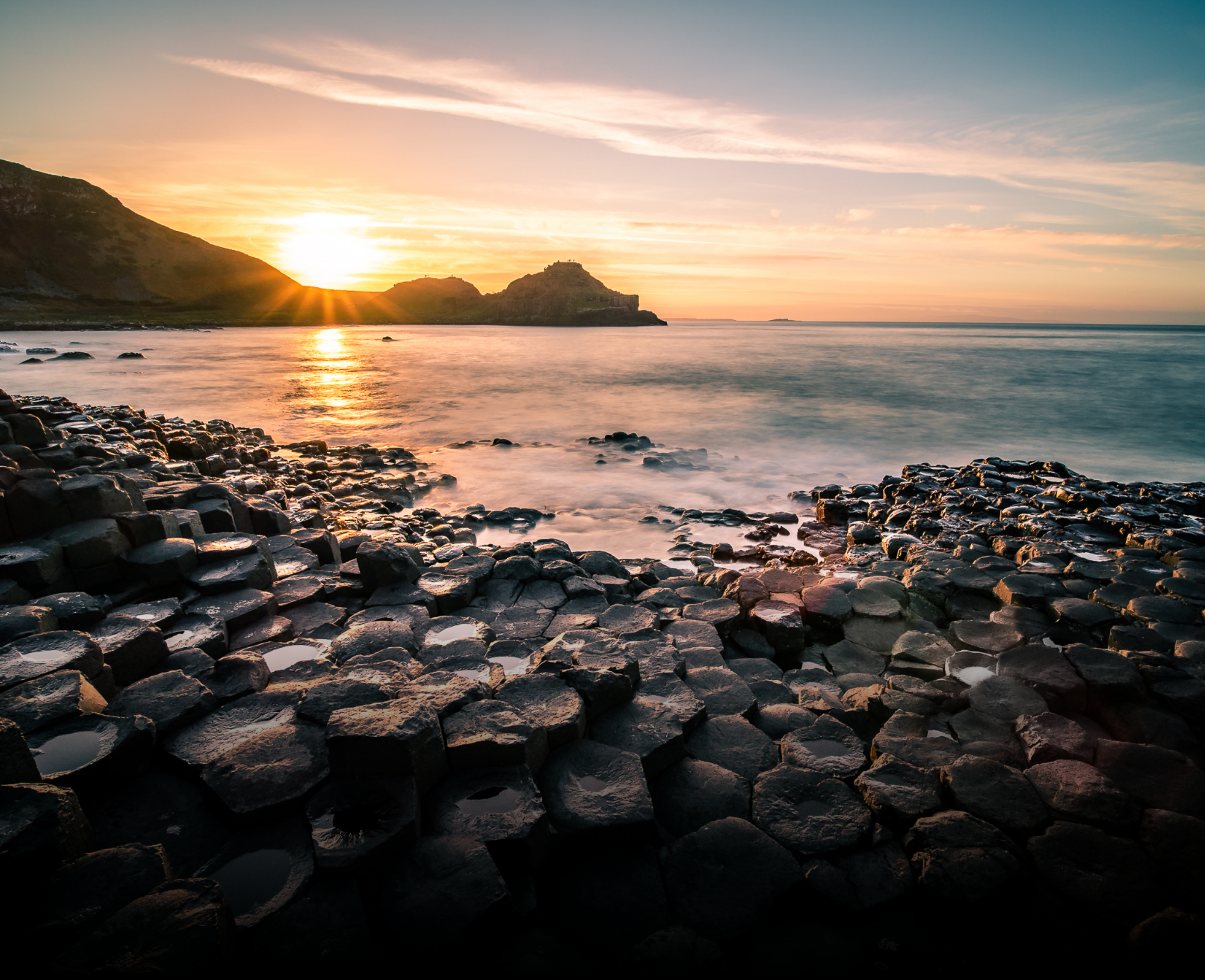

Differential Cohomology

Categories, Characteristic Classes, and Connections

Edited by Araminta Amabel, Arun Debray, and Peter Haine

Differential Cohomology

Categories, Characteristic Classes, and Connections

Edited by Araminta Amabel, Arun Debray, and Peter Haine

January 18, 2023

Abstract

We give an overview of differential cohomology from a modern, homotopy-theoretic perspective in terms of sheaves on manifolds. Although modern techniques are used, we base our discussion in the classical precursors to this modern approach, such as Chern–Weil theory and differential characters, and include the necessary background to increase accessibility. Special treatment is given to differential characteristic classes, including a differential lift of the first Pontryagin class. Multiple applications, including to configuration spaces, invertible field theories, and conformal immersions, are also discussed. This book is based on talks given at MIT's Juvitop seminar run jointly with UT Austin in the Fall of 2019.

Contents

1 Preface

Differential cohomology begins with the observation that many naturally occurring differential forms have integrality properties. One example is the curvature Ω of a connection on a complex vector bundle over a closed manifold M; if $N \subset M$ is a closed, oriented, two-dimensional submanifold, then $\int_N \Omega$ is an integer multiple of 2π. Analogous statements are true, though with different normalization constants, for other Chern–Weil forms of a vector bundle with connection. The first explanation given is typically that the cohomology classes represented by these forms are in the image of the map $\mathrm{H}^*(-;\mathbb{Z}) \to \mathrm{H}^*(-;\mathbb{R})$, but in a way this fails to capture the entire picture: that the de Rham class of the Chern–Weil form has a canonical lift to $\mathrm{H}^*(-;\mathbb{Z})$. For example, $(1/2\pi)\Omega$ lifts to the first Chern class of a complex vector bundle. Differential cohomology is built to house this kind of data: a closed differential form, an integer-valued cohomology class, and an identification of their images in de Rham cohomology.

A similar situation can happen in quantum physics: abelian gauge fields give rise to differential forms such as field strengths and currents, and quantization imposes strong integrality properties on these objects. For example, in the classical theory of electromagnetism, the electric field E is a 1-form, and the magnetic field B is a 2-form. Maxwell's equations on a closed 4-manifold M imply that the field strength $F = B - \mathrm{d}t \wedge E$ is a closed 2-form. But in the quantum theory, the possible values of electric and magnetic fluxes and charges are discretized; there is a minimum magnetic charge q_B, and the integral of F on a closed, oriented surface must be an integer multiple of $2\pi q_B$. Again we have closed forms with integrality conditions, and so the field strength B refines to a cocycle representative of a differential cohomology class $\hat{B} \in \hat{\mathrm{H}}^2(M; q_B\mathbb{Z})$.

Another perspective on differential cohomology is that it does for geometric objects what ordinary cohomology does for their topological analogues. Vector bundles and principal bundles have characteristic classes in cohomology; vector bundles with connection and principal bundles with connection have characteristic classes in differential cohomology. Analogously, topological K-theory is built out of vector bundles, and differential K-theory is built out of vector bundles with connection.

The goal of this book is to provide an introduction to differential cohomology, including both foundational aspects of generalized differential cohomology theories and applications. We follow Bunke–Nikolaus–Völkl, defining differential (generalized) cohomology theories as sheaves of spectra on the site of smooth manifolds. We go over the basics of the theory, including defining the cup product and integration maps. We spend time with characteristic classes: as hinted above, Chern–Weil forms refine to characteristic classes in differential cohomology, but there are additional classes which have no topological counterparts. We also go over several applications of differential cohomology. Often, these are geometric analogues of a well-known application of cohomology to topological questions. For example, characteristic classes obstruct smooth embeddings of manifolds into $\mathbb{R}^n$, and differential characteristic classes can obstruct conformal embeddings into $\mathbb{R}^n$. Some of these applications are angled towards physics; for example, we revisit the idea above that differential cohomology has something to say about quantization.

This book began as lectures given in a graduate student seminar joint between MIT and UT Austin in fall 2019, initiated by Dan Freed and Mike Hopkins. Most chapters are notes from talks given by various speakers at the seminar and a few chapters were written afterwards.

1.1 Assumed background

We hope that these notes are accessible to readers with a wide range of background knowledge. The talks included here were part of a topology seminar, and are therefore biased toward the homotopy theoretic perspective. This is evidenced by the fact that we review the definition of a connection and not that of an ∞-category. However, knowledge of ∞-categories is not a prerequisite for making use of these notes. Comfort with sheaves, spectra, and simplicial sets will make reading easier. The reader will also benefit, both in motivation and understanding, from a familiarity with basic differential geometry; this includes connections, curvature, and de Rham cohomology. Part III of these notes includes talks on several different applications of differential cohomology. Enjoyment of these sections should not require any background other than interest in the section title.

1.2 Linear overview

We give a brief overview of the three parts of these notes. A more detailed introduction is given at the beginning of each part.

1.2.a Part I: Basics of the theory

The purpose of this part is to introduce the basics of and develop the general theory behind differential cohomology. In Chapter 2, we start with some motivation to the approach we take to differential cohomology coming from work of Cheeger–Simons [CS85] and Simons–Sullivan [SS08] on differential characters and ordinary differential cohomology. The perspective we take on differential cohomology theories is as sheaves of spectra on the category Mfld of manifolds; since we also want to consider sheaves that come from chain complexes, we'll work in the framework of sheaves with values in a general ∞-category. While this might sound somewhat daunting, there are many familiar examples:

(1) The functor sending a manifold M to the complex $\Omega^\bullet(M)$ of de Rham cochains on M.

(2) The functor sending a manifold M to the complex $\mathrm{C}^\bullet_{\mathrm{sing}}(M)$ of singular cochains on M.

(3) Given a Lie group G, the functor sending a manifold M to the groupoid $\mathrm{Bun}_G(M)$ (or $\mathrm{Bun}^\nabla_G(M)$) of principal G-bundles on M (with connection).

The new example of differential cohomology is essentially built from these ones in a nontrivial way.

In Chapter 3, we introduce the basics of sheaves on the category of manifolds, how to manipulate sheaves on Mfld, and any the category of sheaves (of sets) on Mfld contains the standard category of infinite-dimensional manifolds (*Fréchet manifolds*) as a full subcategory. One important class of sheaves on Mfld are those that invert all homotopy equivalences of manifolds. Chapter 4 is dedicated to explaining why all sheaves with this property have a very simple and concrete description. In Chapter 5, we explain how to resolve a sheaf by one that inverts all homotopy equivalences of manifolds. This provides a way of decomposing a sheaf of spectra on Mfld into one that inverts all homotopy equivalences and another that "comes from geometry". Chapter 6 explains this decomposition as well as how this gives rise to the Simons–Sullivan "differential cohomology hexagon" [SS08, §1]) relating ordinary cohomology, differential forms, and differential cohomology.

The remainder of this part is dedicated to important examples of differential cohomology theories and refining important constructions with ordinary cohomology. Chapter 7 explains Cheeger–Simons differential characters, differential K-theory, and examples coming from G-bundles in the framework of sheaves on Mfld. Chapter 8 refines the cup product to differential cohomology and explains how to calculate it in many examples. Chapter 9 refines fiber integration to differential cohomology. Chapter 10 finishes the main text of this part with a digression proving Quillen's Transfer Conjecture. Though not directly related to differential cohomology, this result states that connective spectra can be realized as homotopy-invariant sheaves on the category of correspondences of manifolds where the backwards maps are finite covering maps (i.e., connective spectra have natural *transfers* along finite covering maps). Our exposition follows work of Bachmann–Hoyois [BH21, Appendix C].

Part I also has an appendix (Appendix A). In this appendix, we prove a few technical category theory results that we need to get the foundations of sheaves on Mfld on a solid framework in Chapters 3 and 4.

1.2.b Part II: Characteristic classes

Just as one ordinary cohomology is a natural home for characteristic classes, differential cohomology offers its own invariants of bundles. These invariants, known as "differential characteristic classes," are refinements of the classical characteristic classes in cohomology. More explicitly, we will investigate lifts of well-known characteristic classes, such as Chern classes, under the map from differential cohomology to ordinary cohomology.

This part begins be reviewing a few classical techniques and results that will be useful in studying differential characteristic classes, see Chapter 11 and Chapter 12.

Differential characteristic classes where first studied by Cheeger–Simons [CS85]. We discuss differential characters in Chapter 14. Building on work of Bott [Bot73], Freed and Hopkins [FH13] classified all differential characteristic classes for bundles equipped with a flat connection. This refines the classical Chern–Weil story, which we review in Chapter 11. The contents of [FH13] are covered in Chapter 15. A closer look at the methods used in [Bot73] reveal that one can remove the connection data with some alterations. In Chapter 16, we delve into Bott's paper and the theorems it relies upon. In particular, we discuss van Est's theorem relating continuous cohomology to Lie algebra cohomology. Using the results of [Bot73], Hopkins, in Chapter 17, discusses how to lift ordinary Chern classes to a form of differential cohomology, without the presence of a connection. The existence of a differential version of the Cartan formula is also considered.

This part of the notes concludes with an interesting application of differential lifts of Chern classes to a possible construction of the Virasoro group. The Virasoro group is a certain central extension of orientation preserving diffeomorphisms $\mathrm{Diff}^+(S^1)$ of S^1. As Hopkins outlines in Chapter 17, one can obtain central extensions of $\mathrm{Diff}^+(S^1)$ from a certain differential cohomology group. The details of this construction, as well as a review of the Virasoro algebra and group, appear in Chapter 18.

1.2.c Part III: Applications

In this part we discuss some uses of differential cohomology in topology, geometry, and physics. Some, but not all, of these applications are part of the idea that what ordinary cohomology can do for topological questions, differential cohomology can do for geometric ones, and many of these applications are related to various aspects of quantum field theory.

One of the key links between differential cohomology and geometry is through Chern–Simons invariants, invariants of connections which can be defined either in terms of integration of differential characteristic classes or directly using geometric information. Because of this, several applications of differential cohomology to geometry or physics pass through Chern–Simons theory. We introduce and apply Chern–Simons invariants in Chapter 19 and also use them in Chapter 20.

Our first two applications of differential cohomology are in geometry and topology. In Chapter 19, we discuss work of Evans-Lee–Saveliev [ES16], who use Chern–Simons invariants to study the homotopy types of two-point configuration spaces of lens spaces. Then in Chapter 20, we use differential Pontryagin classes and Chern–Simons forms to obstruct conformal immersions of conformal manifolds into Euclidean space, following Chern–Simons [CS74]; along the way we spend some time getting to know the geometry of Chern–Weil and Chern–Simons forms.

The next two applications are to physics. Chapter 21 applies differential cohomology to the quantization of abelian gauge fields, using electromagnetism as an example. In classical physics, the field strength of an abelian gauge field is a closed differential form; quantization lifts from closed forms to cocycles for a differential cohomology group. The other physics application we discuss, in Chapter 22, is quite different: a conjecture of Freed–Hopkins [FH21b] using differential generalized cohomology to classify invertible, non-topological field theories. This is a geometric conjecture modeled on a topological theorem of Freed–Hopkins (*ibid.*) classifying invertible topological field theories using Madsen–Tillmann spectra. We discuss this conjecture and several examples, including classical Chern–Simons theory.

Our final two chapters are about the representation theory of loop groups. Loop groups are infinite-dimensional Lie groups whose representation theory is strikingly similar to that of compact Lie groups, so long as one works with what are called positive energy representations. In Chapter 23, we survey this theory, defining and motivating positive energy representations and sketching a proof of a theorem of Pressley–Segal [PS86], which says that positive energy representations admit projective intertwining actions of $\mathrm{Diff}^+(S^1)$. In Chapter 24, we study the Pressley–Segal theorem at the Lie algebra level, where this intertwining projective action can be made more explicit. Since projective representations are equivalent to representations of a central extension, the Virasoro algebra makes an appearance here.

1.3 What's not included

One original approach to differential cohomology is presented by Hopkins and Singer in [HS05]. While we look to this reference for motivation and intuition, we do not take this as our definition of a differential cohomology theory. Instead, we work with the more modern approach using sheaves on manifolds. We also make use of [HS05] for constructions of the cup product and fiber integration in differential cohomology, see Chapters 8 and 9.

Several examples of differential cohomology theories, such as differential K-theory, are discussed in Chapter 7; however, there are many more examples that we do not mention. Moreover, for most of these notes, we focus our attention on the specific example of the differential version of ordinary cohomology. This leaves several interesting areas of study, such as differential

K-theory characteristic classes, untouched.

We do not present Schreiber's elegant and very general theory of *differential cohomology in a cohesive* ∞*-topos* [Sch13b]. Schreiber's work requires background that we do not assume; we decided to stick with the setting of sheaves on the category of manifolds to make the material accessible to the graduate students attending the seminar.

There are also many applications of differential cohomology to physics which we do not discuss in detail here. See Part III for a discussion of related work.

1.4 Cover image

One of the theses of this book is that differential cohomology has applications to physics. It therefore seems apt to choose a cover image of another example of hexagons in the real world. Our cover image is a picture of Giant's Causeway, a part of the coastline in Northern Ireland consisting of tens of thousands of tessellating hexagonal basalt columns. This image is by Giuseppe Milo and can be found at `flickr.com/photos/giuseppemilo/46587488041/in/photostream/`; we cropped it slightly. It is licensed under the CC BY 2.0 license.

1.5 Acknowledgements

These notes are a compilation of talks given in the Juvitop seminar at MIT in the fall of 2019. The seminar was run jointly with UT Austin and we thank the participants of both cities for their comments and discussion. We would also like to thank the speakers, Dexter Chua, Sanath Devalapurkar, Dan Freed, Mike Hopkins, Greg Parker, Charlie Reid, and Adela Zhang, both for volunteering to speak, as well as writing up notes to appear here.

The seminar, and these notes, could not have existed without the help of Dan Freed and Mike Hopkins. We thank both Freed and Hopkins for their mathematical help, their organizational help, and for giving talks in the seminar. We also would like to express our appreciation for Freed and Hopkins generosity in sharing new mathematical ideas and older insights along the way.

Extra thanks are due to Hopkins for buying us fancy video equipment to help with our half-virtual seminar.

We thank Kiran Luecke and Zhouhang Mao for pointing out a number of typos and errors in the first draft of this text.

PH gratefully acknowledges support from the MIT Dean of Science Fellowship, UC President's Postdoctoral Fellowship, and NSF Mathematical Sciences Postdoctoral Research Fellowship under Grant #DMS-2102957. PH and AA acknowledge support from the National Science Foundation Graduate Research Fellowship under Grant #112237. This text is partially based upon work supported by the National Science Foundation under Grant #DMS-1440140, while AD and PH were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester.

Part I

Basics of the theory

The goal of this first part of the text is to introduce and study *differential cohomology theories*. The term "differential cohomology" was first coined by Hopkins and Singer in [HS05]. In Chapter 2, we introduce the ideas of differential cohomology theories following Cheeger–Simons [CS85] and Simons–Sullivan [SS08]. The basic point is that given a manifold M, we can consider both the "homotopy-theoretic" complex of singular cochains on M, and the "geometric" complex of differential forms on M. These are related by the de Rham isomorphism, and we would like to combine them together into a "cohomology theory" that captures both the features of M as a homotopy type as well as the geometry of M. The thing to notice is that both the complex of singular cochains and differential forms are *sheaves* (in the homotopy-theoretic sense) on the category of all manifolds. So this category of sheaves on manifolds is the setting in which both these homotopy-theoretic and geometric objects live.

Thus the perspective that we take in this text is that differential cohomology theories are *sheaves of spectra on the category* Mfld *of manifolds*. It will also be useful to consider sheaves of spaces on Mfld or sheaves with values in the derived ∞-category of a ring; Chapter 3 starts with introducing sheaves on the category of manifolds with values in any ∞-category. While the phrase "sheaf on Mfld" may sound somewhat daunting, it is surprisingly concrete: a sheaf F on Mfld consists of a functor $\mathsf{Mfld} \to C$ such that for each manifold M, the restriction of F to open subsets of M defines a sheaf on M.

Let C be a presentable ∞-category (e.g, spaces, spectra, or the derived ∞-category of a ring). One of the basic features of the category $\mathsf{Sh}(\mathsf{Mfld}; C)$ of C-valued sheaves on Mfld is that the full subcategory $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ spanned by those sheaves that invert homotopy equivalences is already familiar:

I.1 Theorem (Proposition 4.3.1)**.** *Evaluation on the point defines an equivalence*

$$\begin{aligned} \Gamma_* \colon \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) &\xrightarrow{\sim} C \\ F &\mapsto F(*) \,. \end{aligned}$$

Moreover, the inverse equivalence is given by the constant sheaf functor $\Gamma^* \colon C \to \mathsf{Sh}(\mathsf{Mfld}; C)$. *That is,* $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ *coincides with the full subcategory of* $\mathsf{Sh}(\mathsf{Mfld}; C)$ *spanned by the constant sheaves.*

We call objects of $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ $\mathbb{R}$-*invariant* sheaves. Chapter 4 is dedicated to proving Theorem I.1. In Chapter 4 we also give an explicit formula for the constant sheaf functor $C \to \mathsf{Sh}(\mathsf{Mfld}; C)$:

I.2 Proposition (Proposition 4.3.12)**.** *The constant sheaf functor*

$$\Gamma^* \colon C \xrightarrow{\sim} \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \subset \mathsf{Sh}(\mathsf{Mfld}; C)$$

is given by the assignment

$$X \mapsto \left[M \mapsto X^{\Pi_\infty(M)}\right].$$

Here, $X^{\Pi_\infty(M)}$ denotes the cotensor *of the object $X \in C$ by the underlying homotopy type $\Pi_\infty(M)$ of the manifold M (see Recollection 4.3.4).*

The cotensor in Proposition I.2 might look a bit mystifying, but it is actually a familiar object in the specific values of C that we're most interested in:

(1) Let $C = \mathrm{Spc}$ be the ∞-category of spaces. In this case, the constant sheaf functor is given by

$$X \mapsto \left[M \mapsto \mathrm{Map}_{\mathrm{Spc}}(\Pi_\infty(M), X)\right].$$

(2) Let $C = \mathrm{Spt}$ be the ∞-category of spectra. In this case, the constant sheaf functor is given by

$$E \mapsto \left[M \mapsto \mathrm{Hom}_{\mathrm{Spt}}(\Sigma^\infty_+ \Pi_\infty(M), E)\right],$$

where $\mathrm{Hom}_{\mathrm{Spt}}$ is the mapping *spectrum*.

(3) Let R be a ring and let $C = \mathrm{D}(R)$ be the *derived ∞-category* of R obtained from the category of chain complexes of R-modules by formally inverting the quasi-isomorphisms [HA, Definition 1.3.5.8, Proposition 1.3.5.15, & Remark 7.1.1.16]. In this case, the constant sheaf functor is given by

$$A_* \mapsto \left[M \mapsto \mathrm{RHom}_R(\mathrm{C}_*(M; R), A_*)\right].$$

Here $\mathrm{C}_*(M; R)$ is the complex of singular chains on M, and RHom_R is the derived Hom functor of chain complexes of R-modules.

As a consequence of Proposition I.2 (and some simple observations), we show that there is a chain of four adjoints

$$\mathrm{Sh}(\mathrm{Mfld}; C) \underset{\Gamma^\sharp}{\overset{\Gamma_\sharp}{\substack{\longrightarrow \\ \xleftarrow{\Gamma^*} \\ \xrightarrow{\Gamma_*} \\ \longleftarrow}}} C. \tag{I.3}$$

Here functors lie above their right adjoints. The extreme right adjoint $\Gamma^\sharp$ has an explicit formula (see Lemma 4.1.3), but is not particularly useful. On the other hand, under the identification

$$\Gamma^* : C \simeq \mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C)$$

the extreme left adjoint $\Gamma_\sharp$ corresponds to the left adjoint to the inclusion

$$\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C) \hookrightarrow \mathrm{Sh}(\mathrm{Mfld}; C)$$

We initially construct the left adjoint $\Gamma_\sharp$ abstractly via the Adjoint Functor Theorem, but since it plays a very important role throughout this text, it is useful to have an explicit formula for

$\Gamma_\sharp$. Chapter 5 is dedicated to showing that $\Gamma_\sharp(F)$ is computed by a simple geometric realization. Write Δ^n_{alg} for the hyperplane

$$\Delta^n_{\mathrm{alg}} \coloneqq \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1 \} \subset \mathbb{R}^{n+1} .$$

I.4 Theorem (Corollary 5.1.4)**.** *The left adjoint* $\Gamma_\sharp \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$ *is given by the formula*

$$\Gamma_\sharp(F) \simeq |F(\Delta^\bullet_{\mathrm{alg}})| .$$

Chapter 5 also explores some important consequences of Theorem I.4. For example, we give differential refinements of classifying spaces for G-bundles (see §5.1.b).

Some of the proofs in Chapters 3 to 5 rely on technical results about ∞-topoi or presentable ∞-categories. To avoid distracting the reader from the main point of the text, we have relegated many of these details to Appendix A.

Chapter 6 specializes to sheaves with values in a presentable stable ∞-category like spectra or the derived ∞-category of a ring. Using the many adjoint functors (I.3) constructed in Chapter 4, we prove the existence of a *fracture square* that shows that every sheaf on Mfld can be glued together from an $\mathbb{R}$-invariant sheaf and a sheaf with vanishing global sections (§6.2). Using this fracture square, we provide a version of the Simons–Sullivan differential cohomology diagram (Theorem 2.3.2) for any differential cohomology theory (§ 6.2.a). We also begin the study of differential refinements of spectra (§6.2.b).

With the basic foundations out of the way, Chapter 7 is dedicated to examples of differential cohomology theories. These include *ordinary differential cohomology* after Cheeger–Simons and Delgine (§7.3), and *differential* K*-theory* after Hopkins–Singer (§7.4).

In Chapter 8 we further analyze ordinary differential cohomology by giving it a product structure called the *Deligne cup product*.

I.5 Definition. Let $k \geq 0$ be an integer. The *Deligne complex* $\mathbb{Z}(k)$ is the pullback

$$\begin{tikzcd} \mathbb{Z}(k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\ \Sigma^k \Omega^k_{\mathrm{cl}} \arrow[r] & \mathbb{R} \end{tikzcd}$$

in the ∞-category $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{D}(\mathbb{Z}))$ of sheaves on Mfld with values in the derived ∞-category of $\mathbb{Z}$.

The Deligne complex $\mathbb{Z}(k)$ used to define ordinary differential cohomology. The Deligne cup product

$$\mathbb{Z}(m) \otimes_{\mathbb{Z}} \mathbb{Z}(n) \to \mathbb{Z}(m+n)$$

is defined by combining the cup product on integral cohomology with the wedge product on differential forms. We conclude Chapter 8 with an analysis of the Deligne cup product in detail in the lowest dimensions (§8.3).

Chapter 9 refines fiber integration to differential cohomology. After reviewing fiber integration for ordinary cohomology, we introduce differential versions of Thom classes and orientations (§ 9.1). We then use these notions to define *differential fiber integration* and explain how this works for S^1-bundles.

Chapter 10 is a digression explaining Bachmann and Hoyois' proof of Quillen's *Transfer Conjecture* [BH21, Appendix C]. This identifies the ∞-category of $\mathbb{E}_\infty$-spaces with $\mathbb{R}$-invariant sheaves on a 2-category of manifolds with morphisms *correspondences*

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ M_0 & & M_1 \end{tikzcd},$$

where the "backwards" maps are finite covering maps. Restricting to grouplike objects on both sides gives a description of the ∞-category of connective spectra in terms of sheaves on this 2-category of manifolds and correspondences. Chapter 10 is not used later in the text; we have included it because of its connection to Chapters 3 to 5, but the uninterested reader can safely skip it.

2 Introduction

by Peter Haine

The purpose of this chapter is to give some motivation for the perspective we take on differential cohomology. We do this by giving an overview of the work of Cheeger–Simons [CS85], Deligne [Del71, §2.2; Voi07, §12.3], and Simons–Sullivan [SS08] on differential cohomology.

2.1 Motivation for differential cohomology

2.1.1 Observation (Simons–Sullivan [SS08, §1])**.** Let M be a manifold. Then we have exact sequences

(2.1.2)
$$\begin{tikzcd} & \mathrm{H}^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r, "-\mathrm{Bock}"] & \mathrm{H}^k(M;\mathbb{Z}) \arrow[dr] & \\ \mathrm{H}^{k-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & & \mathrm{H}^k_{\mathrm{dR}}(M) \\ & \Omega^{k-1}(M)/\mathrm{im}(\mathrm{d}) \arrow[r, "\mathrm{d}"'] & \Omega^k_{\mathrm{cl}}(M) \arrow[ur] & \end{tikzcd},$$

where the top sequence is the Bockstein sequence associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0\,,$$

and we are identifying singular and de Rham cohomology via the de Rham isomorphism

$$\mathrm{H}^*_{\mathrm{dR}}(M) \cong \mathrm{H}^*(M;\mathbb{R})\,.$$

The top sequence is "purely homotopy-theoretic" in nature, while the bottom sequence is "purely geometric" in nature (i.e., the functor Ω^k_{cl} is not homotopy-invariant).

2.1.3 Question. Can we fill (2.1.4) in with an invariant $\hat{\mathrm{H}}^k(M;\mathbb{Z})$ in maroon

(2.1.4)
$$\begin{tikzcd} & \mathrm{H}^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r, "-\mathrm{Bock}"] \arrow[dr] & \mathrm{H}^k(M;\mathbb{Z}) \arrow[dr] & \\ \mathrm{H}^{k-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^k(M;\mathbb{Z}) \arrow[u] \arrow[d] & \mathrm{H}^k_{\mathrm{dR}}(M) \\ & \Omega^{k-1}(M)/\mathrm{im}(\mathrm{d}) \arrow[r, "\mathrm{d}"'] \arrow[ur] & \Omega^k_{\mathrm{cl}}(M) \arrow[ur] & \end{tikzcd},$$

that better blends homotopy theory and geometry, and makes the diagonals exact?

Now let us attempt to provide a satisfactory answer to Question 2.1.3 when $k = 1$.

2.1.5 Attempt (for $k = 1$)**.** Let M be a manifold. Consider the abelian group $\mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z})$ of smooth functions to the circle (with the group structure defined pointwise). We should really think of $\mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z})$ as an infinite-dimensional abelian Lie group. Recall that the inclusion

$$\mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z}) \subset \mathrm{Map}(M, \mathbb{R}/\mathbb{Z})$$

from the space of smooth maps to the space of continuous maps is a homotopy equivalence. Since the circle is 1-truncated,[1] this implies that $\mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z})$ is also 1-truncated.

Since $\mathbb{R}/\mathbb{Z}$ is a $\mathrm{K}(\mathbb{Z}, 1)$, we see that

$$\pi_0 \mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z}) \cong \mathrm{H}^1(M; \mathbb{Z}) .$$

In particular, we have a surjection $\pi_0 \colon \mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z}) \twoheadrightarrow \mathrm{H}^1(M; \mathbb{Z})$. Also notice that

$$\begin{aligned} \pi_1 \mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z}) &\cong \pi_0 \operatorname{Map}_*(\mathrm{S}^1, \mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z})) \\ &\cong \pi_0 \operatorname{Map}_*(\mathrm{S}^1, \operatorname{Map}(M, \mathbb{R}/\mathbb{Z})) \\ &\cong \pi_0 \operatorname{Map}(M, \operatorname{Map}_*(\mathrm{S}^1, \mathbb{R}/\mathbb{Z})) \\ &\cong \pi_0 \operatorname{Map}(M, \Omega(\mathbb{R}/\mathbb{Z})) \\ &\cong \mathrm{H}^0(M; \mathbb{Z}) . \end{aligned}$$

2.1.6 Construction. Let vol denote the standard volume form on the circle $\mathrm{S}^1 \cong \mathbb{R}/\mathbb{Z}$. Define a *curvature* map $\mathrm{curv} \colon \mathrm{C}^\infty(M, \mathbb{R}/\mathbb{Z}) \to \Omega^1_{\mathrm{cl}}(M)$ by

$$\mathrm{curv}(f) \coloneqq f^*(\mathrm{vol}) .$$

2.1.7. The kernel of curv consists of the locally constant maps $M \to \mathbb{R}/\mathbb{Z}$, i.e.,

$$\ker(\mathrm{curv}) \cong \mathrm{H}^0(M; \mathbb{R}/\mathbb{Z}) .$$

Note that the curvature map is not surjective:

$$\mathrm{im}(\mathrm{curv}) = \{\alpha \in \Omega^1_{\mathrm{cl}}(M) \mid \textstyle\int_{\mathrm{S}^1} \alpha \in \mathbb{Z} \text{ for every embedding } \mathrm{S}^1 \hookrightarrow M\} .$$

That is, the image of curv is the group of *closed* 1*-forms with integral periods*.

2.1.8 Definition (integral periods)**.** Let M be a manifold and $k \geq 0$ an integer. A closed k-form ω on M *has integral periods* if for every smooth k-cycle c in M the integral $\int_c \omega$ is an integer. We write

$$\Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}} \subset \Omega^k_{\mathrm{cl}}(M)$$

for the subgroup of k-forms with integral periods.

2.1.9 Remark. A closed k-form ω has integral periods if and only if the class of ω lies in the

[1] I.e., only has nontrivial homotopy groups in degrees ≤ 1.

image of the change-of-coefficients map

$$\mathrm{H}^k(M;\mathbb{Z}) \to \mathrm{H}^k(M;\mathbb{R}) \cong \mathrm{H}^k_{\mathrm{dR}}(M)\,.$$

2.1.10. We also have a map

$$\iota\colon\ \Omega^0(M) = \mathrm{C}^\infty(M,\mathbb{R}) \to \mathrm{C}^\infty(M,\mathbb{R}/\mathbb{Z})$$

given by post-composition with the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$. The map ι has kernel the integer-valued smooth functions $M \to \mathbb{R}$, i.e., the locally constant functions with integer values. That is, $\mathrm{im}(\iota) = \Omega^0_{\mathrm{cl}}(M)_{\mathbb{Z}}$.

2.1.11. These maps give rise to a commutative diagram with exact diagonals

$$\begin{tikzcd}
 & \mathrm{H}^0(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\mathrm{Bock}"] \arrow[dr, hook] & & \mathrm{H}^1(M;\mathbb{Z}) \arrow[dr] & \\
\mathrm{H}^0_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr, hook] & & \mathrm{C}^\infty(M,\mathbb{R}/\mathbb{Z}) \arrow[ur, two heads, "\pi_0"] \arrow[dr, "\mathrm{curv}"] & & \mathrm{H}^1_{\mathrm{dR}}(M) \\
 & \Omega^0(M) \arrow[rr, "\mathrm{d}"'] \arrow[ur, "\iota"] & & \Omega^1_{\mathrm{cl}}(M) \arrow[ur] &
\end{tikzcd}\,.$$

The diagonals become short exact sequences if we replace $\Omega^0(M)$ by $\Omega^0(M)/\Omega^0_{\mathrm{cl}}(M)_{\mathbb{Z}}$ and $\Omega^1_{\mathrm{cl}}(M)$ by $\Omega^1_{\mathrm{cl}}(M)_{\mathbb{Z}}$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
 & \mathrm{H}^0(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\mathrm{Bock}"] \arrow[dr, hook] & & \mathrm{H}^1(M;\mathbb{Z}) \arrow[dr] \arrow[ur] & \\
\mathrm{H}^0_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr, hook] & & \mathrm{C}^\infty(M,\mathbb{R}/\mathbb{Z}) \arrow[ur, two heads, "\pi_0"] \arrow[dr, two heads, "\mathrm{curv}"] & & \mathrm{H}^1_{\mathrm{dR}}(M) \\
 & \Omega^0(M)/\Omega^0_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[rr, "\mathrm{d}"'] \arrow[ur, hook, "\iota"] & & \Omega^1_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}\,.$$

2.1.12. The takeaway is that in Question 2.1.3, we should really replace $\Omega^{k-1}(M)/\mathrm{im}(d)$ by $\Omega^{k-1}(M)/\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}$ and $\Omega^k_{\mathrm{cl}}(M)$ by $\Omega^0_{\mathrm{cl}}(M)_{\mathbb{Z}}$ and ask for the diagonal sequences to be short exact.

2.2 Differential characters

We now present a unified approach to defining the "differential cohomology" groups $\hat{\mathrm{H}}^*(M;\mathbb{Z})$ due to Cheeger–Simons [CS85]. We follow Bär and Becker's exposition on *differential characters* [BB14, Part I, §5].

2.2.1 Notation. Let M be a manifold and $i \geq 0$ an integer. We write $\mathrm{C}_i^{\mathrm{sm}}(M;\mathbb{Z})$ for the abelian group of smooth (integer-valued) chains on M. We write $\mathrm{Z}_i^{\mathrm{sm}}(M;\mathbb{Z}) \subset \mathrm{C}_i^{\mathrm{sm}}(M;\mathbb{Z})$ for the subgroup of smooth cycles.

2.2.2 Definition (Cheeger–Simons [CS85, §1])**.** Let $k \geq 1$ be an integer and M a manifold. A *degree k differential character* on M is a homomorphism $\chi\colon \mathrm{Z}_{k-1}^{\mathrm{sm}}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ such that there exists a k-form $\omega(\chi) \in \Omega^k(M)$ with the property that for every $c \in \mathrm{C}_k^{\mathrm{sm}}(M;\mathbb{Z})$,

$$\chi(\partial c) = \int_c \omega(\chi) \mod \mathbb{Z} \,.$$

We write

$$\hat{\mathrm{H}}^k(M;\mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Z}_{k-1}^{\mathrm{sm}}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

for the abelian group of degree k differential characters on M.

