INVERTIBLE PHASES FOR MIXED SPATIAL SYMMETRIES AND THE FERMIONIC CRYSTALLINE EQUIVALENCE PRINCIPLE

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Abstract. Freed-Hopkins [FH19a] give a mathematical ansatz for classifying gapped invertible phases of matter with a spatial symmetry in terms of Borel-equivariant generalized homology. We propose a slight generalization of this ansatz to account for cases where the symmetry type mixes nontrivially with the spatial symmetry, such as crystalline phases with spin-1/2 fermions. From this ansatz, we prove as a theorem a "fermionic crystalline equivalence principle," as predicted in the physics literature. Using this and the Adams spectral sequence, we compute classifications of some classes of phases with a point group symmetry; in cases where these phases have been studied by other methods, our results agree with the literature.

Contents

0. Introduction

The classification of topological phases of matter has been the subject of intensive research in condensed-matter physics and nearby areas of mathematics for the last decade, but difficult problems still remain: for example, there is not yet an accepted mathematical definition of a topological phase of matter, so researchers must study these systems using ansatzes or heuristic definitions of phases. Restricting to invertible phases, also known as *symmetry-protected topological (SPT) phases*, simplifies the classification question, but defining these phases precisely is also still an open problem. Freed-Hopkins [FH16a] make an ansatz modeling SPT phases using reflection-positive invertible field theories (IFTs), then classify these IFTs using homotopy theory. This approach has been successfully employed in several cases to study examples of SPTs, as in [FH16a, Cam17, WWW18, FHHT20, GOP+20, PW20].

Condensed-matter physicists are also interested in invertible phases in more general settings, including invertible phases on a particular space Y, as in [Ran10], or invertible phases symmetric for a group G acting on space, such as phases on the plane which have a rotation symmetry and the examples in [SMJZ13]. These spatial symmetries are often present in real-world examples of topological phases of matter (see [WACB16, MYL+17] for one example), and can be modeled by lattice Hamiltonian systems in which the symmetry group also acts on the lattice, though again providing precise definitions is still open. In the case where G is a crystallographic group acting on $Y = \mathbb{R}^d$, these systems are called *crystalline SPT phases*. Freed-Hopkins' field-theoretic approach does not directly generalize to this setting, but there is a general ansatz of Kitaev [Kit13, Kit15] that groups of phases on Y for a fixed symmetry type should define a generalized homology theory. Freed-Hopkins [FH19a] apply this to propose a classification of invertible phases in the presence of a G-action on space using equivariant generalized homology.

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Researchers interested in computing groups of crystalline SPT phases provide *crystalline equivalence principles*, including the first such proposal of Thorngren-Else [TE18] and subsequent work in [JR17, CW18, FH19a, ZWY+20, ZYQG20]. Crystalline equivalence principles are arguments that groups of crystalline SPT phases are isomorphic to groups of ordinary SPT phases, where the symmetry type is modified. The theory is well-understood for symmetry types such as O_n and SO_n, corresponding to the physicists' notion of bosonic SPT, but for fermionic SPTs, corresponding to symmetry types such as Spin_n, Spin_n^c, $\mathrm{Pin}_n^\pm$, etc., the story is more complicated. Cheng-Wang [CW18], Zhang-Wang-Yang-Qi-Gu [ZWY+20], and Zhang-Yang-Qi-Gu [ZYQG20] study examples of fermionic crystalline SPTs, and show cases of a fermionic crystalline equivalence principle. Crucially, their work implies any fermionic crystalline equivalence principle must address fermionic phases in which the spatial symmetry mixes with fermion parity, which goes beyond the scope of Freed-Hopkins' ansatz.

The purpose of this paper is to formulate and prove such a fermionic crystalline equivalence principle (FCEP). To do so, we provide an ansatz expressing groups of invertible phases on a G-space Y in which the symmetry type can be merely locally constant over space and can mix with G, including as a special case spatial symmetries mixing with fermion parity. Given data $\mathcal{L}$ expressing this mixing and variance of the symmetry type, we define *phase homology* groups of Y, denoted $Ph_*^G(Y,\mathcal{L})$, and our ansatz predicts that the group of such invertible phases is isomorphic to $Ph_0^G(Y,\mathcal{L})$. Providing this ansatz is an additional goal of this paper, and is necessary input for our FCEP: the ansatz reexpresses the FCEP as an isomorphism between certain phase homology groups and groups of IFTs, as we state and prove in Theorem 2.8. This is the first homotopy-theoretic account of an FCEP, and to the best of our knowledge is the first fully general version of the FCEP in the literature.

As a corollary of the FCEP, the computation of phase homology groups that represent groups of point-group-equivariant fermionic phases reduces to computations of bordism groups; this paper's third goal is to make these computations in several examples, both for the purpose of testing our ansatz by comparing it to established predictions in physics, and for making additional predictions of groups of crystalline SPT phases in as yet unstudied settings. For symmetry types that have been studied before by other methods, our computations agree with the literature, bolstering our ansatz.

Now we go into a little more detail about these ansatzes and theorems. Freed-Hopkins [FH19a] formulate an ansatz for invertible phases of matter on a topological space Y equipped with an action of a compact Lie group G. First, specify the *symmetry type* of the theory as a map $\rho\colon H\to\mathrm{O}$, where $\mathrm{O}\coloneqq\varinjlim_n \mathrm{O}_n$ is the infinite orthogonal group and H is a topological group. From this data we can form a Madsen-Tillmann spectrum MTH, whose homotopy groups compute the bordism groups of manifolds with an H-structure on the tangent bundle. Let $I_\mathbb{Z}$ denote the Anderson dual of the sphere spectrum and $E\coloneqq\mathrm{Map}(MTH,\Sigma^2 I_\mathbb{Z})$.

Ansatz 0.1 (Freed-Hopkins [FH19a, Ansatz 3.3])**.** The abelian group of isomorphism classes of phases on Y equivariant for a G-symmetry that does not mix with the symmetry type H is the Borel-equivariant Borel-Moore homology group $E_{0,\mathrm{BM}}^{hG}(Y)$.

We will define equivariant Borel-Moore homology in the generality we need in Definition 1.17.

When G is trivial and $Y=\mathbb{R}^n$, the group of phases in Ansatz 0.1 is naturally isomorphic to $[MTH,\Sigma^{d+2}I_\mathbb{Z}]$, which Freed-Hopkins [FH16a] show is the classification of invertible field theories with symmetry type H.[1] When $Y=\mathbb{R}^d$ and G is a crystallographic group, this group of phases is expected to model the classification of crystalline SPT phases with this symmetry type, and indeed, Freed-Hopkins [FH19a, Example 3.5] prove a version of the bosonic crystalline equivalence principle of Thorngren-Else [TE18] as a consequence of their ansatz, matching physicists' predictions.

For fermionic phases, Ansatz 0.1 is not the full answer, and providing the full answer is a major goal of this paper. Physicists distinguish between phases with "spinless fermions" and "spin-1/2 fermions", asking how the spatial symmetry group G mixes with fermion parity. For example, one could consider phases on the plane equivariant for a C_4 rotation symmetry, and either ask that fermions' spin is unaffected by the spatial rotations, or that a full spatial rotation flips the spin on the fermion. This is reminiscent of the better-understood dichotomy of fermionic phases with a time-reversal symmetry T: one may have $T^2=1$ or

[1]This result is conditioned on a conjecture about non-topological invertible theories; at present, we have as a theorem only that the invertible TFTs are classified by the torsion subgroup of this group. This is discussed by Freed-Hopkins [FH16a, §5.4] and Freed [Fre19, Lecture 9].

T^2 equal to the fermion parity operator. These two classes of phases are modeled with different symmetry types, and similarly we use different data to model crystalline phases with spinless vs. spin-1/2 fermions.

To accommodate this mixing between the internal symmetry type H and the spatial symmetry group G, we generalize Freed-Hopkins' setup slightly using parametrized homotopy theory, considering local systems f of symmetry types over the base Y. These give rise to local systems of Thom spectra; if Y has a G-action we obtain from f a local system $\mathcal{L}'$ of Borel-equivariant Thom spectra, modeled as a functor from Y, thought of as an ∞-groupoid, to the ∞-category $\mathcal{S}p^G$ of Borel-equivariant spectra. Let $\mathcal{L} \coloneqq \mathrm{Map}(-, \Sigma^2 I_{\mathbb{Z}}) \circ \mathcal{L}'$ as maps $Y \to \mathcal{S}p^G$, where $I_{\mathbb{Z}}$ has trivial G-action. We define the *equivariant phase homology* $Ph_*^G(Y; f)$ to be the equivariant Borel-Moore homology of the local system $\mathcal{L}\colon Y \to \mathcal{S}p^G$.

Ansatz 1.22. The group of G-equivariant invertible phases on Y for this data is isomorphic to the equivariant phase homology group $Ph_0^G(Y; f)$.

When f is trivializable, this reduces to Ansatz 0.1; in general, though, it allows the symmetry type to mix with the spatial symmetry, or to be merely locally constant on Y.

Now we specialize to the cases of spinless and spin-1/2 fermions. For spinless fermions, G and H do not mix, so we use the data of a constant local system of symmetry types and recover Freed-Hopkins' original ansatz. For spin-1/2 fermions, we specify data of an extension of G by H

$$0 \longrightarrow H \longrightarrow \widetilde{H} \longrightarrow G \longrightarrow 0, \tag{0.2}$$

together with a representation $\lambda\colon G \to \mathrm{O}_d$ dictating how G acts on space.[2] In the cases we consider in this paper, $H = \mathrm{Spin}$ or $H = \mathrm{Spin}^c$, and we specify $\widetilde{H}$ by way of the central extension

$$0 \longrightarrow \mu_2 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0 \tag{0.3}$$

whose isomorphism class is picked out by $w_2(V_\lambda) + w_1(V_\lambda)^2 \in H^2(BG; \mu_2)$, where $V_\lambda \to BG$ is the associated vector bundle to the representation λ and μ_2 is the group of square roots of unity. Then, $\widetilde{H} \coloneqq H \times_{\mu_2} \widetilde{G}$. Using this data, we build an equivariant local system f of symmetry types, obtaining a phase homology group $Ph_0^G(\mathbb{R}^d, f)$ that we predict is isomorphic to the group of invertible phases for this data.

The FCEP, previously studied in special cases by [CW18, TE18, ZWY+20, ZYQG20], identifies groups of crystalline SPT phases with groups of fermionic SPT phases with an internal G-symmetry — but exchanging symmetry types: spinless crystalline phases correspond to spin-1/2 internal phases, and vice versa. Freed-Hopkins [FH16a] model groups of SPT phases with an internal G-symmetry using IFTs, and following Freed-Hopkins [FH16a] and the excellent overview by Beaudry-Campbell [BC18], these groups of TFTs can be expressed in terms of bordism groups of certain Thom spectra. Standard techniques in algebraic topology, notably the Adams spectral sequence over $\mathcal{A}(1)$, can be used to compute these bordism groups, so one application of a general version of the FCEP is to provide access to tractable tools for computing groups of crystalline SPT phases.

One of the major aims of this paper is to state and prove as a theorem a version of the FCEP, identifying phase homology groups with groups of IFTs; then Ansatz 1.22 translates this into a statement about crystalline SPTs and ordinary SPTs. In Definitions 2.3 and 2.4, we define the symmetry types for spinless and spin-1/2 fermions for a purely internal G-symmetry. In general these definitions are a little technical, but when the spatial representation λ factors through $\mathrm{SO}_d \subset \mathrm{O}_d$, the spinless internal symmetry type is $H \times G \to \mathrm{O}$ and the spin-1/2 symmetry type is $H \times_{\mu_2} \widetilde{G} \to \mathrm{O}$, with the maps induced by the projection onto the first factor.

Theorem 2.8 (Fermionic crystalline equivalence principle)**.** *Fixing data of G, H, λ, etc. as above, let $f_0, f_{1/2}$ denote the local systems of symmetry types for the case of spinless, resp. spin-1/2 fermions. Then $Ph_0^G(\mathbb{R}^d; f_0)$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spin-1/2 internal symmetry type, and $Ph_0^G(\mathbb{R}^d, f_{1/2})$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spinless internal symmetry type.*

The proof has two key steps.

(1) Phase homology groups are defined using equivariant parametrized homotopy theory. Proposition 1.32 reexpresses them using ordinary homotopy theory, as homotopy groups of a Thom spectrum

[2] We also specify some additional data; see Data 2.1 in §2 for the full details.

built from a virtual vector bundle over $B\widetilde{H}$. The proof uses the Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14a, ABG+14b] approach to Thom spectra.

(2) Then, in Theorems 2.11 and 2.24, we "shear" this Thom spectrum, writing down a map $\widetilde{H}_n \to H_{n+d} \times G$ and showing that it induces a homotopy equivalence on Thom spectra, implying that phase homology groups are determined by H-bordism groups of a Thom spectrum over BG. Our proof is modeled on a fairly general shearing theorem in Freed-Hopkins [FH16a, §10].

After these two steps, the proof of Theorem 2.8 amounts to looking at the Thom spectra for the internal symmetry types and noticing that we end up with equivalent Thom spectra over BG in the cases we want to equate.

With this tool in hand, we can compute phase homology groups for point groups acting on $\mathbb{R}^d$, which are our model for groups of fermionic phases equivariant for point group symmetries. We do these computations for many 2d and 3d point groups, for both spinless and spin-1/2 fermions, and in Altland-Zirnbauer classes D and A (corresponding to $H = \mathrm{Spin}$, resp. Spin^c). Our computations use two avatars of the Adams spectral sequence. It is well-known that low-dimensional spin bordism can be computed using connective *ko*-homology and the Adams spectral sequence over $\mathcal{A}(1)$, and there is an excellent introduction to this technique by Beaudry-Campbell [BC18], but we also use a variant, computing spin^c bordism via *ku*-homology and the Adams spectral sequence over $\mathcal{E}(1)$, e.g. in §4.4.3. This is hardly a new idea, but there appear to be no examples of this specific kind of computation in the literature before now. We hope that our computations serve as useful examples of how to use this version of the Adams spectral sequence for spin^c bordism; this could be of independent interest.

For 2d point groups, these phases have been studied in the physics literature using very different methods. We compare our results with those of other researchers in §4.1.4, §4.2.4, §4.3.4, and §4.4.5, and find agreement, providing evidence in favor of Freed-Hopkins' ansatz and our generalization. However, there is not yet work on fermionic crystalline SPT phases for most 3d point groups, so our computations are predictions. We do many computations and make many predictions, and in §3.1 we collect a few that we think are relatively interesting or accessible. For example:

Theorem. *Let A_4 act on $\mathbb{R}^3$ as the orientation-preserving symmetries of a tetrahedron. Then $Ph_0^{A_4}(\mathbb{R}^3; f)$ vanishes, where f is the local system of symmetry types for either spinless or spin-1/2 fermions in both Altland-Zirnbauer classes D and A.*

This is a combination of Theorems 5.4, 5.6, and 5.8. Therefore, assuming Ansatz 1.22, there are no nontrivial spinless nor spin-1/2 fermionic SPT phases equivariant for a chiral tetrahedral symmetry in Altland-Zirnbauer classes D or A. It would be interesting to see this prediction studied using lattice methods for fermionic crystalline phases.

In §6, we leave behind the FCEP and consider a different class of examples, SPTs equivariant for a glide reflection symmetry, providing a test for Freed-Hopkins' ansatz for a crystallographic group that is not a point group. Lu-Shi-Lu [LSL17] conjecture a general classification of these SPTs: that if $TP_d(H)$ denotes the group of d-dimensional SPT phases with symmetry type H, then the group of d-dimensional glide SPTs is isomorphic to $TP_{d-1}(H) \otimes \mathbb{Z}/2$. Xiong-Alexandradinata [XA18] derive this classification using physics-based arguments. We use Freed-Hopkins' ansatzes [FH16a, FH19a] to translate Lu-Shi-Lu's conjecture into a statement about phase homology groups and prove it.

Recall $E := \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$ and let $\widehat{Ph}_*^{\mathbb{Z}}(\mathbb{R}^d, \underline{E})$ denote the kernel of the forgetful map from $\mathbb{Z}$-equivariant phase homology to nonequivariant phase homology, where $\mathbb{Z}$ acts on $\mathbb{R}^d$ by glide translations, and $\underline{E} \to \mathbb{R}^d$ is the constant local system. This kernel models Lu-Shi-Lu's group of glide SPTs, as they require glide SPTs to be trivial in the absence of the glide symmetry.

Theorem 6.4. *There is a natural isomorphism $\widehat{Ph}_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$.*

This provides additional evidence in favor of the ansatz.

We want to mention that there are other homotopy-theoretic approaches to the study of phases of matter with a spatial symmetry, including those of Antolín Camarena, Sheinbaum, and collaborators [AACSS16, SC20] and Cornfeld-Carmeli [CC21]. These authors deal with free fermion phases, which are out of scope of this paper, though see §7.1.

0.1. **Reader's guide to the different sections.** Overview:

- In §§1–2 we discuss general aspects of our model for phases on a G-space Y and prove the FCEP. These sections involve the most homotopy theory.
- In §§3–5 we make phase homology calculations which according to Ansatz 1.22 calculate groups of fermionic crystalline SPT phases for which the symmetry group is a point group. We collect the results of these computations in Tables 1, 2, 3, 4, 5, and 6, and summarize the methods of computation in §3.2.
- In §6 we consider phases on $\mathbb{R}^d$ with a glide symmetry, and prove a theorem computing the corresponding phase homology classification.

Now a little more detail. In §1, we use Borel-equivariant parametrized homotopy theory to state a mild generalization of Freed-Hopkins' ansatz on invertible phases with spatial symmetry. In §1.1, we consider phases on a space Y without a group action, using local systems of symmetry types (Definition 1.3). We define phase homology and in Ansatz 1.10 express the group of invertible phases for such a local system in terms of phase homology. This is a slight generalization of [FH19a, Ansatz 2.1]. In §1.2, we allow group actions, defining equivariant local systems of symmetry types and equivariant Borel-Moore homology for a local system for the purpose of formulating Ansatz 1.22 expressing groups of invertible phases for a spatial symmetry in terms of equivariant phase homology. This is a minor generalization of Freed-Hopkins' ansatz [FH19a, Ansatz 3.3] to the parametrized setting. Then, in §1.3, we specialize to the case relevant to the FCEP, defining the local systems of symmetry types for spatial symmetries that mix with fermion parity. We prove Proposition 1.32 expressing the phase homology groups for this data in terms of nonequivariant, nonparametrized homotopy theory, and do not need equivariant or parametrized homotopy theory in the rest of the paper.

Next, §2, whose goal is to state and prove the FCEP. We begin in Definitions 2.2, 2.3, and 2.4 by defining the spinless and spin-1/2 local systems of symmetry types for both equivariant (i.e. G acting on space) and internal (G not acting on space) symmetries, and use these definitions to state our FCEP theorem in Theorem 2.8, identifying phase homology groups for these local systems in terms of groups of IFTs. As mentioned, the nontrivial part of the proof runs a shearing argument to simplify a Thom spectrum over $B\widetilde{H}$ into a smash product of MTSpin and a Thom spectrum over BG. In §2.1, we prove Theorem 2.11 accomplishing this in class D, for which $H = \mathrm{Spin}$. Then, in §2.2, we prove Theorem 2.24, which is the analogous theorem in class A, i.e. for $H = \mathrm{Spin}^c$, via a similar proof. Finally, in §2.3, we combine these arguments to prove Theorem 2.8.

In §3, we address a few generalities related to the FCEP before studying it in examples. First, in §3.1, we provide a summary of some phases or phenomena newly predicted by our computations which might be interesting to investigate further. In §3.2, we introduce and review the tools from algebraic topology we need to make these computations: the Adams and Atiyah-Hirzebruch spectral sequences. In §3.3, we discuss how to use the Adams filtration to detect when an invertible TFT of $\widetilde{H}$-manifolds only depends on the underlying $\mathrm{SO} \times G$-structure, which is believed to correspond to detecting which fermionic phases are really bosonic phases that are fermionic in a trivial way. Finally, in §3.4, we state and prove several lemmas needed in the computations in the next sections.

Then, in §§4–5, we implement this in examples, computing phase homology groups of $\mathbb{R}^d$ equivariant for two- and three-dimensional point-group symmetries, which in Ansatz 1.22 are interpreted as groups of point group equivariant fermionic phases on $\mathbb{R}^d$. In all cases we consider Altland-Zirnbauer classes D and A (corresponding to symmetry types spin and spin^c, respectively), and consider phases with spinless fermions and spin-1/2 fermions. These computations amount to computing spin and spin^c bordism groups of Thom spectra of vector bundles over BG, where G is the point group of interest; we use the Adams and Atiyah-Hirzebruch spectral sequences to determine these bordism groups.

In §4, we consider $\mathbb{Z}/2$ acting by a reflection (§4.1) and by an inversion (§4.2), as well as C_n acting by rotations (§4.3) and D_{2n} acting by rotations and reflections on $\mathbb{R}^2$ (§4.4) or purely by rotations on $\mathbb{R}^3$ (§4.5). The results of these computations can be found in Tables 1, 2, 3, 4, and 5. Most of these symmetry types have been studied in the physics literature, and we compare our results with other researchers'.

In §5, we study many 3d point groups, including chiral tetrahedral symmetry (§5.1), pyritohedral symmetry (§5.2), full tetrahedral symmetry (§5.3), chiral octahedral symmetry (§5.4), full octahedral symmetry (§5.5), chiral icosahedral symmetry (§5.6), and full icosahedral symmetry (§5.7). In all cases, we study phases with spinless and spin-1/2 fermions in Altland-Zirnbauer types D and A. Our predictions in this section are new as far as we can determine. See Table 6 for the results of the computations.

In §6, we discuss phases equivariant for a glide reflection symmetry. Lu-Shi-Lu [LSL17] conjecture a general classification of such phases, and we translate their conjecture into a statement on phase homology groups using Freed-Hopkins' ansatz, then prove that statement. Finally, in §7, we suggest some directions for further research.

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1. Phases on a G-space: the general principle

We reprise the ansatz of Freed-Hopkins [FH19a, Ansatzes 2.1, 3.3] on invertible phases on a G-space, though we need to generalize it: physicists often consider crystalline phases in which the symmetry acting on spacetime mixes with the internal symmetry (e.g. a reflection squaring to $(-1)^F$), leading us to generalize from homology to twisted homology.

What we do not do is define a phase of matter. Precisely defining topological phases of matter, even in the absence of spatial symmetries, is a difficult open question. Our ansatz is a heuristic that these objects can be classified with what we call *phase homology*, which we do define.

1.1. Invertible phases on a space.

Let Y be a locally compact topological space and $\mathcal{C}$ an ∞-category.[3] Following Ando-Blumberg-Gepner [ABG10, ABG18], we say a $\mathcal{C}$*-valued local system* on Y is a functor $\mathcal{L}\colon \pi_{\le\infty}Y \to \mathcal{C}$ here $\pi_{\le\infty}Y$ is the fundamental ∞-groupoid of Y.[4] If $\mathcal{L}\colon Y \to \mathcal{S}p$ is a local system of spectra, the homology of Y valued in $\mathcal{L}$ is $\mathcal{L}_*(Y) \coloneqq \pi_*(\operatorname{hocolim}\mathcal{L})$, and the cohomology of Y valued in $\mathcal{L}$ is $\mathcal{L}^*(Y) \coloneqq \pi_*(\operatorname{holim}\mathcal{L})$; this generalizes (co)homology with local coefficients.

Given a subspace $j\colon Y' \hookrightarrow Y$, we also define relative homology groups: j induces a map $j_*\colon \operatorname{hocolim}_{Y'}\mathcal{L}|_{Y'} \to \operatorname{hocolim}_Y \mathcal{L}$, and we define $\mathcal{L}(Y, Y') \coloneqq \pi_*(\operatorname{cofib}(j_*))$. Relative cohomology is analogous.

Definition 1.1. Assume that the one-point compactification $\overline{Y}$ of Y is a finite CW complex and $\mathcal{L}$ extends to a local system $\overline{\mathcal{L}}\colon \overline{Y} \to \mathcal{S}p$. Choose such an extension $\overline{\mathcal{L}}$ over the basepoint $*$. The *Borel-Moore homology* of Y valued in $\mathcal{L}$ is

$$\mathcal{L}_{\mathrm{BM},*}(Y) \coloneqq \overline{\mathcal{L}}_*(\overline{Y}, *). \tag{1.2}$$

Definition 1.1 appears to depend on the choice of extension of $\mathcal{L}$ to $\overline{Y}$, but given two choices of extension, the cofibers of the induced maps $\operatorname{hocolim}\overline{\mathcal{L}}|_* \to \operatorname{hocolim}\overline{\mathcal{L}}$ are equivalent, hence compute the same Borel-Moore homology groups.

When $\mathcal{L}$ is constant, this recovers the usual notion of Borel-Moore (generalized) homology [BM60, Mil95].

Recall that a *symmetry type* is a space B with a map $f\colon B \to B\mathrm{O}$.

Definition 1.3. A *local system of symmetry types* over the space Y is a local system on Y valued in the ∞-category of spaces with a map to $B\mathrm{O}$.

This is closely related to Raptis-Steimle's definition of parametrized tangential structures [RS17, §2].

Symmetry types often arise as the stabilizations in n of maps $B\rho_n\colon BH_n \to B\mathrm{O}_n$ induced from representations $\rho_n\colon H_n \to \mathrm{O}_n$; see [FH16a, §2] for a general discussion. Likewise, the local systems of symmetry types we consider arise from BH-bundles over Y.

We repeatedly use the notion of *Thom spectra*; the definition given by Freed-Hopkins [FH16a, §6.1.4] covers the cases we need.[5]

[3]There are different definitions of ∞-categories; we work with *quasicategories* as developed by Joyal [Joy02] and Lurie [Lur09], so as to follow [ABG10, ABG18]. However, this paper does not depend on implementation-specific details. See [ABG18, §2] for more information and some useful references.

[4]This is not the only approach to parametrized homotopy theory; see also May-Sigurdsson [MS06] and Braunack-Mayer [BM19].

[5]Thom spectra have been heavily studied in homotopy theory; key references include Thom [Tho54], Atiyah [Ati61], May-Quinn-Ray-Tornehave [May77], and Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14a, ABG+14b].

Definition 1.4. Given a representation $\rho_n\colon H_n \to \mathrm{O}_n$ or $\rho\colon H \to \mathrm{O}$, where $\mathrm{O} \coloneqq \varinjlim_n \mathrm{O}_n$, we introduce notation for several Thom spectra. Let $V_n \to B\mathrm{O}_n$ and $V \to B\mathrm{O}$ denote the tautological vector bundle, resp. the tautological stable vector bundle. By convention, $V \to B\mathrm{O}$ has rank zero.

(1) The Thom spectra MH_n, resp. MH, are the Thom spectra of $(B\rho_n)^*V_n \to BH_n$, resp. $(B\rho)^*V \to BH$.
(2) The *Madsen-Tillmann spectra* [MT01, MW07] MTH_n, resp. MTH, are the Thom spectra of $(B\rho_n)^*(-V) \to BH_n$, resp. $(B\rho)^*(-V) \to BH$.

We will use $H \in \{\mathrm{O}, \mathrm{SO}, \mathrm{Spin}, \mathrm{Spin}^c, \mathrm{Pin}^\pm, \mathrm{Pin}^c\}$; in all of these cases, ρ is the usual map $H \to \mathrm{O}$ used in, e.g., [FH16a].

Remark 1.5. Some Thom spectra go by many names. The notation $\mathbb{RP}^\infty_n$ denotes $(B\mathrm{O}_1)^{nV_1}$, and similarly $\mathbb{CP}^\infty_n \coloneqq (B\mathrm{SO}_2)^{nV_2}$. Thus, for example, $\Sigma^2 MTSO_2$, $\Sigma^2 MTU_1$, and $\Sigma^2\mathbb{CP}^\infty_{-1}$ all refer to $(B\mathrm{SO}_2)^{2-V_2}$.

Definition 1.6. The *Anderson dual of the sphere spectrum* [And69, Yos75] is a spectrum $I_\mathbb{Z}$ satisfying the universal property that for any spectrum X, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{n-1}(X), \mathbb{Z}) \longrightarrow [X, \Sigma^n I_\mathbb{Z}] \longrightarrow \mathrm{Hom}(\pi_n(X), \mathbb{Z}) \longrightarrow 0. \tag{1.7}$$

As all such spectra are equivalent, we refer to "the" Anderson dual of the sphere spectrum to mean any particular choice of $I_\mathbb{Z}$.

(1.7) splits, but not naturally, implying a non-natural isomorphism from $[X, \Sigma^n I_\mathbb{Z}]$ to the direct sum of the torsion summand of $\pi_{n-1}(X)$ and the free summand of $\pi_n(X)$. We often use this fact implicitly, calculating $\pi_*(X)$ but depending on the reader to rearrange it into $[X, \Sigma^* I_\mathbb{Z}]$. For more on $I_\mathbb{Z}$ and its appearance in this context, see Freed-Hopkins [FH16a, §5.3, §5.4].

Let $\mathrm{Th}\colon \mathcal{T}op_{/B\mathrm{O}} \to \mathcal{S}p$ denote the Thom spectrum functor and $\mathcal{I}\colon \mathcal{S}p^{\mathrm{op}} \to \mathcal{S}p$ denote the functor $\mathrm{Map}(-, \Sigma^2 I_\mathbb{Z})$.

Definition 1.8. Let Y be a locally compact space and $f\colon Y \to \mathcal{T}op_{/B\mathrm{O}}$ be a parametrized symmetry type on Y. The *phase homology* of this data, denoted $Ph_*(Y; f)$, is the Borel-Moore homology

$$Ph_*(Y; f) \coloneqq (\mathcal{I} \circ \mathrm{Th} \circ f)_{\mathrm{BM},*}(Y). \tag{1.9}$$

Ansatz 1.10. With Y and f as in Definition 1.8, the group of invertible topological phases on Y for the local system of symmetry types f is the phase homology group $Ph_0(Y; f)$.

Again, this is not a mathematical definition, but rather a heuristic.

Remark 1.11. When f is constant, Ansatz 1.10 is the original ansatz of Freed-Hopkins [FH19a, Ansatz 2.1]. In that case, the ansatz builds on the idea that invertible phases on Y are related to families of reflection-positive invertible field theories on Y. The generalization to nonconstant f allows one to prescribe how the symmetry type of the family varies along Y. For example, one might want to consider families of phases in which the monodromy around a loop in Y acts by orientation reversal.

1.2. Invertible phases on a G-space.

Our model for invertible crystalline phases requires considering the case where a compact Lie group G acts on Y. Again we closely follow Freed-Hopkins [FH19a, §3] but using twisted Borel-Moore homology.

Throughout this section, G is a Lie group; unlike in [FH16a, FH19a], we do not need G to be compact. Indeed, in the study of crystalline phases, G is often an infinite discrete subgroup of $\mathrm{Isom}(\mathbb{E}^n)$, and we will consider one such example in §6. We work with the ∞-category $\mathcal{S}p^G$ of *Borel G-equivariant spectra*, whose objects can be modeled by data of a sequence of G-spaces X_n together with G-equivariant maps $\Sigma X_n \to X_{n+1}$.[6] Notions of homotopy equivalence, etc., are as in [FH16a, 6.1], and do not require their compactness assumption on G.

[6] There are a few different notions of G-spectra in the equivariant homotopy theory literature, and their names can be confusing. Borel G-equivariant spectra can be thought of as "spectra with a G-action" or "spectra living over BG," and are different from *genuine G-spectra*, which have a richer structure. To a geometer, "equivariant (generalized) cohomology" usually means the Borel theory, but to a homotopy theorist, it means the genuine theory. See [Sul20, §2.1] for a detailed introduction into the different names and notions of G-spaces and G-spectra.

Definition 1.12. Suppose G admits a finite-dimensional, real orthogonal representation $\lambda\colon G\to \mathrm{O}_d$. The one-point compactification of $\mathbb{R}^d$ with this G-action is a G-space denoted S^λ and called a *representation sphere*.

The suspension functor $\Sigma^\lambda \coloneqq S^\lambda\wedge -$ is not invertible in G-spaces, but upon stabilization is invertible in Borel G-spectra; we denote its inverse by $\Sigma^{-\lambda}$. Given a virtual G-representation $V=\lambda-\mu$ (i.e. a formal difference of two finite-dimensional real orthogonal representations), we define the Borel G-spectrum $\mathbb{S}^V \coloneqq \Sigma^{-\mu}\Sigma^\infty S^\lambda$. We will let $\mathbb{S}$ denote the sphere spectrum with trivial G-action.

Definition 1.13. Let Y be a G-space and $\mathcal{L}\colon Y\to \mathcal{S}p^G$ be a local system. The *(Borel-)equivariant homology* of Y with respect to $\mathcal{L}$ is denoted $\mathcal{L}_*^G(Y)$ and defined to be

$$\mathcal{L}_*^G(Y) \coloneqq \pi_*(\mathrm{Map}_{\mathcal{S}p^G}(\mathbb{S}, \operatorname{hocolim}\mathcal{L})^{hG}), \tag{1.14}$$

where $(-)^{hG}\colon \mathcal{S}p^G\to\mathcal{S}p$ denotes the homotopy fixed-points functor.

If $j\colon Y'\hookrightarrow Y$ is an inclusion of G-spaces, it induces a map

$$j_*\colon \mathrm{Map}_{\mathcal{S}p^G}(\mathbb{S}, \operatorname*{hocolim}_{Y'}\mathcal{L}|_{Y'})^{hG}\longrightarrow \mathrm{Map}_{\mathcal{S}p^G}(\mathbb{S}, \operatorname*{hocolim}_{Y}\mathcal{L})^{hG}, \tag{1.15}$$

and we define the *relative (Borel-)equivariant homology*

$$\mathcal{L}_*^G(Y,Y') \coloneqq \pi_*(\mathrm{cofib}(j_*)) \tag{1.16}$$

as in the nonequivariant case.

Definition 1.17. Let Y be a G-space and $\mathcal{L}\colon Y\to\mathcal{S}p^G$ be an $\mathcal{S}p^G$-valued local system. Assume that the one-point compactification $\overline{Y}$ of Y is a CW complex and $\mathcal{L}$ extends to a local system $\overline{\mathcal{L}}\colon \overline{Y}\to\mathcal{S}p^G$. Choose such an extension $\overline{\mathcal{L}}$. The *equivariant Borel-Moore homology* of Y valued in $\mathcal{L}$ is

$$\mathcal{L}^G_{\mathrm{BM},*}(Y) \coloneqq \overline{\mathcal{L}}_*^G(\overline{Y},*). \tag{1.18}$$

Just like Definition 1.1, this does not actually depend on the choice of extension.

Definition 1.19. Let Y be a G-space. A *G-equivariant local system of symmetry types* is a G-space B and a G-equivariant map $f\colon B\to Y\times B\mathrm{O}$, where $B\mathrm{O}$ has a trivial G-action.

Taking the Thom spectrum of the map to $B\mathrm{O}$ defines a local system $\mathrm{Th}\circ f\colon Y\to\mathcal{S}p^G$.

Definition 1.20. Let Y be a G-space whose one-point compactification is a finite CW complex, and let $f\colon B\to Y\times B\mathrm{O}$ be a G-equivariant local system of symmetry types for Y. The *G-equivariant phase homology* of this data, denoted $Ph_*^G(Y;f)$, is the equivariant Borel-Moore homology

$$Ph_*^G(Y;f) \coloneqq (\mathcal{I}\circ\mathrm{Th}\circ f)^G_{\mathrm{BM},0}(Y). \tag{1.21}$$

Ansatz 1.22. With Y and f as in Definition 1.20, the group of invertible topological phases on Y for the equivariant local system of symmetry types f is the G-equivariant phase homology group $Ph_0^G(Y;f)$.

Again, this is a heuristic and not a definition. When G is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ (e.g. a wallpaper or space group) acting on $Y=\mathbb{E}^n$, these phases are called *crystalline SPT phases* in the physics literature.

1.3. **Mixing internal and crystalline symmetries.** The fermionic crystalline equivalence principle is about invertible topological phases in which an internal symmetry mixes with the symmetry group acting on space. In this section, we construct the equivariant local systems of symmetry types for these phases. First, we review how mixing of symmetries is handled in the purely internal case in Example 1.23; then we address the case of spatial symmetries in Proposition 1.32, showing how to reduce the computation of the relevant equivariant phase homology groups to a nonparametrized question. We will simplify these computations further in §2 when we discuss the FCEP in more detail, then study several examples in §§4–5.

Example 1.23 (Mixing for internal symmetries)**.** In the study of SPTs, one commonly encounters symmetry types where there are two different symmetries present, such as time reversal and fermion parity, but they mix, meaning the group they generate is not a product of the individual symmetry groups, but rather an extension. For example, we could ask for a generator T of the group of time-reversal symmetries to square to the fermion parity $(-1)^F$, via the extension $0\to\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/2\to 0$, rather than considering phases where $T^2=1$, corresponding to the split extension $0\to\mathbb{Z}/2\to\mathbb{Z}/2\times\mathbb{Z}/2\to\mathbb{Z}/2\to 0$.

Freed-Hopkins [FH16a] make the ansatz that SPT phases are classified up to equivalence by their low-energy limits, which are invertible field theories. The symmetry type is expressed as an H_n-structure, where H_n is a group with a map to O_n; mixing manifests as an extension involving the base symmetry type (e.g. Spin_n for fermionic phases) and the additional symmetry. For example, the two cases of time-reversal symmetry squaring to the identity or to fermion parity are represented by the extensions

$$1 \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{Pin}_n^+ \longrightarrow \mathbb{Z}/2 \longrightarrow 1 \tag{1.24a}$$

$$1 \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{Pin}_n^- \longrightarrow \mathbb{Z}/2 \longrightarrow 1, \tag{1.24b}$$

respectively, together with the standard maps $\mathrm{Pin}_n^\pm \to \mathrm{O}_n$.

When one of the groups we want to mix acts on space, we can specify a mixed symmetry type by the following data:

- a symmetry type $\rho_n\colon H_n \to \mathrm{O}_n$, called the *base symmetry type*,
- the point group symmetry $\lambda\colon G \to \mathrm{O}_d$,
- an extension

$$1 \longrightarrow H_n \longrightarrow \widetilde{H}_n \longrightarrow G \longrightarrow 1 \tag{1.25}$$

  specifying how they mix, and
- an extension $\widetilde{\rho}_n\colon \widetilde{H}_n \to \mathrm{O}_n$ of $\rho_n\colon H_n \to \mathrm{O}_n$.

Freed-Hopkins [FH16a, §9.2] relate Altland-Zirnbauer's symmetry classes of condensed-matter systems [Zir96, AZ97] to ten symmetry types in topology.[7] Using this, we call the case $H = \mathrm{Spin}$ the *class D case* and $H = \mathrm{Spin}^c$ the *class A case*.

Let Y be a G-space. Then the map

$$Y \times E\widetilde{H}_n/H_n \longrightarrow Y \tag{1.26}$$

is a G-equivariant fiber bundle with fiber BH_n, and the total space maps to $B\mathrm{O}_n$ as specified by the virtual vector bundle

$$f\colon -(Y \times (E\widetilde{H}_n \times_{H_n} \mathbb{R}^n)) \longrightarrow Y \times E\widetilde{H}_n/H_n. \tag{1.27}$$

After stabilizing (i.e. letting $n \to \infty$), this is an equivariant local system of symmetry types over Y, so has equivariant phase homology groups $Ph_*^G(Y; f)$. Under Ansatz 1.22, $Ph_0^G(Y; f)$ models the group of invertible topological phases on Y in which fermion parity mixes with the spatial symmetry as specified by (1.25). The notion of G-equivariant phases for this symmetry type (without a reference space Y) is taken to mean G-equivariant phases on $\mathbb{R}^d$, where G acts on $\mathbb{R}^d$ through λ.

Remark 1.28 (Change of symmetry type). We would like to be able to move information between instances of this data: for example, there should be forgetful maps from equivariant phases on a space to nonequivariant ones, and we model them with maps between phase homology groups for the two local systems of symmetry types.

Suppose we are given two instances of the data above. That is, we ask for a commutative diagram of Lie groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_n & \longrightarrow & \widetilde{H}_n & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\widetilde{\varphi}} & & \big\downarrow{\scriptstyle\varphi_G} & & \\ 1 & \longrightarrow & H_n' & \longrightarrow & \widetilde{H}_n' & \longrightarrow & G' & \longrightarrow & 1 \end{array} \tag{1.29}$$

together with maps $\rho_n\colon H_n \to \mathrm{O}_n$ and $\rho_n'\colon H_n \to \mathrm{O}_n$, $\lambda\colon G \to \mathrm{O}_d$ and $\lambda'\colon G' \to \mathrm{O}_d$, and $\widetilde{\rho}_n\colon \widetilde{H}_n \to \mathrm{O}_n$ and $\widetilde{\rho}_n'\colon \widetilde{H}_n' \to \mathrm{O}_n$ which commute with the vertical maps in (1.29). Fix a G'-space Y; then through (1.27) this defines equivariant local systems of symmetry types f for G, resp. f' for G'. The maps between the data induce a pullback or forgetful map $\varphi^*\colon Ph_*^{G'}(Y; f') \to Ph_*^G(Y; f)$, where G acts on Y through φ_G. Using

[7] This "tenfold way" is a relativistic version of Dyson's threefold way [Dys62], and appears in many contexts in physics, including [Kit09, RSFL10, FM13, WS14, FH16a, KZ16, GM20, IT20].

Ansatz 1.22, we interpret this pullback map realizing an invertible phase on Y with a G'-symmetry to a phase with a G-symmetry.

The construction of φ^* amounts to checking that diagrams you would expect to commute do in fact commute. The data we gave induces a commutative diagram

$$\begin{array}{ccc} -(Y \times (E\widetilde{H}_n \times_{H_n} \mathbb{R}^n)) & \longrightarrow & Y \times E\widetilde{H}_n/H_n \\ \downarrow & & \downarrow \\ -(Y \times (E\widetilde{H}'_n \times_{H'_n} \mathbb{R}^n)) & \longrightarrow & Y \times E\widetilde{H}'_n/H'_n. \end{array} \tag{1.30}$$

The rows define equivariant local systems symmetry types; then f and f' are the maps to $Y \times BO$. Let $\varphi^\circ \colon \mathcal{S}p^{G'} \to \mathcal{S}p^G$ be the map in which G acts on Borel G'-spectra through φ; then, upon applying $\mathcal{I} \circ \mathrm{Th}$, we obtain local systems $\mathcal{L}$, resp. $\mathcal{L}'$ of Borel G-, resp. G'-spectra. To define phase homology, we assumed that an extension $\overline{\mathcal{L}}$ of $\mathcal{L}$ to $\overline{Y}$ exists, so choose such an extension; then $\overline{\mathcal{L}}' := \overline{\mathcal{L}} \circ \varphi^\circ$ is an extension of $\mathcal{L}$. We obtain from the inclusion $* \hookrightarrow \overline{Y}$ a commutative diagram of spectra

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{S}p^{G'}}(\mathbb{S}, \operatorname{hocolim}_{*} \overline{\mathcal{L}}'|_*)^{hG'} & \longrightarrow & \mathrm{Map}_{\mathcal{S}p^{G}}(\mathbb{S}, \operatorname{hocolim}_{*} \overline{\mathcal{L}}|_*)^{hG} \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}p^{G'}}(\mathbb{S}, \operatorname{hocolim}_{\overline{Y}} \overline{\mathcal{L}}')^{hG'} & \longrightarrow & \mathrm{Map}_{\mathcal{S}p^{G}}(\mathbb{S}, \operatorname{hocolim}_{\overline{Y}} \overline{\mathcal{L}})^{hG}. \end{array} \tag{1.31}$$

Thus, we get a map between the cofibers of the vertical arrows, and π_* of that map is the desired map on phase homology.

For us there are two particularly important examples.

1. Let $H'_n = H_n$ and $G = 1$, which forces $\widetilde{\varphi} \colon \widetilde{H}_n \to \widetilde{H}'_n$ to be the inclusion $H'_n \to \widetilde{H}'_n$. The above construction produces a map from H-equivariant phase homology to nonequivariant phase homology on Y, which we interpret as modeling the forgetful map from phases with a G-symmetry to phases without a G-symmetry.
2. Let $G' = G$, $H'_n = \mathrm{SO}_n$. and H_n be either Spin_n or Spin^c_n, with φ the usual map. In this case the pullback map goes from equivariant phase homology where the base symmetry type is SO to equivariant phase homology where the base symmetry type is Spin or Spin^c. We interpret this in physics as modeling the procedure that regards a bosonic phase as a fermionic phase by adding some trivial fermionic degrees of freedom. This is analogous to the procedure which regards an oriented TFT as a spin TFT that does not depend on the spin structure.

Crucially for computations, we can simplify the equivariant phase homology groups for the symmetry types in (1.27) into a description not requiring equivariant or parametrized stable homotopy theory.

Proposition 1.32. *There is an isomorphism*

$$Ph_0^G(\mathbb{R}^d; f) \xrightarrow{\cong} [(B\widetilde{H})^{d-\lambda-\widetilde{\rho}}, \Sigma^{d+2} I_{\mathbb{Z}}] \tag{1.33}$$

natural for changing the symmetry type in the sense of Remark 1.28.

Proof. We want to compute the twisted equivariant Borel-Moore homology for this equivariant local system of symmetry types, where $Y = \mathbb{R}^d$ with G acting through λ. This amounts to the following: one-point compactify to a local system over S^λ; take the colimit of the local system and call it E; then compute $[\mathbb{S}, E]^G$ (in the notation of [FH19a]; this means $\pi_0(\mathrm{Map}(\mathbb{S}, E)^{hG})$). Now, the local system $(\mathcal{I} \circ \mathrm{Th} \circ f) \colon S^\lambda \to \mathcal{S}p^G$ is nonequivariantly the trivial local system with fiber $\mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$, so $E \simeq S^\lambda \wedge \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$; in general, G can act nontrivially on both S^λ and MTH, but always acts trivially on $\Sigma^2 I_{\mathbb{Z}}$. Therefore we may follow [FH19a, (3.6)] and identify

$$\mathrm{Map}(\mathbb{S}, S^\lambda \wedge \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})) \simeq \mathrm{Map}(\mathbb{S}^{d-\lambda} \wedge MTH, \Sigma^{d+2} I_{\mathbb{Z}}), \tag{1.34}$$

though the G-action on $\mathbb{S}^{d-\lambda} \wedge MTH$ is not the diagonal action, but rather the induced G-action on the Thom spectrum of the G-equivariant virtual bundle $(d - \underline{\lambda} - \rho) \to BH$ (see [FH16a, §6.2.2]).

Since G acts trivially on $\Sigma^{d+2}I_{\mathbb{Z}}$,

$$\mathrm{Map}(\mathbb{S}^{d-\lambda}\wedge MTH, \Sigma^{d+2}I_{\mathbb{Z}})^{hG} \simeq \mathrm{Map}((\mathbb{S}^{d-\lambda}\wedge MTH)_{hG}, \Sigma^{d+2}I_{\mathbb{Z}}). \tag{1.35}$$

It now suffices to show that

$$(\mathbb{S}^{d-\lambda}\wedge MTH)_{hG} \simeq (B\widetilde{H})^{-\widetilde{\rho}-\lambda+d}. \tag{1.36}$$

Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14a, Proposition 1.20] show that the Thom spectrum of a virtual bundle $V\to X$, identified with a map $V\colon X\to B\mathrm{O}$, is the homotopy colimit

$$X^V \simeq \mathrm{hocolim}\big(X \xrightarrow{V} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathcal{S}p\big), \tag{1.37}$$

where the notation means to interpret X as, through its fundamental ∞-groupoid, providing a diagram in the ∞-category $\mathcal{S}p$ of spectra. Here $B\mathrm{GL}_1(\mathbb{S})$ is the classifying space of stable spherical fibrations [Sta63, May77] and $BJ\colon B\mathrm{O}\to B\mathrm{GL}_1(\mathbb{S})$ is a form of the J-homomorphism [Whi42, May77]. Heuristically, (1.37) says that the virtual vector bundle V defines a local system of $\wedge$-invertible spectra, with the fiber at a point $x\in X$ given by $\mathbb{S}^{V_x}$, and that the Thom spectrum is obtained from an associated bundle construction. See [ABG+14a, ABG+14b] for more detail on this approach to Thom spectra.

