θ_Q, and
- several additional fields (the dilaton, dilatino, gravitino, and gaugino) which will not be directly relevant to this paper.

Let us say more about the B-field, since its model as a gerbe with connection may be less familiar. A *gerbe* is a categorification of the idea of a principal $\mathbb{T}$-bundle; here $\mathbb{T}$ is the circle group. Thus, for example, a principal $\mathbb{T}$-bundle $P \to M$ is classified by its first Chern class $c_1(P) \in H^2(M; \mathbb{Z})$, and a gerbe $Q \to M$ is classified by its *Dixmier-Douady class* $\mathrm{DD}(Q) \in H^3(M; \mathbb{Z})$ [**DD63, Bry93**]. A connection on a principal $\mathbb{T}$-bundle has holonomy around loops; a connection on a gerbe has holonomy on closed surfaces. And so on.

Gerbes were first introduced by Giraud [**Gir71**]. There are several different and equivalent ways to precisely define gerbes and their connections; heuristically you can think of a gerbe on M as a sheaf of groupoids on M locally equivalent to the trivial sheaf with fiber $\mathrm{pt}/\mathbb{T}$. One way to make this precise is the following.

If $f\colon Y \to X$ is a map, we let $Y^{[n]} \coloneqq Y \times_X Y \times_X \cdots \times_X Y$; $Y^{[n]}$ is the space of n-simplices in the Čech nerve for f.

Definition 1.1 (Murray [**Mur96**])**.** A *bundle gerbe* over a manifold M is a surjective submersion $\pi\colon Y \to M$, a $\mathbb{T}$-bundle $P \to Y^{[2]}$, and an isomorphism $\mu\colon \pi_{12}^*P \otimes \pi_{23}^*P \xrightarrow{\cong} \pi_{13}^*P$ of $\mathbb{T}$-bundles over $Y^{[3]}$ satisfying the natural associativity condition (see below) over $Y^{[4]}$.

Given two $\mathbb{T}$-bundles $P_1, P_2 \to X$, their *tensor product* $P_1 \otimes P_2$ is the unit circle bundle inside the tensor product of the Hermitian line bundles $L_1, L_2 \to X$ associated to P_1, resp. P_2. The maps $\pi_{12}, \pi_{23}, \pi_{13}\colon Y^{[3]} \rightrightarrows Y^{[2]}$ are the three face maps in the Čech nerve $Y^\bullet$ associated to f, given explicitly by contracting two of the three copies of Y via $Y \times_X Y \to Y$.

The associativity condition in Definition 1.1 is a little unwieldy to state explicitly, but can be found in in [**Mur10**, Definition 4.1(2)].

Definition 1.2 ([**Mur96**])**.** A *connection* Θ_Q on a bundle gerbe $Q = (Y, P, \mu)$ is data of a 2-form $B \in \Omega^2(Y)$ and a connection Θ_P on P such that if $\Omega_P \in \Omega^2(P)$ denotes the curvature of P and $\pi_1, \pi_2\colon Y^{[2]} \to Y$ are the two projections, then

$$\Omega_P = \pi_2^*B - \pi_1^*B. \tag{1.3}$$

The *curvature* of Θ_Q is $\Omega_Q \coloneqq \mathrm{d}B$, which is a closed 3-form.

The key thing to know about this definition is that, just like a principal $\mathbb{T}$-bundle $P \to M$ with connection locally has a connection 1-form A and globally has a curvature 2-form Ω_P which locally satisfies $\Omega_P = \mathrm{d}A$, a gerbe with connection Q locally has a connection 2-form B and globally has a curvature 3-form Ω_Q which locally satisfies $\Omega_Q = \mathrm{d}B$. For more information, see, e.g., Brylinski [**Bry93**, §5.3].

Definition 1.4. Because E_8 is a simple, connected, simply connected, compact Lie group, there is a canonical isomorphism $H^4(B\mathrm{E}_8;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$. Let c denote the generator corresponding to $1 \in \mathbb{Z}$. In $B(\mathrm{E}_8\times\mathrm{E}_8) \simeq B\mathrm{E}_8 \times B\mathrm{E}_8$, let c_1 and c_2 denote the copies of c coming from the first, resp. second copies of $B\mathrm{E}_8$ via the Künneth map.

The class $c_1 + c_2$ is invariant under the $\mathbb{Z}/2$ swapping action, so descends via the Serre spectral sequence to a class in $H^4(B((\mathrm{E}_8\times\mathrm{E}_8)\rtimes\mathbb{Z}/2);\mathbb{Z})$, which we also call $c_1 + c_2$.

Definition 1.5. Spin_n is also a compact, connected, simply connected simple Lie group when $n \geq 3$, and the generator of $H^4(B\mathrm{Spin}_n;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ corresponding to 1 is denoted λ.

The class λ is preserved under the standard embeddings $\mathrm{Spin}_n \hookrightarrow \mathrm{Spin}_{n+k}$, so we often work with its stabilized avatar $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z})$. We use this to define λ for Spin_n when $n < 3$. Because $2\lambda = p_1$, the class λ is often denoted $\frac{1}{2}p_1$. The mod 2 reduction of λ is the Stiefel-Whitney class w_4.

Lemma 1.6 (Whitney sum formula)**.** *Let X be a topological space and $E_1, E_2 \to X$ be two vector bundles with spin structure. Then $\lambda(E_1\oplus E_2) = \lambda(E_1) + \lambda(E_2)$.*

Proof. It suffices to prove the universal case, which amounts to the calculation of the pullback of λ by the map

$$\oplus\colon B\mathrm{Spin}_{k_1} \times B\mathrm{Spin}_{k_2} \longrightarrow B\mathrm{Spin}_{k_1+k_2}. \tag{1.7}$$

For $n \geq 3$, Spin_n is a connected, simply connected, compact simple Lie group, so $H^\ell(B\mathrm{Spin}_n;\mathbb{Z})$ vanishes for $\ell = 1,2,3$ and is isomorphic to $\mathbb{Z}$ for $\ell = 0,4$. For $n < 3$, $H^*(B\mathrm{Spin}_n;\mathbb{Z})$ is still trivial or free abelian in degrees 4 and below. Therefore by the Künneth formula, for all k_1, k_2, $H^4(B\mathrm{Spin}_{k_1}\times B\mathrm{Spin}_{k_2};\mathbb{Z})$ is a free abelian group, meaning that if we can show $2\lambda(E_1\oplus E_2) = 2\lambda(E_1) + 2\lambda(E_2)$, then we can deduce $\lambda(E_1\oplus E_2) = \lambda(E_1)+\lambda(E_2)$.

As $2\lambda = p_1$, we have reduced to the Whitney sum formula for p_1. The Whitney sum formula $p_1(E_1\oplus E_2) = p_1(E_1) + p_1(E_2)$ does not actually hold for all vector bundles, but Brown [**Bro82**, Theorem 1.6] (see also Thomas [**Tho62**]) showed that the difference $p_1(E_1\oplus E_2) - p_1(E_1) - p_1(E_2)$ vanishes when E_1 and E_2 are orientable, so in our setting of spin vector bundles, we can conclude. ☐

Remark 1.8. There are other ways to prove Lemma 1.6: for example, it follows immediately from a result of Jenquin [**Jen05**, Corollary 4.9] in a simple generalized cohomology theory. Johnson-Freyd and Treumann [**JFT20**, §1.4] sketch another proof of Lemma 1.6.

Next, we introduce the Chern-Weil homomorphism. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $f \in \mathrm{Sym}^k(\mathfrak{g}^\vee)$, i.e. f is a degree-k polynomial function on $\mathfrak{g}$ which is invariant under the adjoint G-action on $\mathfrak{g}$. Given a manifold M, a principal G-bundle $P \to M$, and a connection Θ on P, let $\Omega \in \Omega^2_P(\mathfrak{g})$ denote the curvature 2-form. Then one can evaluate f on $\Omega^{\wedge k} \in \Omega^{2k}_P(\mathfrak{g}^{\otimes k})$, producing a form $f(\Omega^{\wedge k}) \in \Omega^{2k}_P$; because f is Ad-invariant, $f(\Omega^{\wedge k})$ descends to a form $w(\Theta) \in \Omega^{2k}_M$, which is always closed. This defines a ring homomorphism, called the *Chern-Weil homomorphism* [**Car50, Che52**],

$$w\colon \mathrm{Sym}^\bullet(\mathfrak{g}^\vee) \longrightarrow H^*_{\mathrm{dR}}(M), \tag{1.9a}$$

which doubles the degree and is natural in M; moreover, the de Rham class of $w(\Theta)$ depends on P but not on the connection. Using de Rham's theorem and naturality, w upgrades to a ring homomorphism

$$w\colon \mathrm{Sym}^\bullet(\mathfrak{g}^\vee) \longrightarrow H^*(BG;\mathbb{R}), \tag{1.9b}$$

which Chern and Weil showed is an isomorphism when G is compact [**Che52, Wei49**]. Thus, when G is compact, a class $x \in H^{2*}(BG;\mathbb{Z})$ defines a polynomial $\mathrm{CW}_x \in \mathrm{Sym}^*(\mathfrak{g}^\vee)$, the w-preimage of the de Rham class of x. We will also write $\mathrm{CW}_x(\Theta)$ to denote the form defined by evaluating the polynomial CW_x on the curvature form of Θ.

Returning to 10d $\mathcal{N} = 1$ supergravity, Green-Schwarz [**GS84**] noticed that in order to trivialize an anomaly, one has to impose a relation between P and Q and their connections, so that Q is not quite a gerbe, but instead something twisted. Specifically, the curvature Ω_Q is no longer closed, but instead satisfies the equation

$$\mathrm{d}\Omega_Q = \mathrm{CW}_{c_1+c_2}(\Theta_P) - \mathrm{CW}_\lambda(\Theta^{\mathrm{LC}}), \tag{1.10}$$

where Θ^{LC} is the Levi-Civita connection on the principal Spin_n-bundle of frames of M.[5] This is called a *Bianchi identity* in the physics literature, motivating the following definition.

Definition 1.11. Given data of a compact Lie group G and a class $\mu \in H^4(BG;\mathbb{Z})$, the *twisted Bianchi identity* is the equation

$$\mathrm{d}H = \mathrm{CW}_\mu(\Theta_P), \tag{1.12}$$

where H is a 3-form and Θ_P is a connection on a principal G-bundle.

As in the case of (1.10), we think of this as mixing the data of two connections, one on a principal G-bundle and one on a gerbe. In the next section, we interpret twisted Bianchi identities as coming from connections on higher groups.

1.2. From the Bianchi identity to higher groups. In this section, we show that the twisted Bianchi identity (1.12) is a natural consequence of combining a principal G-bundle and a gerbe, each with connections, into a principal $\mathbb{G}$-bundle, where $\mathbb{G}$ is a certain Lie 2-group built from G and μ, together with additional data that we think of as a connection on $\mathbb{G}$. First we introduce 2-groups and their principal bundles; then, following [**FSS12, SSS12**], we recover the twisted Bianchi identity. As a result, we can precisely define the tangential structure for the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string, i.e. the topological part of the data which, when put on a manifold M, allows one to study $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory on that manifold.

Definition 1.13. A 2-*group* $\mathbb{G}$ is a group object in the bicategory of small categories.

Definition 1.14. A *Lie* 2-*group* is a 2-group $\mathbb{G}$ whose underlying category has been given the structure of a category object in smooth manifolds.

This means that the sets of objects and morphisms are smooth manifolds, and assignments such as the source of a map or the composition of two maps are smooth.

[5] Before Green-Schwarz, it was already known that $\mathrm{CW}_{c_1+c_2}(\Theta_P)$ and $\mathrm{d}\Omega_Q$ had to mix in order to preserve supersymmetry, thanks to work of Bergshoeff-de Roo-de Wit-van Nieuwenhuizen [**BdRdWvN82**] and Chapline-Manton [**CM83**].

2-groups were first introduced by Hoàng Xuân Sính in her thesis [**Hoà75**], and Lie 2-groups were introduced by Baez [**Bae02**, §2].

We call a 2-group *strict* if it is strict as a monoidal category, i.e. its associators and unitors are all identity maps. Mac Lane's coherence theorem [**Mac71**, Chapter 7] implies every 2-group is equivalent to a strict 2-group, but the analogous statement is false for Lie 2-groups; see Remark 1.25.

Example 1.15. If G is a group, it defines a monoidal groupoid with G as its set of objects, tensor product $g \otimes h \coloneqq gh$, and only the identity morphisms. This is a 2-group, and inherits the structure of a Lie 2-group if G is a Lie group.

This procedure embeds the bicategory of groups, group homomorphisms, and identity 2-morphisms into the bicategory of 2-groups, and we will therefore abuse notation and call this 2-group G again.

Example 1.16. Let A be an abelian group, and let $A[1]$ denote the monoidal groupoid with a single object $*$ and $\mathrm{Hom}_{A[1]}(*, *) \coloneqq A$. This is a 2-group, and if A is Lie, $A[1]$ is a Lie 2-group.

It turns out every 2-group $\mathbb{G}$ factors as an extension of these examples. Let e be the identity object of $\mathbb{G}$ and $\pi_0(\mathbb{G})$ be the group of isomorphism classes of objects in $\mathbb{G}$. Then there is a short exact sequence of 2-groups

$$0 \longrightarrow \mathrm{Aut}_{\mathbb{G}}(e)[1] \longrightarrow \mathbb{G} \longrightarrow \pi_0(\mathbb{G}) \longrightarrow 0. \tag{1.17}$$

The Eckmann-Hilton theorem guarantees $\mathrm{Aut}_{\mathbb{G}}(e)$ is abelian. Extensions (1.17) are classified by the data of:

(1) an action of $\pi_0(\mathbb{G})$ on $\mathrm{Aut}_{\mathbb{G}}(e)$, and
(2) a cohomology class $k \in H^3(B\pi_0(\mathbb{G}); \mathrm{Aut}_{\mathbb{G}}(e))$, called the *k-invariant* of $\mathbb{G}$.

When $\mathbb{G}$ has the discrete topology, this is unambiguous, but when $\mathbb{G}$ is a Lie 2-group, one must be careful what kind of cohomology is used here. The correct notion of cohomology is the *Segal-Mitchison cohomology* [**Seg70, Seg75**] of $\pi_0(\mathbb{G})$ valued in the abelian Lie group $\mathrm{Aut}_{\mathbb{G}}(e)$, as shown by Schommer-Pries [**SP11**, Theorem 1].

Now we want to discuss principal $\mathbb{G}$-bundles. The idea is that if G is a group, a principal G-bundle is a submersion which is locally trivial, and whose fibers are G-torsors. For a Lie 2-group $\mathbb{G}$, we need the fibers to locally look like $\mathbb{G}$, meaning they must be categorified somehow.

Definition 1.18 (Bartels [**Bar06**], Nikolaus-Waldorf [**NW13**, Definition 6.1.5])**.** Let $\mathbb{G}$ be a Lie 2-group. A *principal* $\mathbb{G}$*-bundle* over a smooth manifold M is a Lie groupoid $\mathcal{P}$ with a surjective submersion $\mathrm{obj}(P) \to M$ and a smooth right action ρ of $\mathbb{G}$ on $\mathcal{P}$ such that the map

$$(\mathrm{pr}_1, \rho)\colon \mathcal{P} \times \mathbb{G} \longrightarrow \mathcal{P} \times_M \mathcal{P} \tag{1.19}$$

is a weak equivalence of Lie groupoids.

See Nikolaus-Waldorf [**NW13**, §6] for more details. The principal $\mathbb{G}$-bundles on a manifold M form a 2-groupoid $\mathcal{B}un_{\mathbb{G}}(X)$ [**NW13**, Theorem 6.2.1].

Definition 1.20. Let $\mathbb{G}$ be a 2-group, and let $C_{\mathbb{G}}$ be the bicategory with a single object $*$ and morphism category $\mathrm{Hom}_{C_{\mathbb{G}}}(*,*) \coloneqq \mathbb{G}$. The *classifying space* of $\mathbb{G}$, denoted $B\mathbb{G}$, is the geometric realization of the nerve of $C_{\mathbb{G}}$.[6]

When $\mathbb{G}$ is a Lie 2-group, we make the same definition. This time $C_{\mathbb{G}}$ is a topological bicategory, so its nerve is a simplicial space, and geometrically realizing, we obtain the space $B\mathbb{G}$.

Theorem 1.21 (Nikolaus-Waldorf [**NW13**, Theorems 4.6, 5.3.2, 7.1])**.** *If $\mathbb{G}$ is a strict Lie 2-group, then there is a natural equivalence $[X, B\mathbb{G}] \xrightarrow{\cong} \pi_0(\mathcal{B}un_{\mathbb{G}}(X))$.*

Nikolaus-Waldorf's proof builds on Baez-Stevenson's related but distinct characterization of $[X, B\mathbb{G}]$ [**BS09**, Theorem 1] in terms of nonabelian Čech cohomology.

When G is an ordinary group, if G has the discrete topology, BG has only one nonzero homotopy group, which is $\pi_1(BG) = G$; likewise if $\mathbb{G}$ is a discrete 2-group, $\pi_i(B\mathbb{G})$ is nontrivial only for $i = 1, 2$; $\pi_1(B\mathbb{G}) = \pi_0(\mathbb{G})$ and $\pi_2(B\mathbb{G}) = \mathrm{Aut}_{\mathbb{G}}(e)$. When $\mathbb{G}$ is a Lie 2-group, we have no control over its homotopy groups in general, just like BG when G is positive-dimensional.

If $\mathbb{G}$ has the discrete topology, the data classifying (1.17), namely the action of $\pi_0(\mathbb{G})$ on $\mathrm{Aut}_{\mathbb{G}}(e)$ and the k-invariant, is equivalent to the Postnikov data of $B\mathbb{G}$, worked out by Mac Lane-Whitehead [**MLW50**]: this data classifies fibrations over BG with fiber the Eilenberg-Mac Lane space $K(\mathrm{Aut}_{\mathbb{G}}(e), 2)$. The total space of the fibration with this Postnikov data is homotopy equivalent to $B\mathbb{G}$.

Example 1.22. Let G be a compact Lie group; then, the Segal-Mitchison cohomology group $H^3_{\mathrm{SM}}(G; \mathbb{T})$ classifying Lie 2-group extensions of G by $\mathbb{T}[1]$ is naturally isomorphic to $H^4(BG; \mathbb{Z})$ [**SP11**, Corollary 97]. Therefore given a class $\mu \in H^4(BG; \mathbb{Z})$, we obtain a Lie 2-group $\mathcal{S}tr(G, \mu)$ fitting into a central extension

$$0 \longrightarrow \mathbb{T}[1] \longrightarrow \mathcal{S}tr(G, \mu) \longrightarrow G \longrightarrow 0, \tag{1.23}$$

which is sometimes called the *string 2-group* or *string cover* associated to G and λ. Of all the string covers, the most commonly studied one is $\mathrm{String}_n \coloneqq \mathcal{S}tr(\mathrm{Spin}_n, \lambda)$, which is called *the* string 2-group.

This class of 2-groups was first studied by Baez-Lauda [**BL04**, §8.5].

The sequence (1.23) implies that upon taking classifying spaces,

$$B\mathbb{G} \longrightarrow BG \xrightarrow{\mu} K(\mathbb{Z}, 4) \tag{1.24}$$

is a fibration.

Remark 1.25. Theorem 1.21 classified principal $\mathbb{G}$-bundles when $\mathbb{G}$ is a strict 2-group, but it is a theorem of Baez-Lauda [**BL04**, Corollary 60] that there is no strict Lie 2-group model for $\mathcal{S}tr(G, \mu)$ when G is simply connected and $\mu \neq 0$. However, there is a fix: in the setting of *Fréchet Lie 2-groups*, where we allow the spaces of objects and morphisms of $\mathbb{G}$ to be Fréchet manifolds, there is a strict model for $\mathcal{S}tr(G, \mu)$ [**BSCS07, LW23**], so $B\mathcal{S}tr(G, \mu)$ actually classifies principal $\mathcal{S}tr(G, \mu)$-bundles. This suffices for studying bordism groups.

[6] There are many different definitions of the nerve of a bicategory; the fact that their geometric realizations are canonically homotopy equivalent is a theorem of Carrasco-Cegarra-Garzón [**CCG10**], allowing us to speak about $B\mathbb{G}$ without specifying which kind of bicategorical nerve to use.

Following Sati-Schreiber-Stasheff [**SSS12**], we will now relate $\mathcal{S}tr(G,\mu)$ to the twisted Bianchi identity for G and μ. To do so, we use the language of stacks and differential cohomology, following [**HS05, FH13, Sch13, BNV16, ADH21**]. Make the category $\mathcal{M}an$ into a site by defining the covers to be surjective submersions, and define a *stack* to be a functor of ∞-categories $\mathcal{M}an^{op} \to \mathcal{T}op$ which satisfies descent for hypercovers. This defines a presentable ∞-category $\mathcal{S}t$ of stacks [**Lur09**, Proposition 6.5.2.14], and the Yoneda embedding $h\colon \mathcal{M}an \to \mathcal{S}t$ embeds $\mathcal{M}an$ as a full subcategory. We will often simply write M for the stack $h(M)$; we never compare these two notions directly, so this will not introduce confusion.

For any space X, the functor $\mathrm{Map}(\text{–}, X)\colon \mathcal{M}an \to \mathcal{T}op$ is a sheaf, and this procedure defines a functor of ∞-categories $\Gamma^*\colon \mathcal{T}op \to \mathcal{S}t$. The values of the stacks produced by Γ^* evaluated on manifolds M are homotopy-invariant in M. Γ^* has a left adjoint $\Gamma_\sharp\colon \mathcal{S}t \to \mathcal{T}op$ (see Dugger [**Dug01**, Proposition 8.3], Morel-Voevodsky [**MV99**, Proposition 3.3.3], and [**ADH21**, Proposition 4.3.1]); $\Gamma_\sharp(\mathbf{X})$ for a stack $\mathbf{X}$ can be thought of as the best approximation to $\mathbf{X}$ by a stack whose values on manifolds are homotopy-invariant.

Let $\Delta^n_{\mathrm{alg}} \coloneqq \{(t_0,\dots,t_n) \mid t_0+\dots+t_n = 1\} \subset \mathbb{R}^{n+1}$. These "algebraic n-simplices" assemble into a cosimplicial manifold $\Delta^\bullet_{\mathrm{alg}}$, and [**ADH21**, Corollary 5.1.4] there is a natural homotopy equivalence $\Gamma_\sharp(\mathbf{X}) \simeq |\mathbf{X}(\Delta^\bullet_{\mathrm{alg}})|$, where as usual $|\text{–}|$ denotes geometric realization.

Thus, for a manifold M, there is a natural homotopy equivalence $\Gamma_\sharp(M) \xrightarrow{\simeq} M$, so a map $M \to \mathbf{X}$ naturally induces a map $M \to \Gamma_\sharp(\mathbf{X})$.

Lemma 1.26. *Suppose $\mathbf{X} \to \mathbf{Y} \leftarrow \mathbf{Z}$ is a diagram in $\mathcal{S}t$, and that $\mathbf{Y}(\Delta^n_{\mathrm{alg}})$ and $\mathbf{Z}(\Delta^n_{\mathrm{alg}})$ are connected for all n. Then*

$$\Gamma_\sharp(\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}) \simeq \Gamma_\sharp(\mathbf{X}) \times_{\Gamma_\sharp(\mathbf{Y})} \Gamma_\sharp(\mathbf{Z}). \tag{1.27}$$

Proof. Pullbacks of sheaves can be computed pointwise, then sheafifying, so given a pullback $\mathbf{X} \to \mathbf{Y} \leftarrow \mathbf{Z}$ in $\mathcal{S}t$, for each $n \ge 0$ the pullback of

$$\mathbf{X}(\Delta^n_{\mathrm{alg}}) \longrightarrow \mathbf{Y}(\Delta^n_{\mathrm{alg}}) \longleftarrow \mathbf{Z}(\Delta^n_{\mathrm{alg}}) \tag{1.28}$$

is $(\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z})(\Delta^n_{\mathrm{alg}})$. The Bousfield-Friedlander theorem [**BF78, Bou01**] implies that, given the hypotheses on $\mathbf{Y}$ and $\mathbf{Z}$ in the theorem statement, the homotopy pullback of the geometric realizations of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ is the geometric realization of the levelwise homotopy pullback (1.28) (see [**War20**, p. 14-9] for this specific consequence of the Bousfield-Friedlander theorem). $\square$

Example 1.29. For G a Lie group, there is a stack $B_\nabla G$ whose value on a manifold M is the geometric realization of the nerve of the groupoid of principal G-bundles on M with connection [**FH13**]. This object is denoted $\mathbf{B}G_{\mathrm{conn}}$ in [**FSS12, SSS12, Sch13**], $\mathbb{B}G^\nabla$ in [**BNV16**, §5], and Bun_G^∇ in [**ADH21**].

There is a natural homotopy equivalence $\Gamma_\sharp(B_\nabla G) \xrightarrow{\simeq} BG$ [**ADH21**, Corollary 13.3.29], which can be interpreted as forgetting from a principal bundle with connection to a principal bundle.

Example 1.30. For $k \ge 0$, there is a stack $B^k_\nabla \mathbb{T}$ whose value on a manifold M is the geometric realization of the nerve of the ∞-groupoid of cocycles for the differential cohomology group $\check{H}^{k+1}(M;\mathbb{Z})$. This object is studied in [**FSS12, SSS12, Sch13**], where it is denoted $\mathbf{B}^k U(1)_{\mathrm{conn}}$.

Lemma 1.31. *There is a homotopy equivalence $\Gamma_\sharp(B^k_\nabla \mathbb{T}) \simeq K(\mathbb{Z}, k+1)$.*

Proof. Schreiber [Sch13, Observation 1.2.134] produces the following pullback square in $\mathcal{S}t$:

$$\begin{tikzcd} B^k_\nabla\mathbb{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Omega^{k+1}_{c\ell} \arrow[d] \\ K(\mathbb{Z}, k+1) \arrow[r] & K(\mathbb{R}, k+1), \end{tikzcd} \tag{1.32}$$

where $\Omega^{k+1}_{c\ell}$ is the stack of closed $(k+1)$-forms. For that stack and $K(\mathbb{R}, n+1)$, the values on each Δ^n_{alg} are connected spaces, so Lemma 1.26 identifies $\Gamma_\sharp(B^k_\nabla\mathbb{T}) \simeq K(\mathbb{Z}, k+1) \times_{K(\mathbb{R},k+1)} \Gamma_\sharp(\Omega^{k+1}_{c\ell})$. To finish, observe that, essentially by the de Rham theorem, the map $\Omega^{k+1}_{c\ell} \to K(\mathbb{R}, k+1)$ passes to a homotopy equivalence after applying $\Gamma_\sharp$. This follows from [BNV16, Lemma 7.15] together with the Dold-Kan theorem. ☐

These stacks are the universal setting for the Chern-Weil map.

Theorem 1.33 (Cheeger-Simons [CS85], Bunke-Nikolaus-Völkl [BNV16])**.** *Let G be a compact Lie group and $c \in H^k(BG;\mathbb{Z})$, where k is even. Then there is a map $\check c\colon B_\nabla G \to B^{k-1}_\nabla\mathbb{T}$ natural in (G, c) such that for any manifold M and map $f\colon M \to B_\nabla G$, interpreted as a principal G-bundle $P \to M$ with connection Θ,*

(1) if $\mathit{char}\colon \check H^*(–;\mathbb{Z}) \to H^*(–;\mathbb{Z})$ denotes the characteristic class map, then $\mathit{char}(\check c \circ f) = c(P)$, and
(2) if $\mathit{curv}\colon \check H^*(–;\mathbb{Z}) \to \Omega^*_{c\ell}$ denotes the curvature map, then $\mathit{curv}(\check c \circ f) = \mathrm{CW}_c(\Theta)$.

Cheeger-Simons lifted the Chern-Weil map to differential cohomology; Bunke-Nikolaus-Völkl recast it in terms of $B_\nabla G$. The map *char* in Theorem 1.33 is the map down the left of the square (1.32); *curv* is the map across the top of (1.32).

Definition 1.34 (Fiorenza-Schreiber-Stasheff [FSS12, §6.2])**.** Given a compact Lie group G and a class $\mu \in H^4(BG;\mathbb{Z})$, let $\mathbf{BStr}(G, \mu)$ denote the fiber of the map $\check\mu\colon B_\nabla G \to B^3_\nabla\mathbb{T}$.