It follows that $\omega(\chi)$ is unique and closed. Moreover, $\omega(\chi)$ has integral periods. The form $\omega(\chi)$ is called the *curvature* of χ, and we have a curvature map

$$\begin{aligned} \mathrm{curv}\colon \hat{\mathrm{H}}^k(M;\mathbb{Z}) &\to \Omega^k(M) \\ \chi &\mapsto \omega(\chi) \end{aligned}$$

with image $\Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}}$ those closed k-forms with integral periods.

2.2.3 Warning. The indexing convention used here is off by 1 from the indexing convention in [CS85, §1]. However, this indexing convention is what was later adopted by Simons–Sullivan [SS08, §1]. See also Remark 2.3.3 for why k is the 'right' index rather than $k-1$.

2.2.4 Remark. When $k = 0$, the diagram (2.1.4) is quite degenerate, and it will be convenient to define $\hat{\mathrm{H}}^0(M;\mathbb{Z}) \coloneqq \mathrm{H}^0(M;\mathbb{Z})$.

Now let us construct maps to fill in the "differential cohomology" diagram (2.1.4).

2.2.5 Construction (characteristic class)**.** There is a *characteristic class* map

$$\mathrm{cc}\colon \hat{\mathrm{H}}^k(M;\mathbb{Z}) \to \mathrm{H}^k(M;\mathbb{Z})$$

defined as follows. Since $\mathrm{Z}_{k-1}^{\mathrm{sm}}(M;\mathbb{Z})$ is a free $\mathbb{Z}$-module and the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ is an epimorphism, any homomorphism $\chi\colon \mathrm{Z}_{k-1}^{\mathrm{sm}}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ lifts to a homomorphism

$$\tilde{\chi}\colon \mathrm{Z}_{k-1}^{\mathrm{sm}}(M;\mathbb{Z}) \to \mathbb{R} \,.$$

Now define a homomorphism $I(\tilde{\chi})\colon \mathrm{C}_k^{\mathrm{sm}}(M;\mathbb{Z}) \to \mathbb{Z}$ by the assignment

$$c \mapsto -\tilde{\chi}(\partial c) + \int_c \mathrm{curv}(\chi)\,.$$

Since $\mathrm{curv}(\chi)$ is closed, $I(\tilde{\chi})$ defines a cocycle. Moreover, $I(\tilde{\chi})$ takes integral values, and the cohomology class $[I(\tilde{\chi})] \in \mathrm{H}^k(M;\mathbb{Z})$ does not depend on the choice of lift $\tilde{\chi}$. We define the characteristic class map cc by the assignment

$$\begin{aligned} \mathrm{cc}\colon \hat{\mathrm{H}}^k(M;\mathbb{Z}) &\to \mathrm{H}^k(M;\mathbb{Z}) \\ \chi &\mapsto [I(\tilde{\chi})] \end{aligned}\quad .$$

2.2.6 Warning. Simons and Sullivan [SS08] denote the characteristic class map cc by 'ch'.

2.2.7 Construction. Consider the universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{i-1}(M;\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}^i(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\langle -,-\rangle} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(M;\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \longrightarrow 0\,,$$

where the morphism $\langle -,-\rangle$ is given by sending the class of a cocycle u to the homomorphism

$$\begin{aligned} \langle u,-\rangle\colon \mathrm{H}_i(M;\mathbb{Z}) &\to \mathbb{R}/\mathbb{Z} \\ [z] &\mapsto u(z)\,. \end{aligned}$$

Since the circle $\mathbb{R}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, for any $\mathbb{Z}$-module A and integer $j > 0$, we have $\mathrm{Ext}^j_{\mathbb{Z}}(A,\mathbb{R}/\mathbb{Z}) = 0$. In particular, $\langle -,-\rangle$ is an isomorphism.

Setting $i = k-1$, precomposition with the quotient map $\mathrm{Z}^{\mathrm{sm}}_{k-1}(M;\mathbb{Z}) \twoheadrightarrow \mathrm{H}_{k-1}(M;\mathbb{Z})$ defines an injection

$$\mathrm{H}^i(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(M;\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Z}^{\mathrm{sm}}_{k-1}(M;\mathbb{Z}),\mathbb{R}/\mathbb{Z})\,.$$

It follows from the definitions that this factors through $\hat{\mathrm{H}}^k(M;\mathbb{Z})$. We simply denote this composite by $\langle -,-\rangle\colon \mathrm{H}^{k-1}(M;\mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{\mathrm{H}}^k(M;\mathbb{Z})$.

2.2.8 Construction. Define a map $\iota\colon \Omega^{k-1}(M) \to \hat{\mathrm{H}}^k(M;\mathbb{Z})$ by setting

$$\iota(\omega)(z) \coloneqq \exp\bigl(2\pi i \textstyle\int_z \omega\bigr)$$

for every smooth $(k-1)$-cycle z. By Stokes' Theorem, we see that $\mathrm{curv}(\iota(\omega)) = \mathrm{d}\omega$.

We have an $\mathbb{R}$-valued lift of $\iota(\omega)$ given by setting

$$\tilde{\iota}(\omega)(z) \coloneqq \int_z \omega$$

for every smooth $(k-1)$-cycle z. So by Stokes' Theorem we have

$$
\begin{aligned}
I(\tilde{\iota}(\omega))(c) &= -\tilde{\iota}(\omega)(\partial c) + \int_c \operatorname{curv}(\iota(\omega)) \\
&= -\int_{\partial c} \omega + \int_c \mathrm{d}\omega = 0
\end{aligned}
$$

for every smooth k-chain c. Hence $\operatorname{cc} \circ \iota = 0$.

We see that $\iota\colon \Omega^{k-1}(M) \to \hat{\mathrm{H}}^k(M;\mathbb{Z})$ has kernel those closed forms ω such that $\int_z \omega$ is an integer for all $z \in \mathrm{Z}^{\mathrm{sm}}_{k-1}(M;\mathbb{Z})$. That is,

$$
\ker(\iota) = \Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}
$$

is the group of closed $(k-1)$-forms with integral periods. Hence ι descends to an injection

$$
\iota\colon \Omega^{k-1}(M)/\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}} \rightarrowtail \hat{\mathrm{H}}^k(-;\mathbb{Z}) .
$$

2.3 The differential cohomology hexagon

2.3.1 Notation. Write Mfld for the category of smooth manifolds and GrAb for the category of graded abelian groups.

2.3.2 Theorem (Simons–Sullivan [SS08, Theorem 1.1])**.** *There is an essentially unique functor*

$$
\hat{\mathrm{H}}^*(-;\mathbb{Z})\colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{GrAb}
$$

equipped with natural transformations

(2.3.2.1) $\langle -,- \rangle\colon \mathrm{H}^{*-1}(-;\mathbb{R}/\mathbb{Z}) \to \hat{\mathrm{H}}^*(-;\mathbb{Z})$,

(2.3.2.2) $\iota\colon \Omega^{*-1}(M)/\Omega^{*-1}_{\mathrm{cl}}(M)_{\mathbb{Z}} \to \hat{\mathrm{H}}^*(-;\mathbb{Z})$,

(2.3.2.3) $\operatorname{cc}\colon \hat{\mathrm{H}}^*(-;\mathbb{Z}) \to \mathrm{H}^*(-;\mathbb{Z})$,

(2.3.2.4) *and* $\operatorname{curv}\colon \hat{\mathrm{H}}^*(-;\mathbb{Z}) \to \Omega^*_{\mathrm{cl}}(-)_{\mathbb{Z}}$

filling in the “differential cohomology hexagon”

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{H}^{*-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\mathrm{Bock}"] \arrow[dr] & & \mathrm{H}^{*}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
\mathrm{H}^{*-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^{*}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & & \mathrm{H}^{*}_{\mathrm{dR}}(M) \\
& \dfrac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \arrow[rr, "\mathrm{d}"'] \arrow[ur] & & \Omega^{*}_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

so that the diagonal sequences are exact.

Any functor $\hat{\mathrm{H}}^*(-;\mathbb{Z}) \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{GrAb}$ satisfying these properties is called *ordinary differential cohomology*.

2.3.3 Remark (Deligne’s model)**.** Motivated by Deligne cohomology in Hodge theory [Del71, §2.2; Voi07, §12.3], we can consider the smooth version of the Deligne complex on a manifold M. Write $\mathbb{Z}(k)$ for the complex of sheaves on M

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \longrightarrow \Omega^{k-1} \longrightarrow 0\,,$$

where Ω^i is in degree $i+1$. The *k-th smooth Deligne cohomology group* of M is the sheaf cohomology (i.e., hypercohomology) group $\mathrm{H}^k(M;\mathbb{Z}(k))$. We will see later that smooth Deligne cohomology agrees with ordinary differential cohomology (see Lemma 7.3.4).

2.3.4 Questions. There are a number of questions that naturally arise

(2.3.4.1) Is there differential K-theory?

Yes! Hopkins–Singer [HS05] define differential K-theory. Simons–Sullivan [SS10; SS12] tell a similar story, and define differential K-theory in terms of vector bundles with connection. We study this in §7.4.

(2.3.4.2) What about differential [favorite cohomology theory]?

Also yes, but the theory is more complicated. The fundamental observation is that everything we’ve considered comes from a sheaf of abelian groups or chain complexes (which we regard as spectra) on the category of *all* smooth manifolds. We begin to set up this theory in Chapter 3.

Moreover, the ∞-category $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spt})$ of sheaves of spectra on the category of manifolds has rich structure that gives rise to a “differential cohomology hexagon” associated to every object. We study this in Chapter 6.

2.3.5 Remark. The category $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$ is really the right place for moduli spaces of manifolds to live, and Fréchet manifolds embed as a full subcategory of $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$. See §3.7.

There are many applications of this perspective on differential cohomology that we study throughout this book. See, in particular, Part III.

3 Basics of sheaves on manifolds

by Peter Haine

The purpose of this chapter is to begin to set up the basics of differential cohomology theories as sheaves on the category of all manifolds. Section 3.1 starts with the basic definitions. Section 3.2 gives a reminder on derived ∞-categories and their relation to spectra so that we can give examples of sheaves on the category of manifolds in § 3.3. In § 3.4, we explain why in all situations of interest, we can check equivalences of differential cohomology theories "on stalks". Section 3.5 gives an alternative description of the ∞-category of sheaves on manifolds in terms of sheaves on the smaller category of Euclidean spaces. Section 3.6 is a digression giving a reformulation of the sheaf condition in terms of an *excision* condition (or *Mayer–Vietoris* property) and a "finiteness" condition. We finish the chapter with a digression explaining Losik and Hain's results embedding infinite dimensional manifolds into sheaves of sets on the category of (finite dimensional) manifolds (§ 3.7).

3.1 Definitions

3.1.1 Notation. We write Mfld for the (ordinary) category of smooth manifolds, including the empty manifold. The category Mfld has a Grothendieck topology where the covering families are families of open embeddings

$$\{j_\alpha \colon U_\alpha \hookrightarrow M\}_{\alpha \in A}$$

such that the family of open sets $\{j_\alpha(U_\alpha)\}_{\alpha \in A}$ is an open cover of M. Whenever we regard Mfld as a site, we use this topology.

3.1.2 Remark. Since the category Mfld is equivalent to the category of manifolds with a fixed embedding into $\mathbb{R}^\infty$, the category Mfld is essentially small.

3.1.3 Definition. Let C be a presentable ∞-category. We write

$$\mathsf{PSh}(\mathsf{Mfld}; C) \coloneqq \mathrm{Fun}(\mathsf{Mfld}^{\mathrm{op}}, C)$$

and write

$$\mathsf{Sh}(\mathsf{Mfld}; C) \subset \mathsf{PSh}(\mathsf{Mfld}; C)$$

for the full subcategory spanned by the C-valued sheaves on the site Mfld with respect to the Grothendieck topology given by open covers.

Explicitly, a C-valued presheaf $E \colon \mathsf{Mfld}^{\mathrm{op}} \to C$ is a sheaf if and only if for each manifold M, the restriction $E|_{\mathrm{Open}(M)}$ of E to the site $\mathrm{Open}(M)$ of open submanifolds of M is a sheaf on the topological space M.

3.1.4 Remark (on presentability)**.** Perhaps somewhat surprisingly, if C is an ∞-category with all limits and colimits, then the inclusion of C-valued sheaves into C-valued presheaves need not admit a left adjoint: presentability is the standard assumption which guarantees that this left adjoint exists. For this reason, we essentially always work with sheaved valued in a presentable ∞-category.

3.1.5 Notation. We write $\mathrm{S}_{\mathsf{Mfld}} \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{Sh}(\mathsf{Mfld}; C)$ for the left adjoint to the inclusion, that is, the *sheafification* functor.

3.1.6 Notation. We write Set for the category of sets, Spc for the ∞-category of spaces, Spt for the ∞-category of spectra, and Cat_∞ for the ∞-category of ∞-categories.

3.1.7 Example. Let $よ \colon \mathsf{Mfld} \hookrightarrow \mathsf{PSh}(\mathsf{Mfld}; \mathsf{Set})$ denote the Yoneda embedding. For each manifold M, the representable presheaf $よ(M)$ is a sheaf. Unless noted otherwise, we simply write M for the sheaf $よ(M)$.

The following is the fundamental definition of this text:

3.1.8 Definition. The ∞-category of *differential cohomology theories* is the ∞-category $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spt})$ of sheaves of spectra on Mfld.

For most of this text we work in the generality of sheaves with values in a general presentable ∞-category, or stable presentable ∞-category. The main reason for doing this is because we have reason to consider sheaves of spaces, sheaves of chain complexes, and sheaves of spectra, and want to treat them on the same footing.

3.1.9 Remark. We take the approach of Freed–Hopkins [FH13] and consider sheaves on the category of smooth manifolds. The general setup here is very robust, and one can take the basic objects to be manifolds with corners without essential change to how theory works; this is the approach taken by Hopkins–Singer [HS05] and Bunke–Nikolaus–Völkl [BNV16].

The first basic property we prove about sheaves on Mfld is that morphism is an equivalence if and only if it is when evaluated on each Euclidean space. For this, we use the fact that manifolds admit *good covers*.

3.1.10 Recollection (good covers)**.** Let M be an n-manifold. An open cover $\mathcal{U}$ of M is *good* if for every finite set $U_1, \dots, U_m \in \mathcal{U}$ of opens in $\mathcal{U}$, the intersection $U_1 \cap \cdots \cap U_m$ is either empty or diffeomorphic to $\mathbb{R}^n$.

3.1.11 Notation. Let T be a topological space and $U \subset T$ be open. For every open cover $\mathcal{U}$ of U, write $\mathrm{I}(\mathcal{U}) \subset \mathrm{Open}(T)$ for the full subposet consisting of all nonempty finite intersections of elements in $\mathcal{U}$.

3.1.12 Lemma. *Let C be a presentable ∞-category. A morphism $f \colon E \to E'$ in $\mathsf{Sh}(\mathsf{Mfld}; C)$ is an equivalence if and only if for each integer $n \geq 0$, the morphism $f(\mathbb{R}^n) \colon E(\mathbb{R}^n) \to E'(\mathbb{R}^n)$ is an equivalence in C.*

Proof. Let M be a manifold and $\mathcal{U}$ a good cover of M. The morphism f induces a commutative square

$$\begin{tikzcd} E(M) \arrow[r, "\sim"] \arrow[d, "f(M)"'] & \lim_{U \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} E(U) \arrow[d] \\ E'(M) \arrow[r, "\sim"] & \lim_{U \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} E'(U) , \end{tikzcd}$$

where the horizontal morphisms are equivalences because E and E' are sheaves. Since the cover $\mathcal{U}$ is good and f is an equivalence on Euclidean spaces, we see that the induced morphism

$$f : E|_{\mathrm{I}(\mathcal{U})^{\mathrm{op}}} \to E'|_{\mathrm{I}(\mathcal{U})^{\mathrm{op}}}$$

of $\mathrm{I}(\mathcal{U})^{\mathrm{op}}$-indexed diagrams in C is an equivalence, which proves the claim. □

3.2 Reminder on derived ∞-categories and Eilenberg–MacLane spectra

In order to give some important examples of sheaves on Mfld, we review the basics of derived ∞-categories of rings and their relation to spectra.

3.2.1 Notation (derived ∞-categories)**.** Let R be a ring. We write $\mathsf{Ch}(R)$ for the category of chain complexes of R-modules. We write $\mathsf{D}(R)$ for the *derived ∞-category* of R obtained from the category $\mathsf{Ch}(R)$ by formally inverting the quasi-isomorphisms [HA, Definition 1.3.5.8, Propositon 1.3.5.15, & Remark 7.1.1.16]. That is, $\mathsf{D}(R)$ is the universal ∞-category equipped with a functor $\mathsf{Ch}(R) \to \mathsf{D}(R)$ carrying quasi-isomorphisms in $\mathsf{Ch}(R)$ to equivalences in the ∞-category $\mathsf{D}(R)$. Note that for every map of rings $S \to R$, the forgetful functor $\mathsf{Ch}(R) \to \mathsf{Ch}(S)$ preserves quasi-isomorphisms, hence induces a forgetful functor $\mathsf{D}(R) \to \mathsf{D}(S)$.

3.2.2 Recollection (Eilenberg–MacLane spectra)**.** The inclusion $\mathsf{Ab} \subset \mathsf{Spt}$ of the category of abelian groups into the category of spectra as those spectra with homotopy groups in degree 0 (i.e., ordinary cohomology theories) extends to a right adjoint functor

$$\mathrm{H} : \mathsf{D}(\mathbb{Z}) \to \mathsf{Spt} .$$

The functor H is called the *Eilenberg–MacLane* functor [HA, Example 1.3.3.5]. For a ring R, we also simply write H for the composite

$$\mathsf{D}(R) \longrightarrow \mathsf{D}(\mathbb{Z}) \xrightarrow{\mathrm{H}} \mathsf{Spt}$$

for the composite of the forgetful functor $\mathsf{D}(R) \to \mathsf{D}(\mathbb{Z})$ with the Eilenberg–MacLane functor. The spectrum $\mathrm{H}R$ represents ordinary cohomology with coefficients in R.

3.2.3 Recollection ($\mathrm{H}R$-modules)**.** Every spectrum in the image of $\mathrm{H} : \mathsf{D}(R) \to \mathsf{Spt}$ is a module over the Eilenberg–MacLane spectrum $\mathrm{H}R$ representing ordinary cohomology with coefficients in R. Moreover, the Eilenberg–MacLane functor induces an equivalence

$$\mathsf{D}(R) \xrightarrow{\sim} \mathsf{Mod}(\mathrm{H}R)$$

between the derived ∞-category $\mathsf{D}(R)$ and the ∞-category $\mathsf{Mod}(\mathrm{H}R)$ of $\mathrm{H}R$-module spectra [HA, Proposition 7.1.4.6]. Under this equivalence $\mathsf{D}(R) \xrightarrow{\sim} \mathsf{Mod}(\mathrm{H}R)$, the functor $\mathrm{H} : \mathsf{D}(R) \to \mathsf{Spt}$ corresponds to the forgetful functor $\mathsf{Mod}(\mathrm{H}R) \to \mathsf{Spt}$

3.3 First examples

Now we give some examples of sheaves on manifolds coming from topological spaces, complexes of differential forms, and bundles.

3.3.a Topological spaces

3.3.1 Notation. Write Top for the category of topological spaces.

3.3.2 Construction. Define a restricted Yoneda functor y_{Top} by

$$\begin{aligned} y_{\mathsf{Top}} \colon \mathsf{Top} &\to \mathsf{PSh}(\mathsf{Mfld}; \mathsf{Set}) \\ T &\mapsto [M \mapsto \mathrm{Map}_{\mathsf{Top}}(M, T)] . \end{aligned}$$

Since continuous functions glue over open covers, the assignment $M \mapsto \mathrm{Map}_{\mathsf{Fré}}(M, T)$ is a sheaf on Mfld. That is, y_{Top} factors through $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Set})$. Hence every topological space defines a sheaf on Mfld.

3.3.b Differential forms

3.3.3 Example (differential forms)**.** Let $i \geq 0$ be an integer. The functor

$$\Omega^i \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Vect}(\mathbb{R})$$

sending manifold M to vector space $\Omega^i(M)$ of differential i-forms on M with functoriality given by pullback of bundles is a sheaf. Note that by the Yoneda Lemma, there is a natural isomorphism

$$\mathrm{Map}_{\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})}(M, \Omega^i) \cong \Omega^i(M) .$$

3.3.4 Example (de Rham complex)**.** Putting togther all i at once, the functor

$$\Omega^\bullet \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Ch}(\mathbb{R})$$

sending manifold M to its de Rham complex $\Omega^\bullet(M)$ is a sheaf of chain complexes on Mfld.

Even better, $\Omega^\bullet$ is a sheaf in the derived sense: the composite

$$\mathsf{Mfld}^{\mathrm{op}} \xrightarrow{\Omega^\bullet} \mathsf{Ch}(\mathbb{R}) \longrightarrow \mathsf{D}(\mathbb{R})$$

with the localization functor $\mathsf{Ch}(\mathbb{R}) \to \mathsf{D}(\mathbb{R})$ is a sheaf valued in the ∞-*category* $\mathsf{D}(\mathbb{R})$.

3.3.c Bundles & sheaves

3.3.5 Example (vector bundles)**.** Write

$$\mathrm{Vect}_{\mathbb{R}} \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Gpd}$$

for the functor sending a manifold M to the groupoid of (finite dimensional) real vector bundles on M and bundle isomorphisms, with functoriality given by pullback of bundles. Again, the local nature of the definition of a vector bundle ensures that $\mathrm{Vect}_{\mathbb{R}}$ is a sheaf of groupoids on Mfld.

3.3.6 Example (principal bundles)**.** Let G be a Lie group. Write

$$\mathrm{Bun}_G \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Gpd}$$

for the functor sending a manifold M to the groupoid of (smooth) principal G bundles on M and bundle isomorphisms, with functoriality given by pullback of bundles. The locally triviality of principal bundles implies that Bun_G is a sheaf of groupoids on Mfld.

3.3.7 Example (principal bundles with connection)**.** Let G be a Lie group. Write

$$\mathrm{Bun}_G^{\nabla} \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Gpd}$$

for the functor sending a manifold M to the groupoid of (smooth) principal G bundles on M *with connection* and bundle isomorphisms respecting connections, with functoriality given by pullback of bundles. Explicitly, an object of $\mathrm{Bun}_G^{\nabla}(M)$ consists of a pair (P,θ) of a principal G-bundle $P \to M$ and a connection 1-form $\theta \in \Omega^1(P;\mathfrak{g})$. See Chapter 11 for background on connections. A morphism

$$(P,\theta) \to (P',\theta')$$

in $\mathrm{Bun}_G^{\nabla}(M)$ consists of an isomorphism of principal G-bundles $f \colon P \xrightarrow{\sim} P'$ such that $f^*(\theta') = \theta$.

Again, the local nature of the definition of a bundle with connection ensures that Bun_G^{∇} is a sheaf of groupoids on Mfld.

3.3.8 Warning. In Examples 3.3.6 and 3.3.7, it is very important that we have not passed to *isomorphism classes* of principal G-bundles (with connection). The reason is that isomorphism classes do not glue, i.e., do not form a sheaf.

The sheaves Bun_G and Bun_G^{∇} are of great importance. In Part II, we study them in detail.

3.3.9 Example (sheaves)**.** For each manifold M, write $\mathsf{Sh}(M)$ for the ∞-category of sheaves of spaces on M, and $\mathsf{LC}(M) \subset \mathsf{Sh}(M)$ for the full subcategory spanned by the locally constant sheaves of spaces. The assignment $M \mapsto \mathsf{Sh}(M)$ extends to a functor

$$\mathsf{Sh} \colon \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$$

with functoriality given by pullback of sheaves. The functor Sh is a sheaf of (large) ∞-categories on Mfld [HTT, Theorem 6.1.3.9]. Since locally constant sheaves are preserved by sheaf pullback and local constancy is a local condition, the subfunctor $\mathsf{LC} \subset \mathsf{Sh}$ is also a sheaf of (large) ∞-categories on Mfld.

3.4 Checking equivalences on stalks

We now explain that equivalences of sheaves on Mfld with values in a *compactly generated* ∞-category (e.g., Spc, Spt, $\mathrm{D}(R)$) can be checked on "stalks" at the origins in $\mathbb{R}^n$ for $n \geq 0$. The proof of this requires a few technical detours which we defer to Section A.5.

3.4.1 Notation. Let M be a manifold and $x \in M$. We write $\mathrm{Open}_x(M) \subset \mathrm{Open}(X)$ for the full subposet spanned by the open neighborhoods of $x \in M$.

3.4.2 Definition. Let C be a compactly generated ∞-category, $E \in \mathrm{Sh}(\mathrm{Mfld}; C)$ a C-valued sheaf on Mfld, M a manifold, and $x \in M$. The *stalk* of E at $x \in M$ is the filtered colimit

$$x^*(E) \coloneqq \operatorname*{colim}_{U \in \mathrm{Open}_x(M)^{\mathrm{op}}} E(U) \tag{3.4.3}$$

in C.

3.4.4 Warning. It is important that we have phrased Definition 3.4.2 only for compactly generated coefficients. It is true that for any presentable ∞-category C, manifold M, and point $x \in M$, there is a stalk functor $x^* : \mathrm{Sh}(\mathrm{Mfld}; C) \to C$ (see Construction A.5.1). However, if C is not compactly generated then x^* need not be computed by the filtered colimit (3.4.3).

3.4.5 Notation. For each integer $n \geq 0$ and number $r \in \mathbb{R}_{>0}$, write $0_n \in \mathbb{R}^n$ for the origin, and write

$$\mathrm{B}_{\mathbb{R}^n}(r) \subset \mathbb{R}^n$$

for the open ball in $\mathbb{R}^n$ of radius r centered at the 0_n.

3.4.6. Let $E : \mathrm{Mfld}^{\mathrm{op}} \to C$ be a sheaf on Mfld. Note that the stalk $0_n^*(E)$ can be computed as the colimit

$$0_n^*(E) \simeq \operatorname*{colim}_{k \in \mathbb{N}} E(\mathrm{B}_{\mathbb{R}^n}(1/k)) .$$

The following result comes from the functoriality of a sheaf on Mfld in *all* manifolds, the fact that for ever n-manifold M and point $x \in M$, there exists an open embedding $j : \mathbb{R}^n \hookrightarrow M$ such that $j(0_n) = x$, and that equivalences in sheaves on M can be checked on stalks. In Section A.5 we provide a detailed proof.

3.4.7 Proposition (Proposition A.5.3)**.** *Let C be a compactly generated ∞-category. A morphism f in $\mathrm{Sh}(\mathrm{Mfld}; C)$ is an equivalence if and only if for each integer $n \geq 0$, the morphism $0_n^*(f)$ is an equivalence in C.*

3.4.8 Remark. Proposition 3.4.7 is important from our perspective. Freed and Hopkins work with differential cohomology theories using the language of simplicial sheaves and model categories [FH13]. Combining Proposition 3.4.7 with [HTT, Remark 6.5.2.2 & Proposition 6.5.2.14] shows that the model structure on simplicial presheaves on Mfld considered in [FH13, §5] presents the ∞-category $\mathrm{Sh}(\mathrm{Mfld}; \mathrm{Spc})$.

3.4.9 Warning. Proposition 3.4.7 does *not* hold when C is replaced by an arbitrary presentable ∞-category.

3.5 Sheaves on the Euclidean site

In this section, we refine Lemma 3.1.12 in the following manner. Since every manifold admits an open cover by Euclidean spaces, the category of sheaves of *sets* on Mfld is equivalent to sheaves of sets on the full subcategory spanned by the Euclidean spaces. We prove an analogous result for sheaves of *spaces*; this is not immediate in the higher-categorical setting [SAG, Counterexample 20.4.0.1]. The reason for this subtlety is exactly the failure of Whitehead's Theorem to hold in an arbitrary ∞-category of sheaves of spaces. However, Proposition 3.4.7 implies that Whitehead's Theorem holds in $\mathrm{Sh}(\mathsf{Mfld};\mathrm{Spc})$; a general result [Aok20, Appendix A; BGH20, Corollary 3.12.13] implies that sheaves on the site of Euclidean spaces and sheaves on Mfld coincide.

3.5.1 Definition. The *Euclidean site* is the full subcategory $\mathsf{Euc} \subset \mathsf{Mfld}$ spanned by the Euclidean spaces $\mathbb{R}^n$ for $n \geq 0$, with the induced Grothendieck topology.

The proof of the following is quite short. However, it involves some technical tools we have not yet introduced, so we defer it to §A.5.

3.5.2 Notation. Let C be a presentable ∞-category. Write $j : \mathsf{Euc}^{\mathrm{op}} \hookrightarrow \mathsf{Mfld}^{\mathrm{op}}$ for the inclusion, and

$$j_* : \mathrm{PSh}(\mathsf{Euc}; C) \hookrightarrow \mathrm{PSh}(\mathsf{Mfld}; C)$$

for the fully faithful functor given by right Kan extension along j. Restriction of presheaves is left adjoint to j_*; we denote this left adjoint by j^* or $(-)|_{\mathsf{Euc}^{\mathrm{op}}}$.

3.5.3 Lemma (Corollary A.5.6)**.** *Let C be a presentable ∞-category and $F \in \mathrm{PSh}(\mathsf{Mfld}; C)$. Then:*

(3.5.3.1) *The functors $j_* : \mathrm{PSh}(\mathsf{Euc}; C) \hookrightarrow \mathrm{PSh}(\mathsf{Mfld}; C)$ and $j^* : \mathrm{PSh}(\mathsf{Mfld}; C) \hookrightarrow \mathrm{PSh}(\mathsf{Euc}; C)$ preserve sheaves.*

(3.5.3.2) *The adjoint functors $j^* : \mathrm{Sh}(\mathsf{Mfld}; C) \rightleftarrows \mathrm{Sh}(\mathsf{Euc}; C) : j_*$ are inverse equivalences of ∞-categories.*

(3.5.3.3) *If $j^*(F)$ is a sheaf on Euc, then the unit $F \to j_* j^*(F)$ exhibits $j_* j^*(F)$ as the sheafification $\mathrm{S}_{\mathsf{Mfld}}(F)$.*

(3.5.3.4) *If $j^*(F)$ is a sheaf on Euc, then for all $n \geq 0$, the unit $F \to \mathrm{S}_{\mathsf{Mfld}}(F)$ is an equivalence when evaluated on $\mathbb{R}^n$. In particular, the unit $F \to \mathrm{S}_{\mathsf{Mfld}}(F)$ induces an equivalence on global sections.*

3.6 Digression: excision & the sheaf condition

The goal of this section is to prove a convenient reformulation of the sheaf condition in terms of an *excision* property. We do not make use of the reformation in this text, but present it here because it is the manifold analogue of *Nisnevich excision* from algebraic geometry [SAG, Proposition B.5.1.1; AHW17, §3.2; MV99, §3.1, Proposition 1.4]. Another way of explaining the following result is that it says that a presheaf on Mfld is a sheaf if and only if it satisfies the *Mayer–Vietoris* property and transforms countable increasing chains of open submanifolds to limits.

3.6.1 Theorem [BBP19, Theorem 5.1]**.** *Let C be a presentable ∞-category. A C-valued presheaf $F: \mathsf{Mfld}^{\mathrm{op}} \to C$ on Mfld is a sheaf if and only if F satisfies the following conditions:*

(3.6.1.1) *The object $F(\varnothing)$ is terminal in C.*

(3.6.1.2) *For every manifold M and pair of open subsets $U, V \subset M$ such that $U \cup V = M$, the induced square*

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square in C.

(3.6.1.3) *For every manifold M and $\mathbb{N}$-indexed sequence of open sets*

$$U_0 \subset U_1 \subset \cdots \subset M$$

such that $\bigcup_{n\geq 0} U_n = M$, the induced morphism

$$F(M) \to \lim_{n\geq 0} F(U_n)$$

is an equivalence in C.

3.6.2 Remark. Pavlov [Pav22, Theorem 1.7] proves that for sSet-valued presheaves on Mfld, these conditions can be considerably simplified to two conditions on two-element open covers and pairwise disjoint open covers.

We do not have occasion to use Theorem 3.6.1 in this text, but include it for completeness and because it is useful. For example, Theorem 3.6.1 is crucial to work of Berwick-Evans–Boavida de Brito–Pavlov [BBP19] extending results of Madsen–Weiss [MW07, Appendix A]. See Remark 4.4.13 for more details.

The idea of Theorem 3.6.1 is as follows. Conditions (3.6.1.1) and (3.6.1.2) guarantee that F satisfies the sheaf condition with respect to finite open covers. Given descent with respect to finite open covers, by writing a countable cover as a union of a sequence of finite covers of smaller subspaces, (3.6.1.3) implies descent with respect to *countable* open covers. Note that implicit in Theorem 3.6.1 is the claim that descent with respect to countable open covers implies descent with respect to arbitrary open covers.

Since the sheaf condition on Mfld is defined after restriction to each manifold, Theorem 3.6.1 follows from an analogous rephrasing of the sheaf condition for a presheaf on an individual manifold (Proposition 3.6.6). The manifold structure isn't really used here; all that is necessary is that an open cover of an open subset of a manifold admits a countable subcover. Hence we work at this level of generality.

3.6.3 Observation. Let T be a topological space and C a presentable ∞-category. Since limits of finite cubes can be written as iterated pullbacks, the following are equivalent for a presheaf $F \in \mathsf{PSh}(T; C)$ on T:

(3.6.3.1) The presheaf F satisfies descent with respect to nonempty finite covers.

(3.6.3.2) For all opens $U, V \subset T$, the induced square

$$\begin{tikzcd}
F(U \cup V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\
F(U) \arrow[r] & F(U \cap V)
\end{tikzcd}$$

is a pullback square in C.

3.6.4 Recollection. A topological space T is *Lindelöf* if every open cover of T has a countable subcover.

The following conditions are equivalent for a topological space T:

(3.6.4.1) Every open subspace of T is Lindelöf.

(3.6.4.2) Every subspace of T is Lindelöf.

We say that T is *hereditarily Lindelöf* if T satisfies the equivalent conditions (3.6.4.1)–(3.6.4.2).

Note that every second-countable topological space (e.g., manifold) is hereditarily Lindelöf.

3.6.5 Lemma. *Let T be a hereditarily Lindelöf topological space and C a presentable ∞-category. The following are equivalent for a presheaf $F \in \mathrm{PSh}(T; C)$ on T:*

(3.6.5.1) *The presheaf F is a sheaf on T.*

(3.6.5.2) *The presheaf F satisfies descent with respect to countable open covers.*

Proof. Clearly (3.6.5.1) $\Rightarrow$ (3.6.5.2). To see that (3.6.5.2) $\Rightarrow$ (3.6.5.1), let $U \subset T$ be open and let $\mathcal{U}$ be an open cover of U Since T is hereditarily Lindelöf, there exists a countable subset $\mathcal{U}_0 \subset \mathcal{U}$ that also covers U. To conclude, note that have a commutative triangle

$$\begin{tikzcd}
& F(U) \arrow[dl] \arrow[dr, "\sim"] & \\
\lim_{V \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(V) \arrow[rr, "\sim"] & & \lim_{V \in \mathrm{I}(\mathcal{U}_0)^{\mathrm{op}}} F(V) ,
\end{tikzcd}$$

where the right-hand diagonal morphism is an equivalence by (3.6.5.2) and the horizontal morphism is an equivalence because the inclusion $\mathrm{I}(\mathcal{U}_0)^{\mathrm{op}} \subset \mathrm{I}(\mathcal{U})^{\mathrm{op}}$ is limit-cofinal. $\square$

Now we provide a characterization of sheaves on a hereditarily Lindelöf topological space in terms of an excision property. This characterization immediately implies Theorem 3.6.1.

3.6.6 Proposition. *Let T be a hereditarily Lindelöf topological space and C a presentable ∞-category. A C-valued presheaf $F \in \mathrm{PSh}(T; C)$ on T is a sheaf if and only if F satisfies the following conditions:*

(3.6.6.1) *The object* $F(\varnothing)$ *is terminal in* C.

(3.6.6.2) *For all opens* $U, V \subset T$*, the induced square*

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square in C.

(3.6.6.3) *For every* $\mathbb{N}$*-indexed sequence of open sets* $U_0 \subset U_1 \subset \cdots \subset T$*, the induced morphism*

$$F\left(\bigcup_{n\geq 0} U_n\right) \to \lim_{n\geq 0} F(U_n)$$

is an equivalence in C.

Proof. First note that (3.6.6.1) and (3.6.6.2) are equivalent to saying that F satisfies descent with respect to finite covers. By Lemma 3.6.5, it suffices to show that F satisfies descent with respect to countable covers.