Homotopy quotients are also homotopy colimits, meaning

$$(\mathbb{S}^{d-\lambda}\wedge MTH)_{hG} = \operatorname*{hocolim}_{\mathrm{pt}/G}\Big(\mathrm{hocolim}\big(BH \xrightarrow{d-\lambda-\rho} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathcal{S}p\big)\Big), \tag{1.38a}$$

where G acts on the spectra in the diagram through its action on λ, as well as on BH, as prescribed by the extension (1.25). This in particular implies the double homotopy colimit above simplifies into a single homotopy colimit over a $B\widetilde{H}$-shaped diagram:

$$\simeq \mathrm{hocolim}\Big(B\widetilde{H} \xrightarrow{d-\lambda-\widetilde{\rho}} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathcal{S}p\Big), \tag{1.38b}$$

which by (1.37) is the Thom spectrum for $d-\lambda-\widetilde{\rho}\to B\widetilde{H}$, proving (1.36). ☐

Our next step in §2 is to simplify $(B\widetilde{H})^{d-\lambda-\widetilde{\rho}}$. This allows both for a general formulation of the fermionic crystalline equivalence principle as well as explicit calculations.

The following lemma will be helpful for simplifying Thom spectra.

Theorem 1.39 (Relative Thom isomorphism)**.** *Let $\rho\colon H\to \mathrm{O}$ be a symmetry type with the two-out-of-three property, i.e. an H-structure on any two of E, F, or $E\oplus F$ induces one on the third. If $V,W\to X$ are virtual vector bundles such that V has an H-structure, then there is an equivalence*

$$MTH\wedge X^W \xrightarrow{\simeq} MTH\wedge X^{V\oplus W}. \tag{1.40}$$

Proof. The two-out-of-three property gives MTH an E_∞-ring structure, which is needed for some of the constructions we employ from [ABG+14a, ABG+14b] below.

Up to equivalence, the Thom spectrum of a virtual vector bundle $E\to X$ depends only on the homotopy class of the map $f_E\colon X\to B\mathrm{O}\to B\mathrm{GL}_1(\mathbb{S})$, where the first map is given by E, and the second map is the J-homomorphism, as in (1.37). Smashing with MTH corresponds to composing f_E with the map $B\mathrm{GL}_1(\mathbb{S})\to B\mathrm{GL}_1(MTH)$ induced by the Hurewicz map $\mathbb{S}\to MTH$ [ABG+14b, §1.4], and in particular, up to equivalence, $MTH\wedge X^E$ only depends on the homotopy type of the map $X\to B\mathrm{GL}_1(MTH)$.

Because MTH is an E_∞-ring spectrum, $B\mathrm{GL}_1(MTH)$ is a grouplike E_∞-space, and the composition $\psi\colon B\mathrm{O}\to B\mathrm{GL}_1(\mathbb{S})\to B\mathrm{GL}_1(MTH)$ is a map of grouplike E_∞-spaces, where $B\mathrm{O}$ carries the E_∞ structure coming from direct sum. This means that $[X, B\mathrm{GL}_1(MTH)]$ is naturally an abelian group, and that if we define classes in this group using virtual vector bundles $V,W\to X$ to map to $B\mathrm{O}$ then composing with ψ, the class of $E\oplus F$ is the sum of the classes of V and W.

An H-structure on V trivializes the map $X\to B\mathrm{O}\xrightarrow{\psi} B\mathrm{GL}_1(MTH)$ defined by V, so the class of the map defined by $V\oplus W$ is equal to the class of the map defined by W in the abelian group $[X, B\mathrm{GL}_1(MTH)]$. ☐

2. The fermionic crystalline equivalence principle

In this section, our goal is to state and prove the FCEP, Theorem 2.8, identifying phase homology groups in classes D and A with groups of deformation classes of invertible field theories. Assuming Ansatz 1.22, this leads to the more familiar version of the FCEP: crystalline equivalence principles are first introduced by Thorngren-Else [TE18]: the idea is to equate the classification of crystalline topological phases of matter for some group G acting on spacetime with a classification of a different kind of topological phases of matter, in which G is part of the internal symmetry group. Then one may use preexisting techniques for phases without a spatial symmetry to classify phases with the specified G-action on space.

The best-understood crystalline equivalence principles are for bosonic SPTs, as first considered by Thorngren-Else [TE18]. "Bosonic" does not have a precise mathematical translation here; these are phases for which the symmetry type is built using SO or O rather than Spin, Spin^c, $\mathrm{Pin}^\pm$, and so on. If a group G acts on space by orientation-preserving symmetries and H is SO or O, the classification of crystalline SPTs in dimension n with symmetry type H and this G-action is identified with the classification of SPTs for $H = \mathrm{SO}\times G$. To what extent this is an ansatz or a theorem depends on one's model for crystalline SPTs: Freed-Hopkins [FH19a, Example 3.5] derive it as a corollary of their ansatz.[8] For other derivations of the bosonic crystalline equivalence principle from different ansatzes, see Jiang-Ran [JR17] and Thorngren-Else [TE18, ET19].

The fermionic analogue of this statement is more complicated because there are more ways for G to mix with the symmetry type. Thorngren-Else [TE18, §VII.B], Cheng-Wang [CW18], Zhang-Wang-Yang-Qi-Gu [ZWY+20], and Zhang-Wang-Yang-Gu [ZYQG20, §V] all study examples in which an FCEP holds, and each paper discusses that such a principle would have to account for the different ways in which G mixes with H: crystalline phases for which the spatial G-symmetry does not mix with fermion parity correspond to phases with an internal G-symmetry that does mix with fermion parity, and vice versa. Examples of this twisted correspondence also appear in work of Freed-Hopkins [FH19a, Example 3.5], Guo-Ohmori-Putrov-Wan-Wang [GOP+20], and Mao-Wang [MW20], though until now there was no precise general version of the FCEP.

Our version of the FCEP applies in Altland-Zirnbauer classes A and D (i.e. $H = \mathrm{Spin}$ or $H = \mathrm{Spin}^c$), for all compact Lie groups G acting on faithfully on space, and all ways in which G may mix with fermion parity. The slogan "mixed crystalline goes to unmixed internal, and vice versa" is a little hard to glean from the result when the G-action includes reflections, but we obtain an equivalence from phase homology groups for certain equivariant local systems of symmetry types, which under Ansatz 1.22 stands in for groups of crystalline SPT phases, to groups of deformation classes of IFTs, which under Freed-Hopkins' ansatz [FH16a] model groups of phases without spatial symmetries.

To precisely state our FCEP, we must fix some data.

Data 2.1.

- Let H denote the base symmetry type, which today is either of the infinite-dimensional topological groups Spin or Spin^c.
- Let G be a compact Lie group, $\lambda\colon G\to \mathrm{O}_d$ be a faithful representation, and $V_\lambda \coloneqq EG\times_G \mathbb{R}^d \to BG$ be the associated vector bundle.
- Let $\xi\colon G\to \mathrm{O}_{d'}$ be another faithful representation and $V_\xi\to BG$ be the associated vector bundle. Let $1\to\mu_2\to\widetilde G\to G\to 1$ be the central extension classified by $w_2(V_\lambda)+w_1(V_\lambda)^2\in H^2(BG;\mu_2)$. Here μ_2 denotes the group of square roots of unity.
- Let $\widetilde H \coloneqq H\times_{\mu_2}\widetilde G$. Let ρ be the composition $\widetilde H\to H\to \mathrm{O}$ and $V\to B\widetilde H$ be the associated tautological vector bundle.

For us, ξ and λ are usually the same, but they differ when $G=\mathbb{Z}/2$ acts on $\mathbb{R}^d$ by inversion in the case of spin-1/2 fermions: here ξ is the sign representation $\sigma\colon \mathbb{Z}/2\to \mathrm{O}_1$, but $\lambda = d\sigma$. See §4.2 for more detail.

Definition 2.2. The *spin-1/2 equivariant local system of symmetry types* for the above data is the G-equivariant parametrized symmetry type $f_{1/2}\colon B\widetilde H\to\mathbb{R}^{d'}\times B\mathrm{O}$ which sends $x\mapsto(0,B\rho(x))$, and in which G acts on $\mathbb{R}^d$ through λ. The *spinless equivariant local system of symmetry types* f_0 is defined in the same way, except using $H\times G$ instead of $\widetilde H$.

[8] If G acts by reflections, almost as nice of a story is still true, but the internal G-symmetry mixes with H. Thorngren-Else [TE18] and Freed-Hopkins [FH19a, Example 3.5] discuss this case too.

Definition 2.3. Recall that H is either Spin or Spin^c. Let $\dagger \in \{-, c\}$ be $-$ if $H = \mathrm{Spin}$ and c otherwise. The *spinless internal symmetry type* is the symmetry type

- $(-V, d - V_\lambda)\colon BH \times BG \to B\mathrm{O}$, if λ is pin†, or
- $(-V, V_\xi + \mathrm{Det}(V_\xi) - V_\lambda)\colon BH \times BG \to B\mathrm{O}$, if λ is not pin†.

For shorthand, we denote this symmetry type $\rho(0)\colon BH \times BG \to B\mathrm{O}$.

Definition 2.4. The *spin*-1/2 *internal symmetry type* is the symmetry type

$$(-V, d - V_\lambda)\colon BH \times BG \to B\mathrm{O}. \tag{2.5}$$

For shorthand, we denote this symmetry type $\rho(1/2)\colon BH \times BG \to B\mathrm{O}$.

Remark 2.6. The internal symmetry types probably look pretty arbitrary. This is because of the generality of our setup: in some cases of interest, we can rewrite these symmetry types in ways which more closely resembles the proposals of Thorngren-Else [TE18, §VII.B], Cheng-Wang [CW18], and Zhang-Wang-Yang-Qi-Gu [ZWY+20] for the FCEP in specific cases.

Suppose $\lambda = \xi$ and $\mathrm{Im}(\lambda) \subset \mathrm{SO}_d$ but does not lift across $\mathrm{Spin}_d \twoheadrightarrow \mathrm{SO}_d$. Then, the spinless internal symmetry type simplifies to $BH \times BG \to B\mathrm{O}$, where the map is just projection onto the first factor followed by the usual map $BH \to B\mathrm{O}$. That is, for representations with image contained in SO_d, the FCEP switches the "unmixed" (i.e. $BH \times BG$) and "mixed" (i.e. $B(H \times_{\mu_2} \widetilde{G})$) symmetry types when passing between crystalline and internal phases. This matches predictions by Thorngren-Else [TE18] and Cheng-Wang [CW18].

Freed-Hopkins [FH16a, Corollary 8.21] show that the group of deformation classes of reflection-positive IFTs with symmetry type $\rho'\colon H' \to \mathrm{O}$ in (space) dimension n is naturally isomorphic to[9]

$$[MTH', \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{2.7}$$

Theorem 2.8 (Fermionic crystalline equivalence principle)**.** *There are isomorphisms*

$$Ph_k^G(\mathbb{R}^d; f_0) \xrightarrow{\cong} [MT\rho(1/2), \Sigma^{d+k+2} I_{\mathbb{Z}}] \tag{2.9a}$$

$$Ph_k^G(\mathbb{R}^d; f_{1/2}) \xrightarrow{\cong} [MT\rho(0), \Sigma^{d+k+2} I_{\mathbb{Z}}]. \tag{2.9b}$$

Assuming Ansatz 1.22, the physics implication of this theorem is that the abelian group of crystalline SPT phases for the spinless equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs for the spin-1/2 internal symmetry type; and the classification of crystalline SPT phases for the spin-1/2 equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs of the spinless internal symmetry type.

We break the proof of Theorem 2.8 down into a few steps. First, Proposition 1.32 simplifies the question into one of ordinary stable homotopy theory.[10] We obtain Thom spectra for vector bundles over $B\widetilde{H}$, and to finish we must compare these spectra to $MTH \wedge (BG)^E$, where $E \to BG$ is some rank-zero virtual vector bundle. This comparison, in the form of *shearing arguments*, is the core of the proof: we prove Theorem 2.11 ($H = \mathrm{Spin}$) and Theorem 2.24 ($H = \mathrm{Spin}^c$) establishing the homotopy equivalences we need, and after that proving Theorem 2.8 amounts to verifying that the outputs of Theorems 2.11 and 2.24 simplifying the crystalline symmetry types match the Thom spectra for the internal symmetry types in Definitions 2.3 and 2.4.

The proofs of Theorems 2.11 and 2.24 resemble the proofs of the more standard equivalences

$$MTPin^+ \simeq MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \tag{2.10a}$$

$$MTPin^- \simeq MTSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \tag{2.10b}$$

$$MTPin^c \simeq MTSpin^c \wedge (B\mathbb{Z}/2)^{\pm(1-\sigma)} \tag{2.10c}$$

$$MTSpin^c \simeq MTSpin \wedge (B\mathrm{SO}_2)^{\pm(2-V_2)}, \tag{2.10d}$$

[9]Strictly speaking, Freed-Hopkins' theorem classifies only the invertible *topological* field theories, which form the torsion subgroup of (2.7), and they conjecture that the entire group classifies all reflection-positive IFTs.

[10]For the spinless equivariant symmetry type, this is just [FH19a, Example 3.5].

where $\sigma \to B\mathbb{Z}/2$ and $V_2 \to B\mathrm{SO}_2$ denote the respective tautological line bundles. These decompositions were first proven by Kirby-Taylor [KT90a, Lemma 6] (pin^+), Peterson [Pet68, §7] (pin^-), and Bahri-Gilkey [BG87a, BG87b] (spin^c and pin^c). For a unified proof of all of these equivalences, see Freed-Hopkins [FH16a, §10].

2.1. Case $H = \mathrm{Spin}$.

Theorem 2.11 (Shearing, class D)**.** *Let $V \to B\widetilde{H}$ be the tautological bundle.*

(1) Suppose V_ξ admits a pin^- structure. Then there is an equivalence

$$(B\widetilde{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin \wedge (BG)^{d-V_\lambda}. \tag{2.12}$$

(2) If V_ξ does not admit a pin^- structure, there is an equivalence

$$(B\widetilde{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin \wedge (BG)^{V_\xi+\mathrm{Det}(V_\xi)-V_\lambda-d'-1+d}. \tag{2.13}$$

We will most often consider case (2) with $\lambda = \xi$, in which case we learn $(B\widetilde{H})^{d-\lambda-V} \simeq MTSpin \wedge (BG)^{\mathrm{Det}(V_\lambda)-1}$.

Proof. Case (1) is by far the easier of the two: V_ξ admits a pin^- structure iff $w_2(V_\xi) + w_1(V_\xi)^2 = 0$ iff the extension $1 \to \mu_2 \to \widetilde{G} \to G \to 1$ splits. Since $\mu_2 \subset \widetilde{G}$ is central, a splitting induces isomorphisms $\widetilde{G} \cong \mu_2 \times G$ and $\widetilde{H}_n \cong \mathrm{Spin}_n \times G$. Passing to classifying spaces, this identifies $d - V_\lambda - V \colon B\widetilde{H} \to BO$ with $-V \boxplus (d-\lambda) \colon B\mathrm{Spin} \times BG \to BO$; then take Thom spectra.

On to case (2). In this case, in $H^2(B\widetilde{H};\mu_2)$, $w_2(V_\xi)+w_1(V_\xi)^2 = w_2(V)$, so the map $V+V_\xi+\mathrm{Det}(V_\xi) \colon B\widetilde{H} \to B\mathrm{SO}$ lifts across $B\mathrm{Spin} \to B\mathrm{SO}$. Choose such a lift.

Proposition 2.14. *The induced map*

$$(V + V_\xi + \mathrm{Det}(V_\xi), \xi) \colon B\widetilde{H} \longrightarrow B\mathrm{Spin} \times BG \tag{2.15}$$

is a homotopy equivalence commuting with the maps to $B\mathrm{SO}$.

The proof is due to Freed-Hopkins [FH16a, §10].

Proof. We will show that the commutative square

$$\begin{array}{ccc} B\widetilde{H} & \longrightarrow & B\mathrm{Spin} \\ \downarrow{\scriptstyle B(\pi_1\oplus\pi_2)} & & \downarrow \\ B\mathrm{SO} \times BG & \xrightarrow{B(\mathrm{id}\oplus\xi)} & B\mathrm{SO} \end{array} \tag{2.16a}$$

is homotopy Cartesian. Any two homotopy pullbacks of the same diagram are weakly equivalent, with the weak equivalence intertwining the maps to $B\mathrm{SO}$. Since there is also a homotopy pullback square

$$\begin{array}{ccc} B\mathrm{Spin} \times BG & \longrightarrow & B\mathrm{Spin} \\ \downarrow & & \downarrow \\ B\mathrm{SO} \times BG & \xrightarrow{B(\mathrm{id}\oplus\xi)} & B\mathrm{SO}, \end{array} \tag{2.16b}$$

then $B\widetilde{H} \simeq B\mathrm{Spin} \times BG$; this equivalence is realized by (2.15) because that is the only possibility that intertwines the maps in (2.16a) and (2.16b).

To fulfill the promise that (2.16a) is a homotopy pullback square, begin with the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{H}_n & \xrightarrow{(\pi_1,\pi_2)} & \mathrm{SO}_n \times G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\oplus\xi} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}_{n+d} & \longrightarrow & \mathrm{SO}_{n+d} & \longrightarrow & 1. \end{array} \tag{2.17}$$

This induces a map of fiber sequences

$$\begin{array}{ccccc} B\widetilde{H} & \xrightarrow{B(\pi_1,\pi_2)} & B\mathrm{SO}\times BG & \xrightarrow{w_2} & K(\mu_2,2) \\ \downarrow & & \downarrow{\scriptstyle B(\mathrm{id}\oplus\xi)} & & \| \\ B\mathrm{Spin} & \longrightarrow & B\mathrm{SO} & \xrightarrow{w_2} & K(\mu_2,2), \end{array} \tag{2.18}$$

e.g. $B\widetilde{H}$ is the fiber of $w_2\colon B\mathrm{SO}\times BG\to K(\mu_2,2)$. The left square in such a pullback is always homotopy Cartesian, and in (2.18) the left square can be identified with (2.16a). ☐

Including the maps down to $B\mathrm{SO}$ produces the commutative diagram

$$\begin{array}{ccc} B\widetilde{H} & \xrightarrow[\simeq]{(V+V_\xi+\mathrm{Det}(V_\xi),\xi)} & B\mathrm{Spin}\times BG \\ & {\scriptstyle -V}\searrow \quad \swarrow{\scriptstyle -V+V_\xi+\mathrm{Det}(V_\xi)} & \\ & B\mathrm{SO}. & \end{array} \tag{2.19}$$

Taking Thom spectra of the vertical maps, the shearing map induces a homotopy equivalence

$$(B\widetilde{H})^{-V} \xrightarrow{\simeq} \mathit{MTSpin}\wedge(BG)^{V_\xi+\mathrm{Det}(V_\xi)-d'-1}. \tag{2.20}$$

To finish, we subtract V_λ from the vertical arrows in (2.19), then take Thom spectra again. ☐

2.2. **Case $H=\mathrm{Spin}^c$.** Let $\widetilde{H}_n :=\mathrm{Spin}_n^c\times_{\mu_2}\widetilde{G}$, and define $\widetilde{H}$ similarly. The shearing argument is scarcely different than for Theorem 2.11, but it will be useful to rephrase $\widetilde{H}_n$ using the circle group $\mathbb{T}$ instead of μ_2.

The extension of G by μ_2 defines an extension of G by $\mathbb{T}$ by pushing forward along the inclusion $\mu_2\hookrightarrow\mathbb{T}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \widehat{G} & \longrightarrow & G & \longrightarrow & 1. \end{array} \tag{2.21}$$

In cohomology, this construction is classified by the Bockstein map $H^2(BG;\mu_2)\to H^3(BG;\mathbb{Z})$. Let $\widehat{H}_n :=\mathrm{Spin}_n^c\times_{\mathbb{T}}\widehat{G}$ and $\widehat{H}:=\mathrm{Spin}^c\times_{\mathbb{T}}\widehat{G}$. The map $\widetilde{G}\to\widehat{G}$ induces maps $\varphi_n\colon\widetilde{H}_n\to\widehat{H}_n$ and $\varphi\colon\widetilde{H}\to\widehat{H}$; φ is the colimit of the φ_ns.

Lemma 2.22. *The maps $\varphi_n\colon\widetilde{H}_n\to\widehat{H}_n$ are isomorphisms of Lie groups.*

Proof. Write down the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{H}_n & \longrightarrow & \mathrm{SO}_n\times\mathbb{T}\times G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \widehat{H}_n & \longrightarrow & \mathrm{SO}_n\times\mathbb{T}\times G & \longrightarrow & 1 \end{array} \tag{2.23}$$

and apply the five lemma. ☐

And now we shear. Recall our notation from Data 2.1.

Theorem 2.24 (Shearing, class A)**.**

(1) Suppose V_ξ admits a pin^c structure. Then there is an equivalence

$$(B\widehat{H})^{d-V_\lambda-V}\xrightarrow{\simeq}\mathit{MTSpin}^c\wedge(BG)^{d-V_\lambda}. \tag{2.25}$$

(2) If V_ξ does not admit a pin^c structure, there is an equivalence

$$(B\widehat{H})^{d-V_\lambda-V}\xrightarrow{\simeq}\mathit{MTSpin}^c\wedge(BG)^{V_\xi+\mathrm{Det}(V_\xi)-V_\lambda-d'+1-d}. \tag{2.26}$$

Again, we most often use case (2) when $\lambda=\xi$, in which case the right-hand side simplifies to $\mathit{MTSpin}^c\wedge(BG)^{\mathrm{Det}(V_\lambda)-1}$.

Proof. The proof is barely different than that of Theorem 2.11; we indicate only the differences. In that theorem, the engine of the proof when V_ξ was not pin$^-$ was the map (2.15) from $B(\mathrm{Spin}\times_{\mu_2}\widetilde{G}) \to B\mathrm{Spin}\times BG$. Here, V_ξ is not pin^c, so $V_\xi \oplus \mathrm{Det}(V_\xi)$ is oriented but not spin^c. We have that if $\beta\colon H^2(B\widehat{H};\mu_2)\to H^3(B\widehat{H};\mathbb{Z})$ is the Bockstein, $\beta(w_2(V_\xi)+w_1(V_\xi)^2+w_2(V)) = 0$, so $V+V_\xi+\mathrm{Det}(V_\xi)$, interpreted as a map $B\widehat{H}\to B\mathrm{SO}$, lifts to $B\mathrm{Spin}^c$. Our analogue of (2.15) is

$$(V+V_\xi+\mathrm{Det}(V_\xi),\xi)\colon B\widehat{H} \longrightarrow B\mathrm{Spin}^c\times BG. \tag{2.27}$$

As in Proposition 2.14, this is a homotopy equivalence commuting with the maps down to $B\mathrm{SO}$. The proof is almost the same, though we replace Spin with Spin^c in (2.16a) and (2.16b), μ_2 with $\mathbb{T}$ in (2.17), and $K(\mu_2,2)$ with $K(\mathbb{Z},3)$ in (2.18). ☐

2.3. **Putting it together.** The hard work of the proof is already done.

Proof of Theorem 2.8. By Proposition 1.32,

$$Ph_0^G(\mathbb{R}^d; f_{1/2}) \cong [X, \Sigma^{d+1}I_\mathbb{Z}], \tag{2.28}$$

where $X \coloneqq (B\widetilde{H})^{d-V_\lambda-V}$. Then Theorem 2.11 ($H=\mathrm{Spin}$) and Theorem 2.11 ($H=\mathrm{Spin}^c$) split this into $MTH\wedge(BG)^E$ for some rank-zero virtual vector bundle E. For f_0, because $\widetilde{H}\cong H\times G$, Proposition 1.32 gets us to $MTH\wedge(BG)^E$ without having to shear. The only thing left to do is compare these Thom spectra to Definitions 2.3 and 2.4, and sure enough, they match. ☐

3. Computations in examples: summary of results and some generalities

In the next two sections, we study the fermionic crystalline equivalence principle in many examples where the symmetry is given by a two- or three-dimensional point group. Here, we summarize the results and some takeaways for researchers interested in crystalline phases; for more detailed results of computations of groups of phases, see Tables 1, 2, 3, 4, 5, and 6.

In §3.1, we indicate some example phases predicted by our phase homology calculations that have not been previously studied to our knowledge, and which might have accessible or interesting lattice realizations. We also summarize which of our calculations correspond to phases already studied in the literature. In §3.2, we briefly review the computational techniques we use to study phase homology groups, namely the Adams and Atiyah-Hirzebruch spectral sequences. In §3.3, we use the Adams filtration to characterize which invertible field theories with $\widetilde{H}$-structure actually only require weaker structure, such as an $\mathrm{SO}\times G$-structure; this is believed to model the phenomenon in physics of phases which appear to be fermionic, but are in fact bosonic phases that are not fermionic in an interesting way. Finally, in §3.4, we gather some lemmas we use repeatedly in the coming sections. The reader interested in the computations can read §3.1 and §3.2, returning to the other sections later.

3.1. **Some interesting phases to study.** In §§4–5, we compute equivariant phase homology groups for many 2- and 3-dimensional point groups. Using Ansatz 1.22, these computations yield predictions of groups of invertible topological phases. This is a lot of data, so we take the opportunity here to highlight which of our predictions would be interesting to study by other means, e.g. by arguing on the lattice.

We first study some cases already present in the literature and find agreement, including reflections in Altland-Zirnbauer classes D and A (§4.1), inversions in classes D and A (§4.2), cyclic groups acting by rotations in classes D and A (§4.3), and dihedral groups acting by rotations and reflections in class D (§4.4). In all cases we consider both spinless and spin-1/2 fermions.

In addition, we study rotations in class A and many three-dimensional point group symmetries in classes D and A: dihedral groups acting by rotations, pyritohedral symmetry, and chiral and full tetrahedral, octahedral, and icosahedral symmetries. We consider symmetry types with both spinless and spin-1/2 fermions. To the best of our knowledge, these symmetry types have not been studied in the literature, so we indicate some of our predictions that might be interesting to study.

(1) In §4.4.3 and §4.4.4, we compute phase homology groups for the local systems of symmetry types corresponding to class A phases in which the dihedral group D_{2n} acts by rotations and reflections.
  (a) In dimension $d=2$, we predict using Theorems 4.46 and 4.53 a phase generating a $\mathbb{Z}/2n$ for even n with spinless fermions.

(b) In dimension $d = 3$, we would be interested in the predicted $\mathbb{Z}/8 \oplus \mathbb{Z}/2$ for n odd, with either spin-1/2 or spinless fermions (based on (4.45), Theorem 4.45), as well as a phase generating a $\mathbb{Z}/4$ for n even with spin-1/2 fermions (based on Theorems 4.57 and 4.60).

(2) We predict using §5.2 a $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ of 3d class D phases with a pyritohedral symmetry and spinless fermions. In class A, we predict a phase generating a $\mathbb{Z}/4$ subgroup, again with spinless fermions.

(3) In §5.1, we calculate equivariant phase homology groups on $\mathbb{R}^3$ for A_4 acting by tetrahedral symmetry and find that for classes A and D and the spinless and spin-1/2 cases, the zeroth phase homology groups all vanish. Under our ansatz, this predicts there are no nontrivial fermionic phases equivariant for a tetrahedral symmetry in these cases. Can this be seen using a lattice argument?

(4) We predict in §5.3.1 that for 3d class D phases with a full tetrahedral symmetry (i.e. including reflections) and spinless fermions, there is a phase generating a $\mathbb{Z}/4$ subgroup. This phase homology calculation required the most involved mathematical argument, and it would be interesting to see a physical description. A physical interpretation of Proposition 5.46 specifically or an argument averting it would provide some insight into the meaning in physics of the Adams spectral sequence as a tool for studying fermionic phases.

Our computations predict plenty of other phases, but many of them either have Adams filtration zero (see §3.3) and therefore are not predicted to be intrinsically fermionic, or have more complicated symmetry types, such a full octahedral symmetry, that would be harder to study on the lattice.

Remark 3.1. In the computations we make in the next several sections, we generally report more bordism groups than we need to determine the phase homology groups corresponding to groups of invertible phases: to compute the group of n-dimensional invertible field theories with symmetry type $H \to \mathrm{O}$, we need the torsion subgroup of $\pi_n(MTH)$ and the free summand in $\pi_{n+1}(MTH)$. Bordism has other applications in geometry and physics, so we usually report all bordism groups $\pi_k(MTH)$ that follow from the calculations that we need for crystalline phases. When $k \geq n+1$, these provide information about higher-dimensional crystalline phases; for $k < \dim(\lambda)$, though, it is not clear what a crystalline phase could mean when there are not enough space dimensions for G to act by λ, and we do not give a physical meaning to these computations. See [GOP+20] for some discussion when space*time* is $\dim(\lambda)$-dimensional.

3.2. **Methods of computation.** In this section, we summarize the techniques we use to make these computations, and gather a few auxiliary lemmas we need along the way. Most of our computations can be reframed as computing certain twisted *ko*- and *ku*-homology groups of finite groups in low degrees; the reader interested in learning how to perform such computations is encouraged to refer to the monographs of Bruner-Greenlees [BG03, BG10] on connective *ko*- and *ku*-theory, as well as Beaudry-Campbell's article [BC18] on using the Adams spectral sequence to compute *ko*-theory.

Computing spin bordism: Let *ko* denote the connective real K-theory spectrum. Anderson-Brown-Peterson [ABP67] show that the Atiyah-Bott-Shapiro map $MTSpin \to ko$ [ABS64] is 7-connected, meaning that for any space or spectrum X, the induced map $\Omega_k^{\mathrm{Spin}}(X) \to ko_k(X)$ is an isomorphism for $k \leq 7$. We often pass between spin bordism and *ko*-theory without comment. We compute the free and 2-torsion summands of $ko_*(X)$ using the Adams spectral sequence; see below. The forgetful map $MTSpin \to MTSO$ induces an equivalence on odd-primary torsion, so to compute odd-primary torsion, we typically compute $\Omega_*^{\mathrm{SO}}(X)$ via the Atiyah-Hirzebruch spectral sequence, which we also discuss below.

Computing spin^c bordism: Let *ku* denote connective complex K-theory. Anderson-Brown-Peterson [ABP67] also produce a 7-connected map $MTSpin^c \to ku \vee \Sigma^4 ku$; we will also use the Adams spectral sequence to determine the free and 2-torsion summands of $ku_*(X)$, as described below. The forgetful map $MTSpin^c \to MTSO \wedge (B\mathrm{U}_1)_+$ induces an equivalence on odd-primary torsion, so we compute $\Omega_*^{\mathrm{SO}}(X \times B\mathrm{U}_1)$, typically with the Atiyah-Hirzebruch spectral sequence.

Now we briefly introduce the Adams and Atiyah-Hirzebruch spectral sequences in the ways that we use them.

3.2.1. *The Adams spectral sequence.* The (2-primary) Adams spectral sequence [Ada58, Theorems 2.1, 2.2] computes the 2-completion of the homotopy groups of a pointed space or spectrum X. Its E_2-page is

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\widetilde{H}^*(X;\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow \pi_{t-s}(X)_2^\wedge. \tag{3.2}$$

Here $\mathcal{A}$ is the 2-primary Steenrod algebra.

Remark 3.3. The usual bigrading convention for Adams spectral sequences places $E_r^{s,t}$ at coordinates $(t-s,s)$. We follow this convention. The *topological degree* of an element at coordinates $(t-s,s)$ in an Adams spectral sequence refers to $t-s$, and s is called its *filtration*.

There is a general change-of-rings theorem, where if $\mathcal{B}$ is a graded Hopf algebra, $\mathcal{C}\subset\mathcal{B}$ is a graded Hopf subalgebra, and M and N are graded $\mathcal{B}$-modules, then there is a natural isomorphism

$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(\mathcal{B}\otimes_{\mathcal{C}} M, N) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{C}}^{s,t}(M,N). \tag{3.4}$$

When $X = ko\wedge Y$ or $ku\wedge Y$, this greatly simplifies the E_2-page of (3.2). Inside the mod 2 Steenrod algebra $\mathcal{A}$, define the subalgebras $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$ and $\mathcal{E}(1) \coloneqq \langle Q_0, Q_1\rangle$;[11] then, Stong [Sto63] showed $\widetilde{H}^*(ko;\mathbb{Z}/2)\cong \mathcal{A}\otimes_{\mathcal{A}(1)}\mathbb{Z}/2$ and Adams [Ada61] showed $\widetilde{H}^*(ku;\mathbb{Z}/2)\cong\mathcal{A}\otimes_{\mathcal{E}(1)}\mathbb{Z}/2$. Both $\mathcal{A}(1)$ and $\mathcal{E}(1)$ are Hopf subalgebras of $\mathcal{A}$ so (3.4) says we need only consider

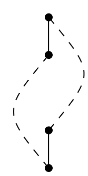
$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\widetilde{H}^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \widetilde{ko}_{t-s}(X)_2^\wedge \tag{3.5a}$$

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\widetilde{H}^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \widetilde{ku}_{t-s}(X)_2^\wedge. \tag{3.5b}$$

This line of reasoning, first used by Davis [Dav74], is by now a standard trick in algebraic topology. For further reading, we recommend the paper of Beaudry-Campbell [BC18], who go into detail about how to define and calculate these Ext groups and work several examples over $\mathcal{A}(1)$. There are fewer worked examples of (3.5b) in the literature; see Bruner-Greenlees [BG03], Nguyen [Ngu09], Francis [Fra11, §5] and Al-Boshmki [AB16] for closely related calculations.

Our notation is standard in the $\mathcal{A}(1)$-case, but since examples for $\mathcal{E}(1)$ are sparser, we record here a few notational conventions for working with $\mathcal{E}(1)$-modules and this spectral sequence. When we draw $\mathcal{E}(1)$-modules, we will use solid straight lines to denote Q_0-actions and dashed curved lines to denote Q_1-actions. Therefore, for example, $\mathcal{E}(1)$ as a module over itself looks like this.

(3.6)



For any $\mathcal{E}(1)$-module M, $H^{*,*}(\mathcal{E}(1)) \coloneqq \mathrm{Ext}_{\mathcal{E}(1)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ acts on $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M,\mathbb{Z}/2)$, analogously to the case of $\mathcal{A}(1)$-modules; if $M=\widetilde{H}^*(X;\mathbb{Z}/2)$, then just as over $\mathcal{A}(1)$, tracking this action through the Adams spectral sequence provides information about the action of ku_* on $\widetilde{ku}_*(X)$. Differentials are equivariant for this action, just like for the Adams spectral sequence over $\mathcal{A}(1)$. Since $\mathcal{E}(1)$ is an exterior algebra, Koszul duality provides an isomorphism of bigraded algebras

$$H^{*,*}(\mathcal{E}(1))\cong \mathbb{Z}/2[h_0,v_1], \tag{3.7}$$

where $|h_0| = (1,1)$ and $|v_1| = (1,3)$ [BC18, Example 4.5.6]. We will denote an h_0-action by a vertical line, and a v_1-action by a lighter diagonal line. Like for ko, h_0 lifts to multiplication by 2; v_1 lifts to the action of the Bott element $\beta\in ku_2$ [BG03, §2.1].

We will often write $\mathrm{Ext}_{\mathcal{A}(1)}(M)$ for $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(M,\mathbb{Z}/2)$, and similarly for $\mathcal{E}(1)$; when it is clear which subalgebra we are working over, we will just write $\mathrm{Ext}(M)$.

By now there is a large body of work using the Adams spectral sequence, especially over $\mathcal{A}(1)$, to compute things related to invertible field theories or invertible phases. This includes [Sto86, Kil88, Hil09, Fra11, FH16a, Cam17, BC18, GPW18, Guo18, FH19b, WW19a, WW19b, WWZ19, DL20a, DL20b, DL20c, GOP$^+$20, KPMT20, LOT20, LT20, WW20a, WW20b, WW20c, WWZ20].

3.2.2. *The Atiyah-Hirzebruch spectral sequence.* The (homological) Atiyah-Hirzebruch spectral sequence [AH61] for oriented bordism has signature

$$E^2_{p,q} = \widetilde{H}_p(X;\Omega_*^{\mathrm{SO}}) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(X). \tag{3.8}$$

[11]These generators are given in two different bases of $\mathcal{A}$; the relations between them are $Q_0=\mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2+\mathrm{Sq}^2\mathrm{Sq}^1$.

In general, using the Atiyah-Hirzebruch spectral sequence can feel different depending on application-specific details, so we point the reference-minded reader to García-Etxebarria-Montero [GEM19, §2.2.2, §3] for an introduction and some examples which may be helpful.

There are many references using the Atiyah-Hirzebruch spectral sequence to compute things related to invertible field theories or invertible phases, such as [Kil88, Edw91, Mon15, Cam17, KT17, Mon17, Hsi18, SdBKP18, SSG18, Ste18, STY18, SXG18, Xio18, ET19, FH19a, GEM19, MM19, OSS19, Shi19, TY19, BLT20, DGL20, DH20, DL20c, ETS20, GOP+20, HH20, HKT20, Hor20, HTY20, JF20a, JF20b, KPMT20, LOT20, LT20, SFQ20, Tho20, TW20, WW20b, Yu20, DGG21, KLST21].

We use a few other spectral sequences in our computations, but only for one-off computations, so we address them when we get to them.

3.3. **Adams filtration 0 phases are secretly bosonic.** In Remark 1.28, we defined a map from phase homology with symmetry type SO to phase homology with symmetry types Spin or Spin^c and interpreted it as regarding bosonic SPT phases as fermionic SPT phases in a trivial way. Physicists studying fermionic SPT phases are often interested in the cokernel of this map, which is thought of as the group of intrinsically fermionic SPT phases. Because bosonic crystalline phases are relatively well-understood, e.g. in the work of Hermele, Huang, Song, and their collaborators [HSHH17, SHFH17, HH18, SHQ+19, SFQ20, SXH20] and via the bosonic crystalline equivalence principle of Thorngren-Else [TE18], we are most interested in intrinsically fermionic SPT phases.

The structure of the Adams spectral sequence allows us to identify the image of this bosonic-to-fermionic map on phase homology with little extra work. For more about the Adams spectral sequence, see §3.2; for now, we need only that phase homology groups, reinterpreted through Theorem 2.8 as groups of invertible field theories, are computed as homotopy groups of spectra, and that the homotopy groups of any spectrum M come with a canonical filtration called the *(mod 2) Adams filtration*

$$\pi_n M = F_n^0 \supseteq F_n^1 \supseteq F_n^2 \supseteq \cdots \tag{3.9}$$

For more information, see [BC18, §4.7]. This has two properties which are important for us.

(1) The Adams spectral sequence computes the Adams filtration: after 2-completing, the associated graded of (3.9) is the E_∞-page of the Adams spectral sequence, in that $E_\infty^{s,t} = \mathrm{gr}_s \pi_{t-s} M$.
(2) If $M = MTH$ is a Thom spectrum whose homotopy groups compute bordism groups, elements of the associated graded in degree 0 correspond to the 2-primary part of the group of deformation classes of invertible TFTs which depend on something weaker than an H-structure, such as a spin IFT which is defined by evaluating an oriented IFT on spin manifolds.

This means we can identify which invertible TFTs really use the H-structure, and which do not.

Now a little more detail. We do not need to say much more about (1): we depict Adams spectral sequences on a grid with coordinates $(t-s, s)$, such as in Figure 1, right, so F_n^0/F_n^1 is found in the E_∞-page at coordinate $(n, 0)$.

For (2), we make a simplifying assumption: that for the specific degree n we are investigating, $\pi_n MTH$ is 2-torsion. This assumption holds in all cases where we want to study the Adams filtration in this article, but if you want to relax it, see Remark 3.18. The assumption implies that up to extension questions on the E_∞-page, the mod 2 Adams spectral sequence fully determines $\pi_n MTH$,[12] and that the natural map

$$(\pi_n(MTH))^\vee := \mathrm{Hom}(\pi_n(MTH), \mathbb{C}^\times) \longrightarrow [MTH, \Sigma^{n+1} I_{\mathbb{Z}}] \tag{3.10}$$

is an isomorphism.

To pass from bordism groups to isomorphism class of invertible field theories, we must take character duals $A \mapsto A^\vee := \mathrm{Hom}(A, \mathbb{C}^\times)$. This is a good thing, actually: a degree-0 element of $\mathrm{gr}_\bullet \pi_n(MTH)$ does not usually uniquely lift to an element of $\pi_n MTH$: the ambiguity is F_n^1. But in $(\pi_n(MTH))^\vee$, we get a subgroup: the surjection

$$\pi_n(MTH) \twoheadrightarrow \pi_n(MTH)/F_n^1 \cong \mathrm{gr}_0 \pi_n(MTH) \tag{3.11a}$$

[12] Some extension questions can be addressed using the $H^{*,*}(\mathcal{A}(1))$-action on the E_∞-page, but there are also *hidden extensions* which are harder to address. None of the calculations we make in this article manifest hidden non-split extensions; one example where they do occur is $H = \mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/8$ [DDHM].

passes under character duality to an inclusion

$$(\mathrm{gr}_0\pi_n(MTH))^\vee \hookrightarrow (\pi_n(MTH))^\vee. \tag{3.11b}$$

Therefore, in a mild abuse of notation, we refer to this subgroup of $(\pi_n(MTH))^\vee$, identified with a subgroup of the group isomorphism classes of invertible TFTs with H-structure, as the group of *Adams filtration* 0 *invertible TFTs with H-structure.*

It is a theorem [FH19b, §8.4] that this subgroup consists of theories closely related to classical Dijkgraaf-Witten theories [FQ93, §1].[13] Isomorphism classes of these invertible TFTs are determined by their partition functions [FH16a, §5.3], so we specify these theories by their partition functions, which are bordism invariants $\Omega_n^H \to \mathbb{C}^\times$.

For the Adams spectral sequence, $E_2^{0,n} = \mathrm{Ext}_{\mathcal{A}}^{0,n}(\widetilde{H}^*(MTH;\mathbb{Z}/2);\mathbb{Z}/2)$ is canonically identified with

$$\mathrm{Hom}_{\mathcal{A}}(\widetilde{H}^*(MTH;\mathbb{Z}/2),\Sigma^n\mathbb{Z}/2), \tag{3.12}$$

which is a subspace of

$$\mathrm{Hom}_{\mathcal{A}b}(\widetilde{H}^n(MTH;\mathbb{Z}/2),\mathbb{Z}/2) \cong (\widetilde{H}^n(MTH;\mathbb{Z}/2))^\vee. \tag{3.13}$$

The fourth quadrant of the Adams spectral sequence is empty, so $E_\infty^{0,n}$ is a subspace of $E_2^{0,n}$. Take the sequence of maps

$$\mathrm{gr}_0\pi_n(MTH) = E_\infty^{0,n} \hookrightarrow E_2^{0,n} \hookrightarrow (\widetilde{H}^n(MTH;\mathbb{Z}/2))^\vee \tag{3.14a}$$

and apply character duality:

$$(\mathrm{gr}_0\pi_n(MTH))^\vee \twoheadleftarrow (E_2^{0,n})^\vee \twoheadleftarrow \widetilde{H}^n(MTH;\mathbb{Z}/2). \tag{3.14b}$$

Now compose with the Thom isomorphism to obtain

$$\zeta\colon H^n(BH;\mathbb{Z}/2) \twoheadrightarrow (\mathrm{gr}_0\pi_n(MTH))^\vee. \tag{3.14c}$$

That is, a degree-n mod 2 cohomology class of BH determines an isomorphism class of Adams filtration 0 invertible TFTs, and all Adams filtration 0 invertible TFTs arise in this way. The map need not be injective, e.g. by the Wu formula when $H = \mathrm{O}$.

Tracing this through Thom's collapse map tells us that given a cohomology class $\theta \in H^n(BH;\mathbb{Z}/2)$, the partition function $\zeta(\theta)$ is the bordism invariant which takes a closed n-manifold with H-structure $(M, f\colon M \to BH)$ and returns

$$\zeta(\theta)(M,f) = (-1)^{\langle f^*\theta,[M]\rangle}. \tag{3.15}$$

That is, use the H-structure to pull θ back to M, then evaluate it on the mod 2 fundamental class. This construction uses some aspects of the H-structure on M, but in the cases relevant to this paper, it is insensitive to the difference between Spin and O, which is believed to pass to the physicists' distinction between fermionic and bosonic phases.

Lemma 3.16. *If $H = \mathrm{Spin}\times_{\mu_2}\widetilde{G}$ or $H = \mathrm{Spin}^c\times_{\mu_2}\widetilde{G}$, where $\widetilde{G}$ is in Data 2.1, and $H' \coloneqq \mathrm{O}\times G$, then the map $H \to H'$ of tangential structures induces a surjective map $H^*(BH';\mathbb{Z}/2) \to H^*(BH;\mathbb{Z}/2)$, and therefore the partition functions (3.15) of the Adams filtration 0 theories only depend on the underlying H'-structure of an H-manifold.*

Proof. First, the Spin case. We established a shearing equivalence $MTH \cong MTSpin \wedge X$, where X is a Thom spectrum of a rank-zero virtual vector bundle over BG, and this equivalence fits into a homotopy commutative diagram

$$\begin{array}{ccc} MTH & \xrightarrow{\simeq} & MTSpin\wedge X \\ \downarrow & & \downarrow \\ MTO\wedge(BG)_+ & \longrightarrow & MTO\wedge X. \end{array} \tag{3.17a}$$

[13] These theories are not quite the same thing as classical Dijkgraaf-Witten theories, which are TFTs of oriented manifolds with a principal G-bundle, and which use $\mathbb{R}/\mathbb{Z}$-valued cohomology, rather than $\mathbb{Z}/2$-valued cohomology. Unoriented generalizations of classical Dijkgraaf-Witten theory are studied in more detail in work of Kim [Kim18, §6], the author [Deb20, §3.1], and You [You20].

Apply mod 2 cohomology and invoke the Thom isomorphism to obtain a commutative diagram

$$\begin{array}{ccc} H^*(BH;\mathbb{Z}/2) & \xleftarrow{\cong} & H^*(B\mathrm{Spin}\times BG;\mathbb{Z}/2) \\ \uparrow & & \uparrow \xi \\ H^*(BO\times BG;\mathbb{Z}/2) & \xleftarrow{\mathrm{id}} & H^*(BO\times BG;\mathbb{Z}/2) \end{array} \tag{3.17b}$$

The map $H^*(B\mathrm{O};\mathbb{Z}/2)\to H^*(B\mathrm{Spin};\mathbb{Z}/2)$ is surjective, so the Künneth formula implies ξ is too, so the left-hand arrow $H^*(BH';\mathbb{Z}/2)\to H^*(BH;\mathbb{Z}/2)$ is as well.

For $H=\mathrm{Spin}^c$, the proof is the same – $B\mathrm{Spin}^c$ has an additional characteristic class $c_1\in H^2(B\mathrm{Spin}^c;\mathbb{Z})$, but its mod 2 reduction is w_2, so ξ is still surjective. ☐

Remark 3.18. In all cases that one might reasonably encounter, the bordism group $\pi_n X$ is finitely generated, so we can ask what happens if it contains p-torsion for an odd prime p or free summands. For a p-torsion summand, the story is very similar: one instead uses the mod p Adams filtration on $\pi_n M_p^\wedge$, which is detected by the $\mathbb{Z}/p$-Adams spectral sequence. This has almost the same signature as the $\mathbb{Z}/2$-Adams spectral sequence we use in this paper, except that $\mathbb{Z}/2$ is replaced with $\mathbb{Z}/p$ and the Steenrod algebra is over $\mathbb{Z}/p$ instead of $\mathbb{Z}/2$. Because the mod p Thom isomorphism requires an orientation, the story is a little more nuanced for tangential structures which do not induce an orientation.

For free summands in $\pi_n M$, there is no analogous story. The invertible field theories in question are not topological, and at present their classification is still a conjecture [Fre19, Lecture 9]. Assuming this conjecture, though, the Adams filtration does not tell the whole story. For example, consider 3d invertible spin field theories, (conjecturally) classified by

$$[\mathit{MTSpin},\Sigma^4 I_{\mathbb{Z}}]\xrightarrow{\cong}\mathrm{Hom}(\Omega_4^{\mathrm{Spin}},\mathbb{Z})\cong\mathbb{Z}, \tag{3.19}$$

generated by the map φ sending a spin 4-manifold to its signature divided by 16 [Roh52]. As the signature does not depend on the spin structure, 16φ generates $\mathrm{Hom}(\Omega_4^{\mathrm{SO}},\mathbb{Z})$,[14] and therefore the image of the forgetful map $[\mathit{MTSO},\Sigma^4 I_{\mathbb{Z}}]\to[\mathit{MTSpin},\Sigma^4 I_{\mathbb{Z}}]$ is identified with the subgroup $16\mathbb{Z}$. That is, assuming the conjecture on the classification of not-necessarily-topological invertible field theories, a 3d spin invertible field theory only depends on the underlying orientation iff it is q times a generator, where $16\mid q$. So for free summands in the abelian group of isomorphism classes of invertible field theories, the Adams filtration approach does not work, and one must use other methods.