We will see momentarily that maps to $\mathbf{BStr}(G, \mu)$ lead to solutions to the twisted Bianchi identity for G and μ.

Proposition 1.35. *There is a natural homotopy equivalence $\Gamma_\sharp(\mathbf{BStr}(G, \mu)) \simeq B\mathcal{S}tr(G, \mu)$.*

For this reason we think of $\mathbf{BStr}(G, \mu)$ as the classifying stack of principal $\mathcal{S}tr(G, \mu)$-bundles with connection, though this is only a heuristic.[7]

Proof. Apply Lemma 1.26 to the diagram

$$B_\nabla G \xrightarrow{\check\mu} B^3_\nabla\mathbb{T} \longleftarrow *, \tag{1.36}$$

as the values of both $*$ and $B^3_\nabla\mathbb{T}$ are connected on Δ^n_{alg} for each n. This implies that $\Gamma_\sharp(\mathbf{BStr}(G, \mu))$ is the fiber of $\mu\colon BG \to K(\mathbb{Z}, 4)$, which we identified with $B\mathbb{G}$ in (1.24). ☐

[7] There are at least *five* notions of a connection on principal $\mathbb{G}$-bundles for $\mathbb{G}$ a 2-group: three are discussed by Waldorf [Wal18, §5], a fourth by Rist-Saemann-Wolf [RSW22], and a fifth, defined only for $\mathbb{G} = \mathrm{String}_n$, by Waldorf [Wal13].

Proposition 1.37 (Fiorenza-Schreiber-Stasheff [**FSS12**, §6.3])**.** *Let G be a compact Lie group, $U \subset \mathbb{R}^n$ be an open set, and $P \to U$ be a principal G-bundle with connection Θ. A lift of the corresponding map $f_{P,\Theta}\colon U \to B_\nabla G$ to a map $\widetilde{f}_{P,\Theta}\colon U \to \mathbf{BStr}(G,\mu)$ induces a form $H \in \Omega^3(U)$ such that H and Θ satisfy the twisted Bianchi identity* (1.12)*.*

The idea here is that we have specified a trivialization of the differential characteristic class $\check{\mu}(P,\theta)$. Applying the curvature map $curv\colon B^3_\nabla\mathbb{T} \to \Omega^4_{c\ell}$, we have also specified a trivialization of $\mathrm{CW}_\mu(\Theta)$, which locally is the data H showing that $\mathrm{CW}_\mu(\Theta)$ is exact.

A map to $\mathbf{BStr}(G,\mu)$ is more data than what we get from Proposition 1.37, as we have trivialized not just the Chern-Weil form, but also the differential characteristic class. This can be interpreted as saying the data H specifying the trivialization is quantized to form a twisted version of a gerbe with connection.

To summarize, given a map $M \to \mathbf{BStr}(G,\mu)$, the stack which we think of as modeling $\mathcal{S}tr(G,\mu)$-bundles with connection, we obtain:

(1) a principal $\mathcal{S}tr(G,\mu)$-bundle $\mathcal{P} \to M$ by Proposition 1.35, and
(2) a "twisted gerbe with connection," i.e. local data of a gerbe $Q \to M$ such that Ω_Q and the G-connection Θ induced by the map $\mathbf{BStr}(G,\mu) \to B_\nabla G$ satisfy the twisted Bianchi identity (1.12) by Proposition 1.37.

Motivated by this, we define of the tangential structure for the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string. This first appears in [**SSS12**, §3.2], with [**Sat11b, FSS15a**] considering some related examples.

Definition 1.38. Let $G \coloneqq (\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$. A *differential ξ_n^{het}-structure* on a manifold M is the following data:

(1) a Riemannian metric and spin structure on M,
(2) a principal G-bundle $P \to M$ with connection Θ, and
(3) a lift of

$$((B_{\mathrm{Spin}}(M), \Theta^{\mathrm{LC}}), (P,\Theta))\colon M \longrightarrow B_\nabla(\mathrm{Spin}_n \times G) \tag{1.39}$$

to a map $M \to \mathbf{BStr}(\mathrm{Spin}_n \times G, c_1 + c_2 - \lambda)$.

Here $B_{\mathrm{Spin}}(M) \to M$ is the principal Spin_n-bundle of frames of M, and Θ^{LC} denotes its Levi-Civita connection.

For bordism groups we want the topological version of this.

Definition 1.40. A *tangential structure* is a space B and a map $\xi\colon B \to B\mathrm{O}$. Given a tangential structure ξ, a *ξ-structure* on a virtual vector bundle $E \to X$ is a lift of the classifying map $f_E\colon X \to B\mathrm{O}$ to a map $\widetilde{f}_E\colon X \to B$ such that $\xi \circ \widetilde{f}_E = f_E$. A *$\xi$-structure* on a manifold M is a ξ-structure on its tangent bundle.

We make the analogous definition with maps $\xi_n\colon B_n \to B\mathrm{O}_n$; in this case, we only refer to ξ_n-structures on n-manifolds.

Lashof [**Las63**] defined bordism groups Ω_*^ξ of manifolds with ξ-structure, and Boardman [**Boa65**, §V.1] defined a Thom spectrum $MT\xi$ whose homotopy groups are naturally isomorphic to Ω_*^ξ via the Pontrjagin-Thom construction.[8] We think

[8]In homotopy theory, it is common to study the Thom spectra $M\xi$ representing ξ-structures on the stable *normal* bundle ν_M of a manifold M, and indeed many of the results we cite about MTSO, $\mathit{MTString}$, etc. are stated for MSO, $\mathit{MString}$, etc., or about Thom spectra $M\xi$ in general.

of the category of tangential structures as the category of spaces over $B\mathrm{O}$, and bordism groups and Thom spectra are functorial in this category. That is, taking bordism groups and Thom spectra is functorial as long as one commutes with the map down to $B\mathrm{O}$.

The following definition is a special case of a definition due to Sati-Schreiber-Stasheff [**SSS12**, Definition 2.8]. See [**Sat11b, FSS15a, FSS21**] for other related examples.

Definition 1.41. Let $G_n \coloneqq \mathrm{Spin}_n \times (\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$ and

$$\mathbb{G}_n^{\mathrm{het}} \coloneqq \mathcal{S}tr(G_n, c_1 + c_2 - \lambda). \tag{1.42}$$

The $\mathrm{E}_8 \times \mathrm{E}_8$ *heterotic tangential structure* is the tangential structure

$$\xi_n^{\mathrm{het}} \colon B\mathbb{G}_n^{\mathrm{het}} \longrightarrow B\mathrm{Spin}_n \longrightarrow B\mathrm{O}_n, \tag{1.43}$$

where the first map comes from the quotient of $\mathbb{G}^{\mathrm{het}}$ by $\mathbb{T}[1]$, followed by projection onto the Spin_n factor in G_n. We also define $\mathbb{G}^{\mathrm{het}}$ and ξ^{het} analogously by stabilizing in n.

In other words: a differential ξ_n^{het}-structure is a lift of a map to $B_\nabla(\mathrm{Spin} \times G)$ to $\mathbf{BStr}(G, \mu)$; by Proposition 1.35, a topological ξ_n^{het}-structure is the image of this data under $\Gamma_\sharp$. In particular, a ξ_n^{het}-structure on an n-manifold M includes data of a principal $\mathbb{G}_n^{\mathrm{het}}$-bundle $\mathcal{P} \to M$.

Taking the quotient of $\mathbb{G}^{\mathrm{het}}$ by $\mathbb{T}[1]$ induces a map of tangential structures

$$\phi \colon B\mathbb{G}^{\mathrm{het}} \longrightarrow B\mathrm{Spin} \times B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2). \tag{1.44}$$

Thus, much like a spin^c manifold M has an associated $\mathbb{T}$-bundle P with $c_1(P)$ mod $2 = w_2(M)$, a ξ^{het}-manifold has associated $(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2)$-bundle P. From this perspective, a ξ^{het}-structure on a manifold M is the following data:

- a spin structure on M,
- a double cover $\pi \colon \widetilde{M} \to M$,
- two principal E_8-bundles $P, Q \to \widetilde{M}$ which are exchanged by the nonidentity deck transformation of π, and
- a trivialization of the class $\lambda(M) - (c(P) + c(Q)) \in H^4(M; \mathbb{Z})$.

By a *trivialization* of a cohomology class $\alpha \in H^n(M; A)$ we mean a null-homotopy of the classifying map $f_\alpha \colon M \to K(A, n)$. Thus orientations are identified with trivializations of w_1, etc. To make the trivialization of $\lambda(M) - (c(P) + c(Q))$ precise, we have to descend the class $c(P) + c(Q)$, a priori an element of $H^4(\widetilde{M}; \mathbb{Z})$, to $H^4(M; \mathbb{Z})$. We can do this because, as noted in Definition 1.4, the class $c_1 + c_2$ descends through the Serre spectral sequence to the base.

Remark 1.45. We can combine some the data of a ξ^{het} structure on M into a twisted characteristic class. Let $\mathbb{Z}^\sigma$ be the $\mathbb{Z}[\mathbb{Z}/2]$-module isomorphic to $\mathbb{Z}^2$ as an abelian group, and in which the nontrivial element of $\mathbb{Z}/2$ swaps the two factors. Then, let $\mathbb{Z}^\sigma_\pi$ denote the local system on M which is the associated bundle $\widetilde{M} \times_{\mathbb{Z}/2} \mathbb{Z}^\sigma$. A pair of classes $x, y \in H^k(\widetilde{M}; \mathbb{Z})$ exchanged by the deck transformation thus define a class in $H^k(M; \mathbb{Z}^\sigma_\pi)$, so, the classes $c(P)$ and $c(Q)$ in $H^4(\widetilde{M}; \mathbb{Z})$ together

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This is not a problem: for any tangential structure ξ, there is a tangential structure $\xi^\perp$ such that a ξ-structure on TM is equivalent data to a $\xi^\perp$-structure on ν_M and vice versa, so that $MT\xi \simeq M\xi^\perp$, so the general theory is the same. And for $\xi = \mathrm{O}$, SO, Spin, Spin^c, and String, $\xi \simeq \xi^\perp$ and in those cases we can ignore the difference between $M\xi$ and $MT\xi$.

define a class $\widetilde{c}(P,Q) \in H^4(M;\mathbb{Z}^\sigma_\pi)$, which is a characteristic class of an $((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$-bundle.

If $\mathbb{Z}$ denotes the $\mathbb{Z}[\mathbb{Z}/2]$-module isomorphic to $\mathbb{Z}$ as an abelian group and with trivial $\mathbb{Z}/2$-action, then taking the quotient of $\mathbb{Z}^\sigma$ by the submodule generated by $(1,-1)$ defines a map of $\mathbb{Z}[\mathbb{Z}/2]$-modules $q\colon \mathbb{Z}^\sigma \to \mathbb{Z}$, hence also a map between the corresponding twisted cohomology groups, and this map sends $\widetilde{c}(P,Q) \mapsto c(P) + c(Q)$. Therefore one could recast a ξ^{het}-structure on a spin manifold M as the data of a principal $((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$-bundle (P,Q,π) together with a trivialization of $\lambda(M) - q(\widetilde{c}(P,Q))$.

Bott-Samelson [**BS58**, Theorems IV, V(e)] showed that the map $B\mathrm{E}_8 \to K(\mathbb{Z},4)$ defined by the characteristic class c is 15-connected. This implies that up to isomorphism, a principal $((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$-bundle on a manifold of dimension 15 or lower is equivalent data to its characteristic class $\widetilde{c}$.

Remark 1.46. One might want to simplify by restricting to the special case where $\pi\colon \widetilde{M} \to M$ is trivial (as done in, e.g., [**Wit86**]), in which case the data of a ξ^{het}-structure simplifies to the data of a spin structure on M, two principal E_8-bundles $P,Q \to M$, and a trivialization of $\lambda(M) - c(P) - c(Q)$. This corresponds to the tangential structure $\xi^{r,\mathrm{het}}\colon B\mathcal{S}tr(\mathrm{Spin} \times \mathrm{E}_8 \times \mathrm{E}_8, c_1 + c_2 - \lambda) \to B\mathrm{Spin} \to B\mathrm{O}$.

1.3. The CHL string. Eleven-dimensional $\mathcal{N} = 1$ supergravity admits a time-reversal symmetry, allowing it to be defined on pin$^+$ 11-manifolds.[9] Therefore we can compactify it on a Möbius strip with certain boundary data to obtain a nine-dimensional supergravity theory; the goal of this subsection is to determine the tangential structure of this theory. Eleven-dimensional $\mathcal{N} = 1$ supergravity is expected to be the low-energy limit of a theory called M-theory,[10] and compactifying M-theory on the Möbius strip is expected to produce a string theory called the *Chaudhuri-Hockney-Lykken (CHL) string* [**CHL95**] whose low-energy limit is the 9-dimensional supergravity theory described above; we study the tangential structure of this supergravity theory in this subsection with the aim of also learning about the CHL string.

However, we do not want our perspective on the CHL string to be overly one-sided. Once and for all, choose a section $s\colon \mathbb{Z}/2 \hookrightarrow (\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$ of the quotient $(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ by the normal $\mathrm{E}_8 \times \mathrm{E}_8$ subgroup. Then there is another way to produce the CHL string by compactifying: consider the circle with its nontrivial principal $\mathbb{Z}/2$-bundle $P \to S^1$. Via the map $\mathbb{Z}/2 \hookrightarrow \mathrm{Spin} \times ((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$ sending $1 \mapsto (\mathrm{id}, s(1))$, this bundle defines a $\mathrm{Spin} \times ((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$-structure on S^1 for which λ and $c_1 + c_2$ are both trivial, so this structure lifts to define a ξ^{het}-structure on S^1. We will call the circle with this ξ^{het}-structure $\mathbb{RP}^1$, as $S^1 \cong \mathbb{RP}^1$ as manifolds and the ξ^{het}-structure comes from the double cover $S^1 \to \mathbb{RP}^1$. The CHL string is precisely what one obtains by compactifying the E_8^2 heterotic string on $\mathbb{RP}^1$.

We want to determine the tangential structure ξ^{CHL} such that the product of $\mathbb{RP}^1$ with a manifold with ξ^{CHL}-structure has an induced ξ^{het}-structure. In general, keeping track of how the tangential structure changes under compactification can

[9] In addition to the pin$^+$ structure, one needs the additional data of a lift of $w_4(TM)$ to $w_1(TM)$-twisted integral cohomology. See [**Wit97, Wit16, FH21a**].

[10] M-theory is expected to require additional data on top of the tangential structure described above for 11-dimensional $\mathcal{N} = 1$ supergravity. See [**FSS20**, Table 1] and the references listed there.

be subtle; for a careful analysis, see Schommer-Pries [**SP18**, §9]. But for the CHL string, we can get away with a more ad hoc approach: following Chaudhuri-Polchinski [**CP95**] (see also [**dBDH$^+$00**, §2.2.1]) we restrict to the case where the principal $\mathbb{Z}/2$-bundle on $\mathbb{RP}^1 \times M$ obtained by the quotient map (1.44) is the pullback of the Möbius bundle $S^1 \to \mathbb{RP}^1$ along the projection $\mathrm{pr}_1 \colon \mathbb{RP}^1 \times M \to \mathbb{RP}^1$.

Proposition 1.47. *Let M be a spin manifold and $P \to M$ be a principal E_8-bundle. The data of a trivialization $\mathfrak{s}$ of $\lambda(M) - 2c(P)$ induces a ξ^{het}-structure on $\mathbb{RP}^1 \times M$ whose associated principal $\mathbb{Z}/2$-bundle is the Möbius bundle $S^1 \times M \to \mathbb{RP}^1 \times M$. Moreover, if $\dim(M) \leq 14$, this assignment is a natural bijection from the set of isomorphism classes of data $(P, \mathfrak{s})$ to the set of ξ^{het}-structures on $\mathbb{RP}^1 \times M$ whose associated $\mathbb{Z}/2$-bundle is $S^1 \times M \to \mathbb{RP}^1 \times M$.*

Proof. Let $\pi \colon S^1 \times M \to M$ be the projection onto the second factor. Given $P \to M$ and $\mathfrak{s}$, the pair of E_8-bundles $(\pi^*P, \pi^*P) \to S^1 \times M$ are exchanged by the deck transformation for $S^1 \times M \to \mathbb{RP}^1 \times M$, and $(c_1 + c_2)$ evaluated on the pair (π^*P, π^*P) is $2c(P) \in H^4(\mathbb{RP}^1 \times M; \mathbb{Z}/2)$. Choosing the string structure on $\mathbb{RP}^1$ induced from the bounding framing, we obtain a canonical trivialization of $\lambda(\mathbb{RP}^1 \times M) - \lambda(M) \in H^4(\mathbb{RP}^1 \times M; \mathbb{Z})$ from the two-out-of-three property of string structures. Putting all of this together, we see that we have data of two E_8-bundles on $S^1 \times M$ exchanged by the deck transformation, and a trivialization of $\lambda - (c_1 + c_2)$ on $\mathbb{RP}^1 \times M$, thus defining a ξ^{het}-structure as claimed.

To see that this produces all ξ^{het}-structures associated with $S^1 \times M \to \mathbb{RP}^1 \times M$, recall from Remark 1.45 that the $((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2)$-bundle associated to a ξ^{het}-structure is classified by a characteristic class in twisted cohomology. The assumption that the associated $\mathbb{Z}/2$-bundle is $S^1 \times M \to \mathbb{RP}^1 \times M$ implies this class belongs to $H^4(\mathbb{RP}^1 \times M; \underline{\mathbb{Z} \oplus \mathbb{Z}})$, where a generator of $\pi_1(\mathbb{RP}^1)$ acts on $\underline{\mathbb{Z} \oplus \mathbb{Z}}$ by swapping the two factors, and $\pi_1(M)$ acts trivially. The twisted Künneth formula [**Gre06**, Theorem 1.7] gives us an isomorphism

$$H^4(\mathbb{RP}^1 \times M; \underline{\mathbb{Z} \oplus \mathbb{Z}}) \xrightarrow{\cong} H^4(M; \mathbb{Z}), \tag{1.48}$$

meaning that the pair of E_8-bundles on the orientation double cover $S^1 \times M$ pull back from bundles on M, which must be isomorphic in order to be exchanged by the $\mathbb{Z}/2$-action. ☐

The Bianchi identity corresponding to this data can therefore be simplified to use a single bundle $P \to M$ and the class $c(P) + c(P)$: we obtain

$$\mathrm{d}H = \mathrm{CW}_{2c}(\Theta_P) - \mathrm{CW}_\lambda(\Theta^{\mathrm{LC}}), \tag{1.49}$$

i.e. the twisted Bianchi identity for $G = \mathrm{Spin} \times \mathrm{E}_8$ and $\mu = 2c - \lambda$. Then, following Definitions 1.38 and 1.41, we make the following definitions.

Definition 1.50. A *differential ξ_n^{CHL}-structure* on a manifold M is the following data:

(1) a Riemannian metric and spin structure on M,
(2) a principal E_8-bundle $P \to M$ with connection Θ, and
(3) a lift of

$$((B_{\mathrm{Spin}}(M), \Theta^{\mathrm{LC}}), (P, \Theta)) \colon M \longrightarrow B_\nabla(\mathrm{Spin}_n \times \mathrm{E}_8) \tag{1.51}$$

to a map $M \to \mathbf{BStr}(\mathrm{Spin}_n \times \mathrm{E}_8, 2c - \lambda)$.

What we call $B\mathbb{G}_n^{\mathrm{CHL}}$ coincides with what Sati-Schreiber-Stasheff call $B\mathrm{String}^{2a}$ [**SSS12**, (2.18), §2.3.3] and which also appears in work of Fiorenza-Sati-Schreiber [**FSS15a**, Remark 4.1.1], though those papers do not discuss its relationship with the CHL string.

Definition 1.52 (Sati-Schreiber-Stasheff [**SSS12**, (2.18), §2.3.3])**.** Let

$$\mathbb{G}_n^{\mathrm{CHL}} := \mathcal{S}tr(\mathrm{Spin}_n \times \mathrm{E}_8, 2c - \lambda). \tag{1.53}$$

The *CHL tangential structure* is the tangential structure

$$\xi_n^{\mathrm{CHL}} \colon B\mathbb{G}_n^{\mathrm{CHL}} \longrightarrow B\mathrm{Spin}_n \longrightarrow B\mathrm{O}_n, \tag{1.54}$$

where the first map comes from the quotient of $\mathbb{G}^{\mathrm{CHL}}$ by $\mathbb{T}[1]$, followed by projection onto the Spin_n factor. Stabilizing in n, we also obtain $\mathbb{G}^{\mathrm{CHL}}$ and a tangential structure ξ^{CHL}.

A ξ^{CHL}-structure on an n-manifold M in particular comes with data of a principal $\mathbb{G}_n^{\mathrm{CHL}}$-bundle $\mathcal{P} \to M$, and can be formulated as the data of a principal E_8-bundle $P \to M$ and a trivialization of $\lambda(M) - 2c(P) \in H^4(M;\mathbb{Z})$.

Remark 1.55. Since a ξ^{CHL} structure includes data identifying λ as twice another class, it induces a trivialization of the mod 2 reduction of λ, which is w_4. That is, a ξ^{CHL} structure induces a $\mathrm{Spin}\langle w_4\rangle$ structure, where $B\mathrm{Spin}\langle w_4\rangle$ is the homotopy fiber of $w_4 \colon B\mathrm{Spin} \to K(\mathbb{Z}, 4)$. This structure has been studied in, e.g. [**Wit97, KS04, FH21a**] for applications to M-theory.

Remark 1.56 (Variation of the tangential structure along the moduli space)**.** There is a moduli space of CHL string theories, not just one, and the gauge group depends on where in the moduli space one is; this moduli space was first studied by Chaudhuri-Polchinski [**CP95**]. At a generic point, the gauge group is broken to $\mathbb{T}^8$, and at various special points the gauge group enhances to E_8 or other nonabelian groups: see [**FFG$^+$21**, Table 3]. We work only at the E_8 point of the moduli space in this paper; it would be interesting to apply the techniques in this paper to other points in the CHL moduli space.

There has been quite a bit of recent research studying the moduli spaces of compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string and the CHL string, and investigating which gauge groups can occur [**FGN18, CDLZ20, FFG$^+$20, CDLZ21, FFG$^+$21, FPDF21, MV21, CDLZ22, CGH22, CMM22, FPDF22, PDF23, MPDF23**].

2. Bordism computations

Now it is time to compute. We will use the Adams spectral sequence to compute $\Omega_*^{\xi^{\mathrm{het}}}$ and $\Omega_*^{\xi^{\mathrm{CHL}}}$; this is a standard tool in computational homotopy theory and more recently appears frequently in the mathematical physics literature, and we point the interested reader to Beaudry-Campbell's introductory article [**BC18**].

Applications of the Adams spectral sequence to mathematical physics questions tend to follow the same formula. Suppose that we want to compute Ω_*^{ξ} for some tangential structure ξ.

(1) First, express ξ as a "twisted ξ'-structure," where ξ' is one of SO, Spin, Spin^c, or String: prove that a ξ-structure on a vector bundle $E \to M$ is equivalent data to an auxiliary vector bundle $V \to M$ and a ξ'-structure on $E \oplus V$.

This implies that $MT\xi \simeq MT\xi' \wedge X$ for some Thom spectrum X that is usually not too complicated.

(2) Next, invoke a change-of-rings theorem to greatly simplify the calculation of the E_2-page for ξ'-bordism of spaces or spectra. Then run the Adams spectral sequence, taking advantage of the extra structure afforded by the change-of-rings theorem.

This recipe goes back to work of Anderson-Brown-Peterson [**ABP69**] and Giambalvo [**Gia73b, Gia73a, Gia76**] computing twisted spin bordism. It is most commonly used in the case $\xi' = \mathrm{Spin}$, where it has been frequently used to compute bordism groups for tangential structures representing field theories with fermions; $\xi' = \mathrm{String}$ is less common but still appears in physically motivated examples, including the tangential structure of the Sugimoto string [**Sug99**] and $\xi = \mathrm{String}^c$ [**CHZ11, Sat11b**].

Unfortunately, ξ^{het} and ξ^{CHL} do not belong to this class of examples: we will see in Lemma 2.2 that there is no way to write these tangential structures as twisted string structures in the sense above.[11,12] So we have to do something different.

At odd primes, we plow ahead with the unsimplified Adams spectral sequence, though since we only care about dimensions 11 and below the computations are very tractable. At $p = 2$, though, we can modify the above strategy to simplify the computation: in §2.1, we generalize the notion of "twisted string bordism" for which the change-of-rings trick works to include string covers (in the sense of Example 1.22) of groups of the form $\mathrm{Spin} \times G$. This applies to both ξ^{het} and ξ^{CHL}, and so we are off to the races.

Remark 2.1. We are far from the first to compute bordism groups for a tangential structure $\xi\colon B \to B\mathrm{O}$ where B is the classifying space of a 2-group. For example, $\Omega_*^{\mathrm{String}}$ has been calculated in a range of degrees by [**Gia71, HR95, MG95, Hov08**]; other examples include [**Hil09, KT17, WW19a, WW19b, WWZ19, Tho20, LT21, Yu21, DL23**].

2.1. Twists of string bordism.

"Started out with a twist, how did it end up like this?
It was only a twist, it was only a twist..."

Once the tangential structure for a bordism question is known, the next step is typically to prove a "shearing" theorem simplifying the tangential structure. For example, the usual route to computing pin$^-$ bordism [**Pet68**, §7] first establishes an isomorphism between pin$^-$ bordism and the spin bordism of the Thom spectrum $\Sigma^{-1}MO_1$, and then computes the latter groups using something like the Adams or Atiyah-Hirzebruch spectral sequence.

There are a few different approaches to shearing theorems, such as those in [**FH21b, DDHM23**], but generally they work with Thom spectra of vector bundles; for example, the above simplification of pin$^-$ bordism begins with the observation that a pin$^-$ structure on a bundle $E \to M$ is equivalent data to a real line bundle $L \to M$ and a spin structure on $E \oplus L$, which follows from a characteristic class computation, and then passes the data of "L and a spin structure on $E \oplus L$" through the Pontrjagin-Thom theorem.

[11] The presence of the B-field, and how the Bianchi identity mixes it with the principal Spin_n-bundle of frames, rules out $\xi' = \mathrm{SO}$, Spin, or Spin^c.

[12] This problem also happens to the tangential structures studied in [**FH21a, DY22**].

This approach does not work for the heterotic and CHL tangential structures.

Lemma 2.2. *There is no spin vector bundle V on $B((\mathrm{E}_8\times\mathrm{E}_8)\rtimes\mathbb{Z}/2)$ such that $\lambda(V)=c_1+c_2$, and there is no spin vector bundle W on $B\mathrm{E}_8$ such that $\lambda(V)=2c$.*

This means there is no way to express a ξ^{het}-structure as "a G-bundle and a string structure on E plus some associated bundle," and likewise for ξ^{CHL}.

Proof. Let G be a compact, simple, simply connected Lie group and $\rho\colon G\to\mathrm{SU}_n$ be a representation. $H^4(BG;\mathbb{Z})$ and $H^4(B\mathrm{SU}_n;\mathbb{Z})$ are both canonically isomorphic to $\mathbb{Z}$, so the pullback map ρ^* on H^4 is a map $\mathbb{Z}\to\mathbb{Z}$, necessarily multiplication by some integer $\delta(\rho)$. Because SU_n is compact, connected, and simply connected, the standard inclusion $\mathrm{SU}_n\to\mathrm{GL}_{2n}(\mathbb{R})$ lifts to a map $\mathrm{SU}_n\to\mathrm{Spin}_{2n}$. Choices of this lift are a torsor over $H^1(B\mathrm{SU}_n;\mathbb{Z}/2)=0$, meaning that the characteristic class λ is uniquely defined for SU_n-representations. Moreover, λ of the defining representation is a generator of $H^4(B\mathrm{SU}_n;\mathbb{Z})$; because $H^4(B\mathrm{SU}_n;\mathbb{Z})$ is torsion-free, it suffices to show $2\lambda=p_1$ is twice a generator, which is standard. The *Dynkin index* of G is the minimum value of $|\delta(\rho)|$ over all such representations ρ. Laszlo-Sorger [**LS97**, Proposition 2.6] show that the Dynkin index of E_8 is 60, meaning that for any vector bundle $V\to B\mathrm{E}_8$ with SU-structure induced from a representation, $\lambda(V)$ is at least 60 times a generator.