Let $V \subset T$ be open and $\mathcal{U} = \{V_i\}_{i\in\mathbb{N}}$ a countable open cover of V. For each $n \in \mathbb{N}$, define

$$U_n := \bigcup_{i=0}^{n} V_i \qquad \text{and} \qquad \mathcal{U}_n := \{V_0, \ldots, V_n\}\,.$$

Then $\mathcal{U}_n$ is a finite open cover of U_n and we have inclusions $U_n \subset U_{n+1}$ and $\mathcal{U}_n \subset \mathcal{U}_{n+1}$. Note that the poset $\mathrm{I}(\mathcal{U})$ is the filtered union

$$\mathrm{I}(\mathcal{U}) = \operatorname*{colim}_{n\geq 0} \mathrm{I}(\mathcal{U}_n)\,.$$

Since F satisfies descent with respect to finite covers, by (3.6.6.3) we see that we have natural equivalences

$$\begin{aligned} F(V) &\simeq \lim_{n\geq 0} F(U_n) \\ &\simeq \lim_{n\geq 0} \lim_{U\in \mathrm{I}(U_n)^{\mathrm{op}}} F(U) \\ &\simeq \lim_{U\in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(U)\,. \end{aligned}$$

Hence F satisfies descent with respect to the countable cover $\mathcal{U}$, as desired. □

Proof of Theorem 3.6.1. Since manifolds are second-countable and open subsets of manifolds are manifolds, the claim is immediate from Proposition 3.6.6 and the definition of what it means to be a sheaf on Mfld (Definition 3.1.3). □

3.7 Digression: relation to infinite dimensional manifolds

We finish this chapter by describing a "Yoneda embedding" of infinite dimensional manifolds into sheaves of *sets* on Mfld.

3.7.1 Recollection (infinite dimensional manifolds)**.** There are two classes of possibly infinite dimensional manifolds that are commonly considered: *Banach manifolds* and *Fréchet manifolds* [GG73, Chapter III, §1; Ham82a, §I.4]. Banach spaces are examples of Fréchet spaces, and the category of Banach manifolds is a full subcategory of the category of Fréchet manifolds.

One reason to consider Fréchet manifolds is that the (smooth) free loop space of a manifold naturally has the structure of a Fréchet manifold:

3.7.2 Example. If M and N are manifolds, and M is compact, then the topological space $\mathrm{C}^\infty(M,N)$ of smooth maps $M \to N$ has a natural Fréchet manifold structure. See [GG73, Chapter III, §1], in particular [GG73, Chapter III, Theorem 1.11], for details.

3.7.3 Notation. We write $\mathsf{Fré}$ for the category of Fréchet manifolds. Note that Mfld is the full subcategory of $\mathsf{Fré}$ spanned by the finite dimensional Fréchet manifolds.

3.7.4 Construction. Define a restricted Yoneda functor $y_{\mathsf{Fré}}$ by

$$\begin{aligned} y_{\mathsf{Fré}} \colon \mathsf{Fré} &\to \mathsf{PSh}(\mathsf{Mfld};\mathsf{Set}) \\ F &\mapsto [M \mapsto \mathrm{Map}_{\mathsf{Fré}}(M,F)] \,. \end{aligned}$$

Notice that since morphisms of Fréchet manifolds are defined locally, for each Fréchet manifold F, the presheaf $y_{\mathsf{Fré}}(F)$ is a sheaf. That is, $y_{\mathsf{Fré}}$ factors through $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$.

3.7.5 Theorem (Hain [Hai79], Losik [Los92; Los94, Theorem 3.1.1; Wal12, Theorem A.1.5])**.** *The functor* $y_{\mathsf{Fré}} \colon \mathsf{Fré} \to \mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$ *is fully faithful.*

The next result about infinite dimensional manifolds is that the embedding $y_{\mathsf{Fré}}$ sends Fréchet manifold of smooth maps from a compact manifold to an arbitrary manifold (Example 3.7.2) to the internal-Hom in $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$. In particular, free loop spaces are correctly represented in $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$. To state this result, let us first recall the internal-Hom in sheaves on Mfld.

3.7.6 Recollection (cartesian closedness)**.** Like any topos, the category $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$ of sheaves of sets on Mfld is *cartesian closed*. In particular, $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$ has an internal-Hom defined by

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})}(E,E') \colon \mathsf{Mfld}^{\mathrm{op}} &\to \mathsf{Set} \\ M &\mapsto \mathrm{Map}_{\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})}(E \times M, E') \,. \end{aligned}$$

3.7.7 Theorem (Waldorf [Wal12, Lemma A.1.7])**.** *Let M and N be manifolds. If M is compact, then there is a natural isomorphism*

$$y_{\mathsf{Fré}}(\mathrm{C}^\infty(M,N)) \cong \mathrm{Hom}_{\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})}(M,N) \,.$$

We finish this section by explaining how a commonly used enlargement of the category of Fréchet manifolds fits into the category $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$.

3.7.8 Remark (diffeological spaces)**.** Souriau introduced [Sou80] *diffeological spaces* as generalization of manifolds to include infinite dimensional manifolds as well as manifold-like spaces appearing in mathematical physics. Diffeological spaces have been extensively studied by Iglesias-Zemmour and collaborators [DI85; Igl86; Igl87; Igl87; Igl07a; Igl07b; Igl13; IK12; IKZ10].

To explain how diffeological spaces fit into sheaves on manifolds, write

$$よ\colon \mathsf{Euc} \hookrightarrow \mathsf{Sh}(\mathsf{Euc};\mathsf{Set}) \simeq \mathsf{Sh}(\mathsf{Mfld};\mathsf{Set})$$

for the Yoneda embedding. A *diffeological space* is a sheaf E on Euc such that for each $n \geq 0$, the natural map

$$E(\mathbb{R}^n) \cong \mathrm{Map}_{\mathsf{Sh}(\mathsf{Euc};\mathsf{Set})}(よ(\mathbb{R}^n), E) \longrightarrow \mathrm{Map}_{\mathsf{Set}}(よ(\mathbb{R}^n)(*), E(*)) = \mathrm{Map}_{\mathsf{Set}}(\mathbb{R}^n, E(*))$$

is injective. This injectivity condition allows a diffeological space to be described as a set X equipped with a collection of "plots"

$$\mathrm{C}^\infty(\mathbb{R}^n, X) \subset \mathrm{Map}_{\mathsf{Set}}(\mathbb{R}^n, X)$$

for each $n \geq 0$, subject to a collection of explicit conditions that are equivalent to saying that the assignment

$$\mathbb{R}^n \mapsto \mathrm{C}^\infty(\mathbb{R}^n, X)$$

is a sheaf on Euc. (To match up notation, $X = E(*)$ and $\mathrm{C}^\infty(\mathbb{R}^n, X) = E(\mathbb{R}^n)$.)

4 $\mathbb{R}$-invariant sheaves

by Peter Haine

In this chapter, we investigate $\mathbb{R}$-*invariant* (or *homotopy invariant*) sheaves on Mfld. These are the sheaves that invert homotopy equivalences of manifolds. The main result of this chapter is Dugger's observation that the global sections functor induces an equivalence from the subcategory $\mathsf{Sh}(\mathsf{Mfld}; C)$ of $\mathbb{R}$-invariant sheaves to C (Proposition 4.3.1). In the case where $C = \mathsf{Spc}$, we show that the constant sheaf functor $\Gamma^* \colon \mathsf{Spc} \to \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$ is given by the the assignment

$$X \mapsto [M \mapsto \mathrm{Map}_{\mathsf{Spc}}(\Pi_\infty(M), X)] \,,$$

where $\Pi_\infty(M)$ denotes the underlying homotopy type of the manifold M. More generally, the constant sheaf functor $\Gamma^* \colon C \to \mathsf{Sh}(\mathsf{Mfld}; C)$ is given by the assignment

$$X \mapsto [M \mapsto X^{\Pi_\infty(M)}] \,,$$

where $X^{\Pi_\infty(M)}$ denotes the *cotensor* of $X \in C$ by $\Pi_\infty(M) \in \mathsf{Spc}$; see Recollection 4.3.4. These results imply that there exists a chain of four (explicit) adjoints

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld}; C) \arrow[r, shift left=3, "\Gamma_\sharp"] & C \arrow[l, shift left=1, "\Gamma^*"'] \arrow[l, phantom] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld}; C) \arrow[r, shift left=4.5, "\Gamma_\sharp"] \arrow[r, shift right=1.5, "\Gamma_*"'] & C \arrow[l, shift right=1.5, "\Gamma^*"'] \arrow[l, shift left=4.5, "\Gamma^\sharp"] \end{tikzcd}$$

relating $\mathsf{Sh}(\mathsf{Mfld}; C)$ and C (4.4.2).

Looking forward, in Chapter 5, we give an explicit formula for $\Gamma_\sharp$ as a geometric realization. In Chapter 6, we use these adjoints and relations between them to construct a "differential cohomology diagram" for sheaves on Mfld with values in any presentable stable ∞-category.

Section 4.1 starts with some preliminary observations about the global sections and constant sheaf functors. In Section 4.2, we define $\mathbb{R}$-invariant sheaves and explore some of their basic properties. Section 4.3 is dedicated to proving that the global sections functor restricts to an equivalence on $\mathbb{R}$-invariant sheaves. In §4.4, we explore some immediate consequences of this result.

4.1 Preliminaries on global sections and constant sheaves

We begin by fixing some notation that we use throughout the rest of this text.

4.1.1 Notation. Write $\Gamma_* \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to C$ for the *global sections* functor, defined by

$$\Gamma_*(E) \coloneqq E(*) \,.$$

Write $\Gamma^{-1} \colon C \to \mathsf{PSh}(\mathsf{Mfld}; C)$ for the *constant presheaf* functor. That is, Γ^{-1} is left adjoint to global sections functor $\Gamma_* \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to C$.

Write $\Gamma^* : C \to \mathrm{Sh}(\mathrm{Mfld}; C)$ for the *constant sheaf* functor. Then Γ^* is the sheafification of Γ^{-1}. Moreover, Γ^* is left adjoint to the restriction $\Gamma_* : \mathrm{Sh}(\mathrm{Mfld}; C) \to C$ of the global sections functor to sheaves.

We use the same notations for the constant (pre)sheaf and global sections functors for the Euclidean site $\mathrm{Euc} \subset \mathrm{Mfld}$ introduced in §3.5.

4.1.2 Observation. Note that we have a natural identification $\Gamma_*\Gamma^{-1} \simeq \mathrm{id}$. Since Γ_* is right adjoint to Γ^{-1}, we conclude that Γ^{-1} is fully faithful [CSY21, Lemma 3.3.1].

The global sections functor also has a right adjoint.

4.1.3 Lemma. *Let C be a presentable ∞-category. Then the functor $\Gamma^\sharp : C \to \mathrm{PSh}(\mathrm{Mfld}; C)$ defined by the formula*

$$\Gamma^\sharp(X)(M) \coloneqq \prod_{m \in M} X$$

is fully faithful and right adjoint to the global sections functor $\Gamma_* : \mathrm{PSh}(\mathrm{Mfld}; C) \to C$. (Here the product is over the underlying set of the manifold M.)

Proof. We define the unit and counit of the adjunction. The unit $\eta_F : F \to \Gamma^\sharp\Gamma_*(F)$ is defined by the natural map

$$F(M) \to \prod_{m \in M} F(\{m\}) \xrightarrow{\sim} \Gamma^\sharp\Gamma_*(F)(M)$$

induced by the inclusions $\{m\} \hookrightarrow M$ for all $m \in M$. The counit $\varepsilon_X : \Gamma_*\Gamma^\sharp(X) \to X$ is given by the natural identification $\prod_* X \simeq X$. The triangle identities are immediate from the definitions.

To conclude, note that since the counit ε is an equivalence, the functor $\Gamma^\sharp$ is fully faithful. □

Before recording the consequences of Lemma 4.1.3 on the level of sheaves, we recall a basic fact from category theory. For a proof see, for example, [MM94, Chapter VII, §4, Lemma 1].

4.1.4 Lemma. *Let $f_* : A \to B$ be a functor between ∞-categories. Assume that f_* admits a left adjoint f^* and right adjoint $f^\sharp$. Then f^* is fully faithful if and only if $f^\sharp$ is fully faithful.*

4.1.5 Corollary. *Let C be a presentable ∞-category.*

(4.1.5.1) *The functor $\Gamma^\sharp$ factors through $\mathrm{Sh}(\mathrm{Mfld}; C)$.*

(4.1.5.2) *The global sections functor $\Gamma_* : \mathrm{Sh}(\mathrm{Mfld}; C) \to C$ is left adjoint to $\Gamma^\sharp : C \to \mathrm{Sh}(\mathrm{Mfld}; C)$.*

(4.1.5.3) *The constant sheaf functor Γ^* is fully faithful.*

Proof. For (4.1.5.1), note that is immediate from Definition 3.1.3 that for each $X \in C$, the presheaf $\Gamma^\sharp(X)$ is a sheaf on Mfld. Lemma 4.1.3 and (4.1.5.1) immediately imply (4.1.5.2). Finally, (4.1.5.3) is a consequence of Lemma 4.1.3, (4.1.5.2), and the full faithfulness of $\Gamma^\sharp$. □

4.2 Basics of $\mathbb{R}$-invariant sheaves

We start by introducing an important subcategory of $\mathsf{Sh}(\mathsf{Mfld}; C)$.

4.2.1 Definition ($\mathbb{R}$-invariant sheaves on Mfld)**.** Let C be a presentable ∞-category. We say that a C-valued presheaf

$$F\colon \mathsf{Mfld}^{\mathrm{op}} \to C$$

is $\mathbb{R}$-*invariant*, *homotopy-invariant*, or *concordance-invariant* if for each manifold M, the first projection $\mathrm{pr}_M\colon M \times \mathbb{R} \to M$ induces an equivalence

$$\mathrm{pr}_M^*\colon F(M) \xrightarrow{\sim} F(M \times \mathbb{R}) .$$

Write

$$\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \subset \mathsf{Sh}(\mathsf{Mfld}; C) \qquad \text{and} \qquad \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C) \subset \mathsf{PSh}(\mathsf{Mfld}; C)$$

for the full subcategories spanned by the $\mathbb{R}$-invariant C-valued sheaves and presheaves, respectively.

We now give some reformulations of $\mathbb{R}$-invariance. The main one is that a presheaf is $\mathbb{R}$-invariant if and only if it inverts all homotopy equivalences between manifolds.

4.2.2 Notation. Let M be a manifold and $t \in \mathbb{R}$. We write $i_{M,t}\colon M \hookrightarrow M \times \mathbb{R}$ for the closed embedding defined by $x \mapsto (x, t)$.

4.2.3 Observation. For each manifold M and $t \in \mathbb{R}$, the map $i_{M\times\mathbb{R},t}$ is given by the composite

$$M \times \mathbb{R} \xhookrightarrow{i_{M,t}\times \mathrm{id}_{\mathbb{R}}} M \times \mathbb{R} \times \mathbb{R} \xrightarrow[\sim]{\mathrm{id}_M \times \mathrm{swap}} M \times \mathbb{R} \times \mathbb{R} ,$$

where $\mathrm{swap}\colon \mathbb{R} \times \mathbb{R} \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}$ is the map that swaps the two factors.

4.2.4 Proposition. *Let C be an ∞-category and $F\colon \mathsf{Mfld}^{\mathrm{op}} \to C$ a presheaf. The following are equivalent:*

(4.2.4.1) *The presheaf F is $\mathbb{R}$-invariant.*

(4.2.4.2) *For each homotopy equivalence of manifolds $f\colon M \to N$, the map $f^*\colon F(N) \to F(M)$ is an equivalence in C.*

(4.2.4.3) *For each manifold M and $t \in \mathbb{R}$, the induced map $i_{M,t}^*\colon F(M \times \mathbb{R}) \to F(M)$ is an equivalence.*

(4.2.4.4) *For each manifold M, the induced maps $i_{M,0}^*, i_{M,1}^*\colon F(M \times \mathbb{R}) \to F(M)$ are equivalent.*

Proof. Since the embeddings

$$i_{M,t}\colon M \hookrightarrow M \times \mathbb{R}$$

are sections of the projection $\mathrm{pr}_M\colon M \times \mathbb{R} \to M$, it is clear that (4.2.4.1) $\Leftrightarrow$ (4.2.4.3) and (4.2.4.1) $\Rightarrow$ (4.2.4.4). It is also clear that (4.2.4.2) $\Rightarrow$ (4.2.4.1)

To see that (4.2.4.1) ⇒ (4.2.4.2), let $g\colon N \to M$ be a homotopy inverse to f, so that there are homotopies of gf and fg with the respective identities fitting into commutative diagrams

$$\begin{tikzcd} M \arrow[d, hook, "i_{M,0}"'] \arrow[dr, equal] & \\ M \times [0,1] \arrow[r, "h_0"] & M \\ M \arrow[u, hook, "i_{M,1}"] \arrow[ur, "gf"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N \arrow[d, hook, "i_{N,0}"'] \arrow[dr, equal] & \\ N \times [0,1] \arrow[r, "h_1"] & N \\ N \arrow[u, hook, "i_{N,1}"] \arrow[ur, "fg"'] & \end{tikzcd} .$$

Since the morphisms $i_{N,0}^*$ and $i_{M,0}^*$ are equivalences and the upper triangles commute, by the 2-of-3 property both h_0^* and h_1^* are equivalences. Since the morphisms $i_{N,1}^*$, $i_{M,1}^*$, h_0^*, and h_1^* are equivalences and the lower triangles commute, we see that

$$(fg)^* \simeq g^* f^* \qquad \text{and} \qquad (gf)^* \simeq f^* g^*$$

are equivalences. By the 2-of-6 property we deduce that f^* and g^* are equivalences.

To complete the proof, we show that (4.2.4.4) ⇒ (4.2.4.1). Assuming (4.2.4.4), since $i_{M,0}$ is a section or the projection $\mathrm{pr}_M\colon M \times \mathbb{R} \to M$, it suffices to show that we have an equivalence

$$\mathrm{pr}_M^*\, i_{M,0}^* \simeq \mathrm{id}_{F(M\times\mathbb{R})} \ .$$

To see this, write $\mathrm{mult}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ for the multiplication map, and notice that we have a commutative diagram in Mfld

(4.2.5)
$$\begin{tikzcd} M \times \mathbb{R} \arrow[r, hook, "i_{M,1}\times \mathrm{id}_{\mathbb{R}}"] \arrow[dr, equal] & M \times \mathbb{R} \times \mathbb{R} \arrow[d, "\mathrm{id}_M \times \mathrm{mult}"] & M \times \mathbb{R} \arrow[l, hook', "i_{M,0}\times \mathrm{id}_{\mathbb{R}}"'] \arrow[d, "\mathrm{pr}_M"] \\ & M \times \mathbb{R} & M \arrow[l, hook', "i_{M,0}"] \end{tikzcd}$$

Combining the assumption that $i_{M\times\mathbb{R},0}^* \simeq i_{M\times\mathbb{R},1}^*$ with Observation 4.2.3 shows that

(4.2.6)
$$(i_{M,0} \times \mathrm{id}_{\mathbb{R}})^* \simeq (i_{M,1} \times \mathrm{id}_{\mathbb{R}})^* \ .$$

Equation (4.2.6) and the commutativity of the diagram (4.2.5) now show that

$$\begin{aligned} \mathrm{pr}_M^*\, i_{M,0}^* &\simeq (i_{M,0} \times \mathrm{id}_{\mathbb{R}})^* \circ (\mathrm{id}_M \times \mathrm{mult})^* \\ &\simeq (i_{M,1} \times \mathrm{id}_{\mathbb{R}})^* \circ (\mathrm{id}_M \times \mathrm{mult})^* \\ &\simeq \mathrm{id}_{F(M\times\mathbb{R})} \ , \end{aligned}$$

as desired. □

4.2.7 Remark. The reformulation of $\mathbb{R}$-invariance given in (4.2.4.4) is due to Voevodsky [MVW06, Lemma 2.16].

Equivalences of $\mathbb{R}$-invariant sheaves can be checked on global sections:

4.2.8 Lemma. *Let C be a presentable ∞-category. A morphism $f\colon E \to E'$ in $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C)$ is an equivalence if and only if $\Gamma_*(f)$ is an equivalence in C.*

Proof. This follows from Lemma 3.1.12 and the assumption that E and E' are $\mathbb{R}$-invariant. □

4.2.a $\mathbb{R}$-invariant sheaves on the Euclidean site

It is convenient to describe $\mathbb{R}$-invariant (pre)sheaves in terms of the Euclidean site $\mathrm{Euc} \subset \mathrm{Mfld}$. We start with the following reformulation of $\mathbb{R}$-invariance:

4.2.9 Lemma. *Let C be a presentable ∞-category and $F \in \mathrm{Sh}(\mathrm{Mfld}; C)$ a sheaf. The following are equivalent:*

(4.2.9.1) *The sheaf F is $\mathbb{R}$-invariant.*

(4.2.9.2) *The functor $F|_{\mathrm{Euc}^{\mathrm{op}}}\colon \mathrm{Euc}^{\mathrm{op}} \to C$ carries every morphism to an equivalence.*

Proof. Since Euclidean spaces are contractible, every map between Euclidean spaces is a homotopy equivalence. Thus applying Proposition 4.2.4 shows that (4.2.9.1) ⇒ (4.2.9.2).

To see that (4.2.9.2) ⇒ (4.2.9.1), assume that $F|_{\mathrm{Euc}^{\mathrm{op}}}\colon \mathrm{Euc}^{\mathrm{op}} \to C$ carries every morphism to an equivalence. Let M be a manifold. To show that pr_M^* is an equivalence, fix a *good* cover $\mathcal{U}$ if M. Then the collection $\{U \times \mathbb{R}\}_{U\in\mathcal{U}}$ is a good cover of $M \times \mathbb{R}$. The projection maps $\mathrm{pr}_U\colon U \times \mathbb{R} \to \mathbb{R}$ induce a commutative square

$$
\begin{tikzcd}
F(M) \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_M^*"'] & \lim_{U\in\mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(U) \arrow[d, "\lim_{U\in\mathrm{I}(\mathcal{U})^{\mathrm{op}}} \mathrm{pr}_U^*"] \\
F(M \times \mathbb{R}) \arrow[r, "\sim"] & \lim_{U\in\mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(U \times \mathbb{R}) \, ,
\end{tikzcd}
\tag{4.2.10}
$$

where the horizontal morphisms are equivalences because F is a sheaf. Since $F|_{\mathrm{Euc}^{\mathrm{op}}}$ carries every morphism to an equivalence and the cover $\mathcal{U}$ consists of Euclidean spaces, we see that the right-hand vertical morphism in (4.2.10) is an equivalence. By the 2-of-3 property, the left-hand vertical morphism in (4.2.10) is an equivalence, as desired. □

Lemma 4.2.9 motivates the following variant of Definition 4.2.1:

4.2.11 Definition ($\mathbb{R}$-invariant sheaves on Euc)**.** Let C be a presentable ∞-category. We say that a C-valued presheaf

$$F\colon \mathrm{Euc}^{\mathrm{op}} \to C$$

is $\mathbb{R}$-*invariant* if F carries every morphism in $\mathrm{Euc}^{\mathrm{op}}$ to an equivalence. Write

$$\mathrm{Sh}_{\mathbb{R}}(\mathrm{Euc}; C) \subset \mathrm{Sh}(\mathrm{Euc}; C) \qquad \text{and} \qquad \mathrm{PSh}_{\mathbb{R}}(\mathrm{Euc}; C) \subset \mathrm{PSh}(\mathrm{Euc}; C)$$

for the full subcategories spanned by the $\mathbb{R}$-invariant C-valued sheaves and presheaves, respectively.

4.2.12 Observation. Let F be an $\mathbb{R}$-invariant presheaf on Mfld. Then the restriction $F|_{\mathsf{Euc}^{\mathrm{op}}}$ is an $\mathbb{R}$-invariant presheaf on Euc. By Lemma 4.2.9, if G is an $\mathbb{R}$-invariant *sheaf* on Euc, then the right Kan extension $j_*(G)\colon \mathsf{Mfld}^{\mathrm{op}} \to C$ is an $\mathbb{R}$-invariant sheaf on Mfld. Hence the inverse equivalences of ∞-categories

$$(-)|_{\mathsf{Euc}^{\mathrm{op}}} : \mathsf{Sh}(\mathsf{Mfld}; C) \rightleftarrows \mathsf{Sh}(\mathsf{Euc}; C) : j_*$$

of Lemma 3.5.3 restrict to inverse equivalences

$$(-)|_{\mathsf{Euc}^{\mathrm{op}}} : \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \rightleftarrows \mathsf{Sh}_{\mathbb{R}}(\mathsf{Euc}; C) : j_* \,.$$

Similarly to Lemma 4.2.8, a morphism of $\mathbb{R}$-invariant presheaves on Euc is an equivalence if and only if it induces an equivalence on global sections.

4.2.13 Lemma. *Let C be a presentable ∞-category. A morphism $f\colon E \to E'$ in $\mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C)$ is an equivalence if and only if $\Gamma_*(f)$ is an equivalence in C.*

Proof. For the nontrivial direction, assume that $\Gamma_*(f)$ is an equivalence, and fix an integer $n \geq 0$; we need to show that $f(\mathbb{R}^n)$ is an equivalence. Consider the commutative square

$$\begin{tikzcd} E(*) \arrow[r] \arrow[d, "f(*)"'] & E(\mathbb{R}^n) \arrow[d, "f(\mathbb{R}^n)"] \\ E'(*) \arrow[r] & E'(\mathbb{R}^n)\,, \end{tikzcd} \tag{4.2.14}$$

where the horizontal morphisms are induced by the unique morphism $\mathbb{R}^n \to *$. Since E and E' carry all morphisms in $\mathsf{Euc}^{\mathrm{op}}$ to equivalences, the horizontal morphisms in (4.2.14) are equivalences. By assumption $f(*) = \Gamma_*(f)$ is an equivalence. Hence the 2-of-3 property shows that $f(\mathbb{R}^n)$ is an equivalence, as desired. □

4.2.15 Remark. Note that in Lemma 4.2.13 we only require that E and E' be *pre*sheaves.

4.3 The constant sheaf functor

The goal of this section is to prove the following result, originally sketched for sheaves of spaces by Dugger [Dug98, Theorem 3.4.3; Dug01, Proposition 8.3] and Morel–Voevodsky [MV99, Proposition 3.3.3]. See the work of Bunk [Bun22b] for another model category-theoretic argument.

4.3.1 Proposition. *Let C be a presentable ∞-category. Then:*

(4.3.1.1) *The constant sheaf functor $\Gamma^*\colon C \to \mathsf{Sh}(\mathsf{Mfld}; C)$ factors through $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$.*

(4.3.1.2) *The global sections functor*

$$\Gamma_* : \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \to C$$

is an equivalence with inverse given by Γ^*.

(4.3.1.3) *A sheaf F on* Mfld *is* $\mathbb{R}$*-invariant if and only if F is constant.*

(4.3.1.4) *The constant sheaf functor* $\Gamma^* : C \to \mathsf{Sh}(\mathsf{Mfld}; C)$ *admits a left adjoint.*

4.3.2 Remark. An analogue of Proposition 4.3.1 holds where the category of manifolds is replaced by the category of smooth complex analytic spaces, and $\mathbb{R}$ is replaced by the open unit disk in $\mathbb{C}$; see [Ayo10, Remarque 1.9]. Similarly, there are many variants of this result where Mfld is replaced by any reasonable category of locally contractible spaces.

4.3.3 Remark. See [HPT22, §2] for related results about constant (hyper)sheaves on locally weakly contractible topological spaces.

4.3.a Background on cotensors

In order to prove Proposition 4.3.1, we give a concrete description of the constant sheaf functor. To do this, we first recall the natural cotensoring of a presentable ∞-category over Spc.

4.3.4 Recollection (cotensoring over Spc)**.** Every presentable ∞-category C is naturally *cotensored over* the ∞-category Spc of spaces [HTT, Remark 5.5.2.6]. That is, there is a functor

$$\begin{aligned} (-)^{(-)} : \mathsf{Spc}^{\mathrm{op}} \times C &\to C \\ (K, X) &\mapsto X^K , \end{aligned}$$

along with natural equivalences

$$\mathrm{Map}_C(X', X^K) \simeq \mathrm{Map}_{\mathsf{Spc}}(K, \mathrm{Map}_C(X', X)) .$$

4.3.5 Example. If $C = \mathsf{Spc}$ is the ∞-category of spaces, then the cotensoring is given by

$$X^K \coloneqq \mathrm{Map}_{\mathsf{Spc}}(K, X) .$$

4.3.6 Example. If $C = \mathsf{Spt}$ is the ∞-category of spectra, then the cotensoring is given by

$$X^K \coloneqq \mathrm{Hom}_{\mathsf{Spt}}(\Sigma^\infty_+ K, X) ,$$

where $\mathrm{Hom}_{\mathsf{Spt}}$ denotes the mapping *spectrum* in Spt.

4.3.7 Example. If R is a ring and let $C = \mathsf{D}(R)$ be the derived ∞-category of R, then the cotensoring is given by

$$A_*^K \coloneqq \mathrm{RHom}_R(\mathrm{C}_*(K; R), A_*) .$$

Here $\mathrm{C}_*(K; R)$ is the complex of singular chains on K, and RHom_R is the derived Hom functor of chain complexes of R-modules. If M is an ordinary R-module regarded as an object of $\mathsf{D}(R)$ concentrated in degree 0, then the complex

$$\mathrm{RHom}_R(\mathrm{C}_*(K; R), M)$$

is the complex $\mathrm{C}^{-*}(K;M)$ of *singular cochains* on K with coefficients in M.

4.3.b Description of the constant sheaf functor

We now give an explicit formula for the constant sheaf functor.

4.3.8 Notation. Recall that we write Top for the category of topological spaces (Notation 3.3.1). Write $\Pi_\infty \colon \mathsf{Top} \to \mathsf{Spc}$ for the functor sending a topological space T to the underlying homotopy type $\Pi_\infty(T)$ of T.

4.3.9 Construction. Write $\Pi_\infty \colon \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{Spc}$ for the left Kan extension of the functor

$$\mathsf{Mfld} \xrightarrow{\text{forget}} \mathsf{Top} \xrightarrow{\Pi_\infty} \mathsf{Spc}$$

along the Yoneda embedding $\mathsf{Mfld} \hookrightarrow \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc})$. By the universal property of the ∞-category of presheaves, the functor $\Pi_\infty \colon \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{Spc}$ is a left adjoint with right adjoint $\mathrm{sm} \colon \mathsf{Spc} \to \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc})$ given by the formula

$$X \mapsto \left[M \mapsto \mathrm{Map}_{\mathsf{Spc}}(\Pi_\infty(M), X)\right] .$$

By the van Kampen Theorem [HA, Proposition A.3.2], the functor $\mathrm{sm} \colon \mathsf{Spc} \to \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc})$ factors through $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$. We use the same notation for the resulting adjunction

$$\Pi_\infty \colon \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \rightleftarrows \mathsf{Spc} \colon \mathrm{sm} .$$

Given a presentable ∞-category C, we write $\Pi_\infty \colon \mathsf{Sh}(\mathsf{Mfld};C) \to C$ for the tensor product

$$\mathsf{Sh}(\mathsf{Mfld};C) \simeq \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes C \xrightarrow{\Pi_\infty \otimes \mathrm{id}_C} \mathsf{Spc} \otimes C \simeq C .$$

We write

$$\mathrm{sm} \colon C \to \mathsf{Sh}(\mathsf{Mfld};C)$$

for the right adjoint of Π_∞. Concretely, sm is defined by sending $X \in C$ to the sheaf

$$M \mapsto X^{\Pi_\infty(M)} .$$

4.3.10 Observation. For each $X \in C$, the sheaf $\mathrm{sm}(X)$ is $\mathbb{R}$-invariant.

4.3.11 Construction (comparison natural transformation)**.** Recall that we write

$$\Gamma^{-1} \colon C \to \mathsf{PSh}(\mathsf{Mfld};C)$$

for the constant presheaf functor. Define a natural transformation $\alpha \colon \Gamma^{-1} \to \mathrm{sm}$ as follows. For each $X \in C$, the component of $\alpha(X)$ at a manifold $M \in \mathsf{Mfld}$ is the map

$$\Gamma^{-1}(X)(M) = X \longrightarrow X^{\Pi_\infty(M)} = \mathrm{sm}(X)(M)$$

induced by the unique map $\Pi_\infty(M) \to *$.

4.3.12 Proposition (formula for the constant sheaf functor)**.** *Let C be a presentable ∞-category. Then:*

(4.3.12.1) *The natural transformation $\alpha \colon \Gamma^{-1} \to \mathrm{sm}$ is an equivalence when restricted to the subcategory $\mathsf{Euc}^{\mathrm{op}} \subset \mathsf{Mfld}^{\mathrm{op}}$. That is, the composite*

$$C \xrightarrow{\ \mathrm{sm}\ } \mathsf{Sh}(\mathsf{Mfld}; C) \xrightarrow[\sim]{(-)|_{\mathsf{Euc}^{\mathrm{op}}}} \mathsf{Sh}(\mathsf{Euc}; C)$$

is naturally identified with the constant presheaf *functor.*

(4.3.12.2) *Every constant C-valued presheaf on* Euc *is a sheaf.*

(4.3.12.3) *The natural transformation $\alpha \colon \Gamma^{-1} \to \mathrm{sm}$ exhibits* sm *as the sheafification of Γ^{-1}. That is, there is a natural equivalence $\mathrm{sm} \simeq \Gamma^*$.*

(4.3.12.4) *The functor $\Pi_\infty \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$ is left adjoint to Γ^*.*

(4.3.12.5) *The functor $\Gamma^* \colon C \hookrightarrow \mathsf{Sh}(\mathsf{Mfld}; C)$ factors through the subcategory*

$$\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \subset \mathsf{Sh}(\mathsf{Mfld}; C)\,.$$

That is, every constant sheaf on Mfld *is $\mathbb{R}$-invariant.*

Proof. To see (4.3.12.1), note that since each Euclidean space $\mathbb{R}^n$ is contractible, the unique map $\Pi_\infty(\mathbb{R}^n) \to *$ is an equivalence. Hence the claim follows from the definition of α.

For (4.3.12.2), combine (4.3.12.1) with the fact that for each $X \in C$, the presheaf

$$\mathrm{sm}(X) \colon \mathsf{Mfld}^{\mathrm{op}} \to C$$

is a sheaf. Note that (4.3.12.3) follows from (4.3.12.1) and Lemma 3.5.3. Statement (4.3.12.4) is immediate from (4.3.12.3) and the definition of sm. Finally, (4.3.12.5) is immediate from (4.3.12.3) and Observation 4.3.10. □

Proposition 4.3.1 now follows with the facts that Γ^* is fully faithful and Γ_* is conservative when restricted to the $\mathbb{R}$-invariant sheaves (Lemma 4.2.8), combined with the following basic lemma from category theory.

4.3.13 Lemma. *Let $f^* \colon A \rightleftarrows B \colon f_*$ be an adjunction between ∞-categories, and assume that the left adjoint f^* is fully faithful. Then f^* is an equivalence if and only if f_* is conservative.*

Proof. If f^* is an equivalence, then f_* is also an equivalence, hence conservative.

On the other hand, assume that f_* is conservative. Since the left adjoint f^* is fully faithful, the unit $\mathrm{id}_A \to f_* f^*$ is an equivalence. Hence f^* is an equivalence if and only if for each object

$X \in B$, the counit $\varepsilon_X \colon f^* f_*(X) \to X$ is an equivalence. Since f_* is conservative, the counit ε_X is an equivalence if and only if

$$f_*(\varepsilon_X) \colon f_* f^* f_*(X) \to f_*(X)$$

is an equivalence. The claim now follows from the fact that the unit $\mathrm{id}_A \to f_* f^*$ is an equivalence and the triangle identity. □

Proof of Proposition 4.3.1. Since $\Gamma^* \colon C \hookrightarrow \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ is fully faithful and

$$\Gamma_* \colon \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \to C$$

is conservative (Lemma 4.2.8), we conclude by Lemma 4.3.13. □

4.3.14 Corollary. *Let C be a presentable ∞-category.*

(4.3.14.1) *The global sections functor $\mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C) \to C$ is an equivalence with inverse given by the constant presheaf functor $\Gamma^{-1} \colon C \to \mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C)$.*

(4.3.14.2) *The inclusion $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Euc}; C) \subset \mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C)$ is an equality. That is, every $\mathbb{R}$-invariant presheaf on Euc is automatically a sheaf.*

Proof. For (4.3.14.1), note that Proposition 4.3.12 shows that the the constant presheaf functor

$$\Gamma^{-1} \colon C \to \mathsf{PSh}(\mathsf{Euc}; C)$$

is fully faithful and factors through $\mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C)$. Since the global sections functor

$$\Gamma_* \colon \mathsf{PSh}_{\mathbb{R}}(\mathsf{Euc}; C) \to C$$

is conservative (Lemma 4.2.13), the claim follows from Lemma 4.3.13.

Note that (4.3.14.2) follows from (4.3.14.1) and the fact that, again by Proposition 4.3.12, the functor $\Gamma^{-1} \colon C \to \mathsf{PSh}(\mathsf{Euc}; C)$ factors through the subcategory $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Euc}; C)$. □

4.4 Consequences of Proposition 4.3.1

4.4.a The four adjoints

Proposition 4.3.1 gives us a chain of four adjoints relating $\mathsf{Sh}(\mathsf{Mfld}; C)$ and C. Let us start by giving a new name to the left adjoint $\Pi_\infty \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$ that is consistent with all of the other functors relating $\mathsf{Sh}(\mathsf{Mfld}; C)$ and C.