3.4. A few utility lemmas.

Definition 3.20. Let A be an abelian group, X be a connected space, and $\alpha\in H^1(X;\mathbb{Z}/2)$. Then A_α denotes the local system on X given by the $\mathbb{Z}[\pi_1(X)]$-module with underlying abelian group A and in which $g\in\pi_1(X)$ acts on A by $(-1)^{\alpha(g)}$, where we interpret α as a map $\pi_1(X)\to\mathbb{Z}/2$ under the identification $H^1(X;\mathbb{Z}/2)\cong\mathrm{Hom}(\pi_1(X),\mathbb{Z}/2)$.

Usually α will be the first Stiefel-Whitney class of a vector bundle, as in the following lemma.

Proposition 3.21. *Let $\sigma\to B\mathbb{Z}/2$ denote the tautological line bundle.*

(1) $H^k(B\mathbb{Z}/2;\mathbb{Z}_{w_1(\sigma)})$ is isomorphic to $\mathbb{Z}/2$ in odd degrees and 0 in even degrees.
(2) If n is odd, $H^k(B\mathbb{Z}/2;(\mathbb{Z}/n)_{w_1(\sigma)})\cong 0$ for all k.
(3) If n is even, $H^k(B\mathbb{Z}/2;(\mathbb{Z}/n)_{w_1(\sigma)})\cong\mathbb{Z}/2$ for all k.

Proof. Use $\mathbb{RP}^\infty:=\varinjlim_n\mathbb{RP}^n$ as our model for $B\mathbb{Z}/2$. Let A be any abelian group. Given k, choose a very large even m; then, the map $\mathbb{RP}^m\hookrightarrow B\mathbb{Z}/2$ induces an isomorphism $H^k(\mathbb{RP}^m;A_{w_1(\sigma)})\xrightarrow{\cong}H^k(B\mathbb{Z}/2;A_{w_1(\sigma)})$. Since m is even, $\mathbb{RP}^m$ is unorientable, and $\mathbb{Z}_{w_1(\sigma)}$ is isomorphic to the orientation local system for $\mathbb{RP}^m$, so there is a Poincaré duality isomorphism $H^k(\mathbb{RP}^m;A_{w_1(\sigma)})\cong H_{m-k}(\mathbb{RP}^m;A)$. ☐

We will repeatedly use the following theorem to show some differentials and extensions are trivial in the Adams spectral sequence.

[14] This follows from the fact that the signature defines an isomorphism $\sigma\colon\Omega_4^{\mathrm{SO}}\to\mathbb{Z}$, which follows from the fact that $\mathbb{CP}^2$, with signature 1, generates Ω_4^{SO} [Tho54, Remarque following Corollaire IV.18].

Theorem 3.22 (Margolis [Mar74])**.** *Let $\mathcal{B}$ be a sub-Hopf algebra of Steenrod algebra and Y be a spectrum with $\widetilde{H}^*(Y;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{B}} \mathbb{Z}/2$ (so that the change-of-rings trick works for computing 2-completed Y-homology). For any spectrum X, there is a splitting*

$$Y \wedge X \simeq F \vee \overline{X}, \tag{3.23}$$

where F is an Eilenberg-Mac Lane spectrum for a graded $\mathbb{Z}/2$-vector space and $\widetilde{H}^*(\overline{X};\mathbb{Z}/2)$ has no free summands as an $\mathcal{A}$-module.

The upshot is that in the Adams spectral sequence for computing $\pi_*(Y \wedge X)^\wedge_2$, the piece of the E_2-page coming from free summands of $\widetilde{H}^*(X;\mathbb{Z}/2)$ as a $\mathcal{B}$-module do not emit or receive nontrivial differentials, and do not participate in nontrivial extensions.

Lemma 3.24. *Let G be a finite group and $E \to BG$ be a rank-zero virtual vector bundle.*

(1) If $4 \mid n$, $\widetilde{ko}_n(BG^E) \otimes \mathbb{Q} \cong H_0(BG;\mathbb{Q}_{w_1(E)})$; if $4 \nmid n$, $\widetilde{ko}_n(BG^E)$ is torsion.
(2) The same is true for $\widetilde{ku}_n(BG^E)$, except divisibility by 4 is replaced by divisibility by 2.

Proof. Atiyah-Hirzebruch [AH61] proved that the Chern character defines an equivalence

$$ch\colon ku \wedge H\mathbb{Q} \xrightarrow{\simeq} \bigvee_{k\ge 0} \Sigma^{2k} H\mathbb{Q}. \tag{3.25}$$

The Thom isomorphism theorem establishes that $\widetilde{H}_*(BG^E;\mathbb{Q}) \cong H_*(BG;\mathbb{Q}_{w_1(E)})$, and since G is finite, this vanishes above degree zero by Maschke's theorem.

The proof for ko-theory is the same, except first using the complexification map $c\colon ko \to ku$:

$$ch \circ c\colon ko \wedge H\mathbb{Q} \xrightarrow{\simeq} \bigvee_{k\ge 0} \Sigma^{4k} H\mathbb{Q}. \tag{3.26}$$ ☐

Choosing E to be the trivial bundle shows the conclusions also hold for the torsion in $\widetilde{ko}_*(BG)$ and $\widetilde{ku}_*(BG)$.

Lemma 3.27 (Adem-Milgram)**.** *Fix a prime p, and let H be a subgroup of a finite group G with $[G:H]$ coprime to p and P be a Sylow p-subgroup of H. Assume P is abelian and that $N_H(P)/P = N_G(P)/P$; then the restriction map $\rho_{H,G}\colon H^*(BG;\mathbb{Z}/p) \to H^*(BH;\mathbb{Z}/p)$ is an isomorphism.*

Proof. This is a slight strengthening of theorems of Swan [Swa60] and Adem-Milgram [AM04, Theorems II.6.6 and II.6.8], who prove that if K is a finite group with abelian p-Sylow subgroup P, then the restriction map $H^*(BK;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^{N_K(P)}$ is an isomorphism. In our setting, the data of P and $N(P)/P$ are identical for G and H, so both restriction maps $r_{P,G}\colon H^*(BG;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^N$ and $r_{P,H}\colon H^*(BH;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^N$ are isomorphisms. Since $r_{P,G} = r_{P,H} \circ \rho_{G,H}$, we are done. ☐

Lemma 3.28 (Bock-to-Sq^1 lemma)**.** *Let $\beta\colon H^k(\text{–};\mathbb{Z}/2) \to H^{k+1}(\text{–};\mathbb{Z})$ denote the integral Bockstein. Then $\beta(x) \bmod 2 = \mathrm{Sq}^1(x)$.*

Proof. The commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{2} & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array} \tag{3.29}$$

induces a commutative diagram of their induced long exact sequences in cohomology; in particular, $\beta \bmod 2$ equals the Bockstein for the bottom short exact sequence, which is Sq^1. ☐

In the mixed unoriented case, Theorems 2.11 and 2.24 ask us to study Thom spectra for determinants of representations. We use the following lemma to simplify them.

Lemma 3.30. *Let* $\lambda\colon G\to \mathrm{O}_d$ *be a faithful representation whose image contains a reflection and* $V_\lambda\to BG$ *be the associated vector bundle. Then the splitting of the surjection*

$$G\xrightarrow{\lambda}\mathrm{O}_d\xrightarrow{\pi_0}\mathbb{Z}/2 \tag{3.31}$$

lifts to a splitting of the Thom spectrum $(BG)^{\mathrm{Det}(V_\lambda)-1}$ *as*

$$(BG)^{\mathrm{Det}(V_\lambda)-1}\xrightarrow{\simeq}(B\mathbb{Z}/2)^{\sigma-1}\vee M, \tag{3.32}$$

and the inclusion $\widetilde{H}^*(M;\mathbb{Z}/2)\hookrightarrow\widetilde{H}^*((BG)^{\mathrm{Det}(V_\lambda)-1};\mathbb{Z}/2)$ *is injective with image a complementary vector space to the subspace spanned by* $\{Uw_1(V_\lambda)^k\mid k\ge 0\}$.

Proof. Let $g\in G$ be an element sent to λ by a reflection. Then $g^2=1$, so the maps $\langle g\rangle\hookrightarrow G\twoheadrightarrow\mathbb{Z}/2$ compose to an isomorphism. Upon taking Thom spectra, these can be identified with maps $(B\mathbb{Z}/2)^{\sigma-1}\to(BG)^{\mathrm{Det}(V_\lambda)-1}\to(B\mathbb{Z}/2)^{\sigma-1}$ composing to (a map homotopy equivalent to) the identity, which splits off $(B\mathbb{Z}/2)^{\sigma-1}$. The image of the map $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1};\mathbb{Z}/2)\to\widetilde{H}^*((BG)^{\mathrm{Det}(V_\lambda)-1};\mathbb{Z}/2)$ is spanned by $\{Uw_1(V_\lambda)^k\mid k\ge 0\}$, and the image of $\widetilde{H}^*(M;\mathbb{Z}/2)$ is a complementary subspace. □

4. Examples: rotations and reflections

4.1. **Warmup: reflections.** The simplest example of the fermionic crystalline equivalence principle occurs when the spatial symmetry is $\mathbb{Z}/2$ acting by a reflection. This symmetry can mix with $\mu_2\subset\mathrm{Spin}_d$, and there are two cases. The following principle is well-established in physics literature; see Shiozaki-Shapourian-Ryu [SSR17b] and Song-Huang-Fu-Hermele [SHFH17, §VII].

- If $\mathbb{Z}/2$ and μ_2 do not mix (often written that the reflection squares to 1), then the classification matches the classification of pin$^+$ invertible field theories.
- Conversely, if $\mathbb{Z}/2$ and μ_2 do mix (often written that the reflection squares to $(-1)^F$), the classification matches that of pin$^-$ invertible field theories.

Condensed-matter theorists also study theories with time-reversal symmetry. Though this is also an antiunitary symmetry that can mix with μ_2, the classification in terms of pin structures is opposite that of reflections: when time-reversal symmetry does not mix with fermion parity, we get pin$^-$, and when it does mix, we get pin$^+$. This is also well-established in physics, and is discussed by Kapustin-Thorngren-Turzillo-Wang [KTTW15], Freed-Hopkins [FH16a], and others.

The difference between these two correspondences is a first hint that the fermionic crystalline equivalence principle must be more complicated than the bosonic version; this point is raised by Thorngren-Else [TE18, §V.A] and Cheng-Wang [CW18, §II.C].

d	Class D, spinless §4.1.1	Class D, spin-1/2 §4.1.2	Class A §4.1.3
1	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/4$
2	$\mathbb{Z}/2$	0	0
3	$\mathbb{Z}/16$	0	$\mathbb{Z}/8\oplus\mathbb{Z}/2$
4	0	0	0

Table 1. $\mathbb{Z}/2$-equivariant phase homology groups for the cases in which $\mathbb{Z}/2$ acts by a reflection. As discussed in §4.1, these arise as the homotopy groups of the Anderson duals of $MTPin^+$, $MTPin^-$, and $MTPin^c$. For this group action, the spinless and spin-1/2 classifications in class A coincide.

4.1.1. *Class D, spinless.* When the reflection does not mix with the internal symmetry group, our ansatz is exactly that of Freed-Hopkins. In this setting, $\mathbb{Z}/2$ acts on $\mathbb{R}^d$ as $(d-1)+\sigma$, where k denotes the rank-k trivial representation and σ denotes the sign representation. Let f_0^D denote the equivariant local system of symmetry types for the class D spinless case. Arguing as in [FH19a, (3.6)], in space dimension d we see that

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d;f_0^D)\cong[MTSpin\wedge(B\mathbb{Z}/2)^{1-\sigma},\Sigma^{d+2}I_{\mathbb{Z}}]. \tag{4.1}$$

Using (2.10a), $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^+$, identifying these phase homology groups as homotopy groups of the Anderson dual of $MTPin^+$, as expected. Finally, to obtain the specific groups in Table 1, we use the preexisting calculations of pin$^+$ bordism from [Gia73b, KT90a, KT90b].

4.1.2. *Class D, spin-1/2.* Again $\mathbb{Z}/2$ acts by $d-1+\sigma$, and this time, reflection mixes with fermion parity. Let $f^D_{1/2}$ denote the equivariant local system of symmetry types for this case. The associated bundle to the $\mathbb{Z}/2$-representation given by reflection is not pin$^-$, so by Theorem 2.11,

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d; f^D_{1/2}) \cong [MTSpin \wedge (B\mathbb{Z}/2)^{\mathrm{Det}(\sigma)-1}, \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.2}$$

Because σ is a line bundle, $\mathrm{Det}(\sigma) = \sigma$. Using (2.10b), $MTSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$, so these phase homology groups are identified with homotopy groups of the Anderson dual of $MTPin^-$ as predicted. These bordism groups are calculated in [ABP69, KT90b].

4.1.3. *Class A.* For spin^c phases (those of Altland-Zirnbauer class A), the spinless and spin-1/2 classifications coincide: V_λ is pin^c, so Theorem 2.24 tells us to consider $MTSpin^c \wedge (B\mathbb{Z}/2)^{1-\sigma}$ in both cases, and by (2.10c), this spectrum is equivalent to $MTPin^c$.

Bahri-Gilkey [BG87a, BG87b] compute pin^c bordism groups,[15] giving us the phase homology groups in Table 1.

4.1.4. *Comparison with prior work.* Reflection-equivariant fermionic phases have been studied by many teams of researchers with many methods. Their results agree with each other, and with us.

Class D, spinless: These phases, especially the $\mathbb{Z}/16$ in $d = 3$, are studied by Song-Huang-Fu-Hermele [SHFH17, §V.A], Hsieh-Cho-Ryu [HCR16, §IV], Shiozaki-Shapourian-Ryu [SSR17b, §II.B, §II.D], Guo-Ohmori-Putrov-Wan-Wang [GOP$^+$20, §10.7], and Mao-Wang [MW20].

Class D, spin-1/2: Song-Huang-Fu-Hermele [SHFH17, §V.B], Shapourian-Shiozaki-Ryu [SSR17a, SSR17b], Guo-Ohmori-Putrov-Wan-Wang [GOP$^+$20, §10.7], and Bultinck-Williamson-Haegeman-Verstraete [BWHV17, §IX].

Class A: These phases have been studied by Isobe-Fu [IF15], Hong-Fu [HF17], Shapourian-Shiozaki-Ryu [SSR17a, SSR17b], Song-Huang-Fu-Hermele [SHFH17, §4], and Shiozaki-Shapourian-Gomi-Ryu [SSGR18, §V].

4.2. **Inversions.** *Inversion symmetry* is the $\mathbb{Z}/2$-symmetry on $\mathbb{R}^d$ acting by $(x_1, \dots, x_d) \mapsto (-x_1, \dots, -x_d)$. This offers another relatively simple example of the FCEP, but with a new feature in the spin-1/2 case: the classes in $H^2(B\mathbb{Z}/2; \mathbb{Z}/2)$ specified by the extension $1 \to \mathbb{Z}/2 \to \widetilde{G} \to \mathbb{Z}/2 \to 1$ and by $w_2(\lambda) + w_1(\lambda)^2$ are not always equal. This does not change very much, as we explain in §4.2.2 below.

d	Class D, spinless §4.2.1	Class D, spin-1/2 §4.2.2	Class A §4.2.3
1	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/4$
2	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/8$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4$
3	0	$\mathbb{Z}/16$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$
4	0	$\mathbb{Z} \oplus \mathbb{Z}/16$	$\mathbb{Z}^2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$

TABLE 2. $\mathbb{Z}/2$-equivariant phase homology groups for the cases where $\mathbb{Z}/2$ acts as inversion. The symmetry type whose Thom spectrum determines these groups depends on d; see the referenced sections for which symmetry types appear.

[15] In low degrees, Beaudry-Campbell [BC18, §5.6] compute low-degree pin^c bordism groups using the Adams spectral sequence over $\mathcal{A}(1)$, using that $MTPin^c \simeq MTSpin \wedge \Sigma^{-2} MU_1 \wedge \Sigma^{-1} MO_1$. One can also compute using the Adams spectral sequence over $\mathcal{E}(1)$, as in §4.4.3; we found this to be a fun and useful exercise for getting comfortable with this variation of the Adams spectral sequence.

4.2.1. *Class D, spinless case.* First, the case for which inversion symmetry and fermion parity do not mix. The $\mathbb{Z}/2$-action on $\mathbb{R}^d$ is a direct sum of d copies of the sign representation σ, so as a $\mathbb{Z}/2$-space, $\mathbb{R}^d$ is denoted $d\sigma$. This case is covered by Freed-Hopkins [FH19a, Example 3.5], and the phase homology groups are

$$[MTSpin \wedge (B\mathbb{Z}/2)^{d-d\sigma}, \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.3}$$

The spectra $MTSpin \wedge (B\mathbb{Z}/2)^{d-d\sigma}$ are periodic in d.

Lemma 4.4. *If $d' - d$ is divisible by 4, $MTSpin \wedge (B\mathbb{Z}/2)^{d(1-\sigma)} \simeq MTSpin \wedge (B\mathbb{Z}/2)^{d'(1-\sigma)}$.*

Proof. This is an instance of Theorem 1.39, using that spin structures satisfy the 2-out-of-3 property and that, since 4σ is spin, so is $(d'-d)(1-\sigma)$. ☐

Thus we have only to determine $MTSpin \wedge (B\mathbb{Z}/2)^{d(1-\sigma)}$ for small d.

- When $d = 0$, we get $MTSpin \wedge (B\mathbb{Z}/2)_+$.
- When $d = 1$, (2.10a) tells us $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^+$.
- For $d = 2$, we have $MTSpin \wedge (B\mathbb{Z}/2)^{2-2\sigma}$.[16]
- When $d = -1$, (2.10b) gives $MTSpin \wedge \Sigma^{-1}(B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$.

The low-degree homotopy groups of these spectra that we need are computed by Giambalvo [Gia73b] and Kirby-Taylor [KT90a, KT90b] (the pin$^+$ case); Anderson-Brown-Peterson [ABP69] and Kirby-Taylor [KT90b] (the pin$^-$ case); Giambalvo [Gia73a] (the case $d = 2$); and Mahowald-Milgram [MM76] (the spin $\times\, \mathbb{Z}/2$ case). Thus we obtain the phase homology groups for the spinless class D case in Table 2.

4.2.2. *Class D, spin-1/2 case.* Now we consider the case where the inversion symmetry and $\mu_2 \subset \mathrm{Pin}_d^-$ mix as specified by the nontrivial extension $1 \to \mu_2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$. This is not classified by $w_2 + w_1^2$ of the associated bundle to the spatial representation: in the language of §2, $\lambda \not\cong \xi$. Instead, this extension is classified by $w_2(\sigma) + w_1(\sigma)^2$, and σ is not pin$^-$, so if $f^D_{1/2}$ denotes the class D spin-1/2 equivariant local system of symmetry types on $\mathbb{R}^d$, Theorem 2.11 computes $Ph_*^{\mathbb{Z}/2}(\mathbb{R}^d; f^D_{1/2})$ using the Thom spectrum of the virtual bundle

$$-V \boxplus (\sigma + \sigma - d\sigma) \cong -V \boxplus (d-2)(1-\sigma). \tag{4.5}$$

Thus

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d; f^D_{1/2}) \cong [MTSpin \wedge (B\mathbb{Z}/2)^{(d-2)(1-\sigma)}, \Sigma^{d+2} I_{\mathbb{Z}}], \tag{4.6}$$

and Lemma 4.4 says the domain is again 4-periodic, but differently from the spinless case.

- When $d = 0$, we have $MTSpin \wedge (B\mathbb{Z}/2)^{2-2\sigma}$.
- When $d = 1$, we have $MTSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$.
- When $d = 2$, we have $MTSpin \wedge (B\mathbb{Z}/2)_+$.
- When $d = -1$, we have $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^+$.

In the degrees we need, these bordism groups are computed in the same references we gave above in §4.2.1, and the relevant phase homology groups appear in Table 2.

Remark 4.7. This fourfold periodicity in the tangential structure appears in a few other contexts in mathematical physics, such as recent work of Hason, Komargodski, and Thorngren [HKT20, §4.4] and Córdova, Ohmori, Shao, and Yan [COSY20] applying it to the study of anomalies of domain wall theories as well as work of Tachikawa and Yonekura [TY19, §3] studying anomalies arising in string theory.

4.2.3. *Class A.* In class A, whether with spinless or spin-1/2 fermions, the FCEP predicts by way of Theorem 2.24 that an inversion symmetry in dimension d leads us to study $MTSpin^c \wedge (B\mathbb{Z}/2)^{d-d\sigma}$. For any vector bundle $V \to X$, $V \oplus V \cong V \otimes \underline{\mathbb{C}}$, and complex vector bundles are spin^c, so by Theorem 1.39, we can remove factors of $2 - 2\sigma$ from $d - d\sigma$ without changing the Thom spectrum, so we want to study $MTSpin^c \wedge (B\mathbb{Z}/2)_+$ when d is even and $MTSpin^c \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^c$ when d is odd.

We discussed pin^c bordism in §4.1.3. Bahri-Gilkey [BG87a, BG87b] also compute $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/2)$: they establish that the *Smith homomorphism* $\widetilde{\Omega}_n^{\mathrm{Spin}^c}(B\mathbb{Z}/2) \to \Omega_{n-1}^{\mathrm{Pin}^c}$, which sends a spin^c manifold M and principal

[16]Campbell [Cam17, §7.8] shows this spectrum is equivalent to $MT(\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4)$. Bordism for this symmetry type, called spin-$\mathbb{Z}/4$ bordism or spin$^{c/2}$ bordism, is used in several places in recent mathematical physics literature, including [Cam17, Hsi18, FH19a, GEM19, TY19, DL20a, GOP$^+$20, HKT20, WW20a, Wan20, MV21].

$\mathbb{Z}/2$-bundle $P \to M$ to the induced pin^c structure on a smooth submanifold representative of the Poincaré dual of $w_1(P) \in H^1(M;\mathbb{Z}/2)$, is an isomorphism for all n; thus we get the groups in Table 2 by applying the universal property (1.7) of $I_{\mathbb{Z}}$ to either $\Omega_*^{\mathrm{Pin}^c}$ or $\Omega_*^{\mathrm{Spin}^c} \oplus \Omega_{*-1}^{\mathrm{Pin}^c}$, depending on dimension.

4.2.4. *Comparison with prior work.* Inversion-symmetric SPT phases are pretty well-studied, even in the fermionic case, and our phase homology calculations reproduce classifications of inversion-symmetric phases in the literature.

Class D, spinless: These phases are studied by Shiozaki-Xiong-Gomi [SXG18, §V.B] and Cheng-Wang [CW18, §III].

Class D, spin-1/2: These phases are studied by You-Xu [YX14, §III], Shiozaki-Shapourian-Ryu [SSR17a, SSR17b], Cheng-Wang [CW18, §III], and Shiozaki-Xiong-Gomi [SXG18, §V.A].

Class A: These phases are studied by You-Xu [YX14, §IV.A.3], Shiozaki-Shapourian-Ryu [SSR17b, §V.B], and Song-Huang-Fu-Hermele [SHFH17, §IV]. Shiozaki-Shapourian-Ryu also study the phases corresponding to the $\mathbb{Z}/2^{k+2}$ summand in $[\mathit{MTPin}^c, \Sigma^{2k+3} I_{\mathbb{Z}}]$ in arbitrary odd dimensions.[17]

Remark 4.8. Guo-Ohmori-Putrov-Wan-Wang [GOP+20, §10.8] also study inversion-symmetric fermionic phases from a bordism-theoretic perspective, in both the spinless and spin-1/2 cases. Their results disagree with ours, and with the rest of the literature, because they use different symmetry types to model inversion-equivariant fermionic phases.

4.3. **Rotations.** We turn to the case of phases equivariant for the cyclic group C_n acting by rotation on a plane. These phases have been studied by several groups of authors, and our results are consistent with prior work; see §4.3.4 for more information.

Let $\lambda\colon C_n \to \mathrm{SO}_2$ denote this representation and $V_\lambda \to BC_n$ be the associated vector bundle. One can directly check that $C_n \to \mathrm{SO}_2$ lifts across $\mathrm{Spin}_2 \to \mathrm{SO}_2$ iff n is odd.

d	n	Class D, spinless §4.3.1	Class D, spin-1/2 §4.3.2	Class A §4.3.3
2	0 mod 4	$\mathbb{Z} \oplus \mathbb{Z}/(n/2)$	$\mathbb{Z} \oplus \mathbb{Z}/2n \oplus \mathbb{Z}/2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2n \oplus \mathbb{Z}/(n/2)$
	2 mod 4	$\mathbb{Z} \oplus \mathbb{Z}/(n/2)$	$\mathbb{Z} \oplus \mathbb{Z}/4n$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2n \oplus \mathbb{Z}/(n/2)$
	$1,3$ mod 4	$\mathbb{Z} \oplus \mathbb{Z}/n$	$\mathbb{Z} \oplus \mathbb{Z}/n$	$\mathbb{Z}^2 \oplus \mathbb{Z}/n \oplus \mathbb{Z}/n$
3	0 mod 4	0	0	0
	2 mod 4	0	0	0
	$1,3$ mod 4	0	0	0

TABLE 3. C_n-equivariant phase homology groups for the cases in which C_n acts by rotations. Classification of fermionic phases with a C_n rotation symmetry. For the spinless class D case, these are classified by $[\mathit{MTSpin} \wedge (BC_n)^{2-V_\lambda}, \Sigma^{d+1} I_{\mathbb{Z}}]$; for spin-1/2 class D, by $[\mathit{MTSpin} \wedge (BC_n)_+, \Sigma^{d+1} I_{\mathbb{Z}}]$; and for class A, both spinless and spin-1/2, by $[\mathit{MTSpin}^c \wedge (BC_n)_+, \Sigma^{d+1} I_{\mathbb{Z}}]$.

4.3.1. *Class D, spinless case.* In this case, C_n does not mix with $\mu_2 \subset \mathrm{Spin}$, and Theorem 2.11 reduces Ansatz 1.22 to the computation of $[\mathit{MTSpin} \wedge (BC_n)^{2-V_\lambda}, \Sigma^{d+2} I_{\mathbb{Z}}]$ if n is even, or $[\mathit{MTSpin} \wedge (BC_n)_+]$, if n is odd.

Lemma 4.9. *$\Omega_3^{\mathrm{SO}}(BC_n) \cong \mathbb{Z}/n$, $\Omega_4^{\mathrm{SO}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{SO}}(BC_n)$ is torsion.*

Proof. Compute with the Atiyah-Hirzebruch spectral sequence for oriented bordism; it collapses for $p+q \le 4$, and the 5-line of the E^2-page is torsion, implying $\Omega_5^{\mathrm{SO}}(BC_n)$ is torsion. ☐

Corollary 4.10 (Bruner-Greenlees [BG10, Example 7.3.2, §12.2.D], García-Etxebarria and Montero [GEM19, §C.2]). *For n odd, $\Omega_3^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}/n$, $\Omega_4^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

[17] The presence of this summand follows from the existence of a $\mathbb{Z}/2^{k+2}$ summand in $\Omega_{2k+2}^{\mathrm{Pin}^c}$, which is proven by Bahri-Gilkey [BG87b].

Proof. Because n is odd, BC_n is stably trivial at 2, and $MTSpin \to MTSO$ is an equivalence away from 2. ☐

Theorem 4.11. *If n is even, $\widetilde{\Omega}_3^{\mathrm{Spin}}((BC_n)^{2-V_\lambda}) \cong \mathbb{Z}/(n/2)$, $\widetilde{\Omega}_4^{\mathrm{Spin}}((BC_n))^{2-V_\lambda}) \cong \mathbb{Z}$, and $\widetilde{\Omega}_5^{\mathrm{Spin}}((BC_n)^{2-V_\lambda})$ is torsion.*

Proof. The computation breaks into 2-primary and odd-primary pieces. The forgetful map $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an odd-primary isomorphism, and because $2 - V_\lambda$ is orientable, there is a Thom isomorphism $\widetilde{\Omega}_*^{\mathrm{SO}}((BC_n)^{2-V_\lambda}) \cong \Omega_*^{\mathrm{SO}}(BC_n)$. Thus, Lemma 4.9 takes care of the odd-primary part.

Write $n = 2^\ell m$, where m is odd. Then the map $BC_{2^\ell} \to BC_n$ is a stable 2-primary equivalence, because it induces an isomorphism on mod 2 cohomology, so for the 2-primary piece it suffices to understand the case $n = 2^\ell$. Campbell [Cam17, Theorem 1.8] studies $\Omega_d^{\mathrm{Spin}}((BC_{2^\ell})^{2-V_\lambda})$, obtaining $\mathbb{Z}/2^{\ell-1}$ when $d = 3$, $\mathbb{Z}$ when $d = 4$, and torsion when $d = 5$, which suffices.[18] ☐

4.3.2. *Class D, spin-1/2 case.* Theorem 2.11 asks us to compute $[MTSpin \wedge (BC_n)_+, \Sigma^{d+2} I_{\mathbb{Z}}]$, which (1.7) tells us in terms of $\Omega_*^{\mathrm{Spin}}(BC_n)$. For n odd, we already saw this in Corollary 4.10.

Proposition 4.12. *Let $n \equiv 2 \bmod 4$. Then $\Omega_3^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}/4n$, $\Omega_4^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

Proof. Inclusion $BC_2 \to BC_n$ is a 2-local equivalence, so the fact that the 2-torsion is $\mathbb{Z}/8$ in degree 3 and vanishes in degree 4 follows as soon as we know that for $\Omega_*^{\mathrm{Spin}}(BC_2)$. This was originally done by Mahowald-Milgram [MM76] but has been computed in a few other places, including Mahowald [Mah82, Lemma 7.3], Bruner-Greenlees [BG10, Example 7.3.1], Siegemeyer [Sie13, Theorem 2.1.5], and García-Etxebarria and Montero [GEM19, (C.18)]. What remains is odd-primary information, which is equivalent to the odd-primary part of oriented bordism, which we computed in Lemma 4.9. ☐

Proposition 4.13. *For $n \equiv 0 \bmod 4$, $\Omega_3^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2n$, $\Omega_4^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

Proof. Write $n = 2^\ell m$, where m is odd. As in the proof of Theorem 4.11, the 2-primary part of the answer is detected by $BC_{2^\ell} \to BC_n$, and the odd-primary part of the answer is detected oriented bordism. Davighi-Lohitsiri [DL20a, §A.3] compute $\Omega_k^{\mathrm{Spin}}(BC_{2^\ell})$ for $k \le 6$, giving the 2-primary summand, and for the odd-primary part we use Lemma 4.9. ☐

Botvinnik-Gilkey-Stolz [BGS97, Theorem 2.4], Bruner-Greenlees [BG10, Example 7.3.3], and Siegemeyer [Sie13, §2.2] do special cases of this computation, by a variety of methods.

4.3.3. *Class A.* The representation of C_n on $\mathbb{R}^2$ by rotations is unitary (under the standard identification $\mathbb{R}^2 = \mathbb{C}$), hence spin^c, so in both the spinless and spin-1/2 cases, we consider $MTSpin^c \wedge (BC_n)_+$: in the spinless case, we have a Thom isomorphism $MTSpin^c \wedge (BC_n)^{2-V_\lambda} \xrightarrow{\simeq} MTSpin^c \wedge (BC_n)_+$, and in the spin-1/2 case, $\mathrm{Det}(V_\lambda)$ is trivial, so Theorem 2.24 also gives us $MTSpin^c \wedge (BC_n)_+$.

Theorem 4.14. *The first few spin^c bordism groups of BC_n are*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}/2k & \Omega_1^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}/(2k+1)\\
\Omega_2^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z} & \Omega_2^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}\\
\Omega_3^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}/4k \oplus \mathbb{Z}/k & \Omega_3^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong (\mathbb{Z}/(2k+1))^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}^2 & \Omega_4^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}^2,
\end{aligned}$$

and $\Omega_5^{\mathrm{Spin}^c}(BC_n)$ is torsion for all n.

[18] There are a few other computations of $\widetilde{\Omega}_*^{\mathrm{Spin}}((BC_{2^\ell})^{2-V_\lambda})$ in low degrees by other methods. For $\ell = 1$, see Giambalvo [Gia73b], García-Etxebarria and Montero [GEM19, (C.21)], and Freed-Hopkins [FH19a, §5]. For $\ell > 1$, see Botvinnik-Gilkey [BG97, §5] and Davighi-Lohitsiri [DL20a, §A.4]; Botvinnik-Gilkey only report the orders of the bordism groups, but their computations show that the groups we need are cyclic. Be aware that Campbell and Davighi-Lohitsiri consider a different vector bundle than $2 - V_\lambda$, though their calculations apply to this case.

Proof. Write $n = 2^\ell \cdot m$, where m is odd. It suffices to compute the 2-primary piece and $\Omega_*^{\mathrm{Spin}^c}(BC_n)\otimes\mathbb{Z}[1/2]$. The inclusion $C_{2^\ell}\to C_n$ is stably a 2-primary equivalence, so for the 2-primary piece it suffices to determine $\Omega_*^{\mathrm{Spin}^c}(BC_{2^\ell})$. Bahri-Gilkey [BG87b, Theorem 1] compute these groups; when $\ell = 0$ they are $\Omega_*^{\mathrm{Spin}^c}(\mathrm{pt})$, which begins $\mathbb{Z}$, 0, $\mathbb{Z}$, 0, $\mathbb{Z}^2$, 0; and when $\ell \ne 0$ we have the same free summands as when $\ell = 0$, but additional torsion summands: $\Omega_1^{\mathrm{Spin}^c}(BC_{2^\ell}) \cong \mathbb{Z}/2^\ell$, and $\Omega_3^{\mathrm{Spin}^c}(BC_{2^\ell}) \cong \mathbb{Z}/2^{\ell-1}\oplus\mathbb{Z}/2^{\ell+1}$.

After smashing with $H\mathbb{Z}[1/2]$, the forgetful map $\mathit{MTSpin}^c \to \mathit{MTSO}\wedge(B\mathrm{U}_1)_+$ is an equivalence, so $\mathit{MTSO}\wedge(B\mathrm{U}_1)_+$ detects all odd-primary torsion in spin^c bordism. To compute this, we use the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(B\mathrm{U}_1\times BC_n;\Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(B\mathrm{U}_1\times BC_n). \tag{4.15}$$

The Künneth theorem implies the first few homology groups of $B\mathrm{U}_1\times BC_n$ are $H_0=\mathbb{Z}$, $H_1 = \mathbb{Z}/n$, $H_2=\mathbb{Z}$, $H_3 = (\mathbb{Z}/n)^{\oplus 2}$, $H_4 = \mathbb{Z}$, and $H_5 = (\mathbb{Z}/n)^{\oplus 3}$. When we feed this to the spectral sequence (4.15), there are no nonzero differentials to or from any element in total degree $p+q<5$: because $\Omega_i^{\mathrm{SO}} = 0$ for $i=1,2,3$, the only possible nonzero differential would be a $d_4\colon E^2_{5,0}\to E^2_{0,4}$, but the splitting $\Omega_*^{\mathrm{SO}}(B\mathrm{U}_1\times BC_n) = \Omega_*^{\mathrm{SO}}(\mathrm{pt})\oplus\widetilde\Omega_*^{\mathrm{SO}}(B\mathrm{U}_1\times BC_n)$ splits off the $q=0$ line splits off from the rest of the spectral sequence, killing this d_4. This tells the odd-primary torsion in degrees 0 through 4, and since the 5-line of the E_2-page is torsion, $\Omega_5^{\mathrm{SO}}(B\mathrm{U}_1\times BC_n)$ is also torsion. □

4.3.4. *Comparison with prior work.* Rotation-equivariant phases in class D have been studied by several groups, including Shiozaki-Shapourian-Ryu [SSR17b, §IV.C], Guo-Ohmori-Putrov-Wan-Wang [GOP+20, §10.9], and Freed-Hopkins [FH19a, §5], who all restrict to the case $n=2$, and most comprehensively by Cheng-Wang [CW18, §IV, §V], who consider arbitrary n and both the spinless and spin-1/2 cases in $d=2,3$. Freed-Hopkins begin from the same ansatz as us so agreement is no surprise. In the remaining cases, there is almost complete agreement: all classifications compute the same torsion summands, but they all miss the free summand in $d=2$. This is not a discrepancy, however: many authors restrict to considering phases whose low-energy effective theories are expected to be topological field theories, which in the ansatz of Freed-Hopkins [FH16a, §§5.3–5.4] amounts to considering the torsion subgroup of the classification using $I_{\mathbb{Z}}\mathit{MTH}$. The non-topological theories corresponding to the free summand have been discussed in a few references, including Freed [Fre19, Lecture 9] and Wan-Wang [WW20a, §7.1]; at present, their mathematical description remains partly conjectural.

Rotation-equivariant phases in class A are studied by Shiozaki-Shapourian-Ryu [SSR17b, §IV.D], Shiozaki-Xiong-Gomi [SXG18, §V.C.1], and Lu-Vishwanath-Khalaf [LVK19]. Shiozaki-Shapourian-Ryu and Lu-Vishwanath-Khalaf's classifications agree with us on torsion but miss the free summand as before, and Shiozaki-Xiong-Gomi's computation completely matches ours. Again, the free summand corresponds to non-topological invertible field theories.

4.4. **Rotations and reflections.** In this section, we compute the phase homology groups corresponding to phases on $\mathbb{R}^d$ equivariant for the D_{2n}-action of rotations and reflections in a given plane. Zhang-Wang-Yang-Qi-Gu [ZWY+20] also study these phases for $d=2$ and in class D; we compare our results to theirs in §4.4.5.

Let λ be the standard real 2-dimensional representation of D_{2n} and $V_\lambda\to BD_{2n}$ be the associated vector bundle. Let s be a reflection in D_{2n} and r a rotation through the angle $2\pi/n$. Then, define $x,y\in H^1(BD_{2n};\mathbb{Z}/2) = \mathrm{Hom}(D_{2n},\mathbb{Z}/2)$ by

$$x(s^\ell r^m) \coloneqq \ell \bmod 2 \tag{4.16a}$$

$$y(s^\ell r^m) \coloneqq m \bmod 2. \tag{4.16b}$$

In the representation λ, $s^\ell r^m\in D_{2n}$ acts by an orientation-reversing endomorphism iff ℓ is odd, so $w_1(V_\lambda) = x$.

Proposition 4.17 ([Sna13, Theorem 4.6], [Tei92, §2.3], [Han93, Theorems 5.5 and 5.6])**.**

(1) If n is odd, $H^*(BD_{2n};\mathbb{Z}/2)\cong\mathbb{Z}/2[x]$.
(2) If $n\equiv 0 \bmod 4$, $H^*(BD_{2n};\mathbb{Z}/2)\cong\mathbb{Z}/2[x,y,w]/(xy+y^2)$, where $|w|=2$ and $w=w_2(V_\lambda)$.
(3) If $n\equiv 2\bmod 4$, $H^*(BD_{2n};\mathbb{Z}/2)\cong\mathbb{Z}/2[x,y]$.

Lemma 4.18. *For $n\equiv 2\bmod 4$, $w_2(V_\lambda) = xy+y^2$.*

d	n	Class D, spinless §4.4.1	Class D, spin-1/2 §4.4.2	Class A, spinless §4.4.3	Class A, spin-1/2 §4.4.4
2	0 mod 4	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2n$	$\mathbb{Z}/(n/2) \oplus (\mathbb{Z}/2)^{\oplus 2}$
	2 mod 4	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2n$	$\mathbb{Z}/n \oplus \mathbb{Z}/2$
	1, 3 mod 4	$\mathbb{Z}/2$	0	$\mathbb{Z}/n$	$\mathbb{Z}/n$
3	0 mod 4	$(\mathbb{Z}/2)^{\oplus 4}$	0	$(\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$
	2 mod 4	$(\mathbb{Z}/2)^{\oplus 3}$	0	$(\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$
	1, 3 mod 4	$\mathbb{Z}/16$	0	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$

TABLE 4. D_{2n}-equivariant phase homology groups, where D_{2n} acts through rotations and reflections. These arise as homotopy groups of Anderson duals of $\mathit{MTSpin} \wedge X_n$ and $\mathit{MTSpin}^c \wedge X_n$, where X_n is one of $(BD_{2n})^{2-V_\lambda}$ or $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$. See §4.4 for details and proofs.

Proof. Since $s, r^{n/2} \in D_{2n}$ commute, there is a map $j\colon \mathbb{Z}/2\times\mathbb{Z}/2 \to D_{2n}$ sending $(1,0) \mapsto s$ and $(0,1) \mapsto r^{n/2}$. The pullback map $j^*\colon H^*(BD_{2n};\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$ sends x and y to linearly independent elements of $H^1(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$: one way to see this is to identify the pullback map with the map $\mathrm{Hom}(D_{2n},\mathbb{Z}/2) \to \mathrm{Hom}(\mathbb{Z}/2\times\mathbb{Z}/2,\mathbb{Z}/2)$ given by precomposing with j. Thus j^* is an isomorphism on $H^1(\text{–};\mathbb{Z}/2)$. For both BD_{2n} and $B\mathbb{Z}/2 \times B\mathbb{Z}/2$, the mod 2 cohomology ring is the free symmetric algebra on $H^1(\text{–};\mathbb{Z}/2)$, so j^* is an isomorphism of cohomology rings.

Thus we can compute $w_2(V_\lambda)$ by regarding λ as a $\mathbb{Z}/2\times\mathbb{Z}/2$ representation. Let $\ell_1 \subset \lambda$ be the fixed locus of s, which is a subspace, and ℓ_2 be its orthogonal complement. Then $\lambda = \ell_1 \oplus \ell_2$ as $(\mathbb{Z}/2\times\mathbb{Z}/2)$-representations. Both s and $r^{n/2}$ act nontrivially on ℓ_2; on ℓ_1, s acts trivially and $r^{n/2}$ acts nontrivially. Thus $w(\ell_1) = 1 + j^*(y)$, $w(\ell_2) = 1 + j^*(x) + j^*(y)$, and

$$w_2(j^*V_\lambda) = w_2(\ell_1) + w_1(\ell_1)w_1(\ell_2) + w_2(\ell_2) = j^*(y(x+y)). \tag{4.19}$$ ☐

Lemma 4.20. *Suppose n is odd and $i\colon \mathbb{Z}/2 \hookrightarrow D_{2n}$ is the inclusion of $\langle s\rangle$. Let $V \to BD_{2n}$ be a virtual vector bundle such that $w_1(V)$, as an element $\mathrm{Hom}(D_{2n},\mathbb{Z}/2)$, is nonzero on s. Then, the induced map of Thom spectra $\widehat{\imath}\colon (B\mathbb{Z}/2)^{i^*V} \to (BD_{2n})^V$ is a 2-primary homotopy equivalence.*

Proof. By the homology Whitehead theorem, it suffices to show $\widehat{\imath}$ induces an isomorphism on mod 2 cohomology. The Thom isomorphism rewords our question to be about the map $H^*(BD_{2n};\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, and Proposition 4.17 tells us that both $H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$ and $H^*(BD_{2n};\mathbb{Z}/2)$ are abstractly isomorphic to $\mathbb{Z}/2[x]$ with $|x| = 1$; we will show $i^*x_{BD_{2n}} = x_{B\mathbb{Z}/2}$, implying i^* is a ring isomorphism. Since x is the only nonzero degree-one element and V and i^*V are both unorientable, $x = w_1(V)$ and $i^*x = w_1(i^*V) \neq 0$. ☐

We will need the next calculations to determine the odd-primary torsion subgroups of the phase homology groups we calculate. Recall that $x \in H^1(BD_{2n};\mathbb{Z}/2)$ is equal to $w_1(V_\lambda)$.

Lemma 4.21 (Handel [Han93, Theorems 5.8, 5.9])**.**

$$H_*(BD_{2n};(\mathbb{Z}[1/2])_x) \cong \begin{cases} \mathbb{Z}/n, & n \equiv 1 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4.22}$$

Handel calculates $H^*(BD_{2n};\mathbb{Z}_x)$; use the universal coefficient theorem to switch to $\mathbb{Z}[1/2]$-homology.

Proposition 4.23. *Suppose $V \to BD_{2n}$ is a rank-zero virtual vector bundle with $w_1(V) = x$. Then the odd-torsion subgroup of $\widetilde{\Omega}_k^{\mathrm{Spin}}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ for $k = 1$, and vanishes for $k = 0, 2, 3,$ and 4.*

Proof. Apply the Atiyah-Hirzebruch spectral sequence for the completion of spin bordism at primes other than 2. Since $w_1(V) = x$, the Thom isomorphism identifies $\widetilde{H}_*((BD_{2n})^V) \cong H_*(BD_{2n};\mathbb{Z}_x)$, and by Lemma 4.21 we know these groups away from 2. The only nonzero entry in the E^2-page of total degree less than 5 is $E^2_{1,0} \cong \mathbb{Z}/n$, so the spectral sequence collapses in the desired range and we conclude. ☐

Proposition 4.24. *With V as in Proposition 4.23, the odd-torsion subgroup of $\widetilde{\Omega}_k^{\mathrm{Spin}^c}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ for $k=1$ and 3, and vanishes for $k=0,2,$ and 4.*

Proof. Use the Atiyah-Hirzebruch spectral sequence for the completion of MTSpin^c at odd primes, just as for Proposition 4.23. ☐

4.4.1. *Class D, spinless case.* Since we are considering spinless fermions, the FCEP tells us to compute $[\mathit{MTSpin}\wedge(BD_{2n})^{2-V_\lambda},\Sigma^{d+1}I_{\mathbb{Z}}]$.

Proposition 4.25. *For n odd, the first few spin bordism groups of X_n are*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/16,\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.20 with $2-V_\lambda$ to get a 2-primary stable equivalence $(BD_{2n})^{2-V_\lambda}\simeq(B\mathbb{Z}/2)^{1-\sigma}$, then (2.10a) to get $\mathit{MTSpin}\wedge(B\mathbb{Z}/2)^{1-\sigma}\simeq\mathit{MTPin}^+$. Low-degree pin$^+$ bordism groups are calculated in [Gia73b, KT90a, KT90b]. For the odd-torsion subgroups, use Proposition 4.23. ☐

Now we turn to the case where $n\equiv 2\bmod 4$.

Theorem 4.26. *When $n\equiv 2\bmod 4$, the first few spin bordism groups of X_n are*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)&\cong(\mathbb{Z}/2)^{\oplus 3},\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

As usual, this together with the universal property (1.7) of $I_{\mathbb{Z}}$ gives the $n\equiv 2\bmod 4$ entries in Table 4.

Proof. We will use the Adams spectral sequence at the prime 2 to compute $\widetilde{\Omega}_d^{\mathrm{Spin}}(X_n)$ for $d\le 7$. This only sees 2-primary information, but we already calculated the odd-torsion subgroup in Proposition 4.23. Recall that $w_1(V_\lambda)=x$ and (from Lemma 4.18) $w_2(V_\lambda)=xy+y^2$; thus $w_1(2-V_\lambda)=x$ and $w_2(2-V_\lambda)=x^2+xy+y^2$. This tells us the Steenrod squares in $\widetilde{H}^*(X_n;\mathbb{Z}/2)$, e.g. $\mathrm{Sq}^1(U)=Ux$ and $\mathrm{Sq}^2(U)=U(x^2+xy+y^2)$. Continuing in this vein determines the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$ in low degrees, as shown in Figure 1, left. We obtain a splitting as $\mathcal{A}(1)$-modules:

$$\widetilde{H}^*(X_n;\mathbb{Z}/2)\cong\mathcal{A}(1)\oplus\Sigma R_0\oplus\Sigma^2\mathcal{A}(1)\oplus\Sigma^4\mathcal{A}(1)\oplus\Sigma^4\mathcal{A}(1)\oplus P. \tag{4.27}$$

The $\mathcal{A}(1)$-module R_0 is defined to be $\widetilde{H}^*((B\mathbb{Z}/2)^{1-\sigma};\mathbb{Z}/2)$; the copy appearing here is the indecomposable summand containing Uy. The submodule P contains no elements of degree below 6, so is irrelevant for our low-degree computations; we need to determine $\mathrm{Ext}(M)$ for the remaining summands. For $\Sigma^k\mathcal{A}(1)$, there is a single $\mathbb{Z}/2$ summand in topological degree k and filtration 0, and for ΣR_0, see [GMM68, §2] or [BC18, Figure 24]. Putting these together, we display the E_2-page of this Adams spectral sequence in Figure 1, right. In this range, a combination of h_1-equivariance and Margolis' theorem (Theorem 3.22) forces all differentials to vanish, and Margolis' theorem implies there are no hidden extensions, so we are done. ☐

Finally, consider the case that $n\equiv 0\bmod 4$.