We would like to generalize to real representations.

Lemma 2.3. *The complexification map $\mathrm{Spin}_n\to\mathrm{O}_n\to\mathrm{U}_n$ has image contained in SU_n.*

Proof. A lift of a representation $\rho\colon G\to\mathrm{U}_n$ has image contained in SU_n if and only if c_1 of the complex vector bundle associated to ρ vanishes. When one pulls back across the complexification map $B\mathrm{O}_n\to B\mathrm{U}_n$, c_1 is sent to the image of w_1 under the Bockstein map $\beta\colon H^1(B\mathrm{O}_n;\mathbb{Z}/2)\to H^2(B\mathrm{O}_n;\mathbb{Z})$; when we pull back further to $B\mathrm{Spin}_n$, $w_1\mapsto 0$, so $c_1=\beta w_1\mapsto 0$ too. ◻

Thus the Dynkin index fact we mentioned above applies to complexifications of representations landing in Spin_n.

If V is a real representation of a group G, $V\otimes\mathbb{C}\cong V\oplus V$ as real representations, so using the Whitney sum formula for λ (Lemma 1.6), $\lambda(V\otimes\mathbb{C})=2\lambda(V)$. Therefore if V is any real spin representation of E_8, $\lambda(V\otimes\mathbb{C})$ is at least 60 times a generator, so $\lambda(V)$ is at least 30 times a generator. Thus the class defining $\mathbb{G}^{\mathrm{CHL}}$, which is twice a generator, is not λ of any spin representation of E_8; likewise for $\mathbb{G}^{\mathrm{het}}$, as one could restrict to either factor of E_8 inside $\mathrm{E}_8^2\rtimes\mathbb{Z}/2$ and obtain a representation with λ equal to the generator.

Finally, the Atiyah-Segal completion theorem extends this from representations to all vector bundles. Because λ is additive (Lemma 1.6), it factors through the Grothendieck group $KSpin(BG)$ of spin vector bundles on BG, and similarly, evaluated on spin representations, λ factors through the corresponding Grothendieck group $RSpin(G)$. Atiyah-Segal [**AS69**, §7, §8] show that taking the associated bundle of an arbitrary representation exhibits the Grothendieck ring $KO^0(BG)$ of all vector bundles on BG as the completion of the representation ring $RO(G)$ at its augmentation ideal. Thus given a $\mathbb{Z}$-valued characteristic class c of arbitrary vector bundles of G which satisfies the Whitney sum formula, passing from representations of G to vector bundles on BG does not decrease the minimal value of $|c|$.

In order to use the Atiyah-Segal theorem, we need to get from spin representations and vector bundles to arbitrary ones. We will do so, at the cost of lowering the minimum value of λ a little bit. For any vector bundle V, $V^{\oplus 4}$ admits a canonical spin structure: the Whitney sum formula for Stiefel-Whitney classes shows a spin structure exists; then choose a spin structure universally over $B\mathrm{O}$. Therefore we can define λ of an arbitrary representation of E_8 or vector bundle on $B\mathrm{E}_8$ by $\lambda(V) \coloneqq \frac{1}{4}\lambda(V^{\oplus 4})$, valued in $\frac{1}{4}\mathbb{Z}$. Therefore passing from $RO(\mathrm{E}_8) \to KO^0(B\mathrm{E}_8)$ to $RSpin(\mathrm{E}_8) \to KSpin(B\mathrm{E}_8)$ divides the minimal value of λ by at most 4, and now we can invoke Atiyah-Segal, so it is still not possible to get $2c$ and ξ^{CHL}; and likewise for $\mathrm{E}_8^2 \rtimes \mathbb{Z}/2$ in place of E_8 to show that the characteristic class for ξ^{het} cannot be achieved. □

So we take a different approach: we cannot get Thom spectra corresponding to vector bundles, but we can still obtain *MTString*-module Thom spectra. We accomplish this using the theory of Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG+14a, ABG+14b**] (ABGHR), which we briefly summarize.

The idea behind the ABGHR perspective on Thom spectra is to generalize the notion of local coefficients to generalized cohomology theories. Given a based, connected space X and a homomorphism $\rho\colon \pi_1(X) \to \mathrm{GL}_1(\mathbb{Z}) \cong \{\pm 1\}$, one obtains a local coefficient system $\mathbb{Z}_\rho$ on X: this is a bundle on X with fiber $\mathbb{Z}$, and whose monodromy around a loop $\gamma \in \pi_1(X)$ is precisely $\rho(\gamma)$. Given $\mathbb{Z}_\rho$, we can take twisted cohomology groups: if $\widetilde{X} \to X$ denotes the universal cover, then the cochain complex $C^*(\widetilde{X};\mathbb{Z})$ has a $\pi_1(X)$-action induced from the $\pi_1(X)$-action on $\widetilde{X}$. If $C^*(X;\mathbb{Z}_\rho)$ denotes the subcomplex of $C^*(\widetilde{X};\mathbb{Z})$ of cochains which transform under this $\pi_1(X)$-action by ρ, then $H^*(X;\mathbb{Z}_\rho) \coloneqq H^*(C^*(X;\mathbb{Z}_\rho))$.

Another way to say this is that if pt/G denotes the category with one object $*$ and $\mathrm{Hom}(*,*) = G$, ρ defines a $\mathrm{pt}/\pi_1(X)$-shaped diagram of chain complexes of abelian groups:

$$\mathrm{pt}/\pi_1(X) \xrightarrow{\rho} \mathrm{pt}/\{\pm 1\} \longrightarrow \mathcal{C}h_{\mathbb{Z}}, \tag{2.4}$$

sending pt to $C^*(\widetilde{X};\mathbb{Z})$, and sending $g \in \pi_1(X)$ to the action by $\rho(g)$. The subcomplex of cochains that transform by ρ is precisely the limit of this diagram. For functoriality reasons, we envision this complex as cochains on some object $\mathcal{X}$ which is a *co*limit of a diagram akin to (2.4).

To summarize, twisted cohomology, i.e. cohomology of the Thom spectrum, is expressed as a colimit of a diagram of chain complexes of $\mathbb{Z}$-modules induced from a map $X \to B\mathrm{Aut}(\mathbb{Z})$. Ando-Blumberg-Gepner-Hopkins-Rezk lift this to spectra. Specifically, given a ring spectrum R, Ando-Blumberg-Gepner-Hopkins-Rezk naturally associate a topological group[13] $\mathrm{GL}_1(R)$, thought of as the group of units or group of automorphisms of R. The classifying space $B\mathrm{GL}_1(R)$ carries the universal local system of R-lines; a local system of R-lines over X is equivalent data to a map $X \to B\mathrm{GL}_1(R)$.

Definition 2.5 (Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG+14a**, Defn. 2.20])**.** The *Thom spectrum* Mf associated to a map $f\colon X \to B\mathrm{GL}_1(R)$ is the colimit of the diagram $X \to B\mathrm{GL}_1(R) \to \mathcal{M}od_R$, where we think of X as its fundamental ∞-groupoid.

[13] $\mathrm{GL}_1(R)$ is not exactly a topological group, but the homotopy-coherent version thereof: a grouplike A_∞-space.

When $R = \mathbb{S}$, this is due to Lewis [**LMSM86**, Chapter IX]. In Definition 2.5, we have to consider the fundamental ∞-groupoid, rather than just π_1, because R can have higher automorphisms, because spectra are derived objects.

The Thom spectrum of a map to $B\mathrm{GL}_1(R)$ is an R-module.

Example 2.6 (Twisted ordinary cohomology)**.** There is a homotopy equivalence $B\mathrm{GL}_1(H\mathbb{Z}) \simeq K(\mathbb{Z}/2, 1)$, so the ABGHR viewpoint recovers $\mathrm{Aut}(\mathbb{Z})$ and the usual notion of cohomology twisted by a local system. To prove this homotopy equivalence, use the homotopy pullback square of E_∞-spaces [**ABG$^+$14b**, Definition 2.1]

$$\begin{tikzcd} \mathrm{GL}_1(H\mathbb{Z}) \arrow[r] \arrow[d, "\varphi"] & \Omega^\infty H\mathbb{Z} \arrow[d, "\psi"] \\ (\pi_0(H\mathbb{Z}))^\times \arrow[r] & \pi_0(H\mathbb{Z}). \end{tikzcd} \tag{2.7}$$

$\Omega^\infty H\mathbb{Z} \simeq \mathbb{Z}$ as E_∞-spaces, and ψ is a homotopy equivalence of E_∞-spaces. Therefore φ is also a homotopy equivalence of E_∞-spaces, and we conclude.

Example 2.8 (Thom spectra from vector bundles)**.** Boardman's original definition of Thom spectra [**Boa65**, §V.1] associates them to virtual vector bundles $V \to X$. Let us connect this to the ABGHR definition. Virtual vector bundles are classified by maps $f_V \colon X \to B\mathrm{O}$, and one avatar of the J-homomorphism [**Whi42**] is a map $J \colon \mathrm{O} \to \mathrm{GL}_1(\mathbb{S})$ [**ABG10**, Example 3.15], which deloops to a map of spaces $BJ \colon B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$. A map with this signature is a natural assignment from virtual vector bundles $V \to X$ to local systems of invertible $\mathbb{S}$-modules, and BJ assigns to V the local system with fiber $\mathbb{S}^{V_x}$ at each $x \in X$. Putting these maps together, we have an X-shaped diagram

$$X \xrightarrow{f_V} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathcal{S}p, \tag{2.9}$$

and the colimit of this diagram, which is a Thom spectrum in the ABGHR sense, coincides with the Thom spectrum X^V in the usual sense. This is a combination of theorems of Lewis [**LMSM86**, Chapter IX] and Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG$^+$14a**, Corollary 3.24].

This approach to Thom spectra plays well with multiplicative structures. If R is an E_∞-ring spectrum, then the grouplike A_∞-structure on $\mathrm{GL}_1(R)$ refines to a grouplike E_∞-structure, making $\mathrm{GL}_1(R)$ and therefore $B\mathrm{GL}_1(R)$ into infinite loop spaces. For $1 \le k \le \infty$, if X is a k-fold loop space and $f \colon X \to B\mathrm{GL}_1(R)$ is a k-fold loop map, then the Thom spectrum Mf inherits the structure of an E_k-ring spectrum. This is a theorem of Lewis [**LMSM86**, Theorem IX.7.1] for $R = \mathbb{S}$ and Ando-Blumberg-Gepner [**ABG18**, Theorem 1.7] for more general R.

$B\mathrm{O}$ has an infinite loop space structure coming from the addition-like operation on $B\mathrm{O}$ of direct sum of vector bundles. The J-homomorphism $BJ \colon B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$ is an infinite loop map, so we get an E_∞-ring structure on $MT\xi$ if ξ is a tangential structure satisfying a *2-out-of-3 property*, i.e. whenever any two of E, F, and $E \oplus F$ have a ξ-structure, the third has an induced ξ-structure. The idea is that the 2-out-of-3 property implies that $\xi \colon B \to B\mathrm{O}$ is an infinite loop map, so passing to $B\mathrm{GL}_1(\mathbb{S})$ and taking the Thom spectrum, we obtain an E_∞-ring spectrum. This applies to MTO, $MTSO$, $MTSpin^c$, $MTSpin$, and $MTString$; however, some commonly considered tangential structures appearing in physics do not have this property, including $B\mathrm{Pin}^{\pm}$.

Proposition 2.10. *Let B and X be infinite loop spaces and $\xi\colon B\to B\mathrm{O}$ and $f\colon B\to X$ be infinite loop maps, so that the fiber $\eta\colon F\to B$ of f is also a map of infinite loop spaces. This data naturally defines twists of the Thom spectrum $M(\xi\circ\eta)$ over X, i.e. a map $X\to B\mathrm{GL}_1(M(\xi\circ\eta))$.*

Proof. The fiber of $\eta\colon F\to B$ is another infinite loop map $\zeta\colon \Omega X\to F$, so the induced map of Thom spectra (where the maps down to $B\mathrm{O}$ are $\xi\circ\eta\circ\zeta$ and $\xi\circ\eta$ respectively) is a map of E_∞-ring spectra. Because $\xi\circ\eta\circ\zeta$ is nullhomotopic, its Thom spectrum is a suspension spectrum, so we have a map of E_∞-ring spectra $\Sigma^\infty_+\Omega X\to M(\xi\circ\eta)$.

Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG**$^+$**14b**, (1.4), (1.7)] prove that Σ^∞_+ and GL_1 are an adjoint pair on the categories of infinite loop spaces and E_∞-ring spectra. Applying this adjunction, we have a map of infinite loop spaces $\Omega X\to \mathrm{GL}_1(M(\xi\circ\eta))$; deloop to obtain the map in the theorem statement. $\square$

Theorem 2.11 (Beardsley [**Bea17**, Theorem 1])**.** *With notation as in Proposition 2.10, the Thom spectrum of the "universal twist" $X\to B\mathrm{GL}_1(M(\xi\circ\eta))$ is canonically equivalent to $M\xi$.*

Corollary 2.12.

(1) *There is a map $\widehat{w}_1\colon K(\mathbb{Z}/2,1)\to B\mathrm{GL}_1(MTSO)$ which, after taking the quotient $MTSO\to H\mathbb{Z}$, passes to the homotopy equivalence $K(\mathbb{Z}/2,1)\to B\mathrm{GL}_1(H\mathbb{Z})$ from Example 2.6.*

(2) *There is a map $\widehat{w}_2\colon K(\mathbb{Z}/2,2)\to B\mathrm{GL}_1(MTSpin)$ which, after composing with the Atiyah-Bott-Shapiro map $MTSpin\to ko$* [**ABS64, Joa04**]*, is the usual map $K(\mathbb{Z}/2,2)\hookrightarrow B\mathrm{GL}_1(ko)$* [**DK70, HJ20**].

(3) *There is a map $\widehat{\beta w_2}\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(MTSpin^c)$ which, after composing with the Atiyah-Bott-Shapiro map $MTSpin^c\to ku$* [**ABS64, Joa04, AHR10**]*, is the usual twist of K-theory by degree-3 classes $K(\mathbb{Z},3)\to B\mathrm{GL}_1(ku)$* [**DK70, Ros89, AS04, ABG10**].

(4) *There is a map $\widehat{\lambda}\colon K(\mathbb{Z},4)\to B\mathrm{GL}_1(MTString)$ which, when composed with the Ando-Hopkins-Rezk orientation $MTString\to tmf$* [**AHR10**]*, is the Ando-Blumberg-Gepner map $K(\mathbb{Z},4)\to B\mathrm{GL}_1(tmf)$* [**ABG10**, Proposition 8.2].

Part (3) is a theorem of Hebestreit-Joachim [**HJ20**, Appendix C]. The other parts are surely known, though we were unable to find them in the literature.

Proof. Apply Proposition 2.10 to the four maps

(1) $w_1\colon B\mathrm{O}\to K(\mathbb{Z}/2,1)$, whose fiber is $B\mathrm{SO}$;
(2) $w_2\colon B\mathrm{SO}\to K(\mathbb{Z}/2,2)$, whose fiber is $B\mathrm{Spin}$;
(3) $\beta\circ w_2\colon B\mathrm{SO}\to K(\mathbb{Z},3)$, whose fiber is $B\mathrm{Spin}^c$, where $\beta\colon H^k(-;\mathbb{Z}/2)\to H^{k+1}(-;\mathbb{Z})$ is the Bockstein; and
(4) $\lambda\colon B\mathrm{Spin}\to K(\mathbb{Z},4)$, whose fiber is $B\mathrm{String}$.

All four of these are infinite loop maps, because these characteristic classes are additive in direct sums. For compatibility with preexisting twists, we use the fact that in the spin^c and string cases, Ando-Blumberg-Gepner [**ABG10**, §7, §8] construct the desired twists $K(\mathbb{Z},3)\to B\mathrm{GL}_1(ku)$ and $K(\mathbb{Z},4)\to B\mathrm{GL}_1(tmf)$ in the same way as we construct the twists of MTSpin^c and $\mathit{MTString}$, so compatibility follows from functoriality. The cases of ko and $H\mathbb{Z}$ are analogous. $\square$

The homotopy groups of the Thom spectra of the twists Corollary 2.12 have bordism interpretations. Looking at $\widehat{w}_2$ for example, a spin structure on an oriented manifold is a trivialization of $w_2(TM)$, but given a space X and a degree-2 cohomology class B, thought of as a map $f_B\colon X \to K(\mathbb{Z}/2, 2)$, the homotopy groups of $MT(\widehat{w}_2 \circ f_B)$ are the bordism groups of oriented manifolds M together with a map $g\colon M \to X$ and a trivialization of $w_2(TM) + g^*B$, as was shown by Hebestreit-Joachim [**HJ20**, Corollary 3.3.8]. The other three cases are analogous; in particular, we have described the Thom spectra for ξ^{het} and ξ^{CHL} as $\mathit{MTString}$-module Thom spectra.

These kinds of twisted bordism have been studied before: spin^c structures twisted by a degree-3 cohomology class were first studied by Douglas [**Dou06**, §5], and they appear implicitly in work of Freed-Witten [**FW99**] on anomaly cancellation. Twisted spin and string structures of the sort appearing in Corollary 2.12 were first considered by B.L. Wang [**Wan08**, Definitions 8.2, 8.4]. See [**DFM11a, DFM11b, Sat11a, Sat11c, Sat12, Sat15, SW15, LSW20, SY21**] for more examples of twisted generalized cohomology theories from a similar point of view and some applications in physics.

The first case, involving twists of MTSO by degree-1 $\mathbb{Z}/2$-cohomology classes, is the notion of a twisted orientation from the beginning of this section: given a real line bundle $L \to X$, we ask for data of a map $g\colon M \to X$ and an orientation on $TM \oplus g^*(L)$. In the ABGHR perspective this says that the map $\widehat{w}_1$ factors through $B\mathrm{O}_1$ as

$$K(\mathbb{Z}/2, 1) \xrightarrow{\simeq} B\mathrm{O}_1 \hookrightarrow B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(\mathit{MTSO}). \tag{2.13}$$

But the others do not factor this way.

Remark 2.14. There is a complex version of (2.13). Let $\mathcal{W}$ denote *Wall's bordism spectrum* [**Wal60**], whose homotopy groups are the bordism groups of manifolds with an integral lift of w_1. Explicitly, if $\xi\colon F \to B\mathrm{O}$ is the fiber of $\beta w_1\colon B\mathrm{O} \to K(\mathbb{Z}, 2)$, then $\mathcal{W} \coloneqq MT\xi$. Proposition 2.10 then produces a map $\widehat{\beta w_1}\colon K(\mathbb{Z}, 2) \to B\mathrm{GL}_1(\mathcal{W})$, but degree-2 cohomology classes are equivalent to complex line bundles, and $\widehat{\beta w_1}$ factors as

$$K(\mathbb{Z}, 2) \xrightarrow{\simeq} B\mathbb{T} \to B\mathrm{O}_2 \to B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(\mathcal{W}). \tag{2.15}$$

Remark 2.16. One consequence of the fact that $\widehat{w}_1$ (resp. $\widehat{\beta w_1}$) factors as in (2.13) (resp. (2.15)), i.e. as a twist associated to a real (resp. complex) line bundle $L \to X$ is that the associated MTSO-module (resp. $\mathcal{W}$-module) Thom spectrum splits as $\mathit{MTSO} \wedge X^{L-1}$ (resp. $\mathcal{W} \wedge X^{L-2}$). Working universally over $B\mathrm{O}_1$ and $B\mathbb{T}$, Theorem 2.11 gives us homotopy equivalences $\mathit{MTSO} \wedge (B\mathrm{O}_1)^{L-1} \simeq \mathit{MTO}$ and $\mathcal{W} \wedge (B\mathbb{T})^{L-2} \simeq \mathit{MTO}$; the former is a theorem of Atiyah [**Ati61**, Proposition 4.1].

We will apply Corollary 2.12 to the degree-4 characteristic classes that the Bianchi identity told us for the heterotic and CHL tangential structures. Given a space X with a class $\mu \in H^4(X; \mathbb{Z})$, let $\mathcal{B}(X)$ denote the homotopy fiber of $\lambda + \mu\colon B\mathrm{Spin} \times X \to K(\mathbb{Z}, 4)$, and let ξ^μ denote the tangential structure

$$\xi^\mu\colon \mathcal{B}(X) \longrightarrow B\mathrm{Spin} \times X \longrightarrow B\mathrm{O}. \tag{2.17}$$

$MT\xi^\mu$ is equivalent to the *MTString*-module Thom spectrum associated to the twist $\widehat{\lambda}\circ\mu\colon X\to B\mathrm{GL}_1(\mathit{MTString})$. If $X = BG$ for a Lie group G, $\mathcal{B}(X)$ is the classifying space of the string 2-group $\mathcal{S}(\mathrm{Spin}\times G, \lambda+\mu)$. Let $\mathcal{A}$ denote the 2-primary Steenrod algebra and for $n\ge 0$, let $\mathcal{A}(n)$ denote the subalgebra of $\mathcal{A}$ generated by $\mathrm{Sq}^1,\dots,\mathrm{Sq}^{2^n}$. In joint work with Matthew Yu [**DY23**], we compute the $\mathcal{A}$-module structure on $H^*(MT\xi^\mu;\mathbb{Z}/2)$.

Definition 2.18. Let R denote the $\mathbb{Z}/2$-algebra $\mathcal{A}(1)[S]$, i.e. the algebra with generators Sq^1, Sq^2, and S, and with Adem relations for Sq^1 and Sq^2. Given X and μ as above, define the $\mathcal{A}(1)$-module $T(X,\mu) \coloneqq H^*(X;\mathbb{Z}/2)$, and give $T(X,\mu)$ an R-module structure by defining

$$S(x) \coloneqq \mu x + \mathrm{Sq}^4(x). \tag{2.19}$$

We want to think of S as Sq^4 and $T(X,\mu)$ as an $\mathcal{A}(2)$-module, but a priori it is not clear that this S-action satisfies the Adem relations.

Theorem 2.20 ([**DY23**])**.**

(1) The R-module structure on $T(X,\mu)$ satisfies the Adem relations for Sq^1, Sq^2, and $\mathrm{Sq}^4 = S$, hence induces an $\mathcal{A}(2)$-module structure on $T(X,\mu)$.

(2) There is an map of $\mathcal{A}$-modules

$$H^*(MT\xi^\mu;\mathbb{Z}/2)\longrightarrow \mathcal{A}\otimes_{\mathcal{A}(2)} T(X,\mu), \tag{2.21}$$

natural in the data (X,μ), which is an isomorphism in degrees 15 and below.

We describe a proof of this theorem in Remark 2.26 different from the one in [**DY23**].

Corollary 2.22. *For $t-s\le 15$, the E_2-page of the Adams spectral sequence computing 2-completed ξ^μ-bordism is*

$$E_2^{t,s} = \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(T(X,\mu),\mathbb{Z}/2). \tag{2.23}$$

As $\mathcal{A}(2)$ is much smaller than $\mathcal{A}$, this is much easier to work with.

Proof. This follows from the change-of-rings formula: if $\mathcal{B}$ is a graded Hopf algebra, $\mathcal{C}$ is a graded Hopf subalgebra of $\mathcal{B}$, and M and N are graded $\mathcal{B}$-modules, then there is a natural isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{C}} M, N)\xrightarrow{\cong}\mathrm{Ext}^{s,t}_{\mathcal{C}}(M,N). \tag{2.24}$$

This you can think of as the derived version of a maybe more familiar isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{C}} M, N)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{C}}(M,N). \tag{2.25}$$

In our example, $\mathcal{B}$ is the Steenrod algebra, which is a Hopf algebra, and $\mathcal{C}$ is $\mathcal{A}(2)$, which is indeed a Hopf subalgebra of $\mathcal{A}$, so we can invoke (2.24) and conclude. $\square$

We will use this simplification in the cases $\xi^\mu = \xi^{\mathrm{het}}, \xi^{\mathrm{CHL}}$ to run the Adams spectral sequences computing $\Omega_*^{\xi^{\mathrm{het}}}$ and $\Omega_*^{\xi^{\mathrm{CHL}}}$ at $p=2$.

Remark 2.26 (Proof sketch of Theorem 2.20)**.** To prove (1), check the Adem relations for $\mathcal{A}(2)$ directly. The first step in proving part (2) is to establish a Thom isomorphism for mod 2 cohomology. We make use of the *Thom diagonal*, a map of $\mathit{MTString}$-modules

$$MT\xi^\mu \xrightarrow{\Delta^t} MT\xi^\mu \wedge \mathit{MTString} \wedge \Sigma^\infty_+ X \tag{2.27}$$

defined as follows: the diagonal map $\Delta\colon X \to X \times X$ is a map of spaces over $B\mathrm{GL}_1(\mathit{MTString})$, if we give X the map $\widehat{\lambda}\circ\mu$ to $B\mathrm{GL}_1(\mathit{MTString})$ and we give $X\times X$ the map $(\widehat{\lambda}\circ\mu, *)$. Applying the $\mathit{MTString}$-module Thom spectrum functor to Δ produces (2.27). Smash (2.27) with $H\mathbb{Z}/2$. The result is the Thom diagonal for a twist of $H\mathbb{Z}/2$, but all such twists are trivializable (i.e. all $H\mathbb{Z}/2$-bundles admit an orientation). Therefore by [**ABG**$^+$**14b**, Proposition 3.26] the following composition is an equivalence:

$$MT\xi^\mu \wedge H\mathbb{Z}/2 \xrightarrow{\Delta^t} MT\xi^\mu \wedge \Sigma^\infty_+ X \wedge H\mathbb{Z}/2 \longrightarrow \mathit{MTString} \wedge \Sigma^\infty_+ X \wedge H\mathbb{Z}/2, \tag{2.28}$$

which is the $\mathbb{Z}/2$-homology Thom isomorphism. The analogous fact is true for mod 2 cohomology.

The Thom diagonal makes $H^*(MT\xi^\mu;\mathbb{Z}/2)$ into a free, rank-1 module over $H^*(\mathcal{B}(X);\mathbb{Z}/2)$, generated by the Thom class U. As the Thom diagonal is a map of spectra, we may use the Cartan formula to compute the Steenrod squares of an arbitrary element of $H^*(MT\xi^\mu;\mathbb{Z}/2)$ in terms of Steenrod squares in $\mathcal{B}(X)$ and $\mathrm{Sq}(U)$. As both $\mathrm{Sq}(U)$ and our desired isomorphism in (2.21) are natural in X and μ, it suffices to understand the universal case, where $X = K(\mathbb{Z},4)$ and μ is the tautological class $\tau \in H^4(K(\mathbb{Z},4);\mathbb{Z})$. In this case, Theorem 2.11 implies $MT\xi^\mu \simeq \mathit{MTSpin}$. By work of Anderson-Brown-Peterson [**ABP67**], if J is the $\mathcal{A}(1)$-module $\mathcal{A}(1)/\mathrm{Sq}^3$ and M is the $\mathcal{A}(1)$-module $\mathbb{Z}/2\oplus\Sigma^8\mathbb{Z}/2\oplus\Sigma^{10}J$, then there is a map of $\mathcal{A}$-modules

$$H^*(\mathit{MTSpin};\mathbb{Z}/2) \longrightarrow \mathcal{A}\otimes_{\mathcal{A}(1)} M \tag{2.29}$$

which is an isomorphism in degrees 15 and below. And Giambalvo [**Gia71**, Corollary 2.3] shows that there is a map $H^*(\mathit{MTString};\mathbb{Z}/2) \to \mathcal{A}\otimes_{\mathcal{A}(2)}\mathbb{Z}/2$ which is also an isomorphism in degrees 15 and below. Therefore by the change-of-rings theorem (2.24) it suffices to exhibit a map of $\mathcal{A}(2)$-modules

$$T(K(\mathbb{Z},4),\tau) \longrightarrow \mathcal{A}(2)\otimes_{\mathcal{A}(1)} M \tag{2.30}$$

which is an isomorphism in degrees 15 and below. This can be verified directly, using as input the $\mathcal{A}(2)$-module structure on $H^*(K(\mathbb{Z},4);\mathbb{Z}/2)$ calculated by Serre [**Ser53**, §10].