4.4.1 Definition. Let C be a presentable ∞-category. We write

$$\Gamma_\sharp \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$$

for the left adjoint to $\Gamma^* \colon C \to \mathsf{Sh}(\mathsf{Mfld}; C)$.

4.4.2. Combining Corollary 4.1.5 and Proposition 4.3.1, we have a chain of four adjoints

$$\mathrm{Sh}(\mathrm{Mfld};\mathcal{C}) \begin{array}{c} \xrightarrow{\Gamma_\sharp} \\ \xleftarrow{\Gamma^*} \\ \xrightarrow{\Gamma_*} \\ \xleftarrow[\Gamma^\sharp]{} \end{array} \mathcal{C}\ , \tag{4.4.3}$$

where functors lie above their right adjoints. Moreover, $\Gamma^\sharp$ and Γ^* are fully faithful.

4.4.4 Observation. The composite

$$\Gamma^*\Gamma_\sharp \colon \mathrm{Sh}(\mathrm{Mfld};\mathcal{C}) \to \mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld};\mathcal{C})$$

is left adjoint to the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld};\mathcal{C}) \subset \mathrm{Sh}(\mathrm{Mfld};\mathcal{C})$. Similarly, the composite

$$\Gamma^*\Gamma_* \colon \mathrm{Sh}(\mathrm{Mfld};\mathcal{C}) \to \mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld};\mathcal{C})$$

is right adjoint to the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld};\mathcal{C}) \subset \mathrm{Sh}(\mathrm{Mfld};\mathcal{C})$.

4.4.5 Definition (homotopification)**.** Let $\mathcal{C}$ be a presentable ∞-category. We write

$$\mathrm{L}_{\mathrm{hi}} \coloneqq \Gamma^*\Gamma_\sharp \qquad \text{and} \qquad \mathrm{R}_{\mathrm{hi}} \coloneqq \Gamma^*\Gamma_*$$

for the left and right adjoint to the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld};\mathcal{C}) \subset \mathrm{Sh}(\mathrm{Mfld};\mathcal{C})$, respectively. We call L_{hi} the *homotopification* functor.

4.4.6 Observation (formulas for L_{hi} and R_{hi})**.** Proposition 4.3.12 and Corollary 5.1.4 show that L_{hi} and R_{hi} is given by the formulas

$$\mathrm{L}_{\mathrm{hi}}(E)(M) \simeq \Gamma_\sharp(E)^{\Pi_\infty(M)} \qquad \text{and} \qquad \mathrm{R}_{\mathrm{hi}}(E)(M) \simeq E(*)^{\Pi_\infty(M)}\ .$$

Also note that we have identifications

$$\Gamma_*\,\mathrm{L}_{\mathrm{hi}} \simeq \Gamma_\sharp \qquad \text{and} \qquad \Gamma_*\,\mathrm{R}_{\mathrm{hi}} \simeq \Gamma_*\ .$$

In particular, when $\mathcal{C} = \mathrm{Spc}$, the functors L_{hi} and R_{hi} are given by the formulas

$$\mathrm{L}_{\mathrm{hi}}(E)(M) \simeq \mathrm{Map}_{\mathrm{Spc}}(\Pi_\infty(M), \Gamma_\sharp(E)) \qquad \text{and} \qquad \mathrm{R}_{\mathrm{hi}}(E)(M) \simeq \mathrm{Map}_{\mathrm{Spc}}(\Pi_\infty(M), E(*))\ .$$

4.4.7 Remark (cohesion)**.** Much of the structure of sheaves on Mfld that we are interested in for studying differential cohomology (particularly Chapter 6) only depends on the existence of the chain of four adjoints (4.4.3). In the case where $\mathcal{C} = \mathrm{Spc}$, the existence of these extra adjoints for the global sections geometric morphism (along with the condition that the extreme left adjoint $\Gamma_\sharp$ preserve finite products; see Corollary 5.1.4) is what Schreiber calls a *cohesive ∞-topos* [Sch13b, Definition 3.4.1]. The primary examples of cohesive ∞-topoi are *global spaces* [Rez14], *orbispaces* [Lur19, Chapter 3], and variants of sheaves on Mfld. Cohesive ∞-topoi are a very general setting in which one can talk about a generalized form of "differential cohomology".

Many of the ideas about cohesive ∞-topoi go back to work of Lawvere [Law94; Law05; Law07; LR03, §C.1] as well as Simpson–Teleman [ST97].

4.4.b $\mathbb{R}$-localization

We now observing that the inclusion of $\mathbb{R}$-invariant (pre)sheaves into all (pre)sheaves admits a left adjoint.

4.4.8 Observation. Notice that the full subcategory $\mathrm{PSh}_{\mathbb{R}}(\mathrm{Mfld}; C) \subset \mathrm{PSh}(\mathrm{Mfld}; C)$ is closed under both limits and colimits. Hence $\mathrm{PSh}_{\mathbb{R}}(\mathrm{Mfld}; C)$ is presentable and by the Adjoint Functor Theorem, the inclusion

$$\mathrm{PSh}_{\mathbb{R}}(\mathrm{Mfld}; C) \subset \mathrm{PSh}(\mathrm{Mfld}; C)$$

admits both a left and a right adjoint. We write $\mathrm{L}_{\mathbb{R}} \colon \mathrm{PSh}(\mathrm{Mfld}; C) \to \mathrm{PSh}_{\mathbb{R}}(\mathrm{Mfld}; C)$ for the left adjoint to the inclusion.

4.4.9 Definition ($\mathbb{R}$-localization)**.** Let C be a presentable ∞-category. We refer to the left adjoint

$$\mathrm{L}_{\mathbb{R}} \colon \mathrm{PSh}(\mathrm{Mfld}; C) \to \mathrm{PSh}_{\mathbb{R}}(\mathrm{Mfld}; C)$$

as the $\mathbb{R}$-*localization* functor.

4.4.10. Chapter 5 is dedicated to providing an explicit formula for the functor $\mathrm{L}_{\mathbb{R}}$.

It is not hard to see that the homotopification functor $\mathrm{L}_{\mathrm{hi}} \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to \mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C)$ is obtained by sheafifying $\mathrm{L}_{\mathbb{R}}$:

4.4.11 Corollary. *Let C be a presentable ∞-category. If $F \in \mathrm{PSh}(\mathrm{Mfld}; C)$ is $\mathbb{R}$-invariant, then the sheafification $\mathrm{S}_{\mathrm{Mfld}}(F)$ is $\mathbb{R}$-invariant and the unit $F \to \mathrm{S}_{\mathrm{Mfld}}(F)$ is an equivalence when restricted to $\mathrm{Euc}^{\mathrm{op}} \subset \mathrm{Mfld}^{\mathrm{op}}$.*

Proof. Since the presheaf $F|_{\mathrm{Euc}^{\mathrm{op}}}$ is $\mathbb{R}$-invariant and a sheaf (4.3.14.2), the claim follows from (3.5.3.4). □

4.4.12 Corollary. *Let C be a presentable ∞-category. Then:*

(4.4.12.1) *The composite*

$$\mathrm{S}_{\mathrm{Mfld}}\mathrm{L}_{\mathbb{R}} \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to \mathrm{Sh}(\mathrm{Mfld}; C)$$

factors through $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C)$ and is left adjoint to the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathrm{Mfld}; C) \hookrightarrow \mathrm{Sh}(\mathrm{Mfld}; C)$. That is, there is a natural equivalence

$$\mathrm{L}_{\mathrm{hi}} \simeq \mathrm{S}_{\mathrm{Mfld}}\mathrm{L}_{\mathbb{R}} .$$

(4.4.12.2) *For each $F \in \mathrm{Sh}(\mathrm{Mfld}; C)$, the natural transformation*

$$\mathrm{L}_{\mathbb{R}}(F)|_{\mathrm{Euc}^{\mathrm{op}}} \to \mathrm{L}_{\mathrm{hi}}(F)|_{\mathrm{Euc}^{\mathrm{op}}}$$

is an equivalence.

Proof. For (4.4.12.1), first note that Corollary 4.4.11 immediately implies that $\mathrm{S}_{\mathsf{Mfld}}\mathrm{L}_{\mathbb{R}}$ factors through $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$. To see that $\mathrm{S}_{\mathsf{Mfld}}\mathrm{L}_{\mathbb{R}}$ is left adjoint to the inclusion, let $F, G \in \mathsf{Sh}(\mathsf{Mfld}; C)$, and assume that G is $\mathbb{R}$-invariant. Using the fact that $\mathrm{L}_{\mathbb{R}} : \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ is left adjoint to the inclusion and that F is a sheaf, we compute

$$\begin{aligned}\mathrm{Map}_{\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld};C)}(\mathrm{S}_{\mathsf{Mfld}}\mathrm{L}_{\mathbb{R}}(F), G) &\simeq \mathrm{Map}_{\mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld};C)}(\mathrm{L}_{\mathbb{R}}(F), G) \\ &\simeq \mathrm{Map}_{\mathsf{Sh}(\mathsf{Mfld};C)}(F, G)\end{aligned}$$

Item (4.4.12.2) follows from Corollary 4.4.11 and (4.4.12.1). □

We finish this section with some remarks on the difference between $\mathrm{L}_{\mathbb{R}}$ and L_{hi} and the notations we have chosen.

4.4.13 Remark ($\mathrm{L}_{\mathbb{R}}$ vs. L_{hi})**.** For a general presentable ∞-category C and C-valued sheaf E on Mfld, the presheaf $\mathrm{L}_{\mathbb{R}}(E)$ need not be a sheaf. Hence L_{hi} is *not* given by simply restricting $\mathrm{L}_{\mathbb{R}}$ to sheaves. However, the main result of the work of Berwick-Evans–Boavida de Brito–Pavlov [BBP19] shows that when $C = \mathsf{Spc}$, the functor $\mathrm{L}_{\mathbb{R}}$ *does* preserve sheaves. That is, for each sheaf $E \in \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$, the natural morphism $\mathrm{L}_{\mathbb{R}}(E) \to \mathrm{L}_{\mathrm{hi}}(E)$ is an equivalence. The keys to their proof are the reformulation of the sheaf condition given in Theorem 3.6.1 and technical results about when geometric realizations commute with infinite products and pullbacks akin to the results in [SAG, §A.5.4]. We do not have occasion to use Berwick-Evans, Boavida de Brito, and Pavlov's result in this text.

4.4.14 Remark (notations)**.** Our notations $\mathrm{L}_{\mathbb{R}}$ and L_{hi} are chosen in analogy with the standard notations in *unstable motivic homotopy theory* [BH21, §2.2; MVW06; Mor06; Mor12; Voe98; MV99]. To explain this, let us give an overview of how motivic spaces are defined.

Let S be a scheme. We say that a presheaf F on the category Sm_S of smooth schemes of finite type over S is $\mathbb{A}^1$*-invariant* if for every $X \in \mathsf{Sm}_S$, the projection $\mathrm{pr}_X : X \times_S \mathbb{A}^1_S \to X$ induces an equivalence

$$\mathrm{pr}_X^* : F(X) \xrightarrow{\sim} F(X \times_S \mathbb{A}^1_S) .$$

Write $\mathsf{PSh}_{\mathbb{A}^1}(\mathsf{Sm}_S) \subset \mathsf{PSh}(\mathsf{Sm}_S)$ for the full subcategory spanned by the $\mathbb{A}^1$-invariant presheaves of spaces on Sm_S. The inclusion $\mathsf{PSh}_{\mathbb{A}^1}(\mathsf{Sm}_S) \subset \mathsf{PSh}(\mathsf{Sm}_S)$ admits a left adjoint, typically denoted by $\mathrm{L}_{\mathbb{A}^1}$ and called $\mathbb{A}^1$*-localization*. The ∞-category of motivic spaces over S is defined as the ∞-category

$$\mathsf{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S) \coloneqq \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S) \cap \mathsf{PSh}_{\mathbb{A}^1}(\mathsf{Sm}_S)$$

of presheaves of spaces on Sm_S that are $\mathbb{A}^1$-invariant as well as sheaves for the *Nisnevich topology* on Sm_S. The inclusion

$$\mathsf{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S) \subset \mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$$

of motivic spaces into Nisnevich sheaves on Sm_S also admits a left adjoint, typically denoted by $\mathrm{L}_{\mathrm{mot}}$ and called *motivic localization*. An important point is that the functor

$$\mathrm{L}_{\mathbb{A}^1} : \mathsf{PSh}(\mathsf{Sm}_S) \to \mathsf{PSh}_{\mathbb{A}^1}(\mathsf{Sm}_S)$$

does not carry Nisnevich sheaves to Nisnevich sheaves, so $\mathrm{L}_{\mathrm{mot}}$ is not given by simply restricting $\mathrm{L}_{\mathbb{A}^1}$ to Nisnevich sheaves.

Here, we should think Mfld as analogous to Sm_S and $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$ as analogous to $\mathsf{Sh}_{\mathrm{Nis}}(\mathsf{Sm}_S)$. In analogy with $\mathrm{L}_{\mathbb{A}^1}$, we have chosen to use the notation $\mathrm{L}_{\mathbb{R}}$ for the functor inverting $\mathbb{R}$ at the level of presheaves. Similarly, we have used letters for the sheaf version of inverting $\mathbb{R}$. The "hi" in L_{hi} stands for "homotopy invariant".

4.4.c Changing coefficients

We conclude this chapter by explaining how changing the coefficient presentable ∞-category C interacts with the four adjoints $\Gamma_\sharp \dashv \Gamma^* \dashv \Gamma_* \dashv \Gamma^\sharp$.

4.4.15 Observation. Let $f^* \colon D \rightleftarrows C : f_*$ be an adjunction between presentable ∞-categories. Since f_* preserves limits, pointwise application of f_* defines a functor

$$\mathsf{Sh}(\mathsf{Mfld}; C) \to \mathsf{Sh}(\mathsf{Mfld}; D)$$

that we also denote by f_*. The functor $f_* \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to \mathsf{Sh}(\mathsf{Mfld}; D)$ preserves limits and accessible, hence admits a left adjoint that we also denote by

$$f^* \colon \mathsf{Sh}(\mathsf{Mfld}; D) \to \mathsf{Sh}(\mathsf{Mfld}; C)\ .$$

4.4.16 Warning. Pointwise application of $f^* \colon D \to C$ need *not* preserve sheaves, hence the functor

$$f^* \colon \mathsf{Sh}(\mathsf{Mfld}; D) \to \mathsf{Sh}(\mathsf{Mfld}; C)$$

is *not* generally given by pointwise application of $f^* \colon D \to C$. The left adjoint to the functor $f_* \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to \mathsf{Sh}(\mathsf{Mfld}; D)$ is given by the composite

$$\mathsf{Sh}(\mathsf{Mfld}; D) \hookrightarrow \mathsf{PSh}(\mathsf{Mfld}; D) \xrightarrow{f^* \circ -} \mathsf{PSh}(\mathsf{Mfld}; C) \xrightarrow{\mathrm{S}_{\mathsf{Mfld}}} \mathsf{Sh}(\mathsf{Mfld}; C)$$

of pointwise application of $f^* \colon D \to C$ with sheafification.

The tensor product of presentable ∞-categories (see §A.1) gives us an alternative description of these functors:

4.4.17 Observation. Let $f^* \colon D \rightleftarrows C : f_*$ be an adjunction between presentable ∞-categories. The equivalences

$$\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc}) \otimes C \xrightarrow{\sim} \mathsf{Sh}(\mathsf{Mfld}; C) \qquad \text{and} \qquad \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc}) \otimes D \xrightarrow{\sim} \mathsf{Sh}(\mathsf{Mfld}; D)$$

of Example A.1.3 fit into canonically commutative squares

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes C \arrow[r, "\sim"] \arrow[d, "\mathrm{id} \otimes f_*"'] & \mathsf{Sh}(\mathsf{Mfld};C) \arrow[d, "f_*"] \\ \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes D \arrow[r, "\sim"] & \mathsf{Sh}(\mathsf{Mfld};D) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes D \arrow[r, "\sim"] \arrow[d, "\mathrm{id} \otimes f^*"'] & \mathsf{Sh}(\mathsf{Mfld};D) \arrow[d, "f^*"] \\ \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes C \arrow[r, "\sim"] & \mathsf{Sh}(\mathsf{Mfld};C)\,. \end{tikzcd}$$

4.4.18 Observation. Let C be a presentable ∞-category. Under the identifications

$$\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \otimes C \simeq \mathsf{Sh}(\mathsf{Mfld};C) \qquad \text{and} \qquad \mathsf{Spc} \otimes C \simeq C\,,$$

the chain of four adjoints

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};C) \arrow[r, shift left=3, "\Gamma_\sharp"] \arrow[r, shift left=1, hook', "\Gamma_*"'] & C \arrow[l, shift left=1, hook', "\Gamma^*"'] \arrow[l, shift left=3, hook, "\Gamma^\sharp"] \end{tikzcd}$$

is given by applying the tensor product of presentable ∞-categories $(-) \otimes C$ to the chain of adjoints

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \arrow[r, shift left=3, "\Gamma_\sharp"] \arrow[r, shift left=1, "\Gamma_*"'] & \mathsf{Spc}\,. \arrow[l, shift left=1, hook', "\Gamma^*"'] \arrow[l, shift left=3, hook, "\Gamma^\sharp"] \end{tikzcd}$$

The general yoga of tensor products of presentable ∞-categories implies that changing coefficients is compatible with the four operations $\Gamma_\sharp \dashv \Gamma^* \dashv \Gamma_* \dashv \Gamma^\sharp$.

4.4.19 Proposition. *Let $f^* \colon D \rightleftarrows C \colon f_*$ be an adjunction between presentable ∞-categories. Then the squares*

$$\begin{tikzcd} C \arrow[r, hook, "\Gamma^\sharp"] \arrow[d, "f_*"'] & \mathsf{Sh}(\mathsf{Mfld};C) \arrow[d, "f_*"] \\ D \arrow[r, hook, "\Gamma^\sharp"'] & \mathsf{Sh}(\mathsf{Mfld};D)\,, \end{tikzcd} \quad \begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};C) \arrow[r, "\Gamma_*"] \arrow[d, "f_*"'] & C \arrow[d, "f_*"] \\ \mathsf{Sh}(\mathsf{Mfld};D) \arrow[r, "\Gamma_*"'] & D\,, \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C \arrow[r, hook, "\Gamma^*"] \arrow[d, "f_*"'] & \mathsf{Sh}(\mathsf{Mfld};C) \arrow[d, "f_*"] \\ D \arrow[r, hook, "\Gamma^*"'] & \mathsf{Sh}(\mathsf{Mfld};D) \end{tikzcd}$$

canonically commute. Equivalently, the squares

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};D) \arrow[r, "\Gamma_*"] \arrow[d, "f^*"'] & D \arrow[d, "f^*"] \\ \mathsf{Sh}(\mathsf{Mfld};C) \arrow[r, "\Gamma_*"'] & C\,, \end{tikzcd} \quad \begin{tikzcd} D \arrow[r, hook, "\Gamma^*"] \arrow[d, "f^*"'] & \mathsf{Sh}(\mathsf{Mfld};D) \arrow[d, "f^*"] \\ C \arrow[r, hook, "\Gamma^*"'] & \mathsf{Sh}(\mathsf{Mfld};C)\,, \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};D) \arrow[r, "\Gamma_\sharp"] \arrow[d, "f^*"'] & D \arrow[d, "f^*"] \\ \mathsf{Sh}(\mathsf{Mfld};C) \arrow[r, "\Gamma_\sharp"'] & C \end{tikzcd}$$

canonically commute.

Proof. The claim for the first collection of three squares follows from the tensor product description of Observations 4.4.17 and 4.4.18 combined with [Hai21, Observation 1.15]. □

There are a few specific instances of Proposition 4.4.19 that we are particularly interested in. Most of them concern the relationship between stable and unstable coefficients.

4.4.20 Notation. We write

$$\Sigma^\infty_+ \colon \mathrm{Spc} \to \mathrm{Spt}$$

for the functor sending a space X to the suspension spectrum of the space $X_+ \coloneqq X \sqcup *$ obtained by adding a disjoint basepoint to X. We write

$$\Omega^\infty \colon \mathrm{Spt} \to \mathrm{Spc}$$

for the right adjoint to Σ^∞_+. Given an $\mathbb{E}_1$-ring spectrum R, we write $\Omega^\infty_R \colon \mathrm{Mod}(R) \to \mathrm{Spc}$ for the composite

$$\mathrm{Mod}(R) \longrightarrow \mathrm{Spt} \xrightarrow{\Omega^\infty} \mathrm{Spc}$$

of the forgetful functor with Ω^∞

4.4.21 Example. The square

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Mfld};\mathrm{Spc}) \arrow[r, "\Gamma_\sharp"] \arrow[d, "\Sigma^\infty_+"'] & \mathrm{Spc} \arrow[d, "\Sigma^\infty_+"] \\
\mathrm{Sh}(\mathrm{Mfld};\mathrm{Spt}) \arrow[r, "\Gamma_\sharp"'] & \mathrm{Spt}
\end{tikzcd}$$

canonically commutes.

4.4.22 Example. Let R be an $\mathbb{E}_1$-ring spectrum. Then the squares

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Mfld};\mathrm{Mod}(R)) \arrow[r, "\Gamma_*"] \arrow[d, "\Omega^\infty_R"'] & \mathrm{Mod}(R) \arrow[d, "\Omega^\infty_R"] \\
\mathrm{Sh}(\mathrm{Mfld};\mathrm{Spc}) \arrow[r, "\Gamma_*"'] & \mathrm{Spc}\,,
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Mod}(R) \arrow[r, hook, "\Gamma^*"] \arrow[d, "\Omega^\infty_R"'] & \mathrm{Sh}(\mathrm{Mfld};\mathrm{Mod}(R)) \arrow[d, "\Omega^\infty_R"] \\
\mathrm{Spc} \arrow[r, hook, "\Gamma^*"'] & \mathrm{Sh}(\mathrm{Mfld};\mathrm{Spc})\,,
\end{tikzcd}$$

canonically commute.

5 $\mathbb{R}$-localization

by Peter Haine

The purpose of this chapter is to provide formulas for the $\mathbb{R}$-localization functor

$$\mathrm{L}_{\mathbb{R}} \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C)$$

(Definition 4.4.5) and left adjoint $\Gamma_\sharp \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$ to the constant sheaf functor (4.4.2). Specifically, write

$$\Delta^n_{\mathrm{alg}} \coloneqq \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1 \} \subset \mathbb{R}^{n+1}$$

for the algebraic n-simplex; the assignment $[n] \mapsto \Delta^n_{\mathrm{alg}}$ defines a cosimplicial manifold. We show that $\mathrm{L}_{\mathbb{R}}$ and $\Gamma_\sharp$ are computed by the geometric realizations

$$\mathrm{L}_{\mathbb{R}}(F)(M) \simeq |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| \qquad \text{and} \qquad \Gamma_\sharp(E) \simeq |E(\Delta^{\bullet}_{\mathrm{alg}})|$$

(Proposition 5.1.2 and Corollary 5.1.4).

In § 5.1, we give a precise statement of the main result of this section (Proposition 5.1.2), but do not prove it. We then explain some consequences of these formulas (§ 5.1.b). In § 5.2, we recall some background on simplicial homotopies in ∞-categories that we need to prove the formula for $\mathrm{L}_{\mathbb{R}}$. Section 5.3 is dedicated to proving this formula.

5.1 The Morel–Suslin–Voevodsky construction

5.1.a The construction

5.1.1 Notation (algebraic simplices)**.** Let $n \geq 0$ be an integer. Write Δ^n_{alg} for the hyperplane in $\mathbb{R}^{n+1}$ defined by

$$\Delta^n_{\mathrm{alg}} \coloneqq \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1 \} \subset \mathbb{R}^{n+1} ,$$

so that as a smooth manifold Δ^n_{alg} is diffeomorphic to $\mathbb{R}^n$. We call Δ^n_{alg} the *algebraic n-simplex.*

In the usual way, the algebraic n-simplices for $n \geq 0$ assemble into a cosimplicial manifold

$$\Delta^{\bullet}_{\mathrm{alg}} \colon \mathbf{\Delta} \to \mathsf{Mfld} .$$

5.1.2 Proposition (Morel–Suslin–Voevodsky construction)**.** *Let C be a presentable ∞-category. The left adjoint*

$$\mathrm{L}_{\mathbb{R}} \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C)$$

is given by the geometric realization

$$\mathrm{L}_{\mathbb{R}}(F)(M) \simeq |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| .$$

5.1.3 Remark. We call the construction $F \mapsto |F(- \times \Delta^{\bullet}_{\mathrm{alg}})|$ the *Morel–Suslin–Voevodsky* construction. Morel and Voevodsky provide a very general version of the Morel–Suslin–Voevodsky construction for "sites with an interval object" [MV99, §2.3], which covers the site Mfld with $\mathbb{R}$ as the interval object (see also [AE17, §4.3; AHW17, §4]). They attribute this argument to Suslin.

Their arguments are model category-theoretic and apply to a more specific coefficient ∞-categories C than we're interested in. Hence we provide separate argument. See the work of Bunk [Bun22a, Proposition 2.5] for another model category-theoretic argument.

So as to not take us too far afield, we settle for working with the site of manifolds rather than a general site with an interval object. Our proof of Proposition 5.1.2 takes the approach used in Brazelton's notes on motivic homotopy theory [Bra18, §3].

5.1.b Consequences of the Morel–Suslin–Voevodsky construction

We defer the proof of Proposition 5.1.2 to §§ 5.2 and 5.3 and first explain the consequences of Proposition 5.1.2 for the functors $\Gamma_\sharp$ and L_{hi}.

5.1.4 Corollary. *Let C be a presentable ∞-category and $F \in \mathsf{Sh}(\mathsf{Mfld}; C)$. Then:*

(5.1.4.1) *For each $n \geq 0$ we have*
$$\mathrm{L}_{\mathrm{hi}}(F)(\mathbb{R}^n) \simeq |E(\Delta^\bullet_{\mathrm{alg}} \times \mathbb{R}^n)| \,.$$

(5.1.4.2) *The left adjoint $\Gamma_\sharp \colon \mathsf{Sh}(\mathsf{Mfld}; C) \to C$ to the constant sheaf functor is given by*
$$\Gamma_\sharp(E) \simeq |E(\Delta^\bullet_{\mathrm{alg}})| \,.$$

Proof. For (5.1.4.1), note that by Corollary 4.4.12, the natural map
$$\mathrm{L}_{\mathbb{R}}(E)|_{\mathsf{Euc}^{\mathrm{op}}} \to \mathrm{L}_{\mathrm{hi}}(E)|_{\mathsf{Euc}^{\mathrm{op}}}$$
is an equivalence. Hence the claim is an immediate consequence of Proposition 5.1.2. Item (5.1.4.2) follows from (5.1.4.1) and the identification $\Gamma_\sharp \simeq \Gamma_* \mathrm{L}_{\mathrm{hi}}$ (Observation 4.4.6). □

5.1.5. Since $\mathrm{L}_{\mathrm{hi}} \simeq \Gamma^*\Gamma_\sharp$, Proposition 4.3.12 and Corollary 5.1.4 show that L_{hi} is given by the formula
$$\mathrm{L}_{\mathrm{hi}}(E)(M) \simeq |E(\Delta^\bullet_{\mathrm{alg}})|^{\Pi_\infty(M)} \,.$$
In particular, when $C = \mathsf{Spc}$, the functor L_{hi} is given by the formula
$$\mathrm{L}_{\mathrm{hi}}(E)(M) \simeq \mathrm{Map}_{\mathsf{Spc}}(\Pi_\infty(M), |E(\Delta^\bullet_{\mathrm{alg}})|) \,.$$

Corollary 5.1.4 also reproves the formula for the underlying homotopy type of a manifold using smooth simplices:

5.1.6 Corollary. *Let M be a manifold. There is a natural equivalence*
$$\Pi_\infty(M) \simeq |\,\mathrm{Map}_{\mathsf{Mfld}}(\Delta^\bullet_{\mathrm{alg}}, M)|$$

in the ∞-category Spc.

Proof. Recall that we write $よ\colon \mathsf{Mfld} \hookrightarrow \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$ for the Yoneda embedding (Example 3.1.7). By definition, $\Gamma_\sharp\colon \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{Spc}$ is the left Kan extension of $\Pi_\infty\colon \mathsf{Mfld} \to \mathsf{Spc}$. Hence we compute

$$\begin{aligned}\Pi_\infty(M) &\simeq \Gamma_\sharp(よ(M)) \\ &\simeq |よ(M)(\Delta^\bullet_{\mathrm{alg}})| && \text{(Corollary 5.1.4)} \\ &= |\operatorname{Map}_{\mathsf{Mfld}}(\Delta^\bullet_{\mathrm{alg}}, M)|\ . \end{aligned}$$ □

The formula for $\Gamma_\sharp$ also proves:

5.1.7 Corollary. *Let C be a presentable ∞-category. If geometric realizations commute with finite products in C (e.g., C is an ∞-topos or stable), then the functor*

$$\Gamma_\sharp\colon \mathsf{Sh}(\mathsf{Mfld};C) \to C$$

preserves finite products.

5.1.8 Remark. The functors $\mathrm{L}_{\mathbb{R}}$, L_{hi}, and $\Gamma_\sharp$ do not generally commute with finite limits. However, general category theory [Hoy17, Proposition 3.4] shows that the functor

$$\mathrm{L}_{\mathbb{R}}\colon \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld};\mathsf{Spc})$$

is *locally cartesian*: for any cospan $E \to G \leftarrow F$ with $E, G \in \mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld};\mathsf{Spc})$, the natural morphism

$$\mathrm{L}_{\mathbb{R}}(E \times_G F) \to E \times_G \mathrm{L}_{\mathbb{R}}(F)$$

is an equivalence. Since the sheafification functor $\mathrm{S}_{\mathsf{Mfld}}\colon \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$ is left exact, Corollary 4.4.12 shows that L_{hi} and $\Gamma_\sharp$ are locally cartesian as well.

5.2 Background on simplicial homotopies in ∞-categories

In order to prove the Morel–Suslin–Voevodsky formula (Proposition 5.1.2), we need to use homotopies of simplicial objects in an arbitrary ∞-category. Since we're working natively to ∞-categories and not in simplicial sets or simplicial presheaves, doing so requires a reformulation of the usual definition of a simplicial homotopy.

5.2.a Motivation from simplicial sets

Recall that a *simplicial homotopy* between morphisms of simplicial sets $f_0, f_1\colon X_\bullet \to Y_\bullet$ consists of a morphism $h\colon X_\bullet \times \Delta^1 \to Y_\bullet$ along with identifications of the restriction of h to $X_\bullet \times \{0\}$ with f_0 and the restriction of h to $X_\bullet \times \{1\}$ with f_1. First we reformulate this notion in terms of morphisms in the overcategory $\mathsf{sSet}_{/\Delta^1}$.

5.2.1 Notation. Write $u^* \colon \mathrm{sSet} \to \mathrm{sSet}_{/\Delta^1}$ for the functor $X_\bullet \mapsto X_\bullet \times \Delta^1$. Note that u^* is right adjoint to the forgetful functor $u_! \colon \mathrm{sSet}_{/\Delta^1} \to \mathrm{sSet}$.

5.2.2 Lemma. *Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. There is a natural bijection*

$$\mathrm{Map}_{\mathrm{sSet}}(X_\bullet \times \Delta^1, Y_\bullet) \cong \mathrm{Map}_{\mathrm{sSet}_{/\Delta^1}}(u^*(X_\bullet), u^*(Y_\bullet)) .$$

Proof. Since $u_!$ is left adjoint to u^*, we have natural bijections

$$\begin{aligned} \mathrm{Map}_{\mathrm{sSet}_{/\Delta^1}}(u^*(X_\bullet), u^*(Y_\bullet)) &\cong \mathrm{Map}_{\mathrm{sSet}}(u_! u^*(X_\bullet), Y_\bullet) \\ &= \mathrm{Map}_{\mathrm{sSet}}(X_\bullet \times \Delta^1, Y_\bullet) . \end{aligned}$$ □

In order to use Lemma 5.2.2 to generalize simplicial homotopies to arbitrary ∞-categories, notice that the functor u^* admits an alternative interpretation that makes sense for simplicial objects in any ∞-category.

5.2.3 Observation (presheaf categories and slice categories)**.** Let S be a small category and $s \in S$. Write $よ \colon S \hookrightarrow \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set})$ for the Yoneda embedding. The colimit-preserving extension of the "sliced Yoneda embedding"

$$\begin{aligned} S_{/s} &\hookrightarrow \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set})_{/よ(s)} \\ [s' \to s] &\mapsto [よ(s') \to よ(s)] \end{aligned}$$

defines an equivalence of categories

$$\mathrm{Fun}((S_{/s})^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\sim} \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set})_{/よ(s)} .$$

Under this identification, the functor $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}((S_{/s})^{\mathrm{op}}, \mathrm{Set})$ given by precomposition with the forgetful functor $(S_{/s})^{\mathrm{op}} \to S^{\mathrm{op}}$ is identified with the functor

$$よ(s) \times (-) \colon \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set})_{/よ(s)} .$$

Moreover, the functor $よ(s) \times (-)$ is right adjoint to the forgetful functor

$$\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set})_{/よ(s)} \to \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Set}) .$$

5.2.4. Specializing to the case $S = \mathbf{\Delta}$ and $s = [1]$ shows that the functor $u^* \colon \mathrm{sSet} \to \mathrm{sSet}_{/\Delta^1}$ is identified with the functor

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, \mathrm{Set})$$

given by precomposition with the forgetful functor $(\mathbf{\Delta}_{/[1]})^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$. We also write

$$u^* \colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, \mathrm{Set})$$

for this functor.

Thus, we have a further reformulation of what a simplicial homotopy is:

5.2.5 Corollary. *Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. There is a natural bijection*

$$\mathrm{Map}_{\mathrm{sSet}}(X_\bullet \times \Delta^1, Y_\bullet) \cong \mathrm{Map}_{\mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}},\mathrm{Set})}(u^*(X_\bullet), u^*(Y_\bullet)) .$$

The benefit of Corollary 5.2.5 is that the right-hand side makes sense in *any* ∞-category.

5.2.6 Notation. Write $u\colon (\mathbf{\Delta}_{/[1]})^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ for the forgetful functor. For $i \in [1]$, write

$$j_i\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}_{/[1]})^{\mathrm{op}}$$

for the fully faithful functor given on objects by the assignment

$$[n] \mapsto \big[[n] \to \{i\} \hookrightarrow [1]\big] ,$$

with the obvious assignment on morphisms. Given an ∞-category D, write

$$u^*\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D) \to \mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, D) \qquad \text{and} \qquad j_i^*\colon \mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, D) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D)$$

for the functors given by precomposition with u and j_i, respectively.

5.2.7 Observation. For each $i \in [1]$, the fully faithful functor $j_i\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}_{/[1]})^{\mathrm{op}}$ is a section of the forgetful functor $u\colon (\mathbf{\Delta}_{/[1]})^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$. That is $uj_0 = uj_1 = \mathrm{id}$.

5.2.8 Definition (simplicial homotopy [HA, Definition 7.2.1.6])**.** Let D be an ∞-category and let

$$f_0, f_1\colon X_\bullet \to Y_\bullet$$

be morphisms in the ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D)$ of simplicial objects in D. A *simplicial homotopy* from f_0 to f_1 consists of the following data:

(5.2.8.1) A morphism $h\colon u^*(X_\bullet) \to u^*(Y_\bullet)$ in $\mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, D)$.

(5.2.8.2) Equivalences $j_0^*(h) \simeq f_0$ and $j_1^*(h) \simeq f_1$ of morphisms $X_\bullet \to Y_\bullet$ in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D)$.

We often write $h\colon u^*(X_\bullet) \to u^*(Y_\bullet)$ for the entire data of a simplicial homotopy from f_0 to f_1.

5.2.b Cofinality & sifted ∞-categories

The fact that we need about simplicial homotopies is that if $h\colon u^*(X_\bullet) \to u^*(Y_\bullet)$ is a simplicial homotopy from f_0 to f_1, then f_0 and f_1 induce the same map $|X_\bullet| \to |Y_\bullet|$ on geometric realizations. To prove this, we need to use some special properties of category $\mathbf{\Delta}^{\mathrm{op}}$, namely that it is *sifted*. The purpose of this subsection is to review the theory of sifted ∞-categories.

5.2.9 Definition. Let $p : I' \to I$ be a functor between ∞-categories. We say that p is *colimit-cofinal* if for every ∞-category D and diagram $F : I \to D$, the following conditions hold:

(5.2.9.1) The colimit $\operatorname{colim}_I F$ exists if and only if the colimit $\operatorname{colim}_{I'} Fp$ exists.

(5.2.9.2) If the colimit $\operatorname{colim}_I F$ exists, then the natural map

$$\operatorname{colim}_I F \to \operatorname{colim}_{I'} Fp$$

is an equivalence.

5.2.10 Definition. An ∞-category I is *sifted* if I is nonempty and the diagonal functor $I \to I \times I$ is colimit-cofinal.