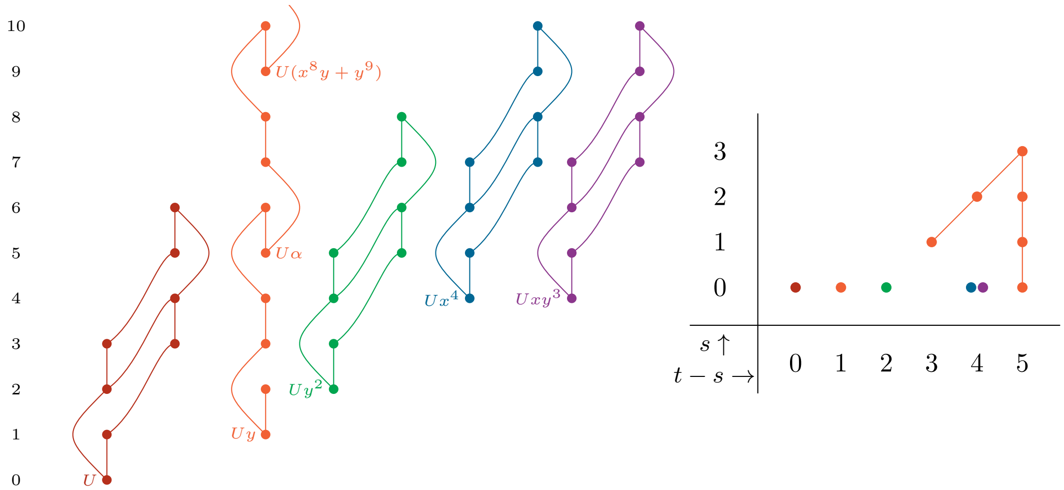


FIGURE 1. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda};\mathbb{Z}/2)$ in low degrees, when $n \equiv 2 \bmod 4$. Here $\alpha := x^4y + y^5$. The submodule pictured here contains all elements of degree at most 5. Right: the E_2-page of the corresponding Adams spectral sequence computing ko-theory.

Theorem 4.28. *Let $n \equiv 0 \bmod 4$.*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/2\\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) &\cong (\mathbb{Z}/2)^{\oplus 2}\\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) &\cong (\mathbb{Z}/2)^{\oplus 4},\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof of Theorem 4.28. First, by Proposition 4.23, the only odd-primary torsion in $\widetilde{\Omega}_k^{\mathrm{Spin}}(X_n)$ for $k \le 4$ is in degree 1. Draw the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = \widetilde{H}_p(X_n;\Omega_q^{\mathrm{Spin}}) \Longrightarrow \widetilde{\Omega}_{p+q}^{\mathrm{Spin}}(X_n). \tag{4.29}$$

After applying the Thom isomorphism, this needs as input $H_*(BD_{2n};\mathbb{Z}_x)$ and $H_*(BD_{2n};\mathbb{Z}/2)$. The former can be determined using Handel's calculation [Han93, Theorem 5.8] of $H^*(BD_{2n};\mathbb{Z}_x)$, and the latter can be determined from Proposition 4.17; in both cases use the universal coefficient theorem to pass from homology to cohomology. We obtain $E^2_{1,0} \cong \mathbb{Z}/n$ and $E^2_{0,1} \cong \mathbb{Z}/2$, so there are three options for $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)$: $\mathbb{Z}/n$, $\mathbb{Z}/n \oplus \mathbb{Z}/2$, or $\mathbb{Z}/2n$. We will address this ambiguity later.

Using Proposition 4.17, $w_1(2 - V_\lambda) = x$ and $w_2(2 - V_\lambda) = w + x^2$. Hence $\mathrm{Sq}^1(U) = Ux$ and $\mathrm{Sq}^2(U) = U(w + x^2)$. We also need the Steenrod squares of x, y, and w. For degree reasons, $\mathrm{Sq}(x) = x + x^2$ and $\mathrm{Sq}(y) = y + y^2$.

Lemma 4.30 ([Mal11, §4.1])**.** $\mathrm{Sq}(w) = w + wx + w^2$.

These and the Cartan formula determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$. In Figure 2, left, we display this structure in low degrees.

In particular,

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \cong \mathcal{A}(1) \oplus \Sigma R_2 \oplus \Sigma^2\mathbb{Z}/2 \oplus \Sigma^4 J \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^5\mathcal{O} \oplus P, \tag{4.31}$$

where P is 5-connected, and we define R_2, J, and $\mathcal{O}$ as follows. First, R_2 is defined to be the kernel of the augmentation map $\mathcal{A}(1) \to \mathbb{Z}/2$; the indecomposable summand in (4.31) isomorphic to ΣR_2 is generated by Uy and Uy^2. The *Joker* is the $\mathcal{A}(1)$-module $J := \mathcal{A}(1)/(\mathrm{Sq}^3)$; here it is generated by Uw^2. Finally,

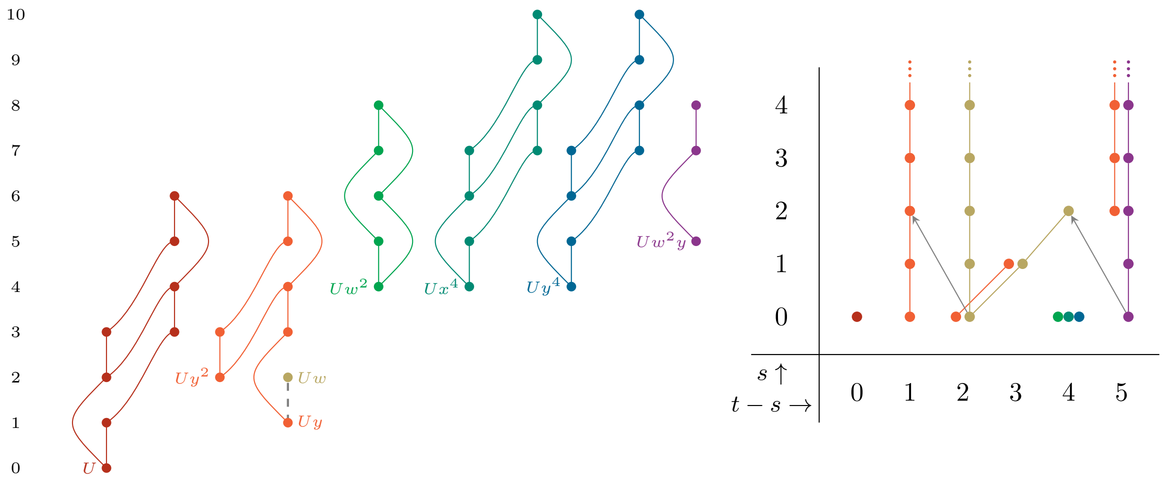


FIGURE 2. Left: the low-degree mod 2 cohomology of $(BD_{2n})^{2-V_\lambda}$ over $\mathcal{A}(1)$, $n \equiv 0 \bmod 4$. This summand contains all elements in degrees 5 and below. The dashed line indicates that the $\mathbb{Z}/2^r$ Bockstein maps Uy to Uw, which we need in Lemma 4.32. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ko}_*((BD_{2n})^{2-V_\lambda})^\wedge_2$. See Lemma 4.32 for how to address the differential in topological degree 2 and Lemma 4.35 to show the differential in topological degree 5 vanishes.

¿ $:= \mathcal{A}(1)/(\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^3)$ and is called the *upside-down question mark*; here it is generated by Uw^2y. For each of these summands M in (4.31), $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$ is known in the degrees relevant to us – except for P, which is too high-degree to affect our calculations anyways.

- For $\Sigma^k\mathcal{A}(1)$ there is a single $\mathbb{Z}/2$ in bidegree $s = 0$, $t = k$.
- For R_2, J, and ¿, see [BC18, Figure 29].[19]
- For $\mathbb{Z}/2$, see [BC18, Figure 20].

Put these together to obtain the E_2-page as in Figure 2, right. Lemma 3.24 tells us the E_∞-page is torsion, so there must be nonzero differentials in the range shown, though not necessarily the d_2s pictured.

The first nonzero differential is a d_r from the 2-line to the 1-line; by h_0-equivariance, it kills the entire yellow tower in the 2-line. Since a d_r differential decreases $t-s$ by 1 and increases s by r, on the E_{r+1}-page, the 2-line contains only the first r summands of the orange tower, and the 3-line contains only the orange $\mathbb{Z}/2$ summand in degree $s = 0$. There can be no further differentials to or from the 1- or 2-lines, so we obtain $\mathbb{Z}/2^r$ in degree 1 and $\mathbb{Z}/2$ in degree 2.

Lemma 4.32. *2^r is the largest power of 2 dividing n, i.e. $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n$.*

Proof. The May-Milgram theorem [MM81] identifies Adams spectral sequence differentials between towers with Bockstein spectral sequence differentials. What it means here is that the lemma statement is equivalent to the statement that the Bockstein map $\beta\colon H^1(-;\mathbb{Z}/2^r) \to H^2(-;\mathbb{Z}/2)$ associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2^{r+1} \longrightarrow \mathbb{Z}/2^r \longrightarrow 0 \tag{4.33}$$

carries a preimage of Uy to Uw. Both of these classes are in the image of the pullback map induced by $(B\mathbb{Z}/n)^{2-V} \to (BD_{2n})^{2-V}$, and the Bockstein is natural with respect to the Thom isomorphism, so we just have to check this in the cohomology of $B\mathbb{Z}/n$, where it is true [Cam17, DL20a]. ∎

The next differential that might be nonzero, and which is the only possibly nonzero differential to or from an element of degree 3 or 4, is $d_2\colon E_2^{0,5} \to E_2^{2,6}$. If this $d_2 = 0$, there is also an extension problem in degree $t-s = 4$ of the form

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 0. \tag{4.34}$$

[19] The first calculations of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(R_2, \mathbb{Z}/2)$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(J, \mathbb{Z}/2)$ that we know of are due to Adams-Priddy [AP76, §3].

Lemma 4.35. *This d_2 vanishes, and the extension (4.34) splits.*

Proof. We will prove this by mapping to a simpler Adams spectral sequence that has already been studied, as depicted in Figure 3.

Because V_λ is the pullback of the tautological bundle $V_2 \to B\mathrm{O}_2$ along $B\lambda\colon BD_{2n} \to B\mathrm{O}_2$, we obtain a map of Thom spectra $f\colon X_n = (BD_{2n})^{2-V_\lambda} \to (B\mathrm{O}_2)^{2-V_2}$; the codomain is often denoted $\Sigma^2 MTO_2$. Under f, our $Uw \in \widetilde{H}^2(X_n;\mathbb{Z}/2)$ is the pullback of $Uw_2 \in \widetilde{H}^2(\Sigma^2 MTO_2)$.

The spin bordism of $\Sigma^2 MTO_2$ is identified with the bordism theory of the group $\mathrm{Pin}^{\tilde{c}+} := (\mathrm{Pin}^+ \ltimes \mathrm{Spin}_2)/\mu_2$. Invertible field theories for this tangential structure are believed to correspond to invertible topological phases of Altland-Zirnbauer type AII [FH16a, (9.25), (10.2)].[20]

Several authors study the Adams spectral sequence for $\Omega_*^{\mathrm{Pin}^{\tilde{c}+}} \cong \widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^2 MTO_2)$ in low degrees, including Freed-Hopkins [FH16a, Figure 5, case $s = -2$], Campbell [Cam17, Example 6.10], and Wan-Wang-Zheng [WWZ20, §6.2.3]. Their work shows $Uw_2 \in \widetilde{H}^2(\Sigma^2 MTO_2;\mathbb{Z}/2)$ generates a $\Sigma^2\mathbb{Z}/2$ summand as an $\mathcal{A}(1)$-submodule of $\widetilde{H}^*(\Sigma^2 MTO_2;\mathbb{Z}/2)$, and therefore f^* restricts to an isomorphism from that $\Sigma^2\mathbb{Z}/2$ summand to our $\Sigma^2\mathbb{Z}/2$ summand generated by Uw. This means the submodule of the E_2-page for $\widetilde{\Omega}_*^{\mathrm{Spin}}(X_n)$ coming from $\Sigma^2\mathbb{Z}/2$ maps isomorphically onto the submodule of the E_2-page for $\widetilde{\Omega}_*^{\mathrm{Spin}}(\Sigma^2 MTO_2)$ coming from the $\Sigma^2\mathbb{Z}/2$ generated by Uw_2 — and crucially, in that spectral sequence, $E_2^{0,5} \cong 0$. See the commutative diagram of pink arrows in Figure 3. Thus the image of our d_2 under f vanishes, and the map between these spectral sequences on $E_2^{2,6}$s (the targets of these d_2s) is an isomorphism, so our d_2 also vanishes.

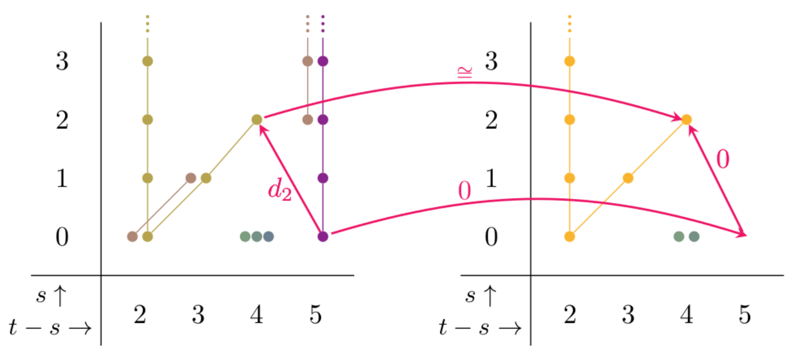


FIGURE 3. The map $X_n \to \Sigma^2 MTO_2$ induces a map between the Adams spectral sequences computing their *ko*-theory groups. We use this in Lemma 4.35 to show the pictured d_2 vanishes, as the square of pink arrows in the above figure is commutative. The right-hand side of this figure, which displays $\mathrm{Ext}(\widetilde{H}^*(\Sigma^2 MTO_2;\mathbb{Z}/2))$, is adapted from Campbell [Cam17, Figure 6.9].

Now suppose (4.34) does not split; then, there are elements $x, y \in \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)$ such that $x = 2y$ and the image of x image in the E_∞-page of the Adams spectral sequence is the nonzero element of $E_\infty^{2,6} \cong \mathbb{Z}/2$. Then f maps this $\mathbb{Z}/2$ isomorphically onto a $\mathbb{Z}/2$ in the E_∞-page for $\Sigma^2 MTO_2$, so $f_*(x) \neq 0$. But $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \cong (\mathbb{Z}/2)^{\oplus 3}$ [FH16a, Theorem 9.87], so no matter where y maps to, $2y = x \mapsto 0$, which is a problem. ☐

We have thus determined $\widetilde{\Omega}_d^{\mathrm{Spin}}(X_n)^\wedge_2$ for $d = 3, 4$, so we are done. ☐

4.4.2. *Class D, spin-1/2 case.*

Lemma 4.36. *V_λ is not pin$^-$.*

Proof. For n even, this follows by pulling back along $BC_n \to BD_{2n}$: we saw in §4.3 that the pullback is not spin, so V_λ cannot be pin$^-$. For n odd, pull back along the map $B\mathbb{Z}/2 \to BD_{2n}$ induced by the inclusion of a reflection; the pullback is not pin$^-$, so neither is V_λ. ☐

Therefore by Theorem 2.11, we consider $X_n := (BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.

[20]For further discussion, see also Metlitski [Met15] and Seiberg-Witten [SW16, §A.4].

Proposition 4.37. *For n odd, the first few spin bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2n\\ \widetilde\Omega_2^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/8\\ \widetilde\Omega_3^{\mathrm{Spin}}(X_n) &\cong 0\\ \widetilde\Omega_4^{\mathrm{Spin}}(X_n) &\cong 0,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.20 with $\mathrm{Det}(V_\lambda)-1$ get a 2-primary stable equivalence $(BD_{2n})^{Det(V_\lambda)-1} \simeq (B\mathbb Z/2)^{\sigma-1}$, then (2.10b) to get $\mathit{MTSpin}\wedge(B\mathbb Z/2)^{1-\sigma}\simeq \mathit{MTPin}^-$. Low-degree pin$^-$ bordism groups are calculated in [ABP69, KT90b]. For the odd-torsion subgroups, use Proposition 4.23. ☐

Theorem 4.38. *When $n\equiv 2 \bmod 4$, the first few bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/n\oplus\mathbb Z/2\\ \widetilde\Omega_2^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/8\oplus\mathbb Z/4\\ \widetilde\Omega_3^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2\oplus\mathbb Z/2\\ \widetilde\Omega_4^{\mathrm{Spin}}(X_n) &\cong 0,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. We establish a 2-primary equivalence $\mathit{MTSpin}\wedge X_n\simeq \mathit{MTPin}^-\wedge(B\mathbb Z/2)_+$, so the free and 2-torsion part of the spin bordism groups of X are isomorphic to the pin$^-$ bordism groups of $B\mathbb Z/2$. Once we finish this, we use work of Guo-Ohmori-Putrov-Wan-Wang [GOP$^+$20, §7.2.1] computing $\Omega_k^{\mathrm{Pin}^-}(B\mathbb Z/2)$ in degrees 5 and below to get the 2-primary part; for the odd-primary torsion, we use Proposition 4.23 as usual.

Lemma 4.39. *The inclusion $i\colon \mathbb Z/2\times\mathbb Z/2\hookrightarrow D_{2n}$ given by a reflection and a half-turn induces a 2-primary equivalence of Thom spectra $(B(\mathbb Z/2\times\mathbb Z/2))^{i^*\mathrm{Det}(V_\lambda)-1}\xrightarrow{\simeq}(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.*

Proof. The map $Bi\colon B(\mathbb Z/2\times\mathbb Z/2)\to BD_{2n}$ induces an equivalence on mod 2 cohomology, and therefore by the Thom isomorphism theorem also induces an equivalence on the mod 2 cohomology of the Thom spectra in question. This suffices by the stable Whitehead theorem. ☐

The stable bundle $i^*\mathrm{Det}(V_\lambda)\to B(\mathbb Z/2\times\mathbb Z/2)$ splits as an exterior direct sum $\sigma\boxplus\underline{0}$, where $\sigma\to B\mathbb Z/2$ is the tautological line bundle. Therefore the Thom spectrum also splits: $(B(\mathbb Z/2\times\mathbb Z/2))^{i^*\mathrm{Det}(V_\lambda)-1}\simeq (B\mathbb Z/2)^{\sigma-1}\wedge(B\mathbb Z/2)_+$. Therefore by (2.10b),

$$\mathit{MTSpin}\wedge(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}\simeq \mathit{MTSpin}\wedge(B\mathbb Z/2)^{\sigma-1}\wedge(B\mathbb Z/2)_+\simeq \mathit{MTPin}^-\wedge(B\mathbb Z/2)_+. \tag{4.40}$$ ☐

Finally, let $n\equiv 0\bmod 4$. Recall $H^*(BD_{2n};\mathbb Z/2)\cong\mathbb Z/2[x,y,w]/(xy+y^2)$ with $|x|=|y|=1$ and $|w|=2$, so $\mathrm{Sq}(x)=x+x^2$ and $\mathrm{Sq}(y)=y+y^2$, and from Lemma 4.30, $\mathrm{Sq}(w)=w+wx+w^2$. The Stiefel-Whitney classes of $\mathrm{Det}(V_\lambda)$ tell us that if U is the Thom class, $\mathrm{Sq}^1(U)=Ux$ and $\mathrm{Sq}^2(U)=0$ in the cohomology of X_n.

Theorem 4.41. *For $n\equiv 0\bmod 4$, the first few bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/n\oplus\mathbb Z/2\\ \widetilde\Omega_2^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/8\oplus\mathbb Z/4\\ \widetilde\Omega_3^{\mathrm{Spin}}(X_n) &\cong \mathbb Z/2\oplus\mathbb Z/2\\ \widetilde\Omega_4^{\mathrm{Spin}}(X_n) &\cong 0,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. First, by Proposition 4.23, the only odd-primary torsion in $\widetilde{\Omega}_k^{\mathrm{Spin}}(X_n)$ for $k \leq 4$ is in degree 1. Draw the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = \widetilde{H}_p(X_n; \Omega_q^{\mathrm{Spin}}) \Longrightarrow \widetilde{\Omega}_{p+q}^{\mathrm{Spin}}(X). \tag{4.42}$$

After applying the Thom isomorphism, this needs as input $H_*(BD_{2n}; \mathbb{Z}_x)$ and $H_*(BD_{2n}; \mathbb{Z}/2)$. The former can be determined using Handel's calculation [Han93, Theorem 5.8] of $H^*(BD_{2n}; \mathbb{Z}_x)$, and the latter can be determined from Proposition 4.17; in both cases use the universal coefficient theorem to pass from homology to cohomology. Since $E^2_{1,0} \cong \mathbb{Z}/n$ and $E^2_{0,1} \cong \mathbb{Z}/2$, there are three options for $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)$: $\mathbb{Z}/n$, $\mathbb{Z}/n \oplus \mathbb{Z}/2$, or $\mathbb{Z}/2n$. We will learn which one is correct in our analysis of the 2-primary part below.

For the 2-primary part, we use the Adams spectral sequence as usual. By Lemma 3.30, a choice of a reflection in D_{2n} induces a splitting

$$X_n \xrightarrow{\simeq} (B\mathbb{Z}/2)^{\sigma-1} \vee M_n, \tag{4.43}$$

such that the map $\widetilde{H}^*(M_n; \mathbb{Z}/2) \to \widetilde{H}^*(X_n; \mathbb{Z}/2)$ is injective with image complementary to the subspace spanned by $\{Ux^i \mid i \geq 0\}$. We focus on $\mathit{MTSpin} \wedge M_n$, adding in the summands arising from $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^-$ at the end. The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M_n; \mathbb{Z}/2)$ is determined by its image in $\widetilde{H}^*(X_n; \mathbb{Z}/2)$, which we know using Sq^1 and Sq^2 of x, y, w, and U via the Cartan formula. Using this, we draw this $\mathcal{A}(1)$-module structure in Figure 4, left.

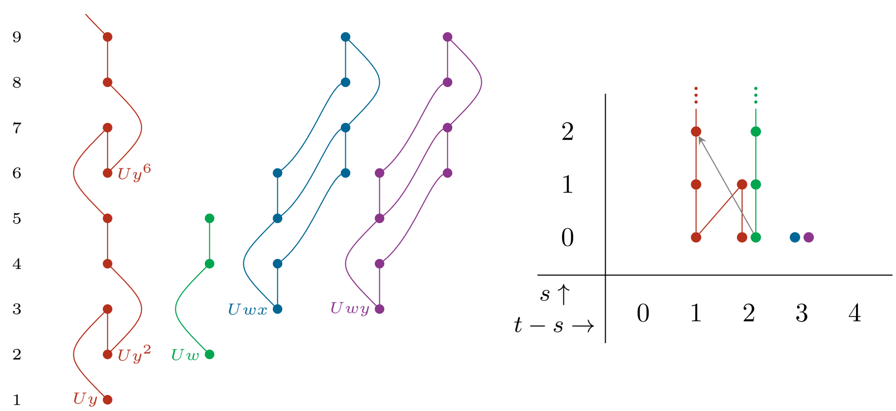


Figure 4. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M_n; \mathbb{Z}/2)$ in low degrees. The pictured summand contains all elements in degrees 4 and below. Right: the Ext of this module, which is the E_2-page of the Adams spectral sequence converging to $\widetilde{ko}_*(M_n)$. See the proof of Theorem 4.41 for more information.

As $\mathcal{A}(1)$-modules,

$$\widetilde{H}^*(M_n; \mathbb{Z}/2) \cong \Sigma R_1 \oplus \Sigma^2 \widetilde{\mathcal{O}} \oplus \Sigma^3 \mathcal{A}(1) \oplus \Sigma^3 \mathcal{A}(1) \oplus P, \tag{4.44}$$

where P is 4-connected; we will see below that the 4-line is empty, so there are no nonzero differentials from $\mathrm{Ext}(P)$ to anything we care about. Here ΣR_1 is the indecomposable summand containing Uy. For the $\Sigma^k \mathcal{A}(1)$ summands, we know the Ext; for ΣR_1, see [BC18, Figure 26], and for $\Sigma^2 \widetilde{\mathcal{O}}$, see [BC18, Figure 29]. Assembling these, we display the E_2-page for $t - s \leq 5$ in Figure 4, right. Lemma 3.24 implies $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion, as claimed, and that there must be a differential d_r from the infinite tower in topological degree 2 to the infinite tower in topological degree 1, though it might not be the d_2 pictured.[21] Margolis' theorem and h_0-equivariance rule out any other nonzero differentials to or from elements with $t - s \leq 4$. Therefore in this range, $E_{r+1} = E_\infty$. The infinite tower in topological degree 2 is killed by the differential, as are all but r of the $\mathbb{Z}/2$ summands of the infinite tower in topological degree 1. The first few 2-completed spin bordism groups of M_n are therefore $\mathbb{Z}/2^r$ in degree 1, $\mathbb{Z}/4$ in degree 2, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in degree 3, and 0 in degrees 0 and 4.

[21] In fact, r is the largest number such that $2^r \mid n$. Like in the proof of Lemma 4.32, one can deduce this using the Bockstein from Uy to Uw and the May-Milgram theorem.

Finally, we add in the pin$^-$ bordism summands as computed in [ABP69, KT90b]: a $\mathbb{Z}/2$ in degrees 0 and 1, a $\mathbb{Z}/8$ in degree 2, and 0 otherwise. In particular, since the 2-torsion subgroup of $\widetilde\Omega_1^{\mathrm{Spin}}(X_n)$ is of the form $\mathbb{Z}/2\oplus\mathbb{Z}/2^r$, $\widetilde\Omega_1^{\mathrm{Spin}}(X_n)\cong\mathbb{Z}/n\oplus\mathbb{Z}/2$. ☐

4.4.3. *Class A, spinless case.* In this case, Theorem 2.24 asks us to consider $X_n := \mathit{MTSpin}^c\wedge(BD_{2n})^{2-V_\lambda}$.

Theorem 4.45. *For n odd, the first few spin bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde\Omega_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde\Omega_2^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/4\\ \widetilde\Omega_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde\Omega_4^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/8\oplus\mathbb{Z}/2,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}^c}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.20 with $2-V_\lambda$ to get a 2-primary stable equivalence $(BD_{2n})^{2-V_\lambda}\simeq(B\mathbb{Z}/2)^{1-\sigma}$, then (2.10c) to get $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{1-\sigma}\simeq\mathit{MTPin}^c$. The pin^c bordism groups we need are calculated by Bahri-Gilkey [BG87a, BG87b]. For the odd-torsion subgroups, use Proposition 4.24. ☐

Theorem 4.46. *Let $n\equiv 2 \bmod 4$; then the low-degree spin^c bordism of X_n is*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde\Omega_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde\Omega_2^{\mathrm{Spin}^c}(X_n)&\cong(\mathbb{Z}/2)^{\oplus 2}\\ \widetilde\Omega_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2n\\ \widetilde\Omega_4^{\mathrm{Spin}^c}(X_n)&\cong(\mathbb{Z}/2)^{\oplus 4},\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}^c}(X_n)$ is torsion.

Proof. First, Proposition 4.24 computes the odd-torsion subgroups: a $\mathbb{Z}/n$ in degrees 1 and 3, and nothing else below degree 5.

To compute the 2-primary information we use the Adams spectral sequence over $\mathcal{E}(1)$, which converges to $\widetilde{ku}_*(X_n)$, together with Anderson-Brown-Peterson's isomorphism $\widetilde\Omega_n^{\mathrm{Spin}^c}(X_n)\xrightarrow{\cong}\widetilde{ku}_n(X_n)\oplus\widetilde{ku}_{n-4}(X_n)$ for $n\le 7$ [ABP67].

The $\mathcal{A}(1)$-module structure on $\widetilde H^*(X_n;\mathbb{Z}/2)$ that we calculated in (4.27) and displayed in Figure 1, left, determines the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1)\cong\mathcal{E}(1)\oplus\Sigma^2\mathcal{E}(1)$. Therefore

$$\widetilde H^*(X_n;\mathbb{Z}/2)\cong\mathcal{E}(1)\oplus\Sigma R_0\oplus\Sigma^2\mathcal{E}(1)\oplus\Sigma^2\mathcal{E}(1)\oplus\Sigma^4\mathcal{E}(1)\oplus\Sigma^4\mathcal{E}(1)\oplus\Sigma^4\mathcal{E}(1)\oplus P, \tag{4.47}$$

where P is 5-connected; we draw a picture of this $\mathcal{E}(1)$-module in Figure 5, left.

Next Ext. For $\Sigma^k\mathcal{E}(1)$, there is a unique $\mathbb{Z}/2$ summand, in degree $s=0$, $t=k$; for ΣR_0, we must work a little harder.

Proposition 4.48. $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(R_0,\mathbb{Z}/2)$ *is given in Figure 6, right.*

Proof. Our proof uses as input $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$, where N_1 is defined to be the $\mathcal{A}$-module $\Sigma^{-1}\widetilde H^*(\mathbb{RP}^2;\mathbb{Z}/2)$, with two $\mathbb{Z}/2$ summands connected by a Sq^1; this in turn defines its $\mathcal{A}(1)$- and $\mathcal{E}(1)$-module structures. Davis-Mahowald [DM81, §2] calculate $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$ as a graded vector space but we also need its $H^{*,*}(\mathcal{E}(1))$-module structure.

Let $\langle Q_1\rangle\subset\mathcal{E}(1)$ denote the subalgebra generated by Q_1, which is a two-dimensional vector space over $\mathbb{Z}/2$. As $\mathcal{E}(1)$-modules, $N_1\cong\mathcal{E}(1)\otimes_{\langle Q_1\rangle}\mathbb{Z}/2$, so by the change-of-rings theorem (3.4), there are isomorphisms of $H^{*,*}(\mathcal{E}(1))$-modules

$$\mathrm{Ext}_{\mathcal{E}(1)}(N_1)\cong\mathrm{Ext}_{\langle Q_1\rangle}(\mathbb{Z}/2)\cong\mathbb{Z}/2[v_1], \tag{4.49}$$

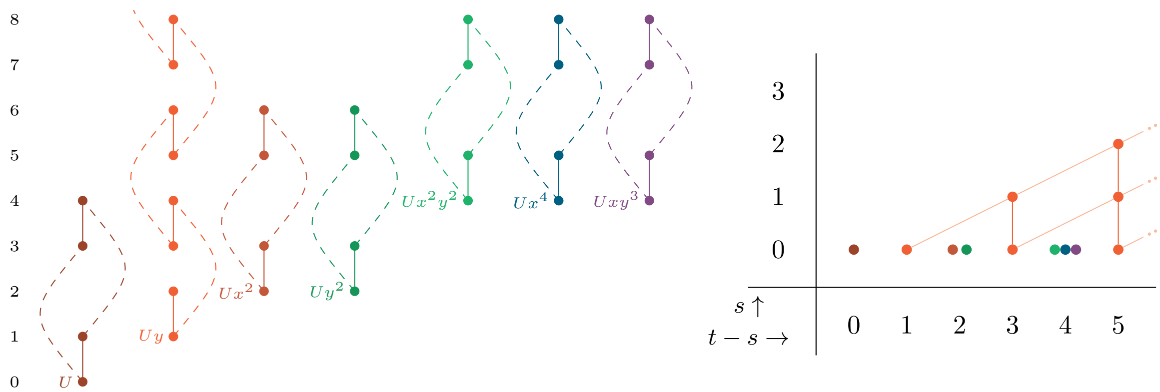


FIGURE 5. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda};\mathbb{Z}/2)$, $n\equiv 2 \bmod 4$, in low degrees. The pictured submodule contains all elements in degrees 5 and below. Right: the Adams E_2-page computing $\widetilde{ku}_*((BD_{2n})^{2-V_\lambda})$.

with $v_1 \in \mathrm{Ext}^{1,3}_{\mathcal{E}(1)}(N_1,\mathbb{Z}/2)$. The rightmost isomorphism in (4.49) uses Koszul duality [BC18, Remark 4.5.4], which applies because $\langle Q_1\rangle$ is an exterior algebra.

Now for R_0, we use the extension of $\mathcal{E}(1)$-modules

$$0 \longrightarrow \Sigma^2 R_0 \longrightarrow R_0 \longrightarrow N_1 \longrightarrow 0, \tag{4.50}$$

drawn in Figure 6, left.

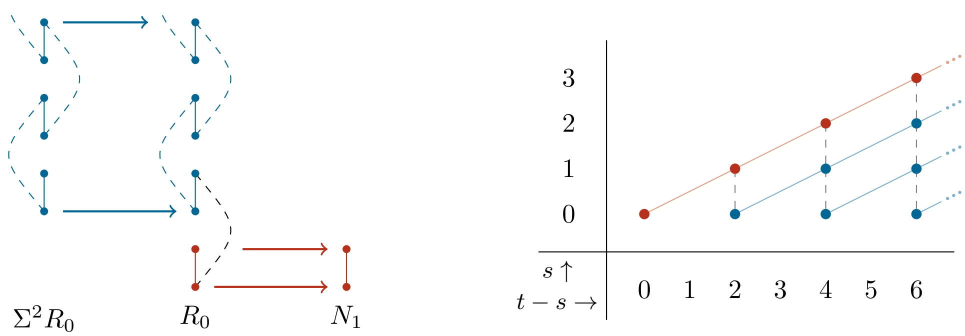


FIGURE 6. Left: the extension (4.50). Right: the long exact sequence it induces of Ext groups. See the proof of Proposition 4.48 for why the long exact sequence looks like this; the key feature is that there are no elements in odd topological degree, so all boundary maps vanish. The dashed lines are h_0-extensions which are not implied by the long exact sequence, but are shown in the proof of Proposition 4.48.

At first, all we know is $\mathrm{Ext}(N_1)$. Because this lives solely in even topological degrees, and $\Sigma^2 R_0$ is 2-connected, the long exact sequence diagram is empty in topological degree 1, so the boundary map

$$\delta\colon \mathrm{Ext}^{s,s+1}_{\mathcal{E}(1)}(R_0,\mathbb{Z}/2) \to \mathrm{Ext}^{s,s}_{\mathcal{E}(1)}(N_1,\mathbb{Z}/2) \tag{4.51}$$

vanishes, which tells us the line $t-s=0$ in $\mathrm{Ext}(R_0)$ consists of a single $\mathbb{Z}/2$ summand in filtration 0. Therefore the line $t-s=2$ in the long exact sequence diagram consists of two $\mathbb{Z}/2$ summands: one in filtration 1 coming from N_1, and one in filtration 0 coming from $\Sigma^2 R_0$. Since the 1-line of the diagram is empty and $\mathrm{Ext}(N_1)$ is concentrated in even degrees, the 3-line of the diagram is empty, so there are no differentials to the 2-line. Continuing in this way produces Figure 6, right.

Finally, acting by $h_0 \in H^{*,*}(\mathcal{E}(1))$ defines an isomorphism

$$\mathrm{Ext}^{0,2}_{\mathcal{E}(1)}(R_0,\mathbb{Z}/2) \to \mathrm{Ext}^{1,3}_{\mathcal{E}(1)}(R_0,\mathbb{Z}/2). \tag{4.52}$$

This can be checked directly from the definition: begin with the unique nontrivial map $R_0 \to \Sigma^2\mathbb{Z}/2$ and act on it by an extension representing h_0 (namely the extension $0 \to \Sigma\mathbb{Z}/2 \to N_1 \to \mathbb{Z}/2 \to 0$); the result is a nontrivial extension. ☐

With $\mathrm{Ext}(\Sigma R_0)$ in hand, we return to our goal of computing $\widetilde{ku}_*(X_n)$. We draw the E_2-page of the Adams spectral sequence in (5), right. Margolis' theorem (Theorem 3.22) forces all differentials in this range to vanish, except possible differentials with target the 7-line, and there can be no hidden extensions in the range depicted. Thus for $n = 2k < 7$, $\widetilde{ku}_*(X_n) \cong (\mathbb{Z}/2)^{\oplus k+1}$ and for $n = 2k+1 < 8$, $\widetilde{ku}_*(X_n) \cong \mathbb{Z}/2^{k+1}$; we finish with the fact that the map $MTSpin^c \to ku \vee \Sigma^4 ku$ is 7-connected, so we can read off the spin^c bordism groups from the ku-homology groups. ☐

Theorem 4.53. *If $n \equiv 0 \bmod 4$, write $n = 2^k m$ with m odd. The first few spin^c bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) \cong (\mathbb{Z}/2)^{\oplus 2}$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2n$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) \cong (\mathbb{Z}/2)^{\oplus 4},$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion.

Proof of Theorem 4.53. By Proposition 4.24, the odd-primary torsion is isomorphic to the odd-primary torsion of $\mathbb{Z}/n$ in degrees 1 and 3 and vanishes in degrees 0, 2, and 4.

At 2, we use the Adams spectral sequence. We described the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ in (4.31) and draw it in Figure 2; this determines the $\mathcal{E}(1)$-module structure, with isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, $R_2 \cong \mathcal{\mathring{O}} \oplus \Sigma\mathcal{E}(1)$ and $J \cong \mathcal{E}(1) \oplus \Sigma^2\mathbb{Z}/2$. Hence as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathcal{E}(1) \oplus \Sigma\mathring{\mathcal{O}} \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2\mathbb{Z}/2 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^5\mathring{\mathcal{O}} \oplus P, \tag{4.54}$$

where P is 5-connected. We draw this $\mathcal{E}(1)$-module in Figure 7, left.

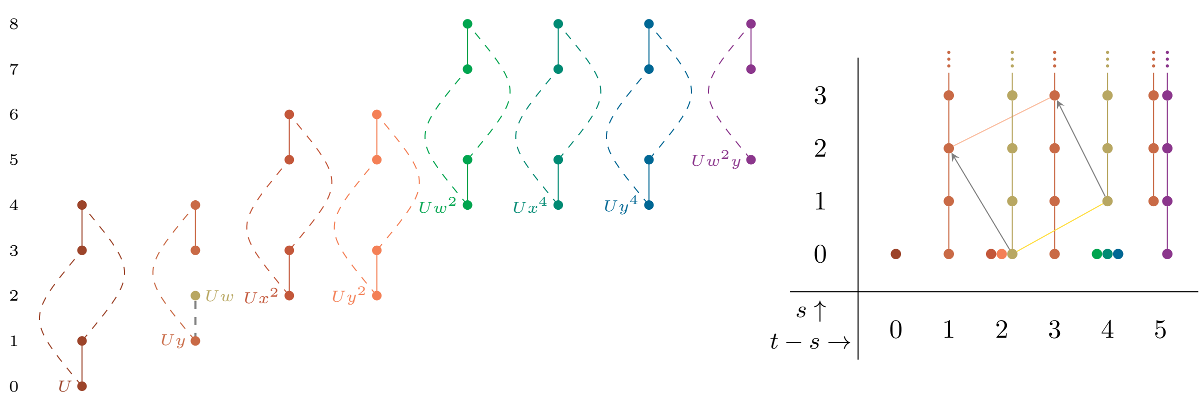


FIGURE 7. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda};\mathbb{Z}/2)$, $n \equiv 0 \bmod 4$, in low degrees. The pictured submodule contains all elements in degrees 5 and below. The gray dashed line indicates that the $\mathbb{Z}/2^r$ Bockstein maps a preimage of Uy to Uw, which we use in the proof of Theorem 4.53. Right: the E_2-page for the Adams spectral sequence computing $\widetilde{ku}_*((BD_{2n})^{2-V_\lambda})$. The two pictured differentials are related by a v_1-action.

We calculated $\mathrm{Ext}(\mathbb{Z}/2)$ in (3.7), and Adams-Priddy [AP76, §3] show

$$\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\mathring{\mathcal{O}}, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{E}(1)}^{s+1,t+1}(\mathbb{Z}/2,\mathbb{Z}/2), \tag{4.55}$$

with the isomorphism intertwining the $H^{*,*}(\mathcal{E}(1))$-actions. We can therefore draw the E_2-page of the Adams spectral sequence in Figure 7, right. We hide most v_1-actions to declutter the diagram.

The first differential that could be nonzero is from the 2-line to the 1-line; as differentials are h_0-equivariant, if a d_r differential is nonzero on one summand in the tower on the 2-line, then it is nonzero on the entire tower, so we refer to differentials between towers. The May-Milgram theorem [MM81] characterizes differentials between towers: there is a d_r differential between those two towers iff the Bockstein $\beta\colon H^1(-;\mathbb{Z}/2^r)\to H^2(-;\mathbb{Z}/2)$ carries a preimage of Uy to Uw. The Thom isomorphism is natural with respect to this Bockstein, so it suffices to know whether $\beta(y)=w$ in $H^2(BD_{2n};\mathbb{Z}/2)$, and we saw this in the proof of Lemma 4.32, where r is the largest number such that $2^r\mid n$. This means that the 2-torsion in $\widetilde\Omega_1^{\mathrm{Spin}^c}(X_n)$ is isomorphic to that of $\mathbb{Z}/n$, so along with our odd-torsion computation we see that $\widetilde\Omega_1^{\mathrm{Spin}^c}(X_n)\cong\mathbb{Z}/n$.

The other differential we need to resolve in range goes from the tower in the 4-line to the tower in the 3-line. Action by $v_1\in ku_2$ carries the tower in the 2-line to the tower in the 4-line, and the tower in the 1-line to the tower in the 3-line, and differentials are v_1-equivariant, so there is also a d_r differential between these towers. As seen in Figure 7, right, on the E_∞-page there are $r+1$ $\mathbb{Z}/2$ summands on the 3-line, all connected, so together with our odd-torsion computation we see that $\widetilde\Omega_3^{\mathrm{Spin}^c}(X_n)\cong\mathbb{Z}/2n$.

There can be no other nonzero differentials in range, and Margolis' theorem precludes any hidden extensions, so we are done. ☐

4.4.4. *Class A, spin-1/2 case.*

Lemma 4.56. *V_λ is pin^c iff n is odd.*

Proof. For n odd, we saw that inclusion of a reflection defines a map $B\mathbb{Z}/2\to BD_{2n}$ which is an isomorphism on mod 2 cohomology. Therefore we can compute Stiefel-Whitney classes of V_λ by pulling back to $B\mathbb{Z}/2$, and we saw that the pullback bundle is stably equivalent to a line bundle, so $w_2=0$.

For n even, recall that V_λ is pin^c iff $\beta(w_2(V_\lambda))=0$, where $\beta\colon H^k(-;\mathbb{Z}/2)\to H^{k+1}(-;\mathbb{Z})$ is the integral Bockstein. Lemma 3.28 means it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda))\neq 0$. In the notation of Proposition 4.17, for $n\equiv 2\bmod 4$, $w_2(V_\lambda)=xy+y^2$, and $\mathrm{Sq}^1(xy+y^2)=x^2y+xy^2\neq 0$. For $n\equiv 0\bmod 4$, $w_2(V_\lambda)=w$, and by Lemma 4.30, $\mathrm{Sq}^1(w)\neq 0$. ☐

Therefore for n odd, we consider $X_n\coloneqq (BD_{2n})^{2-V_\lambda}$. We computed $\Omega_k^{\mathrm{Spin}^c}(X_n)$ for $k\le 4$ in Theorem 4.45. For n even, Theorem 2.24 directs us to the spin^c bordism of $X_n\coloneqq (BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 4.57. *If $n\equiv 2\bmod 4$, the first few spin^c bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde\Omega_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde\Omega_2^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/4\oplus\mathbb{Z}/2\\ \widetilde\Omega_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\oplus\mathbb{Z}/2\\ \widetilde\Omega_4^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2.\end{aligned}$$

Because Lemma 3.24 implies $\widetilde\Omega_5^{\mathrm{Spin}^c}(X_n)$ is torsion, the phase homology groups for this symmetry type are $\mathbb{Z}/n\oplus\mathbb{Z}/2$ for $d=2$ and $\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2$ for $d=3$.

The 2-local equivalence $\mathit{MTSpin}\wedge X_n\simeq \mathit{MTPin}^-\wedge(B\mathbb{Z}/2)_+$ we used in Theorem 4.38 implies a 2-local equivalence $\mathit{MTSpin}^c\wedge X_n\simeq\mathit{MTPin}^c\wedge(B\mathbb{Z}/2)_+$, so when $n=2$, these are also the pin^c bordism groups of $\mathbb{Z}/2$. This may be of independent interest.

Proof. We can read the odd-primary torsion off of Proposition 4.24. For 2-primary torsion we use the Adams spectral sequence over $\mathcal{E}(1)$ as usual. Recall from the proof of Theorem 4.38 that $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}\simeq(B\mathbb{Z}/2)^{\sigma-1}\wedge(B\mathbb{Z}/2)_+$. Guo-Ohmori-Putrov-Wan-Wang [GOP+20, §7.2.1] determine the $\mathcal{A}(1)$-module structure on $\widetilde H^*((B\mathbb{Z}/2)^{\sigma-1}\wedge B\mathbb{Z}/2;\mathbb{Z}/2)$ in low degrees. Using their work, and the isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1)\cong\mathcal{E}(1)\oplus\mathcal{E}(1)$ and $R_5\cong\mathcal{E}(1)\oplus\Sigma R_0$, there is an isomorphism of $\mathcal{E}(1)$-modules

$$\widetilde H^*((B\mathbb{Z}/2)^{\sigma-1}\wedge B\mathbb{Z}/2;\mathbb{Z}/2)\cong\Sigma\mathcal{E}(1)\oplus\Sigma^2R_0\oplus\Sigma^3\mathcal{E}(1)\oplus\Sigma^3\mathcal{E}(1)\oplus P,\tag{4.58}$$

where P is 4-connected. Since we began with $(B\mathbb{Z}/2)^{\sigma-1}\wedge(B\mathbb{Z}/2)_+$, this does not account for everything; the disjoint basepoint gives us another summand equivalent to

$$\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{\sigma-1}\simeq\mathit{MTPin}^c\tag{4.59}$$

by (2.10c). We will add in the pin^c bordism groups coming from this summand, which can be read off from the work of Bahri-Gilkey [BG87a, BG87b], after running the Adams spectral sequence for the other summand.

Returning to (4.58), we will see momentarily that $E_2^{s,t}$ is empty when $t-s=4$ and $s\ge 2$, which precludes differentials from the 5-line to the 4-line and therefore means that P does not affect the calculations we make. In Figure 8, left, we draw (4.58). We computed $\mathrm{Ext}(R_0)$ in Proposition 4.48, so we can draw the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((B\mathbb{Z}/2)^{\sigma-1}\wedge B\mathbb{Z}/2)$, as in Figure 8, right. In the degrees we care about, this collapses, and we deduce the spin^c bordism of $(B\mathbb{Z}/2)^{\sigma-1}\wedge B\mathbb{Z}/2$ and combine it with pin^c bordism to conclude. ☐

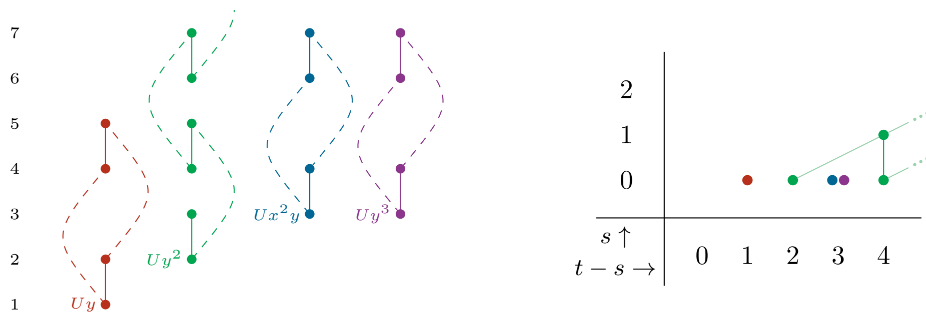


FIGURE 8. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}\wedge B\mathbb{Z}/2;\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements in degrees 4 and below. Right: Ext of this submodule, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M_n)$ for $t-s\le 4$. See the proof of Theorem 4.57 for more information.