2.2. ξ^{het} bordism at $p = 2$. In this section we will first compute $H^*(BG;\mathbb{Z}/2)$ as an $\mathcal{A}(2)$-module in low degrees, where $G \coloneqq \mathrm{E}_8^2\rtimes\mathbb{Z}/2$; then, using Corollary 2.22, we run the Adams spectral sequence computing 2-completed ξ^{het} bordism in degrees 11 and below.

First, though, we reformulate the problem slightly. Consider the tangential structure $\xi^{\mathrm{het}\prime}\colon B^{\mathrm{het}\prime} \to BO$ defined in the same manner as ξ^{het}, but with $K(\mathbb{Z},4)$ replacing $B\mathrm{E}_8$. In a little more detail, $\mathbb{Z}/2$ acts on $K(\mathbb{Z},4)\times K(\mathbb{Z},4)$ by swapping the two factors; taking the Borel construction

$$B \coloneqq (K(\mathbb{Z},4)\times K(\mathbb{Z},4))\times_{\mathbb{Z}/2} E\mathbb{Z}/2 \tag{2.31}$$

produces a fiber bundle

$$K(\mathbb{Z},4)\times K(\mathbb{Z},4) \longrightarrow B \longrightarrow B\mathbb{Z}/2. \tag{2.32}$$

For $i = 1, 2$, let $c_i \in H^4(K(\mathbb{Z},4)\times K(\mathbb{Z},4);\mathbb{Z})$ be the tautological class for the i^{th} $K(\mathbb{Z},4)$ factor. The class c_1+c_2 is invariant under the $\mathbb{Z}/2$-action, so we can follow it through the Serre spectral sequence to learn that it defines a nonzero class $c_1+c_2 \in H^4(B;\mathbb{Z}/2)$. Define $f\colon B^{\mathrm{het}'} \to B\mathrm{Spin}\times B$ to be the fiber of $\lambda - (c_1+c_2)\colon B\mathrm{Spin}\times B \to K(\mathbb{Z},4)$; then the tangential structure $\xi^{\mathrm{het}'}$ is the composition

$$\begin{tikzcd} B^{\mathrm{het}'} \arrow[r, "f"] \arrow[rrr, bend right, "\xi^{\mathrm{het}'}"'] & B\mathrm{Spin}\times B \arrow[r, "\mathrm{pr}_1"] & B\mathrm{Spin} \arrow[r] & B\mathrm{O}. \end{tikzcd} \tag{2.33}$$

That is, a $\xi^{\mathrm{het}'}$ structure on a manifold M is a spin structure, a principal $\mathbb{Z}/2$-bundle $P\to M$, two classes $c_1, c_2 \in H^4(P;\mathbb{Z})$ which are exchanged under the deck transformation, and a trivialization of $\lambda(M)-(c_1+c_2)$ (where the latter class is descended to M). This is the same data as a ξ^{het} structure, except that we do not ask for c_1 or c_2 to come from principal E_8-bundles; therefore there is a map of tangential structures $\widetilde{c}\colon \xi^{\mathrm{het}} \to \xi^{\mathrm{het}'}$, i.e. a map of spaces $B\mathbb{G}^{\mathrm{het}} \to B^{\mathrm{het}'}$ commuting with the maps down to $B\mathrm{O}$. Like for ξ^{het}, a $\xi^{\mathrm{het}'}$-structure is a twisted string structure in the sense of Corollary 2.12, via the class $\lambda-(c_1+c_2)\colon B\to K(\mathbb{Z},4)$.

Bott-Samelson [**BS58**, Theorems IV, V(e)] showed that the characteristic class $c\in H^4(B\mathrm{E}_8;\mathbb{Z})$ we defined in Definition 1.4, interpreted as a map $c\colon B\mathrm{E}_8\to K(\mathbb{Z},4)$, is 15-connected. This means that the homomorphism $\widetilde{c}$ induces on bordism groups, $\widetilde{c}\colon \Omega_k^{\xi^{\mathrm{het}}} \to \Omega_k^{\xi^{\mathrm{het}'}}$, is an isomorphism in degrees 14 and below. For our string-theoretic purposes, we only care about $k\le 12$, so we may as well compute $\xi^{\mathrm{het}'}$-bordism. In the rest of this subsection, we often blur the distinction between ξ^{het} and $\xi^{\mathrm{het}'}$; we will point out where it matters which one we are looking at.

Remark 2.34. Turning off the $\mathbb{Z}/2$ symmetry switching the two E_8 factors, i.e. passing to a $\xi^{r,\mathrm{het}}$-structure as in Remark 1.46, simplifies this story considerably: the bordism groups were known decades ago. Specifically, replace $B\mathrm{E}_8$ with $K(\mathbb{Z},4)$ in the definition of $\xi^{r,\mathrm{het}}$ to define a tangential structure $\xi^{r,\mathrm{het}\prime}$, which on a manifold M consists of a spin structure on M, two classes $c_1,c_2\in H^4(M;\mathbb{Z})$, and a trivialization of $\lambda(M)-c_1-c_2$. As Witten [**Wit86**, §4] noticed, this data is equivalent to a spin structure and the single class c_1, which may be freely chosen; then c_2 must be $\lambda(M)-c_1$. Therefore the tangential structure $\xi^{r,\mathrm{het}\prime}$-structure is simply $B\mathrm{Spin}\times K(\mathbb{Z},4)\to B\mathrm{O}$, and just as for ξ^{het}, the map $MT\xi^{r,\mathrm{het}} \to MT\xi^{r,\mathrm{het}\prime} \simeq MTSpin\wedge K(\mathbb{Z},4)_+$ is an isomorphism on homotopy groups in degrees 14 and below. Stong [**Sto86**] computes $\Omega_*^{\mathrm{Spin}}(K(\mathbb{Z},4))$ in degrees 12 and below.

As we discussed in §1.2, the data of a trivial principal $\mathbb{Z}/2$-bundle on a manifold M and two principal E_8-bundles $P,Q\to M$ define a principal $\mathrm{E}_8^2\rtimes\mathbb{Z}/2$-bundle on M with c_1+c_2 equal to $c(P)+c(Q)$; data trivializing $c(P)+c(Q)-\lambda(M)$ therefore defines a ξ^{het} structure. Analogously, the trivial $\mathbb{Z}/2$-bundle and a pair $c_1,c_2\in H^4(M;\mathbb{Z})$ with a trivialization of $c_1+c_2-\lambda$ define a $\xi^{\mathrm{het}'}$ structure.

Lemma 2.35. *A spin manifold M has a canonical $\xi^{\mathrm{het}'}$ structure specified as above by the trivial principal $\mathbb{Z}/2$-bundle, the cohomology classes $c_1=\lambda$ and $c_2=0$, and the canonical trivialization of $\lambda-\lambda=0\in H^4(M;\mathbb{Z})$.*

This defines a map of tangential structures and therefore a map of Thom spectra $s_1\colon MTSpin\to MT\xi^{\mathrm{het}'}$. A $\xi^{\mathrm{het}'}$-structure includes data of a spin structure;

forgetting the rest of the $\xi^{\mathrm{het}\prime}$-structure defines a map $s_2\colon MT\xi^{\mathrm{het}\prime} \to MTSpin$. The composition of s_1 and s_2 is homotopy equivalent to the identity, because the underlying spin structure of the $\xi^{\mathrm{het}\prime}$ manifold built in Lemma 2.35 is the same spin structure we began with.

Corollary 2.36. *There is a spectrum $\mathcal{Q}$ and a splitting*

$$(s_2, q)\colon MT\xi^{\mathrm{het}\prime} \xrightarrow{\simeq} MTSpin \vee \mathcal{Q}. \tag{2.37}$$

We will use this later to reduce the amount of spectral sequence computations we have to make.

Both Lemma 2.35 and Corollary 2.36 require us to use $\xi^{\mathrm{het}\prime}$ and not ξ^{het}, though of course the consequence on low-degree bordism groups is true for both.

When K is a finite group, Nakaoka [**Nak61**, Theorem 3.3] proved that there is a ring isomorphism from the mod 2 cohomology of $B(\mathbb{Z}/2 \ltimes (K \times K))$ to the E_2-page of the Serre spectral sequence

$$E_2^{p,q} = H^p(B\mathbb{Z}/2; \underline{H^q(BK \times BK; \mathbb{Z}/2)}) \Longrightarrow H^{p+q}(B(\mathbb{Z}/2 \ltimes (K \times K)); \mathbb{Z}/2). \tag{2.38}$$

Here the underline denotes the local coefficient system arising from the $\mathbb{Z}/2$-action on $BK \times BK$ by switching the two factors. Since this local coefficient system can be nontrivial, one has to be careful defining the multiplicative structure on the E_2-page of (2.38), but here it can be made explicit. As a $\mathbb{Z}/2[\mathbb{Z}/2]$-module, $\underline{H^*(BK \times BK; \mathbb{Z}/2)}$ is a direct sum of:

- the subalgebra $\mathcal{H}_1$ of classes fixed by $\mathbb{Z}/2$, which are of the form $x \otimes x$ for $x \in H^*(BK; \mathbb{Z}/2)$; and
- the submodule $\mathcal{H}_2$ spanned by classes of the form $x \otimes y$ where x and y are linearly independent.

Since $\mathbb{Z}/2$ acts trivially on $\mathcal{H}_1$ and $\mathcal{H}_1$ is a ring, $H^*(B\mathbb{Z}/2; \mathcal{H}_1)$ has a ring structure. And as a $\mathbb{Z}/2[\mathbb{Z}/2]$-module, $\mathcal{H}_2$ is of the form $M \oplus M$ where $\mathbb{Z}/2$ acts by swapping the two factors, so $H^*(B\mathbb{Z}/2; \mathcal{H}_2)$ vanishes in positive degrees.[14] In degree zero, we obtain invariants, spanned by elements of the form $x \otimes y + y \otimes x$, with $x, y \in H^*(BK; \mathbb{Z}/2)$. $\mathcal{H}_1 \oplus (\mathcal{H}_2)^{\mathbb{Z}/2} = E_2^{0,\bullet}$ is a subalgebra of $H^*(BK \times BK; \mathbb{Z}/2)$.

So far we have specified ring structures on $H^*(B\mathbb{Z}/2; \mathcal{H}_1) \supsetneq E_2^{>0,\bullet}$ and $\mathcal{H}_1 \oplus (\mathcal{H}_2)^{\mathbb{Z}/2} = E_2^{0,\bullet}$, and these ring structures agree where they overlap. Therefore to specify a ring structure on the entirety of the E_2-page, it suffices to write down the product of an element in $(\mathcal{H}_2)^{\mathbb{Z}/2}$ and an element in positive p-degree. We say that all such products vanish; this is the ring structure that appears in Nakaoka's theorem.

Of course, E_8 is not a finite group. Nakaoka's theorem is true in quite great generality [**Eve65, Kah84, Lea97**]; the version we need is proven by Evens [**Eve65**], who proves the same ring isomorphism when K is a compact Lie group. Thus this applies to ξ^{het}, and not necessarily to $\xi^{\mathrm{het}\prime}$, but since their cohomology rings are isomorphic in degrees 14 and below, it does not matter which one we use in this calculation.

[14] To see this, first observe that mod 2 group cohomology for G is additive in the $\mathbb{Z}/2[G]$-module of coefficients, so it suffices to prove that $H^*(B\mathbb{Z}/2; M \oplus M)$ vanishes in positive degrees when $M = \mathbb{Z}/2$. But $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is isomorphic to $\mathbb{Z}/2[\mathbb{Z}/2]$ as $\mathbb{Z}/2[\mathbb{Z}/2]$-modules (i.e. as vector spaces with $\mathbb{Z}/2$-representations, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is isomorphic to the vector space of functions on the group $\mathbb{Z}/2$), and group cohomology valued in the group ring is trivial, e.g. because the group ring is its own free resolution.

Now we make this ring structure and $\mathcal{A}(2)$-module structure explicit. Since $c\colon B\mathrm{E}_8 \to K(\mathbb{Z}, 4)$ is 15-connected, it induces an isomorphism in cohomology in degrees 14 and below, so we can use the cohomology of $K(\mathbb{Z}, 4)$ as a stand-in for the cohomology of $B\mathrm{E}_8$. Serre [**Ser53**, §10] computed the mod 2 cohomology of $K(\mathbb{Z}, 4)$. It is an infinitely generated polynomial algebra; in degrees 12 and below the generators are: the tautological class $D \in H^4(K(\mathbb{Z}, 4); \mathbb{Z}/2)$, $F \coloneqq \mathrm{Sq}^2 D$, $G \coloneqq \mathrm{Sq}^3 D$, $J \coloneqq \mathrm{Sq}^4 F$, and $K \coloneqq \mathrm{Sq}^5 F$.

If C is one of D, F, G, J, or K, we let C_1 denote the class coming from the first copy of $B\mathrm{E}_8$ and C_2 denote the class coming from the second copy. Thus we have the following additive basis for the low-degree cohomology of BG:

(1) In $\mathcal{H}_1$, $D_1D_2x^k$ and $F_1F_2x^k$ for $k \geq 0$.
(2) In $(\mathcal{H}_2)^{\mathbb{Z}/2}$, $D_1 + D_2$, $F_1 + F_2$, $G_1 + G_2$, $D_1^2 + D_2^2$, $J_1 + J_2$, $D_1F_1 + D_2F_2$, $D_1F_2 + D_2F_1$, $D_1G_1 + D_2G_2$, $D_1G_2 + D_2G_1$, $K_1 + K_2$, $F_1^2 + F_2^2$, $D_1^3 + D_2^3$, and $D_1^2D_2 + D_1D_2^2$.

Next, we determine the $\mathcal{A}(2)$-module structure using a theorem of Quillen.

Theorem 2.39 (Quillen's detection theorem [**Qui71**, Proposition 3.1])**.** *Let X be a space and let $\mathbb{Z}/k$ act on X^k by cyclic permutations. Let $Y \coloneqq E\mathbb{Z}/k \times_{\mathbb{Z}/k} X^k$, which is a fiber bundle over $B\mathbb{Z}/k$ with fiber X^k. Let $i_1\colon X^k \to Y$ be inclusion of the fiber at the basepoint and $i_2\colon B\mathbb{Z}/k \times X \to Y$ be induced by the diagonal map; then*

$$(i_1^*, i_2^*)\colon H^*(Y; \mathbb{Z}/k) \longrightarrow H^*(X^k; \mathbb{Z}/k) \oplus H^*(B\mathbb{Z}/k \times X; \mathbb{Z}/k) \tag{2.40}$$

is injective.

For us, $k = 2$, $X = B\mathrm{E}_8$, and $Y = BG$. Thus, to compute Steenrod squares for classes in $H^*(BG; \mathbb{Z}/2)$, we can assume we are in $B\mathrm{E}_8^2$ if the class is in $(\mathcal{H}_2)^{\mathbb{Z}/2}$; for $\mathcal{H}^1$, we also need to know $\mathrm{Sq}(x)$, and i_2^* tells us $\mathrm{Sq}(x) = x + x^2$. Thus we can compute the $\mathcal{A}(2)$-module structure on $H^*(BG; \mathbb{Z}/2)$, hence also on $T(-(c_1 + c_2))$; we focus on the latter. Like most calculations of this form, it is a little tedious but straightforward, and can be done by hand in a reasonable length of time. After working through the calculation, we have learned the following.

Proposition 2.41. *Let $\mathcal{M}$ be the quotient of $T(-(c_1 + c_2))$ by all elements in degrees 14 and higher. Then $\mathcal{M}$ is the direct sum of the following submodules.*

(1) M_1, the summand containing the Thom class U.
(2) $M_2 \coloneqq \widetilde{H}^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ modulo those elements in degrees 13 and above.
(3) M_3, the summand containing $U(D_1^2 + D_2^2)$.
(4) M_4, the summand containing UD_1D_2.
(5) M_5, the summand containing UD_1D_2x.
(6) M_6, the summand containing $U(D_1F_1 + D_2F_2)$.
(7) M_7, the summand containing $U(D_1D_2^2 + D_1^2D_2)$.

We draw this decomposition in Figure 1.

Recall from Corollary 2.36 that $MT\xi^{\mathrm{het}\prime}$ splits as $MT\mathit{Spin} \vee \mathcal{Q}$. Since $\Omega_*^{\xi^{\mathrm{het}}} \cong \Omega_*^{\xi^{\mathrm{het}\prime}}$ in the range we need and Ω_*^{Spin} is known thanks to work of Anderson-Brown-Peterson [**ABP67**], we focus on $\pi_*(\mathcal{Q})$. To do so, we will identify the submodule of the E_2-page of the Adams spectral sequence for $\xi^{\mathrm{het}\prime}$ coming from spin bordism via $s_1\colon MT\mathit{Spin} \to MT\xi^{\mathrm{het}\prime}$; the E_2-page for $\mathcal{Q}$ is then a complementary submodule.

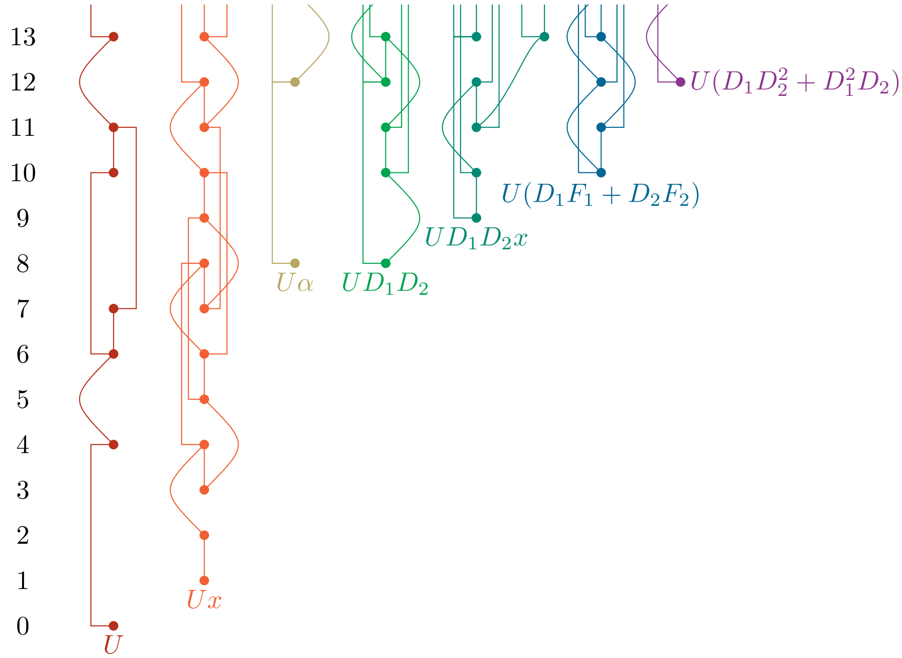

FIGURE 1. The $\mathcal{A}(2)$-module $T(-(c_1+c_2))$ in low degrees. Here $\alpha := D_1^2 + D_2^2$. The pictured submodule contains all classes in degrees 12 and below.

The canonical $\xi^{\mathrm{het}\prime}$-structure on a spin manifold from Lemma 2.35 can be rephrased as follows: a spin structure on a manifold M is equivalent data to: a spin structure on M, a map $c\colon M \to K(\mathbb{Z},4)$, and a trivialization of $c - \lambda(M)$. Thus spin structures are twisted string structures in the sense of Corollary 2.12 (in fact the universal twist in the sense of Remark 2.16), so the map

$$(1,0)\colon K(\mathbb{Z},4) \longrightarrow (K(\mathbb{Z},4)\times K(\mathbb{Z},4)) \times_{\mathbb{Z}/2} E\mathbb{Z}/2 = B \tag{2.42}$$

lifts to a map of *MTString*-module Thom spectra $s_1\colon \mathit{MTSpin} \to MT\xi^{\mathrm{het}\prime}$. Naturality of Theorem 2.20 then tells us the image of s_1^* on mod 2 cohomology, allowing us to determine which of the summands in Proposition 2.41 correspond to *MTSpin* and which correspond to $\mathcal{Q}$. Specifically, the pullback map sends $x \mapsto 0$, is nonzero on D_1, F_1, G_1, etc., and sends D_2, F_2, G_2, etc., to zero. This implies that in the direct-sum decomposition $MT\xi^{\mathrm{het}\prime} \simeq \mathit{MTSpin} \vee \mathcal{Q}$, the summands M_1, M_3, and M_6 come from the cohomology of *MTSpin*, and the remaining summands come from the cohomology of $\mathcal{Q}$.

In order to run the Adams spectral sequence for $\mathcal{Q}$, we need to compute the Ext of M_2, M_4, M_5, and M_7 over $\mathcal{A}(2)$. After we compute this, we will display the E_2-page in Figure 3. For an $\mathcal{A}(2)$-module M, $\mathrm{Ext}_{\mathcal{A}(2)}^{*,*}(M,\mathbb{Z}/2)$, which we will usually denote $\mathrm{Ext}_{\mathcal{A}(2)}(M)$ or $\mathrm{Ext}(M)$, is a bigraded module over the bigraded $\mathbb{Z}/2$-algebra $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$; both the algebra and module structures arise from the *Yoneda product* [**Yon54**, §4] (see [**BC18**, §4.2] for a review). This module structure is helpful for determining differentials in the Adams spectral sequence: differentials are equivariant with respect to the action. The module structure also constrains extensions on its E_∞-page.

May (unpublished) and Shimada-Iwai [**SI67**, §8] determined $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$. We will only need to track the actions of three elements: $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}(2)}(\mathbb{Z}/2)$, $h_1 \in \mathrm{Ext}^{1,2}_{\mathcal{A}(2)}(\mathbb{Z}/2)$, and $h_2 \in \mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$. These elements are in the image of the map $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$ induced by the quotient $\mathcal{A} \to \mathcal{A}(2)$, so we do not have to worry about whether Corollary 2.22 is compatible with the $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$-action on the E_2-page of the Adams spectral sequence. (It is, though.) When we draw Ext charts as in Figure 3, we denote h_0-actions as vertical lines, h_1-actions as diagonal lines with slope 1, and h_2-actions as diagonal lines with slope 1/3. When one of these lines is not present, the corresponding h_i acts as 0.

Often one computes Ext groups of $\mathcal{A}(2)$-modules using computer programs developed by Bruner [**Bru18**] and Chatham-Chua [**CC21**], or tools such as the May spectral sequence [**May66**] or the Davis-Mahowald spectral sequence [**DM82, MS87**] (see also [**BR21**, Chapter 2]) to compute Ext groups of $\mathcal{A}(2)$-modules, but for the four modules we care about, we can get away using simpler calculations by hand and computations already in the literature.

(1) Davis-Mahowald [**DM78**, Table 3.2] compute $\mathrm{Ext}_{\mathcal{A}(2)}(M_2)$ in the degrees we need.
(2) In degrees 13 and below, M_4 is isomorphic to $\Sigma^8(\mathcal{A}(2)\otimes_{\mathcal{A}(0)}\mathbb{Z}/2)$; therefore the Ext groups of these two $\mathcal{A}(2)$-modules, as algebras over $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$, are isomorphic in topological degrees 12 and below. Thus we can compute with the change-of-rings theorem (2.24): as $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$-algebras,

$$\mathrm{Ext}_{\mathcal{A}(2)}(\mathcal{A}(2) \otimes_{\mathcal{A}(0)} \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0], \tag{2.43}$$

with $h_0 \in \mathrm{Ext}^{1,1}$. This identification of $\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2)$ follows from Koszul duality [**BC18**, Example 4.5.5].
(3) M_5 looks a lot like M_2, which gives us a way to compute $\mathrm{Ext}_{\mathcal{A}(2)}(M_5)$. Specifically, if $\tau_{\le k}M$ denotes the quotient of an $\mathcal{A}(2)$-module M by the submodule of elements in degrees greater than k, then there is a short exact sequence of $\mathcal{A}(2)$-modules

$$0 \longrightarrow \Sigma^{13}\mathbb{Z}/2 \longrightarrow \tau_{\le 13}M_5 \longrightarrow \tau_{\le 13}\Sigma^8 M_2 \longrightarrow 0. \tag{2.44}$$

We draw this sequence in Figure 2, left. (2.44) induces a long exact sequence in Ext groups; passage between M and $\tau_{\le 13}M$ does not change Ext groups in degrees 12 and below, and since we only care about degrees 12 and below, we can and do pass between $\tau_{\le 13}M$ and M without comment.

We already know $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$ and $\mathrm{Ext}_{\mathcal{A}(2)}(M_2)$, so we can run the long exact sequence associated to (2.44) to compute $\mathrm{Ext}_{\mathcal{A}(2)}(M_5)$ in degrees 12 and below; we draw this long exact sequence in Figure 2, right. In the range we care about, there is exactly one boundary map that is not forced to be zero for degree reasons, namely

$$\partial\colon \mathrm{Ext}^{0,13}_{\mathcal{A}(2)}(\Sigma^{13}\mathbb{Z}/2) \longrightarrow \mathrm{Ext}^{1,13}_{\mathcal{A}(2)}(\Sigma^8 M_2); \tag{2.45}$$

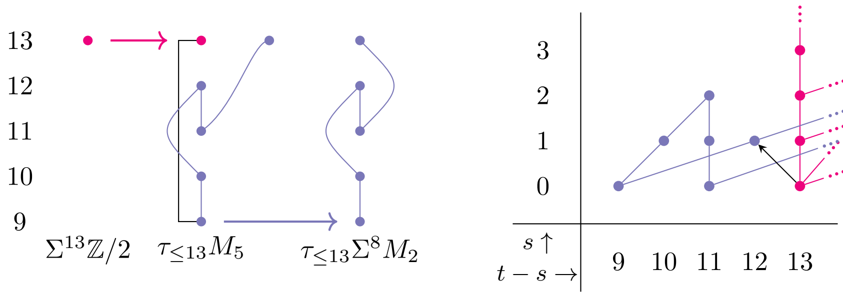

FIGURE 2. Left: the short exact sequence (2.44) of $\mathcal{A}(2)$-modules. Right: the associated long exact sequence in Ext. See the discussion after (2.45) for why the pictured boundary map (black arrow) is nonzero.

it must be nonzero, because that is the only way to obtain $\mathrm{Ext}^{0,13}_{\mathcal{A}(2)}(M_5) = \mathrm{Hom}_{\mathcal{A}(2)}(M_5, \Sigma^{13}\mathbb{Z}/2) = 0$, and by inspection of Figure 1 this Hom group vanishes.

(4) If $C\eta := \Sigma^{-2}\widetilde{H}^*(\mathbb{CP}^2;\mathbb{Z}/2)$, there is a 14-connected quotient map $M_7 \to \Sigma^{12}C\eta$, so $\mathrm{Ext}_{\mathcal{A}(2)}(\Sigma^{12}C\eta)$ and $\mathrm{Ext}_{\mathcal{A}(2)}(M_7)$ do not differ in the range we care about. Bruner-Rognes [**BR21**, Figure 0.15] compute $\mathrm{Ext}_{\mathcal{A}(2)}(C\eta)$.

Using these computations, we obtain the following description of the E_2-page of the Adams spectral sequence for the summand $\mathcal{Q}$ of $MT\xi^{\mathrm{het}\prime}$.

Proposition 2.46. *The E_2-page of the Adams spectral sequence for $\mathcal{Q}$ in topological degrees 12 and below is as given in Figure 3. In particular, in this range, the E_2-page is generated as an $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{Z}/2)$-module by eight elements: $p_1 \in \mathrm{Ext}^{0,1}$, $p_3 \in \mathrm{Ext}^{0,3}$, $p_7 \in \mathrm{Ext}^{0,7}$, $a \in \mathrm{Ext}^{0,8}$, $b \in \mathrm{Ext}^{2,10}$, $c \in \mathrm{Ext}^{0,9}$, $d \in \mathrm{Ext}^{0,11}$, and $e \in \mathrm{Ext}^{0,12}$.*

There are plenty of differentials in this Adams spectral sequence which could be nonzero, even when we take into account the fact that Adams differentials commute with h_0, h_1, and h_2:

(D1) $d_2\colon E_2^{0,8} \to E_2^{2,9}$, whose value on a could be $h_2^2 p_1$, $h_0^2 p_7$, or a linear combination of those two elements.
(D2) $d_2\colon E_2^{1,9} \to E_2^{3,10}$, which could send $h_0 a$ or $h_1 p_7$ to $h_0^3 p_7$.
(D3) $d_2\colon E_2^{0,9} \to E_2^{2,8}$ and $d_2\colon E_2^{1,11} \to E_2^{3,12}$, intertwined by an h_1-action, which could send $c \mapsto b$ and $h_1 c \mapsto h_1 b$.
(D4) $d_2\colon E_2^{0,12} \to E_2^{2,13}$, which could send $e \mapsto h_1^2 c = h_0^2 d$.
(D5) If the differentials in (D1) and (D2) vanish, $d_3\colon E_3^{0,8} \to E_3^{3,10}$ could be nonzero on a.
(D6) If the differential in (D4) vanishes, $d_5\colon E_5^{0,12} \to E_5^{5,16}$ (and its image under h_0) or $d_6\colon E_6^{0,12} \to E_6^{6,17}$ could be nonzero.