That is, siftedness means that we can compute colimits indexed over $I \times I$ as colimits indexed over the diagonal copy of I. There are two key examples of sifted ∞-categories:

5.2.11 Example. Every filtered ∞-category is sifted [HTT, Example 5.5.8.3]. The category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted [HTT, Lemma 5.5.8.4]

It is useful to have an explicit condition to verify in order to see that an ∞-category is sifted. Quillen's Theorem A [HTT, Theorem 4.1.3.1] implies the following reformulations of siftedness. For these, recall that an ∞-category C is *weakly contractible* if the classifying space of C is contractible.

5.2.12 Lemma. *Let I be a nonempty ∞-category. The following conditions are equivalent:*

(5.2.12.1) *The ∞-category I is sifted.*

(5.2.12.2) *For all $i, i' \in I$, the ∞-category $I_{i/} \times_I I_{i'/}$ is weakly contractible.*

(5.2.12.3) *For each $i \in I$, the forgetful functor $I_{i/} \to I$ is colimit-cofinal.*

5.2.13 Example. Since the category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, the forgetful functor

$$u : (\mathbf{\Delta}_{/[1]})^{\mathrm{op}} = (\mathbf{\Delta}^{\mathrm{op}})_{[1]/} \to \mathbf{\Delta}^{\mathrm{op}}$$

is colimit-cofinal.

5.2.c Realizations of simplicial homotopies

Finally, we prove that simplicially homotopic maps induce equivalent maps on geometric realizations. We begin with an observation.

5.2.14 Observation. Let D be an ∞-category that admits geometric realizations of simplicial objects and let $X_\bullet$ be a simplicial object in D. For $i \in [1]$, the induced maps

$$\alpha_i : \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet = \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} (u j_i)^*(X_\bullet) \longrightarrow \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^*(X_\bullet)$$

are equivalences and are homotopic. Specifically, they are both inverses of the equivalence

$$\operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^*(X_\bullet) \xrightarrow{\sim} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet$$

provided by the fact that u is colimit-cofinal (Example 5.2.13).

5.2.15 Lemma. *Let D be an ∞-category that admits geometric realizations of simplicial objects. Let $f_0, f_1 \colon X_\bullet \to Y_\bullet$ be morphisms of simplicial objects in D and let h be a simplicial homotopy from f_0 to f_1. Then the simplicial homotopy h induces an equivalence $|f_0| \simeq |f_1|$ between the induced morphisms*

$$|f_0|, |f_1| \colon |X_\bullet| \to |Y_\bullet|$$

on geometric realizations.

Proof. In light of Observation 5.2.14, for $i \in [1]$ we have a commutative square

$$\begin{tikzcd} {|X_\bullet|} \arrow[r, "\alpha_i", "\sim"'] \arrow[d, "{|j_i^*(h)|}"'] & \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^*(X_\bullet) \arrow[d, "\operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} h"] \\ {|Y_\bullet|} \arrow[r, "\sim", "\alpha_i"'] & \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^*(Y_\bullet) \,. \end{tikzcd}$$

Thus the equivalences

$$j_0^*(h) \simeq f_0 \qquad \text{and} \qquad j_1^*(h) \simeq f_1$$

provided by the simplicial homotopy h induce equivalences

$$|f_0| \simeq |j_0^*(h)| \simeq \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} h \simeq |j_1^*(h)| \simeq |f_1|$$

in the arrow ∞-category of D. □

5.3 Proof of the Morel–Suslin–Voevodsky formula

We prove Proposition 5.1.2 by applying the following recognition principle for localization functors.

5.3.1 Proposition [HTT, Proposition 5.2.7.4]**.** *Let C be an ∞-category and $L \colon D \to D$ a functor with essential image $LD \subset D$. Then the following are equivalent:*

(5.3.1.1) *There exists a functor $F \colon D \to D'$ with fully faithful right adjoint $G \colon D' \hookrightarrow D$ such that $GF \simeq L$.*

(5.3.1.2) *The functor $L \colon D \to LD$ is left adjoint to the inclusion $LD \hookrightarrow D$.*

(5.3.1.3) *There is a natural transformation* $\eta\colon \mathrm{id}_D \to L$ *such that for all* $d \in D$*, the morphisms*

$$\eta_{L(d)}, L(\eta_d)\colon L(d) \to L(L(d))$$

are equivalences.

5.3.2 Notation. Let us temporarily write $\mathrm{H}\colon \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}(\mathsf{Mfld}; C)$ for the Morel–Suslin–Voevodsky construction

$$\mathrm{H}(F)(M) := |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| \,.$$

5.3.3 Construction. Let C be a presentable ∞-category. Define a natural transformation

$$\eta\colon \mathrm{id}_{\mathsf{PSh}(\mathsf{Mfld};C)} \to \mathrm{H}$$

as follows. Let M be a manifold, and also simply write M for the constant cosimplicial manifold at M. Projection onto the first factor defines a morphism of cosimplicial manifolds

$$\mathrm{pr}_M\colon M \times \Delta^{\bullet}_{\mathrm{alg}} \to M$$

from the product cosimplicial manifold $M \times \Delta^{\bullet}_{\mathrm{alg}}$ to the constant cosimplicial manifold at M. For each C-valued presheaf $F \in \mathsf{PSh}(\mathsf{Mfld}; C)$, the morphism $\eta_F\colon F \to \mathrm{H}(F)$ is defined as the geometric realization

$$\eta_F(M) := |\,\mathrm{pr}_M^*\,|\colon F(M) \simeq |F(M)| \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| = \mathrm{H}(F)(M)\,.$$

Equivalently, the morphism $\eta_F(M)$ is the composite

$$F(M) \simeq F(M \times \Delta^{0}_{\mathrm{alg}}) \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})|$$

of the equivalence $F(M) \simeq F(M \times \Delta^{0}_{\mathrm{alg}})$ induced by the projection $M \times \Delta^{0}_{\mathrm{alg}} \simeq M$ with the induced map

$$F(M \times \Delta^{0}_{\mathrm{alg}}) \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})|$$

from the 0-simplices of the simplicial object $F(M \times \Delta^{\bullet}_{\mathrm{alg}})$ to its geometric realization.

5.3.a Proof of $\mathbb{R}$-invariance

In order to apply Proposition 5.3.1, the we first check:

5.3.4 Lemma. *Let* C *be a presentable* ∞*-category. For any presheaf* $F\colon \mathsf{Mfld}^{\mathrm{op}} \to C$*, the presheaf* $\mathrm{H}(F)$ *is* $\mathbb{R}$*-invariant.*

To prove Lemma 5.3.4, we apply the technology of simplicial homotopies.

5.3.5 Lemma. *Let M be a manifold. There is a natural simplicial homotopy in $\mathsf{Mfld}^{\mathrm{op}}$ from the map*

$$i_{M\times\Delta^\bullet_{\mathrm{alg}},0} \circ \mathrm{pr}_{M\times\Delta^\bullet_{\mathrm{alg}}} : M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R} \to M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R}$$

to the identity.

Proof. Define a simplicial homotopy

$$h : u^*(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R}) \to u^*(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R})$$

as follows. For each map $\sigma : [n] \to [1]$ in $\mathbf{\Delta}$, write $h'_\sigma : \Delta^n_{\mathrm{alg}} \times \mathbb{R} \to \Delta^n_{\mathrm{alg}} \times \mathbb{R}$ for the smooth map defined by the formula

$$h'_\sigma(t_0, \dots, t_n, x) \coloneqq \left(t_0, \dots, t_n, x \textstyle\sum_{k\in\sigma^{-1}(1)} t_k\right) .$$

Define $h_\sigma : M \times \Delta^n_{\mathrm{alg}} \times \mathbb{R} \to M \times \Delta^n_{\mathrm{alg}} \times \mathbb{R}$ by setting $h_\sigma \coloneqq \mathrm{id}_M \times h'_\sigma$. It is immediate from the definitions that h defines a simplicial homotopy

$$u^*(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R}) \to u^*(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbb{R}) ,$$

and, moreover,

$$j_0^*(h) = i_{M\times\Delta^\bullet_{\mathrm{alg}},0} \circ \mathrm{pr}_{M\times\Delta^\bullet_{\mathrm{alg}}} \qquad \text{and} \qquad j_1^*(h) = \mathrm{id}_{M\times\Delta^\bullet_{\mathrm{alg}}\times\mathbb{R}} . \qquad \square$$

Proof of Lemma 5.3.4. Let M be a manifold. Since $\mathrm{pr}_M\, i_{M,0} = \mathrm{id}_M$, to see that

$$\mathrm{pr}_M^* : \mathrm{H}(F)(M) \to \mathrm{H}(F)(M \times \mathbb{R})$$

is an equivalence, it suffices to show that $\mathrm{pr}_M^*\, i_{M,0}^* \simeq \mathrm{id}_{\mathrm{H}(F)(M\times\mathbb{R})}$. This follows from combining Lemmas 5.2.15 and 5.3.5. $\square$

5.3.b Proof that the unit is an equivalence

The second thing to check is that for every presheaf G, the morphism $\eta_{\mathrm{H}(G)}$ is an equivalence. Combined with Lemma 5.3.4 this guarantees that the essential image of the functor

$$\mathrm{H} : \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}(\mathsf{Mfld}; C)$$

is $\mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C)$.

5.3.6 Lemma. *Let C be a presentable ∞-category. If $F : \mathsf{Mfld}^{\mathrm{op}} \to C$ is $\mathbb{R}$-invariant, then the map $\eta_F : F \to \mathrm{H}(F)$ is an equivalence.*

Proof. Let M be a manifold. Since F is $\mathbb{R}$-invariant and $\Delta^n_{\mathrm{alg}} \cong \mathbb{R}^n$ for each $n \geq 0$, the projection $\mathrm{pr}_M : M \times \Delta^\bullet_{\mathrm{alg}} \to M$ from the cosimplicial manifold $M \times \Delta^\bullet_{\mathrm{alg}}$ to the constant cosimplicial manifold at M induces an equivalence

$$\mathrm{pr}_M^* \colon F(M) \xrightarrow{\sim} F(M \times \Delta^\bullet_{\mathrm{alg}})$$

of simplicial objects in C. The claim now follows by passing to geometric realizations. □

5.3.7 Corollary. *Let C be a presentable ∞-category. The essential image of the functor*

$$\mathrm{H} \colon \mathsf{PSh}(\mathsf{Mfld}; C) \to \mathsf{PSh}(\mathsf{Mfld}; C)$$

is $\mathsf{PSh}_{\mathbb{R}}(\mathsf{Mfld}; C)$.

Now we complete the proof of Proposition 5.1.2 by showing that $\mathrm{H}(\eta_F)$ is an equivalence.

5.3.8 Lemma. *Let C be a presentable ∞-category. For all $F \in \mathsf{PSh}(\mathsf{Mfld}; C)$, the maps*

$$\eta_{\mathrm{H}(F)}, \mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$$

are equivalences.

Proof. By Lemma 5.3.6 and Corollary 5.3.7, the morphism $\eta_{\mathrm{H}(F)}$ is an equivalence. To see that $\mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$ is an equivalence, note that for each manifold M we have

$$\mathrm{H}(F)(M) = \operatorname*{colim}_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}})$$

and

$$\begin{aligned} \mathrm{H}(\mathrm{H}(F))(M) &= \operatorname*{colim}_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}} \times \Delta^n_{\mathrm{alg}}) \\ &\simeq \operatorname*{colim}_{([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}} \times \Delta^n_{\mathrm{alg}}) \,. \end{aligned}$$

Moreover, the map $\mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$ is induced by restriction of diagrams along the fully faithful functor

$$\begin{aligned} \mathbf{\Delta}^{\mathrm{op}} &\hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \\ [m] &\mapsto ([m], [0]) \,. \end{aligned}$$

First taking the colimit over the variable $[m] \in \mathbf{\Delta}^{\mathrm{op}}$, we see that the map $\mathrm{H}(\eta_F)(M)$ is induced by the map from the 0-simplices $\mathrm{H}(F)(M)$ of the simplicial object $\mathrm{H}(F)(M \times \Delta^\bullet_{\mathrm{alg}})$ to its geometric realization. Since $\mathrm{H}(F)$ is $\mathbb{R}$-invariant (Lemma 5.3.4), the simplicial object $\mathrm{H}(F)(M \times \Delta^\bullet_{\mathrm{alg}})$ is equivalent to the constant simplicial object at $\mathrm{H}(F)(M)$, hence the induced map

$$\mathrm{H}(F)(M) \to \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{H}(F)(M \times \Delta^n_{\mathrm{alg}})$$

from the 0-simplices is an equivalence. □

Proof of Proposition 5.1.2. Combine Corollary 5.3.7, Lemma 5.3.8, and Proposition 5.3.1. □

6 Structures in the stable case

by Peter Haine

In ordinary differential cohomology, we had the Simons–Sullivan “differential cohomology hexagon”

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{H}^{*-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\mathrm{Bock}"] \arrow[dr] & & \mathrm{H}^{*}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
\mathrm{H}^{*-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^{*}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & & \mathrm{H}^{*}_{\mathrm{dR}}(M) \\
& \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \arrow[rr, "\mathrm{d}"'] \arrow[ur] & & \Omega^{*}_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0\,,
\end{tikzcd}$$

which actually characterized ordinary differential cohomology (Theorem 2.3.2). We want to be able to reproduce an analogue of the differential cohomology hexagon for *any* sheaf of spectra on Mfld. To do this, we need to identify how cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{R}$ as well as $\Omega^{*-1}(M)/\Omega^{*-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}$ and $\Omega^{*}_{\mathrm{cl}}(M)_{\mathbb{Z}}$ fit into the story.

One general machine for producing diagrams aesthetically similar to the differential cohomology hexagon is the theory of *recollements*, or ways of “gluing” a category together out of two pieces. It turns out that the differential cohomology hexagon falls exactly into this framework: one of the subcategories that we build $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spt})$ from is the subcategory $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld};C)$ of $\mathbb{R}$-invariant sheaves, and the other piece is the subcategory of sheaves with vanishing global sections. Since this whole story is a special case of the theory of recollements, the first half of the section (§6.1) gives a quick introduction to the theory of recollements and the key results. In §6.2, we apply this general machinery to sheaves on manifolds to obtain the a version of differential cohomology hexagon for any sheaf of spectra on Mfld (see (6.2.14)). We finish the section by making precise what it means for a sheaf of spectra on Mfld to “refine” a cohomology theory.

6.1 Background on recollements

Recollements[2] were introduced by Grothendieck and Verdier in the context of topoi [SGA 4_{I}, Exposé IV, §9] and by Beĭlinson–Bernstein–Deligne in the context of triangulated categories [BBD82, §1.4] to “glue” together sheaves over open-closed decompositions of a space. However, there are many other situations in which an ∞-category can be “glued together” from two subcategories that are in some sense complementary. For example, if R is a ring and $I \subset R$ is a finitely generated ideal, then the derived ∞-category of R can be clued together from its

[2] Roughly, the French verb *recoller* means “to glue back together”.

subcategories of I-nilpotent and I-local objects.

The goal of this section is to explain this general theory and how it can be applied to the context of sheaves of spectra on the category of manifolds. The key insight is that given a stable ∞-category X and a full subcategory $i_* : \mathsf{Z} \hookrightarrow \mathsf{X}$ that is both localizing and colocalizing

$$\begin{tikzcd} \mathsf{Z} \arrow[r, hook, "i_*" description] & \mathsf{X} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd},$$

the ∞-category X can be glued together from the subcategory Z and the subcategory $\mathsf{Z}^{\perp} \subset \mathsf{X}$ *right orthogonal* to X (Proposition 6.1.21 and Corollary 6.1.22). That is, $\mathsf{Z}^{\perp}$ is the subcategory of objects of X that admit no nontrivial maps *from* objects of Z. This applies to the situation of interest because we have both a left and right adjoint

$$\begin{tikzcd} \mathrm{Sh}_{\mathbb{R}}(\mathsf{Mfld};\mathsf{Spt}) \arrow[r] & \mathrm{Sh}(\mathsf{Mfld};\mathsf{Spt}) \arrow[l, bend right, "\mathrm{L}_{\mathrm{hi}}"'] \arrow[l, bend left, "\mathrm{R}_{\mathrm{hi}}"] \end{tikzcd}$$

to the inclusion of $\mathbb{R}$-invariant sheaves on Mfld into all sheaves (4.4.2). We'll apply the general theory studied in this section to the context of sheaves on Mfld in §6.2.

6.1.a Motivation

To explain the motivation for recollements, let X be a topological space and $Z \subset X$ a closed subspace. Write $U := X \smallsetminus Z$ for the open complement of Z in X, and write

$$i : Z \hookrightarrow X \qquad \text{and} \qquad j : U \hookrightarrow X$$

for the inclusions. Any sheaf F of sets on X pulls back to sheaves

$$F_Z := i^*(F) \qquad \text{and} \qquad F_U := j^*(F)$$

on Z and U, respectively. Moreover, the sheaf F is completely determined by the sheaves F_Z and F_U in the following sense. Applying i^* to the unit $\eta : F \to j_* j^*(F)$, we obtain a natural morphism

$$u : F_Z = i^*(F) \to i^* j_* j^*(F) = i^* j_*(F_U) .$$

The triangle identities imply that there is a commutative square

(6.1.1)
$$\begin{tikzcd} F \arrow[r] \arrow[d] & j_*(F_U) \arrow[d] \\ i_*(F_Z) \arrow[r, "i_*(u)"'] & i_* i^* j_*(F_U) \end{tikzcd},$$

where the three morphisms

$$F \to i_* i^*(F) = i_*(F_Z), \quad F \to j_* j^*(F) = j_*(F_U), \qquad \text{and} \qquad j_*(F_U) \to i_* i^* j_*(F_U)$$

are all unit morphisms. One can show that the square (6.1.1) is in a *pullback square*. This provides an explicit way to reconstruct F from the data of the sheaves F_Z and F_U along with the morphism $u : F_Z \to i^* j_*(F_U)$.

In fact, even more is true. The whole *category* $\mathrm{Sh}(X; \mathrm{Set})$ can be reconstructed from the categories $\mathrm{Sh}(Z; \mathrm{Set})$ and $\mathrm{Sh}(U; \mathrm{Set})$ together with the functor $i^* j_* : \mathrm{Sh}(U; \mathrm{Set}) \to \mathrm{Sh}(Z; \mathrm{Set})$ in the following sense. Write $[1]$ for the "walking arrow" poset $\{0 < 1\}$. There is a pullback square of categories

(6.1.2)
$$\begin{tikzcd} \mathrm{Sh}(X; \mathrm{Set}) \arrow[r] \arrow[d, "j^*"'] & \mathrm{Fun}([1], \mathrm{Sh}(Z; \mathrm{Set})) \arrow[d, "\mathrm{target}"] \\ \mathrm{Sh}(U; \mathrm{Set}) \arrow[r, "i^* j_*"'] & \mathrm{Sh}(Z; \mathrm{Set}) . \end{tikzcd}$$

Here the unlabeled top horizontal arrow sends a sheaf $F \in \mathrm{Sh}(X; \mathrm{Set})$ to the morphism given by applying i^* to the unit $F \to j_* j^*(F)$. More explicitly, an object of $\mathrm{Sh}(X; \mathrm{Set})$ is equivalent to the data of a sheaf F_Z on Z, a sheaf F_U on U, and a *gluing morphism* $F_Z \to i^* j_*(F_U)$. Morphisms are morphisms of sheaves on Z and U commuting with the specified gluing morphisms.

In the rest of this section, we explain the general categorical framework for decompositions of this form. We do not explain the proofs of the results presented in this section; for those, the reader should consult [HA, §A.8; SAG, §7.2; BG16].

6.1.b Definitions and general results

Now we generalize the situation for sheaves explained in § 6.1.a. The following are the key features of the situation.

6.1.3 Definition. Let X be an ∞-category with finite limits. Fully faithful functors

$$i_* : \mathsf{Z} \hookrightarrow \mathsf{X} \qquad \text{and} \qquad j_* : \mathsf{U} \hookrightarrow \mathsf{X}$$

exhibit X as the *recollement* of Z and U if:

(6.1.3.1) The functors i_* and j_* admit left exact left adjoints i^* and j^*, respectively.

(6.1.3.2) The functor $j^* i_* : \mathsf{Z} \to \mathsf{U}$ is constant at the terminal object of U.

(6.1.3.3) The functors $i^* : \mathsf{X} \to \mathsf{Z}$ and $j^* : \mathsf{X} \to \mathsf{U}$ are jointly conservative. That is, a morphism f in X is an equivalence if and only if both $i^*(f)$ and $j^*(f)$ are equivalences.

We refer to the subcategory $\mathsf{Z} \subset \mathsf{X}$ as the *closed* subcategory, and $\mathsf{U} \subset \mathsf{X}$ as the *open* subcategory.

6.1.4 Remark. Note that (6.1.3.2) in particular implies that the are *no* nontrivial maps from objects in $\mathsf{Z} \subset \mathsf{X}$ to objects in $\mathsf{U} \subset \mathsf{X}$.

6.1.5 Warning. Note that the condition that X be the recollement of Z and U is *not* symmetric: if X is the recollement of Z and U, then X need note be the recollement of U and Z. For example, the composite $i^* j_*$ is not usually constant at the terminal object of Z.

The two most important examples of recollements from topology and algebraic geometry are the following:

6.1.6 Example. Let X be a topological space, $i \colon Z \hookrightarrow X$ a closed subspace, and $j \colon U \hookrightarrow X$ the open complement of Z in X. Let C be a presentable ∞-category that is compactly generated or stable. Then the pushforward functors

$$i_* \colon \mathrm{Sh}(Z; C) \hookrightarrow \mathrm{Sh}(X; C) \qquad \text{and} \qquad j_* \colon \mathrm{Sh}(U; C) \hookrightarrow \mathrm{Sh}(X; C)$$

exhibit $\mathrm{Sh}(X; C)$ as the recollement of $\mathrm{Sh}(Z; C)$ and $\mathrm{Sh}(U; C)$. See [HA, Remark A.8.16; Hai21, Corollaries 2.12 & 2.23]

6.1.7 Example. Let X be a scheme, $Z \hookrightarrow X$ a closed subscheme, and $U \hookrightarrow X$ the complementary open subscheme in X. Assume that U is quasicompact. We write $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(U)$ for the stable ∞-categories of quasicoherent sheaves on X and U, respectively. We write

$$\mathrm{QCoh}_Z(X) \subset \mathrm{QCoh}(X)$$

for the full subcategory spanned by those quasicoherent sheaves that are set-theoretically supported on Z. In this setting, the pushforward functor $\mathrm{QCoh}(U) \hookrightarrow \mathrm{QCoh}(X)$ and the inclusion $\mathrm{QCoh}_Z(X) \subset \mathrm{QCoh}(X)$ exhibit $\mathrm{QCoh}(X)$ as the recollement of $\mathrm{QCoh}(U)$ and $\mathrm{QCoh}_Z(X)$. See, for example, [SAG, Proposition 7.2.3.1].

6.1.8 Warning. In Example 6.1.7, note that the subcategory $\mathrm{QCoh}(U)$ is the *closed* subcategory, and the subcategory $\mathrm{QCoh}_Z(X)$ is the *open* subcategory. There are thus two competing naming conventions for the "closed" and "open" subcategories: one coming from the theory of sheaves on topological spaces (Example 6.1.6), and one coming from quasicoherent sheaves on schemes (Example 6.1.7). Both are used in the literature, depending on whether one is working in a "topological" or "algebro-geometric" context. In this text we use the "topological" convention.

The following result explains how to reconstruct a recollement from the closed and open subcategories together with *gluing functor* $i^* j_* \colon \mathsf{U} \to \mathsf{Z}$.

6.1.9 Theorem [HA, Corollary A.8.13, Remark A.8.5, & Proposition A.8.17; QS19, 1.17]**.** *Let* $i_* \colon \mathsf{Z} \hookrightarrow \mathsf{X}$ *and* $j_* \colon \mathsf{U} \hookrightarrow \mathsf{X}$ *be functors that exhibit* X *as the recollement of* Z *and* U*. There is a pullback square of* ∞*-categories*

$$\begin{tikzcd}
\mathsf{X} \arrow[r] \arrow[d, "j^*"'] & \mathrm{Fun}([1], \mathsf{Z}) \arrow[d, "\text{target}"] \\
\mathsf{U} \arrow[r, "i^* j_*"'] & \mathsf{Z} .
\end{tikzcd}$$

Here the unlabeled top horizontal arrow sends an object $F \in \mathsf{X}$ to the morphism given by applying i^* to the unit $F \to j_* j^*(F)$.

As a consequence, there is a pullback square of endofunctors of X

$$
\begin{tikzcd}
\mathrm{id}_{\mathsf{X}} \arrow[r] \arrow[d] & j_* j^* \arrow[d] \\
i_* i^* \arrow[r] & i_* i^* j_* j^* .
\end{tikzcd}
\tag{6.1.10}
$$

Here the top horizontal and left vertical morphisms are the unit morphisms, the bottom horizontal morphism is obtained by applying $i_* i^*$ to the unit morphism $\mathrm{id}_{\mathsf{X}} \to j_* j^*$, and the right vertical morphism is obtained by precomposing the unit morphism $\mathrm{id}_{\mathsf{X}} \to i_* i^*$ with $j_* j^*$.

6.1.11 Definition. Let $i_* \colon \mathsf{Z} \hookrightarrow \mathsf{X}$ and $j_* \colon \mathsf{U} \hookrightarrow \mathsf{X}$ be functors that exhibit X as the recollement of Z and U. The pullback square (6.1.10) is referred to as the *fracture square* of the recollement.

Often the functors i_* and j^* admit further adjoints.

6.1.12 Theorem [HA, Corollary A.8.7, Remark A.8.8, & Proposition A.8.11; QS19, Corollary 1.10]. *Let $i_* \colon \mathsf{Z} \hookrightarrow \mathsf{X}$ and $j_* \colon \mathsf{U} \hookrightarrow \mathsf{X}$ be functors that exhibit X as the recollement of Z and U.*

(6.1.12.1) *If the ∞-category Z has an initial object, then j^* admits a fully faithful left adjoint $j_! \colon \mathsf{U} \hookrightarrow \mathsf{X}$.*

(6.1.12.2) *If, moreover, X has a zero object, then i_* admits a right adjoint $i^! \colon \mathsf{X} \to \mathsf{Z}$ characterized by the property that*

$$i_* i^! \simeq \mathrm{fib}(\eta \colon \mathrm{id}_{\mathsf{X}} \to j_* j^*) .$$

In particular, applying i^*, there is a fiber sequence

$$i^! \longrightarrow i^* \xrightarrow{i^* \eta} i^* j_* j^* .$$

(6.1.12.3) *If X is stable, then Z and U are also stable. Moreover, there is a canonical fiber sequence*

$$j_! j^* \longrightarrow \mathrm{id}_{\mathsf{X}} \longrightarrow i_* i^* ,$$

where the first morphism is the counit and the second is the unit.

(6.1.12.4) *If X is presentable and the gluing functor $i^* j_*$ is accessible, then Z and U are presentable.*

6.1.13. Thus, if X is stable, there is a chain of adjunctions

$$
\begin{tikzcd}
\mathsf{Z} \arrow[r, "i_*" description] & \mathsf{X} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*" description] & \mathsf{U} . \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}
$$

We're interested in applying this to the situation where i_* is the inclusion of $\mathrm{Sh}_{\mathbb{R}}(\mathsf{Mfld};\mathsf{Spt})$ into $\mathrm{Sh}(\mathsf{Mfld};\mathsf{Spt})$, i^* is L_{hi}, and $i^!$ is R_{hi}. To get an analogue of the "differential cohomology hexagon", we need to enlarge the fracture square (6.1.10) using the fiber sequences from (6.1.12.2) and (6.1.12.3) along with one more.

6.1.14 Construction (norm map)**.** Let X and U be ∞-categories, and suppose we are given adjunctions

$$\begin{tikzcd} \mathsf{X} \arrow[r, "j^*" description] & \mathsf{U} \arrow[l, hook', bend right, "j_!"'] \arrow[l, hook, bend left, "j_*"] \end{tikzcd}$$

where left adjoint $j_!$ and right adjoint j_* are fully faithful. Write $\varepsilon \colon j^* j_* \to \mathrm{id}_{\mathsf{U}}$ for the counit. Since $j_!$ is left adjoint to j^* and the counit ε is an equivalence, we have equivalences

$$\mathrm{Map}(j_!, j_*) \simeq \mathrm{Map}(\mathrm{id}_{\mathsf{U}}, j^* j_*) \xrightarrow[\varepsilon\circ -]{\sim} \mathrm{Map}(\mathrm{id}_{\mathsf{U}}, \mathrm{id}_{\mathsf{U}})\,. \tag{6.1.15}$$

The *norm* natural transformation

$$\mathrm{Nm} \colon j_! \to j_*$$

is the natural transformation corresponding to the identity $\mathrm{id}_{\mathsf{U}} \to \mathrm{id}_{\mathsf{U}}$ under the equivalence (6.1.15).

6.1.16 Theorem. *Let X be a stable ∞-category and let $i_* \colon \mathsf{Z} \hookrightarrow \mathsf{X}$ and $j_* \colon \mathsf{U} \hookrightarrow \mathsf{X}$ be functors that exhibit X as the recollement of Z and U. Then the sequence*

$$j_! j^* \xrightarrow{\mathrm{Nm}\, j^*} j_* j^* \longrightarrow i_* i^* j_* j^*$$

is a fiber sequence. As a consequence, the fracture square fits into a commutative diagram

$$\begin{tikzcd} & j_! j^* \arrow[r, equal] \arrow[d] & j_! j^* \arrow[d, "\mathrm{Nm}\, j^*"] \\ i_* i^! \arrow[r] \arrow[d, equal] & \mathrm{id}_{\mathsf{X}} \arrow[r] \arrow[d] & j_* j^* \arrow[d] \\ i_* i^! \arrow[r] & i_* i^* \arrow[r] & i_* i^* j_* j^* \end{tikzcd} \tag{6.1.17}$$

where all rows and columns are fiber sequences.

Aside from the explicit identification of the first map in the lower horizontal fiber sequence of (6.1.10) with the norm map, Theorem 6.1.16 can be deduced by applying the following characterization of pullback squares of stable ∞-categories horizontally and vertically to the fracture square (6.1.10).

6.1.18 Recollection. Let C be a pointed ∞-category and

$$\begin{tikzcd} W \arrow[r, "\bar{f}"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \tag{6.1.19}$$

a commutative square in C. Then there is a natural equivalence

$$\mathrm{fib}(W \to X \times_Z Y) \simeq \mathrm{fib}(\mathrm{fib}(\bar{f}) \to \mathrm{fib}(f)) .$$

In particular, if C is stable, then $\mathrm{fib}(\bar{f}) \xrightarrow{\sim} \mathrm{fib}(f)$ if and only if the square (6.1.19) is a pullback square. See [BA14, §2; Nar19] for more details.

6.1.c Orthogonal complements & the stable situation

In the stable case, it turns out that the data of a recollement of X is equivalent to the data of the closed subcategory $\mathsf{Z} \subset \mathsf{X}$. The open subcategory $\mathsf{U} \subset \mathsf{X}$ can be recovered as an *orthogonal complement* to Z in the following sense.

6.1.20 Definition. Let X be an ∞-category and $\mathsf{Z} \subset \mathsf{X}$ a full subcategory.

(6.1.20.1) We say that an object $X \in \mathsf{X}$ is *right orthogonal* to the subcategory Z if for each $Z \in \mathsf{Z}$, the mapping space $\mathrm{Map}_{\mathsf{X}}(Z, X)$ is contractible.

(6.1.20.2) We say that an object $X \in \mathsf{X}$ is *left orthogonal* to the subcategory Z if for each $Z \in \mathsf{Z}$, the mapping space $\mathrm{Map}_{\mathsf{X}}(X, Z)$ is contractible.

The *right orthogonal complement* of Z is the full subcategory $\mathsf{Z}^{\perp} \subset \mathsf{X}$ spanned by those objects right orthogonal to Z. The *left orthogonal complement* of Z is the full subcategory ${}^{\perp}\mathsf{Z} \subset \mathsf{X}$ spanned by those objects right orthogonal to Z.

6.1.21 Proposition [SAG, Proposition 7.2.1.10; BG16, Lemmas 2 & 5 and Proposition 7]**.** *Let X be a stable ∞-category, and $i_* \colon \mathsf{Z} \hookrightarrow \mathsf{X}$ a full subcategory. Assume that the inclusion i_* admits a left adjoint i^* and a right adjoint $i^!$. Then:*

(6.1.21.1) *The inclusion $\mathsf{Z}^{\perp} \subset \mathsf{X}$ admits a left adjoint $j^{\perp} \colon \mathsf{X} \to \mathsf{Z}^{\perp}$ defined as the cofiber*

$$j^{\perp} \coloneqq \mathrm{cofib}(\varepsilon \colon i_* i^! \to \mathrm{id}_{\mathsf{X}}) .$$

(6.1.21.2) *The inclusion ${}^{\perp}\mathsf{Z} \subset \mathsf{X}$ admits a right adjoint ${}^{\perp}j \colon \mathsf{X} \to {}^{\perp}\mathsf{Z}$ defined as the fiber*

$${}^{\perp}j \coloneqq \mathrm{fib}(\eta \colon \mathrm{id}_{\mathsf{X}} \to i^* i_*) .$$

(6.1.21.3) *The composite functors*

$$\mathsf{Z}^{\perp} \hookrightarrow \mathsf{X} \xrightarrow{{}^{\perp}j} {}^{\perp}\mathsf{Z} \qquad \textit{and} \qquad {}^{\perp}\mathsf{Z} \hookrightarrow \mathsf{X} \xrightarrow{j^{\perp}} \mathsf{Z}^{\perp}$$

are inverse equivalences of ∞-categories.

(6.1.21.4) *The stable ∞-category X is the recollement of the stable subcategories Z and $\mathsf{Z}^{\perp}$.*

6.1.22 Corollary. *Let X be a stable ∞-category, and let $i_* : \mathsf{Z} \hookrightarrow \mathsf{X}$ and $j_* : \mathsf{U} \hookrightarrow \mathsf{X}$ be functors that exhibit X as the recollement of Z and U. Then the essential image of the fully faithful functor j_* is the right orthogonal complement $\mathsf{Z}^{\perp}$ of Z.*

Said differently, *every* stable recollement arises via Proposition 6.1.21.

6.1.23 Remark (semiorthognal decompositions)**.** Proposition 6.1.21 and Corollary 6.1.22 say that recollements are special types of *semiorthogonal decompositions* of ∞-categories. Semiorthogonal decompositions were originally introduced (in the context of triangulated categories) by Bondal and Kapranov [BK89] to break apart stable ∞-categories arising in algebraic geometry into more simple pieces. There are many beautiful examples (namely, Beĭlinson's celebrated semiorthogonal decomposition of $\mathsf{Coh}(\mathbb{P}^n)$ [Beĭ78; Beĭ84a]) and connections to other important algebraic structures such as t-structures. The interested reader is encouraged to consult [SAG, §7.2] as well as Antieau and Elmanto's recent work [AE21].

6.2 Decomposing sheaves on manifolds

We now apply the framework of recollements introduced in §6.1 to the case where

$$\mathsf{X} = \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spt}) \qquad \text{and} \qquad \mathsf{Z} = \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; \mathsf{Spt}) .$$

Since we can do so at no extra cost, we'll work in the more general setting of sheaves valued in a presentable stable ∞-category. First, let's align our notation with Proposition 6.1.21.

6.2.1. Let C be a presentable stable ∞-category. Writing $\mathsf{X} = \mathsf{Sh}(\mathsf{Mfld}; C)$ and $\mathsf{Z} = \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$, in the notation of Proposition 6.1.21 we have $i^* = \mathrm{L}_{\mathrm{hi}}$ and $i^! = \mathrm{R}_{\mathrm{hi}}$.

6.2.2 Definition. Let C be a stable presentable ∞-category. A sheaf $\hat{E} : \mathsf{Mfld}^{\mathrm{op}} \to C$ is *pure* if $\hat{E}$ is right orthogonal to $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$. We write

$$\mathsf{Sh}_{\mathrm{pu}}(\mathsf{Mfld}; C) := \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)^{\perp} \subset \mathsf{Sh}(\mathsf{Mfld}; C)$$

for the full subcategory spanned by the pure sheaves.

6.2.3 Observation. Recall that the subcategory $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ is the essential image of the constant sheaf functor $\Gamma^* : C \hookrightarrow \mathsf{Sh}(\mathsf{Mfld}; C)$ (Proposition 4.3.1). Let $X \in C$ and $\hat{E} \in \mathsf{Sh}(\mathsf{Mfld}; C)$. Then

$$\mathrm{Map}_{\mathsf{Sh}(\mathsf{Mfld}; C)}(\Gamma^*(X), \hat{E}) \simeq \mathrm{Map}_C(X, \Gamma_*(\hat{E})) .$$

Thus $\hat{E}$ is right orthogonal to $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ if and only if

$$\Gamma_*(\hat{E}) = \hat{E}(*) = 0 .$$

Said differently, $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C)$ is the kernel of the constant sheaf functor $\Gamma_* \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to C$.

Also note that since the global sections functor Γ_* preserves all limits and colimits, the subcategory of pure sheaves is stable under limits and colimits.