Theorem 4.60. *If $n\equiv 0 \bmod 4$, the first few spin^c bordism groups of X_n are*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/4\oplus\mathbb{Z}/2\\ \widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/(n/2)\oplus(\mathbb{Z}/2)^{\oplus 2}\\ \widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2.\end{aligned}$$

Because Lemma 3.24 implies $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion, the phase homology groups for this symmetry type are $\mathbb{Z}/(n/2)\oplus(\mathbb{Z}/2)^{\oplus 2}$ for $d=2$ and $\mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2$ for $d=3$.

Proof. We closely follow the proof of Theorem 4.41. For odd-primary torsion, use Proposition 4.24 to see that the odd-primary torsion in the range we care about is isomorphic to the odd torsion in $\mathbb{Z}/n$ in degrees 1 and 3, and is 0 in degrees 0, 2, and 4.

On to the prime 2. In Theorem 4.41, we established a splitting $X_n\simeq(B\mathbb{Z}/2)^{\sigma-1}\vee M_n$, allowing us to focus solely on $\widetilde{\Omega}_*^{\mathrm{Spin}}(M_n)$: $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{\sigma-1}\simeq\mathit{MTPin}^c$ (2.10c), and we know pin^c bordism groups thanks to Bahri-Gilkey [BG87a, BG87b]. In (4.44), we determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees, and the isomorphisms of $\mathcal{E}(1)$-modules $R_1\cong\mathbb{Z}/2\oplus\Sigma R_0$ and $\mathcal{A}(1)\cong\mathcal{E}(1)\oplus\Sigma^2\mathcal{E}(1)$ mean that as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(M_n;\mathbb{Z}/2)\cong\Sigma\mathbb{Z}/2\oplus\Sigma^2R_0\oplus\Sigma^2\mathcal{O}\oplus\Sigma^3\mathcal{E}(1)\oplus\Sigma^3\mathcal{E}(1)\oplus P,\tag{4.61}$$

where P is 4-connected. A priori, $\mathrm{Ext}(P)$ could have nonzero differentials to elements of the 4-line, but we will see that this does not happen without needing to compute $\mathrm{Ext}(P)$. In Figure 9, left, we draw (4.61). To determine the E_2-page of the Adams spectral sequence, see (3.7) for $\mathrm{Ext}(\mathbb{Z}/2)$, Proposition 4.48 for $\mathrm{Ext}(R_0)$, and (4.55) for $\mathrm{Ext}(\mathcal{O})$. We draw the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(M_n)$, as in Figure 9, right — though for legibility, most v_1-actions are hidden. Lemma 3.24 implies there must be differentials in this range, though not necessarily the d_2s pictured.

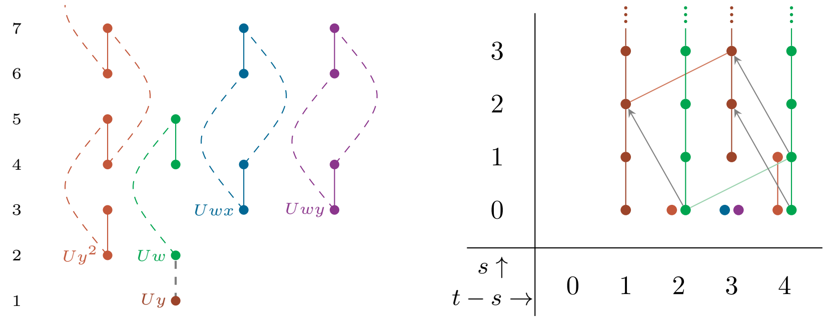


FIGURE 9. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements in degrees 4 and below. The dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, which we use to resolve a differential. Right: Ext of this submodule, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M_n)$ for $t-s\le 4$. Most v_1-actions are hidden for readability. See the proof of Theorem 4.60 for more information.

For $\widetilde\Omega_1^{\mathrm{Spin}^c}(M_n)$ to be torsion, there must be a differential d_r from the 2-line to the 1-line; then, $\widetilde\Omega_1^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/2^r$, and since $\Omega_1^{\mathrm{Pin}^c}\cong 0$, $\widetilde\Omega_1^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/2^r$ as well. Differentials between towers, such as this d_r, are characterized by the May-Milgram theorem [MM81], and just as in the proof of Theorem 4.53, we conclude r is the largest natural number such that $2^r\mid n$. Combining this with our odd-torsion calculation, $\widetilde\Omega_1^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/n$.

Continuing in increasing topological degree, this d_r kills the entire orange tower in the 2-line, and we infer $\widetilde\Omega_2^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/2$. The green and blue summands in the 3-line survive and split off by Margolis' theorem. v_1-equivariance of differentials implies that $d_r\colon E_2^{s,4+s}\to E_2^{s+2,s+3}$ is nonzero, and again maps the orange tower to the dark red tower, leaving a single $\mathbb{Z}/2$ summand in $E_3^{1,4}$. There can be no further differentials to the 3-line, so $\widetilde\Omega_3^{\mathrm{Spin}^c}(M_n)_2^\wedge\cong\mathbb{Z}/2^{r-1}\oplus(\mathbb{Z}/2)^{\oplus 2}$. Our odd-primary calculation then tells us that $\widetilde\Omega_3^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/(n/2)\oplus(\mathbb{Z}/2)^{\oplus 2}$. Finally, the orange tower in the 4-line is killed by the d_r we most recently discussed, and the two light red $\mathbb{Z}/2$ summands in the 4-line cannot emit or receive differentials. Thus as promised $\mathrm{Ext}(P)$ does not have nonzero differentials to the 4-line, so we conclude by adding the pin^c bordism summands back in. □

4.4.5. *Comparison with* [ZWY+20]. Interacting fermionic phases equivariant for a dihedral group D_{2n} acting by rotations and reflections have also been studied by Zhang-Wang-Yang-Qi-Gu [ZWY+20], who considered both spinless and spin-1/2 phases in dimension $2+1$ for all n, and in Altland-Zirnbauer class D. They also study systems without a spatial symmetry, using the extended supercohomology classification of Wang-Gu [WG18, WG20] to classify these phases and discuss the FCEP for dihedral groups. We find complete agreement with their results except for phases with spinless fermions when $n\equiv 0\bmod 4$, where we predict $\mathbb{Z}/2\oplus\mathbb{Z}/2$ and they predict $\mathbb{Z}/2$. This appears to arise from a calculation error: as we note below in Remark 4.62, the comparison map between supercohomology and the Anderson dual of spin bordism is an isomorphism for this symmetry type.

Remark 4.62. The phases we classify are realized by the extended supercohomology classifications of Wang-Gu [WG18, WG20] and Kapustin-Thorngren [KT17].[22] Gaiotto-Johnson-Freyd [GJF19, §§5.4–5.6] determine that the extended supercohomology classification à la [KT17, WG18] is the cohomology of $(BD_{2n})^{2-V_\lambda}$ or $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$ with respect to a spectrum they call $\mathrm{fGP}^\times_{\le 2}$, which is equivalent to the (-3)-connected cover of $I_\mathbb{Z}\mathit{MTSpin}$. Wang-Gu's refinement in [WG20] corresponds instead to the spectrum $\mathrm{fGP}^\times$, equivalent to the (-7)-connected cover of $I_\mathbb{Z}\mathit{MTSpin}$.[23]

[22]These classifications concern phases with an internal D_{2n} symmetry, but the fermionic crystalline equivalence principle allows us to pass back and forth.

[23]The reader may at this point wonder why our classification is a generalized *homology* theory, while these extended supercohomology classifications are generalized *cohomology* theories. This is a subtle point. The passage between homology and cohomology occurs because in these dimensions, we may approximate *MTSpin* by *KO* due to Anderson-Brown-Peterson's [ABP67] study of the connectivity of the Atiyah-Bott-Shapiro map [ABS64], then use that *KO* is shifted Anderson self-dual [And69,

The connective covering maps induce comparison maps from the classifications of fermionic phases using extended supercohomology to the classification of fermionic phases under our ansatz. For $\mathrm{fGP}^\times$, the map is sufficiently connected as to be an isomorphism between the classifications of $(d+1)$-dimensional phases for all $d \le 5$. For $\mathrm{fGP}^\times_{\le 2}$, the map is not always an isomorphism even for $d=2$: the cokernel when computing supercohomology of X is $\widetilde H^0(X;\mathbb Z)$, and this is nonzero e.g. for $X = (BC_n)^{2-V_\lambda}$ from §4.3. But for dihedral groups, $\widetilde H^0((BD_{2n})^\xi;\mathbb Z)$ vanishes whenever $\xi \to BD_{2n}$ is a rank-0 unorientable virtual vector bundle, so in this case the comparison map is an isomorphism.

4.5. D_{2n} **acting by rotations.** The dihedral group D_{2n} can act on $\mathbb R^3$ in an orientation-preserving manner, where $C_n \subset D_{2n}$ acts by rotations in a plane and a preimage of the generator of $D_{2n}/C_n \cong \mathbb Z/2$ acts by a rotation perpendicular to that plane. Said differently, this point group is defined by a representation $\lambda\colon D_{2n} \to \mathrm{SO}_3$ which decomposes as $\rho \oplus \sigma$, where ρ is the standard two-dimensional representation by rotations and reflections, and $\sigma\colon D_{2n} \to \mathrm O_1$ is the sign representation, which is the determinant of ρ. Confusingly, this point group is sometimes called "three-dimensional dihedral symmetry;" in this convention, the three-dimensional action by $\rho \oplus \underline{\mathbb R}$ is called *pyramidal symmetry*.

As far as we know, interacting fermionic phases for this D_{2n} symmetry have not been studied in the literature before.

n	Class D, spinless §4.5.1	Class D, spin-1/2 §4.5.2	Class A, spinless §4.5.3	Class A, spin-1/2 §4.5.4
0 mod 4	$\mathbb Z/2$	$(\mathbb Z/2)^{\oplus 2}$	0	$(\mathbb Z/2)^{\oplus 2}$
2 mod 4	0	$(\mathbb Z/2)^{\oplus 2}$	0	$(\mathbb Z/2)^{\oplus 3}$
1, 3 mod 4	0	0	0	0

TABLE 5. D_{2n}-equivariant phase homology groups, where D_{2n} acts faithfully on $\mathbb R^3$ by rotations. These arise as homotopy groups of Anderson duals of $\mathit{MTSpin}\wedge X_n$ and $\mathit{MTSpin}^c\wedge X_n$, where X_n is one of $(BD_{2n})^{3-V_\lambda}$ or $(BD_{2n})_+$. See §4.5 for details and proofs.

For any representation $\phi\colon D_{2n} \to \mathrm O_d$, let $V_\phi \to BD_{2n}$ denote the associated vector bundle.

Lemma 4.63.

(1) If n is odd, V_λ is pin^c but not pin$^-$.
(2) If n is even, V_λ is not pin^c.

Proof. For (2), we show that if β is the integral Bockstein, $\beta w_2(V_\lambda) \ne 0$. By Lemma 3.28, it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda)) \ne 0$. For $n \equiv 2 \bmod 4$,

$$w_2(V_\lambda) = w_2(V_\rho) + w_1(V_\rho)w_1(V_\sigma) + w_2(V_\sigma) = x^2 + xy + y^2, \tag{4.64a}$$

and $\mathrm{Sq}^1(x^2+xy+y^2) = x^2y + xy^2$. For $n \equiv 0 \bmod 4$,

$$w_2(V_\lambda) = w_2(V_\rho) + w_1(V_\rho)w_1(V_\sigma) + w_2(V_\sigma) = w + x^2, \tag{4.64b}$$

and $\mathrm{Sq}^1(w+x^2) = wx$, so in neither case is V_λ pin^c.

Now (1). Choose $i\colon \mathbb Z/2 \hookrightarrow D_{2n}$ given by a splitting of $D_{2n} \twoheadrightarrow D_{2n}/C_n \cong \mathbb Z/2$; restricting to $\mathbb Z/2$ along i, λ decomposes as $2\sigma \oplus \mathbb R$. Therefore $i^*V_\lambda \to B\mathbb Z/2$ is spin^c but not spin: $w_2(2V_\sigma) = w_1(V_\sigma)^2 = x^2$, and for any vector bundle V, $V\oplus V$ admits a complex structure, hence a spin^c structure. In particular, $\beta(w_2(i^*V_\lambda)) \ne 0$. The maps $\mathbb Z/2 \hookrightarrow D_{2n} \twoheadrightarrow \mathbb Z/2$ compose to the identity, so the induced maps on cohomology also compose to the identity. Therefore $\beta(w_2(V_\lambda)) \ne 0$ too. ☐

These propositions are the analogues of Propositions 4.23 and 4.24, helping us calculate odd-primary torsion in phase homology groups.

FMS07, HS14, Ric16, HLN20] to pass between $I_{\mathbb Z}KO$-homology and $\Sigma^4 KO$-cohomology. See Freed-Hopkins [FH19a, §5.1] for further discussion.

Lemma 4.65 (Handel [Han93, Theorems 5.2, 5.3])**.**

$$\widetilde{H}_k(BD_{2n};\mathbb{Z}[1/2]) \cong \begin{cases} \mathbb{Z}/n, & k\equiv 3 \bmod 4\\ 0, & \text{otherwise.}\end{cases} \tag{4.66}$$

As usual, Handel computes $H^*(BD_{2n};\mathbb{Z})$, and it is up to us to change to homology with $\mathbb{Z}[1/2]$ coefficients.

Proposition 4.67. *Suppose V is a rank-zero oriented virtual vector bundle.*

1. *The odd-torsion subgroup of $\widetilde{\Omega}_k^{\mathrm{Spin}}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ when $k=3$ and vanishes for all other $k\le 6$.*
2. *The odd-torsion subgroup of $\widetilde{\Omega}_k^{\mathrm{Spin}^c}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ when $k=3$ and $k=5$ and vanishes for all other $k\le 6$.*

Proof. It suffices to work at odd primes. There are odd-primary equivalences $\mathit{MTSpin}\to\mathit{MTSO}$ and $\mathit{MTSpin}^c\to \mathit{MTSO}\wedge(B\mathrm{U}_1)_+$; moreover, since V is oriented, there is a Thom isomorphism $\mathit{MTSO}\wedge(BD_{2n})_+\xrightarrow{\cong}\mathit{MTSO}\wedge(BD_{2n})^V$. Therefore it suffices to study $\widetilde{\Omega}_*^{\mathrm{SO}}(BD_{2n})$ for (1) and $\widetilde{\Omega}_*^{\mathrm{SO}}(BD_{2n}\wedge B\mathrm{U}_1)$ for (2) after completing at an odd prime p. Using Proposition 4.67 for input, as well as the Künneth formula to determine $H_*(BD_{2n}\wedge B\mathrm{U}_1)_p^\wedge$, one sees that the Atiyah-Hirzebruch spectral sequences computing these bordism groups collapse for degree reasons in total degree 6 and below. □

4.5.1. *Class D, spinless case.* Let f_0^D denote the equivariant local system of symmetry types for this case. Theorem 2.11 tells us that to compute $Ph_*^{D_{2n}}(\mathbb{R}^3, f_0^D)$, we should study the spin bordism of $X_n\coloneqq(BD_n)^{3-V_\lambda}$.

Proposition 4.68. *Suppose n is odd. Then*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/4\\
\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n)&\cong 0\\
\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/n\\
\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}\\
\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/16\\
\widetilde{\Omega}_6^{\mathrm{Spin}}(X_n)&\cong 0,
\end{aligned}$$

and therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D)\cong 0$.

Proof. Proposition 4.67 shows that $\widetilde{\Omega}_k^{\mathrm{Spin}}(X_n)$ lacks odd-primary torsion for $k=4,5$, so it suffices to work at 2. The inclusion $\mathbb{Z}/2\hookrightarrow D_{2n}$ induces an isomorphism $H^*(BD_{2n};\mathbb{Z}/2)\to H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, as we saw in the proof of Lemma 4.20, hence by naturality of the Thom isomorphism gives an isomorphism

$$\widetilde{H}^*(X_n;\mathbb{Z}/2)\xrightarrow{\cong}\widetilde{H}^*((B\mathbb{Z}/2)^{3-V_\lambda|_{B\mathbb{Z}/2}};\mathbb{Z}/2). \tag{4.69}$$

Restricted to $\mathbb{Z}/2$, $\lambda\cong 2\sigma\oplus\mathbb{R}$, so by the stable Whitehead theorem, (4.69) gives a stable 2-primary equivalence $X_n\simeq(B\mathbb{Z}/2)^{2-2\sigma}$. Campbell [Cam17, §7.8] computes $\widetilde{\Omega}_k^{\mathrm{Spin}}((B\mathbb{Z}/2)^{2-2\sigma})$, obtaining the free and 2-torsion summands we claim in the theorem statement.[24] □

Proposition 4.70 (Pedrotti [Ped17, Theorem 8.0.8])**.** *For $n\equiv 2 \bmod 4$, $\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)\cong\mathbb{Z}$, and by Lemma 3.24 $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D)$ vanishes.*

Remark 4.71. Pedrotti reports this computation in terms of w_1 and w_2 of $3-V_\lambda$, rather than λ itself, so we should check that our characteristic classes agree with his: we want $w_1(3-V_\lambda)=0$ and $w_2(3-V_\lambda)=x^2+xy+y^2$. Indeed $\mathrm{Im}(\lambda)\subset\mathrm{SO}_3$, so V_λ is orientable, and from (4.64a) that $w_2(V_\lambda)=x^2+xy+y^2$. Since $w_1(V_\lambda)=0$, these are also w_1 and w_2 of $3-V_\lambda$, as desired.

Proposition 4.72 (Pedrotti [Ped17, Theorem 9.0.14])**.** *For $n\equiv 0 \bmod 4$, $\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n)\cong\mathbb{Z}\oplus\mathbb{Z}/2$, and by Lemma 3.24 $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D)\cong\mathbb{Z}/2$.*

[24] Campbell computes only through dimension 5, and Beaudry-Campbell [BC18, Figure 26] shows how to extend Campbell's computation to dimension 6.

Pedrotti takes as input $w_1(3-V_\lambda)=0$ and $w_2(3-V_\lambda)=w+x^2$, which agrees with the classes of V_λ (e.g. (4.64b)). Beware that what we call x he calls y, and vice versa!

4.5.2. *Class D, spin-1/2 case.* Let $f^D_{1/2}$ denote the equivariant local system of symmetry types for this case. Lemma 4.63 and Theorem 2.11 tell us that to compute $Ph_*^{D_{2n}}(\mathbb{R}^3, f^D_{1/2})$, we should study the spin bordism of $(BD_n)^{\mathrm{Det}(V_\lambda)-1}$. Since V_λ is orientable, this is isomorphic to $\Omega_4^{\mathrm{Spin}}(BD_{2n})$.

Proposition 4.73. *Suppose n is odd. Then $\Omega_4^{\mathrm{Spin}}(BD_{2n})\cong\mathbb{Z}$ and $\Omega_5^{\mathrm{Spin}}(BD_{2n})\cong 0$, so $Ph_0^{D_{2n}}(\mathbb{R}^3, f^D_{1/2})\cong 0$.*

Proof. The proof is almost the same as that of Proposition 4.68: by Proposition 4.67, there is no odd-primary torsion, and $B\mathbb{Z}/2\to BD_{2n}$ induces an isomorphism on mod 2 cohomology, hence also on 2-local spin bordism, and Mahowald-Milgram [MM76] show $\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}/2)\cong\mathbb{Z}$ and $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}/2)\cong 0$.[25] □

Proposition 4.74. *For n even, $\Omega_4^{\mathrm{Spin}}(BD_{2n})\cong\mathbb{Z}\oplus(\mathbb{Z}/2)^{\oplus 2}$.*

Proof. Pedrotti [Ped17, Theorems 8.0.4 and 9.0.3] shows $\Omega_4^{\mathrm{Spin}}(BD_{2n})\cong\mathbb{Z}\oplus H_4(BD_{2n};\mathbb{Z})$, and the latter is computed by Handel [Han93, Theorem 5.2]. □

Bruner-Greenlees [BG10, Corollary 8.5.9] also compute this when n is a power of 2.

By Lemma 3.24, $\Omega_5^{\mathrm{Spin}}(BD_{2n})$ is torsion, so $Ph_0^{D}(\mathbb{R}^3, f^D_{1/2})\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$.

4.5.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types in the spinless type A case. By Lemma 4.63, we should compute $\widetilde\Omega_*^{\mathrm{Spin}^c}(X_n)$, where $X_n := (BD_{2n})^{3-V_\lambda}$.

When n is odd, V_λ is spin^c, so there is a Thom isomorphism $MTSpin^c\wedge X_n\simeq MTSpin^c\wedge(BD_{2n})_+$.

Theorem 4.75. *Suppose n is odd. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}/2\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}/4n\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}^2\\
\Omega_5^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}/8n\oplus\mathbb{Z}/2\\
\Omega_6^{\mathrm{Spin}^c}(BD_{2n})&\cong\mathbb{Z}^2.
\end{aligned}$$

Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^A)\cong 0$.

Proof. Proposition 4.67 accounts for the odd-primary torsion, so we just have to work at 2. The map $\mathbb{Z}/2\hookrightarrow D_{2n}$ induced by a choice of reflection defines an isomorphism on mod 2 cohomology, therefore by the stable Whitehead theorem is a 2-local stable equivalence. Therefore it defines an isomorphism $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/2)^\wedge_2\to\Omega_*^{\mathrm{Spin}^c}(BD_{2n})^\wedge_2$, and the spin^c bordism of $B\mathbb{Z}/2$ is computed by Bahri-Gilkey [BG87a, BG87b]. □

Theorem 4.76. *Suppose $n\equiv 2\bmod 4$. Then the first few spin^c bordism groups of X_n are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}(X)&\cong\mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}^c}(X)&\cong(\mathbb{Z}/2)^{\oplus 2}\\
\widetilde\Omega_2^{\mathrm{Spin}^c}(X)&\cong\mathbb{Z}\\
\widetilde\Omega_3^{\mathrm{Spin}^c}(X)&\cong\mathbb{Z}/n\oplus(\mathbb{Z}/2)^{\oplus 2}\\
\widetilde\Omega_4^{\mathrm{Spin}^c}(X)&\cong\mathbb{Z}^2,
\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion.

[25]This is also been computed by other methods by Mahowald [Mah82, Lemma 7.3], Bruner-Greenlees [BG10, Example 7.3.1], Siegemeyer [Sie13, Theorem 2.1.5], and García-Etxebarria and Montero [GEM19, (C.18)].

Proof. The odd-torsion subgroups can be read off of (4.67). For the 2-primary part, we use the Adams spectral sequence over $\mathcal{E}(1)$. Letting U denote the Thom class, we saw $w_1(V_\lambda) = 0$, so $\mathrm{Sq}^1(U) = 0$, and (4.64a) $w_2(V_\lambda) = x^2 + xy + y^2$, so $\mathrm{Sq}^2(U) = U(x^2 + xy + y^2)$. Using this and the Cartan formula, we have an $\mathcal{E}(1)$-module isomorphism

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \cong \mathcal{O} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus P, \tag{4.77}$$

where P is 4-connected. We draw this in Figure 10, left. A priori $\mathrm{Ext}(P)$ could have nonzero differentials to the 4-line and therefore affect our computation, but we will see that this cannot happen without needing to determine $\mathrm{Ext}(P)$.

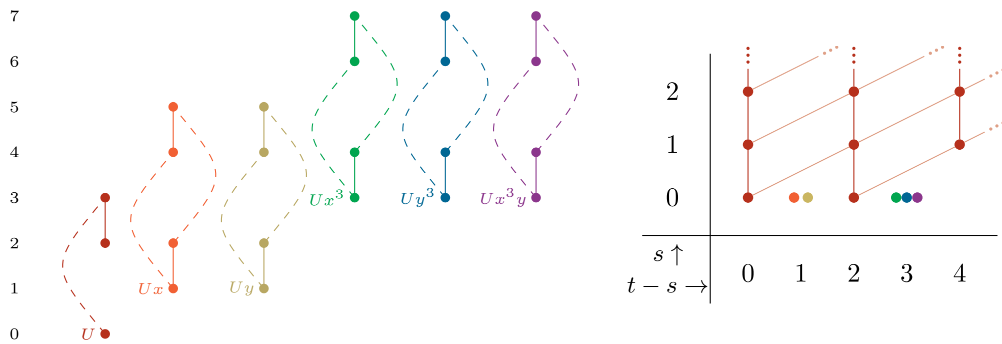


FIGURE 10. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$ when $n \equiv 2 \bmod 4$. The pictured summand contains all elements in degrees 4 and below. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ku}_*(X_n)^\wedge_2$.

We calculated $\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{O})$ in (4.55), so we can draw the E_2-page of the Adams spectral sequence in Figure 10, right. h_0-equivariance rules out nonzero differentials in degrees 3 and below, but a priori there could be a nonzero differential from the 5-line to then 4-line. To rule this out, use Lemma 3.24 to see that $\widetilde{ku}_4(X_n)$ has one free summand. Therefore there cannot be any nonzero differentials to the 4-line: h_0-equivariance would mean that if there were such a differential, it would kill all but finitely many summands in the 4-line of the E_2-page, preventing $\widetilde{ku}_4(X_n)$ from having a free part. ☐

Theorem 4.78. *When $n \equiv 0 \bmod 4$, the first few spin^c bordism groups of X_n are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/n \oplus (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}^2,
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^A) \cong 0$.

Proof. The odd-torsion subgroups are calculated in Proposition 4.67. For the 2-torsion, we use the Adams spectral sequence over $\mathcal{E}(1)$. Recall that $w_1(V_\lambda) = 0$ and (from (4.64b)) $w_2(V_\lambda) = w + x^2$, so $w_1(3 - V_\lambda) = 0$ and $w_2(3 - V_\lambda) = w + x^2$. Thus in $\widetilde{H}^*(X_n;\mathbb{Z}/2)$, $\mathrm{Sq}^1(U) = 0$ and $\mathrm{Sq}^2(U) = U(w + x^2)$. Using this and the Cartan formula, we can compute the $\mathcal{E}(1)$-action on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$, and find that

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \cong \mathcal{O} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2 \oplus \Sigma^4\mathcal{O} \oplus P, \tag{4.79}$$

where P is 4-connected. We draw this in Figure 11, left. We will see in a moment that $\mathrm{Ext}(P)$ has no nonzero differentials to elements in degree 4 and below, which means we can ignore it in our computations. We calculated $\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{O})$ in (4.55), so we can draw the E_2-page of the Adams spectral sequence in Figure 10, right. Margolis' theorem (Theorem 3.22) implies the only possible nonzero differentials from an element of topological degree 4 or below are the differentials from a tower in the 4-line to the blue tower in the 3-line,

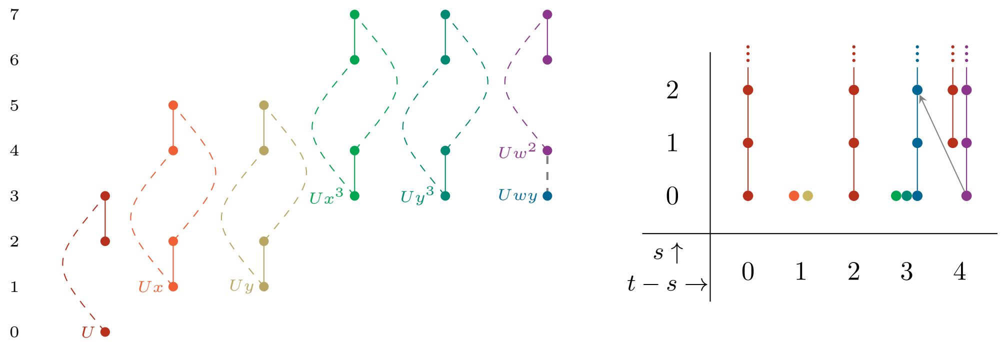


Figure 11. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$ when $n\equiv 0 \bmod 4$. The pictured summand contains all elements in degrees 4 and below. The gray dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, where r is the largest number for which $2^r\mid n$; this is not part of the $\mathcal{E}(1)$-module structure, but we use it in Theorem 4.78 to resolve a differential. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ku}_*(X_n)^\wedge_2$; v_1-actions are hidden for legibility. We will see in Theorem 4.78 that there is a d_r from the purple tower in the 4-line to the 3-line, though it is not always the d_2 pictured.

and Lemma 3.24 implies $\widetilde{ku}_4(X_n)$ has free rank 1, so for some r this differential d_r is nonzero. Moreover, its source must be the purple tower: the red tower is in the image of $v_1\colon E_2^{s,s+2}\to E_2^{s+1,s+5}$, so if $d_r(x)=y$ for any element x of the red tower in degree 4, then y is also in the image of v_1, but the blue tower is not in this image. Therefore we know that d_r kills the entire purple tower in degree 4, and the red tower survives to the E_∞-page: the red tower supports no nonzero differentials to the 3-line, and if there were a differential from the 5-line to the red tower, h_0-linearity guarantees it would kill all but finitely many summands of the red tower, contradicting Lemma 3.24.

It remains only to determine the value of r. In $H^*(BD_{2n};\mathbb{Z}/2)$, the $\mathbb{Z}/2^k$ Bockstein carries (a preimage of) wy to w^2, where k is the largest number such that $2^k\mid n$. This can be checked by, e.g., pulling back to BC_n, where this Bockstein is discussed by [Cam17, DL20a]. The Thom isomorphism theorem implies the $\mathbb{Z}/2^k$ Bockstein sends (a preimage of) Uwy to Uw^2, and therefore by the May-Milgram theorem [MM81], $r=k$. ◻

4.5.4. *Class A, spin-1/2 case.* Let $f^A_{1/2}$ denote the equivariant local system of symmetry types in the spin-1/2 type A case. In this case the ansatz tells us to study $\Omega_*^{\mathrm{Spin}^c}(BD_{2n})$.

Proposition 4.80. *For n odd, $Ph_0^{D_{2n}}(\mathbb{R}^3,f^A_{1/2})=0$.*

Proof. This follows from our computation of $\Omega_k^{\mathrm{Spin}^c}(BD_{2n})$ in Theorem 4.75. ◻

Theorem 4.81. *Suppose $n\equiv 2 \bmod 4$. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n\oplus(\mathbb{Z}/4)^{\oplus 2}\oplus(\mathbb{Z}/2)^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 3}\\
\Omega_5^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n\oplus(\mathbb{Z}/8)^{\oplus 2}\oplus\mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 5}\\
\Omega_6^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 6}.
\end{aligned}$$

Proof. We calculated the odd-primary torsion in these bordism groups in Proposition 4.67; now the 2-primary part. The inclusion $\mathbb{Z}/2\times\mathbb{Z}/2\to D_{2n}$ given by a reflection and a rotation by π induces an isomorphism on mod 2 cohomology, so by the stable Whitehead theorem, $BD_{2n}\to B(\mathbb{Z}/2\times\mathbb{Z}/2)$ is an equivalence after stabilizing and 2-completing. Therefore $\Omega_*^{\mathrm{Spin}^c}(BD_{2n})_2^\wedge \xrightarrow{\cong} \Omega_*^{\mathrm{Spin}^c}(B(\mathbb{Z}/2\times\mathbb{Z}/2))_2^\wedge$; since $MTSpin^c \to ku\vee\Sigma^4 ku$ is an isomorphism in degrees 8 and below, it suffices to know $ku_*(B(\mathbb{Z}/2\times\mathbb{Z}/2))$.

Ossa [Oss89, Proposition 3] computes $ku_*(B(\mathbb{Z}/2\times\mathbb{Z}/2))$ by establishing an equivalence

$$ku\wedge B\mathbb{Z}/2\wedge B\mathbb{Z}/2 \simeq (ku\wedge\Sigma^2 B\mathbb{Z}/2)\vee\Sigma^2 H(\mathbb{Z}/2[u,v]), \tag{4.82}$$

where the third term refers to a generalized Eilenberg-Mac Lane spectrum on the graded abelian group $\mathbb{Z}/2[u,v]$.[26] Using the stable splitting

$$\Sigma^\infty(B\mathbb{Z}/2\times B\mathbb{Z}/2)_+ \simeq \mathbb{S}\vee\Sigma^\infty B\mathbb{Z}/2\vee\Sigma^\infty B\mathbb{Z}/2\vee\Sigma^\infty(B\mathbb{Z}/2\wedge B\mathbb{Z}/2), \tag{4.83}$$

we see that $ku_*(B(\mathbb{Z}/2\times\mathbb{Z}/2))$ can be assembled from the following pieces.

(1) $ku_*(\mathrm{pt})$, which contributes $\mathbb{Z}$ in even degrees and 0 in odd degrees.
(2) Two copies of $\widetilde{ku}_*(B\mathbb{Z}/2)$. Hashimoto [Has83, Theorem 3.1] shows each copy vanishes in even degrees and is isomorphic to $\mathbb{Z}/2^{k+1}$ in odd degree $2k+1$.
(3) $\widetilde{ku}_*(\Sigma^2 B\mathbb{Z}/2)$. Hashimoto (*ibid.*) shows this vanishes in even degrees and is isomorphic to $\mathbb{Z}/2^k$ in odd degree $2k+1$.
(4) $\pi_*(\Sigma^2 H\mathbb{Z}/2[u,v])$, which contributes 0 in degrees 0 and 1 and $(\mathbb{Z}/2)^{\oplus(k-1)}$ in degrees $k\geq 2$.

Putting this together and adding in the odd-primary torsion, we obtain $ku_k(BD_{2n})$ for $k\leq 6$; using the Anderson-Brown-Peterson isomorphism $\Omega_k^{\mathrm{Spin}^c}(BD_{2n})\cong ku_k(BD_{2n})\oplus ku_{k-4}(BD_{2n})$, valid for $k<8$, we obtain the bordism groups in the theorem statement. ☐

Theorem 4.84. *Suppose $n\equiv 0 \bmod 4$. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n\oplus(\mathbb{Z}/4)^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 2},
\end{aligned}$$

and $\Omega_5^{\mathrm{Spin}^c}(BD_{2n})$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_{1/2}^A)\cong(\mathbb{Z}/2)^{\oplus 2}$.

Proof. Since $\Omega_*^{\mathrm{Spin}^c}(BD_{2n})\cong\Omega_*^{\mathrm{Spin}^c}(\mathrm{pt})\oplus\widetilde\Omega_*^{\mathrm{Spin}^c}(BD_{2n})$, we will focus on $\widetilde\Omega_*^{\mathrm{Spin}^c}(BD_{2n})$, adding on $\Omega_*^{\mathrm{Spin}^c}(\mathrm{pt})$ at the end. We also focus on the 2-primary story: the odd-primary torsion is calculated in Proposition 4.67.

Recall from Proposition 4.17 that $H^*(BD_{2n};\mathbb{Z}/2)\cong\mathbb{Z}/2[x,y,w]/(xy+y^2)$, with $|x|=|y|=1$ and $|w|=2$. A choice of reflection induces a section of $BD_{2n}\to B\mathbb{Z}/2$, and therefore there is a spectrum M_n and a splitting

$$\Sigma^\infty BD_{2n}\xrightarrow{\cong} M_n\vee\Sigma^\infty B\mathbb{Z}/2, \tag{4.85}$$

such that as a subspace of $\widetilde H^*(BD_{2n};\mathbb{Z}/2)$, $\widetilde H^*(M_n;\mathbb{Z}/2)$ is complementary to the subspace S spanned by $\{x^n \mid n\geq 0\}$, because S is the image of the pullback map $\widetilde H^*(B\mathbb{Z}/2;\mathbb{Z}/2)\to\widetilde H^*(BD_{2n};\mathbb{Z}/2)$. Bahri-Gilkey [BG87a, BG87b] show that $\widetilde\Omega_4^{\mathrm{Spin}^c}(B\mathbb{Z}/2)\cong 0$, so we just have to understand $\widetilde\Omega_4^{\mathrm{Spin}^c}(M_n)$.

We will use the Adams spectral sequence over $\mathcal{E}(1)$ to show that $\widetilde{ku}_0(M_n)\cong 0$ and $\widetilde{ku}_4(M_n)\cong(\mathbb{Z}/2)^{\oplus 2}$, which suffices to prove the theorem. For degree reasons, $\mathrm{Sq}(x)=x+x^2$ and $\mathrm{Sq}(y)=y+y^2$, and in Lemma 4.30 we saw $\mathrm{Sq}(w)=w+wx+w^2$. Using this, we find that as $\mathcal{E}(1)$-modules,

$$\widetilde H^*(M_n;\mathbb{Z}/2)\cong\Sigma R_0\oplus\Sigma^2\mathcal{E}(1)\oplus\Sigma^3\mathring{\circlearrowright}\oplus\Sigma^4\mathbb{Z}/2\oplus\Sigma^4\mathcal{E}(1)\oplus\Sigma^4\mathcal{E}(1)\oplus P, \tag{4.86}$$

where P is 5-connected, and therefore too highly connected to affect our calculations. We draw this in Figure 12, left. We have already computed $\mathrm{Ext}(M)$ for the remaining summands M: see Proposition 4.48 for $\mathrm{Ext}(R_0)$, (4.55) for $\mathrm{Ext}(\mathring{\circlearrowright})$, and (3.7) for $\mathrm{Ext}(\mathbb{Z}/2)$. Therefore we obtain the E_2-page of the Adams spectral sequence in Figure 12, right.

[26] Ossa's splitting (4.82) or its analogue on homotopy groups has also been proven in several other ways: see Johnson-Wilson [JW97], Bruner [Bru99, Corollary 3.3], Bruner-Greenlees [BG03, Example 4.11.2], Powell [Pow14], and Bruner-Mira-Stanley-Snaith [BMSS15, Theorem 2.12].

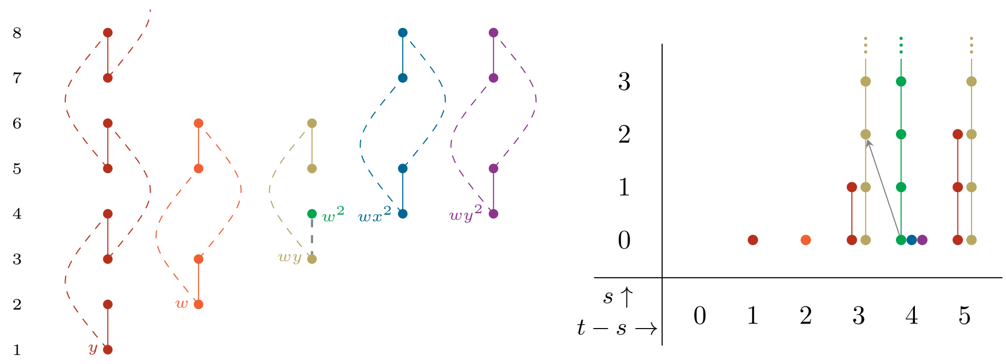


FIGURE 12. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees. The gray dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, where r is the largest number such that $2^r \mid n$; this is not part of the $\mathcal{E}(1)$-module structure, but we will use it in Theorem 4.84 to resolve a differential. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ku}(M_n)^\wedge_2$. v_1-actions are hidden for readability. We will see in Theorem 4.84 that there is a nonzero differential from the 4-line to the 3-line, though it is not necessarily the d_2 pictured.

The 0-line is empty, so $\widetilde{ku}_0(M_n) \cong 0$, as promised. Lemma 3.24 implies $\widetilde{ku}_3(M_n)$ is torsion; therefore there must be a nonzero differential d_r from the purple tower in the 4-line to the yellow tower in the 3-line. As in previous examples (Lemma 4.32 and Theorems 4.53 and 4.60), k is the largest number such that $2^k \mid n$: the $\mathbb{Z}/2^k$ Bockstein sends a preimage of wy to w^2, which can be checked after pulling back to BC_n as usual. The May-Milgram theorem [MM81] then identifies $r = k$. Therefore from the E_{r+1}-page onward, the green tower is gone, and the 4-line consists only of the two $\mathbb{Z}/2$ summands in Adams filtration zero, so $\widetilde{ku}_4(M_n) \cong (\mathbb{Z}/2)^{\oplus 2}$. □

5. EXAMPLES: TETRAHEDRAL, OCTAHEDRAL, AND ICOSAHEDRAL SYMMETRIES

Point group	Ref.	Class D, spinless	Class D, spin-1/2	Class A, spinless	Class A, spin-1/2
Chiral tet. (A_4, T)	§5.1	0	0	0	0
Pyrit. ($A_4 \times \mathbb{Z}/2$, T_h)	§5.2	$(\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$
Full tet. (S_4, T_d)	§5.3	$\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$	0	$(\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$
Chiral oct. (S_4, O)	§5.4	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
Full oct. ($S_4 \times \mathbb{Z}/2$, O_h)	§5.5	$(\mathbb{Z}/2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}$
Chiral icos. (A_5, I)	§5.6	0	0	0	0
Full icos. ($A_5 \times \mathbb{Z}/2$, I_h)	§5.7	$(\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$

TABLE 6. Phase homology groups in dimension $3+1$ equivariant with respect to various tetrahedral, octahedral, and icosahedral symmetries and the ways they can mix with fermion parity. See the referenced sections for how the fermionic crystalline equivalence principle associates this data with symmetry types for invertible TFTs.

5.1. Chiral tetrahedral symmetry.

We compute phase homology groups equivariant for a chiral tetrahedral symmetry $\lambda\colon A_4 \to \mathrm{SO}_3$. As far as we know, this point group has not yet been considered by physicists in the setting of fermionic phases. We will show that our ansatz implies there are no nontrivial phases with either spinless or spin-1/2 fermions in both class D and class A. As usual, $V_\lambda \to BA_4$ denotes the vector bundle associated to λ.

Proposition 5.1. *$H^*(BA_4;\mathbb{Z}/2) \cong \mathbb{Z}/2[u,v,w]/(u^3+v^2+w^2+vw)$, where $|u| = 2$ and $|v| = |w| = 3$.* $\mathrm{Sq}(u) = u + v + w + u^2$, $\mathrm{Sq}(v) = v + u^2 + uw + v^2$, *and* $\mathrm{Sq}(w) = w + u^2 + uv + w^2$.

Except for the Steenrod operations, this result can be found in several places, such as [Kin] and [AM04, Theorem III.1.3], so we will be brief.

Proof sketch. Use the Lyndon-Hochschild-Serre spectral sequence [Lyn48, Ser50, HS53] for the short exact sequence $1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to A_4 \to \mathbb{Z}/3 \to 1$; the mod 2 cohomology of $\mathbb{Z}/3$ is trivial, so the spectral sequence collapses, and

$$H^*(BA_4;\mathbb{Z}/2) \cong H^0(B\mathbb{Z}/3; H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)) = H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)^{\mathbb{Z}/3}. \tag{5.2}$$

We can choose this $\mathbb{Z}/3$-action to be such that a generator of $\mathbb{Z}/3$ acts on $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{1, \alpha, \beta, \alpha + \beta\}$ by $\alpha \mapsto \alpha + \beta$, $\beta \mapsto \alpha$, and $\alpha + \beta \mapsto \beta$. In a mild abuse of notation, we identify $\mathbb{Z}/2 \times \mathbb{Z}/2$ with $H^1(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$: these are dual $\mathbb{Z}/2$-vector spaces, and we have a basis for one, which we identify with the dual basis vectors of the other. Thus $H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha, \beta]$.

The unique nonzero degree-2 cohomology class fixed by $\mathbb{Z}/3$ is $u := \alpha^2 + \alpha\beta + \beta^2$, and two linearly independent degree-3 classes fixed by $\mathbb{Z}/3$ are $v := \alpha^3 + \alpha^2\beta + \beta^3$ and $w := \alpha^3 + \alpha\beta^2 + \beta^3$, whence the relation.

For the Steenrod squares, the identification in (5.2) of $H^*(BA_4;\mathbb{Z}/2)$ as a subalgebra of $H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$ is the pullback map for $B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to BA_4$, hence $\mathcal{A}$-equivariant, so we can compute $\mathrm{Sq}(u)$ in $H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$; the computation follows from $\mathrm{Sq}(\alpha) = \alpha + \alpha^2$ and $\mathrm{Sq}(\beta) = \beta + \beta^2$. ☐

Lemma 5.3. *$w_1(V_\lambda) = 0$ and $w_2(V_\lambda) = u$.*

Proof. Since V_λ is orientable, $w_1(V_\lambda) = 0$, and since V_λ is not spin, $w_2(V_\lambda) \neq 0$. Since $H^2(BA_4;\mathbb{Z}/2) \cong \mathbb{Z}/2\cdot u$, $w_2(V_\lambda) = u$. ☐

One way to see that this representation is not spin is to look at the *binary tetrahedral group $2T$*, defined to be the preimage of $A_4 \subset \mathrm{SO}_3$ under the double cover $\mathrm{Spin}_3 \twoheadrightarrow \mathrm{SO}_3$. If V_λ were spin, $2T$ would be a split extension of A_4 by μ_2, but it is not split.

5.1.1. *Class D, spinless case.* If A_4 does not mix with the symmetry type, our ansatz reduces to that of Freed-Hopkins, which reduces the computation of these A_4-equivariant phase homology groups to the computation of $[MTSpin \wedge (BA_4)^{3-V_\lambda}, \Sigma^5 I_{\mathbb{Z}}]$.

Theorem 5.4. *The first few spin bordism groups of $X := (BA_4)^{3-V_\lambda}$ are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/3\\
\widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong 0\\
\widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/6\\
\widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong \mathbb{Z}\\
\widetilde{\Omega}_5^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/18 \oplus \mathbb{Z}/2\\
\widetilde{\Omega}_6^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_7^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/9.
\end{aligned}$$

Thus if f_0^D denotes the A_4-equivariant local system of symmetry types for this case, $Ph_0^{A_4}(\mathbb{R}^3, f_0^D) = 0$.

Proof. At the prime 2, we use the Adams spectral sequence; if p is an odd prime, the map $\widetilde{\Omega}_*^{\mathrm{Spin}}(X) \to \widetilde{\Omega}_*^{\mathrm{SO}}(X)$ is an isomorphism on p-torsion, and we will determine the p-torsion part of $\widetilde{\Omega}_*^{\mathrm{SO}}(X)$.

First, the 2-primary piece. Letting U denote the mod 2 Thom class as usual, $\mathrm{Sq}^1(U) = 0$ and $\mathrm{Sq}^2(U) = Uu$. This and Proposition 5.1 allow us to determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ in low degrees, as depicted in Figure 13, left.

Hence as $\mathcal{A}(1)$-modules,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathcal{O} \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^5\mathcal{A}(1) \oplus P, \tag{5.5}$$

where P is 8-connected. Because we only care about degrees 6 and below, P is irrelevant for us, and for the remaining summands in (5.5), $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(-,\mathbb{Z}/2)$ has already been computed. For $\Sigma^k\mathcal{A}(1)$, there's a single $\mathbb{Z}/2$

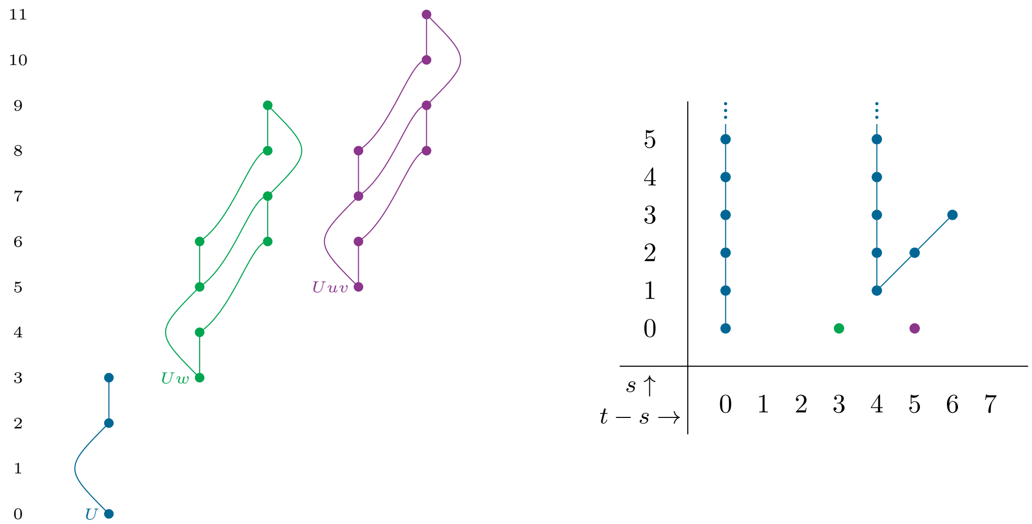


FIGURE 13. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. This submodule contains all elements of degree at most 8. Right: the E_2-page of the Adams spectral sequence calculating $\widetilde{ko}_*((BA_4)^{3-V_\lambda})$, given by $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2),\mathbb{Z}/2)$.

with $s=0$, $t=k$; for $\mathcal{\mathring{O}}$, see [BC18, Figure 29]. We put this together and display the E_2-page for our spectral sequence in Figure 13, right. A combination of h_0-equivariance and Margolis' theorem (Theorem 3.22) rules out nontrivial differentials and hidden extensions. Therefore the 2-primary part of $\widetilde{\Omega}_k^{\mathrm{Spin}}(X)$ has a single free summand each in degrees 0 and 4, is 0 in degrees 1 and 2, is $\mathbb{Z}/2$ in degrees 3 and 6, and is $\mathbb{Z}/2\oplus\mathbb{Z}/2$ in degree 5.