Lemma 2.47. *The differentials (D2), (D5), and (D6) vanish.*

PROOF. Our strategy is to use the fact that $\mathbb{G}^{\mathrm{het}} \to \mathbb{Z}/2$ splits to zero out differentials. This splitting does not extend to a splitting of $MT\xi^{\mathrm{het}}$, but it will be close enough.

The inclusion $\iota\colon \mathbb{Z}/2 \hookrightarrow \mathbb{G}^{\mathrm{het}}$ defines a map $\iota'\colon MTString \wedge B\mathbb{Z}/2 \to MT\xi^{\mathrm{het}}$ which on Adams E_2-pages is precisely the inclusion of the summand $\mathrm{Ext}(M_2)$.

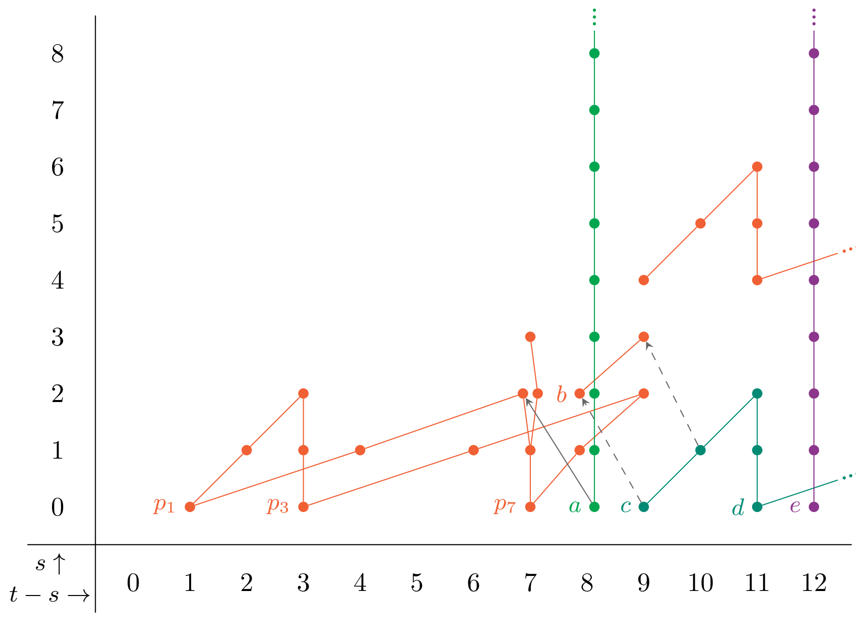

FIGURE 3. In Corollary 2.36, we showed $MT\xi^{\mathrm{het}\prime} \simeq MTSpin \vee \mathcal{Q}$; this figure denotes the E_2-page of the Adams spectral sequence computing $\pi_*(\mathcal{Q})$ in degrees 12 and below. This corresponds to a subset of the summands in Figure 1. In Lemma 2.50, we show that the solid gray differential beginning at a is nonzero; we leave open the other two differentials, which are dashed in this figure.

Quotienting $\mathbb{G}^{\mathrm{het}}$ by $\mathbb{T}[1]$, then by $\mathrm{E}_8 \times \mathrm{E}_8$, produces a map

$$p\colon MT\xi^{\mathrm{het}} \xrightarrow[(1.44)]{\phi} MTSpin \wedge (B((\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2))_+ \longrightarrow MTSpin \wedge (B\mathbb{Z}/2)_+, \tag{2.48}$$

and $p \circ \iota\colon MTString \wedge (B\mathbb{Z}/2) \to MTSpin \wedge (B\mathbb{Z}/2)_+$ is the usual map $MTString \to MTSpin$ together with the addition of a basepoint. This means that any element of $\widetilde\Omega_*^{\mathrm{String}}(B\mathbb{Z}/2)$ whose image in $\widetilde\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/2)$ is nonzero must also be nonzero in $\Omega_*^{\xi^{\mathrm{het}}}$, which kills many differentials to or from $\mathrm{Ext}(M_2)$. To produce such elements, study the map of Adams spectral sequences induced by $p \circ \iota$, which on E_2-pages is the map

$$\mathrm{Ext}_{\mathcal{A}(2)}(\widetilde H^*(B\mathbb{Z}/2;\mathbb{Z}/2)) \longrightarrow \mathrm{Ext}_{\mathcal{A}(1)}(H^*(B\mathbb{Z}/2;\mathbb{Z}/2)). \tag{2.49}$$

Davis-Mahowald [**DM78**, Table 3.2] compute $\mathrm{Ext}_{\mathcal{A}(2)}(H^*(B\mathbb{Z}/2;\mathbb{Z}/2))$ in the degrees we need, and Gitler-Mahowald-Milgram [**GMM68**, §2] provide a computation of $\mathrm{Ext}_{\mathcal{A}(1)}(H^*(B\mathbb{Z}/2;\mathbb{Z}/2))$. We draw the map (2.49) in Figure 4. All differentials in the spectral sequence over $\mathcal{A}(1)$ vanish using h_0- and h_1-equivariance, and by inspection there are no hidden extensions. Therefore we can identify some classes which survive $p \circ \iota$ and use this to trivialize some differentials in Figure 3.

- By computing the image of $p \circ \iota$ on Ext groups, we learn that the map $\widetilde\Omega_7^{\mathrm{String}}(B\mathbb{Z}/2) \to \widetilde\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}/2)$ can be identified with the map $\mathbb{Z}/16 \oplus$

$\mathbb{Z}/2 \to \mathbb{Z}/16$ sending $(1,0) \mapsto 1$ and $(0,1) \mapsto 0$.[15] Therefore, any differential to or from the four summands in topological degree 7 linked by h_0-actions must vanish, including (D2) and (D5).
- Similarly, the map $\widetilde{\Omega}_{11}^{\mathrm{String}}(B\mathbb{Z}/2) \to \widetilde{\Omega}_{11}^{\mathrm{Spin}}(B\mathbb{Z}/2)$ can be identified with the inclusion $\mathbb{Z}/8 \hookrightarrow \mathbb{Z}/128 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$ sending $1 \mapsto (16,0,0)$, which follows either by computing $p \circ \iota$ on Ext groups or computing η-invariants on the generator of $\widetilde{\Omega}_{11}^{\mathrm{String}}(B\mathbb{Z}/2)$, which can be taken to be the product of $\mathbb{RP}^3$ with a Bott manifold.[16] Thus (D6) vanishes. □

Lemma 2.50. *The differential (D1) is nonzero; specifically, $d_2(a) = h_2^2 p_1$.*

We will deduce this from the following fact.

Proposition 2.51. *The map $\Omega_4^{\xi^{r,\mathrm{het}}} \to \Omega_4^{\xi^{\mathrm{het}}}$ is surjective after 2-completion.*

Recall that $\xi^{r,\mathrm{het}}$ is the analogue of ξ^{het} but with $(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}/2$ replaced with $\mathrm{E}_8 \times \mathrm{E}_8$.

Proof of Lemma 2.50 assuming Proposition 2.51. In this proof, implicitly 2-complete all abelian groups. If $d_2(a) = 0$, then $h_2^2 p_1 \in E_2^{2,9}$ survives to the E_∞-page, so the h_2-action $E_\infty^{1,5} \to E_\infty^{2,9}$ is nonzero. This lifts to imply that taking the product with S^3 with string structure induced from its Lie group framing, which defines a map $\Omega_4^{\xi^{\mathrm{het}}} \to \Omega_7^{\xi^{\mathrm{het}}}$, is also nonzero. Direct products with framed manifolds correspond to action by elements of $\pi_*(\mathbb{S})$ on homotopy groups, so this product with S^3 is natural with respect to maps of spectra.

Since $\Omega_4^{\xi^{r,\mathrm{het}}} \to \Omega_4^{\xi^{\mathrm{het}}}$ is surjective, we may compute the product with S^3 as a map

$$- \times S^3 \colon \Omega_4^{\xi^{r,\mathrm{het}}} \longrightarrow \Omega_7^{\xi^{r,\mathrm{het}}} \tag{2.52}$$

and then map back to $\Omega_7^{\xi^{\mathrm{het}}}$. However, as we noted in Remark 2.34, $\Omega_7^{\xi^{r,\mathrm{het}}} \cong \Omega_7^{\mathrm{Spin}}(K(\mathbb{Z},4))$, and Stong [**Sto86**] showed $\Omega_7^{\mathrm{Spin}}(K(\mathbb{Z},4)) = 0$. Thus taking the product with S^3 is the zero map $\Omega_4^{\xi^{\mathrm{het}}} \to \Omega_7^{\xi^{\mathrm{het}}}$, which is incompatible with $d_2(a)$ vanishing. □

Proof of Proposition 2.51. Let F be the fiber of the map $\phi\colon MT\xi^{r,\mathrm{het}} \to MT\xi^{\mathrm{het}}$, so that there is a long exact sequence

$$\cdots \longrightarrow \Omega_4^{\xi^{r,\mathrm{het}}} \xrightarrow{\phi} \Omega_4^{\xi^{\mathrm{het}}} \xrightarrow{\partial} \pi_3(F) \longrightarrow \Omega_3^{\xi^{r,\mathrm{het}}} \longrightarrow \cdots \tag{2.53}$$

We will show $\pi_3(F)_2^\wedge = 0$, which implies the proposition statement by exactness. To do so, we must understand F.

(F is the fiber.)

[15] Alternatively, one could show that the $\mathbb{Z}/16 \subset \widetilde{\Omega}_7^{\mathrm{String}}(B\mathbb{Z}/2)$ is mapped injectively into $\widetilde{\Omega}_7^{\mathrm{Spin}}(B\mathbb{Z}/2)$ by checking on a generator. One can show that $\mathbb{RP}^7$ admits a string structure; then the generator of that $\mathbb{Z}/16$ subgroup of $\widetilde{\Omega}_7^{\mathrm{String}}(B\mathbb{Z}/2)$ is $\mathbb{RP}^7$ with its nontrivial principal $\mathbb{Z}/2$-bundle. Its image in $\widetilde{\Omega}_7^{\mathrm{Spin}}(B\mathbb{Z}/2)$ has order at least 16, because the η-invariant of a suitable twisted Dirac operator associated to the $\mathbb{Z}/2$-bundle defines a bordism invariant $\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}/2) \to \mathbb{R}/\mathbb{Z}$, and on $(\mathbb{RP}^7, S^7 \to \mathbb{RP}^7)$, this η-invariant is $\ell/16 \bmod 1$ for some odd ℓ, as follows from a formula of Donnelly [**Don78**, Proposition 4.1].

[16] All orientable 3-manifolds have trivializable tangent bundles, hence string structures; for a construction of a Bott manifold with string structure, see [**FH21a**, §5.3].

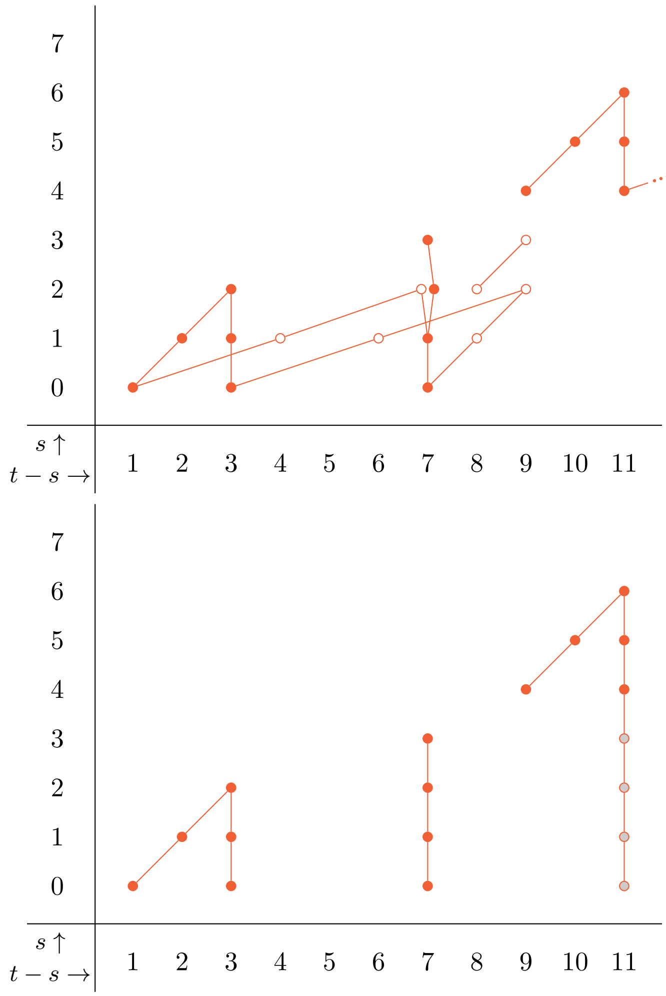

FIGURE 4. Top: $\mathrm{Ext}_{\mathcal{A}(2)}(\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2),\mathbb{Z}/2)$, the E_2-page of the Adams spectral sequence computing $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2)_2^\wedge$. Filled dots have nonzero image in $\mathrm{Ext}_{\mathcal{A}(1)}$; unfilled dots are the kernel. Bottom: $\mathrm{Ext}_{\mathcal{A}(1)}(\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2),\mathbb{Z}/2)$, a summand of the E_2-page of the Adams spectral sequence computing $\widetilde{\Omega}_*^{\mathrm{Spin}}(B\mathbb{Z}/2)_2^\wedge$. Filled dots are in the image of the map from $\mathrm{Ext}_{\mathcal{A}(2)}$; gray dots are the cokernel. This map of spectral sequences is used in the proof of Lemma 2.47.

We use a standard technique.

Let V be the rank-zero stable vector bundle on $B\mathbb{G}^{\mathrm{het}}$ classified by the map $\xi^{\mathrm{het}}\colon B\mathbb{G}^{\mathrm{het}} \to B\mathrm{O}$ and let $\sigma \to B\mathbb{G}^{\mathrm{het}}$ be the line bundle classified by the map quotienting by $\mathbb{T}[1]$, then by Spin, then by E_8^2:

$$B\mathbb{G}^{\mathrm{het}} \longrightarrow B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2) \longrightarrow B\mathbb{Z}/2. \tag{2.54}$$

Then, inclusion of the zero section of σ defines a map of spaces over $\mathbb{Z} \times BO$: $\phi\colon (B\mathbb{G}^{\mathrm{het}}, V) \to (B\mathbb{G}^{\mathrm{het}}, V \oplus \sigma)$. Here we use the notation (B, ξ) to denote a space B and a map $\xi\colon B \to \mathbb{Z} \times BO$, and we use $\mathbb{Z} \times BO$ instead of BO because σ is not rank 0. Let M^- denote the Thom spectrum of $V \oplus \sigma\colon B\mathbb{G}^{\mathrm{het}} \to \mathbb{Z} \times BO$, and let $\widetilde{\phi}\colon MT\xi \to M^-$ denote the map of Thom spectra induced by ϕ; we claim $F \simeq \Sigma^{-1}M$. To see this, we will use a theorem in [**DDK$^+$**] which identifies the fiber of $\widetilde{\phi}$ as the map $MT\xi^{r,\mathrm{het}} \to MT\xi^{\mathrm{het}}$. Specifically, [**DDK$^+$**] shows that the fiber of $\widetilde{\phi}$ is the Thom spectrum of the pullback of V to the sphere bundle $S(\sigma)$ of σ. This sphere bundle is the pullback of the universal sphere bundle over $B\mathbb{Z}/2$ by the classifying map of σ:

$$\begin{tikzcd} S(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S(L) \simeq E\mathbb{Z}/2 \arrow[d] \\ B\mathbb{G}^{\mathrm{het}} \arrow[r] & B\mathbb{Z}/2 \end{tikzcd} \tag{2.55}$$

The sphere bundle of the tautological line bundle $L \to B\mathbb{Z}/2$ is $E\mathbb{Z}/2 \to B\mathbb{Z}/2$, which is contractible, so the pullback diagram (2.55) simplifies to a fiber diagram, and the sphere bundle is the fiber of (2.54). Since (2.54) was induced from a group homomorphism by taking classifying spaces, one can compute its fiber by taking the classifying space of the kernel of the homomorphism, which is $\mathcal{S}(\mathrm{Spin} \times \mathrm{E}_8^2, c_1 + c_2 - \lambda)$. In Remark 1.46 we saw that applying the Thom spectrum functor to $B\mathcal{S}(\mathrm{Spin} \times \mathrm{E}_8^2, c_1 + c_2 - \lambda) \to B\mathbb{G}^{\mathrm{het}}$, i.e. to the map $S(\sigma) \to B\mathbb{G}^{\mathrm{het}}$, produces the map $MT\xi^{r,\mathrm{het}} \to MT\xi^{\mathrm{het}}$, and therefore the fiber of this map is $\Sigma^{-1}M^-$.

To finish the proof, attack F with the Adams spectral sequence, using its description as the Thom spectrum $\Sigma^{-1}M$ to get a description in terms of Ext of an $\mathcal{A}(2)$-module by using [**DY23**] again. Recall from Figure 2, left, the $\mathcal{A}(2)$-module $\tau_{\le 13}M_5$; the result of the computation here is that the $\mathcal{A}(2)$-module relevant for computing $\pi_*(F)^\wedge_2$ agrees with $\Sigma^{-9}(\tau_{\le 13}M_5)$ in degrees 4 and below. Then, Figure 2, right, computes $\mathrm{Ext}_{\mathcal{A}(2)}(\Sigma^{-9}(\tau_{\le 13}M_5))$, which is the E_2-page of the Adams spectral sequence computing $\pi_*(F)^\wedge_2$, in degrees 3 and below (shift the topological degree of everything in Figure 2, right, down by 9). The E_2-page vanishes in topological degree 3, which implies $\pi_3(F)^\wedge_2 = 0$. $\square$

Lemma 2.56. *The differential (D4) vanishes.*

Proof. The source of this differential is $E_2^{0,12} \cong \mathbb{Z}/2 \cdot e$ in Adams filtration zero. Classes α in Adams filtration 0 are canonically identified with classes c_α forming a subgroup of mod 2 cohomology, and α survives to the E_∞-page if and only if the bordism invariant $\int c_\alpha$ is nonzero. Here, $\alpha = e$ and $c_\alpha = D_1D_2^2 + D_1^2D_2$, so our differential vanishes if and only if e survives to the E_∞-page if and only if the following invariant is nonzero:

$$\int \left(D_1D_2^2 + D_1^2D_2\right)\colon \Omega_{12}^{\xi^{\mathrm{het}}} \longrightarrow \mathbb{Z}/2. \tag{2.57}$$

We will produce a manifold on which this invariant is nonzero.

The quaternionic projective plane $\mathbb{HP}^2$ has $H^*(\mathbb{HP}^2;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$ with $|x| = 4$ and $\lambda(\mathbb{HP}^2) = x$ [**BH58**, §15.5, §15.6] (see also [**FH21a**, §5.2]). The Künneth formula tells us $H^*(\mathbb{HP}^2 \times S^4;\mathbb{Z}) \cong \mathbb{Z}[x,y]/(x^3, y^2)$, with $|y| = 4$; since TS^4 is stably trivial, $\lambda(S^4)$ vanishes and the Whitney sum formula (Lemma 1.6) implies $\lambda(\mathbb{HP}^2 \times S^4) = x$.

To define a ξ^{het}-structure on $\mathbb{HP}^2 \times S^4$, it suffices to produce two E_8-bundles $P, Q \to \mathbb{HP}^2 \times S^4$ and a trivialization of $\lambda(\mathbb{HP}^2 \times S^4) - c(P) - c(Q)$. Since we can freely prescribe $c(P)$ and $c(Q)$, choose P and Q such that $c(P) = y$ and $c(Q) = x - y$; then $\lambda(\mathbb{HP}^2 \times S^4) - c(P) - c(Q) = 0$, so we can choose a trivialization. Since $D_1 = c(P) \bmod 2$ and $D_2 = c(Q) \bmod 2$,

$$\int_{\mathbb{HP}^2 \times S^4} \left(D_1 D_2^2 + D_1^2 D_2\right) = \left(\int_{\mathbb{HP}^2 \times S^4} (yx^2 + xy^2)\right) \bmod 2 = 1. \qquad \square \tag{2.58}$$

Now we have to tackle extension questions. In this part of the computation, it will be helpful to reference Figure 3, as we will use the description of the E_∞-page of this spectral sequence several times while addressing extension questions.

Lemma 2.59. *In degrees* 10 *and below, all extension questions in the Adams spectral sequence for $\pi_*(\mathcal{Q})^\wedge_2$ either split or are detected by h_0 on the E_∞-page, except possibly for the extensions involving the classes $c \in E_\infty^{0,9}$, $h_1^2 p_7 \in E_\infty^{2,11}$, and $h_1 b \in E_\infty^{3,12}$.*

The classes $h_1 b$ and c may vanish on the E_∞-page, depending on the fate of the differentials in (D3).

Proof. The h_0-action alone solves all extensions in this range except in degrees 8, 9, and 10.

If the d_2s in (D3) vanish, there is an extension question in degree 8. The h_0-actions in the tower generated by $h_0 a$ lift to produce a $\mathbb{Z}$ in $\Omega_8^{\xi^{\mathrm{het}}}$, so the only question is whether there is an extension involving $h_1 p_7$ and b. Suppose this extension does not split, so $\pi_8(\mathcal{Q})^\wedge_2 \cong \mathbb{Z} \oplus \mathbb{Z}/4$. We can choose a generator x of this $\mathbb{Z}/4$ such that the image of x in the Adams E_∞-page is $h_1 p_7 \in E_\infty^{1,9}$; since this is h_1 times another class on the E_∞-page, x is η times a class $y \in \pi_7(\mathcal{Q})^\wedge_2$, where η is the generator of $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. Since $2\eta = 0$, $2x = 2\eta y = 0$; since x was supposed to generate a $\mathbb{Z}/4$, this is a contradiction, and therefore this extension splits.

The same trick splits all extensions in degree 10, and all extensions involving the class in $E_\infty^{4,13}$. $\square$

Proposition 2.60. *All extension questions in $\pi_9(\mathcal{Q})^\wedge_2$ split, so $\pi_9(\mathcal{Q})^\wedge_2 \cong (\mathbb{Z}/2)^{\oplus 4}$ if the differentials in (D3) vanish, and $\pi_9(\mathcal{Q})^\wedge_2 \cong (\mathbb{Z}/2)^{\oplus 2}$ if they do vanish.*

Proof. If the differentials in (D3) do not vanish, this is a consequence of Lemma 2.59, so assume that those differentials vanish.

First suppose we can split all extensions involving c. Then the only extension remaining is between $h_1^2 p_7$ and $h_1 b$. In Lemma 2.59, we split the extension between $h_1 p_7$ and b, so the classes $h_1 p_7$ and b lift to classes $\underline{h_1 p_7}$, resp. $\underline{b}$, which generate a $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \pi_8(\mathcal{Q})^\wedge_2$. The action by h_1 lifts to imply that the images of $\eta \cdot \underline{h_1 p_7}$ and $\eta \cdot \underline{b}$ in the E_∞-page are $h_1^2 p_7$, resp. $h_1 b$, and η carries the $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\underline{h_1 p_7}$ and $\underline{b}$ to a $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \pi_9(\mathcal{Q})^\wedge_2$ generated by $\eta \underline{h_1 p_7}$ and $\eta \underline{b}$, thus splitting the extension between $h_1^2 p_7$ and $h_1 b$.

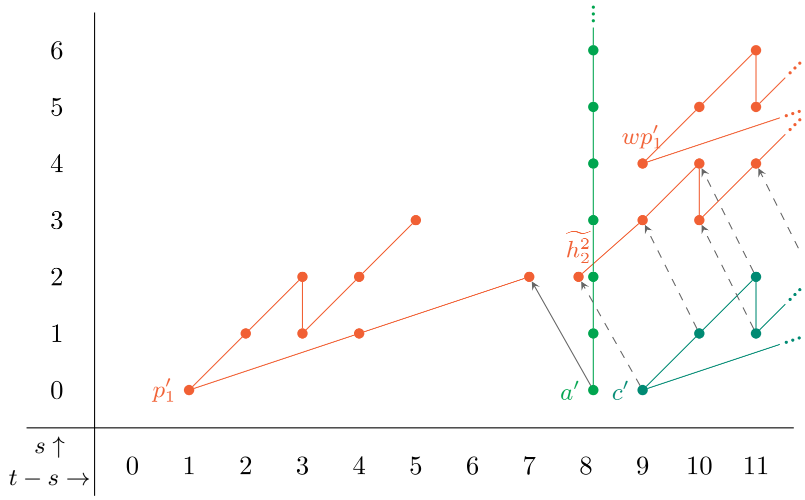

FIGURE 5. The E_2-page of the Adams spectral sequence computing $\pi_*(\mathcal{Q}')_2^\wedge$, where $\mathcal{Q}'$ is the spectrum defined in the proof of Proposition 2.60. By comparing with the Adams spectral sequence for $\mathcal{Q}$, we learn $d_2(a') = h_2^2 p_1'$ from Lemma 2.50, and that the dashed differentials (e.g. $d_2(c')$, $d_2(h_1 c')$) vanish if and only if the differentials in (D3) vanish.

Now we need to prove that c lifts to a class $\underline{c}$ such that $2\underline{c} = 0$. Let X be the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B\mathbb{G}^{\mathrm{het}} \arrow[d] \\ \mathbb{RP}^2 \arrow[r] & B\mathbb{Z}/2 \end{tikzcd} \tag{2.61}$$

and let $\xi\colon X \to BO$ be the pullback of ξ^{het} to X. Both vertical arrows in (2.61) are fibrations with fiber BE_8^2; using the induced map of Serre spectral sequences, we learn $H^*(X;\mathbb{Z}/2) \cong H^*(B\mathbb{G}^{\mathrm{het}};\mathbb{Z}/2)/(x^3)$, where $x \in H^1(B\mathbb{G}^{\mathrm{het}};\mathbb{Z}/2)$ is the generator. One can replay the whole argument we ran with ξ in place of ξ^{het}, defining ξ' analogously to $\xi^{\mathrm{het}\prime}$, and deduce the following.

(1) The map $c\colon BE_8 \to K(\mathbb{Z},4)$ induces an isomorphism $\Omega_*^\xi \to \Omega_*^{\xi'}$ in degrees 14 and below,
(2) there is a spectrum $\mathcal{Q}'$ and a splitting $MT\xi \simeq MTSpin \vee \mathcal{Q}'$, and
(3) the map $X \to B\mathbb{G}^{\mathrm{het}}$ induces a map $MT\xi' \to MT\xi^{\mathrm{het}\prime}$ which is the identity on the $MTSpin$ factors and sends $\mathcal{Q} \to \mathcal{Q}'$.

The analogue of Proposition 2.41 for ξ' is exactly the same, except replacing M_2 with $\Sigma C2$ and M_5 with $\Sigma^9 C2$, where $C2$ is the $\mathcal{A}(2)$-module $\Sigma^{-1}\widetilde{H}^*(\mathbb{RP}^2;\mathbb{Z}/2)$. Bruner-Rognes [**BR21**, §6.1] compute $\mathrm{Ext}_{\mathcal{A}(2)}(C2)$, and using that we can draw the E_2-page of the Adams spectral sequence computing $\pi_*(\mathcal{Q}')_2^\wedge$ in Figure 5. For the classes p_1, a, and c we considered in the E_2-page of the Adams spectral sequence for $\mathcal{Q}$, let p_1', a', and c' be the corresponding classes in the E_2-page for $\mathcal{Q}'$: they live in the same bidegrees and the map $\mathcal{Q}' \to \mathcal{Q}$ carries $x' \to x$ for $x \in \{p_1, a, c\}$.