Now we introduce the left adjoint to the inclusion $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C) \subset \mathrm{Sh}(\mathrm{Mfld}; C)$ following the prescription of (6.1.21.1). In the following, we think of $\mathrm{R}_{\mathrm{hi}}(\hat{E})$ as playing the role of cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$ in the differential cohomology hexagon (Theorem 2.3.2).

6.2.4 Definition. Let C be a stable presentable ∞-category. Define a functor

$$\mathrm{Cyc} \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to \mathrm{Sh}(\mathrm{Mfld}; C)$$

and a *curvature* natural transformation $\mathrm{curv} \colon \mathrm{id} \to \mathrm{Cyc}$ by the cofiber sequence

$$\mathrm{R}_{\mathrm{hi}} \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\mathrm{curv}} \mathrm{Cyc} ,$$

where $\varepsilon \colon \mathrm{R}_{\mathrm{hi}} \to \mathrm{id}$ is the counit. For a C-valued sheaf $\hat{E}$ on Mfld, we call $\mathrm{Cyc}(\hat{E})$ the sheaf of *differential cycles* associated to $\hat{E}$.

6.2.5. As a consequence of Proposition 6.1.21, Cyc factors through $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C)$ and is left adjoint to the inclusion $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C) \subset \mathrm{Sh}(\mathrm{Mfld}; C)$.

6.2.6 Observation. Since the global sections functor Γ_* preserves all limits and colimits, the subcategory of pure sheaves is stable under *both* limits and colimits. Since $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C)$ is presentable, the inclusion $\mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C) \hookrightarrow \mathrm{Sh}(\mathrm{Mfld}; C)$ also admits a right adjoint.

To do this, we identify the left adjoint to the functor $\mathrm{Cyc} \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to \mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C)$.

6.2.7 Definition. Let C be a stable presentable ∞-category. Define a functor

$$\mathrm{Def} \colon \mathrm{Sh}(\mathrm{Mfld}; C) \to \mathrm{Sh}(\mathrm{Mfld}; C)$$

by the fiber sequence

$$\mathrm{Def} \longrightarrow \mathrm{id} \xrightarrow{\eta} \mathrm{L}_{\mathrm{hi}} ,$$

where $\eta \colon \mathrm{id} \to \mathrm{L}_{\mathrm{hi}}$ is the unit. For a C-valued sheaf $\hat{E}$ on Mfld, we call $\mathrm{Def}(\hat{E})$ the sheaf of *differential deformations* associated to $\hat{E}$.

6.2.8 Observations. In light of Theorem 6.1.12, the functor

$$\mathrm{Def} \colon \mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C) \to \mathrm{Sh}(\mathrm{Mfld}; C)$$

is left adjoint to the functor Cyc. In particular, $\mathrm{Def} \colon \mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C) \to \mathrm{Sh}(\mathrm{Mfld}; C)$ is fully faithful (Lemma 4.1.4).

6.2.9. We have chains of adjunctions

$$\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C) \underset{\mathrm{R_{hi}}}{\overset{\mathrm{L_{hi}}}{\leftrightarrows\!\!\!\!\!\leftarrow}} \mathsf{Sh}(\mathsf{Mfld}; C) \underset{}{\overset{\mathrm{Def}}{\leftrightarrows\!\!\!\!\!\leftarrow}}_{\mathrm{Cyc}} \mathsf{Sh}_{\mathrm{pu}}(\mathsf{Mfld}; C)\,.$$

To align notation with (6.1.13), we have $\mathsf{X} = \mathsf{Sh}(\mathsf{Mfld}; C)$, $\mathsf{Z} = \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$, and $\mathsf{U} = \mathsf{Sh}_{\mathrm{pu}}(\mathsf{Mfld}; C)$. The functors $i_* : \mathsf{Z} \hookrightarrow \mathsf{X}$ and $j_* : \mathsf{U} \hookrightarrow \mathsf{X}$ are the two unlabeled inclusions. We also have $i^! = \mathrm{R_{hi}}$, $i^* = \mathrm{L_{hi}}$, $j^* = \mathrm{Cyc}$, and $j_! = \mathrm{Def}$.

6.2.a The differential cohomology hexagon

Now we explain how the extended fracture diagram of a stable recollement (Theorem 6.1.16) gives rise to a "differential cohomology hexagon".

6.2.10 Notation. We write $\mathrm{d} : \mathrm{Def} \to \mathrm{Cyc}$ for the composite

$$\mathrm{d} : \mathrm{Def} \longrightarrow \mathrm{id} \xrightarrow{\mathrm{curv}} \mathrm{Cyc}\,.$$

6.2.11 Corollary (fracture square)**.** *Let C be a stable presentable ∞-category. The ∞-category $\mathsf{Sh}(\mathsf{Mfld}; C)$ is the recollement of the subcategories $\mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; C)$ and $\mathsf{Sh}_{\mathrm{pu}}(\mathsf{Mfld}; C)$. In particular, there is a commutative diagram*

(6.2.12)
$$\begin{tikzcd}
 & \mathrm{Def} \arrow[r, equal] \arrow[d] & \mathrm{Def} \arrow[d, "\mathrm{d}"] \\
\mathrm{R_{hi}} \arrow[r] \arrow[d, equal] & \mathrm{id}_{\mathsf{Sh}(\mathsf{Mfld}; C)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Cyc} \arrow[d] \\
\mathrm{R_{hi}} \arrow[r] & \mathrm{L_{hi}} \arrow[r] & \mathrm{L_{hi}}\,\mathrm{Cyc}
\end{tikzcd}$$

of functors $\mathsf{Sh}(\mathsf{Mfld}; C) \to \mathsf{Sh}(\mathsf{Mfld}; C)$, where the lower right-hand square is a pullback and all rows and columns are fiber sequences.

6.2.13. Informally, $\mathsf{Sh}(\mathsf{Mfld}; C)$ is the ∞-category of triples

$$(\hat{E}_{\mathbb{R}}, \hat{E}_{\mathrm{pu}}, \phi : \hat{E}_{\mathbb{R}} \to \mathrm{L_{hi}}\,\hat{E}_{\mathrm{pu}})\,,$$

where $\hat{E}_{\mathbb{R}}$ is a $\mathbb{R}$-invariant sheaf, $\hat{E}_{\mathrm{pu}}$ is a pure sheaf, and ϕ is any morphism.

6.2.14 (differential cohomology hexagon)**.** With some rearrangement, Corollary 6.2.11 and the fact that pullback squares compose, we see that there is a diagram of pullback squares

$$
\begin{tikzcd}
\Sigma^{-1} \mathrm{L}_{\mathrm{hi}} \mathrm{Cyc} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{R}_{\mathrm{hi}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\mathrm{Def} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{id}_{\mathsf{Sh}(\mathsf{Mfld};\mathcal{C})} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Cyc} \arrow[d] \\
0 \arrow[r] & \mathrm{L}_{\mathrm{hi}} \arrow[r] & \mathrm{L}_{\mathrm{hi}} \mathrm{Cyc} \, .
\end{tikzcd}
\tag{6.2.15}
$$

Rearranging the diagram (6.2.15), for each $\hat{E} \in \mathsf{Sh}(\mathsf{Mfld};\mathcal{C})$ we get the following *differential cohomology hexagon*

$$
\begin{tikzcd}
 & \mathrm{R}_{\mathrm{hi}}(\hat{E}) \arrow[rr] \arrow[dr] & & \mathrm{L}_{\mathrm{hi}}(\hat{E}) \arrow[dr] & \\
\Sigma^{-1} \mathrm{L}_{\mathrm{hi}} \mathrm{Cyc}(\hat{E}) \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur] \arrow[dr, "\mathrm{curv}"] & & \mathrm{L}_{\mathrm{hi}} \mathrm{Cyc}(\hat{E}) \\
 & \mathrm{Def}(\hat{E}) \arrow[rr, "\mathrm{d}"'] \arrow[ur] & & \mathrm{Cyc}(\hat{E}) \arrow[ur] &
\end{tikzcd}
\tag{6.2.16}
$$

Here the diagonals are fiber sequences, the top and bottom rows are extensions of fiber sequences by one term, and both squares are pullback squares. The "top row" consists of $\mathbb{R}$-invariant sheaves, whereas the "bottom row" consists of sheaves that are, in some sense, more geometric.

Since $\mathrm{L}_{\mathrm{hi}} \simeq \Gamma^*\Gamma_\sharp$ and $\mathrm{R}_{\mathrm{hi}} \simeq \Gamma^*\Gamma_*$ (4.4.2), the differential cohomology hexagon (6.2.16) can be rewritten as

$$
\begin{tikzcd}
 & \Gamma^*\Gamma_*\hat{E} \arrow[rr] \arrow[dr] & & \Gamma^*\Gamma_\sharp\hat{E} \arrow[dr] & \\
\Sigma^{-1}\Gamma^*\Gamma_\sharp\mathrm{Cyc}(\hat{E}) \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur] \arrow[dr] & & \Gamma^*\Gamma_\sharp\mathrm{Cyc}(\hat{E}) \\
 & \mathrm{Def}(\hat{E}) \arrow[rr, "\mathrm{d}"'] \arrow[ur] & & \mathrm{Cyc}(\hat{E}) \arrow[ur] &
\end{tikzcd}
$$

6.2.b Differential refinements

We finish this section by making precise what it means for a differential cohomology theory $\hat{E} \in \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spt})$ to refine a cohomology theory $E \in \mathsf{Spt}$.

6.2.17 Definition. Let $\mathcal{C}$ be a presentable stable ∞-category. A *differential refinement* of a an object $E \in \mathcal{C}$ is pair $(\hat{E}, \phi)$ of a sheaf $\hat{E} \in \mathsf{Sh}(\mathsf{Mfld};\mathcal{C})$ together with an equivalence $\phi \colon \Gamma_\sharp(\hat{E}) \xrightarrow{\sim} E$ in $\mathcal{C}$.

6.2.18. From the fracture square (Corollary 6.2.11), a differential refinement of $E \in C$ is equivalently the data of a pure sheaf $\hat{P} \in \mathrm{Sh}_{\mathrm{pu}}(\mathrm{Mfld}; C)$ along with a morphism $E \to \Gamma_\sharp(\hat{P})$ in C. Given this data, we can construct a differential refinement $\hat{E}$ in the sense of Definition 6.2.17 as the pullback

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{P} \arrow[d] \\
\Gamma^*(E) \arrow[r] & \Gamma^*\Gamma_\sharp(\hat{P}) .
\end{tikzcd}$$

In this case, we have:

(6.2.18.1) $\mathrm{Def}(\hat{E}) \xrightarrow{\sim} \mathrm{Def}(\hat{P})$.

(6.2.18.2) $\mathrm{Cyc}(\hat{E}) \xrightarrow{\sim} \hat{P}$.

(6.2.18.3) $\Gamma_*(\hat{E})$ fits into a fiber sequence

$$\Gamma_*(\hat{E}) \to E \to \Gamma_\sharp(\hat{P}) .$$

6.2.19 Construction (pullback of a differential refinement)**.** Let C be a presentable stable ∞-category, $f : E \to E'$ a morphism in C, and $(\hat{E}', \phi')$ a differential refinement of E'. Form the pullback

(6.2.20)
$$\begin{tikzcd}
\hat{E} \arrow[r, "\hat{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{E}' \arrow[d] \\
\Gamma^*(E) \arrow[r, "\Gamma^*(f)"'] & \Gamma^*(E') ,
\end{tikzcd}$$

where the morphism $\hat{E}' \to \Gamma^*(E')$ is adjoint to the given equivalence $\phi' : \Gamma_\sharp(\hat{E}') \xrightarrow{\sim} E'$. Since $\Gamma_\sharp$ is exact, applying $\Gamma_\sharp$ to the square (6.2.20) gives a pullback square

$$\begin{tikzcd}
\Gamma_\sharp(\hat{E}) \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & \Gamma_\sharp(\hat{E}') \arrow[d, "\phi'"', "\wr"] \\
E \arrow[r, "f"'] & E' ,
\end{tikzcd}$$

which provides an equivalence $\phi : \Gamma_\sharp(\hat{E}) \xrightarrow{\sim} E$. The *pullback differential refinement* of $(\hat{E}', \phi')$ along f is the differential refinement $(\hat{E}, \phi)$ of E.

6.2.21 Lemma. *In the notation of Construction 6.2.19, the following*

(6.2.21.1) *The morphism* $\mathrm{Def}(\hat{f})\colon \mathrm{Def}(\hat{E}) \to \mathrm{Def}(\hat{E}')$ *is an equivalence.*

(6.2.21.2) *The morphism* $\mathrm{Cyc}(\hat{f})\colon \mathrm{Cyc}(\hat{E}) \to \mathrm{Cyc}(\hat{E}')$ *is an equivalence.*

(6.2.21.3) *The global sections of* $\hat{E}$ *is given by the pullback*

$$\begin{tikzcd} \Gamma_*(\hat{E}) \arrow[r, "\Gamma_*(\hat{f})"] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_*(\hat{E}') \arrow[d] \\ E \arrow[r, "f"'] & E' . \end{tikzcd}$$

7 Examples

by Araminta Amabel

The purpose of this section is to construct examples of differential cohomology theories, i.e., sheaves of spectra on the category Mfld. We'll construct these examples by using the method of *differential refinements* introduced in §6.2.b. Note that given a spectrum E, there are possibly many differential refinements of E. We will construct differential cohomology theories refining the cohomology theory E by the following process:

(1) Choose a pure sheaf $\hat{P}$ (Definition 6.2.2).

(2) Compute $\Gamma_\sharp \hat{P}$ using the formula $\Gamma_\sharp \hat{P} = \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \hat{P}(\Delta^\bullet_{\mathrm{alg}})$ of Corollary 5.1.4.

(3) Find a map of spectra $f : E \to \Gamma_\sharp \hat{P}$.

(4) Define $\hat{E}$ as in the pullback

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{P} \arrow[d] \\ \Gamma^*(E) \arrow[r, "\Gamma^*(f)"'] & \Gamma^*\Gamma_\sharp(\hat{P})\,. \end{tikzcd}$$

We start in §7.1 with differential refinements of 0 and what the differential cohomology hexagon looks like in this case. In §7.2, we refine this simplest example by adding a filtration. Section 7.3 explains how the Cheeger–Simons theory of differential characters fits into this story, and §7.4 studies differential refinements of K-theory.

7.1 The simplest example

To start off, let's try to construct a differential refinement where the pure sheaf $\hat{P}$ is zero. That is, $\hat{P} = 0 = \Gamma^* 0$. In this case, since the functor $\Gamma_\sharp$ is exact, $\Gamma_\sharp(\hat{P}) = 0$. Any spectrum E maps uniquely to 0. Thus for any spectrum E we have a differential refinement $\hat{E}$ defined by the pullback

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d] & \Gamma^* E \arrow[d] \\ \Gamma^* 0 \arrow[r, equal] & \Gamma^* 0 \end{tikzcd}$$

Since the bottom horizontal arrow is an equivalence, the top horizontal arrow is as well: $\hat{E} = \Gamma^* E$. The rest of the differential cohomology diagram looks as follows

$$\begin{tikzcd}
 & \Gamma^* E \arrow[rr] \arrow[dr, equal] & & \Gamma^* E \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \Gamma^* E \arrow[ur, equal] \arrow[dr] & & 0 \\
 & \mathrm{Def}(\Gamma^* E) \arrow[rr] \arrow[ur] & & 0 \arrow[ur] &
\end{tikzcd} .$$

Since the upwards diagonal sequence is a fiber sequence, we also have $\mathrm{Def}(\Gamma^* E) = 0$.

This example is just saying that that E-cohmology is a special case of differential cohomology. We're really just reformulating the fact that the constant sheaf functor $\Gamma^* : \mathsf{Spt} \to \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spt})$ is fully faithful with essential image the $\mathbb{R}$-invariant sheaves (Proposition 4.3.1).

7.2 The simplest example, but with a filtration

We give an alternative differential refinement of the zero spectrum which comes with a natural filtration.

7.2.1. Let $\Omega^\bullet \in \mathsf{Sh}(\mathsf{Mfld}; \mathsf{D}(\mathbb{R}))$ the sheaf of de Rham forms with cohomological grading; so Ω^k is in degree $-k$. Consider the resulting functor of spectra, $\mathrm{H}\Omega^\bullet$. By the Poincaré Lemma, $\Omega^\bullet$ is quasi-isomorphic to the constant sheaf at $\mathbb{R}[0]$. Thus $\mathrm{H}\Omega^\bullet \xrightarrow{\sim} \Gamma^*\mathrm{H}\mathbb{R}$. In particular, $\mathrm{H}\Omega^\bullet$ is not pure.

However, since $\mathrm{H}\Omega^\bullet \simeq \Gamma^*\mathrm{H}\mathbb{R}$ is $\mathbb{R}$-invariant, the purification $\mathrm{Cyc}(\mathrm{H}\Omega^\bullet)$ is equivalent to zero. Now $\Omega^\bullet$ has a filtration by degree. For $k \in \mathbb{N}$, let $\Omega^{\geq k}$ denote the stunted piece of the chain complex $\Omega^\bullet$ where we have replaced everything in degrees $< k$ by 0. We get induced filtrations of $\mathrm{H}\Omega^\bullet$ and of $\mathrm{Cyc}(\mathrm{H}\Omega^\bullet) \simeq \Gamma^* 0$.

For $k \geq 1$, there is an equivalence $\Omega^{\geq k}(*) \simeq 0$ of chain complexes. Thus the global sections of $\mathrm{H}\Omega^{\geq k}$ is 0,

$$\Gamma_* \mathrm{H}\Omega^{\geq k} = \mathrm{H}\Omega^{\geq k}(*) = 0 .$$

By definition, this means that $\mathrm{H}\Omega^{\geq k}$ is a pure sheaf if (and only if) $k \geq 1$. The purification functor Cyc is the identity on pure sheaves, so we obtain a filtration of the pure sheaf $\Gamma^* 0$ by pure sheaves

$$\Gamma^* 0 \to \mathrm{H}\Omega^{\geq 1} \to \cdots \to \mathrm{H}\Omega^{\geq k} \to \cdots .$$

Now for each $k \geq 1$, we can choose the pure sheaf $\mathrm{H}\Omega^{\geq k}$ and follow our procedure.

We need to compute the homotopification of our chosen pure sheaf.

7.2.2 Lemma. *For any $k \in \mathbb{N}$, there is an equivalence $\Gamma_\sharp \mathrm{H}\Omega^{\geq k} \simeq \mathrm{H}\mathbb{R}$.*

Proof. For $k = 0$,we have seen that $\mathrm{H}\Omega^{\geq 0} \simeq \Gamma^*\mathrm{H}\mathbb{R}$, which is already homotopy invariant. Thus $\Gamma_\sharp \Gamma^* \mathrm{H}\mathbb{R} \simeq \mathrm{H}\mathbb{R}$. For $k \geq 1$, see [BNV16, Lemma 7.15]. □

The following family of differential refinements was introduced by Hopkins and Singer, [HS05].

7.2.3 Definition. Let E be a spectrum and $f : E \to \mathrm{H}\mathbb{R}$ a map of spectra. For each $k \geq 1$, write $\hat{E}(k)$ for the pullback

$$\begin{tikzcd} \hat{E}(k) \arrow[r] \arrow[d] & \Gamma^* E \arrow[d, "f"] \\ \mathrm{H}\Omega^{\geq k} \arrow[r] & \mathrm{H}\mathbb{R}\,. \end{tikzcd}$$

The differential cohomology diagram (6.2.14) for $\hat{E}(k)$ looks like

$$\begin{tikzcd} & \Gamma^*\Gamma_*\hat{E} \arrow[rr] \arrow[dr] & & \Gamma^* E \arrow[dr] & \\ \Sigma^{-1}\Gamma^*\mathrm{H}\mathbb{R} \arrow[ur] \arrow[dr] & & \hat{E}(k) \arrow[ur] \arrow[dr] & & \Gamma^*\mathrm{H}\mathbb{R} \\ & \mathrm{H}\Omega^{\leq k-1}[-1] \arrow[rr] \arrow[ur] & & \mathrm{H}\Omega^{\geq k} \arrow[ur] & \end{tikzcd} .$$

7.3 Ordinary differential cohomology

Take $E = \mathrm{H}\mathbb{Z}$ and the map $\mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{R}$ induced from the inclusion $\mathbb{Z} \subset \mathbb{R}$.

7.3.1 Definition. The k-th *ordinary differential cohomology group* of a manifold M, denoted $\hat{\mathrm{H}}^k(M)$ is the $(-k)$-th homotopy group

$$\hat{\mathrm{H}}^k(M) = \pi_{-k}\widehat{\mathrm{H}\mathbb{Z}}(k)(M)$$

where $\widehat{\mathrm{H}\mathbb{Z}}(k)$ is defined by the homotopy pullback square

$$\begin{tikzcd} \widehat{\mathrm{H}\mathbb{Z}}(k) \arrow[r] \arrow[d] & \Gamma^*\mathrm{H}\mathbb{Z} \arrow[d] \\ \mathrm{Cyc}(\mathrm{H}\Omega^{\geq k}) \arrow[r] & \Gamma^*\mathrm{H}\mathbb{R}\,. \end{tikzcd}$$

7.3.2. Note that $\mathrm{Cyc}(\mathrm{H}\Omega^{\geq k}) \simeq \mathrm{H}\Omega^{\geq k}$ if $k \geq 1$ and is $\mathrm{H}\mathbb{R}$ if $k = 0$.

7.3.3 Remark. The group $\hat{\mathrm{H}}^k(M)$ is also known as the *Cheeger–Simons differential characters*, or the *smooth Deligne cohomology*.

The following gives an explicit complex computing ordinary differential cohomology. This complex first appeared in the setting of complex manifolds in Deligne's work on Hodge theory (see [Del71, §2.2; Voi07, §12.3]), and is why differential cohomology is also called smooth Deligne cohomology.

7.3.4 Lemma. *Let $k \geq 1$. The sheaf of spectra $\widehat{\mathrm{H}\mathbb{Z}}(k)$ is given by applying the Eilenberg–MacLane functor* $\mathrm{H}\colon \mathsf{D}(\mathbb{Z}) \to \mathsf{Spt}$ *(Recollection 3.2.2) pointwise to the sheaf of chain complexes*

$$(\Gamma^*\mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{k-1} \to 0 \to \cdots).$$

Here Ω^i is in degree $-i-1$. Moreover, the group $\hat{\mathrm{H}}^k(M)$, for a manifold M, can be computed as the k-th sheaf cohomology group of this sheaf of chain complexes.

Proof. By construction, $\widehat{\mathrm{H}\mathbb{Z}}(k)$ comes from applying H of the sheaf of chain complexes F given by the homotopy pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Gamma^*\mathbb{Z}[0] \arrow[d] \\ \Omega^{\geq k} \arrow[r] & \Omega^\bullet . \end{tikzcd}$$

Since the bottom horizontal arrow is an inclusion, its cofiber is given by the cokernel. We have a cofiber sequence in $\mathsf{D}(\mathbb{Z})$

$$\Omega^{\geq k} \to \Omega^\bullet \to \Omega^{\leq k-1}$$

where $\Omega^{\leq k-1}$ has Ω^i in degree $-i$, and 0 above $k-1$. The cofiber of the top horizontal map is equivalent to the cofiber of the bottom horizontal map. Since we are in a stable setting, these cofiber sequences are also fiber sequences. Thus, we have a fiber sequence

$$F \to \Gamma^*\mathbb{Z}[0] \to \Omega^{\leq k-1}$$

where $\mathbb{Z}[0] \to \Omega^{\leq k-1}$ includes $\mathbb{Z}$ and Ω^0. The fiber of this inclusion is a shift of the mapping cone, which is

$$(\Gamma^*\mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{k-1} \to 0 \to \cdots)$$

Finally, note that $\pi_{-k}(\mathrm{H}F) = \mathrm{H}^k(F)$. □

7.3.5 Example. Take $k = 0$. Then $\widehat{\mathrm{H}\mathbb{Z}}(k) \simeq \Gamma^*\mathrm{H}\mathbb{Z}$ and

$$\Gamma^*\mathrm{H}\mathbb{Z}(M) = \mathrm{Hom}_{\mathsf{Spt}}(\Sigma^\infty_+\Pi_\infty(M), \mathrm{H}\mathbb{Z})$$

(Example 4.3.6 and Proposition 4.3.12). Hence $\Gamma^*\mathrm{H}\mathbb{Z}(M)$ has 0-th homotopy group $\mathrm{H}^0(M;\mathbb{Z})$.

The following two computations from Kumar's notes [Kum18].

7.3.6 Example. Take $k = 1$. We compute $\hat{\mathrm{H}}^1(M)$. By Lemma 7.3.4, we can compute $\hat{\mathrm{H}}^1(M)$ as the 1-st sheaf cohomology group of the sheaf of chain complexes $(\Gamma^*\mathbb{Z} \to \Omega^0)$. After choosing a good cover of M, we can compute this sheaf cohomology as Čech cohomology. The Čech cohomology will be the cohomology of the total complex of the following bicomplex,

$$\begin{tikzcd}
\check{\mathrm{C}}^0(\Gamma^*\mathbb{Z}) \arrow[r] \arrow[d] & \check{\mathrm{C}}^0(\Omega^0) \arrow[d] \\
\check{\mathrm{C}}^1(\Gamma^*\mathbb{Z}) \arrow[r] \arrow[d] & \check{\mathrm{C}}^1(\Omega^0) \arrow[d] \\
\check{\mathrm{C}}^2(\Gamma^*\mathbb{Z}) \arrow[r] \arrow[d] & \check{\mathrm{C}}^1(\Omega^0) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

with $\check{\mathrm{C}}^i(\Gamma^*\mathbb{Z})$ in bidegree $(0,-i)$ and $\check{\mathrm{C}}^i(\Omega^0)$ in bidgree $(-1,-i)$. The differential on this bicomplex is

$$D = d^{\mathrm{hor}} + (-1)^p d^{\mathrm{ver}} ,$$

where p is the horizontal degree. The piece of the total complex that we are interested looks like

$$\check{\mathrm{C}}^0(\Gamma^*\mathbb{Z}) \xrightarrow{D_0} \check{\mathrm{C}}^0(\Omega^0) \oplus \check{\mathrm{C}}^1(\Gamma^*\mathbb{Z}) \xrightarrow{D_1} \check{\mathrm{C}}^1(\Omega^0) \oplus \check{\mathrm{C}}^2(\Gamma^*\mathbb{Z}) .$$

If our good cover of M is $\{U_\alpha\}$ with intersections $U_{\alpha\beta}$, then an element of $\check{\mathrm{C}}^0(\Omega^0) \oplus \check{\mathrm{C}}^1(\Gamma^*\mathbb{Z})$ looks like a collection of smooth maps $f_\alpha : U_\alpha \to \mathbb{R}$ and integers $n_{\alpha\beta} \in \mathbb{Z}$. The map D_1 sends

$$D_1(f_\alpha, n_{\alpha\beta}) = (f_\alpha - f_\beta + n_{\alpha\beta}, n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha\beta})$$

In particular, an element of $\ker D_1$ consists of maps f_α that agree on intersections up to an integer. These glue together to give a (smooth) map $f : M \to \mathrm{S}^1 = \mathrm{U}_1$.

The map D_0 sends a collection (n_α) to

$$D_0(n_\alpha) = (c_{n_\alpha}, n_\alpha - n_\beta)$$

where c_{n_α} is the constant function $U_\alpha \to \mathbb{R}$ at the integer n_α. As a map $M \to \mathrm{S}^1$, these glue together to the constant map at the base point.

Thus we have an isomorphism

$$\hat{\mathrm{H}}^1(M) \cong \mathrm{Map}_{\mathrm{sm}}(M, \mathrm{U}_1) .$$

In ordinary cohomology, we have

$$\mathrm{H}^1(M;\mathbb{Z}) = \pi_0 \,\mathrm{Map}_{\mathsf{Spc}}(M, \mathrm{K}(\mathbb{Z},1)) = \pi_0 \,\mathrm{Map}_{\mathsf{Spc}}(M, \mathrm{U}_1) .$$

In this sense, differential cohomology replaced homotopy maps with smooth maps.

7.3.7 Example. Take $k = 2$. Then we have an isomorphism

$$\hat{\mathrm{H}}^2(M) \cong \{\text{line bundles on } M \text{ with connection}\}/\sim \ .$$

In ordinary cohomology, we have

$$\begin{aligned} \mathrm{H}^2(M;\mathbb{Z}) &= \pi_0 \operatorname{Map}_{\mathsf{Spc}}(M, \mathrm{K}(\mathbb{Z},2)) \\ &= \pi_0 \operatorname{Map}_{\mathsf{Spc}}(M, \mathrm{BU}_1) \\ &= \{\text{line bundles on } M\}/\sim \ . \end{aligned}$$

In this sense, the new geometric information encoded in differential cohomology is the connection.

7.4 Differential K-theory

7.4.1. Consider de Rham forms with $\mathbb{C}[u^{\pm 1}]$ coefficients, with u in degree 2. We obtain a family of pure sheaves $\mathrm{H}\Omega^{\geq k}(-;\mathbb{C}[u^{\pm 1}])$. As in Lemma 7.2.2, we have an equivalence,

$$\Gamma_\sharp \mathrm{H}\Omega^{\geq k}(-;\mathbb{C}[u^{\pm 1}]) \simeq \mathrm{H}\mathbb{C}[u^{\pm 1}]$$

7.4.2. Take $E = \mathrm{ku}$ to be the spectrum defining connective complex K-theory. The *Chern character* defines a map of spectra

$$\mathrm{ch} \colon \ \mathrm{ku} \to \mathrm{H}\mathbb{C}[u^{\pm 1}] \,.$$

The resulting family of differential cohomology theories defined by pullback squares,

$$\begin{tikzcd} \widehat{\mathrm{ku}}(k) \arrow[r] \arrow[d] & \Gamma^* \mathrm{H}\Omega^{\geq k}(-;\mathbb{C}[u^{\pm 1}]) \arrow[d] \\ \Gamma^*(\mathrm{ku}) \arrow[r, "\Gamma^*(\mathrm{ch})"'] & \Gamma^* \mathrm{H}\mathbb{C}[u^{\pm 1}] \,. \end{tikzcd}$$

first studied by Hopkins and Singer in [HS05] is called *differential* K*-theory*.

7.4.3. There are other interesting differential refinements of ku that do not arise from the pure sheaves $\mathrm{H}\Omega^{\geq k}(-;\mathbb{C}[u^{\pm 1}])$.

8 The Deligne cup product

by Araminta Amabel

Let M be a manifold. Recall that the Deligne complex $\mathbb{Z}(k)$ is the homotopy pullback

$$\begin{tikzcd} \mathbb{Z}(k) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Sigma^k \Omega^k_{\mathrm{cl}} \arrow[r] & \mathbb{R} . \end{tikzcd}$$

The goal of this section is to combine the cup product on $\mathrm{H}\mathbb{Z}$ and the wedge product on differential forms to put a ring structure on differential cohomology.

8.1 Combining the cup and wedge products

8.1.1. Notice that the cup product on $\mathrm{H}\mathbb{Z}$ and $\mathrm{H}\mathbb{R}$ and the wedge product on differential forms fit into a commutative digram.

$$\begin{tikzcd} \mathbb{Z}(n) \otimes \mathbb{Z}(m) \arrow[r] \arrow[d] & \mathrm{H}\mathbb{Z}[n] \otimes \mathrm{H}\mathbb{Z}[m] \arrow[r, "\smile"] \arrow[d] & \mathrm{H}\mathbb{Z}[m+n] \arrow[dd] \\ \Omega^n_{\mathrm{cl}} \otimes \Omega^m_{\mathrm{cl}} \arrow[r] \arrow[d, "\wedge"'] & \mathrm{H}\mathbb{R}[n] \otimes \mathrm{H}\mathbb{R}[m] \arrow[dr, "\smile"] & \\ \Omega^{n+m}_{\mathrm{cl}} \arrow[rr] & & \mathrm{H}\mathbb{R}[m+n] . \end{tikzcd}$$

8.1.2. By the definition of $\mathbb{Z}(k)$ as a pullback, we can represent $\mathbb{Z}(k)(M)$ as a triple (c, h, ω) where c is an integral degree k cocycle on M, ω is a closed k form on M, and h is a degree $k-1$ real cochain on M so that $\mathrm{d}x = \omega - c$.

8.1.3. In particular, if we represent an element of $\mathrm{C}^n(M; \mathbb{Z}(n))$ by a triple (c_1, h_1, ω_1) and an element of $\mathrm{C}^m(M; \mathbb{Z}(m))$ by a triple (c_2, h_2, ω_2) we would like the product to be a triple

$$(c_1, h_1, \omega_1) \smile (c_2, h_2, \omega_2) = (c_3, h_3, \omega_3) \in \mathrm{C}^{m+n}(M; \mathbb{Z}(m+n)) .$$

Saying that this product comes from combining the cup product and the wedge product, means that $c_3 = c_1 \smile c_2$ and $\omega_3 = \omega_1 \wedge \omega_2$. We are only left with figuring out what h_3 should be. Heuristically, h_3 should be a homotopy between c_3 and ω_3; i.e., a homotopy between the cup product and the wedge product.

8.1.4. Given forms $\omega \in \Omega^n(M)$ and $\eta \in \Omega^m(M)$, we can form the wedge product $\omega \wedge \eta \in \Omega^{n+m}(M)$ and view that as a real cochain under the map

$$\Omega^{n+m}(M) \to \mathrm{C}^{n+m}(M; \mathbb{R}) .$$

We could also map the forms ω, η to real cochains on M and then take their cup product. Let $B(\omega, \eta) \in \mathrm{C}^{n+m-1}(M; \mathbb{R})$ be a choice of natural homotopy between these two cochains so that

$$\mathrm{d}B(\omega, \eta) + B(d\omega, \eta) + (-1)^{|\omega|} B(\omega, d\eta) = \omega \wedge \eta - \omega \smile \eta\ .$$

Note that we can take $B(\omega, 0) = 0$.

8.1.5. Then the product of $(c_1, h_1, \omega_1) \in \mathrm{C}^n(M; \mathbb{Z}(n))$ and $(c_2, h_2, \omega_2) \in \mathrm{C}^m(M; \mathbb{Z}(m))$ is given by

$$(c_3, h_3, \omega_3) = (c_1 \smile c_2, (-1)^{|c_1|} c_1 \smile h_2 + h_1 \smile \omega_2 + B(\omega_1, \omega_2), \omega_1 \wedge \omega_2)\ .$$

For this to be a differential cocycle, we need to have

$$\mathrm{d}((-1)^{|c_1|} c_1 \smile h_2 + h_1 \smile \omega_2 + B(\omega_1, \omega_2) = \omega_1 \wedge \omega_2 - c_1 \smile c_2\ .$$

This will only work if (c_1, h_1, ω_1) and (c_2, h_2, ω_2) are themselves cocycles; i.e., $\mathrm{d}c_i = 0 = \mathrm{d}\omega_i$. In this case, we have

$$\omega_1 \wedge \omega_2 - \omega_1 \smile \omega_2 = \mathrm{d}B(\omega_1, \omega_2) = B(0, \omega_2) + (-1)^{|\omega_1|} B(\omega_1, 0) = \mathrm{d}B(\omega_1, \omega_2)\ .$$

Thus

$$\begin{aligned}
\mathrm{d}\big((-1)^{|c_1|} c_1 \smile h_2 + h_1 \smile \omega_2 + B(\omega_1, \omega_2)\big) &= (-1)^{|c_1|}\mathrm{d}(c_1 \smile h_2) + \mathrm{d}(h_1 \smile \omega_2) + \mathrm{d}B(\omega_1, \omega_2) \\
&= (-1)^{|c_1|}\big(\mathrm{d}c_1 \smile h_2 + (-1)^{|c_1|} c_1 \smile \mathrm{d}h_2\big) + \mathrm{d}h_1 \smile \omega_2 \\
&\qquad + (-1)^{|h_1|} h_1 \smile \mathrm{d}\omega_2 + \mathrm{d}B(\omega_1, \omega_2) \\
&= c_1 \smile \mathrm{d}h_2 + \mathrm{d}h_1 \smile \omega_2 + dB(\omega_1, \omega_2) \\
&= c_1 \smile (\omega_2 - c_2) + (\omega_1 - c_1) \smile \omega_2 + \omega_1 \wedge \omega_2 - \omega_1 \smile \omega_2 \\
&= c_1 \smile \omega_2 - c_1 \smile c_2 + \omega_1 \smile \omega_2 - c_1 \smile \omega_2 \\
&\qquad + \omega_1 \wedge \omega_2 - \omega_1 \smile \omega_2 \\
&= \omega_1 \wedge \omega_2 - c_1 \smile c_2\ .
\end{aligned}$$

8.1.6 Remark. In fact we can get $\mathbb{E}_\infty$-structure from the homotopy pullback diagram. View $\mathrm{H}\mathbb{Z}$ as a (trivially) filtered $\mathbb{E}_\infty$-algebra. View the de Rham complex $\Omega^\bullet$ as a filtered $\mathbb{E}_\infty$-algebra with filtration $\{\Omega^{\geq k}\}_{k \geq 0}$. Then the homotopy pullback of two $\mathbb{E}_\infty$-algebras is again an $\mathbb{E}_\infty$-algebra.