For the odd-primary part, we use the fact that $\Omega_*^{\mathrm{Spin}}\to\Omega_*^{\mathrm{SO}}$ is an equivalence after inverting 2. Moreover, because λ factors through SO_3, $V_\lambda\to BA_4$ is orientable, so there is a Thom isomorphism $\widetilde{\Omega}_*^{\mathrm{SO}}(X)\xrightarrow{\cong}\Omega_*^{\mathrm{SO}}(BA_4)$. Hence we just need the odd-primary part of $\Omega_*^{\mathrm{SO}}(BA_4)$, which is isomorphic to the odd-primary part of $\Omega_*^{\mathrm{Spin}}(BA_4)$. In the degrees we care about, this is isomorphic to $ko_*(BA_4)$, and Bruner-Greenlees [BG10, §7.7.E] show that the odd-primary torsion in $ko_*(BA_4)$ below degree 6 consists of $\mathbb{Z}/3$ summands in degrees 1 and 3 and $\mathbb{Z}/9$ summands in degrees 5 and 7. □

5.1.2. *Class D, spin-1/2 case.* In this case, the symmetries mix as specified by the group extension giving the binary tetrahedral group.

Theorem 5.6. *The A_4-equivariant phase homology group for the class D, spin-1/2 symmetry type in 3d is trivial.*

Proof. Let $f_{1/2}^D$ denote the local system on $\mathbb{R}^3$ assigned to this symmetry type. Since V_λ is not pin$^-$ (if it were, it would be pin$^-$ and orientable, hence spin), Theorem 2.11 says $Ph_0^{A_4}(\mathbb{R}^3;f_{1/2}^D)\cong[\mathit{MTSpin}\wedge(BA_4)_+,\Sigma^5 I_{\mathbb{Z}}]$. Bruner-Greenlees [BG10, §7.7.E] show $ko_4(BA_4)\cong\mathbb{Z}$ and $ko_5(BA_4)$ is torsion, so this phase homology group vanishes. □

5.1.3. *Class A.* Let f_0^A and $f_{1/2}^A$ be the A_4-equivariant local systems of symmetry types for spinless, resp. spin-1/2 fermions in class A.

Lemma 5.7. *$V_\lambda\to BA_4$ is not pin^c.*

Proof. If $\beta\colon H^2(-;\mathbb{Z}/2)\to H^3(-;\mathbb{Z})$ denotes the integral Bockstein, we want to show $\beta w_2(V_\lambda)\neq 0$. By Lemma 3.28, it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda))\neq 0$. Lemma 5.3 gives $w_2(V_\lambda)=b$, and $\mathrm{Sq}^1 b=ab+c$. □

Therefore for spin-1/2 fermions, Theorem 2.24 computes $Ph_*^{A_4}(\mathbb{R}^3;f_{1/2}^A)$ in terms of the spin^c bordism of $(BA_4)^{\mathrm{Det}(V_\lambda)-1}$. Since V_λ is orientable, this is isomorphic to the spin^c bordism of BA_4. For spinless fermions, we use $(BA_4)^{3-V_\lambda}$, as usual.

Theorem 5.8. *The low-degree spin^c bordism groups of $X := (BA_4)^{3-V_\lambda}$ and BA_4 are*

$$\begin{aligned} \widetilde\Omega_0^{\mathrm{Spin}^c}(X) &\cong \mathbb Z & \Omega_0^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb Z\\ \widetilde\Omega_1^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/3 & \Omega_1^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb Z/3\\ \widetilde\Omega_2^{\mathrm{Spin}^c}(X) &\cong \mathbb Z & \Omega_2^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb Z\oplus\mathbb Z/2\\ \widetilde\Omega_3^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/6\oplus\mathbb Z/3 & \Omega_3^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb Z/6\oplus\mathbb Z/3\\ \widetilde\Omega_4^{\mathrm{Spin}^c}(X) &\cong \mathbb Z^2 & \Omega_4^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb Z^2, \end{aligned}$$

and in both cases, $\Omega_5^{\mathrm{Spin}^c}$ is torsion. Hence both $Ph_0^{A_4}(\mathbb R^3; f_0^A)$ and $Ph_0^{A_4}(\mathbb R^3; f_{1/2}^A)$ vanish.

Proof. We use the equivalence $MTSpin^c \simeq ku \vee \Sigma^4 ku$ in degrees below 8, then the Adams spectral sequence over $\mathcal E(1)$ to compute ku-homology at the prime 2.

For the case of spinless fermions, use the $\mathcal A(1)$-module structure on $\widetilde H^*(X;\mathbb Z/2)$ from (5.5) (drawn in Figure 13, left) to compute that the $\mathcal E(1)$-module structure is

$$\widetilde H^*(X;\mathbb Z/2) \cong \mathcal{\mathring{O}} \oplus \Sigma^3\mathcal E(1) \oplus \Sigma^5\mathcal E(1) \oplus \Sigma^5\mathcal E(1) \oplus P, \tag{5.9}$$

where P is 6-connected. We draw this in Figure 14, left. We computed $\mathrm{Ext}_{\mathcal E(1)}^{s,t}(\mathcal{\mathring{O}}, \mathbb Z/2)$ in (4.55), and P is too high-degree to be relevant to us, so the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(X)$ is given in Figure 14, right. Margolis' theorem (Theorem 3.22) implies this spectral sequence collapses and there are no extension problems, so we conclude.

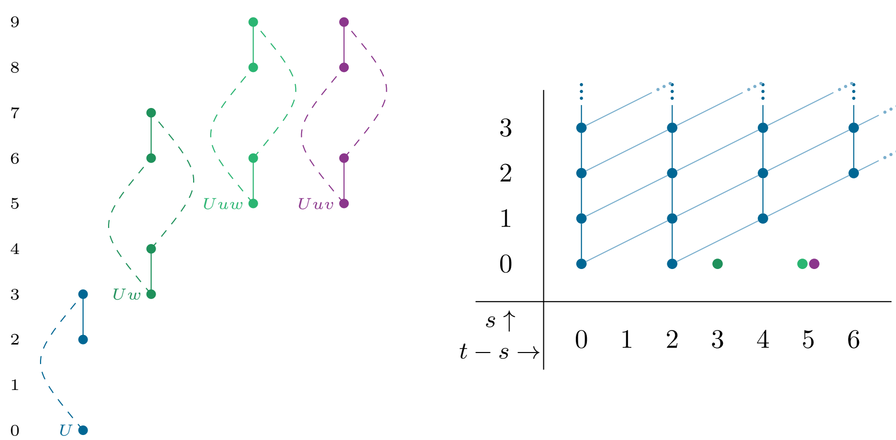


FIGURE 14. Left: the $\mathcal E(1)$-module structure on $\widetilde H^*((BA_4)^{3-V_\lambda};\mathbb Z/2)$ in low degrees. The picture includes all elements in degrees 6 and below. Right: $\mathrm{Ext}_{\mathcal E(1)}^{s,t}(\widetilde H^*((BA_4)^{3-V_\lambda};\mathbb Z/2),\mathbb Z/2)$, the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((BA_4)^{3-V_\lambda})$.

On to the spin-1/2 case. As before, $ku_*(BA_4)$ splits as $ku_*(\mathrm{pt}) \oplus \widetilde{ku}_*(BA_4)$, and we focus on the latter. Bruner-Greenlees [BG03, §2.6] show that 2-locally, there is an equivalence

$$ku \wedge BA_4 \simeq (ku \wedge \Sigma^2 BC_2) \vee \bigvee_\alpha \Sigma^{n_\alpha} H\mathbb Z/2 \tag{5.10}$$

for some collection of integers α; moreover, their calculation of $ku^*(BA_4)$ [BG03, Theorem 2.6.3] implies the only $n_\alpha < 8$ (i.e. the ones relevant for us) are $n_1 = 2$ and $n_2 = 6$. This, together with Hashimoto's computation of $\widetilde{ku}_*(B\mathbb Z/2)$ [Has83, Theorem 3.1], tells us $ku_*(BA_4)^\wedge_2$ in the degrees we need.

We still need to determine the odd-primary torsion.

Lemma 5.11. *Let p be an odd prime; then, the inclusion $\mathbb Z/3 \hookrightarrow A_4$ sending a generator to $(1\ 2\ 3)$ induces a p-primary stable equivalence $\Sigma^\infty(B\mathbb Z/3)_+ \to \Sigma^\infty(BA_4)_+$.*

Proof. Since $|A_4| = 2^2 \cdot 3$, for any $p \geq 5$, the maps $BA_4 \to \mathrm{pt}$ and $B\mathbb{Z}/3 \to \mathrm{pt}$ are p-local stable equivalences, so we only have to address $p = 3$. In this case, Lemma 3.27 implies the inclusion $j\colon \mathbb{Z}/3 \hookrightarrow A_4$ as the subgroup generated by (1 2 3) induces an isomorphism $H^*(BA_4;\mathbb{Z}/3) \to H^*(B\mathbb{Z}/3;\mathbb{Z}/3)$, so we conclude by the mod p Whitehead theorem [Ser53, Chapitre III, Théorème 3]. ☐

The Thom isomorphism theorem then implies $\widetilde{H}^*(X;\mathbb{Z}[1/2]) \to \widetilde{H}^*((B\mathbb{Z}/3)^{3-j^*V_\lambda};\mathbb{Z}[1/2])$ is an isomorphism, so arguing in a similar way, there is a p-primary stable equivalence $X_\lambda \simeq (B\mathbb{Z}/3)^{3-V_\lambda}$. Thus, for the purpose of computing the odd-torsion subgroups of $\Omega_*^{\mathrm{Spin}^c}(BA_4)$ and $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$, we can just work with $\mathbb{Z}/3$.

As a $\mathbb{Z}/3$-representation, j^*V_λ is isomorphic to the direct sum of a trivial representation and the real 2-dimensional representation given by rotation. Each of these is spin^c, the latter because it is unitary, so there is a Thom isomorphism $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/3)^{3-j^*V_\lambda} \cong \mathit{MTSpin}^c \wedge (B\mathbb{Z}/3)_+$, so in both the spinless and spin-1/2 cases, we just need the 3-torsion in $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/3)$, which we computed in Theorem 4.14. ☐

5.2. **Pyritohedral symmetry.** *Pyritohedral symmetry* is the action of $G \coloneqq A_4 \times \mathbb{Z}/2$ on $\mathbb{R}^3$ in which A_4 acts as the orientation-preserving symmetries of a tetrahedron and $\mathbb{Z}/2$ acts through inversion; let λ denote this representation and $V_\lambda \to BG$ be the associated vector bundle. Because G splits as a direct product, it is easier to analyze than full tetrahedral symmetry (i.e. chiral tetrahedral symmetry together with a reflection), as we will see in this and the next section.

5.2.1. *Spinless case.* Let $X \coloneqq (BG)^{3-V_\lambda}$. By the twisted Künneth formula, $H^*(X)$ is 2-torsion; therefore $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ also lacks odd-primary torsion. so we just have to work with the Adams spectral sequence at $p = 2$. In the rest of this section, all cohomology is with $\mathbb{Z}/2$ coefficients unless otherwise stated.

Proposition 5.12. *The first several spin bordism groups of $(BG)^{3-V_\lambda}$ are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_4^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_5^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_6^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_7^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 2}.
\end{aligned}$$

Proof. We employ a trick to reduce the amount of direct computations. We will replace $(3 - V_\lambda) \to BG$ with a virtual vector bundle $E \to BG$ with the same first two Stiefel-Whitney classes, but which splits as an exterior sum over BA_4 and $B\mathbb{Z}/2$. The Thom spectrum $(BG)^E$ has two nice properties: the Adams E_2-page for calculating $\widetilde{ko}_*((BG)^E)$ is isomorphic to that of $\widetilde{ko}_*(X)$, but $(BG)^E$ also splits as a smash product of Thom spectra over BA_4 and $B\mathbb{Z}/2$, simplifying the calculation of said E_2-page. Because we do not construct a map from $\widetilde{ko}_*((BG)^E)$ to $\widetilde{ko}_*((BG)^{3-V_\lambda})$ or vice versa, this isomorphism does not allow us to deduce any differentials, but we will see that all differentials in range vanish for formal reasons, so this is no problem.

The Künneth formula and Proposition 5.1 together imply

$$H^*(BG) \cong \mathbb{Z}/2[x,u,v,w]/(u^3 + v^2 + w^2 + vw), \tag{5.13}$$

where $|x| = 1$, $|u| = 2$, and $|v| = |w| = 3$, and that $\mathrm{Sq}(x) = x + x^2$ and the Steenrod squares of u, v, and w are as in Proposition 5.1.

Lemma 5.14. *The first two Steifel-Whitney classes of V are $w_1(V_\lambda) = x$ and $w_2(V_\lambda) = u + x^2$.*

Proof. Since this representation contains orientation-reversing symmetries, $w_1(V_\lambda)$ must be nonzero, so is x. For w_2, we saw in Lemma 5.3 that when one restricts to $A_4 \subset A_4 \times \mathbb{Z}/2$, one has $w_2(V_\lambda|_{BA_4}) = u$; when one restricts to $\mathbb{Z}/2$, this is 3 copies of the sign representation, hence has $w_2(V_\lambda|_{B\mathbb{Z}/2}) = x^2$. ☐

Let $E \to BG$ be the virtual vector bundle

$$E := 4 - (V_\lambda|_{BA_4} \boxplus -\sigma), \tag{5.15}$$

where $\sigma \to B\mathbb{Z}/2$ is the tautological line bundle. The Whitney sum formula implies for $i = 1, 2$, $w_i(E) = w_i(3 - V_\lambda)$. Feeding this to the Thom isomorphism gives isomorphisms of $\mathcal{A}(1)$-modules

$$\widetilde{H}^*((BG)^{3-V_\lambda}) \cong \widetilde{H}^*((BG)^E) \tag{5.16}$$

hence also isomorphisms of the E_2-pages of the corresponding Adams spectral sequences. Because $E \to BG$ is an external sum,

$$(BG)^E \simeq (BA_4)^{3-V_\lambda} \wedge (B\mathbb{Z}/2)^{\sigma-1}. \tag{5.17}$$

We know the $\mathcal{A}(1)$-module structures on the low-degree cohomology of both summands, and the Künneth formula tells us to tensor them together (over $\mathbb{Z}/2$) to determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BG)^E)$.

In (5.5), we computed the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BA_4)^{3-V_\lambda})$ in low degrees, and split off two $\Sigma^k\mathcal{A}(1)$ summands. Margolis' theorem (Theorem 3.22) promotes that to a splitting of spectra

$$ko \wedge (BA_4)^{3-V_\lambda} \simeq \Sigma^3 H\mathbb{Z}/2 \vee \Sigma^5 H\mathbb{Z}/2 \vee Y, \tag{5.18}$$

such that as an $\mathcal{A}$-module,

$$\widetilde{H}^*(Y) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (\mathcal{O} \oplus P), \tag{5.19}$$

where P is 7-connected. When we smash $(B\mathbb{Z}/2)^{\sigma-1}$ back in, each $\Sigma^k H\mathbb{Z}/2 \wedge (B\mathbb{Z}/2)^{\sigma-1}$ contributes a summand of $\widetilde{H}_{n-k}((B\mathbb{Z}/2)^{\sigma-1})$ to $\widetilde{ko}_n((BG)^E)$, i.e. a $\mathbb{Z}/2$-summand in each degree $\ell \geq k$. The upshot for $\mathcal{A}(1)$-modules is

$$\Sigma^k\mathcal{A}(1) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \cong \bigoplus_{\ell\geq k} \Sigma^\ell H\mathbb{Z}/2. \tag{5.20}$$

By (5.16), these summands are also present in $\widetilde{H}^*((BG)^{3-V_\lambda})$, and Margolis' theorem lifts this to split off corresponding $\Sigma^\ell H\mathbb{Z}/2$ summands. Therefore there is a spectrum Y' such that

$$\widetilde{ko}_n((BG)^E) \cong \pi_n(Y') \oplus \widetilde{H}_{n-3}((B\mathbb{Z}/2)^{\sigma-1}) \oplus \widetilde{H}_{n-5}((B\mathbb{Z}/2)^{\sigma-1}) \tag{5.21}$$

and as $\mathcal{A}$-modules,

$$\widetilde{H}^*(Y') \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (\mathcal{O} \oplus P) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}). \tag{5.22}$$

The change-of-rings theorem (3.4) thus applies to the E_2-page of the Adams spectral sequence calculating $\pi_*(Y')$, yielding

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}((\mathcal{O} \oplus P) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{5.23}$$

We will work with this spectral sequence, adding in the summands corresponding to $\Sigma^3 H\mathbb{Z}/2$ and $\Sigma^5 H\mathbb{Z}/2$ afterwards.

Our first order of business is to compute the tensor product in (5.23). The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1})$ can be found in [BC18, Figure 4].

Lemma 5.24. *There is an isomorphism of $\mathcal{A}(1)$-modules $\mathcal{O} \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \cong \mathcal{A}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^4\mathcal{A}(1) \oplus P$, where P is 7-connected.*

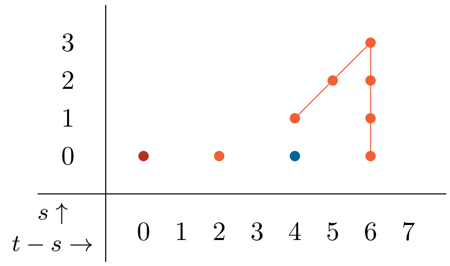
Proof. Compute directly, by hand or by computer. □

By (5.21) and (5.22), we can work with (5.23), then add in the $\mathbb{Z}/2$ summands coming from the $\Sigma^k H\mathbb{Z}/2$ summands at the end. Lemma 5.24 tells us the E_2-page of (5.23) is

$$E_2^{s,t} \cong \mathrm{Ext}(\mathcal{A}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^4\mathcal{A}(1) \oplus P). \tag{5.25}$$

Since P is 7-connected, its Ext is concentrated in degrees irrelevant to us, and we ignore it. $\mathrm{Ext}(\Sigma^2 R_0)$ is computed in the degrees we need by Beaudry-Campbell [BC18, Figures 23, 24]; using this, the E_2-page of (5.23) is

(5.26)



Margolis' theorem and h_1-equivariance of differentials immediately imply there are no nontrivial differentials or extension problems below degree 8, so we conclude. $\square$

5.2.2. *Class D, spin-*$1/2$ *case.* Let $f^D_{1/2}$ denote the equivariant local system of symmetry types corresponding to spin-$1/2$ fermions for a pyritohedral symmetry in class D. Theorem 2.11 computes the equivariant phase homology associated to $f^D_{1/2}$ in terms of the spin bordism of $X := (BA_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$. The isomorphism $\mathrm{Det}(V_\lambda) \cong 0 \boxplus \sigma$ provides an isomorphism $X \simeq (BA_4)_+ \wedge (B\mathbb{Z}/2)^{\sigma-1}$, Lemma 3.30 thus implies the spin bordism of this spectrum computes the pin$^-$ bordism of BA_4, which could be independently interesting.

Theorem 5.27. *The first few spin bordism groups of* X *are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\ \widetilde\Omega_2^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/8\oplus\mathbb{Z}/2\\ \widetilde\Omega_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2\\ \widetilde\Omega_4^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2.\end{aligned}$$

Since $\widetilde\Omega_5^{\mathrm{Spin}}(X)$ *is torsion by Lemma 3.24,* $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f^D_{1/2}) \cong \mathbb{Z}/2$.

Proof. By the twisted Künneth formula, $\widetilde H^*(X)$ has no odd-primary torsion, and therefore neither does $\widetilde\Omega_*^{\mathrm{Spin}}(X)$, so it suffices to work at the prime 2, which we do.

Use Lemma 3.30 to split $X \simeq (B\mathbb{Z}/2)^{\sigma-1} \vee M$, where the map $\widetilde H^*(M;\mathbb{Z}/2) \to \widetilde H^*(X;\mathbb{Z}/2)$ is injective with image a complimentary subspace to $\mathbb{Z}/2\cdot\{Ux^k \mid k \ge 0\}$.

As usual, $w_1(\mathrm{Det}(V_\lambda)-1) = w_1(V_\lambda) = x$ and $w_2(\mathrm{Det}(V_\lambda)-1) = 0$. We also need to know the $\mathcal{A}$-action on $H^*(BG;\mathbb{Z}/2)$; the Künneth formula determines this using as input the $\mathcal{A}$-action on $H^*(BA_4;\mathbb{Z}/2)$, which we computed in Proposition 5.1, and the $\mathcal{A}$-action on $H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, which is standard. Using this, we can determine the $\mathcal{A}(1)$-module structure on $\widetilde H^*(M;\mathbb{Z}/2)$. We obtain an isomorphism of $\mathcal{A}(1)$-modules

$$\widetilde H^*(M;\mathbb{Z}/2) \cong \Sigma^2 R_3 \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus P, \tag{5.28}$$

where P is 4-connected. We will see in Figure 15, right, that for $t-s \le 4$, $E_2^{s,t}$ is concentrated in Adams filtration 0; this and the 4-connectedness of P imply its contribution to the E_2-page cannot affect the spectral sequence in degrees $t-s\le 4$, which is all we need. We draw these summands, except for P, in Figure 15, left.

Freed-Hopkins [FH16a, Figure 5, case $s = 3$] and Beaudry-Campbell [BC18, Figures 32, 33] calculate $\mathrm{Ext}(R_3)$ in the range we need, and we can draw the E_2-page of the Adams spectral sequence in Figure 15, right. This collapses, so we add in the pin$^-$ bordism summands we need from [ABP69, KT90b] to obtain the groups in the theorem. $\square$

5.2.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types corresponding to spinless fermions in class A and $X := (BA_4 \times B\mathbb{Z}/2)^{3-V_\lambda}$; then we saw that $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A)$ is determined by $\widetilde\Omega_*^{\mathrm{Spin}^c}(X)$.

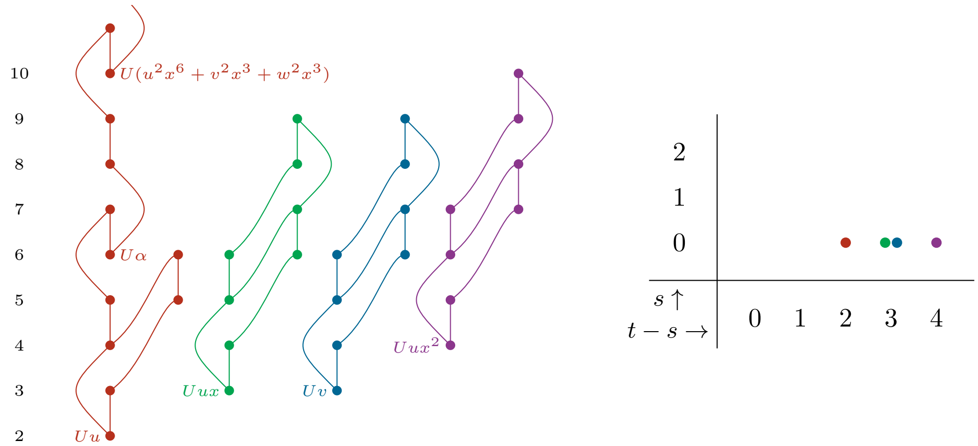


FIGURE 15. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. This picture includes all summands in degrees 4 and below. Here $\alpha \coloneqq u^2x^2 + v^2 + w^2$. Right: the E_2-page of the corresponding Adams spectral sequence.

Theorem 5.29. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_6^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 6}\\
\widetilde{\Omega}_7^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 5},
\end{aligned}$$

so $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}$.

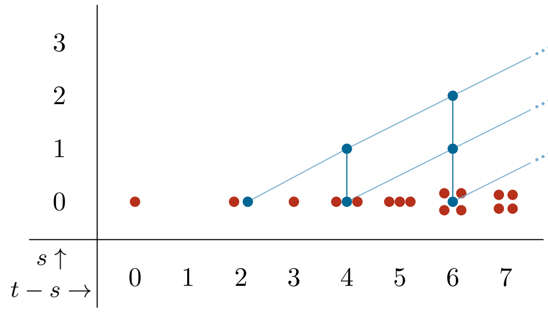
Proof. The twisted Thom isomorphism and twisted Künneth formula imply $\widetilde{H}^*(X;\mathbb{Z})$ is 2-torsion. Therefore for any odd prime p, the mod p Whitehead theorem [Ser53, Chapitre III, Théorème 3] implies $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$ also has no p-torsion. This leaves only $p = 2$, for which we use the Adams spectral sequence over $\mathcal{E}(1)$.

We determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BG)^{3-V_\lambda})$ as given in (5.25), together with an $\Sigma^\ell\mathcal{A}(1)$ for $\ell = 3, 4$, and two $\Sigma^\ell\mathcal{A}(1)$ summands for $\ell \geq 5$. This determines the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, and $R_0 \cong H$, so

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \Sigma^2 H \oplus V' \otimes_{\mathbb{Z}/2} \mathcal{E}(1) \oplus P, \tag{5.30}$$

where V' is a graded $\mathbb{Z}/2$-vector space with a basis of homogeneous elements in degrees 0, 2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, and 7, and P is 7-connected. Therefore the E_2-page is

(5.31)



By Margolis' theorem, there are no nontrivial differentials or extension problems in this range. □

5.2.4. *Class A, spin-1/2 case.* To compute the $(A_4 \times \mathbb{Z}/2)$-equivariant phase homology groups for the local system $f_{1/2}^A$ specified by the spin-1/2 extension in class A Theorem 2.24 asks us to investigate the spin^c bordism of $X \coloneqq (BA_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1} \simeq (BA_4 \times B\mathbb{Z}/2)^{0 \boxplus \sigma - 1}$; we know V_λ is not pin^c because we saw in Lemma 5.7 that the pullback of V_λ along $BA_4 \to BA_4 \times B\mathbb{Z}/2$ is not pin^c.

Theorem 5.32. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}
\end{aligned}$$

By Lemma 3.24, $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X)$ is torsion, so $Ph_0^{A_4 \times \mathbb{Z}/2}(\mathbb{R}^3; f_{1/2}^A) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$.

Proof. We reuse our work from §5.2.2. We saw that $X \simeq (B\mathbb{Z}/2)^{\sigma-1} \vee M$, and we gave the low-degree cohomology of M as an $\mathcal{A}(1)$-module in (5.28) (and drew it in Figure 15, left). This determines the $\mathcal{E}(1)$-module structure on it, so we can calculate spin^c bordism of M using the Adams spectral sequence. For the other summand, we have $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^c$, so we direct-sum in the pin^c bordism groups computed by Bahri-Gilkey [BG87a, BG87b].

There are isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$ and $R_3 \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^4 R_0$. Therefore as an $\mathcal{E}(1)$-module,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4 R_0 \oplus \Sigma^4\mathcal{E}(1) \oplus P, \tag{5.33}$$

where P is 4-connected. As usual for these cases, we will see that $\mathrm{Ext}(\widetilde{H}^*(M;\mathbb{Z}/2), \mathbb{Z}/2)$ has no nonzero elements with $t-s=4$ and $s>1$, so P does not affect our calculations. See Figure 16, left, for a picture of the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$. Look up $\mathrm{Ext}(\Sigma^4 R_0)$ in Proposition 4.48 to obtain the E_2-page of the Adams spectral sequence as in Figure 16, right. This collapses, so we add in the pin^c bordism summands and conclude. □

I could get used to Adams spectral sequences like this one. But alas, they are not all this easy, as we will see in the next section.

5.3. **Full tetrahedral symmetry.** The full group of symmetries of the tetrahedron, including reflections, is the symmetric group S_4, acting via the representation $\lambda\colon S_4 \to \mathrm{O}_3$, which is isomorphic to the quotient of the four-dimensional real permutation representation by the fixed line $\mathbb{R} \cdot (1,1,1,1)$.

Proposition 5.34 ([Ngu09, §2.3])**.** *$H^*(BS_4;\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b,c]/(ac)$, with $|a|=1$, $|b|=2$, and $|c|=3$. The Steenrod squares of the generators are $\mathrm{Sq}(a) = a + a^2$, $\mathrm{Sq}(b) = b + ab + c + b^2$, and $\mathrm{Sq}(c) = c + bc + c^2$.*[27]

[27]The ring structure on $H^*(BS_4;\mathbb{Z}/2)$ was known earlier, due to Cardenas [Car65]; see [AM04, Example VI.1.13].

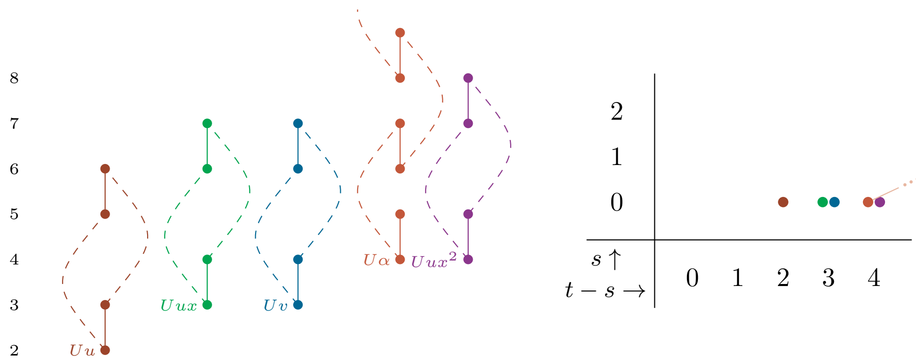


FIGURE 16. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. This picture includes all summands in degrees 4 and below. Here $\alpha := u^2 + vx + wx$. Right: the E_2-page of the corresponding Adams spectral sequence.

Let $V_\lambda \to BS_4$ denote the associated vector bundle to λ.

Proposition 5.35. *$w_1(V_\lambda) = a$, $w_2(V_\lambda) = b$, and $w_3(V_\lambda) = c$.*

Proof. Since λ does not factor through $\mathrm{SO}_3 \subset \mathrm{O}_3$, V_λ is unorientable. Thus $w_1(V_\lambda) \neq 0$, and a is the only nonzero element of $H^1(BS_4;\mathbb{Z}/2)$, so $w_1(V_\lambda) = a$. For w_2, we calculated in Lemma 5.3 that $w_2(V_\lambda|_{A_4}) \neq 0$, so w_2 cannot vanish in BS_4. Our options are a^2, b, and a^2+b. Let $\mathbb{Z}/2 \subset S_4$ be generated by a transposition; then as a $\mathbb{Z}/2$-representation $\lambda \cong \mathbb{R}^2 \oplus \sigma$, so $w_2(V_\lambda|_{\mathbb{Z}/2}) = 0$. The map $H^*(BS_4;\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ sends $b, c \mapsto 0$ and $a \mapsto x$, so the constraint $w_2(V_\lambda|_{\mathbb{Z}/2}) = 0$ rules out $w_2(V_\lambda) = a^2$ and $w_2(V_\lambda) = a^2 + b$, forcing us to conclude $w_2(V_\lambda) = b$. Finally, $w_3(V_\lambda) = c$ follows from the Wu formula. $\square$

We need the next calculation to determine the odd-primary torsion subgroups of the phase homology groups that we calculate.

Lemma 5.36. *Suppose $V \to BS_4$ is a rank-zero virtual vector bundle with $w_1(V) = x$. Then the inclusion $i\colon S_3 \hookrightarrow S_4$ defines an isomorphism*

$$\widetilde{H}_*((BS_3)^{i^*V}) \otimes \mathbb{Z}[1/2] \to \widetilde{H}_*((BS_4)^{V}) \otimes \mathbb{Z}[1/2]. \tag{5.37}$$

Proof. The commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_3 & \longrightarrow & S_3 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \big\| & & \\ 1 & \longrightarrow & A_4 & \longrightarrow & S_4 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \end{array} \tag{5.38}$$

induces a map between their Lyndon-Hochschild-Serre spectral sequences with signatures

$$E^2_{p,q} = H_p(B\mathbb{Z}/2; \underline{H_q(BA_k;\mathbb{Z}[1/2])} \otimes (\mathbb{Z}[1/2])_x) \Longrightarrow H_{p+q}(BS_k; (\mathbb{Z}[1/2])_{w_1(V)}), \tag{5.39}$$

where $\underline{H_q(BA_4;\mathbb{Z}[1/2])}$ means the local system on $B\mathbb{Z}/2$ induced by the action of $\mathbb{Z}/2$ on A_k as specified by the extension $1 \to A_k \to S_k \to \mathbb{Z}/2 \to 1$, and x is the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.

We claim the map on these spectral sequences is an isomorphism on E^2-pages. By Lemma 5.11, the map $H_*(BA_4;\mathbb{Z}[1/2]) \to H_*(BA_3;\mathbb{Z}[1/2])$ is an isomorphism, and this isomorphism intertwines the $\mathbb{Z}/2$-actions on $\underline{H_*(BA_k;\mathbb{Z}[1/2])} \otimes (\mathbb{Z}[1/2])_x$, because (5.38) commutes. Therefore it induces an isomorphism on all E^r-pages, hence also on what these spectral sequences converge to. $\square$

The top row in (5.38) can be identified with $1 \to \mathbb{Z}/3 \to D_6 \to \mathbb{Z}/2 \to 1$, so by the same lines of reasoning as in Propositions 4.23 and 4.24 we deduce

$$\widetilde{\Omega}_k^{\mathrm{Spin}}((BS_4)^V) \otimes \mathbb{Z}[1/2] \cong \begin{cases} \mathbb{Z}/3, & k = 1 \\ 0, & k = 0, 2, 3, 4 \end{cases} \tag{5.40a}$$

$$\widetilde{\Omega}_k^{\mathrm{Spin}^c}((BS_4)^V) \otimes \mathbb{Z}[1/2] \cong \begin{cases} \mathbb{Z}/3, & k = 1, 3 \\ 0, & k = 0, 2, 4. \end{cases} \tag{5.40b}$$

5.3.1. *Class D, spinless case.* As usual in the spinless case for unorientable representations, the ansatz asks us to let $X \coloneqq (BS_4)^{3-V_\lambda}$ and consider $MTSpin \wedge X$.

Theorem 5.41. *The first few spin bordism groups of X are*

$$\begin{aligned} \widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2 \\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/3 \\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2 \\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2 \\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}, \end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion,

Proof. For odd-primary information, see Equation (5.40a). For 2-primary information, we will again use the Adams spectral sequence over $\mathcal{A}(1)$. Our first task is to write down $\widetilde{H}^*(X;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, using Proposition 5.35 to deduce $w_1(3 - V_\lambda) = a$ and $w_2(3 - V_\lambda) = a^2 + b$. We describe this $\mathcal{A}(1)$-module structure in low degrees in Figure 17, left.

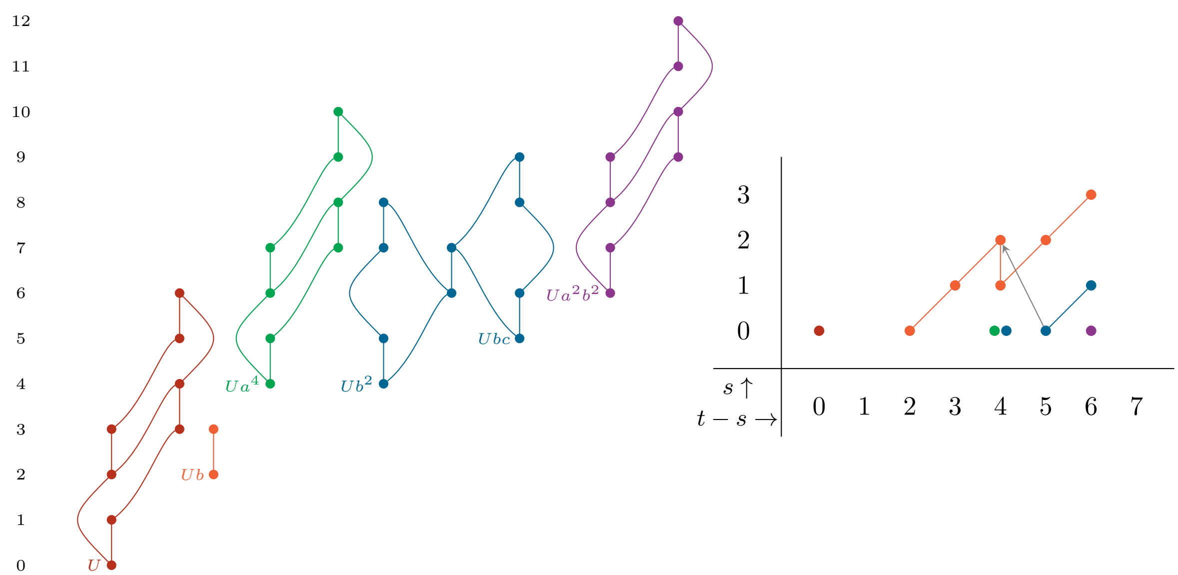


FIGURE 17. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. This submodule contains all elements of degree at most 7. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ko}_*((BS_4)^{3-V_\lambda})$.

Let $\Sigma^4 N_2$ denote the submodule generated by Ub^2 and Ubc, which is a nontrivial extension of J by ΣJ.[28] Then there is an isomorphism

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathcal{A}(1) \oplus \Sigma^2 N_1 \oplus \Sigma^4 \mathcal{A}(1) \oplus \Sigma^4 N_2 \oplus \Sigma^6 \mathcal{A}(1) \oplus P, \tag{5.42}$$

[28]We propose calling N_2 the *butterfly*; it also appears in [WWZ20, Figure 16].

The indecomposable summand isomorphic to $\Sigma^2 N_1$ is generated by Ub, and P has no elements in degrees below 8, and therefore is irrelevant for our low-degree computations. As before, we know what a $\Sigma^k\mathcal{A}(1)$ summand contributes to the E_2-page. To compute $\mathrm{Ext}(N_1)$, we use a well-known explicit (12-shifted) 4-periodic minimal resolution[29]

(5.43)
$$N_1 \xleftarrow{f_0} \mathcal{A}(1) \xleftarrow{f_1} \Sigma^2\mathcal{A}(1)\oplus\Sigma^3\mathcal{A}(1) \xleftarrow{f_2} \Sigma^4\mathcal{A}(1)\oplus\Sigma^5\mathcal{A}(1) \xleftarrow{f_3} \Sigma^7\mathcal{A}(1) \xleftarrow{f_4} \Sigma^{12}\mathcal{A}(1) \xleftarrow{\Sigma^{12}f_1} \Sigma^{14}\mathcal{A}(1)\oplus\Sigma^{15}\mathcal{A}(1) \xleftarrow{\Sigma^{12}f_2} \dots$$

The dimension of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_1,\mathbb{Z}/2)$ is the number of summands of $\Sigma^t\mathcal{A}(1)$ in the s^{th} module in the extension.[30] This (shifted up by 2 for $\Sigma^2 N_1$) gives the orange summands in Figure 17, right.

For $\Sigma^4 N_2$, we use a convenient shortcut: the kernel of the map f_2 in (5.43) is isomorphic to $\Sigma^4 N_2$. Thus, the sequence (5.43) except for the first two terms forms a minimal resolution for $\Sigma^4 N_2$, so for every $s,t\geq 0$, there is an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_2,\mathbb{Z}/2)\cong\mathrm{Ext}^{s+2,t+4}_{\mathcal{A}(1)}(N_1,\mathbb{Z}/2) \tag{5.45}$$

equivariant for the $H^{*,*}(\mathcal{A}(1))$-actions on both sides. This gives us the blue summands in Figure 17, right. Now we can draw the E_2-page for the Adams spectral sequence for $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$, and do so in Figure 17, right.

Margolis' theorem and h_i-equivariance of differentials imply there is a single differential in this range that could be nonzero, namely the pictured $d_2\colon E_2^{0,5}\to E_2^{2,6}$.

Proposition 5.46. *$d_2\colon E_2^{0,5}\to E_2^{2,6}$ vanishes; equivalently, $\widetilde{ko}_4(X)$ has more than eight elements.*

We will prove this using the Atiyah-Hirzebruch spectral sequence in Theorem 5.56.

Assuming Proposition 5.46 for now, there are no further differentials in the range we care about, but we must address four extension questions in degrees 4, 5, and 6:

$$0\longrightarrow\mathbb{Z}/2\longrightarrow A\longrightarrow\mathbb{Z}/2\oplus\mathbb{Z}/2\longrightarrow 0 \tag{5.47a}$$

$$0\longrightarrow\mathbb{Z}/2\longrightarrow\widetilde{ko}_4(X)\longrightarrow A\longrightarrow 0 \tag{5.47b}$$

$$0\longrightarrow\mathbb{Z}/2\longrightarrow\widetilde{ko}_5(X)\longrightarrow\mathbb{Z}/2\longrightarrow 0 \tag{5.47c}$$

$$0\longrightarrow\mathbb{Z}/2\longrightarrow\widetilde{ko}_6(X)\longrightarrow\mathbb{Z}/2\oplus\mathbb{Z}/2\longrightarrow 0. \tag{5.47d}$$

(In fact, a priori, there are five extension problems, but Margolis' theorem splits $E_\infty^{0,6}\cong\mathbb{Z}/2$ off from the rest of the $t-s=6$ line.)

Both (5.47a) and (5.47c) split for the same reason. For $k=4,5$, assume the sequence does not split; then, $\widetilde{ko}_k(X)$ has an element x such that $2x\neq 0$ and if y is the image of $2x$ in the E_∞-page, then $h_1y\neq 0$. This fact lifts to a nonzero action by $\eta\in ko_1$ carrying $2x$ to some element $z\in\widetilde{ko}_{k+1}(X)$ such that $z=2\eta x$ and $z\neq 0$, but $2\eta=0$, causing a contradiction.

Because (5.47a) splits and $(h_0\cdot)\colon E_\infty^{1,5}\to E_\infty^{2,6}$ is an isomorphism, all possible extensions in (5.47b) give $\widetilde{ko}_4(X)\cong\mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 2}$.

Lastly, (5.47d). Action by h_1 defines isomorphisms $E_\infty^{0,5}\to E_\infty^{1,7}$ and $E_\infty^{2,7}\to E_\infty^{3,9}$, and this lifts to imply $(\eta\cdot)\colon\widetilde{ko}_6(X)\to\widetilde{ko}_7(X)$ is injective, splitting (5.47d). ☐

We return to Proposition 5.46. Our proof strategy is to compute $\widetilde{ko}_4(X)$ a different way. First, we pass to $\tau_{0:4}ko$-cohomology, following a strategy of Campbell [Cam17, §7.4] and Freed-Hopkins [FH19a, §5.1], by

[29] After some practice with $\mathcal{A}(1)$-modules, writing this minimal resolution down is straightforward, if a little tedious; we found it a helpful exercise when learning this material and the interested reader might too. Though this minimal resolution is certainly known, it is not explicitly written in many places; the resolution will not be televised.

[30] The $H^{*,*}(\mathcal{A}(1))$-action on $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_1,\mathbb{Z}/2)$ is a little obscure from this perspective; one can show that all h_0- and h_1-actions that could be nonzero for degree reasons are in fact nonzero, as stated in [BB96, §3] and [WWZ20, Figure 15]. One way to see this would be to use the long exact sequences in Ext associated to the two short exact sequences

$$0\longrightarrow\Sigma\mathbb{Z}/2\longrightarrow N_1\longrightarrow\mathbb{Z}/2\longrightarrow 0 \tag{5.44a}$$

$$0\longrightarrow\Sigma^2 N_1\longrightarrow\mathcal{U}\longrightarrow\mathbb{Z}/2\longrightarrow 0, \tag{5.44b}$$

together with the fact that the boundary maps in the long exact sequences commute with the $H^{*,*}(\mathcal{A}(1))$-action.

way of Lemma 5.48. We then run the Atiyah-Hirzebruch spectral sequence computing the $\tau_{0:4}ko$-cohomology of X. As input, we need $\widetilde{H}^*(X;\mathbb{Z})$, which we compute in Theorem 5.49 using a Lyndon-Hochschild-Serre spectral sequence [Lyn48, Ser50, HS53].

Lemma 5.48 (Campbell [Cam17, (7.35), (7.36)]). *There is a noncanonical equivalence $I_\mathbb{Z}(\tau_{0:4}ko) \simeq \Sigma^{-4}\tau_{0:4}ko$. Thus, if $\tau_{0:4}\widetilde{ko}_k(Y)$ is torsion, $\tau_{0:4}\widetilde{ko}_k(Y) \cong \tau_{0:4}\widetilde{ko}^{k-3}(Y)$.*

This is a corollary of the shifted self-equivalence $I_\mathbb{Z}KO \simeq \Sigma^4 KO$ [And69, Theorem 4.16].[31]

By Lemma 3.24, $\widetilde{ko}_4(X) \cong \tau_{0:4}\widetilde{ko}_4(X)$[32] is torsion, so is isomorphic to $\tau_{0:4}\widetilde{ko}^1(X)$. We study this group with the Atiyah-Hirzebruch spectral sequence. As input, we compute $\widetilde{H}^*(X;\mathbb{Z}_{(2)})$, which the Thom isomorphism equates with $H^*(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)})$.

Theorem 5.49.

$$\begin{aligned}
H^0(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong 0\\
H^1(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong \mathbb{Z}/2\\
H^2(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong 0\\
H^3(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2\\
H^4(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong \mathbb{Z}/2\\
H^5(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2.
\end{aligned}$$

Proof. Let $R \coloneqq \mathbb{Z}_{(2)}[x]/(x^2-1)$, which is a $\mathbb{Z}[C_2]$-module in which the nontrivial element of C_2 sends $1\mapsto 1$ and $x\mapsto -x$. As $\mathbb{Z}[C_2]$-modules, $R\cong \mathbb{Z}_{(2)}\oplus(\mathbb{Z}_{(2)})_\sigma$, so we will recover $H^*(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)})$ from $H^*(BS_4;R)$. The Lyndon-Hochschild-Serre spectral sequence

$$E_2^{*,*} = H^*(BC_2;H^*(BA_4;R)) \Longrightarrow H^*(BS_4;R) \tag{5.50}$$

is multiplicative; here S_4 acts on R through *sign*: $S_4\to C_2$ and A_4 acts trivially. R is a $\mathbb{Z}/2$-graded ring, where x is in odd degree, and hence R-valued cohomology is $\mathbb{Z}\times\mathbb{Z}/2$-graded. We use $\{+,-\}$ to denote the $\mathbb{Z}/2$-grading.