The point of all of this is that if the differentials in (D3) vanish, then both c and $h_1^2p_7$ live to the E_∞-page for $\mathcal{Q}$, then both c and $h_1^2p_7$ are in the image of the map Φ on E_∞-pages induced by $\mathcal{Q}' \to \mathcal{Q}$: $c = \Phi(c')$, and Bruner-Rognes [**BR21**, Corollary 4.3] define a class $\widetilde{h_2^2} \in \mathrm{Ext}_{\mathcal{A}(2)}^{2,9}(C2) = \mathrm{Ext}_{\mathcal{A}(2)}^{2,10}(\Sigma C2)$ such that $h_1^2p_7 = \Phi(\widetilde{h_2^2})$. And looking at Figure 5, in the E_∞-page for $\mathcal{Q}'$, $h_1(\widetilde{h_2^2}) \neq 0$ and $h_1(wp_1') \neq 0$, so the $2\eta = 0$ trick from Lemma 2.59 splits the extensions in $\pi_9(\mathcal{Q}')_2^\wedge$. Thus there is a class $\underline{c}' \in \pi_9(\mathcal{Q}')_2^\wedge$ such that $2\underline{c}' = 0$ and the image of $\underline{c}'$ in the E_∞-page is c'. Applying $\Phi(c') = c$, we learn c lifts to $\Phi(\underline{c}')$ in $\pi_9(\mathcal{Q})_2^\wedge$, and twice this class is 0, as we wanted to prove. □

We have therefore proven the following theorem.

Theorem 2.62. *Ignoring odd-primary torsion, there are isomorphisms*

$$\begin{aligned}
\Omega_0^{\xi^{\mathrm{het}}} &\cong \mathbb{Z} & \Omega_6^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2\\
\Omega_1^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \Omega_7^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/16\\
\Omega_2^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \Omega_8^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^{\oplus i}\\
\Omega_3^{\xi^{\mathrm{het}}} &\cong \mathbb{Z}/8 & \Omega_9^{\xi^{\mathrm{het}}} &\cong (\mathbb{Z}/2)^{\oplus j}\\
\Omega_4^{\xi^{\mathrm{het}}} &\cong \mathbb{Z} \oplus \mathbb{Z}/2 & \Omega_{10}^{\xi^{\mathrm{het}}} &\cong (\mathbb{Z}/2)^{\oplus k}\\
\Omega_5^{\xi^{\mathrm{het}}} &\cong 0 & \Omega_{11}^{\xi^{\mathrm{het}}} &\cong A,
\end{aligned}$$

where:

- *A is an abelian group of order 64 isomorphic to one of $\mathbb{Z}/8 \oplus \mathbb{Z}/8$, $\mathbb{Z}/16 \oplus \mathbb{Z}/4$, $\mathbb{Z}/32 \oplus \mathbb{Z}/2$, or $\mathbb{Z}/64$, and*
- *either $i = 1$, $j = 4$, and $j = 4$, or $i = 2$, $j = 6$, and $k = 5$.*

2.2.1. *Some manifold generators.* We finish this section by giving manifold representatives for all the generators for the groups we found in dimensions 10 and below, except possibly for two classes in degrees 9 and 10 if the differentials in (D3) vanish. We also give partial information in dimension 11. In this list, we implicitly localize at 2, though we will soon see in Theorem 2.74 that this does not lose any information.

The map $MTSpin \vee (MTString \wedge B\mathbb{Z}/2) \to MT\xi^{\mathrm{het}}$ is surjective on homotopy groups in degrees 7 and below, quickly giving us many of the generators we need. The low-dimensional generators of spin bordism are standard; for $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2)$, we use the h_2-action on the E_∞-page together with the map $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2) \to \widetilde{\Omega}_*^{\mathrm{Spin}}(B\mathbb{Z}/2)$, as in the proof of Lemma 2.47 (see Figure 4), to deduce generators.

(0) $\Omega_0^{\xi^{\mathrm{het}}} \cong \mathbb{Z}$, generated by the point.

(1) $\Omega_1^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The first summand comes from Ω_1^{Spin}, hence is generated by S^1_{nb}, the circle with ξ^{het}-structure induced from its nonbounding framing. The other summand, corresponding to $p_1 \in E_\infty^{0,1}$ of the Adams spectral sequence for $\mathcal{Q}$, is in Adams filtration zero, hence corresponds to a mod 2 cohomology class and is detected by that class. Looking at Figure 1, this class is the generator of $H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ evaluated on the principal $\mathbb{Z}/2$-bundle associated to a ξ^{het}-structure. Thus we can take as our generator S^1 with ξ^{het}-structure induced by the nontrivial $\mathbb{Z}/2$-bundle

and the inclusion $\mathbb{Z}/2 \hookrightarrow \mathrm{E}_8^2 \rtimes \mathbb{Z}/2$. We will call this generator $\mathbb{RP}^1$, so that we can represent its $\mathbb{Z}/2$-bundle by $S^1 \to \mathbb{RP}^1$.

(2) An action by h_1 in the E_∞-page of an Adams spectral sequence calculating bordism lifts to taking the product with S^1_{nb} on manifold generators. Acting by h_1 defines an isomorphism from the 1-line of the E_∞-page to the 2-line, so we can take $S^1_{nb} \times S^1_{nb}$ and $\mathbb{RP}^1 \times S^1_{nb}$ to be our two generators of $\Omega_2^{\xi^{\mathrm{het}}}$.

(3) $\Omega_3^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/8$; there is a generator whose image in the Adams E_∞-page is p_3. The sequence of maps

$$\widetilde{\Omega}_3^{\mathrm{String}}(B\mathbb{Z}/2) \xrightarrow{\iota} \Omega_3^{\xi^{\mathrm{het}}} \xrightarrow{p} \Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2) \tag{2.63}$$

consists of two isomorphisms $\mathbb{Z}/8 \xrightarrow{\cong} \mathbb{Z}/8 \xrightarrow{\cong} \mathbb{Z}/8$, so it suffices to find a generator of $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2)$ that admits a string structure. The standard generator is $\mathbb{RP}^3$ with principal $\mathbb{Z}/2$-bundle $S^3 \to \mathbb{RP}^3$, and because $\mathbb{RP}^3$ is parallelizable, it admits a string structure.

(4) $\Omega_4^{\xi^{\mathrm{het}}} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The free summand comes from Ω_4^{Spin}, hence is generated by the K3 surface with trivial $\mathbb{Z}/2$-bundle, and E_8-bundles with characteristic classes $-\lambda(\mathrm{K3})$ and 0. $\mathbb{Z}/2$ corresponds to $E_\infty^{1,5} \cong \mathbb{Z}/2 \cdot h_2 p_1$. Action by h_2 lifts to the product with S^3 with its Lie group framing, so we can generate this summand with $S^3 \times \mathbb{RP}^1$.

Remark 2.64. In Proposition 2.51, we established that the homomorphism $\Omega_4^{\mathrm{Spin}}(K(\mathbb{Z},4)) \cong \Omega_4^{\xi^{r,\mathrm{het}}} \to \Omega_4^{\xi^{\mathrm{het}}}$ is surjective; using this, we can replace $S^3 \times \mathbb{RP}^1$, which we will need later. Stong [Sto86] showed $\Omega_4^{\mathrm{Spin}}(K(\mathbb{Z},4)) \cong \mathbb{Z} \oplus \mathbb{Z}$; one $\mathbb{Z}$ factor comes from Ω_4^{Spin}, hence is represented by the K3 surface with trivial map to $K(\mathbb{Z},4)$. The other is detected by the bordism invariant which, given a 4-dimensional spin manifold X and a map $f\colon X \to K(\mathbb{Z},4)$, sends $X \mapsto \int_X f^*c$, where $c \in H^4(K(\mathbb{Z},4);\mathbb{Z})$ is the tautological class. For example, this invariant equals 1 on S^4 with its standard orientation and unique spin structure inducing that orientation, with the map to $K(\mathbb{Z},4)$ given by the class $1 \in H^4(S^4;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$.

The images of the two classes $(\mathrm{K3}, 0)$ and $(S^4, 1)$ in $\Omega_4^{\xi^{\mathrm{het}}}$ must generate. Unsurprisingly, the K3 surface is sent to a generator of the $\mathbb{Z}$ summand we described above; this summand is detected by $\int p_1$. As this invariant vanishes on $(S^4, 1)$, surjectivity of the map on Ω_4 implies that $(S^4, 1)$ maps to the class of $\mathbb{RP}^1 \times S^3$.[17] Thus the $\mathbb{Z}/2$ summand in $\Omega_4^{\xi^{\mathrm{het}}}$ can be generated by S^4 with trivial $\mathbb{Z}/2$-bundle and two E_8-bundles with characteristic classes $c = \pm 1 \in H^4(S^4;\mathbb{Z})$.

The map on Adams spectral sequences induced from the map of spectra $MT\xi^{r,\mathrm{het}} \to MT\xi^{\mathrm{het}}$ sends the class in the E_∞-page representing $(S^4, 1)$ to 0 (see Francis [Fra11, §2] or Lee-Yonekura [LY22, §3.5] for the Adams spectral sequence for $\Omega_*^{\xi^{r,\mathrm{het}}} = \Omega_*^{\mathrm{Spin}}(K(\mathbb{Z},4)))$, so the fact that the image of $(S^4, 1)$ is nonzero in $\Omega_4^{\xi^{\mathrm{het}}}$ is analogous to a hidden extension.

(5) $\Omega_5^{\xi^{\mathrm{het}}} = 0$.

[17] We thank Justin Kaidi for informing of us of this fact.

(6) $\Omega_6^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/2$, and the image of a generator on the E_∞-page is h_2p_3, which lifts to imply that we can take $S^3 \times \mathbb{RP}^3$ as a generator.

(7) $\Omega_7^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/16$. This $\mathbb{Z}/16$ is detected by $\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}/2)$ much like $\mathbb{RP}^3$ was, and we learn that this summand is generated by $\mathbb{RP}^7$ with $\mathbb{G}^{\mathrm{het}}$-bundle induced from the $\mathbb{Z}/2$-bundle $S^7 \to \mathbb{RP}^7$, and is detected in the E_∞-page by p_7.

(8) $\Omega_8^{\xi^{\mathrm{het}}} \cong \mathbb{Z}^2 \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ together with an additional $\mathbb{Z}/2$ summand if the differentials in (D3) do not vanish.

- The first two free summands come from Ω_*^{Spin}; their generators may be taken to be the quaternionic projective plane $\mathbb{HP}^2$ and a Bott manifold B. One can choose B to have a string structure [**FH21a**, §5.3] and we do so. In both cases, the $\mathbb{Z}/2$-bundle associated to the ξ^{het}-structure is trivial; since B is string, we give it the ξ^{het}-structure in which both principal E_8-bundles are trivial. For $\mathbb{HP}^2$, $H^4(\mathbb{HP}^2;\mathbb{Z}) \cong \mathbb{Z}$ with generator x, as we discussed in the proof of Lemma 2.56; we choose a ξ^{het}-structure on $\mathbb{HP}^2$ with principal E_8-bundles $P, Q \to \mathbb{HP}^2$ with $c(P) = -x$ and Q trivial.
- The third free summand comes from the green h_0-tower in topological degree 8 in the Adams spectral sequence for $\pi_*(\mathcal{Q})$. This summand is detected by the bordism invariant

$$f \coloneqq \int c(P)c(Q) \colon \Omega_8^{\xi^{\mathrm{het}}} \longrightarrow \mathbb{Z}, \tag{2.65}$$

because this quantity can be nonzero (as we show below), it vanishes on the two generators we discovered for the other two free summands, and because it must vanish on the remaining, torsion summand. It is a consequence of Lemma 2.50 that the mod 2 reduction of (2.65), which is $\int D_1D_2$, vanishes. This is because every class $x \in E_2^{0,t}$ has an associated degree-t $\mathbb{Z}/2$ cohomology class c_x, and x lives to the E_∞-page if and only if the bordism invariant $\int c_x$ is nonvanishing. Thus the minimum nonzero value of $|f(M)|$, where M is a closed, 8-dimensional ξ^{het}-manifold, is at least 2.
Recall from the proof of Lemma 2.56 that $H^*(\mathbb{HP}^2;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$ with $|x| = 4$ and $\lambda(\mathbb{HP}^2) = x$. Consider the two E_8-bundles $P, Q \to \mathbb{HP}^2$ prescribed by $c(P) = 2x$ and $c(Q) = -x$; then $\lambda(\mathbb{HP}^2) - c(P) - c(Q) = 0$, so this data lifts to a ξ^{het}-structure, and

$$\int_{\mathbb{HP}^2} c(P)c(Q) = 2, \tag{2.66}$$

achieving the minimum. Therefore $\mathbb{HP}^2$ with these two principal E_8-bundles generates the final free summand.
- The $\mathbb{Z}/2$ summand that we know is present independent of any unresolved differentials is generated by h_1p_7, so as usual lifts to $S^1_{nb} \times \mathbb{RP}^7$.
- If $d_2(c) \neq 0$, there is an additional $\mathbb{Z}/2$ summand represented in the E_∞-page by b. We will discuss this summand, and its generator X_8, in §2.2.2.

(9) $\Omega_9^{\xi^{\mathrm{het}}} \cong (\mathbb{Z}/2)^{\oplus 2} \oplus (\mathbb{Z}/2)^{\oplus 2}$, and if the differentials in (D3) vanish, there is an additional $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ summand.

- Two of the $\mathbb{Z}/2$ summands come from $\Omega_9^{\mathrm{Spin}} \cong (\mathbb{Z}/2)^{\oplus 2}$, where they are represented by the generators $\mathbb{HP}^2 \times S^1_{nb}$ and $B \times S^1_{nb}$, with ξ^{het}-structure induced from the corresponding generators in $\Omega_8^{\xi^{\mathrm{het}}}$.
- The other two $\mathbb{Z}/2$ summands that are present no matter the value of the undetermined differentials are in the image of the homomorphism $\iota\colon \widetilde{\Omega}_9^{\mathrm{String}}(B\mathbb{Z}/2) \to \Omega_9^{\xi^{\mathrm{het}}}$. The generator of the summand in lower Adams filtration has image in the E_∞-page equal to $h_1^2 p_7$, so we obtain $S^1_{nb} \times S^1_{nb} \times \mathbb{RP}^7$.
  The summand in higher Adams filtration has nonzero image in the group $\widetilde{\Omega}_9^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, by inspection of Figure 4. The two generators of $\widetilde{\Omega}_9^{\mathrm{Spin}}(B\mathbb{Z}/2)$ can be taken to be $\mathbb{HP}^2 \times \mathbb{RP}^1$ and $B \times \mathbb{RP}^1$; to determine which we get, compose further with the Atiyah-Bott-Shapiro [**ABS64**] map $\widetilde{\Omega}_9^{\mathrm{Spin}}(B\mathbb{Z}/2) \to \widetilde{ko}_9(B\mathbb{Z}/2) \cong \mathbb{Z}/2$, which sends $[\mathbb{HP}^2 \times \mathbb{RP}^1] \mapsto 0$ and $[B \times \mathbb{RP}^1]$ to the generator. The image of the map of Adams spectral sequence in Figure 4 is contained in the summand whose image under the Atiyah-Bott-Shapiro map is nonzero, the image of our generator in $\widetilde{\Omega}_9^{\mathrm{Spin}}(B\mathbb{Z}/2)$ is bordant to $B \times \mathbb{RP}^1$; finally, since B and $\mathbb{RP}^1$ are both string, we can take $B \times \mathbb{RP}^1$ as our last generator in this dimension.
- If $d_2(h_1 c) = 0$, there is another $\mathbb{Z}/2$ summand whose image in the E_∞-page is $h_1 b$. Thus as usual it lifts to $S^1_{nb} \times X_8$, where X_8 is the manifold we describe in §2.2.2.
- If $d_2(c) = 0$, there is another $\mathbb{Z}/2$ summand whose image in the E_∞-page is c. We were unable to find a manifold X_9 representing this generator. Because c is in Adams filtration 0, corresponding to the mod 2 cohomology class $D_1 D_2 x$, if X_9 exists then one can detect it by showing $\int_{X_9} D_1 D_2 x = 1$.

(10) $\Omega_{10}^{\xi^{\mathrm{het}}} \cong (\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/2$, together with potentially another $\mathbb{Z}/2$ summand if the differentials in (D3) vanish.

- Three of the $\mathbb{Z}/2$ summands in $\Omega_{10}^{\xi^{\mathrm{het}}}$ come from $\Omega_{10}^{\mathrm{Spin}} \cong (\mathbb{Z}/2)^{\oplus 3}$. Their generators are known to be $B \times S^1_{nb} \times S^1_{nb}$, $\mathbb{HP}^2 \times S^1_{nb} \times S^1_{nb}$, and a *Milnor hypersurface* X_{10}, defined to be a smooth degree-$(1,1)$ hypersurface in $\mathbb{CP}^2 \times \mathbb{CP}^4$. Milnor [**Mil65**, §3] showed that X_{10} generates the last $\mathbb{Z}/2$ summand in $\Omega_{10}^{\mathrm{Spin}}$.
- The next $\mathbb{Z}/2$ summand is detected by the maps $\widetilde{\Omega}_{10}^{\mathrm{String}}(B\mathbb{Z}/2) \to \Omega_{10}^{\xi^{\mathrm{het}}}$ and $\Omega_{10}^{\xi^{\mathrm{het}}} \to \Omega_{10}^{\mathrm{Spin}}(B\mathbb{Z}/2)$, and by a similar argument to the one we gave for the higher-filtration orange $\mathbb{Z}/2$ summand in degree 9, we may choose $B \times \mathbb{RP}^1 \times S^1_{nb}$ as the generator.
- If $d_2(h_1 c) = 0$, then there is an additional $\mathbb{Z}/2$ summand whose image in the E_∞-page is $h_1 c$. Thus we can take $S^1_{nb} \times X_9$ for a manifold representative, though as discussed above we do not know what X_9 is.

(11) We have not determined generators for $\Omega_{11}^{\xi^{\mathrm{het}}}$, nor even its isomorphism type. This is a question whose answer would be useful for anomaly cancellation for the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string; see Question 0.3 and §3.2.1. Nonetheless, the Adams argument we gave above implies $\Omega_{11}^{\xi^{\mathrm{het}}}$ contains

a $\mathbb{Z}/8$ subgroup, the image of $\iota\colon \widetilde{\Omega}_{11}^{\mathrm{String}}(B\mathbb{Z}/2) \to \Omega_{11}^{\xi^{\mathrm{het}}}$. By comparing with the map $\widetilde{\Omega}_{11}^{\mathrm{String}}(B\mathbb{Z}/2) \to \widetilde{\Omega}_{11}^{\mathrm{Spin}}(B\mathbb{Z}/2)$ as in Figure 4, one learns that the class of $B \times \mathbb{RP}^3$ generates this $\mathbb{Z}/8$.

2.2.2. X_8*, a potentially nonzero class in* $\Omega_8^{\xi^{\mathrm{het}}}$. Though we were unable to determine if the class $b \in E_2^{2,10}$ survives to the E_∞-page, we are able to write down a manifold representative X_8 of the class it determines in $\Omega_8^{\xi^{\mathrm{het}}}$; if b does survive, X_8 should be added to the list of generators above.

Definition 2.67. Let $\mathbb{Z}/2$ act on $S^3 \times S^3 \times S^2$ by the antipodal map on S^2 and the first copy of S^3, and a reflection through a plane on the second S^3. This is a free action; let X_8 denote the quotient, which is a smooth manifold.

X_8 is a generalized Dold manifold of the sort studied by Nath-Sankaran [**NS19**]. Manifolds similar to X_8 frequently appear as generators of bordism groups: see [**FH21a**, §5.5.1] and [**DDHM23**, §14.3.3] for related examples.

Lemma 2.68. *X_8 admits a string structure, and one can choose a string structure on X_8 so that the induced string structure on $S^3 \times S^3$ is the one induced by the Lie group framing on $S^3 \times S^3 \cong \mathrm{SU}_2 \times \mathrm{SU}_2$.*

Proof. Adding the normal bundles for $S^{k-1} \hookrightarrow \mathbb{R}^k$ defines an isomorphism

$$T(S^3 \times S^3 \times S^2) \oplus \underline{\mathbb{R}}^3 \xrightarrow{\cong} \underline{\mathbb{R}}^4 \oplus \underline{\mathbb{R}}^4 \oplus \underline{\mathbb{R}}^3. \tag{2.69}$$

To understand TX_8, we will study (2.69) when we introduce the $\mathbb{Z}/2$-action on $S^3 \times S^3 \times S^2$ whose quotient is X_8. Since the outward unit normal vector field on S^k is O_{k+1}-invariant, $\mathbb{Z}/2$ acts trivially on the $\underline{\mathbb{R}}^3$ on the left side of (2.69), since the outward unit normal vector field provides the trivializations of the normal bundles giving that $\underline{\mathbb{R}}^3$ factor. On the right-hand side, $\mathbb{Z}/2$ by the antipodal map on the first factor of S^3, so acts by -1 on each $\underline{\mathbb{R}}$ summand of the first $\underline{\mathbb{R}}^4$. The reflection on the second S^3 factor means $\mathbb{Z}/2$ acts on the second $\underline{\mathbb{R}}^4$ by -1, 1, 1, and 1 on the four $\underline{\mathbb{R}}$ summands. Finally, the antipodal map on S^2 implies $\mathbb{Z}/2$ acts by -1 on the remaining three $\underline{\mathbb{R}}$ summands.

Passing from equivariant vector bundles on $S^3 \times S^3 \times S^2$ to nonequivariant vector bundles on the quotient, (2.69) induces an isomorphism

$$TX_8 \oplus \underline{\mathbb{R}}^3 \xrightarrow{\cong} \sigma^{\oplus 8} \oplus \underline{\mathbb{R}}^3, \tag{2.70}$$

where $\sigma \to X_9$ is pulled back from the tautological line bundle $\sigma \to \mathbb{RP}^2$. The Whitney sum formula implies $\sigma^{\oplus 8} \to \mathbb{RP}^2$ is spin, and since the string obstruction lives in $H^4(\mathbb{RP}^2;\mathbb{Z}) = 0$, $\sigma^{\oplus 8}$ is string. Thus the pullback to X_8 is also string, so TX_8 is string.

For the Lie group framing string structure, use the fact that the involutions on each S^3 summand can be described in terms of Lie groups: since the quotient of $S^3 \cong \mathrm{SU}_2$ by the antipodal map is $\mathbb{RP}^3 \cong \mathrm{SO}_3$, the Lie group framing on S^3 is equivariant for the antipodal map. Compatibility for the reflection comes from the action of a reflection in $\mathrm{Pin}_3^+ \supset \mathrm{SU}_2$. ∎

Proposition 2.71. *With the string structure described in Lemma 2.68 and the $\mathbb{Z}/2$-bundle $\sigma \to X_8$, $[X_8]$ is linearly independent from $[S^1_{nb} \times \mathbb{RP}^7]$ in $\widetilde{\Omega}_8^{\mathrm{String}}(B\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so the image of $[X_8]$ in the E_∞-page for $\Omega_*^{\mathrm{String}}(B\mathbb{Z}/2)$ is the nonzero class in $E_\infty^{2,10} \cong \mathbb{Z}/2$ (perhaps plus a term in lower filtration).*

Proof. Let $f\colon \widetilde{\Omega}_8^{\mathrm{String}}(\mathbb{RP}^2) \to \widetilde{\Omega}_8^{\mathrm{String}}(B\mathbb{Z}/2)$ be the map induced by $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^\infty \simeq B\mathbb{Z}/2$. The map this induces on Adams spectral sequences is not hard to analyze: Bruner-Rognes [**BR21**, §4.4, Chapter 6, §12.1] run the whole Adams spectral sequence for $\widetilde{tmf}_*(\mathbb{RP}^2)$, using the identification $\Sigma^\infty\mathbb{RP}^2 \simeq \Sigma\mathbb{S}/2$, and as discussed above *tmf*- and *MTString*-homology agree in degrees 14 and below.[18] Likewise, Davis-Mahowald [**DM78**, Table 3.2] compute the E_2-page of the Adams spectral sequence for $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2)$ in the range we need, and with their calculation, h_i-linearity of differentials, and the $2\eta = 0$ trick from the proof of Lemma 2.59 one sees that $\widetilde{\Omega}_8^{\mathrm{String}}(B\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. As discussed in §2.2.1, one of the $\mathbb{Z}/2$ summands is detected by $\mathbb{RP}^7 \times S^1_{nb}$, whose image in the E_∞-page is in filtration 1. Consider the map

$$\Psi\colon \mathrm{Ext}_{\mathcal{A}(2)}(\widetilde{H}^*(\mathbb{RP}^2;\mathbb{Z}/2)) \longrightarrow \mathrm{Ext}_{\mathcal{A}(2)}(\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2)) \tag{2.72}$$

induced by $\mathbb{RP}^2 \to \mathbb{RP}^\infty \simeq B\mathbb{Z}/2$; we draw this map in Figure 6. Ψ is also the map between the E_2-pages of these two Adams spectral sequences; looking at Figure 6, Ψ is injective in topological degree 8, with image containing the nonzero element of $E_2^{2,10}$ but not the nonzero class in $E_2^{1,9}$. As both of these elements survive to the E_∞-page, this lifts to imply that $f\colon \widetilde{\Omega}_8^{\mathrm{String}}(\mathbb{RP}^2) \to \widetilde{\Omega}_8^{\mathrm{String}}(B\mathbb{Z}/2)$ is injective and that if one wants to find a class in $\widetilde{\Omega}_8^{\mathrm{String}}(B\mathbb{Z}/2)$ linearly independent from $\mathbb{RP}^7 \times S^1_{nb}$, it suffices to find a nonzero class in $\widetilde{\Omega}_8^{\mathrm{String}}(\mathbb{RP}^2)$.

The map $\sigma\colon X_8 \to B\mathbb{Z}/2$ factors through $\mathbb{RP}^2$ by definition, so we are done if we can show X_8, with its map to $\mathbb{RP}^2$, is nonbounding. To do so, consider the transfer map $\Sigma^\infty\mathbb{RP}^2 \to \Sigma^\infty S^2$ associated to the double cover $S^2 \to \mathbb{RP}^2$; this induces on string bordism a map $\widetilde{\Omega}_*^{\mathrm{String}}(\mathbb{RP}^2) \to \widetilde{\Omega}_*^{\mathrm{String}}(S^2)$ sending $(M, f\colon M \to \mathbb{RP}^2)$ to the double cover $M' \to M$ associated to the line bundle $f^*\sigma$, together to the map $M' \to S(\sigma) = S^2$.

The map $\Omega_k^\xi \to \widetilde{\Omega}_{k+\ell}^\xi(S^\ell)$ sending $M \mapsto (M \times S^\ell, \mathrm{proj}_2\colon M \times S^\ell \to S^\ell)$ (where S^ℓ carries the bounding stable framing, which with the ξ-structure on M induces a ξ-structure on $M \times S^\ell$) is always an isomorphism (e.g. check this with the Atiyah-Hirzebruch spectral sequence), and $\Omega_6^{\mathrm{String}} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ [**Gia71**, §3, §4], generated by $S^3 \times S^3$ with its Lie group framing, because it is represented by h_2^2 in the Adams spectral sequence. Therefore $\widetilde{\Omega}_8^{\mathrm{String}}(S^2) \cong \mathbb{Z}/2$ is generated by $S^3 \times S^3 \times S^2$, with the map to S^2 given by projection onto the third factor. The image of X_8 under the transfer is its double cover, which is $S^3 \times S^3 \times S^2$, with the correct string structure and map to S^2, so $[X_8] \neq 0$ in $\widetilde{\Omega}_8^{\mathrm{String}}(\mathbb{RP}^2)$, which suffices to prove the theorem. □

Finally, by looking at the map $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2) \to \Omega_*^{\xi^{\mathrm{het}}}$, we conclude:

Corollary 2.73. *Suppose $d_2(c) = 0$ in the Adams spectral sequence for ξ^{het}. Then $[X_8] \neq 0$ in $\Omega_8^{\xi^{\mathrm{het}}}$, and its image in the E_∞-page is the class $b \in E_\infty^{2,10}$ (perhaps plus some elements in lower filtration).*

2.3. ξ^{het} bordism at odd primes.