8.2 The Deligne cup product

Recall that we have an identification of the homotopy pullback $\widehat{\mathrm{H}\mathbb{Z}}(k)$ with the complex of sheaves

$$\mathbb{Z}(k) = \left(\ \Gamma^* \mathbb{Z} \xrightarrow{\iota} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{k-1} \ \right).$$

Under this identification, we can describe the product in differential cohomology more explicitly. This is sometimes called the *Deligne cup product*.

Let M be a manifold and $U \subset M$ an open set. Then $\mathbb{Z}(k)(U)$ is a chain complex that is $\mathrm{C}^0(U;\mathbb{Z})$ in degree 0 and $\Omega^p(U)$ in degree $p+1$.

8.2.1 Proposition. *The Deligne cup product*

$$\smile : \mathbb{Z}(k)(U) \otimes \mathbb{Z}(\ell)(U) \to \mathbb{Z}(k+\ell)(U)$$

is given by

$$x \smile y = \begin{cases} x \cdot y\,, & \deg(x) = 0 \\ x \wedge \iota y\,, & \deg(x) > 0, \deg(y) = 0 \\ x \wedge \mathrm{d}y\,, & \deg(x) > 0, \deg(y) = \ell > 0 \\ 0\,, & \text{otherwise} \end{cases} .$$

8.2.2 Remark. This is only commutative up to homotopy.

8.3 Examples

We analyze the Deligne cup product in detail in the lowest dimensions. Let M be a manifold. Recall the following computations.

- $\mathbb{Z}(0) = \Gamma^*\mathbb{Z}[0]$ is the complex with $\Gamma^*\mathbb{Z}$ in degree zero. Thus $\check{\mathrm{H}}^0(M) = \mathrm{H}^0(M;\mathbb{Z})$.
- $\check{\mathrm{H}}^1(M) = \mathrm{Map}_{\mathrm{sm}}(M, \mathrm{U}_1)$.
- $\check{\mathrm{H}}^2(M) = \{\text{line bundles on } M \text{ with connection}\}/\sim$.

Let $\mathbb{Z}(k)^\ell$ denote the degree ℓ term of the complex $\mathbb{Z}(k)$. For example, $\mathbb{Z}(3)^2 = \Omega^1$. Let $\mathcal{U}$ be a good cover for M. Using Čech cohomology for this good cover, the Deligne cup product gives a map

$$\left(\bigoplus_{i+j=k} \check{\mathrm{C}}^i(\mathcal{U};\mathbb{Z}(k)^j) \right) \otimes \left(\bigoplus_{i+j=l} \check{\mathrm{C}}^i(\mathcal{U};\mathbb{Z}(\ell)^j) \right) \longrightarrow \left(\bigoplus_{i+j=k+\ell} \check{\mathrm{C}}^i(\mathcal{U};\mathbb{Z}(k+\ell)^j) \right).$$

8.3.1 Example. The Deligne cup product

$$\mathbb{Z}(0) \otimes \mathbb{Z}(0) \to \mathbb{Z}(0)$$

should give us a way of taking two locally constant functions of $M \to \mathbb{Z}$ and producing a third. By Proposition 8.2.1, the Deligne cup product of two elements in degree 0 agrees with the ordinary cup product in $\mathrm{H}^0(M;\mathbb{Z})$; i.e., the product of the two locally constant functions.

8.3.2 Example. The Deligne cup product

$$\mathbb{Z}(0) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1)$$

should give us a way of taking a locally constant function $M \to \mathbb{Z}$ and a smooth map $g\colon M \to \mathrm{U}_1$ and producing a new smooth map $M \to \mathrm{U}_1$. In the Čech complex, we are looking at a map

$$\check{C}^0(\mathcal{U};\mathbb{Z}(0)^0) \otimes \left(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\right) \to \left(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\right)$$

Identifying these terms, we have

$$\check{C}^0(\mathcal{U};\mathbb{Z}) \otimes \left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right) \to \left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right)$$

This sends $n \otimes (f, m)$ to $(n \cdot f, n \cdot m)$.

8.3.3 Example. The Deligne cup product

$$\mathbb{Z}(1) \otimes \mathbb{Z}(0) \to \mathbb{Z}(1)$$

should give us a way of taking a locally constant function $M \to \mathbb{Z}$ and a smooth map $g\colon M \to \mathrm{U}_1$ and producing a new smooth map $M \to \mathrm{U}_1$. In the Čech complex, we are looking at a map

$$\left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right) \otimes \check{C}^0(\mathcal{U};\mathbb{Z}) \to \left(\check{C}^0(\mathcal{U};\Omega^0) \oplus \check{C}^1(\mathcal{U};\mathbb{Z})\right)$$

This map sends $(f, m) \otimes n)$ to $(f \cdot \iota n, m \cdot n)$.

More geometrically, we can describe the Deligne cup product as follows. Given a pair (n, f) where $n\colon M \to \mathbb{Z}$ is a locally constant function and $f\colon M \to \mathrm{S}^1$ is a smooth map, the Deligne cup product of n with f is the smooth function $g\colon M \to \mathrm{S}^1$ given by $g(x) = e^{2\pi i n(x)} f(x)$.

8.3.4 Remark. We can note that the Deligne cup product commutes up to homotopy,

$$\begin{tikzcd}
\mathbb{Z}(1) \otimes \mathbb{Z}(0) \arrow[r] \arrow[d] & \mathbb{Z}(1) \\
\mathbb{Z}(0) \otimes \mathbb{Z}(1) \arrow[ur] &
\end{tikzcd}$$

since $(f \cdot \iota n = n \cdot f)$ as functions to $\mathbb{R}$.

8.3.5 Example. The Deligne cup product

$$\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(2)$$

should give us a way of taking two smooth maps $M \to \mathrm{U}_1$ and producing a line bundle on M with connection. In the Čech complex, we are looking at a map

$$\left(\check{C}^0(\mathcal{U};\mathbb{Z}(1)^1) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(1)^0)\right)^{\otimes 2} \to \left(\check{C}^0(\mathcal{U};\mathbb{Z}(2)^2) \oplus \check{C}^1(\mathcal{U};\mathbb{Z}(2)^1) \oplus \check{C}^2(\mathcal{U};\mathbb{Z}(2)^0)\right).$$

Then the Deligne cup product sends

$$(f, n) \otimes (g, m) \mapsto (n_{\alpha\beta} \cdot m_{\beta\gamma}, n_\alpha\beta \cdot g_\beta + 0, f_\alpha \mathrm{d}g_\alpha)\,.$$

If we think of (f, n) and (g, m) as smooth maps $M \to \mathrm{U}_1$, then $(n_{\alpha\beta} \cdot m_{\beta\gamma}, n_{\alpha\beta} \cdot g_\beta, f_\alpha \mathrm{d}g_\alpha)$ corresponds to the line bundle with transition function $n_{\alpha\beta} \cdot g_\beta$ and connection given by one form $(2\pi i) f_\alpha \mathrm{d}g_\alpha$.

By [Beĭ84b, Lemma 1.3.1], the curvature of $f \smile g$ is $\mathrm{dlog}(f) \wedge \mathrm{dlog}(g)$.

9 Fiber integration

by Araminta Amabel

The goal of this section is to define a refinement of *fiber integration* (along with its usual properties) in the setting of differential cohomology. In ordinary cohomology, we get a fiber integration map from combining the Thom isomorphism and the suspension isomorphism. Let $E \to B$ be an oriented fiber bundle with fiber a compact manifold of dimension k. Let $E \hookrightarrow \mathbb{R}^N$ be an embedding with normal bundle ν, and let E^ν denote the Thom space of ν. Then fiber integration is given by the composite

$$\mathrm{H}^{q+k}(E) \xrightarrow{\sim} \mathrm{H}^{q+N}(E^\nu) \xrightarrow{\mathrm{PT}} \mathrm{H}^{q+N}(B_+ \wedge \mathrm{S}^N) \simeq \mathrm{H}^q(B_+)\,,$$

where the first map is the Thom isomorphism, the second map is the Pontryagin–Thom collapse map, and the third map is the suspension isomorphism. Recall that the Thom isomorphism is given by taking the cup product with the Thom class.

To do fiber integration in differential cohomology, we need to provide differential refinements of the following:

(1) Thom classes/orientations.

(2) The suspension isomorphism.

To do this, we combine fiber integration in ordinary cohomology with integration of forms.

9.1 Differential integration

The input will be a fiber bundle of manifolds

$$M \to E \to X\,,$$

where M is a closed, smooth manifold of dimension d. The output will be a map of spectra

$$\mathbb{Z}(k)(E) \to \Sigma^d \mathbb{Z}(k-d)(X)$$

where $\mathbb{Z}(k)$ is the pullback

$$\begin{tikzcd} \mathbb{Z}(k) \arrow[r] \arrow[d] & \Gamma^* \mathrm{H}\mathbb{Z} \arrow[d] \\ \Sigma^{-k}\mathrm{H}\Omega^k_{\mathrm{cl}} \arrow[r] & \Gamma^* \mathrm{H}\mathbb{R} \end{tikzcd}$$

in $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spt})$ and, similarly, $\mathbb{Z}(k-d)$ is the pullback

$$\begin{tikzcd} \mathbb{Z}(k-d) \arrow[r] \arrow[d] & \Gamma^* \mathrm{H}\mathbb{Z} \arrow[d] \\ \Sigma^{d-k}\mathrm{H}\Omega^{k-d}_{\mathrm{cl}} \arrow[r] & \Gamma^* \mathrm{H}\mathbb{R}\,. \end{tikzcd}$$

To produce a map $\mathbb{Z}(k) \to \Sigma^d \mathbb{Z}(k-d)$, it therefore suffices to produce maps $\mathrm{H}\mathbb{Z} \to \Sigma^d \mathrm{H}\mathbb{Z}$ and $\Omega^k_{\mathrm{cl}} \to \Omega^{k-d}_{\mathrm{cl}}$ together with a path between their images in $\Sigma^d \Gamma^* \mathrm{H}\mathbb{R}$.

9.2 Differential Thom classes and orientations

9.2.1 Definition. Let M be a smooth compact manifold and $V \to M$ a real vector bundle of dimension k. A *differential Thom cocycle* on V is a cocycle

$$U = (c, h, \omega) \in \check{Z}(k)^k_{\mathrm{c}}(V)$$

such that, for each $m \in M$

$$\int_{V_m} \omega = \pm 1$$

9.2.2 Remark. A differential Thom class determines a ordinary Thom class in integral cohomology $\mathrm{H}^k_{\mathrm{c}}(V; \mathbb{Z})$.

9.2.3 Definition [HS05, Definition 2.9]**.** An $\hat{\mathrm{H}}$-*orientation* of $p\colon E \to B$ consists of the following three pieces of data:

(1) A smooth embedding $E \subset B \times \mathbb{R}^N$ for some N.

(2) A tubular neighborhood $W \subset B \times \mathbb{R}^N$.

(3) A differential Thom cocycle U on W.

9.3 Differential fiber integration

Our hope is to get an analogue of the suspension isomorphism

$$\mathrm{H}^{q+N}_{\mathrm{c}}(B \times \mathbb{R}^N) \simeq \mathrm{H}^q(B)\,.$$

To understand the correct analogue of the suspension isomorphism in the differential setting, let us consider the most simple case.

9.3.1 Example. Consider the case when B is a point and $N = 1$. Then the ordinary suspension isomorphism says that

$$\mathrm{H}^1(\mathrm{S}^1; \mathbb{Z}) \cong \mathrm{H}^0(\mathrm{pt}; \mathbb{Z}) \simeq \mathbb{Z}$$

The calculation $\mathrm{H}^1(\mathrm{S}^1; \mathbb{Z}) \cong \mathbb{Z}$ is by degree:

$$\mathrm{H}^1(\mathrm{S}^1; \mathbb{Z}) = \pi_0 \operatorname{Map}_{\mathrm{Spc}}(\mathrm{S}^1, \mathrm{K}(\mathbb{Z}, 1)) = \pi_0 \operatorname{Map}_{\mathrm{Spc}}(\mathrm{S}^1, \mathrm{S}^1) \xrightarrow[\deg]{\sim} \mathbb{Z}\,.$$

In differential cohomology, we have an isomorphism

$$\hat{\mathrm{H}}^1(\mathrm{S}^1) \cong \operatorname{Map}_{\mathrm{sm}}(\mathrm{S}^1, \mathrm{S}^1)\,.$$

We still have a degree map

$$\deg\colon\ \mathrm{Map}_{\mathrm{sm}}(\mathrm{S}^1, \mathrm{S}^1) \to \mathbb{Z}\,,$$

but it is no longer an isomorphism.

The upshot is that we are looking for a suspension *map* not an isomorphism.

9.3.2. We start by working with the trivial bundle $B \times \mathbb{R}^N \to B$ and defining integration for compactly-supported forms. This is [HS05, §3.4]. Define the map

$$\int_{B\times\mathbb{R}^N/B}\colon\ \check{\mathrm{C}}(p+N)^{q+N}_{\mathrm{c}}(B\times\mathbb{R}^N) \to \check{\mathrm{C}}(p)^q(B)$$

by the slant product with a fundamental cycle $Z_N \in \mathrm{C}_N(\mathbb{R}^N;\mathbb{Z})$,

$$(c,h,\omega) \mapsto \left(c/Z_N, h/Z_N, \int_{B\times\mathbb{R}^N/B}\omega\right)$$

Note that this is simply a map, *not* an isomorphism.

9.3.3 Remark. Checking that the slant product goes through to differential cohomology seems to require some work. See [HS05, §3.4].

9.3.4 Definition [HS05, Definition 3.11]**.** Let $p\colon E \to B$ be an $\hat{\mathrm{H}}$-oriented map of smooth manifolds with boundary of relative dimension k. The *integration map* is the map

$$\int_{E/B}\colon\ \check{\mathrm{C}}(p+k)^{q+k}(E) \to \check{\mathrm{C}}(p)^q(B)$$

given by the composite

$$\check{\mathrm{C}}(p+k)^{q+k}(E) \xrightarrow{\ \smile U\ } \check{\mathrm{C}}(p+N)^{q+N}_{\mathrm{c}}(B\times\mathbb{R}^N) \xrightarrow{\int_{\mathbb{R}^N}(-)} \check{\mathrm{C}}(p)^q_{\mathrm{c}}(B)\,.$$

9.3.5 Example. In dimension 1, the only closed manifold is S^1. If $E \to B$ is an oriented S^1-bundle, then integration along the fibers defines a map

$$\int_{E/B}\colon\ \hat{\mathrm{H}}^2(E) \to \hat{\mathrm{H}}^1(E)$$

If $x \in \hat{\mathrm{H}}^2(E)$ corresponds to a line bundle with connection, then

$$\int_{E/B} x$$

represents the function $B \to \mathrm{S}^1$ sending $b \in B$ to the monodromy of x computed around the fiber E_b.

10 Digression: the Transfer Conjecture

by Peter Haine

10.1 Introduction

Let X be a space. We have seen that the constant sheaf of spaces $\Gamma^*(X)$ on Mfld is given by the formula

$$\Gamma^*(X) = \mathrm{Map}_{\mathsf{Spc}}(\Pi_\infty(M), X)$$

(Proposition 4.3.12). If $X = \Omega^\infty E$ is the infinite loop space of a spectrum E, then the sheaf $\Gamma^*(X)$ acquires additional functoriality: for any finite covering map between manifolds $f : N \to M$, the *Becker–Gottlieb transfer*

$$\Sigma^\infty_+ \Pi_\infty(M) \to \Sigma^\infty_+ \Pi_\infty(N)$$

[Hau13, Definition 3.11] induces a *transfer* map

$$f_* : \Gamma^*(X)(N) \to \Gamma^*(X)(M)\,.$$

This enhanced functoriality can be used to make $\Gamma^*(X)$ into a copresheaf on a 2-category $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$ with objects smooth manifolds and morphisms *correspondences*

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr, "f"] & \\ M_0 & & M_1\,, \end{tikzcd}$$

where f is a finite covering map. Composition in $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$ is given by pullback.

For a sheaf F on Mfld, Quillen conjectured that an extension of F to $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$ is just another way of encoding an $\mathbb{E}_\infty$-structure on F. However, when Quillen originally formulated this *Transfer Conjecture*, the language to express the higher coherences necessary for the validity of the result was not available. Moreover, Quillen's original formulation was *dis*proven by Kraines and Lada [KL79; Noe14].

The goal of this section is to explain how to deduce the following corrected version of the Transfer Conjecture from very general results of Bachmann–Hoyois on commutative algebras and ∞-categories of spans [BH21, Appendix C].

10.1.1 Theorem (Transfer Conjecture; Corollaries 10.4.5 and 10.4.6)**.** *Let C be a presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathsf{Sh}(\mathsf{Mfld}; \mathsf{CMon}(C))$$

between functors $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}) \to C$ whose restriction to $\mathsf{Mfld}^{\mathrm{op}}$ is a sheaf and sheaves of commutative monoids in C. This further restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{loc},\mathbb{R}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathsf{CMon}(C)$$

between functors $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}) \to C$ *whose restriction to* $\mathsf{Mfld}^{\mathrm{op}}$ *is an* $\mathbb{R}$*-invariant sheaf and commutative monoids in* C.

10.1.2 Example. Setting $C = \mathsf{Spc}$ in Theorem 10.1.1 gives an equivalence between functors

$$\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}) \to \mathsf{Spc}$$

whose restriction to $\mathsf{Mfld}^{\mathrm{op}}$ is an $\mathbb{R}$-invariant sheaf and $\mathbb{E}_\infty$-spaces. Restricting to grouplike objects on both sides and applying the Segal's Recognition Principle for connective spectra [HA, Remark 5.2.6.26] provides an equivalence between grouplike objects of

$$\mathrm{Fun}_{\mathrm{loc},\mathbb{R}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), \mathsf{Spc})$$

and the ∞-category $\mathsf{Spt}_{\geq 0}$ of connective spectra.

10.1.3 Remark. The Becker–Gottlieb transfer is defined in more generality than finite covering maps; for example, for proper submersions. It is possible to modify Theorem 10.1.1 to encode this additional generality. However, since pullbacks along proper submersions do not exist in the category of manifolds, in order for composition of correspondences where one leg is proper to be defined, one needs to work with *derived* manifolds [CS19b; Spi10]. For the sake of simplicity, we will satisfy ourselves with just working with manifolds and finite covering maps.

In order to give a more precise formulation of Theorem 10.1.1, we'll first review constructing 2-categories of *correspondences* or *spans* from 1-categories (§10.2). We then briefly recall the role that ∞-categories of spans play in encoding $\mathbb{E}_\infty$-structures (§ 10.3). Finally, we walk through [BH21, Appendix C] in the case of interest and explain how to deduce the Transfer Conjecture from their results (§ 10.4).

10.2 Categories of spans

In this section we explain how to construct the 2-category $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$ of correspondences of manifolds appearing in the Transfer Conjecture. This is a special case of a general construction for ∞-categories due to Barwick [Bar17, §§3–5]. If D is an n-category, then Barwick's ∞-category of spans in D is an $(n+1)$-category. In order to avoid explaining how to deal with the homotopy coherence problems that arise, we only present the 1-categorical case as we can give a simple definition as a 2-category.

10.2.1 Construction (2-category of spans)**.** Let D be a 1-category, and let $L, R \subset \mathrm{Mor}(D)$ be two classes of morphisms in D satisfying the following properties:

(10.2.1.1) The classes L and R contain all isomorphisms.

(10.2.1.2) The classes L and R are each stable under composition.

(10.2.1.3) Given a morphism $\ell : X \to Z$ in L and morphism $r : Y \to Z$ in R, there exists a pullback diagram

$$\begin{tikzcd} W \arrow[r, "\bar{r}"] \arrow[d, "\bar{\ell}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] \\ X \arrow[r, "\ell"'] & Z \end{tikzcd}$$

in D where $\bar{\ell} \in L$ and $\bar{r} \in R$.

Define a 2-category $\mathsf{Span}(D; L, R)$ as follows. The objects of $\mathsf{Span}(D; L, R)$ are the objects of D. Given objects $X_0, X_1 \in D$, the groupoid $\mathrm{Map}_{\mathsf{Span}(D;L,R)}(X_0, X_1)$ has objects diagrams

$$\begin{tikzcd} & Y \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ X_0 & & X_1 , \end{tikzcd}$$

in D where $\ell \in L$ and $r \in R$, and morphisms isomorphisms of diagrams. Composition is given by pullback of spans: given morphisms $X_0 \to X_1$ and $X_1 \to X_2$ corresponding to spans

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & & & & Z \arrow[dl] \arrow[dr] & \\ X_0 & & X_1 & \text{and} & X_1 & & X_2 , \end{tikzcd}$$

the composite morphism $X_0 \to X_2$ in $\mathsf{Span}(D; L, R)$ is defined as the large pullback span

$$\begin{tikzcd} & & Y \times_{X_1} Z \arrow[dl] \arrow[dr] & & \\ & Y \arrow[dl] \arrow[dr] & & Z \arrow[dl] \arrow[dr] & \\ X_0 & & X_1 & & X_2 . \end{tikzcd}$$

10.2.2 Notation. Let D be a 1-category. We write $\mathrm{all} \coloneqq \mathrm{Mor}(D)$ for the class of all morphisms in D. If D has pullbacks, we write

$$\mathsf{Span}(D) \coloneqq \mathsf{Span}(D; \mathrm{all}, \mathrm{all})$$

for the 2-category of spans of arbitrary morphisms in D.

10.2.3 Observation. Let D be a category and R a class of morphisms in D such that the pullback of a morphism in R along an arbitrary morphism of D exists, and the class R is stable under pullback. Then there is a natural faithful functor

$$D^{\mathrm{op}} \to \mathsf{Span}(D; \mathrm{all}, R)$$

given by the identity on objects, and on morphisms by sending a morphism $f \colon X \to Y$ to the span

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, equal] & \\
Y & & X .
\end{tikzcd}
$$

10.2.4 Example. Write fcov $\subset$ Mor(Mfld) for the class of finite covering maps of manifolds. Note that the pullback of a finite covering map of manifolds along any morphism exists, and the class of finite covering maps is stable under pullback. We write

$$\mathrm{Cor}_{\mathrm{fcov}}(\mathrm{Mfld}) \coloneqq \mathrm{Span}(\mathrm{Mfld}; \mathrm{all}, \mathrm{fcov})$$

for the 2-category with objects manifolds and morphisms *correspondences*[3] of manifolds

$$
\begin{tikzcd}
& N \arrow[dl] \arrow[dr, "f"] & \\
M_0 & & M_1 ,
\end{tikzcd}
$$

where f is a finite covering map.

10.2.5 Example. Write fold $\subset$ Mor(Mfld) for the class of maps that are finite coproducts of fold maps of manifolds, i.e., finite coproducts of fold maps $\nabla \colon M^{\sqcup i} \to M$ from a finite disjoint union of copies of M to M. Note that coproduct decompositions are stable under all pullbacks that exist in the category of manifolds, hence the class fold is stable under pullback. We write

$$\mathrm{Cor}_{\mathrm{fold}}(\mathrm{Mfld}) \coloneqq \mathrm{Span}(\mathrm{Mfld}; \mathrm{all}, \mathrm{fold})$$

for the 2-category with objects manifolds and morphisms correspondences of manifolds

$$
\begin{tikzcd}
& N \arrow[dl] \arrow[dr, "f"] & \\
M_0 & & M_1 ,
\end{tikzcd}
$$

where f is a finite coproduct of fold maps.

Note that fold $\subset$ fcov, so that $\mathrm{Cor}_{\mathrm{fold}}(\mathrm{Mfld})$ defines a subcategory of $\mathrm{Cor}_{\mathrm{fcov}}(\mathrm{Mfld})$ that contains all objects.

10.3 Spans and commutative monoids

In this section we briefly recall the role that ∞-categories of spans play in encoding $\mathbb{E}_\infty$-structures. We begin by introducing the relevant 2-category of spans.

10.3.1 Notation. Write Fin for the category of finite sets. Given an ∞-category C with a terminal object, we write $*$ for the terminal object.

[3] The term "correspondence" is just another name for a span; "correspondence" seems to be the more common term in geometry.

10.3.2 Recollection. Let C be an ∞-category with finite products. A *commutative monoid* or $\mathbb{E}_\infty$*-monoid* in C is a functor $M\colon \mathsf{Fin}_* \to C$ such that $M(*) \xrightarrow{\sim} *$ and for each integer $n \geq 1$, the collapse maps $\{1, \dots, n\}_+ \to \{i\}_+$ induce an equivalence

$$M(\{1, \dots, n\}_+) \xrightarrow{\sim} \prod_{i=1}^{n} M(\{i\}_+) .$$

We write $\mathsf{CMon}(C) \subset \mathrm{Fun}(\mathsf{Fin}_*, C)$ for the full subcategory spanned by the commutative monoids.

10.3.3 Remark. By induction, a functor $M\colon \mathsf{Fin}_* \to C$ is a commutative monoid if and only if $M(*) \xrightarrow{\sim} *$ and for every pair $S, T \in \mathsf{Fin}_*$, the functor M carries the pushout square

$$\begin{tikzcd} S \vee T \arrow[r, "* \vee \mathrm{id}_T"] \arrow[d, "\mathrm{id}_S \vee *"'] & T \arrow[d] \\ S \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & * \end{tikzcd}$$

to a *pullback* square in C.

10.3.4 Observation. The 2-category $\mathsf{Span}(\mathsf{Fin})$ is *semiadditive*: the direct sum in $\mathsf{Span}(\mathsf{Fin})$ is given by disjoint union of finite sets. See [BH21, Lemma C.3; Bar17, Proposition 4.3] for more general results on the semiadditivity of ∞-categories of spans.

10.3.5 Observation (finite pointed sets via spans)**.** Write inj for the class of injective maps in Fin. The functor

$$\mathsf{Fin}_* \to \mathsf{Span}(\mathsf{Fin}; \mathrm{inj}, \mathrm{all})$$

given by sending $X_+ \mapsto X$ and a morphism $f\colon X_+ \to Y_+$ to the span

$$\begin{tikzcd} & f^{-1}(Y) \arrow[dl, hook'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

is an equivalence of categories.

The category $\mathsf{Span}(\mathsf{Fin}; \mathrm{inj}, \mathrm{all})$ is often referred to as the category of finite sets and *partially defined maps*.

The importance of transfers in $\mathbb{E}_\infty$-structures is explained by the following universal property of the 2-category $\mathsf{Span}(\mathsf{Fin})$ of spans of finite sets.

10.3.6 Proposition (Cranch [BH21, Proposition C.1; Cra10, §5])**.** *Let C be an ∞-category with finite products. Then the restriction*

$$\mathrm{Fun}(\mathsf{Span}(\mathsf{Fin}), C) \to \mathrm{Fun}(\mathsf{Fin}_*, C)$$

along the inclusion $\mathsf{Fin}_* \to \mathsf{Span}(\mathsf{Fin})$ *restricts to an equivalence between:*

(10.3.6.1) *Functors $M\colon \mathsf{Span}(\mathsf{Fin}) \to C$ that preserve finite products (equivalently, $M|_{\mathsf{Fin}^{\mathrm{op}}}$ preserves finite products).*

(10.3.6.2) *Commutative monoids in C.*

The inverse is given by right Kan extension.

The 2-category $\mathsf{Span}(\mathsf{Fin})$ has a second (related) universal property: $\mathsf{Span}(\mathsf{Fin})$ is the free semiadditive ∞-category generated by a single object.

10.3.7 Proposition (Harpaz [Har20, Theorem 1.1])**.** *Let C be a semiadditive ∞-category. Then evaluation at $* \in \mathsf{Span}(\mathsf{Fin})$ defines an equivalence*

$$\mathrm{Fun}^{\oplus}(\mathsf{Span}(\mathsf{Fin}), C) \xrightarrow{\sim} C\,.$$

10.4 The Transfer Conjecture after Bachmann–Hoyois

In this section we outline work of Bachmann–Hoyois that implies the Transfer Conjecture [BH21, Appendix C]. The perspective on commutative monoids in C as finite product-preserving functors $\mathsf{Span}(\mathsf{Fin}) \to C$ (Proposition 10.3.6) is fundamental to proving the Transfer Conjecture.

The first step is to relate finite product-preserving functors $\mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld}) \to C$ to presheaves of commutative monoids on Mfld. Then we impose the sheaf condition to pass from $\mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld})$ to $\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$.

10.4.1 Notation. Write $\Theta\colon \mathsf{Mfld}^{\mathrm{op}} \times \mathsf{Span}(\mathsf{Fin}) \to \mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld})$ for the functor given on objects by the assignment

$$(M, I) \mapsto M^{\sqcup I}$$

and on morphisms by the assignment

$$(M \to N, I_0 \leftarrow J \to I_1) \quad \mapsto \quad \begin{tikzcd} & M^{\sqcup J} \arrow[dl] \arrow[dr] & \\ N^{\sqcup I_0} & & M^{\sqcup I_1}\,. \end{tikzcd}$$

The functor Θ is the universal functor that preserves finite products in each variable:

10.4.2 Proposition [BH21, Proposition C.5]**.** *Let C be an ∞-category with finite products. Then the restriction functor*

$$\Theta^*\colon\ \mathrm{Fun}(\mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld}), C) \to \mathrm{Fun}(\mathsf{Mfld}^{\mathrm{op}} \times \mathsf{Span}(\mathsf{Fin}), C)$$

restricts to an equivalence

$$\mathrm{Fun}^{\times}(\mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathsf{Mfld}^{\mathrm{op}}, \mathsf{CMon}(C))\,.$$

The inverse is given by right Kan extension along Θ.

Since every finite covering map is locally a fold map, we see:

10.4.3 Proposition [BH21, Proposition C.11]**.** *Let C be an ∞-category with finite products. Then the restriction functor*

$$\mathrm{Fun}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \to \mathrm{Fun}(\mathsf{Cor}_{\mathrm{fold}}(\mathsf{Mfld}), C)$$

restricts to an equivalence between the full subcategories of those functors whose restrictions to $\mathsf{Mfld}^{\mathrm{op}}$ *are sheaves. The inverse is given by right Kan extension.*

10.4.4 Notation. Write

$$\mathrm{Fun}_{\mathrm{loc}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \subset \mathrm{Fun}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C)$$

for the full subcategory spanned by those functors F whose restrictions to $\mathsf{Mfld}^{\mathrm{op}}$ are sheaves.

We now arrive at Quillen's Transfer Conjecture:

10.4.5 Corollary (Transfer Conjecture)**.** *Let C be an ∞-category with all limits. Restriction along the inclusion* $\mathsf{Mfld}^{\mathrm{op}} \hookrightarrow \mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$ *defines an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathsf{Sh}(\mathsf{Mfld}; \mathsf{CMon}(C)) .$$

Combining Proposition 4.3.1 and Corollary 10.4.5 shows:

10.4.6 Corollary. *Let C be a presentable ∞-category. Restriction along the inclusion*

$$\mathsf{Mfld}^{\mathrm{op}} \hookrightarrow \mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$$

defines an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathrm{loc},\mathbb{R}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld}; \mathsf{CMon}(C)) .$$

Post-composing with the global sections functor Γ_* defines an equivalence

$$\mathrm{Fun}_{\mathrm{loc},\mathbb{R}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathsf{CMon}(C) .$$

10.4.7. Unwinding the definitions we see that restriction along the inclusion

$$\mathsf{Span}(\mathsf{Fin}) \subset \mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld})$$

defines an equivalence

$$\mathrm{Fun}_{\mathrm{loc},\mathbb{R}}(\mathsf{Cor}_{\mathrm{fcov}}(\mathsf{Mfld}), C) \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathsf{Span}(\mathsf{Fin}), C) \simeq \mathsf{CMon}(C) .$$

A Technical details from topos theory

by Peter Haine

The purpose of this appendix is to prove a number of technical results used throughout the text. We have relegated these proofs to this appendix because of one of the following reason:

(1) They are lengthy and, while the result is important, the proof is not important to know.

(2) They require some knowledge from the theory of ∞-topoi.

In §A.1, we explain a formal procedure to get from sheaves of spaces to sheaves valued in another presentable ∞-category C. This lets us deduce many results about sheaves on Mfld valued in a general presentable ∞-category C from the case $C = \mathsf{Spc}$. Section A.2 explains the important properties of the functor given by restricting a sheaf defined on Mfld to a sheaf defined on only a single manifold. Section A.3 explains why this restriction procedure commutes with sheafification. In §A.4, we give some background on notions of "completeness" for ∞-topoi. Section A.5 shows that equivalences in $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$ can be checked on stalks and uses this to show that $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$ satisfies the strongest of these completeness notions (Proposition A.5.4). This also implies that $\mathsf{Sh}(\mathsf{Mfld}; C)$ is equivalent to the category of C-valued sheaves on the subcategory $\mathsf{Euc} \subset \mathsf{Mfld}$ spanned by the Euclidean spaces (Corollary A.5.6).

Since we are mostly interested in sheaves of spaces in this appendix, we adopt the following notational convention.

A.0.1 Notation. We write $\mathsf{Sh}(\mathsf{Mfld}) \coloneqq \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$ for the ∞-topos of sheaves of spaces on Mfld.

A.0.2 Remark. For this appendix, it is sufficient to know that the ∞-category of sheaves of spaces on a site is an ∞-topos, and that a *geometric morphism* of ∞-topoi is a right adjoint functor $f_* \colon \mathsf{X} \to \mathsf{Y}$ whose left adjoint f^* is left exact.

A.1 From sheaves of spaces to C-valued sheaves

Let C be a presentable ∞-category. In this section we explain a formal procedure that allows us to pass from the ∞-category $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})$ of sheaves of spaces on Mfld to the ∞-category $\mathsf{Sh}(\mathsf{Mfld}; C)$ of C-valued sheaves on Mfld. We'll also recall the basics of tensor products of presentable ∞-categories and explain how to describe $\mathsf{Sh}(\mathsf{Mfld}; C)$ as the tensor product

$$\mathsf{Sh}(\mathsf{Mfld}; C) \simeq \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc}) \otimes C .$$

The first thing to observe is that if $G \colon \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc})^{\mathrm{op}} \to C$ is a functor that preserves limits, then the restriction $G \colon \mathsf{Mfld}^{\mathrm{op}} \to C$ is a sheaf. It turns out that all C-valued sheaves arise in this way.

A.1.1 Proposition [SAG, Proposition 1.3.1.7]**.** *Let (S, τ) be an ∞-site and C an ∞-category with all limits. Write $よ_\tau \colon S \to \mathsf{Sh}_\tau(S; \mathsf{Spc})$ for the τ-sheafification of the Yoneda embedding. Then* *pre-composition with $よ_\tau$ defines an equivalence*

$$\operatorname{Fun}^{\lim}(\mathsf{Sh}_\tau(S;\mathsf{Spc})^{\mathrm{op}}, C) \xrightarrow{\sim} \mathsf{Sh}_\tau(S;C)\,.$$

Now we give the ∞-category $\operatorname{Fun}^{\lim}(\mathsf{Sh}(\mathsf{Mfld})^{\mathrm{op}}, C)$ a description in terms of a universal property of presentable ∞-categories.

A.1.2 Recollection [HA, Proposition 4.8.1.17]**.** Let C and D be presentable ∞-categories. The *tensor product* of presentable ∞-categories $C \otimes D$ along with the functor $\otimes\colon C \times D \to C \otimes D$ are characterized by the following universal property: for any presentable ∞-category E, restriction along $\otimes$ defines an equivalence

$$\operatorname{Fun}^{\mathrm{colim}}(C \otimes D, E) \xrightarrow{\sim} \operatorname{Fun}^{\mathrm{colim},\mathrm{colim}}(C \times D, E)\,.$$

Here the right-hand side is the full subcategory of $\operatorname{Fun}(C \times D, E)$ spanned by those functors $C \times D \to E$ that preserve colimits separately in each variable. The tensor product of presentable ∞-categories defines a functor

$$\otimes\colon \mathsf{Pr}^{\mathrm{L}} \times \mathsf{Pr}^{\mathrm{L}} \to \mathsf{Pr}^{\mathrm{L}}$$

and can be used to equip Pr^{L} with the structure of a symmetric monoidal ∞-category.

The tensor product $C \otimes D$ admits the following useful (seemingly asymmetric) description:

$$C \otimes D \simeq \operatorname{Fun}^{\lim}(C^{\mathrm{op}}, D)\,.$$

If $F\colon D \to D'$ is a right adjoint functor of presentable ∞-categories, then the induced right adjoint

$$\mathrm{id}_C \otimes F\colon C \otimes D \simeq \operatorname{Fun}^{\lim}(C^{\mathrm{op}}, D) \to \operatorname{Fun}^{\lim}(C^{\mathrm{op}}, D') \simeq C \otimes D'$$

is given by post-composition with F. Unfortunately, the left adjoint to $\mathrm{id}_C \otimes F$ does not generally admit a simple description. However, if C is compactly assembled and the left adjoint to F is left exact, then the left adjoint to $\mathrm{id}_C \otimes F$ admits a simple description; see [Hai21, §2.2].

A.1.3 Example. For any presentable ∞-category C, we have a natural equivalence

$$\mathsf{Sh}(\mathsf{Mfld}) \otimes C \xrightarrow{\sim} \mathsf{Sh}(\mathsf{Mfld}; C)\,.$$

A.2 Restriction to a manifold

We now give an alternative description of the functor $\mathsf{Sh}(\mathsf{Mfld}; C) \to C$ that sends a sheaf to its value on a manifold M.