Proposition 5.51 (Čadek [Čad99, Lemma 3.1]). *There is an isomorphism of $\mathbb{Z}\times\mathbb{Z}/2$-graded rings $H^*(BC_2;R)\cong\mathbb{Z}_{(2)}[y]/(2y)$ with $|y|=(1,-)$.*

Proposition 5.52 (Bruner-Greenlees [BG03, §2.6]). *There is a presentation of $H^*(BA_4;\mathbb{Z}_{(2)})$ whose only generators and relations below degree 6 are generators α and β in degrees 3 and 4, respectively, and relations $2\alpha=2\beta=0$.*

Corollary 5.53. *As $\mathbb{Z}\times\mathbb{Z}/2$-graded rings,*

$$H^*(BA_4;R)\cong\mathbb{Z}_{(2)}[\alpha_+,\alpha_-,\beta_+,\beta_-,\dots]/(2\alpha_\pm,2\beta_\pm,\dots) \tag{5.54}$$

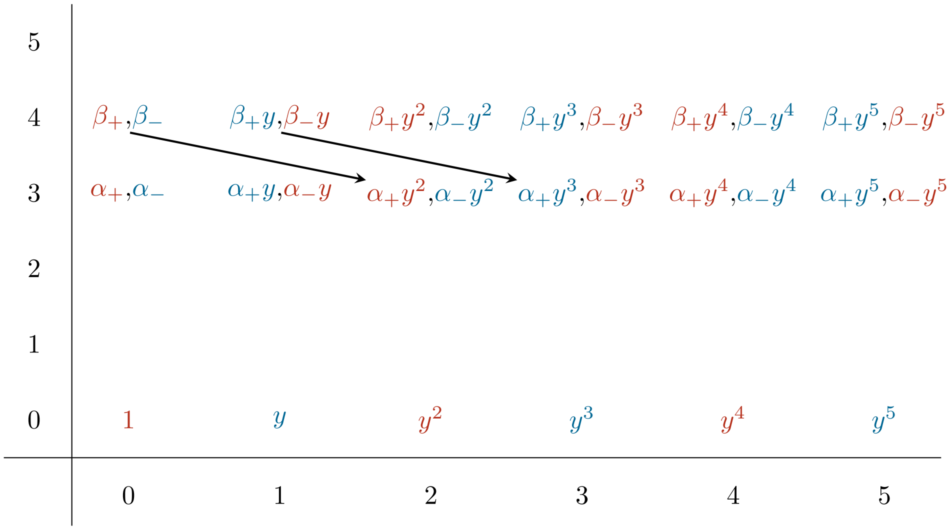
where the generators and relations not displayed are in $\mathbb{Z}$-degrees ≥ 6, $|\alpha_\pm|=(2,\pm)$, and $|\beta_\pm|=(3,\pm)$.

[31] Anderson gives this proof in unpublished lecture notes; see Yosimura [Yos75, Theorem 4] for Anderson's proof. There are at least four additional proofs that $I_\mathbb{Z}KO\simeq\Sigma^4KO$, due to Freed-Moore-Segal [FMS07, Proposition B.11], Heard-Stojanoska [HS14, Theorem 8.1], Ricka [Ric16, Corollary 5.8], and Hebestreit-Land-Nikolaus [HLN20, Example 2.8], all by different methods.

[32] We abuse notation slightly to let $\tau_{0:4}\widetilde{ko}$ denote reduced $\tau_{0:4}ko$-cohomology, rather than $\widetilde{\tau_{0:4}ko}$.

We can now display the E_2-page. Elements with + grading are colored red, and elements with − grading are colored blue; differentials are even in this $\mathbb{Z}/2$-grading.

(5.55)



The map $S_4 \to C_2$ admits a section given by $\{1, (1\ 2)\} \subset S_4$, so the $q = 0$ line supports no nonzero differentials and does not participate in nontrivial extension problems. Looking just at elements graded −, we are done if we can show that $d_2(\beta_-) = \alpha_- y^2$ and $d_2(\beta_+ y) = 0$. Fortunately, Thomas [Tho74] has computed $H^*(BS_4; \mathbb{Z}_{(2)})$: since $H^4(BS_4; \mathbb{Z}_{(2)}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, $d_2(\beta_+) = 0$, so the Leibniz rule implies $d_2(\beta_+ y) = 0$ too. And since $H^5(BS_4; \mathbb{Z}_{(2)}) \cong \mathbb{Z}/2$, $d_2(\beta_- y) \neq 0$, so $d_2(\beta_-) \neq 0$, hence must be $\alpha_- y^2$. □

Thus equipped, we tackle the Atiyah-Hirzebruch spectral sequence.

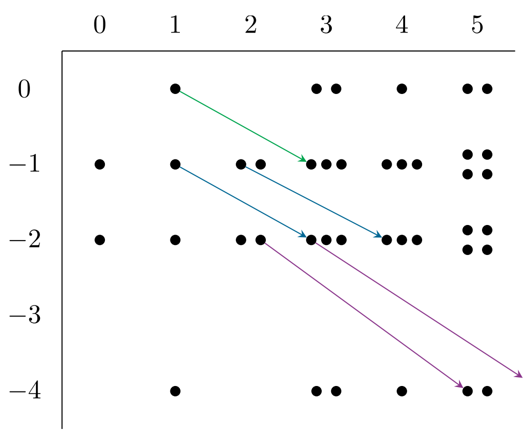
Theorem 5.56. $|\widetilde{ko}_4(X)| \geq 16$ *(thus implying Proposition 5.46).*

Proof. After using Lemma 5.48, we want to compute $\tau_{0:4}\widetilde{ko}^1(X)$, which we attack with the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = \widetilde{H}^p(X; (\tau_{0:4} ko)^q) \Longrightarrow \tau_{0:4}\widetilde{ko}^{p+q}(X). \tag{5.57}$$

Using Proposition 5.34 and Theorem 5.49 as input, the E_2-page is

(5.58)



Maunder [Mau63, Theorem 3.4] identifies the first nonzero differential in the cohomological Atiyah-Hirzebruch spectral sequence with a k-invariant; this includes all differentials shown in (5.58). Let $r\colon H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}/2)$ denote reduction mod 2 and $\beta\colon H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z})$ be the Bockstein. Then, Bruner-Greenlees [BG10, Corollary A.5.2] determine the k-invariants we need for ko-cohomology:

(1) The green $d_2\colon E_2^{p,0} \to E_2^{p+2,-1}$ is $\mathrm{Sq}^2 \circ r\colon \widetilde{H}^p(X;\mathbb{Z}) \to \widetilde{H}^{p+2}(X;\mathbb{Z}/2)$.
(2) Each blue $d_2\colon E_2^{p,-1} \to E_2^{p+2,-1}$ is $\mathrm{Sq}^2\colon \widetilde{H}^p(X;\mathbb{Z}/2) \to \widetilde{H}^{p+2}(X;\mathbb{Z}/2)$.

(3) Each purple $d_3\colon E_3^{p,-2}\to E_3^{p+3,-4}$ is $\beta\circ\mathrm{Sq}^2\colon \widetilde H^p(X;\mathbb Z/2)\to\widetilde H^{p+3}(X;\mathbb Z)$.

We computed the $\mathcal A(1)$-module structure on $\widetilde H^*(X;\mathbb Z/2)$ in (5.42) (and drew it in Figure 17, left), and r and β follow from this and a few facts we just calculated for $\widetilde H^*(X;\mathbb Z)$. For $k\le 5$, we proved $2\widetilde H^k(X;\mathbb Z)=0$, so r is injective in these degrees. Moreover, combining this with Lemma 3.28, that $r\circ\beta=\mathrm{Sq}^1$, we conclude for $k\le 2$ and $x\in\widetilde H^k(X;\mathbb Z/2)$, $\beta\mathrm{Sq}^2(x)=0$ iff $\mathrm{Sq}^1\mathrm{Sq}^2(x)=0$.

All together, these allow us to resolve almost all of the indicated differentials — a priori, we do not know $\beta\mathrm{Sq}^2(x)$ when $x\in E_2^{3,-2}\cong\widetilde H^3(X;\mathbb Z/2)$, but for all x not in the image of $d_2\colon E_2^{1,-1}\to E_2^{3,-2}$, $\mathrm{Sq}^2(x)=0$, so this is fine. We find the 1-line of the E_4-page has five $\mathbb Z/2$ summands, one in $E_4^{2,-1}$, two in $E_4^{3,-2}$, and two in $E_4^{5,-4}$. There could be a nonzero $d_4\colon E_4^{2,-1}\to E_4^{6,-5}$, but the remaining four summands are generated by permanent cycles. ☐

5.3.2. *Class D, spin-1/2 case.* As V_λ is not pin$^-$, Theorem 2.11 tells us to compute the spin bordism of $X\coloneqq (BS_4)^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 5.59. *The first few spin bordism groups of X are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X)&\cong\mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X)&\cong\mathbb Z/6\\ \widetilde\Omega_2^{\mathrm{Spin}}(X)&\cong\mathbb Z/8\oplus\mathbb Z/2\\ \widetilde\Omega_3^{\mathrm{Spin}}(X)&\cong\mathbb Z/2\\ \widetilde\Omega_4^{\mathrm{Spin}}(X)&\cong 0,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X)$ is torsion.

Proof. Odd-primary information is computed in the range we need by (5.40a). For 2-primary information, we use the Adams spectral sequence as usual. Recall the $\mathcal A(1)$-module structure on $H^*(BS_4;\mathbb Z/2)\cong\mathbb Z/2[a,b,c]/(ac)$ from Propositions 5.34 and 5.35. Lemma 3.30 shows that inclusion of a transposition extends to a splitting

$$X\xrightarrow{\simeq}(B\mathbb Z/2)^{\sigma-1}\vee M,\tag{5.60}$$

and the map $\widetilde H^*(M;\mathbb Z/2)\to\widetilde H^*(X;\mathbb Z/2)$ is injective, with image a complementary subspace to the span of $\{Ua^n\mid n\ge 0\}$. As usual, we write down $\widetilde H^*(M;\mathbb Z/2)$ as an $\mathcal A(1)$-module in low degrees, using $w_1(\mathrm{Det}(V_\lambda)-1)=a$ and $w_2(\mathrm{Det}(V_\lambda)-1)=0$, and give the answer in Figure 18, left.

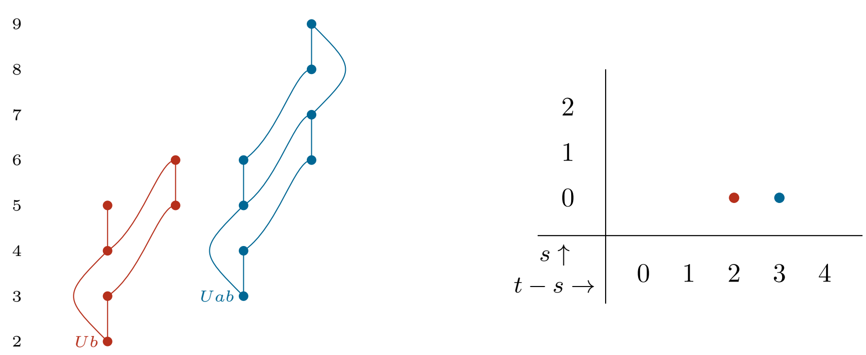


FIGURE 18. Left: The $\mathcal A(1)$-module structure on $\widetilde H^*(M;\mathbb Z/2)$ in low degrees. This submodule contains all elements of degree at most 4. Right: the Ext of this module, which is the beginning of the Adams spectral sequence computing $\widetilde{ko}_*(M)$. More information in the proof of Theorem 5.59.

Let Σ^2N_3 denote the $\mathcal A(1)$-submodule generated by Ub; this module is studied by Baker [Bak18, §5], who calls it the "whiskered Joker." There is an isomorphism of $\mathcal A(1)$-modules

$$\widetilde H^*(M;\mathbb Z/2)\cong\Sigma^2N_3\oplus\Sigma^3\mathcal A(1)\oplus P,\tag{5.61}$$

where P contains no elements of degree less than 4. Therefore if the 4-line of the E_2-page is empty, P does not enter into our calculations — and we will see momentarily that the 4-line is in fact empty. We know what $\Sigma^3\mathcal{A}(1)$ summand contributes to the E_2-page of the Adams spectral sequence. For N_3, we leverage what we learned from N_1 in §5.3.1. Specifically, the unique nonzero $\mathcal{A}(1)$-module map $\mathcal{A}(1) \to N_3$ has kernel isomorphic to $\Sigma^5 N_1$, so a minimal resolution for $\Sigma^5 N_1$ induces a minimal resolution for N_3 which has an additional copy of $\mathcal{A}(1)$ in topological degree 0 and filtration 0, and in which everything else is shifted up one in filtration, giving the red summands in Figure 18, right.

Thus the E_2-page for this Adams spectral sequence is as in Figure 18, right. In this range, the spectral sequence collapses. Combine this with the pin$^-$ bordism summands from [ABP69, KT90b] as usual to obtain the groups in the theorem statement, and Lemma 3.24 finishes us off by telling us $\widetilde\Omega_5^{\mathrm{Spin}}(X)$ is torsion. ☐

5.3.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types for class A with spinless fermions. In this case, the ansatz asks us to consider the spin^c bordism of $X \coloneqq (BS_4)^{3-V_\lambda}$.

Theorem 5.62. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/2\\
\widetilde\Omega_1^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/3\\
\widetilde\Omega_2^{\mathrm{Spin}^c}(X) &= (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde\Omega_3^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/3\\
\widetilde\Omega_4^{\mathrm{Spin}^c}(X) &= (\mathbb{Z}/2)^{\oplus 4},
\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion. Therefore $Ph_0^{S_4}(\mathbb{R}^3, f_0^A) \cong (\mathbb{Z}/2)^{\oplus 4}$.

Proof. We will use the Adams spectral sequence over $\mathcal{E}(1)$ as usual to capture the 2-primary information; for odd-primary information, see (5.40b).

We use the $\mathcal{A}(1)$-module structure on $\widetilde H^*(X;\mathbb{Z}/2)$ that we determined in (5.42) and drew in Figure 17 to determine the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, and $N_2 \cong \mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^2 N_1$, so as $\mathcal{E}(1)$-modules,

$$\widetilde H^*(X;\mathbb{Z}/2) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2 N_1 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus P, \tag{5.63}$$

where P is 5-connected. We draw this in Figure 19, left. Recalling $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$ from (4.49), the E_2-page of

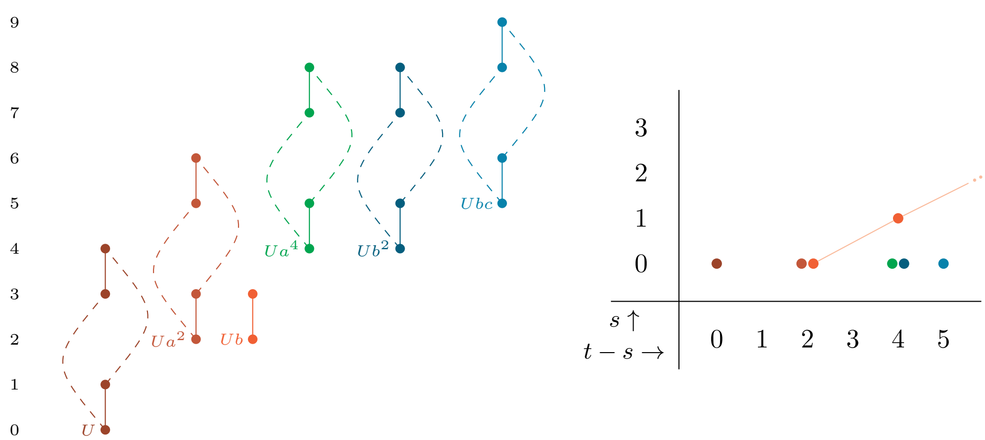


FIGURE 19. Left: the $\mathcal{E}(1)$-module structure on $\widetilde H^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degree at most 5. Right: the Ext of this module, which is the beginning of the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*((BS_4)^{3-V_\lambda})$.

the Adams spectral sequence is in Figure 19, right. There can be no differentials in the range drawn for degree reasons, and Margolis' theorem (Theorem 3.22) implies there are no nontrivial extensions, either, so we are done. □

5.3.4. *Class A, spin-1/2 case.* Theorem 2.24 says that to compute the S_4-equivariant phase homology groups in class A with spin-1/2 fermions, given by the equivariant local system of symmetry types $f^A_{1/2}$, we should investigate the spin^c bordism of $X \coloneqq (BS_4)^{\mathrm{Det}V_\lambda - 1}$: we know V_λ is not pin^c because its pullback along $BA_4 \to BS_4$ is not pin^c, as we established in Lemma 5.7.

Theorem 5.64. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde\Omega_1^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/3\\
\widetilde\Omega_2^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/4\oplus\mathbb{Z}/2\\
\widetilde\Omega_3^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/6\\
\widetilde\Omega_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/8\oplus(\mathbb{Z}/2)^{\oplus 2}.
\end{aligned}$$

By Lemma 3.24, $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion, so $\mathit{Ph}_0^{S_4}(\mathbb{R}^3; f^A_{1/2}) \cong \mathbb{Z}/8\oplus(\mathbb{Z}/2)^{\oplus 2}$.

Proof. See (5.40b) for the odd-primary torsion in $\widetilde\Omega_*^{\mathrm{Spin}^c}(X)$. For 2-torsion, we reuse our work from §5.3.2. First, $X \simeq (B\mathbb{Z}/2)^{\sigma-1}\vee M$, and we gave the low-degree cohomology of M as an $\mathcal{A}(1)$-module in (5.61), and drew it in Figure 18, left. This determines the $\mathcal{E}(1)$-module structure on it, so we can calculate spin^c bordism of M using the Adams spectral sequence. For the other summand, we have $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{\sigma-1}\simeq \mathit{MTPin}^c$, so we direct-sum in the pin^c bordism groups computed by Bahri-Gilkey [BG87a, BG87b].

There are isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1)\cong\mathcal{E}(1)\oplus\Sigma^2\mathcal{E}(1)$ and $N_3\cong\mathcal{E}(1)\oplus\Sigma^2 N_1$. Therefore as an $\mathcal{E}(1)$-module,

$$\widetilde H^*(M;\mathbb{Z}/2)\cong \Sigma^2\mathcal{E}(1)\oplus\Sigma^3\mathcal{E}(1)\oplus\Sigma^4 N_1\oplus P, \tag{5.65}$$

where P is 4-connected. As usual for these cases, we will see that $\mathrm{Ext}(\widetilde H^*(M;\mathbb{Z}/2),\mathbb{Z}/2)$ has no nonzero elements with $t-s=4$ and $s>1$, so P does not affect our calculations. See Figure 20, left, for a picture of the $\mathcal{E}(1)$-module structure on $\widetilde H^*(M;\mathbb{Z}/2)$. We calculated $\mathrm{Ext}(\Sigma^4 N_1)$ in (4.49), so can draw the E_2-page of

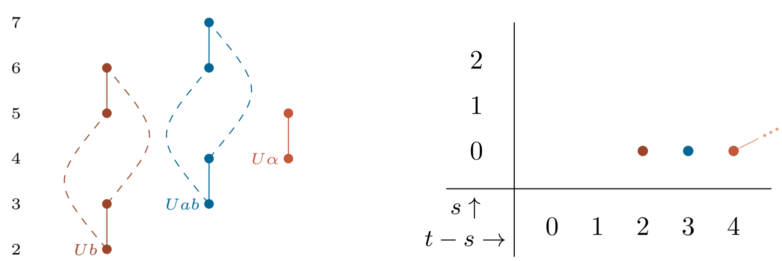


FIGURE 20. Left: the $\mathcal{E}(1)$-module structure on $\widetilde H^*(M;\mathbb{Z}/2)$ in low degrees; the pictured summands include all elements in degrees 4 and below. Here $\alpha\coloneqq a^2b+b^2$. Right: the Ext of this module, which is the beginning of the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M)$.

the Adams spectral sequence in Figure 20, right. This collapses, so we add in the pin^c bordism summands and conclude. □

5.4. **Chiral octahedral symmetry.** Let $\lambda\colon S_4\to \mathrm{O}_3$ denote the representation as symmetries of an octahedron and $V_\lambda\to BS_4$ denote the associated vector bundle. Recall from Proposition 5.34 the mod 2 cohomology of BS^4.

Lemma 5.66. $w_1(V_\lambda)=0$ *and* $w_2(V_\lambda)=b$.

Proof. Since $\mathrm{Im}(\lambda) \subset \mathrm{SO}_3$, $w_1(V_\lambda) = 0$. We know $w_2(V_\lambda)$ restricts to $u \in H^2(BA_4; \mathbb{Z}/2)$ by considering tetrahedral symmetry inside octahedral symmetry and using Lemma 5.3, so $w_2(V_\lambda)$ could be $a^2 + b$ or b. The fact that λ splits as $\sigma \oplus \mathbb{R}^2$ when restricted to a $\mathbb{Z}/2$ subgroup given by a transposition tells us $w_2(V_\lambda)$ is b, not $a^2 + b$. ☐

By Lemma 5.7, the pullback of V_λ to BA_4 is not pin^c, so V_λ is not pin^c, and hence V_λ is also not pin$^-$.

Lemma 5.67.

$$\widetilde{\Omega}_k^{\mathrm{SO}}(BS_4) \otimes \mathbb{Z}[1/2] \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & k = 0, 1, 2, 4, 5, 6. \end{cases} \tag{5.68}$$

Proof. Let ℓ be an odd prime and consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(BS_4; (MTSO^\wedge_\ell)_q) \Longrightarrow (MTSO^\wedge_\ell)_{p+q}(BS_4) = \Omega^{\mathrm{SO}}_{p+q}(BS_4)^\wedge_\ell. \tag{5.69}$$

If $\ell \neq 3$, then $\ell \nmid |S_4|$, so the $\mathbb{Z}_\ell$-cohomology of BS_4 vanishes in positive degrees and (5.69) is trivial, contributing no ℓ-torsion to $\widetilde{\Omega}_*^{\mathrm{SO}}(BS_4) \otimes \mathbb{Z}[1/2]$. For $\ell = 3$, use Thomas' calculation of $H^*(BS_4; \mathbb{Z})$ [Tho74] and the universal coefficient theorem to show that $H_*(BS_4; \mathbb{Z}_3)$ consists of $\mathbb{Z}_3$ in degree 0, $\mathbb{Z}/3$ in degree 2, and nothing else nonzero in degrees 5 and below. Therefore (5.69) collapses, giving us the desired result. ☐

5.4.1. *Class D, spinless case.* Let f_0^D denote the equivariant local system of symmetry types for the spinless class D case. Theorem 2.11 identifies

$$Ph_k^{S_4}(\mathbb{R}^3; f_0^D) \cong [MTSpin \wedge (BS_4)^{3-V_\lambda}, \Sigma^{k+4} I_\mathbb{Z}], \tag{5.70}$$

so we study the spin bordism of $X := (BS_4)^{3-V_\lambda}$.

Theorem 5.71. *There is an $r \geq 2$ such that the first few spin bordism groups of X are*

$$\begin{aligned} \widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z} \\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2 \\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong 0 \\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/6 \oplus \mathbb{Z}/2^k \\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong \mathbb{Z}, \end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion. Hence $Ph_0^{S_4}(\mathbb{R}^3; f_0^D) = 0$.

The Atiyah-Hirzebruch spectral sequence allows one to show $k = 1$, so $\widetilde{\Omega}_3^{\mathrm{Spin}}(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. As usual, we will not need this, so do not prove it.

Proof. For odd-primary torsion, use the fact that $MTSpin \to MTSO$ is an isomorphism, so it suffices to understand $\widetilde{\Omega}_*^{\mathrm{SO}}(X)$, and that $V_\lambda \to BS_4$ is orientable, so there is a Thom isomorphism $\Omega_k^{\mathrm{SO}}(BS_4) \to \widetilde{\Omega}_k^{\mathrm{SO}}(X)$, and we can read off the odd-primary torsion from Lemma 5.67.

On to the prime 2. From Propositions 5.34 and 5.35 we know the mod 2 cohomology of BS_4 and the action of the Steenrod algebra, and using Lemma 5.66 we can draw $\widetilde{H}^*(X; \mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, which we do in Figure 21, left.

Let N_4 denote the $\mathcal{A}(1)$-submodule of $\widetilde{H}^*(X; \mathbb{Z}/2)$ generated by U and Ua. Then,

$$\widetilde{H}^*(X; \mathbb{Z}/2) \cong N_4 \oplus \Sigma^3 \mathcal{A}(1) \oplus \Sigma^4 N_4 \oplus \Sigma^5 \mathcal{A}(1) \oplus P, \tag{5.72}$$

where P is 6-connected. We have not seen N_4 before, and need to calculate its Ext. Fortunately, there is a short exact sequence of $\mathcal{A}(1)$-modules

$$0 \longrightarrow \Sigma J \longrightarrow N_4 \longrightarrow \mathring{O} \longrightarrow 0, \tag{5.73}$$

which induces a long exact sequence in Ext. In Figure 22, we display a picture both of this extension and of the Adams chart for computing the boundary map in the long exact sequence.

We draw the E_2-page in Figure 21, right. Because differentials must be h_0-equivariant, they all vanish in the range pictured except possibly for those from the 4-line to the 3-line, one of which is indicated in the chart. By Lemma 3.24, $\widetilde{ko}_4(X) \otimes \mathbb{Q} \cong \widetilde{ko}_0(X) \otimes \mathbb{Q}$, and from Figure 21, right, the latter group is isomorphic

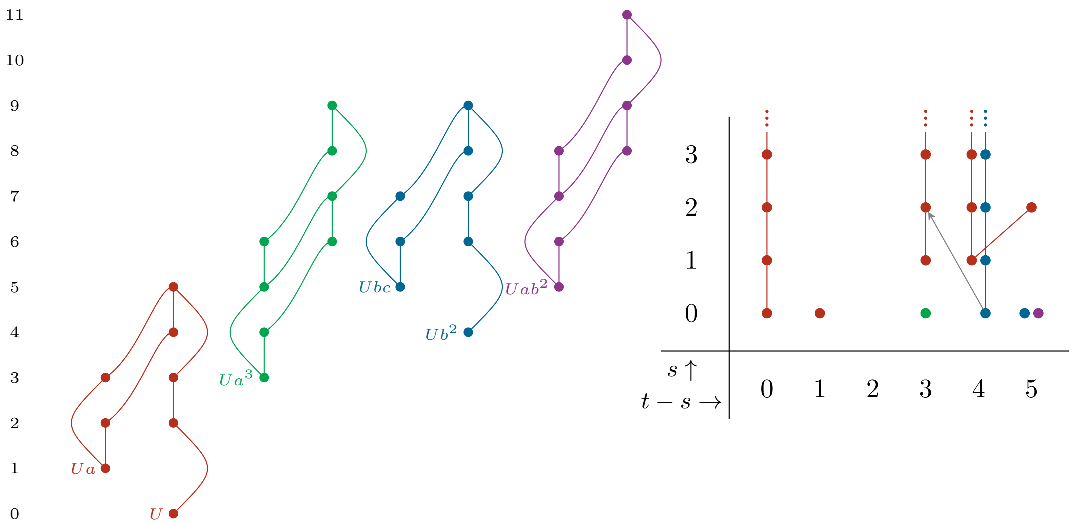


FIGURE 21. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees 6 and below. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ko}_*((BS_4)^{3-V_\lambda})^\wedge_2$. We will see there is a differential from the 4-line to the 3-line; it is in fact the d_2 depicted, though we do not prove that.

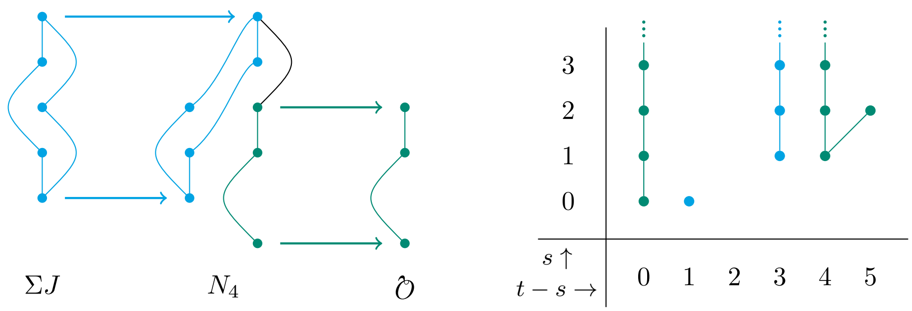


FIGURE 22. Left: the extension (5.73) of $\mathcal{A}(1)$-modules. Right: the long exact sequence in Ext induced from that extension.

to $\mathbb{Q}$. Thus $\widetilde{\Omega}_4^{\mathrm{Spin}}(X)$ has exactly one free summand, so one of the two towers in the 4-line lives to the E_∞-page, and the other admits a nonzero d_r differential to the tower in degree 3. Thus, on the 3-line of the E_{r+1}-page, there is a single green $\mathbb{Z}/2$ summand in degree $s=0$, together with a red tower with finitely many $\mathbb{Z}/2$ summands, giving $\mathbb{Z}/2\oplus\mathbb{Z}/2^k$ in degree 3 as promised.[33] □

5.4.2. *Class D, spin-*1/2 *case.* Let $f^D_{1/2}$ be the S_4-equivariant local system of symmetry types for the case of spin-1/2 fermions in class D. Theorem 2.11 computes the equivariant phase homology of this local system in terms of $\Omega_*^{\mathrm{Spin}}(BS_4)$.

[33] We have not determined which elements of the 4-line the differential is nonzero on. One way to determine this is to use that the generator of $H^{3,7}(\mathcal{A}(1))\cong\mathbb{Z}/2$ carries the summands in the 0-line onto a subset of the red tower in the 4-line. Differentials are equivariant for this action, and differentials emerging from the 0-line vanish, so all differentials must vanish on the red tower too.

Theorem 5.74. *There is an $r \geq 2$ such that the first several spin bordism groups of BS_4 are*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}}(BS_4) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}}(BS_4) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\Omega_2^{\mathrm{Spin}}(BS_4) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\Omega_3^{\mathrm{Spin}}(BS_4) &\cong \mathbb{Z}/24 \oplus \mathbb{Z}/2^{r+1},\\
\Omega_4^{\mathrm{Spin}}(BS_4) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\\
\Omega_5^{\mathrm{Spin}}(BS_4) &\cong 0\\
\Omega_6^{\mathrm{Spin}}(BS_4) &\cong \mathbb{Z}/2.
\end{aligned}$$

Therefore $Ph_0^{S_4}(\mathbb{R}^3; f_{1/2}^D) \cong \mathbb{Z}/2$.

One can use the Atiyah-Hirzebruch spectral sequence to show $r = 2$ in Theorem 5.74; we do not need this so do not present the proof.

Proof. First, we use the Adams spectral sequence to determine the free and 2-primary parts. Since $ko_*(BS_4)$ splits as $ko_*(\mathrm{pt}) \oplus \widetilde{ko}_*(BS_4)$, we focus on $\widetilde{ko}_*(BS_4)$ and add the Bott-song summands in at the end. There is a section s of the parity map $S_4 \to \mathbb{Z}/2$, which stably splits BS_4. That is, there is a spectrum M, a map $t\colon M \to \Sigma^\infty BS_4$, and a weak equivalence

$$(s,t)\colon \Sigma^\infty B\mathbb{Z}/2 \vee M \xrightarrow{\simeq} \Sigma^\infty BS_4. \tag{5.75}$$

This also splits the $\mathcal{A}$-module structure of $\widetilde{H}^*(BS_4;\mathbb{Z}/2)$ as $\widetilde{H}^*(M;\mathbb{Z}/2) \oplus \widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, where $\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2)$ is embedded via the parity map. Therefore $\widetilde{H}^*(M;\mathbb{Z}/2)$ is isomorphic to a complimentary subspace of $\mathbb{Z}/2 \cdot \{a^k \mid k \geq 0\} \subset \widetilde{H}^*(BS_4;\mathbb{Z}/2)$. As this isomorphism is realized by a map of spectra, it is an isomorphism of $\mathcal{A}$-modules, hence $\mathcal{A}(1)$-modules. We will run the Adams spectral sequence for $\widetilde{ko}_*(M)$, and add the $\widetilde{ko}_*(B\mathbb{Z}/2)$ summands in at the end.

The mod 2 cohomology of BS_4 is given in Proposition 5.34, and the action of the Steenrod squares in Proposition 5.35. We can therefore draw $\widetilde{H}^*(M;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, which we do in Figure 23, left. We have

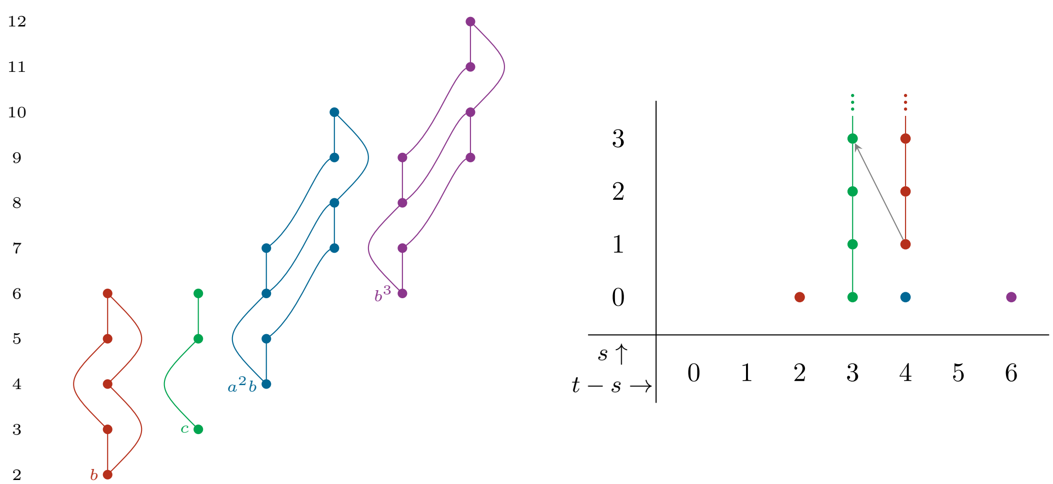


FIGURE 23. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. The submodule pictured here contains all elements of degree at most 6. Right: the corresponding Ext, which is the E_2-page for the Adams spectral sequence converging to the 2-primary part of $\widetilde{ko}_*(M)$.

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2 J \oplus \Sigma^3 \mathcal{O} \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^6\mathcal{A}(1) \oplus P, \tag{5.76}$$

where P is 6-connected. Names of $\mathcal{A}(1)$-modules are as in previous sections; for all these modules except for P, we have already seen $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(-,\mathbb{Z}/2)$, and P is irrelevant for degree reasons. We display the E_2-page in Figure 23, right.

In the range pictured, h_0-equivariance of differentials implies the only possible nontrivial differentials are from the infinite tower in degree 4 to the infinite tower in degree 3; a d_2 is pictured as an example. In fact, those towers must support a nonzero d_r for some r; by h_0-equivariance, d_r is either zero for every element of the tower in degree 4, or nonzero for every element. Hence, if all d_r were zero for all r, then $\widetilde{ko}_3(BS_4)$ would contain a free summand, contradicting Lemma 3.24. Therefore there is some $r\geq 2$ for which all d_r differentials from the tower in degree 4 to the tower in degree 3 are nontrivial (not necessarily the d_2s pictured). On the E_∞-page, the tower in degree 4 vanishes, and only $r+1$ summands of the degree-3 tower remain. Thus we have computed the 2-primary part of $ko_*(BS_4)$ in degrees 6 and lower:

- From $ko_*(\mathrm{pt})$, we have a $\mathbb{Z}$ summand in degrees 0 and 4 and a $\mathbb{Z}/2$ summand in degrees 1 and 2.
- From $\widetilde{ko}_*(B\mathbb{Z}/2)$, we have $\mathbb{Z}/2$ summands in degrees 1 and 2 and a $\mathbb{Z}/8$ summand in degree 3 [MM76].
- From Figure 23, right, we have $\mathbb{Z}/2$ in degree 2, $\mathbb{Z}/2^{r+1}$ in degree 3, and a $\mathbb{Z}/2$ each in degrees 4 and 6.

To determine the odd-primary torsion, use first that the forgetful map $\Omega_*^{\mathrm{Spin}}(-)\to\Omega_*^{\mathrm{SO}}(-)$ is an isomorphism on odd-primary torsion, so we just have to determine the odd-primary torsion in $\Omega_k^{\mathrm{SO}}(BS_4)$ for $k\leq 6$, which we did in Lemma 5.67. ☐

5.4.3. *Class A.* As in the case of chiral tetrahedral symmetry, V_λ does not admit a pin^c structure, since we saw in Lemma 5.7 that its pullback along $BA_4\to BS_4$ also does not admit a pin^c structure. Let f_0^A, resp. $f_{1/2}^A$, denote the equivariant local systems of spectra associated to the class A spinless, resp. spin-1/2 cases. Theorem 2.24 expresses $Ph_0^{S_4}(\mathbb{R}^3;f_0^A)$ and $Ph_0^{S_4}(\mathbb{R}^3;f_{1/2}^A)$ in terms of the spin^c bordism of $(BS_4)^{3-V_\lambda}$ for spinless fermions and BS_4 for spin-1/2 fermions.

Theorem 5.77. *There are integers $r,r'\geq 2$ such that the low-degree spin^c bordism groups of $(BS_4)^{3-V_\lambda}$ and BS_4 are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z}/2 & \Omega_1^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}/2\\
\widetilde\Omega_2^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_2^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\widetilde\Omega_3^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z}/6\oplus\mathbb{Z}/2^r & \Omega_3^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}/12\oplus\mathbb{Z}/2^{r'}\\
\widetilde\Omega_4^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z}^2 & \Omega_4^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}^2\oplus\mathbb{Z}/2\\
\widetilde\Omega_5^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z}/2^{r-1}\oplus\mathbb{Z}/6\oplus(\mathbb{Z}/2)^{\oplus 3} & \Omega_5^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/24\oplus\mathbb{Z}/2^{r'+1}\\
\widetilde\Omega_6^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) &\cong \mathbb{Z}^2 & \Omega_6^{\mathrm{Spin}^c}(BS_4) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 3}.
\end{aligned}$$

One can use the Atiyah-Hirzebruch spectral sequence to show $r=r'=2$. We do not need this, so do not go into the details.

Proof. As usual, the calculation separates into odd-primary and 2-primary parts.

Lemma 5.78. *The only odd-primary torsion in the spin^c bordism of $(BS_4)^{3-V_\lambda}$ and BS_4 in degrees 6 and below consists of two $\mathbb{Z}/3$ summands in degrees 3 and 5.*

Proof. Since $|S^4|=2^3\cdot 3$, we only have to check 3-torsion: if $\ell\geq 5$ is prime, the maps $BS_4\to\mathrm{pt}$ and $(BS_4)^{3-V}\to\mathrm{pt}$ are stable ℓ-primary equivalences by the Whitehead theorem [Ser53, Chapitre III, Théorème 3]. The forgetful map $MTSpin^c\to MSO\wedge(B\mathrm{U}_1)_+$ is an odd-primary equivalence, and since $3-V_\lambda$ is orientable, there is a Thom isomorphism

$$MSO\wedge(B\mathrm{U}_1)_+\wedge(BS_4)^{3-V}\xrightarrow{\simeq} MSO\wedge(B\mathrm{U}_1)_+\wedge(BS_4)_+, \tag{5.79}$$

so in both the spinless and spin-1/2 cases, we can glean the 3-torsion from $\Omega_*^{\mathrm{Spin}^c}(B\mathrm{U}_1\times BS_4)$. As the homology of $B\mathrm{U}_1$ is torsion-free, the Künneth map $H_*(B\mathrm{U}_1)\otimes H_*(BS_4)\to H_*(B\mathrm{U}_1\times BS_4)$ is an isomorphism of graded abelian groups. Using this together with Thomas' [Tho74] calculation of $H_*(BS_4)$, we conclude that the only odd-primary torsion in $H_*(BU_1 \times BS_4)$ in degrees below 7 is $\mathbb{Z}/3 \subset H_3(BU_1 \times BS_4)$ and $\mathbb{Z}/3 \subset H_5(BU_1 \times BS_4)$.

Now feed this to the Atiyah-Hirzebruch spectral sequence with signature

$$E^2_{p,q} = H_p(BU_1 \times BS_4, \Omega_q^{SO}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{SO}(BU_1 \times BS_4). \tag{5.80}$$

The coefficients are sums of $\mathbb{Z}$ and $\mathbb{Z}/2$; since we only care about 3-torsion, we can ignore the $\mathbb{Z}/2$ summands, whose differentials cannot map nontrivially to or from any 3-torsion element. The only 3-torsion on the E^2-page in total degree less than 7 is a single $\mathbb{Z}/3$ summand in each of $E^2_{3,0}$ and $E^2_{5,0}$, coming from our calculation above of 3-torsion in homology. These 3-torsion summands cannot participate in any nonzero differentials: they do not map to each other, and cannot receive any differentials from free summands, or from the 7-line (which we have not calculated). Thus they persist to the E^∞-page. It is a priori possible more 3-torsion is created from free summands on the E^2-page, which could happen if a differential maps from a free summand to another free summand. All free summands are in even total degree, though, so this does not happen, and the only 3-torsion in $\Omega_k^{SO}(BU_1 \times BS_4)$, for $k < 7$, is two $\mathbb{Z}/3$ summands in degrees 3 and 5. ☐

Next, we compute the 2-torsion using the Adams spectral sequence over $\mathcal{E}(1)$.

For the spinless case, recall from (5.72) (drawn in Figure 21) the calculation of $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module. There are isomorphisms of $\mathcal{E}(1)$-modules $N_4 \cong \mathcal{O}^\circ \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2$ and $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, so as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2) \cong \mathcal{O}^\circ \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2 \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4\mathcal{O}^\circ \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus P, \tag{5.81}$$

where P is 6-connected. We draw this in Figure 24, left.

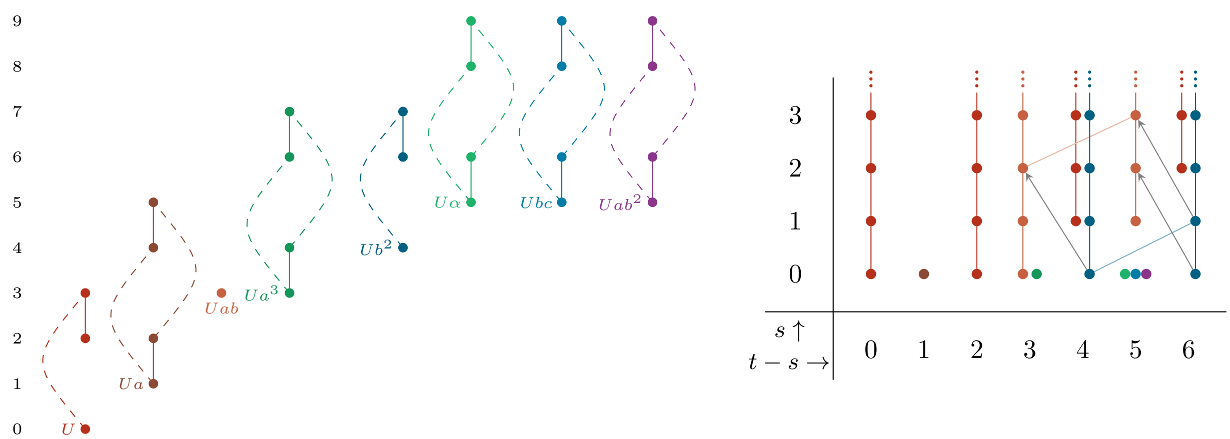


FIGURE 24. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees at most 6. Here $\alpha := a^5 + a^3b$. Right: the corresponding Ext, which is the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((BS_4)^{3-V})$. Some nonzero v_1-actions are hidden for clarity.

To draw the E_2-page of the Adams spectral sequence, use the computations of $\mathrm{Ext}(\mathcal{O}^\circ)$ from (4.55) and $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$ from (3.7) to obtain Figure 24, right. For clarity, we do not draw most v_1-actions. There may be differentials in this range, though we do not determine whether they are the d_2s pictured.

From Figure 24, right, $\widetilde{ku}_0((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}$, so Lemma 3.24 implies there is a single free summand in each even degree and the odd-degree ku-groups are torsion. Therefore, one of the towers on the 4-line must admit a nontrivial d_r differential to the tower on the 3-line, and in fact, v_1-equivariance of the differentials implies that tower on the 4-line must be the blue one coming from $\Sigma^4\mathcal{O}^\circ$. The remaining tower must survive, so on the E_∞-page, the 3-line has its $\mathbb{Z}/2$ summand and a $\mathbb{Z}/2^r$ summand coming from the red tower, and the 4-line has a single $\mathbb{Z}$ summand left. The results on $\widetilde{ku}_5$ and $\widetilde{ku}_6$ follow from v_1- and h_0-equivariance of d_r.

On to the spin-1/2 case. We factor $ku_*(BS_4) \cong ku_*(\mathrm{pt}) \oplus \widetilde{ku}_*(BS_4)$. In the proof of Theorem 5.74, we split $\Sigma^\infty BS_4 \simeq \Sigma^\infty B\mathbb{Z}/2 \vee M$ and determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$. Combining this with Nguyen's computation [Ngu09, Theorem 2.3.1] of the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(BS_4;\mathbb{Z}/2)$, we have that as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{O} \oplus \Sigma^4\mathbb{Z}/2 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^6\mathcal{E}(1) \oplus \Sigma^6\mathcal{E}(1) \oplus P, \tag{5.82}$$

where P is 6-connected. We draw this in Figure 25, left.

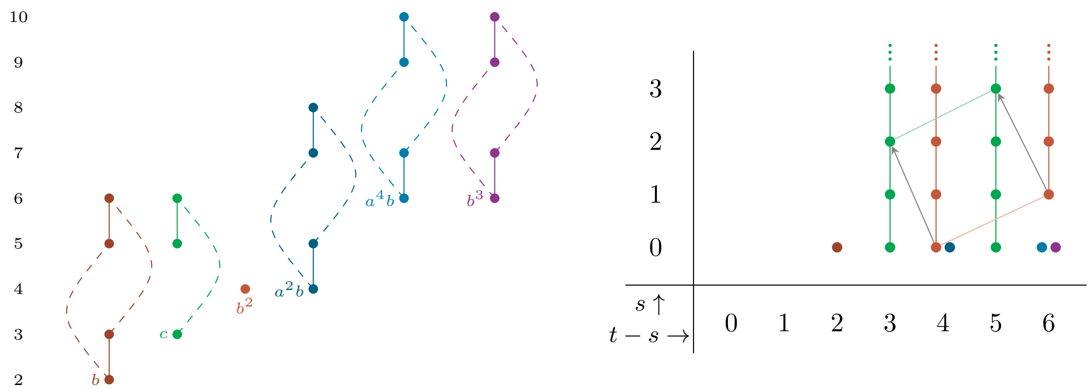


FIGURE 25. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees at most 6. Right: the corresponding Ext, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M)$. Some v_1-actions are hidden to declutter the diagram.

For each of these modules N (except P, which as usual is too high-degree to be relevant), we already calculated $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(N,\mathbb{Z}/2)$: for $\mathcal{O}$, see (4.55), and for $\mathbb{Z}/2$, see (3.7). Therefore the E_2-page for the Adams spectral sequence is as drawn in Figure 25, right. Most of the v_1-actions are hidden to make the diagram clearer. We indicate locations of some possible differentials, but they are not necessarily d_2s.

Lemma 3.24 implies $\widetilde{ku}_*(BS_4)$ is torsion, so all towers present on the E_2-page must emit or receive differentials. Thus there is some $r' \geq 2$ such that the green tower on the 3-line is killed by a $d_{r'}$ emerging from the orange tower on the 4-line; therefore on the E_∞-page, the 4-line contains only the $\mathbb{Z}/2$ summand in $E_\infty^{0,4}$, and the 3-line contains r' $\mathbb{Z}/2$ summands, the remains of the tower. For $\widetilde{ku}_5$ and $\widetilde{ku}_6$, v_1-equivariance of $d_{r'}$ determines the E_∞-page in the same way.

It remains to add in the summands corresponding to $ku_*(\mathrm{pt})$ and $\widetilde{ku}_*(B\mathbb{Z}/2)$; the former contributes a $\mathbb{Z}$ summand in each even dimension, and the latter contributes $\mathbb{Z}/2$ in degree 1, $\mathbb{Z}/4$ in degree 3, and $\mathbb{Z}/8$ in degree 5, by work of Hashimoto [Has83, Theorem 3.1]. □

5.5. **Full octahedral symmetry.** The full group of symmetries of the octahedron, including orientation-reversing ones, is isomorphic to $G := A_4 \times \mathbb{Z}/2$. Let $\lambda\colon G \to \mathrm{O}_3$ denote the corresponding three-dimensional real representation of G, and $V_\lambda \to BG$ denote the associated vector bundle. We saw in §5.4 the pullback of V_λ along $BS_4 \to BG$ is not pin^c, so V_λ is also not pin^c, and therefore is also not pin$^-$.

The Künneth formula and Proposition 5.34 together imply

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[x,a,b,c]/(ac), \tag{5.83}$$

where $|x| = |a| = 1$, $|b| = 2$, and $|c| = 3$.