Theorem 2.74. *$\Omega_*^{\xi^{\mathrm{het}}}$ has no odd-primary torsion in degrees 11 and below.*

[18] See also the closely related work of Beaudry-Bobkova-Pham-Xu [**BBPX22**], who compute $tmf_*(\mathbb{RP}^2)$ using the elliptic spectral sequence.

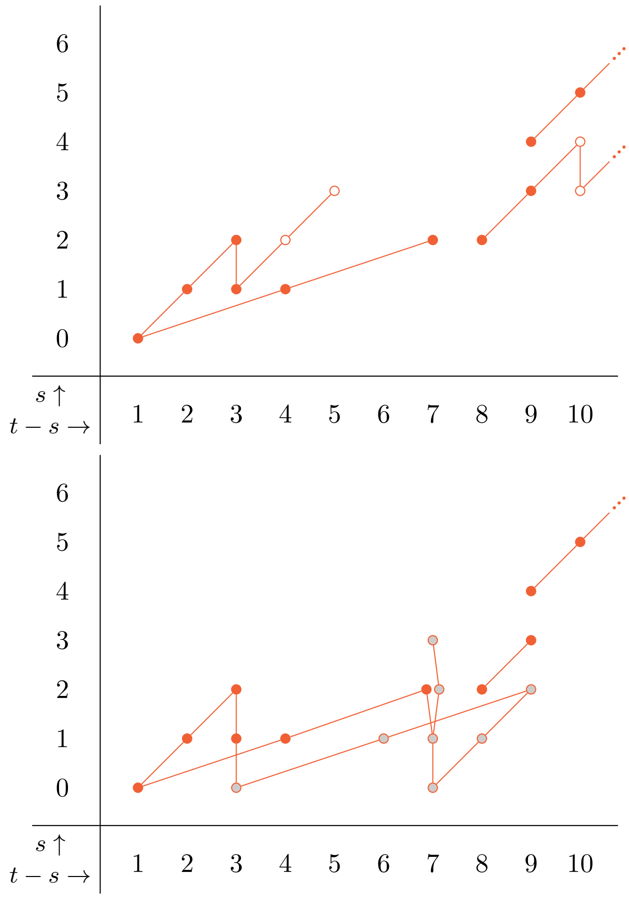

FIGURE 6. The map between the Adams spectral sequences for reduced string bordism induced by the map $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^\infty \simeq B\mathbb{Z}/2$, which we use in the proof of Proposition 2.71. Top: $\mathrm{Ext}_{\mathcal{A}(2)}(\widetilde{H}^*(\mathbb{RP}^2;\mathbb{Z}/2),\mathbb{Z}/2)$, the E_2-page of the Adams spectral sequence computing $\widetilde{\Omega}_*^{\mathrm{String}}(\mathbb{RP}^2)^\wedge_2$. Filled dots have nonzero image after mapping to $\mathbb{RP}^\infty$; unfilled dots are the kernel. Bruner-Rognes [**BR21**, §4.4, Chapter 6, §12.1] compute these Ext groups and run this Adams spectral sequence; from their work we learn there are no differentials in this range (though there are hidden ν-extensions that do not enter into our argument; see (*ibid.*, Theorem 12.5)). Bottom: $\mathrm{Ext}_{\mathcal{A}(2)}(\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2),\mathbb{Z}/2)$, a summand of the E_2-page of the Adams spectral sequence computing $\widetilde{\Omega}_*^{\mathrm{String}}(B\mathbb{Z}/2)^\wedge_2$. Filled dots are in the image of the map from $\mathbb{RP}^2$; gray dots are the cokernel. The E_2-page was computed by Davis-Mahowald [**DM78**, Table 3.2], and during the proof of Proposition 2.71 we argue that there are no differentials or hidden extensions by 2 in the range depicted.

Proof. This amounts to a direct computation with the Adams spectral sequence. We will go over the case $p = 3$ in detail; for $p = 5, 7$ the story is similar but easier, and for $p \geq 11$ it is trivial because the degrees of the Steenrod powers are too high for the Adams spectral sequence to produce torsion.

First we compute $H^*(B\mathbb{G}^{\mathrm{het}}; \mathbb{Z}/3)$ as a module over the Steenrod algebra $\mathcal{A}$ in low degrees in Proposition 2.77, then we do the same for $H^*(MT\xi^{\mathrm{het}}; \mathbb{Z}/3)$ in Proposition 2.83. Once we have this, we can run the Adams spectral sequence, and do so in Proposition 2.86.

Throughout this subsection, $\mathcal{P}^i$ refers to the i^{th} Steenrod power, a degree-$4i$ operation on mod 3 cohomology, and β is the Bockstein homomorphism for the sequence $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 0$.

Lemma 2.75. *Let $C \in H^3(K(\mathbb{Z}, 3); \mathbb{Z}/3)$ denote the mod 3 reduction of the tautological class. Then*

$$H^*(K(\mathbb{Z}, 3); \mathbb{Z}/3) \cong \mathbb{Z}/3[C, \mathcal{P}^1 C, \beta\mathcal{P}^1 C, \dots]/(C^2, \dots), \tag{2.76}$$

where all missing generators and relations are in degrees 14 and above.

Proof. This is a standard application of the Serre spectral sequence for the fibration $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$, so we will be succinct. $E_2^{0,*} \cong H^*(K(\mathbb{Z}, 2); \mathbb{Z}/3) \cong \mathbb{Z}/3[x]$, with $|x| = 2$; by the E_∞-page, all powers of x must be killed by differentials.

The only way to kill x is with a transgressing $d_3\colon E_3^{0,2} \to E_3^{3,0}$. Let $C \coloneqq d_3(x)$. $C^2 = 0$ follows by graded commutativity. The Leibniz rule for differentials means that when $3 \nmid k$, $d_3(x^k) = \pm x^{k-1}C$, and if $3 \mid k$, $d_3(x^k) = 0$.

So x^3 survives to the E_4-page. The only remaining differential that can kill x^3 is the transgressing $d_7\colon E_7^{0,6} \to E_7^{7,0}$, so $d_7(x^3) \neq 0$; by the Kudo transgression theorem [Kud56], because $x^3 = \mathcal{P}^1(x)$, $d_7(x^3) = \mathcal{P}^1 C$. The Leibniz rule then implies $d_7(x^6) = x^3\mathcal{P}^1 C$, so by the E_8-page, everything on the line $p = 0$ in total degree less than 18 has been killed.

Because $d_3(x^3) = 0$, x^2C survives to the E_4-page; the only remaining way for it to support a differential is to have a new class $w \in H^8(K(\mathbb{Z}, 3); \mathbb{Z}/3)$ such that $d_5(x^2C) = w$. To see that $\beta(\mathcal{P}^1 C) = \pm w$, compare with the analogous spectral sequence for $\mathbb{Z}/9$-valued cohomology to see that $\mathcal{P}^1 C$ is not in the image of the mod 3 reduction map from $\mathbb{Z}/9$ cohomology to $\mathbb{Z}/3$ cohomology.[19] ☐

Proposition 2.77. *Let $D \in H^4(B\mathrm{E}_8; \mathbb{Z}/3)$ be the mod 3 reduction of the class c from Definition 1.4, and let D_1 and D_2 be the two copies of D in $H^*(B\mathrm{E}_8^2; \mathbb{Z}/3)$ coming from the two factors of $B\mathrm{E}_8$. In degrees 13 and below, the pullback map on $\mathbb{Z}/3$ cohomology induced by $\phi\colon B\mathbb{G}^{\mathrm{het}} \to B\mathrm{Spin} \times B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2)$ is the quotient ring homomorphism sending $\lambda - D_1 - D_2 \mapsto 0$, $-p_2 - \mathcal{P}^1(D_1 + D_2) \mapsto 0$, and $\beta\mathcal{P}^1(D_1 + D_2) \mapsto 0$.*

Here ϕ is the map we constructed in (1.44) which forgets the B-field.

Proof. Throw the Serre spectral sequence at the fibration

$$K(\mathbb{Z}, 3) \longrightarrow B\mathbb{G}^{\mathrm{het}} \longrightarrow B\mathrm{Spin} \times B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2). \tag{2.78}$$

[19] Alternatively, one could deduce this Bockstein by setting up the Serre spectral sequence for $K(\mathbb{Z}, 3) \to * \to K(\mathbb{Z}, 4)$ and Hill's calculation [Hil09, Corollary 2.9, Figure 1(a)] of the low-degree mod 3 cohomology of $K(\mathbb{Z}, 4)$ as an $\mathcal{A}$-module: C transgresses to a degree-4 class D, and Hill shows $\beta(\mathcal{P}^1 D) \neq 0$, so by the Kudo transgression theorem [Kud56], $\beta(\mathcal{P}^1 C) \neq 0$ in $H^*(K(\mathbb{Z}, 3); \mathbb{Z}/3)$.

The base space is not simply connected, so we might have to worry about local coefficients, but this turns out not to be the case, because the $\mathbb{Z}/2$ symmetry swapping the two E_8 factors, which is the origin of the π_1 in the base, acts trivially on the B-field, which gives us the fiber in (2.78).

In order to run the Serre spectral sequence for (2.78), we need to know the cohomology of $B\mathrm{Spin}$ and $B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2)$. The former is the polynomial ring on the mod 3 reductions of the Pontrjagin classes, which is a theorem of Borel-Hirzebruch [**BH59**, §30.2]; for the latter, run the Serre spectral sequence for the fibration

$$B\mathrm{E}_8^2 \longrightarrow B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2) \longrightarrow B\mathbb{Z}/2. \tag{2.79}$$

Because $H^*(B\mathbb{Z}/2;\mathbb{Z}/3)$ vanishes in positive degrees, this Serre spectral sequence collapses to imply

$$H^*(B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2);\mathbb{Z}/3) \xrightarrow{\cong} H^*(B\mathrm{E}_8^2;\mathbb{Z}/3)^{\mathbb{Z}/2}. \tag{2.80}$$

The answer now follows from the Künneth formula, the fact that we can replace $B\mathrm{E}_8$ with $K(\mathbb{Z},4)$ in the range we need by the result of Bott-Samelson [**BS58**, Theorems IV, V(e)] we mentioned in §2.2, and the mod 3 cohomology of $K(\mathbb{Z},4)$ in low degrees, worked out by Cartan [**Car54**] and Serre [**Ser52**], and stated explicitly by Hill [**Hil09**, Corollary 2.9].

Now back to (2.78) and its Serre spectral sequence. The fibration (2.78) is classified by the degree-4 cohomology class $\lambda - D_1 - D_2$, i.e. it is the pullback of the universal $K(\mathbb{Z},3)$-bundle

$$K(\mathbb{Z},3) \longrightarrow EK(\mathbb{Z},3) \longrightarrow BK(\mathbb{Z},3) \simeq K(\mathbb{Z},4) \tag{2.81}$$

by the map $B\mathrm{Spin} \times B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2) \to K(\mathbb{Z},4)$ classified by $\lambda - D_1 - D_2$.[20] In the Serre spectral sequence for (2.81), the class $C \in E_2^{0,3} = H^3(K(\mathbb{Z},3);\mathbb{Z}/3)$ must transgress to the generator of $E_2^{4,0} = H^4(K(\mathbb{Z},4);\mathbb{Z}/3)$, and this generator pulls back to $\lambda - D_1 - D_2$, enforcing the relation $\lambda - D_1 - D_2 = 0$ in the E_5-page.

The other two pullbacks to zero in the theorem statement then follow from the Kudo transgression theorem [**Kud56**]: $\mathcal{P}^1 C \in E_2^{0,7} = H^7(K(\mathbb{Z},3);\mathbb{Z}/3)$ must transgress to $\mathcal{P}^1(\lambda - D_1 - D_2)$, and analogously for $\beta\mathcal{P}^1 C$. To compute these, we must determine how $\mathcal{P}^1$ acts on the mod 3 reductions of Pontrjagin classes. Shay [**Sha77**] proves a formula for Steenrod powers of Chern classes, which yields the formula for Pontrjagin classes by pullback. Hence, as worked out by Nordström [**Nor**], $\mathcal{P}^1 p_1 = p_2$; then an Adem relation tells us

$$\mathcal{P}^1 p_2 = \mathcal{P}^1\mathcal{P}^1 p_1 = -\mathcal{P}^2 p_1 = p_1^3, \tag{2.82}$$

the last equality because $\mathcal{P}^i$ is the cup product cube on classes of degree $2i$. Thus we see that $\mathcal{P}^1 C$ transgresses to $-p_2 - \mathcal{P}^1(D_1 + D_2)$ and $\beta\mathcal{P}^1 C$ transgresses to $\beta\mathcal{P}^1(D_1 + D_2)$, killing those classes by the E_{10}-page.

Now, the Leibniz rule cleans up the rest of the Serre spectral sequence in total degree at most 13: by the E_{10}-page, everything in this range is concentrated on the line $q = 0$. Therefore on the E_∞-page, the extension question is trivial in this range, and we conclude. □

Proposition 2.83. *Let $\mathcal{M}_3$ denote the quotient of $H^*(MT\xi^{\mathrm{het}};\mathbb{Z}/3)$ by all elements of degree 14 or higher, $\mathcal{M}_3^{\mathrm{SO}}$ denote the quotient of $H^*(MTSO;\mathbb{Z}/3)$ by all*

[20] This map, and hence also the fibration, is only determined up to homotopy, but any two choices of representative give isomorphic answers.

elements of degree 14 *or higher, and* $C\alpha$ *denote the* $\mathcal{A}$*-module which consists of two* $\mathbb{Z}/3$ *summands in degrees* 0 *and* 4 *linked by* $\mathcal{P}^1$*. Then, there is an isomorphism of* $\mathcal{A}$*-modules*

$$\mathcal{M}_3 \cong \mathcal{M}_3^{\mathrm{SO}} \oplus \Sigma^8 C\alpha \oplus \Sigma^{12}\mathbb{Z}/3. \tag{2.84}$$

Proof. In Proposition 2.77, we discovered that the map $\phi\colon B\mathbb{G}^{\mathrm{het}} \to B\mathrm{Spin} \times B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2))$ induces a surjection on mod 3 cohomology in degrees 13 and below. As ϕ commutes with the maps down to $B\mathrm{O}$ that are part of the definition of these tangential structures, ϕ induces a map on Thom spectra

$$MT\xi^{\mathrm{het}} \to MTSpin \wedge B(\mathrm{E}_8^2 \rtimes \mathbb{Z}/2)_+. \tag{2.85}$$

Both of these tangential structures' maps to $B\mathrm{O}$ factor through $B\mathrm{SO}$, so the Thom isomorphism for mod 3 cohomology untwists. The Thom isomorphism is natural for maps of tangential structures, so we conclude that the pullback map on mod 3 cohomology induced by (2.85) is a surjection in degrees 13 and below — and therefore that we can compute Steenrod powers in the cohomology of the latter Thom spectrum. And the map $MTSpin \to MTSO$ is an equivalence away from 2, so we may work with $MTSO$ in place of $MTSpin$. Milnor [**Mil60**, Theorem 4] computed the Steenrod module structure on $H^*(MTSO;\mathbb{Z}/3)$, showing that it is a free $\mathcal{A}/\beta$-module. Using this, we can determine the Steenrod powers of Up_i, where U is the Thom class; and this and the Cartan formula finish the proof. $\square$

Proposition 2.86. *In topological degrees* 12 *and below, the Adams* E_2*-page computing* $(\Omega_*^{\xi^{\mathrm{het}}})_3^\wedge$ *consists of* h_0*-towers concentrated in even topological degrees, and therefore this Adams spectral sequence collapses in degrees* 12 *and below.*

Proof. The direct-sum decomposition in Proposition 2.83 means that it suffices to prove the statement about h_0-towers for $\mathcal{M}_3^{\mathrm{SO}}$, $\Sigma^8 C\alpha$, and $\Sigma^{12}\mathbb{Z}/3$ separately. As usual, with M an $\mathcal{A}$-module, we write $\mathrm{Ext}(M)$ to denote $\mathrm{Ext}_{\mathcal{A}}^{*,*}(M,\mathbb{Z}/3)$. The first ingredient we need is $\mathrm{Ext}(\mathbb{Z}/3)$ itself; the computation of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/3)$ in degrees $t-s \le 11$ is due to Gershenson [**Ger63**]; May [**May65, May66**] expanded this computation to $t-s \le 88$. In topological degrees 2 and below, $\mathrm{Ext}(\mathbb{Z}/3)$ consists of a single h_0-tower in topological degree 0, implying the conclusion for $\Sigma^{12}\mathbb{Z}/3$.

Next, we compute $\mathrm{Ext}(C\alpha)$ using the fact that a short exact sequence of $\mathcal{A}$-modules induces a long exact sequence in Ext groups. Specifically, factor $C\alpha$ as an extension of $\mathcal{A}$-modules

$$0 \longrightarrow \Sigma^4\mathbb{Z}/3 \longrightarrow C\alpha \longrightarrow \mathbb{Z}/3 \longrightarrow 0, \tag{2.87}$$

which we draw in Figure 7, left, and compute the corresponding long exact sequence in Ext in Figure 7, right. There is one potentially nonzero boundary map in range: $\partial\colon \mathrm{Ext}_{\mathcal{A}}^{0,4}(\mathbb{Z}/3) \to \mathrm{Ext}_{\mathcal{A}}^{1,4}(\mathbb{Z}/3)$. This map must be nonzero because $\mathrm{Ext}_{\mathcal{A}}^{0,4}(C\alpha) = \mathrm{Hom}_{\mathcal{A}}(C\alpha, \Sigma^4\mathbb{Z}/3) = 0$. We see that in degrees 6 and below, $\mathrm{Ext}(C\alpha)$ consists solely of h_0-towers in even degrees, which implies the part of the corollary statement coming from $\Sigma^8 C\alpha$.

Finally, $\mathcal{M}_3^{\mathrm{SO}}$. Milnor [**Mil60**, Theorem 4] showed that this module coincides with a free $\mathcal{A}/\beta$-module in degrees 13 and below, and proves (*ibid.*, Lemma 5) that the Ext groups of such a module consist solely of h_0-towers in even topological degree. Therefore in topological degrees 12 and below, $\mathrm{Ext}(\mathcal{M}_3^{\mathrm{SO}})$ also consists solely of h_0-towers in even topological degrees. $\square$

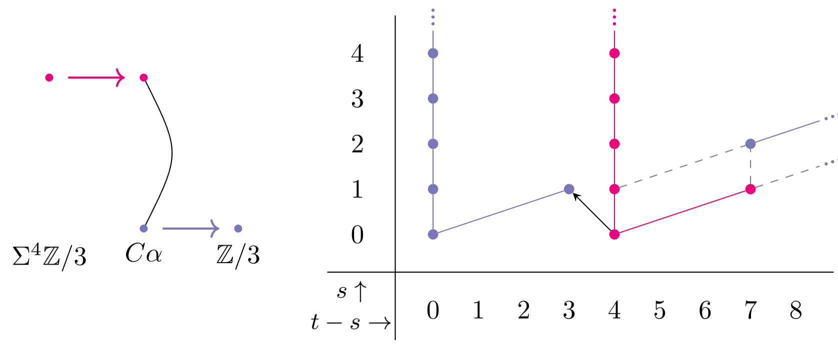

FIGURE 7. Left: the extension (2.87) of $\mathcal{A}$-modules at $p = 3$. Right: the associated long exact sequence in Ext. The dashed gray lines are actions by elements of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/3)$ that cannot be seen from this long exact sequence and must be deduced another way; we do not need them in this paper, so do not go into the details.

This suffices to prove Theorem 2.74 for $p = 3$: h_0-towers on the E_∞-page lift to $\mathbb{Z}_3$ (i.e. the 3-adic integers) summands in $(\Omega_*^{\xi^{\mathrm{het}}})^\wedge_3$, so there is no 3-torsion in this range. $\square$

Remark 2.88. The change-of-rings technique we used at $p = 2$ has an analogue at $p = 3$ for twists of *tmf* (hence also 3-local twisted string bordism in degrees 15 and below, because the Ando-Hopkins-Rezk map [**AHR10**] $\mathit{MTString}_{(3)} \to \mathit{tmf}_{(3)}$ is 15-connected [**HR95, Hil09**]): using Baker-Lazarev's version of the Adams spectral sequence [**BL01**], we can take Ext over the algebra

$$\mathcal{A}^{tmf} := \pi_{-*}\mathrm{Map}_{tmf}(H\mathbb{Z}/3, H\mathbb{Z}/3), \tag{2.89}$$

where $H\mathbb{Z}/3$ is made into a *tmf*-algebra spectrum by the ring spectrum maps $\mathit{tmf} \overset{\tau_{\le 0}}{\to} H\mathbb{Z} \to H\mathbb{Z}/3$, where the first map is the Postnikov 0-connected quotient and the second map is induced from $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/3$. The algebra $\mathcal{A}^{tmf}$ was explicitly calculated by Henriques and Hill, using work of Behrens [**Beh06**] and unpublished work of Hopkins-Mahowald; see Henriques [**DFHH14**, §13.3], Hill [**Hil07**], and Bruner-Rognes [**BR21**, §13] for computations with this Adams spectral sequence.

Just like at $p = 2$, there is a little more work to do apply this spectral sequence to twisted string bordism when the twist does not arise from a vector bundle. We take up this question in joint work with Matthew Yu [**DY23**], where we see how to work over $\mathcal{A}^{tmf}$ for non-vector-bundle twists and that it simplifies the 3-primary computation of $\Omega_*^{\xi^{\mathrm{het}}}$ in degrees relevant to string theory.

2.4. ξ^{CHL} bordism. In this section, we compute the ξ^{CHL} bordism groups. Just like for the ξ^{het} bordism groups, we use the change-of-rings trick from Corollary 2.22 at $p = 2$ and work more directly with the Adams spectral sequence at odd primes. This time, however, we can deduce a lot of information from abstract isomorphisms with the Adams spectral sequences for the string bordism of $B\mathrm{E}_8$, which has been studied by Hill [**Hil09**].

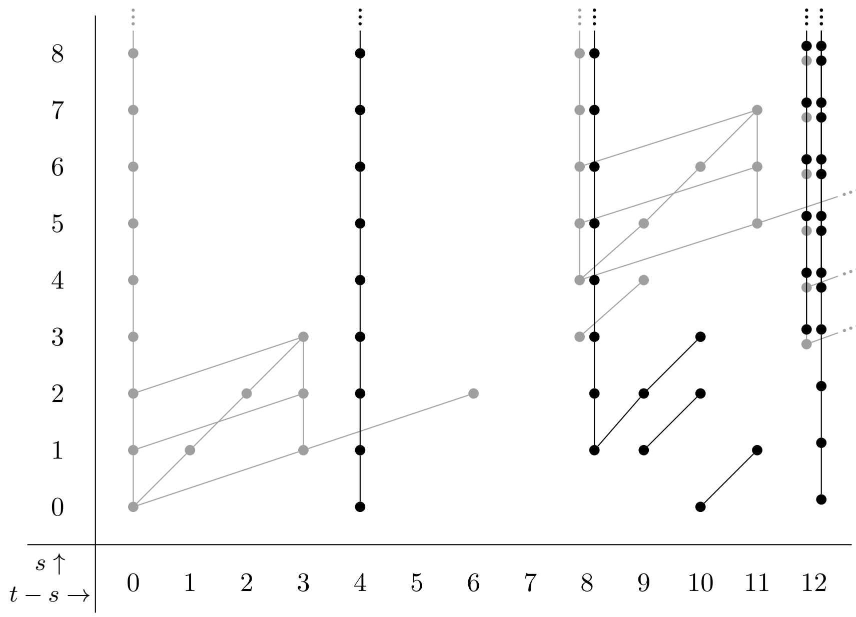

FIGURE 8. The E_2-page for the Adams spectral sequence computing 2-completed ξ^{CHL} bordism. The gray summands correspond to classes with trivial E_8-bundle. See Theorem 2.90 for more information. This figure is adapted from [Hil09, Figure 3].

2.4.1. 2-*primary computation.*

Theorem 2.90. *In degrees* 11 *and below, the* 2*-completions of the abelian groups* $\Omega_*^{\xi^{\mathrm{CHL}}}$ *and* $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$ *are abstractly isomorphic.*

PROOF. By Corollary 2.22, the Adams E_2-page in this range coincides with the Ext of $T(-2c)$ over $\mathcal{A}(2)$. The $\mathcal{A}(2)$-module structure on $T(\mu)$ only depends on the underlying group BG and on μ mod 2, and $2c$ mod $2 = 0$, so as $\mathcal{A}(2)$-modules, $T(-2c) \cong T(0) = H^*(B\mathrm{E}_8; \mathbb{Z}/2)$. So the Adams E_2-page coincides in the range we care about with the E_2-page for $MT\xi^0 = MTString \wedge (B\mathrm{E}_8)_+$. Hill [Hil09, Figure 3] computes the E_2-page corresponding for the reduced string bordism of $B\mathrm{E}_8$, which we use to draw the full E_2-page for $\Omega_*^{\xi^{\mathrm{CHL}}}$ in Figure 8.

This is an abstract isomorphism and does not a priori tell us about differentials or extensions. However, quotienting by $\mathbb{T}[1]$ defines a map $\mathbb{G}^{\mathrm{CHL}} \to \mathrm{Spin} \times \mathrm{E}_8$, which induces a map on Adams spectral sequences for Thom spectra of classifying spaces, and this map of Adams spectral sequences is identified with the map induced by $MTString \wedge (B\mathrm{E}_8)_+ \to MTSpin \wedge (B\mathrm{E}_8)_+$, so any differential for the string bordism of $B\mathrm{E}_8$ deduced by pulling back from the Adams spectral sequence for $MTSpin \wedge B\mathrm{E}_8$ remains valid in our Adams spectral sequence for ξ^{CHL} bordism.

Moreover, we can realize the part of $\Omega_*^{\xi^{\mathrm{CHL}}}$ corresponding to the gray summands in Figure 8 by string manifolds with trivial E_8-bundle, so the gray summands split off of the rest of the Adams spectral sequence.

Looking at the black summands in Figure 8, linearity of differentials with respect to the $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2)$-action on the E_2-page means the only possible nonzero differentials in the range we care about are $d_2\colon E_2^{0,10} \to E_2^{2,11}$ and $d_2\colon E_2^{1,12} \to E_2^{3,13}$. Hill [**Hil09**, §3.3] uses the map to $\mathit{MTSpin} \wedge (B\mathrm{E}_8)_+$ to show that these two differentials are nontrivial, so as we noted above, the same is true for ξ^{CHL} bordism.

As there are no more differentials, and all extensions by 2 in range follow from $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2)$-action without additional information, we have proven the theorem. $\square$

Remark 2.91. As described in Remark 1.55, the map $c\colon B\mathrm{E}_8 \to K(\mathbb{Z}, 4)$ defines a map from ξ^{CHL} structures to $\mathrm{Spin}\langle w_4\rangle$ structures, i.e. the data of a spin structure and a trivialization of w_4. This is the CHL analogue of the passage from ξ^{het} structures to $\xi^{\mathrm{het}'}$ structures from §2.2 — and just as in that case, because c is 15-connected, the induced map $\Omega_k^{\xi^{\mathrm{CHL}}} \to \Omega_k^{\mathrm{Spin}\langle w_4\rangle}$ is an isomorphism for $k \le 14$, so the computations in this section also give $\mathrm{Spin}\langle w_4\rangle$ bordism groups.

An alternate point of view due to Sati-Schreiber-Stasheff [**SSS12**, (2.17)] is that $\mathrm{Spin}\langle w_4\rangle$ structures are twisted string structures in the sense of Corollary 2.12: the trivialization of $w_4(M)$ is equivalent data to a class $\mu \in H^4(M;\mathbb{Z})$ and an identification of 2μ and $\lambda(M)$, so a $\mathrm{Spin}\langle w_4\rangle$-structure is a twisted string structure for the map $-2\colon K(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ (corresponding to the classifying space Sati-Schreiber-Stasheff denote $B\mathrm{String}^{2\mathrm{DD}_2}$). See also [**FH21a**, Remark C.18].