Lemma 5.84. *$w_1(V_\lambda) = x$ and $w_2(V_\lambda) = b + x^2$.*

Proof. For w_1, we know $w_1(V_\lambda) \neq 0$ because V_λ is unorientable, but because $V_\lambda|_{BS_4}$ is orientable, $w_1(V_\lambda)$ cannot be a or $x+a$, leaving $w_1(V_\lambda) = x$.

For w_2, we know the pullback of V_λ to BS_4 has $w_2(V|_{S_4}) = b$. If $i\colon B\mathbb{Z}/2 \to BG$ is induced by the inclusion of a reflection in G, then $i^*\lambda$ decomposes as a direct sum of three copies of the sign representation, so $i^*V_\lambda \cong 3\sigma$. Therefore $i^*w_2(V_\lambda) = x^2$, uniquely constraining $w_2(V_\lambda) = b + x^2$. □

5.5.1. *Class D, spinless case.* The FCEP says we should study the spin bordism of $(BG)^{3-V_\lambda}$. We will argue as we did in the case of pyritohedral symmetry in §5.2, replacing $3 - V_\lambda$ with a virtual vector bundle whose Adams E_2-page is isomorphic to that of $(BG)^{3-V_\lambda}$, but which is easier to calculate. This isomorphism did not come from a map of spectra, so cannot tell us anything about differentials or hidden extensions, but just as for pyritohedral symmetry, we will see that for entirely formal reasons, all differentials vanish and all hidden extensions split in the range we need. Using the twisted Künneth formula, $\widetilde{H}^*((BG)^{3-V_\lambda})$ contains no odd-primary torsion, so neither does $\widetilde{\Omega}_*^{\mathrm{Spin}}((BG)^{3-V_\lambda})$, so using the 2-primary Adams spectral sequence suffices.

For the rest of this section, all homology and cohomology is understood to be with $\mathbb{Z}/2$ coefficients.

Lemma 5.85. *Let $E \to BG$ denote the virtual vector bundle induced from the virtual representation*

$$2 - (V_\lambda|_{S_4} \boxplus (-\sigma)). \tag{5.86}$$

Then, there is an isomorphism of $\mathcal{A}(1)$-modules $\widetilde{H}^*((BG)^{3-V_\lambda}) \cong \widetilde{H}^*((BG)^E)$, hence an isomorphism between the E_2-pages of the Adams spectral sequences for $ko \wedge (BG)^{3-V_\lambda}$ and $ko \wedge (BG)^E$.

Proof. The E_2-pages of these Adams spectral sequences are determined by the $\mathcal{A}(1)$-module structures on cohomology, which are in turn determined by w_1 and w_2 of the virtual bundles $3-V_\lambda$ and E. Since $w_1(E) = x$ and $w_2(E) = u$, then for $i = 1, 2$, $w_i(3 - V_\lambda) = w_i(E)$. ☐

Because E is induced from a representation which is an exterior sum, its Thom spectrum splits as

$$(BG)^E \simeq (BS_4)^{3-V_\lambda|_{S_4}} \wedge (B\mathbb{Z}/2)^{\sigma-1} \tag{5.87}$$

The Künneth theorem then simplifies the E_2-page:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\widetilde{H}^*((BS_4)^{3-V_\lambda|_{S_4}}) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{5.88}$$

Campbell [Cam17, Figure 6.1] computes the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1})$, and we computed $\widetilde{H}^*((BS_4)^{3-V_\lambda})$ in (5.72) (drawn in Figure 21).

Proposition 5.89. *There is an isomorphism of $\mathcal{A}(1)$-modules*

$$\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \otimes_{\mathbb{Z}/2} N_4 \cong \Sigma N_5 \oplus (V_2 \otimes_{\mathbb{Z}/2} \mathcal{A}(1)) \oplus P_2, \tag{5.90}$$

where N_5 is as in Figure 26, V_2 is a graded vector space with a homogeneous basis in degrees $\{0, 2, 3, 4\}$, and P_2 is 4-connected.

Proof. Compute directly, by hand or by computer. ☐

Recall from (5.72) (drawn in Figure 21) the $\mathcal{A}(1)$-module structure on $(BS_4)^{3-V_\lambda}$. Margolis' theorem (Theorem 3.22) splits off a $\Sigma^k H\mathbb{Z}/2$ summand from $ko \wedge (BS_4)^{3-V_\lambda}$ for every direct summand of $\Sigma^k \mathcal{A}(1)$ in $\widetilde{H}^*((BS_4)^{3-V_\lambda})$; below degree 8, this occurs for $k = 3, 5$. Therefore, by the same line of reasoning as in §5.2, there is a spectrum Y' such that

$$\widetilde{ko}_n((BG)^{3-V_\lambda}) \cong \pi_n(Y') \oplus \widetilde{H}_{n-3}((B\mathbb{Z}/2)^{\sigma-1}) \oplus \widetilde{H}_{n-5}((B\mathbb{Z}/2)^{\sigma-1}), \tag{5.91}$$

and as $\mathcal{A}$-modules,

$$\widetilde{H}^*(Y') \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (N_4 \oplus \Sigma^4 N_4 \oplus P_3) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \tag{5.92}$$

where P_3 is a 4-connected $\mathcal{A}(1)$-module. Therefore the change-of-rings formula (3.4) applies to the E_2-page of the Adams spectral sequence for $\pi_*(Y')$, showing

$$E_2^{s,t}(Y') \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}((N_4 \oplus \Sigma^4 N_4 \oplus P_3) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{5.93}$$

To calculate the spin bordism groups of $(BG)^{3-V_\lambda}$, we will work with this spectral sequence, adding the summands corresponding to $\Sigma^3 H\mathbb{Z}/2$ and $\Sigma^5 H\mathbb{Z}/2$ at the end.

Theorem 5.94. *The first few spin bordism groups of* $(BG)^{3-V_\lambda}$ *are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/2\\ \widetilde\Omega_2^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 2}\\ \widetilde\Omega_3^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 2}\\ \widetilde\Omega_4^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 4},\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}((BG)^{3-V_\lambda})$ *is torsion, so the* 0^{th} $(S_4\times\mathbb Z/2)$*-equivariant phase homology group for this case is isomorphic to* $(\mathbb Z/2)^{\oplus 4}$.

Proof. Proposition 5.89 and (5.93) together imply the E_2-page for Y' is

$$E_2^{s,t}(Y') \cong \mathrm{Ext}^{s,t}_{\mathcal A(1)}(\Sigma N_5 \oplus V_2\otimes_{\mathbb Z/2}\mathcal A(1)\oplus \Sigma^5 N_5\oplus \Sigma^4 V_2\otimes_{\mathbb Z/2}\mathcal A(1)\oplus P, \mathbb Z/2), \tag{5.95}$$

where P is 4-connected. We will see that the E_2-page in $t-s\le 4$ is empty for $s\ge 2$, so there can be no differentials involving $\mathrm{Ext}(P)$ in the range we care about.

Our first order of business is therefore to determine $\mathrm{Ext}^{s,t}_{\mathcal A(1)}(N_5,\mathbb Z/2)$ for small s,t. There is an extension of $\mathcal A(1)$-modules

$$0 \longrightarrow R_3 \longrightarrow N_5 \longrightarrow \Sigma R_0 \longrightarrow 0, \tag{5.96}$$

which we draw in Figure 26, left, fitting $\mathrm{Ext}^{s,t}_{\mathcal A(1)}(N_5,\mathbb Z/2)$ into a long exact sequence (Figure 26, right). The $\mathcal A(1)$-module R_3 and its Ext are calculated in the range we need by Freed-Hopkins [FH16a, Figure 5, case $s=3$] and Beaudry-Campbell [BC18, Figures 32, 33]. In the range pictured, there are two boundary maps in Figure 26, right, which could be nonzero; the existence of a nonzero map $N_5\to\Sigma^4\mathbb Z/2$ forces the boundary map $\delta\colon \mathrm{Ext}^{0,4}(R_3)\to\mathrm{Ext}^{1,4}(\Sigma R_0)$ to vanish. We do not need to know whether the other pictured boundary map vanishes.

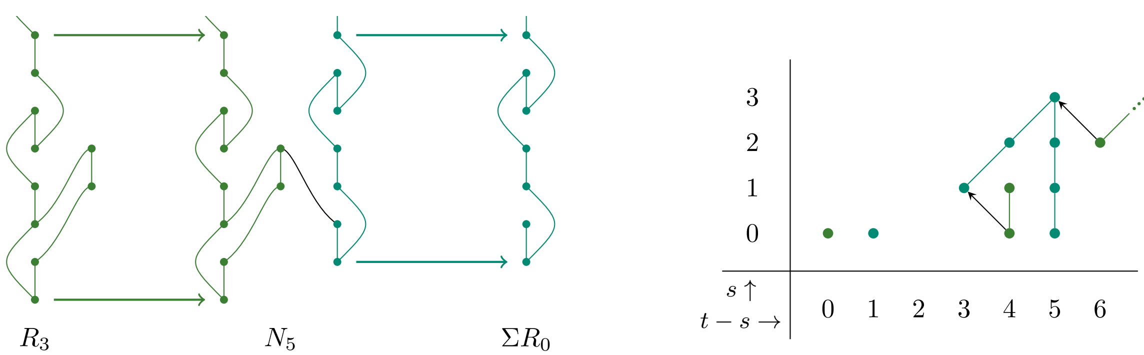
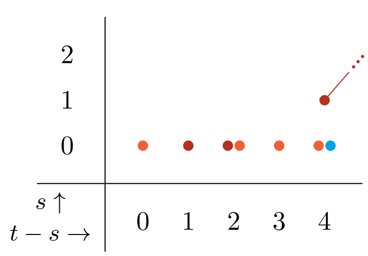


Figure 26. Left: the $\mathcal A(1)$-module N_5 in the extension (5.96). Right: the corresponding long exact sequence in Ext.

Hence the E_2-page for computing $\pi_*(Y')$ is

(5.97)



The 4-line is concentrated in filtration 0 and 1, and so there can be neither nonzero differentials nor nontrivial extension problems involving elements of degree 4 or less. This accounts for $\pi_*(Y')$; for the each factor of $\widetilde{H}_{*-\ell}((B\mathbb{Z}/2)^{\sigma-1})$, add a single $\mathbb{Z}/2$ summand in degrees ℓ and above. ☐

5.5.2. *Class D, spin-*1/2 *case.* Now we ask for the symmetries to mix. Let $f^D_{1/2}$ denote the local system of symmetry types for this case. By Theorem 2.11, we consider the spin bordism of $X := (BS_4\times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$, because V_λ is not pin$^-$. The isomorphism $\mathrm{Det}V_\lambda \cong 0\boxplus\sigma$ provides an isomorphism $X\simeq (BS_4)_+\wedge(B\mathbb{Z}/2)^{\sigma-1}$, so (2.10b) implies the spin bordism of this spectrum computes the pin$^-$ bordism of BS_4, which could be independently interesting.

Theorem 5.98. *The first few spin bordism groups of* X *are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/8\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong (\mathbb{Z}/2)^{\oplus 4}\\
\widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}.
\end{aligned}$$

Since $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ *is torsion by Lemma 3.24,* $Ph_0^{S_4\times\mathbb{Z}/2}(\mathbb{R}^3; f^D_{1/2})\cong\mathbb{Z}/2$.

Proof. As usual, Lemma 3.30 spits X as a sum of $(B\mathbb{Z}/2)^{\sigma-1}$ and another spectrum M, where $\widetilde{H}^*(M;\mathbb{Z}/2)$ is complementary in $\widetilde{H}^*(X;\mathbb{Z}/2)$ to the space spanned by $\{Uw_1(\lambda)^k\}$. The $(B\mathbb{Z}/2)^{\sigma-1}$ summand gives us pin$^-$ bordism, and we focus on M.

We have $w_1(\mathrm{Det}(V_\lambda)-1) = w_1(V_\lambda) = x$ and $w_2(\mathrm{Det}V_\lambda - 1) = 0$; this and the $\mathcal{A}$-module structure on $BS_4\times B\mathbb{Z}/2$ calculated in (5.83) determine the $\mathcal{A}(1)$-module structure on M. We obtain an isomorphism of $\mathcal{A}(1)$-modules

$$\widetilde{H}^*(M;\mathbb{Z}/2)\cong \Sigma R_5\oplus\Sigma^2R_3\oplus\Sigma^3\mathcal{A}(1)\oplus\Sigma^3\mathcal{A}(1)\oplus\Sigma^3\mathcal{A}(1)\oplus\Sigma^3\mathcal{A}(1)\oplus\Sigma^4\mathcal{A}(1)\oplus\Sigma^4\mathcal{A}(1)\oplus P, \tag{5.99}$$

where P is 4-connected. We will see momentarily that for $t-s\le 4$, $E_2^{s,t}$ is empty for $s\ge 2$; this and the 4-connectedness of P imply its contribution to the E_2-page cannot affect the spectral sequence in degrees $t-s\le 4$, which is all we need. We draw these summands, except for P, in Figure 27.

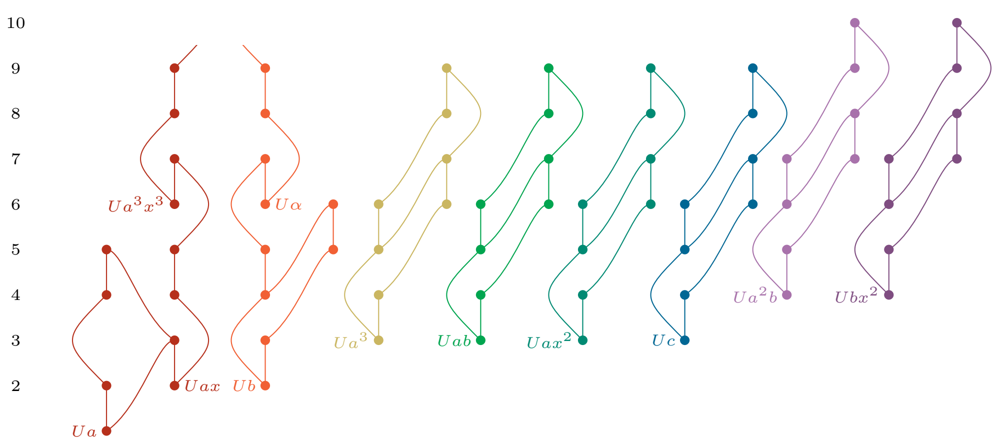
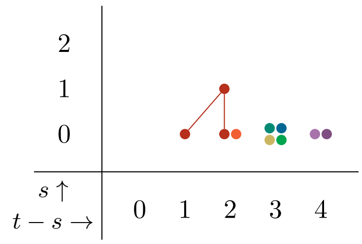


FIGURE 27. The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. The pictured summand contains all classes in degrees 4 and below. Here $\alpha := b^2x^2+a^2b^2+c^2$.

Freed-Hopkins [FH16a, Figure 5, cases $s = \pm 3$] and Beaudry-Campbell [BC18, Figures 32, 33, 37] calculate $\mathrm{Ext}(R_5)$ and $\mathrm{Ext}(R_3)$ in the degrees we need, and we can draw the E_2-page of the Adams spectral sequence:

(5.100)



This collapses, and it and the pin$^-$ bordism groups from the $(B\mathbb{Z}/2)^{\sigma-1}$ summand, which are computed in [ABP69, KT90b], sum together to the groups in the theorem. ☐

5.5.3. *Class A, spinless case.* Let f_0^A denote the local system of symmetry types for this case. We want to calculate $\widetilde{\Omega}_*^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$. Using the twisted Künneth formula, $\widetilde{H}^*((BG)^{3-V_\lambda};\mathbb{Z}/2)$ is 2-torsion, and therefore $\widetilde{\Omega}_*^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$ is too, so it suffices to use the 2-primary Adams spectral sequence.

Theorem 5.101. *The first few spin^c bordism groups of $(BG)^{3-V_\lambda}$ are:*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4},
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$ is torsion. Hence $Ph_0^{S_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}$.

Proof. There is an isomorphism of $\mathcal{E}(1)$-modules

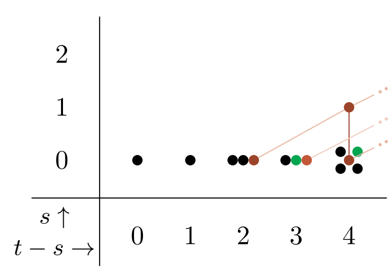
$$N_5 \cong \mathcal{E}(1) \oplus \Sigma R_0 \oplus \Sigma^2 R_0, \tag{5.102}$$

hence another isomorphism of $\mathcal{E}(1)$-modules

$$\widetilde{H}^*((BG)^{3-V_\lambda}) \cong (V_c \otimes_{\mathbb{Z}/2} \mathcal{A}(1)) \oplus \Sigma^2 R_0 \oplus \Sigma^3 R_0 \oplus P_c, \tag{5.103}$$

where P_c is 4-connected and V_c is a graded vector space with a homogeneous basis of elements in degrees $\{0, 1, 2, 2, 3, 3, 4, 4, 4, 4\}$. Therefore the E_2-page of the Adams spectral sequence is

(5.104)



Below degree 5, there are no nonzero differentials, because there is nothing in Adams filtration 2 or higher. And degree considerations rule out hidden extensions, so we are done. ☐

5.5.4. *Class A, spin-1/2 case.* Because V_λ is not pin^c, Theorem 2.24 tells us to compute the spin^c bordism groups of $X := (BS_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 5.105. *The first few spin^c bordism groups of X are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/2\\ \widetilde\Omega_1^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/2\\ \widetilde\Omega_2^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/4\oplus(\mathbb Z/2)^{\oplus 2}\\ \widetilde\Omega_3^{\mathrm{Spin}^c}(X) &\cong (\mathbb Z/2)^{\oplus 4}\\ \widetilde\Omega_4^{\mathrm{Spin}^c}(X) &\cong \mathbb Z/8\oplus\mathbb Z/4\oplus(\mathbb Z/2)^{\oplus 4}.\end{aligned}$$

As $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion, the 0^{th} $(S_4\times\mathbb Z/2)$-equivariant phase homology group for this case is isomorphic to $\mathbb Z/8\oplus\mathbb Z/4\oplus(\mathbb Z/2)^{\oplus 4}$.

Proof. By Lemma 3.30, X splits as $(B\mathbb Z/2)^{\sigma-1}\vee M$, where $\widetilde H^*(M;\mathbb Z/2)$ is isomorphic to a complementary subspace to the subspace $\mathbb Z/2\cdot\{Ux^k\}$ inside $\widetilde H^*(X;\mathbb Z/2)$. As usual, the $(B\mathbb Z/2)^{\sigma-1}$ summand contributes pin^c bordism groups to the final answer, so we focus on M. The $\mathcal A(1)$-module structure we computed in (5.99) and drew in Figure 27 tells us the $\mathcal E(1)$-structure; here, we use that $R_5\cong\mathcal E(1)\oplus\Sigma R_0$ and $R_3\cong\mathcal E(1)\oplus\Sigma^2R_0$ as $\mathcal E(1)$-modules. Therefore, there is an $\mathcal E(1)$-module isomorphism

$$\widetilde H^*(M;\mathbb Z/2)\cong\Sigma\mathcal E(1)\oplus\Sigma^2R_0\oplus\Sigma^2\mathcal E(1)\oplus\Sigma^3\mathcal E(1)\oplus\Sigma^3\mathcal E(1)\oplus\Sigma^3\mathcal E(1)\oplus\Sigma^4R_0\oplus\Sigma^4\mathcal E(1)\oplus\Sigma^4\mathcal E(1)\oplus P, \tag{5.106}$$

where P is 4-connected. Therefore to infer anything about $\widetilde\Omega_4^{\mathrm{Spin}^c}(M)$ from this spectral sequence, we must argue that P does not affect it; this will follow when we see the $t-s=4$ line of the E_2-page is empty in Adams filtration 2 and above, so there can be no nonzero differentials from the 5-line to the 4-line. We draw (5.106) in Figure 28.

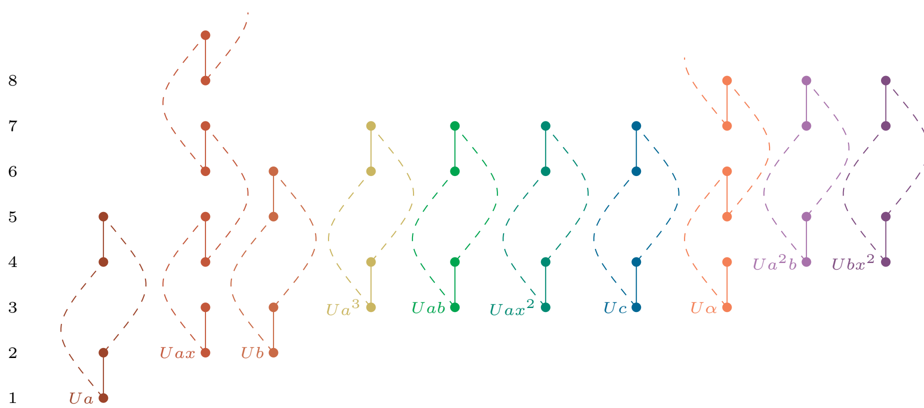
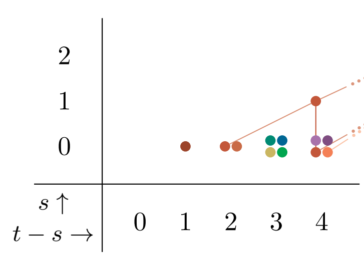


FIGURE 28. The $\mathcal E(1)$-module structure on $\widetilde H^*(M;\mathbb Z/2)$ in low degrees. Here $\alpha\coloneqq abx+b^2+cx$. This submodule contains all elements in degrees 4 and below.

Recalling $\mathrm{Ext}(R_0)$ from Proposition 4.48, the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(M)$ is

(5.107)



In this range, the spectral sequence collapses, so we read off $\widetilde\Omega_*^{\mathrm{Spin}^c}(M)$ and combine it with pin^c bordism as computed in [BG87a, BG87b] to conclude. ☐

5.6. **Chiral icosahedral symmetry.** Let $\lambda\colon A_5 \to \mathrm{SO}_3$ denote the representation given by chiral icosahedral symmetry, and as usual let $V_\lambda \to BA_5$ denote the associated vector bundle.

Remark 5.108. Unlike the previous symmetry groups we studied, icosahedral symmetry is incompatible with translations, and there are no space groups whose underlying point group is either the chiral icosahedral group or the full icosahedral group. This means one should not expect to realize any phases equivariant for these symmetry groups as a lattice Hamiltonian system on a periodic lattice on $\mathbb{R}^3$. This does not rule out the possibility of interesting phases with an icosahedral symmetry: there are examples of phases studied via lattice Hamiltonian realizations on lattices in great generality, such as the toric code model in [Fre19, §2.3], the GDS model in [FH16b, Deb20, FHHT20], and the phases on aperiodic lattices studied by Huang-Wu-Liu [HWL20]. In a similar vein, it may be possible for a Hamiltonian on an aperiodic lattice with icosahedral symmetry to model a nontrivial crystalline SPT. See [VLP+19] for an example of how such an implementation might look.

For icosahedral symmetry, the hard work is behind us. Let $\lambda\colon A_5 \to \mathrm{O}_3$ denote the representation as the orientation-preserving symmetries of the icosahedron. The restriction to $A_4 \subset A_5$ corresponds to symmetries that preserve a concentric tetrahedron. Let $V_\lambda \to BA_5$ be the associated bundle to λ.

Lemma 5.109. *The inclusion $\varphi\colon A_4 \hookrightarrow A_5$ induces an equivalence on mod 2 cohomology. Hence φ induces 2-primary equivalences $\Sigma^\infty(BA_4)_+ \to \Sigma^\infty(BA_5)_+$ and $(BA_4)^{3-\varphi^*(V_\lambda)} \to (BA_5)^{3-V_\lambda}$.*

Proof. The first part is Lemma 3.27: here $[A_5 : A_4] = 5$, $P = \mathbb{Z}/2\times\mathbb{Z}/2$, and for both A_4 and A_5, $N(P)/P \cong \mathbb{Z}/3$.

For the second part, the Thom isomorphism theorem tells us $\varphi'\colon (BA_4)^{3-\varphi^*(V_\lambda)} \to (BA_5)^{3-V_\lambda}$ induces an isomorphism on mod 2 cohomology. The desired 2-primary equivalences then follow from the mod 2 Whitehead theorem [Ser53, Chapitre III, Théorème 3]. ☐

We can therefore reuse the calculations we made at the prime 2 in §5.1 to obtain the 2-primary parts of $\widetilde\Omega_k^{\mathrm{Spin}}((BA_5)^{3-V_\lambda})$ and $\Omega_k^{\mathrm{Spin}}(BA_5)$; the odd-primary pieces are different, but not hard.

Proposition 5.110. *The only odd-primary torsion in $H_k(BA_5)$ for $k<7$ is contained in $H_3(BA_5)\cong\mathbb{Z}/30$.*

Proof sketch. One can compute this using GAP; we also indicate how to do it by hand. Since $|A_5| = 60 = 2^2\cdot 3\cdot 5$, there is no p-primary torsion for $p>5$, so it suffices to determine $H^k(BA_5;\mathbb{Z}/3)$ and $H^k(BA_5;\mathbb{Z}/5)$ in low degrees. This can be done using the theorem of Adem-Milgram [AM04, Theorem II.6.8] mentioned above, since the Sylow 3- and 5-subgroups of A_5 are abelian. ☐

Corollary 5.111. *In $\widetilde\Omega_k^{\mathrm{Spin}}((BA_5)^{3-V_\lambda})$ and $\Omega_k^{\mathrm{Spin}}(BA_5)$, the only odd-primary torsion for $k<7$ is a $\mathbb{Z}/15$ in degree 3.*

Proof. As usual, we use the fact that $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an isomorphism on odd-primary torsion, together with the Thom isomorphism $\widetilde\Omega_*^{\mathrm{SO}}((BA_5)^{3-V_\lambda}) \cong \Omega_*^{\mathrm{SO}}(BA_5)$, to reduce to showing the claim for $\Omega_k^{\mathrm{SO}}(BA_5)$. For this, use the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(BA_5;\Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(BA_5). \tag{5.112}$$

On the E^2-page, the only odd-primary torsion in total degree below 7 is $\mathbb{Z}/15 \subset E^2_{3,0} = H_3(BA_5)$. In all differentials involving $E^r_{3,0}$, the other group is zero, so this odd-primary torsion lives to the E^∞-page.

We also must check that the free summands in total degree below 7 do not receive differentials that produce more odd-primary torsion. There are only two such free summands, in $E^2_{0,0}$ and $E^2_{0,4}$, and they can only receive differentials from 2-torsion abelian groups, so that does not happen. ☐

Now we need to combine this with the 2-primary summands. For $(BA_5)^{3-V_\lambda}$, we need $\Omega_*^{\mathrm{Spin}}((BA_4)^{3-\varphi^*V_\lambda})$, which we computed in Theorem 5.4. For BA_5, we need $\Omega_*^{\mathrm{Spin}}(BA_4)$; in the degrees we need, this is isomorphic to $ko_*(BA_4)$, which Bruner-Greenlees compute in [BG10, §7.7.E].

Theorem 5.113. *The low-degree spin bordism groups of* $(BA_5)^{3-V}$ *and* BA_5 *are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z & \Omega_0^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z\\
\widetilde\Omega_1^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_1^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z/2\\
\widetilde\Omega_2^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_2^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z/2\oplus\mathbb Z/2\\
\widetilde\Omega_3^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z/30 & \Omega_3^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z/60\\
\widetilde\Omega_4^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z & \Omega_4^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z\\
\widetilde\Omega_5^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z/2\oplus\mathbb Z/2 & \Omega_5^{\mathrm{Spin}}(BA_5) &\cong 0\\
\widetilde\Omega_6^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z/2 & \Omega_6^{\mathrm{Spin}}(BA_5) &\cong \mathbb Z/2.
\end{aligned}$$

Hence the 0^{th} A_5*-equivariant phase homology groups vanish for both spinless and spin-*$1/2$ *fermions.*

Finally, class A. Since V_λ is not pin^c, because its restriction to A_4 is not (Lemma 5.7), we care about $(BA_5)^{\mathrm{Det}(V_\lambda)-1} \cong (BA_5)_+$ in the spin-1/2 case, because V_λ is orientable. Let f_0^A, resp. $f_{1/2}^A$, denote the equivariant local systems of symmetry types for the class A spinless, resp. spin-1/2 cases.

Theorem 5.114. *The low-degree spin*^{c} *bordism groups of* $(BA_5)^{3-V_\lambda}$ *and* BA_5 *are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z & \Omega_0^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb Z\\
\widetilde\Omega_1^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_1^{\mathrm{Spin}^c}(BA_5) &\cong 0\\
\widetilde\Omega_2^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z & \Omega_2^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb Z\oplus\mathbb Z/2\\
\widetilde\Omega_3^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z/30 & \Omega_3^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb Z/30\\
\widetilde\Omega_4^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb Z^2 & \Omega_4^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb Z^2,
\end{aligned}$$

and in both cases, $\Omega_5^{\mathrm{Spin}^c}$ *is torsion. Hence both* $\mathit{Ph}_0^{A_5}(\mathbb R^3; f_0^A)$ *and* $\mathit{Ph}_0^{A_5}(\mathbb R^3; f_{1/2}^A)$ *vanish.*

Proof. The calculation separates into 2-primary and odd-primary computations; by Lemma 5.109, the 2-primary pieces are exactly as in Theorem 5.8.

The calculation of the odd-primary parts follows the same line of reasoning as the proof of Lemma 5.78: as usual, use the odd-primary equivalence $\mathit{MTSpin}^c \to \mathit{MTSO}\wedge(B\mathrm U_1)_+$. We know from Proposition 5.110 that the only odd-primary torsion in $H_k(BA_5)$ for $k \le 6$ is $\mathbb Z/15 \subset H_3$; feeding that to the Künneth formula, the only odd-primary torsion in $H_k(B\mathrm U_1\times BA_5)$ is two $\mathbb Z/15$ summands in H_3 and H_5. Then the Atiyah-Hirzebruch argument is identical to the argument in Lemma 5.78. ☐

5.7. **Full icosahedral symmetry.** If one includes orientation-reversing symmetries of the icosahedron, the symmetry group enlarges to $A_5\times\mathbb Z/2$, with the $\mathbb Z/2$ generated by an inversion. This symmetry group is also incompatible with translations, so Remark 5.108 applies. This calculation also quickly reduces to something we already know: restricting the representation to $A_4\times\mathbb Z/2$ yields the pyritohedral symmetry representation we studied in §5.2.

Theorem 5.115. *Let* ρ *be a virtual* $A_5\times\mathbb Z/2$*-representation with rank zero, and let* $V_\rho\to BG$ *denote the associated virtual vector bundle. Suppose that* $w_1(V_\rho)=x$*, where* x *denotes the generator of* $H^1(B\mathbb Z/2;\mathbb Z/2)\subset H^1(B(A_5\times\mathbb Z/2);\mathbb Z/2)$*. Then inclusion of the pyritohedral symmetry subgroup* $\varphi\colon A_4\times\mathbb Z/2\hookrightarrow A_5\times\mathbb Z/2$ *induces a homotopy equivalence* $B(A_4\times\mathbb Z/2)^{V_\rho}\xrightarrow{\simeq} B(A_5\times\mathbb Z/2)^{V_\rho}$*.*

Proof. By the Whitehead theorem, it suffices to establish that φ induces an isomorphism $\widetilde H^*(B(A_5\times\mathbb Z/2)^{V_\rho};k)\to\widetilde H^*(B(A_4\times\mathbb Z/2)^{V_\rho};k)$ for $k=\mathbb Q$ and $k=\mathbb Z/p$ for all primes p.

Lemma 5.109 and the Künneth theorem imply that $\varphi^*\colon H^*(B(A_5\times\mathbb Z/2);\mathbb Z/2)\to H^*(B(A_4\times\mathbb Z/2);\mathbb Z/2)$ is an isomorphism. Together with the Thom isomorphism theorem, this takes care of the case $k=\mathbb Z/2$.

Let G be either of $A_4\times\mathbb Z/2$ or $A_5\times\mathbb Z/2$; the map $B\varphi\colon B(A_4\times\mathbb Z/2)\to B(A_5\times\mathbb Z/2)$ allows us to think of V_ρ as over BG for either G, and make sense of the statement $w_1(V_\rho)=x$. The Thom isomorphism implies $\widetilde H^*((BG)^{V_\rho};\mathbb Z)\cong H^*(BG;\mathbb Z_x)$, and since $\mathbb Z_x$ arises as a pullback local system along $BG\to B\mathbb Z/2$, the twisted Künneth formula proves $\widetilde H^*(BG;\mathbb Z)$ is 2-torsion. The universal coefficient theorem then implies that when we take coefficients in $k = \mathbb{Q}$ or $k = \mathbb{Z}/p$ for p odd, $\widetilde{H}^*(B(A_4 \times \mathbb{Z}/2)^{V_\rho}; k)$ and $H^*(B(A_5 \times \mathbb{Z}/2)^{V_\rho}; k)$ vanish, so the map between them is vacuously an isomorphism. □

Let $\lambda\colon A_5 \times \mathbb{Z}/2 \to \mathrm{O}_3$ denote the representation as the group of symmetries of an icosahedron and $V_\lambda \to B(A_5 \times \mathbb{Z}/2)$ denote the associated vector bundle. Then $w_1(V_\lambda) = x$. Let f_0^D and $f_{1/2}^D$ denote the spinless, resp. spin-1/2 class D equivariant local systems of symmetry types, and f_0^A and $f_{1/2}^A$ denote their analogues in class A.

Corollary 5.116. *φ induces homotopy equivalences*

$$B(A_4 \times \mathbb{Z}/2))^{3-V_\lambda} \xrightarrow{\cong} (B(A_5 \times \mathbb{Z}/2))^{3-V_\lambda} \tag{5.117a}$$

$$(B(A_4 \times \mathbb{Z}/2))^{\mathrm{Det}(V_\lambda)-1} \xrightarrow{\cong} (B(A_5 \times \mathbb{Z}/2))^{\mathrm{Det}(V_\lambda)-1}. \tag{5.117b}$$

Therefore

1. *Proposition 5.12 implies that* $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^D) \cong (\mathbb{Z}/2)^{\oplus 3}$*;*
2. *Theorem 5.27 implies that* $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_{1/2}^D) \cong \mathbb{Z}/2$*;*
3. *Theorem 5.29 implies that* $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}$*; and*
4. *Theorem 5.32 implies that* $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_{1/2}^A) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$*.*

6. Glide symmetry protected phases

Though we have focused on point group symmetries thus far, Freed-Hopkins' ansatz [FH19a] also applies to crystallographic groups. In this section, we apply their ansatz to the group of glide symmetries; invertible phases equivariant for this symmetry have been studied by Lu-Shi-Lu [LSL17] and Xiong-Alexandradinata [XA18], and our results agree with theirs. In particular, Lu-Si-Lu make a conjecture classifying certain glide-symmetric phases in all symmetry types, and we prove that their conjecture follows from Freed-Hopkins' ansatz.

The group of *glide symmetries* acting on $\mathbb{R}^d$, $d \geq 2$, is the free group on the single generator

$$(x_1, x_2, \dots, x_d) \longmapsto (x_1 + 1, -x_2, x_3, \dots, x_d). \tag{6.1}$$

In previous sections, when the symmetry type is $H = \mathrm{Spin}$, Spin^c, $\mathrm{Pin}^\pm$, etc., the symmetry type can mix with the group action on spacetime, corresponding physically to spinless or spin-1/2 fermions. Here, this cannot happen: if μ_2 denotes the kernel of the map $\mathrm{Spin}_n \to \mathrm{SO}_n$ or $\mathrm{Pin}_n^\pm \to \mathrm{O}_n$, all extensions

$$0 \longrightarrow \mu_2 \longrightarrow \widetilde{G} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{6.2}$$

split, so given one of these symmetry types, there is a unique equivariant symmetry type for this $\mathbb{Z}$-action with respect to mixing with fermion parity, corresponding to the trivial local system $\underline{E} \to \mathbb{R}^d$ with value $E \coloneqq \mathrm{Map}(MTH, \Sigma^2 I_\mathbb{Z})$.

Definition 6.3. Recall from Remark 1.28 that we defined a "forgetful map" $\varphi\colon Ph_*^\mathbb{Z}(\mathbb{R}^d; \underline{E}) \to Ph_*(\mathbb{R}^d; \underline{E})$. The *intrinsically $\mathbb{Z}$-equivariant phase homology*, denoted $\widehat{Ph}_*^\mathbb{Z}(\mathbb{R}^d; \underline{E})$, is the kernel of this map.

This corresponds under Freed-Hopkins' ansatz to what Lu-Shi-Lu call a *glide SPT*: an invertible phase equivariant for a $\mathbb{Z}$ glide symmetry which is trivializable when one forgets the symmetry.

Let $TP_d(H)$ denote the abelian group of SPT phases in (spatial) dimension d; Freed-Hopkins' ansatz [FH16a] classifying these phases in terms of invertible field theories predicts $TP_d(H) \cong E_{-d}$.

Lu-Shi-Lu [LSL17] studied groups of glide SPTs and conjectured a formula classifying them in terms of the classification of ordinary SPTs. We prove the corresponding statement on phase homology groups.

Theorem 6.4. *For a given symmetry type $\rho_n\colon H_n \to \mathrm{O}_n$, there is a natural isomorphism $\widehat{Ph}_0^\mathbb{Z}(\mathbb{R}^d; \underline{E}) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$.*

Passing this through the ansatz, this predicts that the group of glide SPTs is naturally isomorphic to $TP_{d-1}(H) \otimes \mathbb{Z}/2$, which is Lu-Shi-Lu's original conjecture [LSL17, Conjecture 1]. Xiong-Alexandradinata [XA18] also obtain this result using physics-based arguments.

Proof of Theorem 6.4. We calculate the 0^{th} $\mathbb{Z}$-equivariant Borel-Moore E-homology of $\mathbb{R}^d$. As the $\mathbb{Z}$-action is free, this is the 0^{th} (nonequivariant) Borel-Moore E-homology of the fundamental domain $X := \mathbb{R}^d/\mathbb{Z}$. Since the one-point compactification $\overline{X}$ of X is a finite CW complex, this Borel-Moore homology is isomorphic to $\widetilde{E}_0(\overline{X})$.

If $\sigma \to S^1$ denotes the Möbius bundle, then X is diffeomorphic to the total space of $\sigma \oplus \underline{\mathbb{R}}^{d-2} \to S^1$, so $\overline{X}$ is the Thom space $(S^1)^{\sigma+d-2}$. The identification $(S^1)^\sigma \cong \mathbb{RP}^2$ induces $\overline{X} \cong \Sigma^{d-2}\mathbb{RP}^2$, and therefore

$$Ph_*^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \cong \widetilde{E}_0(\Sigma^{d-2}\mathbb{RP}^2) \cong \widetilde{E}_{2-d}(\mathbb{RP}^2). \tag{6.5}$$

Lemma 6.6. *Let $p\colon S^2 \to \mathbb{RP}^2$ be the double cover map and $s\colon \widetilde{E}_k(S^1) \to \widetilde{E}_{k+1}(S^2)$ be the suspension isomorphism. The composition $p_* \circ \delta \circ s\colon \widetilde{E}_{-1}(S^2) \to \widetilde{E}_{-1}(S^2)$ is multiplication by 2.*

Proof. This follows because the suspension is the cofiber of the cofiber; then one explicitly checks what happens on mapping cylinders. ☐

Lemma 6.7. *Under these isomorphisms, the forgetful map $Ph_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \to Ph_0(\mathbb{R}^d; \underline{E})$ is identified with δ.*

Proof. Because $\mathbb{Z}$ acts freely on $\mathbb{R}^d$, $E_{0,\mathrm{BM}}^{\mathbb{Z}}(\mathbb{R}^d)$ is identified with $\widetilde{E}_0$ of the one-point compactification of $\mathbb{R}^d/\mathbb{Z}$, which we saw above is homeomorphic to $\Sigma^{d-2}\mathbb{RP}^2$. The codomain of the forgetful map is $E_{0,\mathrm{BM}}(\mathbb{R}^d) \cong \widetilde{E}_0(\Sigma^{d-2}S^2)$, so we have identified δ with a map $\widetilde{E}_0(\Sigma^{d-2}\mathbb{RP}^2) \to \widetilde{E}_0(\Sigma^{d-2}S^2)$. But tracing through the construction in Remark 1.28, this map comes from applying $\widetilde{E}_0$ to an actual map $\Sigma^{d-2}\mathbb{RP}^2 \to \Sigma^{d-2}S^2$.

Next, precompose with $\Sigma^{d+2}p\colon \Sigma^{d-2}S^2 \to \Sigma^{d-2}\mathbb{RP}^2$ and check that this map has degree 2, agreeing with Lemma 6.6. This suffices to identify the maps because $p^*\colon [\mathbb{RP}^2, S^2] \to [S^2, S^2]$ is injective. ☐

$\mathbb{RP}^2$ is homeomorphic to the cofiber of a degree-2 map $S^1 \to S^1$. Hence there is a long exact sequence in reduced E-homology

$$\cdots \longrightarrow \widetilde{E}_{2-d}(S^1) \xrightarrow{m} \widetilde{E}_{2-d}(S^1) \xrightarrow{r} \widetilde{E}_{2-d}(\mathbb{RP}^2) \xrightarrow{\delta} \widetilde{E}_{1-d}(S^1) \longrightarrow \cdots \tag{6.8}$$

where m is multiplication by 2. Exactness implies $\ker(\delta) = \mathrm{Im}(r) = \mathrm{coker}(m)$. Using the suspension isomorphism, $\widetilde{E}_k(S^1) \cong \widetilde{E}_{k-1}$, and therefore $\mathrm{coker}(m) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$, and 6.7 identifies δ with the forgetful map from equivariant to nonequivariant phase homology for $\mathbb{R}^d$. In particular, $\widehat{Ph}_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \cong \ker(\delta)$, which we have naturally identified with $E_{-(d-1)} \otimes \mathbb{Z}/2$. ☐

Remark 6.9. Using the long exact sequence (6.8), we observe that $Ph_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E})$ has exponent 4. This is because for any long exact sequence of abelian groups

$$\cdots \longrightarrow A \xrightarrow{\cdot 2} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\cdot 2} C \longrightarrow \dots \tag{6.10}$$

in which A and C are finitely generated, $\mathrm{Im}(f) \cong A/2$, hence has exponent 2, and $\ker(g)$ is isomorphic to the subgroup of order-2 elements of C, which also has exponent 2. Since B is an extension of $\ker(g)$ by $\mathrm{Im}(f)$, B has exponent 4.

Passing this observation through Freed-Hopkins' ansatz, this recovers an observation of Xiong-Alexandradinata [XA18]: that *any* phase equivariant with respect to glide symmetry, whether a glide SPT or not, has order dividing 4.

Example 6.11. In Altland-Zirnbauer class AII, corresponding to the symmetry type $\mathrm{pin}^{\tilde{c}+}$, the ansatz predicts a unique nontrivial glide SPT in dimension $2+1$, coming from the classification

$$[MTPin^{\tilde{c}+}, \Sigma^4 I_{\mathbb{Z}}] \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \tag{6.12}$$

(the calculation of $[MTPin^{\tilde{c}+}, \Sigma^4 I_{\mathbb{Z}}]$ is due to Freed-Hopkins [FH16a, §9.3]). Physicists are particularly interested in this nontrivial glide SPT phase, which is predicted to have unusual surface states called "hourglass fermions" [WACB16], and which has been studied experimentally [MYL+17].

7. Conclusion and outlook

We conclude by indicating a few directions of potential further research.

7.1. **From free fermions to interacting phases.** Free fermion phases are a rich source of examples of invertible phases in the physics literature, at least for symmetry types spin, $\mathrm{pin}^\pm$, spin^c, etc. The classification of free fermion systems uses K-theory: see Kitaev [Kit09] for the original proposal and Freed-Moore [FM13] for a comprehensive classification. However, for a given dimension and symmetry type, the map from free fermion systems to invertible phases of matter can in general have both kernel (as first observed by Fidkowski-Kitaev [FK10, FK11] and Turner-Pollmann-Berg [TPB11]) and cokernel (as first observed by Wang-Potter-Senthil [WPS14] and Wang-Senthil [WS14]). Researchers are also interested in the free-to-interacting map for phases with spatial symmetries, and this map has been studied from a physics point of view for crystalline phases in several works, including [YR13, IF15, MFM15, LTH16, SS17, RL18, Zou18, LVK19, RS20, ZYQG20, ACR+21].

Freed-Hopkins [FH16a, §9.2, §9.3] mathematically model the map from free to interacting systems using the Atiyah-Bott-Shapiro map $\mathit{MTSpin} \to \mathit{KO}$ [ABS64], but they do not consider spatial symmetries. In view of the large bodies of research on free fermions with spatial symmetries and invertible phases with spatial symmetries, it would be nice to understand the map between them in the presence of spatial symmetry from the low-energy field theory perspective, and to make contact with the work of Adem, Antolín Camarena, Semenoff, and Sheinbaum [AACSS16], Sheinbaum and Antolín Camarena [SC20], and Cornfeld-Carmeli [CC21] studying free fermion phases with spatial symmetries using methods from homotopy theory. This is something we hope to tackle in future work.

7.2. **Other symmetry types.** We investigated two of the ten Altland-Zirnbauer classes, and it would be interesting to know whether a version of the FCEP holds for other classes. One starting point could be class C, corresponding to a spin^h structure [FH16a, (9.25)];[34] the calculations in §2.8 could be applied to Spin_n^h to obtain a fermionic crystalline equivalence principle for class C and hopefully phase homology calculations predicting the existence of additional crystalline SPT phases.

Several teams of researchers have studied or classified interacting fermionic crystalline SPTs for other Altland-Zirnbauer types, including [YR13, YX14, CHMR15, LTH16, WF17, CW18, RL18, SXG18, MSH19, ZXXS20, ZYQG20]. It would be good to compare their computations with the predictions made by an FCEP in other symmetry types.

Another interesting potential connection with preexisting work is the case of class A phases with a spatial reflection interacting with the internal U_1 symmetry. Depending on how the symmetries mix, Shiozaki-Shapourian-Gomi-Ryu [SSGR18, §V.C, §V.E] and Thorngren-Else [TE18, §VII.B] obtain classifications in terms of $\mathrm{pin}^{\tilde c\pm}$ bordism, and we would be interested in knowing whether that can also be obtained from our ansatz. Similarly, can one begin with class C phases and a reflection acting on the internal SU_2 symmetry and obtain a classification in terms of $\mathrm{pin}^{h\pm}$ bordism?

7.3. **Crystallographic groups.** Though we discussed glide symmetries in §6, we have barely touched upon the rich world of crystallographic groups. Free-fermion phases equivariant for these groups have been studied, e.g. in [SMJZ13, KdBvW+17, SSG18, OSS19], but much less is known about the interacting case, even though the our ansatz applies to it. There are some classifications by other methods for various classes of crystallographic groups; for example, Ouyang-Wang-Gu-Qi [OWGQ20] study wallpaper group symmetries, and Sheinbaum-Antolín Camarena [SC20] provide a general framework and a few examples. There is also work by Wang-Alexandradinata-Cava-Bernevig [WACB16] and Guo-Ohmori-Putrov-Wan-Wang [GOP+20] studying interacting phases for specific crystallographic groups that are not point groups.

7.4. **Lattice realizations.** Modeling topological phases as lattice Hamiltonian systems can make any crystallographic symmetries acting on space very explicit, using a lattice and Hamiltonian invariant under the symmetry of interest. Our predictions of point group SPTs should correspond to actual lattice models of phases. We listed several specific predicted phases of interest in §3.1, and these would make for good starting points for lattice realizations.

[34] Spin^h is the symmetry type $\mathrm{Spin} \times_{\mu_2} \mathrm{SU}_2 \to \mathrm{O}$. Freed-Hopkins [FH16a, Proposition 9.16] call this symmetry type G^0; it is sometimes also called spin-SU_2, e.g. in [WWW19]. Likewise, the symmetry types $\mathrm{pin}^{h\pm}$ we refer to later in this section are defined to be $\mathrm{Pin}^\pm \times_{\mu_2} \mathrm{SU}_2$, and are called $G^\pm$ by Freed-Hopkins [FH16a, Proposition 9.16].

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