The proof of Theorem 2.90 took advantage of an abstract isomorphism, so it tells us nothing about the generators. The elements of $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$ coming from $\Omega_*^{\mathrm{String}}(\mathrm{pt})$ are represented by string manifolds with trivial E_8-bundle; these vacuously satisfy the condition $2c = \lambda$, so define classes in $\Omega_*^{\xi^{\mathrm{CHL}}}$ representing the same elements under the abstract isomorphism with $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$.

That leaves a few elements left: copies of $\mathbb{Z}$ in degrees 4 and 8, and copies of $\mathbb{Z}/2$ in degrees 9 and 10. We can represent the generator of $\Omega_4^{\xi^{\mathrm{CHL}}} \cong \mathbb{Z}$ by a K3 surface with an E_8-bundle chosen to satisfy the Bianchi identity; it would be interesting to determine generators of $\Omega_k^{\xi^{\mathrm{CHL}}}$ for $k = 8, 9, 10$.

2.4.2. *Odd-primary computation.*

Theorem 2.92. *For $k \le 12$, $\Omega_k^{\xi^{\mathrm{CHL}}}$ has no odd-primary torsion.*

Proof. First we show the result for $p = 3$. The mod 3 cohomology, as an $\mathcal{A}$-module, of the string cover $\mathcal{S}(G, \lambda)$ only depends on λ mod 3. Therefore in the CHL case, where $\lambda = 2c$, we might as well work with $\lambda = -c$ — or replacing our $K(\mathbb{Z}, 4)$ class with its opposite, $\lambda = c$. This string cover corresponds to the universal twist of $\mathit{MTString}$ over $K(\mathbb{Z}, 4)$ from Corollary 2.12, which means that by Theorem 2.11, the Thom spectrum for this twist is MTSpin again! That is, the E_2-page of the 3-primary Adams spectral sequence for CHL bordism coincides with the E_2-page for spin bordism — or for oriented bordism, because the forgetful map $\mathit{MTSpin} \to \mathit{MTSO}$ is a 3-primary equivalence.

Milnor [**Mil60**, Theorem 4] shows that the mod 3 cohomology of MTSO is free as an $\mathcal{A}/\beta$-module on even-degree generators, where β is the mod 3 Bockstein; then, he proves (*ibid.*, Theorem 1) that for any spectrum with that property and satisfying a finiteness condition, there is no odd-primary torsion in homotopy. The CHL bordism spectrum satisfies these conditions, so we conclude.

For $p \ge 5$, the argument is essentially the same as in Theorem 2.74. $\square$

3. Consequences in string theory

There are a few different uses of bordism groups in theories of quantum gravity. In this section, we discuss applications and questions raised by the computations in the previous section. Though we stay mostly mathematical, some of what we state in this section is only known at a physical level of rigor.

3.1. The cobordism conjecture. As part of the Swampland program in quantum gravity, McNamara-Vafa [**MV19**] made the following conjecture, a consequence of the generally believed fact that theories of quantum gravity should not have global symmetries:

Conjecture 3.1 (McNamara-Vafa cobordism conjecture [**MV19**])**.** Suppose we have a consistent n-dimensional theory of quantum gravity in which the spacetime backgrounds that are summed over carry a ξ-structure. Then, for $3 \le k \le n-1$, $\Omega_k^\xi = 0$.

The key here is the meaning of "the spacetime backgrounds carry a ξ-structure" — we do not mean just that one could sum over ξ-manifolds, but that ξ is in some to-be-specified sense the maximally general structure for which the theory makes sense. String theorists often work with singular manifolds and even Deligne-Mumford stacks on $\mathcal{M}an$ [**PS05, PS06a, PS06b, DFM11a, DFM11b**], and the notion of ξ-bordism appearing in Conjecture 3.1 is expected to take this into account, as some sort of bordism theory of generalized manifolds.

The tangential structures ξ currently known for various theories of quantum gravity do not satisfy the vanishing criterion in Conjecture 3.1, so there must be additional data or conditions on these theories' backgrounds modifying ξ so as to kill its bordism groups. These modifications often take the form of additional extended objects in the theory.

This leads to a common application of the cobordism conjecture: compute the bordism groups for the tangential structure ξ as we currently understand it, and use any nonvanishing groups as beacons illuminating novel objects in the theory, which one then studies. This idea has been applied in [**MV19, BKRU20, GEMSV20, DH21, MV21, Sch21, ACC22, BC22, BCKM22, DHMT22, MR22, Wit22, DDHM23, MDL23**];[21] in this subsection, we will use our computations from §2 and see what we can learn about the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string and the CHL string.

Despite the $k \ge 3$ bound in Conjecture 3.1, modifying ξ to kill classes in Ω_1^ξ and Ω_2^ξ is often physically meaningful, and can predict useful new objects in the theory. This is a common technique in the study of the cobordism conjecture, and we will do this too.

3.1.1. *The* $\mathrm{E}_8 \times \mathrm{E}_8$ *heterotic string.* McNamara-Vafa [**MV19**, §4.5] discussed predictions of their conjecture to the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory, but after making the simplifying assumption that the gauge $(\mathrm{E}_8\times\mathrm{E}_8)\rtimes\mathbb{Z}/2$-bundle is trivial; the corresponding tangential structure is then $B\mathrm{String}$. For example, their conjecture

[21]Despite all of this work, there are still plenty of already-worked-out computations of bordism groups relevant to various string and supergravity theories whose corresponding defects have not been determined. This includes $\Omega_*^{\mathrm{Spin}}(B\mathrm{E}_8)$ [**Sto86, Edw91**], applicable to the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string in the absence of the $\mathbb{Z}/2$ swapping symmetry; Ω_*^{DPin} [**KPMT20**, Appendices E, F], relevant for type I string theory; and $\Omega_*^{\mathfrak{m}_c}$ [**FH21a**], useful for the low-energy limit of M-theory.

must account for $\Omega_3^{\mathrm{String}} \cong \mathbb{Z}/24$, generated by S^3 with its Lie group framing, and they explain how this is trivialized by taking into account the NS5-brane.

With $\Omega_*^{\xi^{\mathrm{het}}}$ in hand, we can predict more objects. Recall the generators we found for ξ^{het}-bordism groups, and our notation for them, from §2.2.1.

Example 3.2. $\Omega_1^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, with generators S^1_{nb} and $\mathbb{RP}^1$. McNamara-Vafa already considered S^1_{nb}, but the latter is new. If one allows manifolds with singularities, $\mathbb{RP}^1$ bounds $D^2/(\mathbb{Z}/2)$, i.e. the disc with a principal $\mathbb{Z}/2$-bundle that is singular at the origin, inflated to a singular $\mathbb{G}^{\mathrm{het}}$-bundle via the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{G}^{\mathrm{het}}$.

This class corresponds to a 7-brane in the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string. The worldvolume of this brane is eight-dimensional, so the link around it in ten-dimensional spacetime is a circle. The monodromy around this circle exchanges the two E_8-bundles. This is exactly the non-supersymmetric 7-brane recently introduced and discussed by Kaidi-Ohmori-Tachikawa-Yonekura [**KOTY23**].

Related 7-branes in different string theories are studied by Distler-Freed-Moore [**DFM11a**] and Dierigl-Heckman-Montero-Torres [**DHMT22**]; the latter study a 7-brane in type IIB string theory, called an R7-brane, which in the cobordism conjecture corresponds to $[\mathbb{RP}^1] \in \Omega_1^{\mathrm{Spin}\text{-}\mathrm{GL}_2^+(\mathbb{Z})}$.

As a way of better understanding Kaidi-Ohmori-Tachikawa-Yonekura's 7-brane, we can try to identify where it is sent under dualities between different string theories. For example, Hořava-Witten [**HW96b, HW96a, Wit96**] identified (a certain limit) of $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory with a theory predicted to be the low-energy limit of a compactification of M-theory on the unit interval. Under this identification, the Kaidi-Ohmori-Tachikawa-Yonekura 7-brane ought to correspond to a defect in M-theory associated to a 2-dimensional bordism class by the cobordism conjecture. Because the passage from M-theory to heterotic string theory requires compactifying on the interval, which is a manifold with boundary, one should use a theory of bordism of compact manifolds which are not necessarily closed.[22] The bordism class should be represented by an interval bundle over $\mathbb{RP}^1$, so we conjecture that the bordism class of the Möbius strip corresponds to the avatar of this brane in M-theory. As a check, M-theory compactified on a Möbius strip is expected to coincide with $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory compactified on $\mathbb{RP}^1$ — they are both predicted to be the CHL string, as we discussed in §1.3, though as usual only a statement about low-energy supergravity limits is known. We will not attempt to fully resolve this question in this paper: among other things, this would require finding “the right” notion of bordism for manifolds with boundary for this application.

Before we leave heterotic/M-duality behind, we point out a notion of bordism of manifolds with boundary, due to Conner-Floyd [**CF66**, §16], for which the Möbius strip is nonbounding; we optimistically conjecture that this is the correct kind of bordism of manifolds with boundary for applications to the cobordism conjecture.

Definition 3.3. Let $\xi_1 \colon B_1 \to B\mathrm{O}$ and $\xi_2 \colon B_2 \to B\mathrm{O}$ be tangential structures and $\eta \colon B_1 \to B_2$ be a *map of tangential structures*, i.e. η commutes with the maps ξ_i. A ξ_2/ξ_1-*manifold* is a compact manifold M with ξ_2-structure together with

[22] McNamara-Vafa [**MV19**, §5] hint at this generalization, though from the perspective of manifolds with singularities rather than manifolds with boundary.

(1) a ξ_1-structure $\mathfrak{x}$ on ∂M, and
(2) an identification of the ξ_2-structure $\eta(\mathfrak{x})$ on ∂M with the ξ_2-structure induced by taking the boundary on M.

Conner-Floyd [**CF66**, §16] introduce a notion of bordism for ξ_2/ξ_1-manifolds,[23] which we write $\Omega_*^{\xi_2/\xi_1}$, such that the Thom spectrum corresponding to this notion of bordism is $MT\xi_2/MT\xi_1$, the cofiber of $\eta\colon MT\xi_1 \to MT\xi_2$. This implies the existence of a long exact sequence

$$\cdots \longrightarrow \Omega_k^{\xi_1} \xrightarrow{\eta} \Omega_k^{\xi_2} \xrightarrow{j} \Omega_k^{\xi_1/\xi_2} \xrightarrow{\partial} \Omega_{k-1}^{\xi_1} \longrightarrow \dots \tag{3.4}$$

where j regards a ξ_2-manifold as a ξ_2/ξ_1-manifold with empty boundary.

Lemma 3.5. *The class of the Möbius strip M is nonzero in $\Omega_2^{\mathrm{Pin}^+/\mathrm{Spin}}$.*[24]

Proof. By (3.4), it suffices to prove that $[\partial M] \neq 0$ in Ω_1^{Spin}. The boundary of the Möbius strip is a circle, and for any pin$^+$ structure on M, the boundary circle has the nonbounding spin structure, i.e. is nonzero in Ω_1^{Spin}. This is because if ∂M had the bounding spin structure, one could glue the disc with its standard pin$^+$ structure to M along ∂M and thereby obtain a pin$^+$ structure on $\mathbb{RP}^2$, but $\mathbb{RP}^2$ does not admit a pin$^+$ structure. $\square$

Lemma 3.5 suggests that Conner-Floyd's notion of bordism of manifolds with boundary could be the correct one for our application in heterotic/M-theory duality.

Example 3.6. Moving onto higher-codimension objects predicted by higher-dimensional bordism groups, $\Omega_2^{\xi^{\mathrm{het}}}$ is nonzero, but can be generated by products of S^1_{nb} and $\mathbb{RP}^1$. This means that if we trivialize $[\mathbb{RP}^1], [S^1_{nb}] \in \Omega_1^{\xi^{\mathrm{het}}}$ in the sense above, namely by allowing $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory to be defined on singular manifolds whose boundaries are $\mathbb{RP}^1$ and S^1_{nb}, then we can realize our chosen generators of $\Omega_2^{\xi^{\mathrm{het}}}$ as boundaries of singular 3-manifolds: for example, we used $D^2/(\mathbb{Z}/2)$ to realize $\mathbb{RP}^1$ as a boundary, so we can use $S^1_{nb} \times D^2/(\mathbb{Z}/2)$ to realize $S^1_{nb} \times \mathbb{RP}^1$ as a boundary. Thus accounting for $\Omega_2^{\xi^{\mathrm{het}}}$ does not require adding any new kinds of defects or singularities beyond what we used for $\Omega_1^{\xi^{\mathrm{het}}}$.

Example 3.7. $\Omega_3^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/8$, generated by $\mathbb{RP}^3$. As in Example 3.2, we can bound $\mathbb{RP}^3$ by $B^4/(\mathbb{Z}/2)$ by allowing a singularity at the origin. This bordism class should correspond to a 5-brane distinct from the NS5-brane.

Example 3.8. $\Omega_4^{\xi^{\mathrm{het}}} \cong \mathbb{Z}\oplus\mathbb{Z}/2$. The $\mathbb{Z}/2$ summand is generated by $S^3\times\mathbb{RP}^1$, where S^3 carries the Lie group framing, so its bordism class can be trivialized using the objects we have already discussed, like in Example 3.6. By Remark 2.64, because $S^3 \times S^1$ is bordant as ξ^{het}-manifolds to S^4 with trivial $\mathbb{Z}/2$-bundle and E_8-bundles with characteristic classes $\pm 1 \in H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$, this bordism class corresponds to the 4-brane recently found by Kaidi-Ohmori-Tachikawa-Yonekura [**KOTY23**].

[23] Conner-Floyd only consider a few examples of ξ_1 and ξ_2. The works [**Sto68, Ale75, Mit75, RST77, Lau00, Bun15**] consider some more tangential structures.

[24] Strictly speaking, $\mathrm{Pin}^+/\mathrm{Spin}$ is not the correct tangential structure: one should replace Pin^+ with something like $\mathfrak{m}_c$ [**Wit97, Wit16, FH21a**], and should replace Spin with something like ξ^{het}, though $\mathfrak{m}_c$- and pin$^+$ structures on 2-manifolds are equivalent data [**FH21a**, §8.5.1].

The $\mathbb{Z}$ summand in $\Omega_4^{\xi^{\mathrm{het}}}$ is new to us. It is generated by the K3 surface with trivial $\mathbb{Z}/2$-bundle; one E_8-bundle is trivial, and the other cancels $\lambda(\mathrm{K3})$. McNamara-Vafa [**MV19**, §4.2.1] address the K3 surface without data of E_8-bundles or a nontrivial B-field, using it to exhibit a higher-form $\mathbb{T}$-symmetry. Our K3 surface corresponds to a different bordism class, but McNamara-Vafa's argument still applies: as the K3 surface is believed to be a valid background for $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory, this higher-form $\mathbb{T}$-symmetry must be broken or gauged in some way. We do not know what this would look like.

$\Omega_5^{\xi^{\mathrm{het}}}$ vanishes and $\Omega_6^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/2$ is generated by $\mathbb{RP}^3 \times S^3$, so as in Example 3.6 we can realize it as a boundary without introducing any new kinds of singularities.

Example 3.9. $\Omega_7^{\xi^{\mathrm{het}}} \cong \mathbb{Z}/16$, generated by $\mathbb{RP}^7$. This bordism class is closely analogous to Examples 3.2 and 3.7; this time, we have a 1-brane, i.e. a string.

Remark 3.10 (Relating bordism classes by compactification[25])**.** For the cobordism conjecture for type IIB string theory considered on spin-$\mathrm{GL}_2^+(\mathbb{Z})$ manifolds, $[\mathbb{RP}^k] \in \Omega_k^{\mathrm{Spin}\text{-}\mathrm{GL}_2^+(\mathbb{Z})}$ is nonzero for $k = 1$, 3, and 7 [**DDHM23**, §14.3.2], so we would expect these classes to correspond to three different kinds of extended objects, akin to Examples 3.2, 3.7, and 3.9. However, in [**DDHM23**, §7], it is shown that the two higher-codimension objects can be expressed as compactifications of the R7-brane corresponding to $\mathbb{RP}^1$, so there is really only one novel object. We suspect something similar happens here: that in $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory, the extended objects corresponding to $\mathbb{RP}^3$ and $\mathbb{RP}^7$ can be accounted for using previously known branes and Kaidi-Ohmori-Tachikawa-Yonekura's 7-brane from Example 3.2 corresponding to $\mathbb{RP}^1$.

From a bordism point of view, we are saying that if we allow singular ξ^{het}-manifolds which locally look like $\mathbb{R}^k \times D^2/(\mathbb{Z}/2)$, it should be possible to not just bound $\mathbb{RP}^1$, but also to bound $\mathbb{RP}^3$ and $\mathbb{RP}^7$. We leave this as a conjecture.

Example 3.11. $\Omega_8^{\xi^{\mathrm{het}}}$, which corresponds to codimension-9 objects, is isomorphic to either $\mathbb{Z}^3 \oplus \mathbb{Z}/2$ or $\mathbb{Z}^3 \oplus (\mathbb{Z}/2)^{\oplus 2}$, depending on the fate of the differential (D3). The generators of these four or five summands that we found are:

- $\mathbb{HP}^2$ with two different ξ^{het}-structures, giving two $\mathbb{Z}$ summands;
- the Bott manifold, generating another free summand;
- $\mathbb{RP}^7 \times S^1_{nb}$ generating the $\mathbb{Z}/2$ summand that is present even if (D3) does not vanish; and
- the manifold X_8 that we discussed in §2.2.2, an $S^3 \times S^3$-bundle over $\mathbb{RP}^2$. If the differential (D3) is nonzero, then X_8 bounds as a ξ^{het}-manifold.

$\mathbb{RP}^6 \times S^1_{nb}$ is already accounted for in the sense of Example 3.6, so we focus on the other generators.

Both B and $\mathbb{HP}^2$ are nonbounding in the bordism group $\Omega_8^{\mathrm{Spin}\text{-}\mathrm{Mp}_2(\mathbb{Z})}$, which appears in the study of the cobordism conjecture for type IIB string theory; see [**DDHM23**, §6.9] for a discussion of defects in type IIB corresponding to these bordism classes. Like in Example 3.8, the story in $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory is presumably not exactly the same, but it may be analogous.

Finally, X_8. Following the arguments in [**MV19**, §4.5] and [**DDHM23**, §7.6, §7.8] the description of X_8 as a fiber bundle over $\mathbb{RP}^2$ with fiber $S^3 \times S^3$ suggests

[25] We thank Markus Dierigl for pointing this out to us.

the following string-theoretic construction: use the singular manifold corresponding to the NS5-brane to bound for the first S^3, compactify on the second S^3, and then fiber over $D^3/(\mathbb{Z}/2)$ to make X_8 a boundary of a singular manifold. We do not know whether this is a valid background for the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string; an argument for or against it could provide an example of a use of Conjecture 3.1 to make a mathematical conjecture for the fate of X_8 based on string-theoretic predictions.

Example 3.12. $\Omega_9^{\xi^{\mathrm{het}}}$ corresponds to zero-dimensional objects, i.e. point defects, and is isomorphic to either $(\mathbb{Z}/2)^{\oplus 4}$ or $(\mathbb{Z}/2)^{\oplus 6}$, depending on the fate of (D3). Three of the generators we found in §2.2.1 are of the form S^1_{nb} times a ξ^{het}-manifold, so have already been accounted for in the sense of Example 3.6. The fourth generator is $B \times \mathbb{RP}^1$, so it is also already accounted for.

The remaining two manifolds that might or might not be necessary are $X_8 \times S^1_{nb}$, which as usual is already taken care of, and a manifold X_9 which we did not determine.

3.1.2. *The CHL string.* In Theorems 2.90 and 2.92, we saw that $\Omega_*^{\xi^{\mathrm{CHL}}}$ is abstractly isomorphic to $\Omega_*^{\mathrm{String}}(B\mathrm{E}_8)$. Thus there is a summand corresponding to $\Omega_*^{\mathrm{String}}(\mathrm{pt})$, and as we saw above, these classes can be represented by string manifolds with trivial E_8-bundle. Some of these manifolds were accounted for by McNamara-Vafa [**MV19**, §4.5] in heterotic string theory, e.g. killing S^3 with its nonbounding framing using the fivebrane, and presumably a similar defect is present in the CHL string. McNamara-Vafa leave plenty of string bordism classes' interpretations in terms of defects open to address, and this would be interesting to understand more in the setting of the CHL string.

We also found a few more classes in $\Omega_*^{\xi^{\mathrm{CHL}}}$. For example, $\Omega_4^{\xi^{\mathrm{CHL}}} \cong \mathbb{Z}$, generated by a K3 surface with E_8-bundle chosen to satisfy the Bianchi identity. Like in Example 3.8, this corresponds to some codimension-4 object, though we do not know what it will look like.

3.2. Is the $\mathbb{Z}/2$ symmetry on the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string anomalous? Quantum field theories can come with the data of an *anomaly*, a mild inconsistency in which key quantities in the field theory are not defined absolutely without fixing additional data. For example, one wants the partition function of a QFT on a manifold M to be a complex number, but an anomaly signals that the partition function is only an element of a complex line which has not been trivialized. The process of resolving this inconsistency, when necessary, is called *anomaly cancellation*.

Freed-Teleman [**FT14**] describe anomaly cancellation for a broad class of quantum field theories as follows: an n-dimensional quantum field theory Z lives at the boundary of an $(n+1)$-dimensional invertible field theory α, called the *anomaly field theory* of Z. The tangential structures of Z and α match. Anomaly cancellation is the procedure of trivializing α, i.e. establishing an isomorphism from α to the trivial theory.

We think of this from Atiyah-Segal's approach [**Ati88, Seg88**] that field theories are symmetric monoidal functors from (potentially geometric) bordism categories into categories such as $\mathcal{V}ect_{\mathbb{C}}$. The perspective of extended field theory means these are often (∞, n)-categories. If $\mathcal{C}$ and $\mathcal{D}$ are two symmetric monoidal (∞, n)-categories, the (∞, n)-category of symmetric monoidal functors $F\colon \mathcal{C} \to \mathcal{D}$ acquires the symmetric monoidal structure of "pointwise tensor product," specified by the formula

$$(F_1 \otimes F_2)(x) \coloneqq F_1(x) \otimes_{\mathcal{D}} F_2(x), \tag{3.13}$$

where x is an object, morphism, higher morphism, etc.

Definition 3.14 (Freed-Moore [**FM06**, Definition 5.7])**.** Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. An *invertible field theory* is a field theory $\alpha\colon \mathcal{B}ord_n^{\xi} \to \mathcal{C}$ such that there is another field theory $\alpha^{-1}\colon \mathcal{B}ord_n^{\xi} \to \mathcal{C}$ such that $\alpha \otimes \alpha^{-1} \simeq \mathbf{1}$, the trivial theory.

The trivial theory $\mathbf{1}\colon \mathcal{B}ord_n^{\xi} \to \mathcal{C}$ is defined to send all objects to the monoidal unit in $\mathcal{C}$ and all morphisms and higher morphisms to identity morphisms, resp. identity higher morphisms.

Therefore the classification of anomalies follows from the classification of invertible field theories, and anomaly cancellation is an isomorphism from an invertible field theory to $\mathbf{1}$. Freed-Hopkins-Teleman [**FHT10**] classify invertible *topological* field theories using stable homotopy theory, and Grady-Pavlov [**GP21**, §5] generalize this in the nontopological setting.

In most cases, including the supergravity theories studied in this paper, the QFT under study is unitary, so their anomaly theories have the Wick-rotated analogue of unitarity, *reflection positivity.* Freed-Hopkins [**FH21b**] classify reflection-positive invertible field theories.

Let $I_{\mathbb{Z}}$ denote the *Anderson dual of the sphere spectrum* [**And69, Yos75**].

Theorem 3.15 (Freed-Hopkins [**FH21b**, Theorem 1.1])**.** *Let ξ be a tangential structure. There is a natural isomorphism from the group of deformation classes of $(n+1)$-dimensional reflection-positive invertible topological field theories on ξ-manifolds to the torsion subgroup of $[MT\xi, \Sigma^{n+2} I_{\mathbb{Z}}]$.*

Freed-Hopkins then conjecture (*ibid.*, Conjecture 8.37) that the entire group classifies all reflection-positive invertible field theories, topological or not.

$I_{\mathbb{Z}}$ satisfies a universal property which leads to the existence of a natural short exact sequence

$$0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^{\xi}, \mathbb{T})) \longrightarrow [MT\xi, \Sigma^{n+2} I_{\mathbb{Z}}] \longrightarrow \mathrm{Hom}(\Omega_{n+2}^{\xi}, \mathbb{Z}) \longrightarrow 0, \tag{3.16}$$

and this sequence carries physical meaning for the classification of possible anomalies for an n-dimensional QFT Z. For example, $\mathrm{Hom}(\Omega_{n+2}^{\xi}, \mathbb{Z})$ is a group of $\mathbb{Z}$-valued degree-$(n+2)$ characteristic classes of ξ-manifolds, and the quotient map in (3.16) sends the anomaly field theory of Z to its *anomaly polynomial.* This data can often be computed using perturbative techniques for Z, and is referred to as the *local anomaly.* Consequently, one can use bordism computations to assess what the group of possible anomalies of a QFT is, and whether a specific anomaly field theory is trivializable; see [**FH21a, TY21, DDHM22, LY22, DY22, Tac22, DOS23**] for recent anomaly cancellation theorems in string and supergravity theories using this technique.

3.2.1. *Anomalies for the* $\mathrm{E}_8 \times \mathrm{E}_8$ *heterotic string.* For the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, the anomaly field theory is an element of the group $[MT\xi^{\mathrm{het}}, \Sigma^{12} I_{\mathbb{Z}}]$: the free part is noncanonically isomorphic to the free part of $\Omega_{12}^{\xi^{\mathrm{het}}}$, and the torsion part is noncanonically isomorphic to the torsion subgroup of $\Omega_{11}^{\xi^{\mathrm{het}}}$. Though we have not completely determined these groups, $\Omega_{11}^{\xi^{\mathrm{het}}}$ is nonzero, as we showed in Theorem 2.62, so there is the possibility of a nontrivial anomaly to cancel. One generally expects that the anomaly field theory itself is trivial, because physicists have undertaken many consistency checks on $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory, but sometimes there is a surprise: in joint work with Dierigl, Heckman, and Montero [**DDHM22**], we found that the anomaly theory for the duality symmetry in type IIB string theory is nonzero, and requires a modification of the theory to be trivialized.

For the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, there has been a fair amount of work already cancelling the anomaly in special cases, but for the full tangential structure ξ^{het}, the question of anomaly cancellation is open. The original work of Green-Schwarz [**GS84**] shows that the anomaly polynomial vanishes, so by (3.16), we only need to look at bordism invariants out of $\Omega_{11}^{\xi^{\mathrm{het}}}$. If one ignores the $\mathbb{Z}/2$ swapping symmetry, the anomaly is known to be trivial: Witten [**Wit86**, §4] showed that the global anomaly is classified by a bordism invariant $\Omega_{11}^{\mathrm{Spin}}(B\mathrm{E}_8) \to \mathbb{C}^\times$, and Stong [**Sto86**] showed that $\Omega_{11}^{\mathrm{Spin}}(B\mathrm{E}_8) = 0$ (see Remark 2.34). Sati [**Sat11a**] studies a closely related question in terms of $\Omega_{11}^{\mathrm{String}}(B\mathrm{E}_8)$.

Recent work of Tachikawa-Yamashita [**TY21**] (see also Tachikawa [**Tac22**] and Yonekura [**Yon22**, §4]) cancels anomalies in a large class of compactifications of heterotic string theory using an ingenious *TMF*-based argument. Their work does not take into account the $\mathbb{Z}/2$ swapping symmetry. It would be interesting to address the full anomaly on ξ^{het}-manifolds, either by directly computing it on generators of $\Omega_{11}^{\xi^{\mathrm{het}}}$ or by adapting Tachikawa-Yonekura's argument. If this symmetry does have a nontrivial anomaly, this would have consequences for the CHL string, either requiring a modification of the theory or showing that it is inconsistent.

3.2.2. *Anomalies for the CHL string.* Anomaly cancellation for the CHL string has been studied less. In Theorems 2.90 and 2.92, we saw that $\Omega_{11}^{\xi^{\mathrm{CHL}}}$ is torsion, so the anomaly polynomial vanishes; and we saw $\Omega_{10}^{\xi^{\mathrm{CHL}}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so there is a potential for the anomaly field theory to be nontrivial, which would be interesting to check.

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