A.2.1 Notation. Let T be a topological space and C a presentable ∞-category. Write

$$\mathsf{PSh}(T;C) \coloneqq \operatorname{Fun}(\operatorname{Open}(T)^{\mathrm{op}}, C)$$

and write $\mathrm{Sh}(T;C) \subset \mathrm{PSh}(T;C)$ for the ∞-category of C-valued sheaves on T. Write

$$\Gamma_{T,*} \colon \mathrm{Sh}(T;C) \to C$$

for the *global sections* functor, defined by $\Gamma_{T,*}(F) \coloneqq F(T)$, and write $\Gamma_T^* \colon C \to \mathrm{Sh}(T;C)$ for the left adjoint to $\Gamma_{T,*}$, i.e., the *constant sheaf* functor.

A.2.2 Observation. Let C be a presentable ∞-category and M a manifold. The forgetful functor $\mathrm{Open}(M) \to \mathrm{Mfld}$ preserves finite limits and is a morphism of sites. Moreover, the forgetful functor satisfies the *covering lifting property* [Pst18, Definition A.12]. In particular:

(A.2.2.1) The presheaf retriction functor $(-)|_M \colon \mathrm{PSh}(\mathrm{Mfld};C) \to \mathrm{PSh}(M;C)$ carries sheaves to sheaves.

(A.2.2.2) The functor $(-)|_M \colon \mathrm{Sh}(M;C) \to \mathrm{Sh}(\mathrm{Mfld};C)$ is both a left and right adjoint [Pst18, Proposition A.18].

A.2.3. Note that the functor given by sending a sheaf E on Mfld to its value on M is given by the composite

$$\mathrm{Sh}(\mathrm{Mfld};C) \xrightarrow{(-)|_M} \mathrm{Sh}(M;C) \xrightarrow{\Gamma_{M,*}} C\ .$$

A.2.4. Moreover, if $p \colon N \to M$ is a morphism in Mfld, then there is a canonical natural transformation fitting into the triangle

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Mfld};C) \arrow[rr, "(-)|_M"] \arrow[dr, "(-)|_N"'] & \arrow[d, phantom, "\Uparrow \mathrm{can}_p"] & \mathrm{Sh}(M;C) \\
& \mathrm{Sh}(N;C) \arrow[ur, "p_*"'] &
\end{tikzcd}$$

defined as follows: given a sheaf E on Mfld and an open subset $U \subset M$, the morphism

$$E(U) \to E(p^{-1}(U))$$

is induced by the projection $p^{-1}(U) \to U$ by the functoriality of E. In particular, upon taking global sections, the morphism

$$\mathrm{can}_p \colon E(M) = \Gamma_{M,*}(E|_M) \to \Gamma_{M,*}(p_*(E|_N)) = E(N)$$

is the morphism $E(M) \to E(N)$ induced by p by the functoriality of E.

A.3 Sheafification

Next we show that restriction from $\mathrm{Sh}(\mathrm{Mfld};C)$ to $\mathrm{Sh}(M;C)$ commutes with sheafification.

A.3.1. Consider the commutative square

$$\begin{tikzcd} \mathsf{Sh}(\mathsf{Mfld};C) \arrow[r, hook] \arrow[d, "(-)|_M"'] & \mathsf{PSh}(\mathsf{Mfld};C) \arrow[d, "(-)|_M"] \\ \mathsf{Sh}(M;C) \arrow[r, hook] & \mathsf{PSh}(M;C)\,. \end{tikzcd}$$

Using the unit and counit of the sheafification-inclusion adjunctions for Mfld and M, one can define an *exchange transformation*

$$\mathrm{Ex}\colon \mathrm{S}_M\circ(-)|_M \to (-)|_M\circ\mathrm{S}_{\mathsf{Mfld}}\,.$$

See [HA, Definition 4.7.4.13; Hai21, Definition 1.1]. The exchange morphism Ex fits into a diagram

$$\begin{tikzcd} \mathsf{PSh}(\mathsf{Mfld};C) \arrow[r, "\mathrm{S}_{\mathsf{Mfld}}"] \arrow[d, "(-)|_M"'] \arrow[dr, phantom, "\overset{\mathrm{Ex}}{\Nearrow}"] & \mathsf{Sh}(\mathsf{Mfld};C) \arrow[d, "(-)|_M"] \\ \mathsf{PSh}(M;C) \arrow[r, "\mathrm{S}_M"'] & \mathsf{Sh}(M;C)\,. \end{tikzcd}$$

A.3.2 Lemma. *Let C be a presentable ∞-category and M a manifold. Then the exchange transformation*

$$\mathrm{Ex}\colon \mathrm{S}_M\circ(-)|_M \to (-)|_M\circ\mathrm{S}_{\mathsf{Mfld}}$$

is an equivalence. That is, there is a commuative square of ∞-categories

$$\begin{tikzcd} \mathsf{PSh}(\mathsf{Mfld};C) \arrow[r, "\mathrm{S}_{\mathsf{Mfld}}"] \arrow[d, "(-)|_M"'] & \mathsf{Sh}(\mathsf{Mfld};C) \arrow[d, "(-)|_M"] \\ \mathsf{PSh}(M;C) \arrow[r, "\mathrm{S}_M"'] & \mathsf{Sh}(M;C)\,. \end{tikzcd}$$

Proof. In the case $C = \mathsf{Spc}$, the claim follows from the fact that the forgetful functor

$$\mathrm{Open}(M) \to \mathsf{Mfld}$$

satisfies the covering lifting property; see [CM21, Proposition 7.1; Pst18, Proposition A.12]. The claim for general C follows from the claim for sheaves of spaces by applying the tensor product of presentable ∞-categories and [Hai21, Lemma 1.18]. □

A.3.3 Corollary. *Let C be a presentable ∞-category, $X \in C$, and M a manifold. Then we have a natural identification $\Gamma^*(X)|_M = \Gamma^*_M(X)$ of the restriction of $\Gamma^*(X)$ to M with the constant sheaf on M at X.*

Proof. Note that by tensoring with the presentable ∞-category C, it suffices to prove the claim for $C = \mathsf{Spc}$. In this case, note that by Lemma A.3.2 the functors

$$(-)|_M\circ\Gamma^*, \Gamma^*_M\colon \mathsf{Spc} \to \mathsf{Sh}(M)$$

are both left exact left adjoints. The claim follows from the fact that for an ∞-topos X, the constant sheaf functor is the unique left exact left adjoint $\mathsf{Spc} \to \mathsf{X}$ [HTT, Proposition 6.3.4.1]. $\square$

A.4 Background on notions of completeness for higher topoi

There are three notions of "completeness" for an ∞-topos X:

(1) *Hypercompleteness:* Whitehead's Theorem holds in X.

(2) *Convergence of Postnikov towers:* Every object of X is the limit of its Postnikov tower.

(3) *Postnikov completeness:* X can be recovered as the limit $\lim_n \mathsf{X}_{\leq n}$ of its subcategories $\mathsf{X}_{\leq n} \subset \mathsf{X}$ of n-truncated objects along the truncation functors $\tau_{\leq n} : \mathsf{X}_{\leq n+1} \to \mathsf{X}_{\leq n}$.

While all of these properties hold for the ∞-topos Spc of spaces, they need not hold for a general ∞-topos. We have implications (3) $\Rightarrow$ (2) $\Rightarrow$(1), and none of the implications are reversible in general. In this section we give a brief overview of hypercompletness as it plays a role in relating the Freed–Hopkins approach to differential cohomology from [FH13] to the ∞-categorical approach we have taken here. Detailed accounts of hypercompleteness and Postnikov completeness can be found in [HTT, §6.5] and [SAG, §A.7], respectively.

A.4.1 Definition. Let X be an ∞-topos. An object $U \in \mathsf{X}$ is *∞-connected* if for every integer $n \geq -2$ the n-truncation $\tau_{\leq n}(U)$ of U is the terminal object of X. A morphism $f : U \to V$ is *∞-connected* if $f : U \to V$ is an ∞-connected object of the ∞-topos $\mathsf{X}_{/V}$.

A.4.2 Definition. Let X be an ∞-topos. An object $U \in \mathsf{X}$ is *hypercomplete* if U is local with respect to the class of ∞-connected morphisms in X. We write $\mathsf{X}^{\mathrm{hyp}} \subset \mathsf{X}$ for the full subcategory spanned by the hypercomplete objects of X. An ∞-topos is *hypercomplete* if $\mathsf{X}^{\mathrm{hyp}} = \mathsf{X}$.

A.4.3. The ∞-category $\mathsf{X}^{\mathrm{hyp}} \subset \mathsf{X}$ is a left exact localization of X, hence an ∞-topos [HTT, p. 699]. Moreover, the ∞-topos $\mathsf{X}^{\mathrm{hyp}}$ is hypercomplete [HTT, Lemma 6.5.2.12].

A.4.4. The ∞-topos $\mathsf{X}^{\mathrm{hyp}}$ is the universal hypercomplete ∞-topos equipped with a geometric morphism to X [HTT, Proposition 6.5.2.13]. For this reason we call $\mathsf{X}^{\mathrm{hyp}}$ the *hypercompletion* of X.

A.4.5 Observation. Let X be an ∞-topos. Then X is hypercomplete if and only if the pullback functor $p^* : \mathsf{X} \to \mathsf{X}^{\mathrm{post}}$ is conservative.

The standard way of working with sheaves of spaces on a site (S, τ) in the language of model-categories is to use the Brown–Joyal–Jardine model structure on simplicial presheaves [Bro73; Jar87]. However, this model structure only presents the hypercompletion of the ∞-topos of sheaves of spaces on (S, τ).

A.4.6 Proposition [HTT, Proposition 6.5.2.14]**.** *Let (S, τ) be a site. Then the underlying ∞-category of the category of simplicial presheaves on S in the Brown–Joyal–Jardine model structure is equivalent to the ∞-topos $\mathsf{Sh}_\tau(S; \mathsf{Spc})^{\mathrm{hyp}}$ of hypercomplete sheaves of spaces on S.*

A.4.7 Definition. Let X be an ∞-topos. A *point* of X is a left exact left adjoint $x^* \colon \mathsf{X} \to \mathsf{Spc}$. Given an object $U \in \mathsf{X}$ and point x^* of X, we call $x^*(U)$ the *stalk* of U at x^*.

A.4.8 Example. Let T be a topological space and $t \in T$. Then the stalk functor

$$(-)_t \colon \mathsf{Sh}(T) \to \mathsf{Spc}$$

defines a point of $\mathsf{Sh}(T)$.

A.4.9 Definition. An ∞-topos X *has enough points* if a morphism f in X is an equivalence if and only if for every point x^* of X, the stalk $x^*(f)$ is an equivalence in Spc.

A.4.10 Example. An ∞-topos with enough points is hypercomplete.

A.4.11 Remark. The existence of enough points is incomparable with the convergence of Postnikov towers and is also incomparable with Postnikov completeness (both of which imply hypercompleteness).

A.4.12 Example. Let M be a manifold. Then the ∞-topos $\mathsf{Sh}(M)$ is Postnikov complete [HTT, Proposition 7.2.1.10 & Theorem 7.2.3.6].

A.5 A conservative family of points

In this section we show the stalks at the origins in $\mathbb{R}^n$ for $n \geq 0$ form a conservative family of points for the ∞-topos $\mathsf{Sh}(\mathsf{Mfld})$ (Proposition A.5.3). This implies that the model structure on simplicial presheaves on Mfld considered by Freed–Hopkins in [FH13, §5] presents the ∞-topos $\mathsf{Sh}(\mathsf{Mfld})$. We also present an observation of Hoyois that shows that the ∞-topos $\mathsf{Sh}(\mathsf{Mfld})$ is Postnikov complete (Proposition A.5.4).

We begin by discussing the stalk of a sheaf on Mfld at a point of a manifold.

A.5.1 Construction. Let M be a manifold and $x \in M$. In light of Lemma A.3.2, the composition of the restriction to M with the stalk at x defines a left exact left adjoint

$$\mathsf{Sh}(\mathsf{Mfld}; C) \xrightarrow{(-)|_M} \mathsf{Sh}(M; C) \xrightarrow{(-)_x} C\ ,$$

which we denote by x^*. Given a sheaf E on Mfld, we call $x^*(E)$ the *stalk* of E at $x \in M$.

A.5.2 Observation. Let M be a manifold and $j \colon U \hookrightarrow M$ an open embedding. Then, by definition, the triangle

$$\begin{tikzcd}
\mathsf{Sh}(\mathsf{Mfld}; C) \arrow[r, "(-)|_M"] \arrow[dr, "(-)|_U"'] & \mathsf{Sh}(M; C) \arrow[d, "j^*"] \\
& \mathsf{Sh}(U; C)
\end{tikzcd}$$

commutes. Thus for any $x \in U$, then there is a canonical identification of the stalk functor $\mathsf{Sh}(\mathsf{Mfld}; C) \to C$ at $x \in U$ with the stalk functor at $j(x) \in M$.

Recall that for each integer $n \geq 0$, write $0_n \in \mathbb{R}^n$ for the origin (Notation 3.4.5).

A.5.3 Proposition. *Let C be a compactly assembled ∞-category. Then the set of stalk functors*

$$\{0_n^* : \mathrm{Sh}(\mathrm{Mfld}; C) \to C\}_{n \geq 0}$$

is jointly conservative. In particular, the ∞-topos $\mathrm{Sh}(\mathrm{Mfld})$ is hypercomplete.

Proof. In light if [Hai21, Lemma 2.12], it suffices to treat the case $C = \mathrm{Spc}$. In this case, first note that the family of restriction functors

$$(-)|_M : \mathrm{Sh}(\mathrm{Mfld}) \to \mathrm{Spc}$$

for $M \in \mathrm{Mfld}$ is jointly conservative (Observation A.2.2). For each manifold M, the ∞-topos $\mathrm{Sh}(M)$ is a hypercomplete ∞-topos and the points of M provide conservative family of points for $\mathrm{Sh}(M)$ [HTT, Corollary 7.2.1.17]. Thus the stalk functors

$$x^* : \mathrm{Sh}(\mathrm{Mfld}) \to \mathrm{Spc}$$

for all $M \in \mathrm{Mfld}$ and $x \in M$ form a conservative family of points for $\mathrm{Sh}(\mathrm{Mfld})$. To conclude, note that for every manifold M and point $x \in M$, there exists an open embedding $j : \mathbb{R}^n \hookrightarrow M$ such that $j(0_n) = x$ and apply Observation A.5.2. ☐

We now give a quick argument showing that the ∞-topos $\mathrm{Sh}(\mathrm{Mfld})$ is Postnikov complete. We learned the following argument from Hoyois; it is a slight refinement of the argument for the convergence of Postnikov towers that Hoyois gave in [Hoy13].

A.5.4 Proposition. *The ∞-topos $\mathrm{Sh}(\mathrm{Mfld})$ is Postnikov complete.*

Proof. Since $\mathrm{Sh}(\mathrm{Mfld})$ is hypercomplete, by Observation A.4.5 it suffices to show that the right adjoint $p_* : \mathrm{Sh}(\mathrm{Mfld})^{\mathrm{post}} \to \mathrm{Sh}(\mathrm{Mfld})$ is fully faithful. That is, we need to show that for every collection of objects $\{F_n\}_{n \geq -2}$ of $\mathrm{Sh}(\mathrm{Mfld})$ equipped with compatible equivalences $\tau_{\leq n}(F_{n+1}) \xrightarrow{\sim} F_n$, and integer $k \geq -2$, the natural morphism

$$\tau_{\leq k}\left(\lim_{n \geq -2} F_n\right) \to F_k \tag{A.5.5}$$

is an equivalence. To see this, note that since the restriction functors

$$\{(-)|_M : \mathrm{Sh}(\mathrm{Mfld}) \to \mathrm{Sh}(M)\}_{M \in \mathrm{Mfld}}$$

are jointly conservative and commute with limits and truncations, it suffices to show that the morphism (A.5.5) becomes an equivalence after restriction to each manifold M. This last claim follows from the fact that the ∞-topos $\mathrm{Sh}(M)$ is Postnikov complete (Example A.4.12). ☐

We finish this section by proving that $\mathrm{Sh}(\mathrm{Mfld})$ is equivalent to the ∞-topos of sheaves on the smaller site $\mathrm{Euc} \subset \mathrm{Mfld}$ spanned by the Euclidean spaces (Definition 3.5.1). Note that since every manifold admits a cover by Euclidean spaces, the Euclidean site is a *basis* for the Grothendieck topology on Mfld (see [Lur18, §B.6] for more about bases for Grothendieck topologies).

A.5.6 Corollary. *Let C be a presentable ∞-category. Then restriction of presheaves*

$$(-)|_{\mathsf{Euc}^{\mathrm{op}}} : \mathrm{Sh}(\mathsf{Mfld}; C) \to \mathrm{Sh}(\mathsf{Euc}; C)$$

is an equivalence of ∞-categories. The inverse is given by right Kan extension along the inclusion $\mathsf{Euc}^{\mathrm{op}} \hookrightarrow \mathsf{Mfld}^{\mathrm{op}}$.

Proof. Since $\mathrm{Sh}(\mathsf{Euc}; C)$ and $\mathrm{Sh}(\mathsf{Mfld}; C)$ are the tensor products of presentable ∞-categories

$$\mathrm{Sh}(\mathsf{Euc}; C) \simeq \mathrm{Sh}(\mathsf{Euc}) \otimes C \qquad \text{and} \qquad \mathrm{Sh}(\mathsf{Mfld}; C) \simeq \mathrm{Sh}(\mathsf{Mfld}) \otimes C\ ,$$

it suffices to treat the case where $C = \mathsf{Spc}$ is the ∞-category of spaces. In this case, since the ∞-topos $\mathrm{Sh}(\mathsf{Mfld})$ is hypercomplete (Proposition A.5.3), the claim follows from the fact that $\mathsf{Euc} \hookrightarrow \mathsf{Mfld}$ is a basis for the topology on Mfld; see [Aok20, Appendix A] or [BGH20, Corollary 3.12.13]. $\square$

Part II

Characteristic classes

The objective of this portion of the notes is to construct, study, and use refinements of standard characteristic classes to differential cohomology.

Historically, differential characteristic classes were studied by Cheeger and Simons [CS85]. This view is covered in Chapter 14.

The modern approach uses the machinery of sheaves on manifolds developed in Part I of these notes. Given a Lie group G, we consider three different, but related, sheaves $\mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Spc}$ on Mfld:

(1) The constant sheaf at the classifying space $\mathrm{B}G$ of G (Example 13.1.28). We simply denote this sheaf by $\mathrm{B}G$.

(2) The sheaf Bun_G sending a manifold M to the groupoid of principal G-bundles on M (Example 3.3.6).

(3) The sheaf Bun_G^{∇} sending a manifold M to the groupoid of principal G-bundles on M with connection (Example 3.3.7)..

Characteristic classes live in the de Rham cohomology of these sheaves.

II.1 Definition. Let S be a sheaf on manifolds. The *de Rham cohomology* of S is $\Omega^{\bullet}(S)$.

For example, the de Rham cohomology $\Omega^{\bullet}(\mathrm{B}G)$ of the constant sheaf $\mathrm{B}G$ is where ordinary characteristic classes live.

II.2 Remark. Given a manifold M, one can recover the differential cohomology $\check{\mathrm{H}}^k(M)$ by taking the k-th de Rham cohomology.

The de Rham cohomology of Bun_G^{∇} is studied in Chapter 15. The de Rham cohomology $\Omega^{\bullet}(\mathrm{Bun}_G^{\nabla})$ classifies characteristic classes for G-bundles with connections. In Chapter 15, we give a proof of the main theorem of [FH13]. The theorem is as follows,

II.3 Theorem (Freed–Hopkins)**.** *The Chern–Weil homomorphism induces an isomorphism*

$$(\mathrm{Sym}^{\bullet}\, \mathfrak{g}^{\vee})^G \xrightarrow{\sim} \Omega^{\bullet}(\mathrm{Bun}_G^{\nabla})\,.$$

Thus the Chern–Weil construction, reviewed in Chapter 11, produces all characteristic classes for bundles with connection. The set up for the proof of Theorem II.3 uses tools similar to the Cartan model for equivariant de Rham cohomology, which we review in Chapter 12. Related work on Borel equivariant differential cohomology was done by Redden [Red17].

The de Rham cohomology of Bun_G is a bit more complicated. The tools we use to compute $\Omega^{\bullet}\mathrm{Bun}_G$ originate in Bott's paper [Bot73]. In Chapter 16, we review the techniques used in [Bot73] including continuous cohomology and the van Est theorem. The takeaway of Chapter 16 is the following theorem of Bott:

II.4 Theorem (Bott). *The continuous cohomology* $\mathrm{H}^{p-q}_{\mathrm{cont}}(G;\mathrm{Sym}^q(\mathfrak{g}^\vee))$ *is isomorphic to the de Rham cohomology group* $\mathrm{H}^p(\mathrm{Bun}_G;\Omega^q)$*:*

$$\mathrm{H}^p(\mathrm{Bun}_G;\Omega^q) \cong \mathrm{H}^{p-q}_{\mathrm{cont}}(G;\mathrm{Sym}^q(\mathfrak{g}^\vee)) .$$

We will really only use Bott's theorem in degrees $p - q \leq 0$.

In Chapter 17, the results of [Bot73] are applied to provide lifts of Chern classes to differential cohomology. In particular, we will see there exists multiple lifts of each Chern class c_i to $\mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{C})};\mathbb{Z}_{\mathbb{C}}(n))$. The collection of lifts is determined by the following result, credited by Hopkins to Bott:

II.5 Theorem. *There is a pullback square*

$$\begin{array}{ccc}
\mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_k(\mathbb{C})};\mathbb{Z}_{\mathbb{C}}(n)) & \longrightarrow & \mathrm{H}^{2n}(\mathrm{BU}_k;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}^{n}(\mathrm{BU}_k \times \mathrm{BU}_k;\mathbb{C}) & \xrightarrow[\text{diagonal}^*]{} & \mathrm{H}^{2n}(\mathrm{BU}_k;\mathbb{C}) .
\end{array}$$

A real analogue of this theorem provides lifts of the Pontryagin classes.

II.6 Remark. Note that differential cohomology $\mathrm{H}^i(-;\mathbb{Z}(j))$ is bigraded. The differential lifts of characteristic classes discussed in Chapter 15 live in bidegree where $i = j$. We refer to these classes as "on-diagonal." The classes defined in Chapter 17 live in bidegree where $i = 2j$, and we call these "off-diagonal" classes. Notationally, for a class c, we use $\hat{c}$ to denote an on-diagonal differential lift and $\tilde{c}$ for an off-diagonal lift.

As an application of this construction, in Chapter 18 we explain how a differential lift of the first Pontryagin class $\tilde{p}_1 \in \mathrm{H}^4(\mathrm{BSL}(\mathbb{R});\mathbb{Z}(2))$ can be used to produce the Virasoro group. The Virasoro group is a certain central extension of $\mathrm{Diff}^+(\mathrm{S}^1)$ by U_1,

$$\mathrm{U}_1 \to \widetilde{\mathrm{Diff}}^+(\mathrm{S}^1) \to \mathrm{Diff}^+(\mathrm{S}^1) .$$

The construction of $\widetilde{\mathrm{Diff}}^+(\mathrm{S}^1)$ uses the fiber integration for differential cohomology covered in Chapter 9 and pullback along the classifying map of a certain bundle. This process is outlined in Chapter 17 and covered in depth in Chapter 18. Note that there are multiple lifts of p_1 to differential cohomology. We obtain criterion for which lift $\tilde{p}_1$ could correspond to the Virasoro algebra central extension, but we do not pin down which lift works.

As far as we know, the material in Chapter 17 and Chapter 18 does not appear elsewhere in the literature, aside from the underpinning in [Bot73]. The new ideas here are due to Dan Freed, Mike Hopkins, and Constantin Teleman.

11 Chern–Weil theory

by Greg Parker

As this talk is a review of standard material, many technical results are stated without proof. For more detailed review, including proofs, the reader should consult [MS74, Appendix C] for a review of connections and Chern–Weil theory for vector bundles, [KN96a, Chapter II] or [Roe98, Chapter 2] for the theory of connections on principal bundles, and [KN96b, Chapter XII] for Chern–Weil theory for principal bundles.

11.1 Motivation

To begin, let's recall

11.1.1 Theorem (Gauss–Bonnet)**.** *Let (Σ, g) be a compact, oriented, Riemannian 2-manifold without boundary. Let $\chi(\Sigma)$ be its Euler characteristic. Then*

$$\int_{\Sigma} \kappa \mathrm{d}A = 2\pi\chi(\Sigma)\,.$$

Here κ is the Gaussian curvature defined as follows. If $R_{ij}\mathrm{d}x_i\mathrm{d}x_j$ is the Riemann curvature tensor, locally

$$R = \begin{pmatrix} 0 & R_{12} \\ -R_{21} & 0 \end{pmatrix} \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

and $\kappa = R_{12}$. So we can rewrite the above as

$$\langle [\sqrt{\det(R)}], [\Sigma] \rangle = \int_{\Sigma} \sqrt{\det(R)} = 2\pi\chi(\Sigma) = \langle 2\pi e(\mathrm{T}\Sigma), [\Sigma] \rangle\,,$$

where $e(\mathrm{T}\Sigma)$ is the Euler class of Σ and the brackets on the right-hand side denote the pairing

$$\mathrm{H}^2(\Sigma; \mathbb{R}) \otimes \mathrm{H}_2(\Sigma; \mathbb{R}) \to \mathbb{R}\,.$$

Thus we observe $\sqrt{\det(R)}$, a polynomial in the curvature, captures information about the topology of Σ and its tangent bundle $\mathrm{T}\Sigma$. Chern–Weil theory (which was actually the original formulation/theory of characteristic classes) generalizes the above to higher dimension and arbitrary bundles.

11.2 Connections and curvature

In order to formulate things correctly, we will need to recall some facts about connections and curvature, both for vector bundles and for principal bundles.

11.2.1 Convention. Throughout this talk, let M be a closed n-manifold, $\pi : E \to M$ a rank k real or complex vector bundle with structure group $G = \mathrm{O}_k$ or $G = \mathrm{U}_k$. Denote the real (or Hermitian) inner product by $\langle -, - \rangle$. Let $\mathfrak{g}$ denote the Lie algebra of G.

Also, let K be a Lie group and $p : Q \to X$ be a principal K-bundle. Let $\mathfrak{k}$ denote the Lie algebra of K.

11.2.a For vector bundles

We would like to differentiate sections $\psi : M \to E$. The problem is $\psi_{x(t)}$ for a path $x(t) \subset M$ all live in different vector spaces: $E_{x(t)}$, respectively, so we must find a way to "connect" them.

View $\psi_{x(t)}$ as a path in the total space. The derivative (intuitively) is the vertical component of $\frac{\partial \psi}{\mathrm{d}t}$. Think of $f : \mathbb{R} \to \mathbb{R}$, then $\frac{\mathrm{d}f}{\mathrm{d}t}$ is the y-coordinate of the graph inside $\mathbb{R}^2$. To define this precisely we need to choose a splitting

$$\mathrm{T}E \simeq VE \oplus HE$$

into the "vertical" and "horizontal" subbundles. The vertical piece $VE = \ker \mathrm{d}\pi$ is canonical, and the horizontal piece HE is not. Such a splitting is called a *connection*. Once we choose a connection, we get an isomorphism $\mathrm{d}\pi : HE \to TM$. So given $e \in E_{x(t)}$ we can lift $\dot{x}$ (a vector field along $x(t)$) to one $X_H^e \subset HE$. Then the flow is a path in E projecting to $x(t)$, which is the *parallel transport*, denoted $\varphi_t e \in E_{x(t \times I)}$. Then

$$\varphi_{-t}\psi(t) \in E_{x(0)}$$

for all t, so we can differentiate. The covariant derivative (with respect to our chosen connection) in the $\dot{x}(0)$ direction at $x(0)$ is $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_{-t}\psi_{x(t)}$. Thus we get an operator

$$\mathrm{d}_A \text{ or } \nabla^A : \Gamma(M; E) \to \Gamma(M; \mathrm{T}^*M \otimes E)$$

associated to a connection A, called the *covariant derivative*. Here, ∇^A eats a vector field $X \in \Gamma(M; \mathrm{T}M)$ and gives the derivative in that direction at each point. It satisfies

- C^∞*-linearity in the direction of the derivative:* $\nabla^A_{fX}\psi = f\nabla^A_X\psi$, and
- *Leibniz rule:* $\nabla^A f\psi = \mathrm{d}f \otimes \psi + f\nabla\psi$.

The existence of connections is preserved under various bundle constructions.

11.2.2 Proposition. *Given ∇^A on E, ∇^B on F we get connections:*

- *∇^{A^*} on the dual bundle E^*.*
- *∇^{AB} on the tensor product $E \otimes F$ by the formula*

$$\nabla^{AB}(\varphi \otimes \psi) = \nabla^A\varphi \otimes \psi + \varphi \otimes \nabla^B\psi\,.$$

- *if $F : M \to N$ and $E \to N$ then $F^*(\nabla^A)$ is a connection on f^*E by*

$$(F^*\nabla^A)_X\psi(m) := \nabla^A_{F_*X}\psi(f(m)) \in E_{F(m)} = F^*E_m\,.$$

11.2.3 Proposition. *Two connections differ by a* 1*-form valued in* $\mathrm{End}(E)$*. In particular, the set of connections form an affine, and hence contractible, space.*

11.2.4 Remark. Thus one might expect invariants defined using them (if discrete) to not depend on the choice of connection.

Proof. Let A and A' be two connections on the bundle $\pi : E \to M$. For $f \in C^\infty(X)$ and ψ a section of π, we have

$$\begin{aligned}(\nabla^A - \nabla^{A'})(f\psi) &= \mathrm{d}f \otimes \psi + f\nabla^A\psi - \mathrm{d}f \otimes \psi - f\nabla^{A'}\psi \\ &= f(\nabla^A - \nabla^{A'})\psi\end{aligned}$$

is C^∞-linear with values in $\Gamma(\mathrm{T}^*M \otimes E)$ so $\nabla^A - \nabla^{A'} \in \Omega^1(\mathrm{End}(E))$. □

11.2.5 Example. On the trivial rank k-bundle $\underline{\mathbb{R}}_M$ on M, the exterior derivative

$$\mathrm{d} : \Gamma(M; \underline{\mathbb{R}}_M) \to \Omega^1(M)$$

is a connection.

11.2.6 Example. In a local trivialization (by Proposition 11.2.3) we can always write $\nabla = \mathrm{d} + A$, where $A \in \Omega^1(\mathrm{End}(E))$. That is $A = A_1 \mathrm{d}x_1 + \cdots A_n \mathrm{d}x_n$ for A_i matrices, and

$$\nabla_i \psi = \frac{\partial \psi}{\partial x_i} + A_i \psi \,.$$

11.2.7 Example. On $\mathrm{End}(E) = E^* \otimes E$, the induced ∇ from Proposition 11.2.2 is

$$\nabla B = \mathrm{d}B + [A, B]$$

in a trivialization.

Define a connection as *compatible with* $\langle -, - \rangle$ if

$$\mathrm{d}\langle \psi, \varphi \rangle = \langle \nabla\psi, \varphi \rangle + \langle \psi, \nabla\varphi \rangle \,.$$

Note that for compatible ∇, A will be in $\Omega^1(\mathfrak{o}(E))$ or $\mathfrak{u}(E)$.

11.2.8 Remark. A fancy way of saying this is $\langle -, - \rangle \in E^* \otimes E^*$ has $\nabla = 0$.

11.2.9 Lemma. *Every bundle E has a connection compatible with* $\langle -, - \rangle$.

Proof. Locally, connections of the form $\mathrm{d} + A$ are compatible with $\langle -, - \rangle$ if A is in $\Omega^1(\mathfrak{o}(E))$ or $\Omega^1(\mathfrak{u}(E))$. This gives existence locally. Using a partition of unity, one obtains the desired connection globally. □

11.2.b For principal bundles

For a principal K-bundle $p\colon Q \to X$, the space of vertical tangent vectors $\ker(\mathrm{d}p)$ of Q gives a short exact sequence

$$0 \to \ker(p_*) \to \mathrm{T}Q \to p^*\mathrm{T}X \to 0 \tag{11.2.10}$$

of vector bundles over P. As in the vector bundle situation, a connection will be a way of considering *horizontal* tangent vectors.

11.2.11 Definition. A *principal connection* on $p\colon Q \to X$ is a splitting of the exact sequence (11.2.10).

The kernel $\ker(\mathrm{d}p)$ can be identified with the trivial bundle with fiber the tangent space of the fiber K of p. That is, we have an isomorphism

$$\ker(\mathrm{d}p) \cong Q \times \mathfrak{k}\,.$$

A splitting of equation (11.2.10) is equivalent to a section of the map $\ker(\mathrm{d}p) \to \mathrm{T}Q$. Using the identification $\ker(\mathrm{d}p) \cong Q\times\mathfrak{k}$, a section $\mathrm{T}Q \to \ker(\mathrm{d}p)$ is equivalent to a section of $\mathrm{T}^*Q\otimes(\mathfrak{k}\times Q)$; i.e., a one form with coefficients in $\mathfrak{k}$.

11.2.12 Definition. Let $p\colon Q \to X$ a principal K-bundle with principal connection. The *connection* 1*-form* is the principal connection is the one form $\omega \in \Omega^1(Q;\mathfrak{k})$ corresponding the splitting of equation (11.2.10).

Note that $\mathfrak{k}$ acts on $\mathfrak{k}$ in two ways: by right m_r, and by conjugation $\mathrm{Ad}_{\mathfrak{k}}$.

11.2.13 Lemma. *Let $p\colon Q \to X$ a principal K-bundle with principal connection* 1*-form ω. Then ω is K-equivariant,*

$$\mathrm{Ad}_{\mathfrak{k}}(m_r(\omega)) = \omega$$

and for $\xi \in \mathfrak{k}$ with associated vector field X_ξ, we have $\omega(X_\xi) = \xi$.

11.2.14 Remark. A connection on a principal bundle gives rise to a vector bundle connection on any associated vector bundle. Likewise, a K-compatible connection on a vector bundle E gives rise to a connection on the K-frame bundle, and these operations are inverses. The horizontal distribution on $\mathrm{T}Q$ complementing $\ker(p_*)$ in equation (11.2.10) is obtained from a vector bundle connection as directions of the infinitesmal parallel transport at a point. In the opposite direction, the parallel transport of frames on Q naturally gives a parallel transport of section of the vector bundle. Alternatively, in local coordinates the connection form is just $\mathrm{d} + \omega$ for ω the ad-equivariant connection form on Q.

11.3 Curvature

11.3.a For vector bundles

Given two vector fields $X, Y \in \Gamma(M; \mathrm{T}M)$, the maps ∇_X and ∇_Y need not commute; i.e.,

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X \neq 0$$

Geometrically, since these were defined by flowing along horizontal lifts, $\tilde{X}, \tilde{Y}$, this is a question about non-commuting flows; i.e., $[\tilde{X}, \tilde{Y}]$. In particular, if the horizontal bundle HE is integrable, then $[\tilde{X}, \tilde{Y}] = 0$ so the flows (and hence ∇_X, ∇_Y) commute. Thus the *curvature*

$$F_A(X, Y)(\psi) := [\nabla_X, \nabla_Y](\psi) - \nabla_{[X,Y]}(\psi)$$

is a measure of the integrability of $HE \subseteq E$. Here A is such that locally we have $\nabla = d + A$.

We get a local description of the curvature by

$$F_A = \mathrm{d}A + A \wedge A \,.$$

In other words,

$$F_A = F_A^{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$$

with

$$F_A^{ij} = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i \,.$$

11.3.1 Claim. *The curvature F_A defines a 2-form with values in the endomorphism bundle,*

$$F_A \in \Omega^2(M; \mathrm{End}(E)) \,.$$

In particular, the curvature is C^∞-linear, $F_A(f\psi) = f F_A \psi$.

Proof. This follows from the Leibniz rule for connections. □

For $F_A \in \Omega^2(\mathrm{End}(E))$. We have

$$\mathrm{d}_A : \Omega^2(\mathrm{End}(E)) \to \Omega^3(\mathrm{End}(E))$$

by $\alpha \otimes B \mapsto \mathrm{d}\alpha \otimes B + \alpha \otimes \nabla B$.

11.3.2 Theorem (Bianchi identity)**.** *The exterior derivative of the curvature vanishes,* $\mathrm{d}_A F_A = 0$.

11.3.b For principal bundles

The wedge product of $\omega \in \Omega^1(Q; \mathfrak{k} \otimes \mathfrak{k})$ with itself is an element of $\Omega^2(Q; \mathfrak{k})$. The Lie bracket on $\mathfrak{g}$ induces a map

$$[-] : \Omega^2(Q; \mathfrak{k} \otimes \mathfrak{k}) \to \Omega^2(Q; \mathfrak{k}) \,.$$

11.3.3 Definition. Let $p : Q \to X$ a principal K-bundle with principal connection 1-form ω. The *curvature* of ω is

$$\Omega = \mathrm{d}\omega + [\omega \wedge \omega]$$

in $\Omega^2(Q; \mathfrak{k})$.

Consider $\mathfrak{k}$ as a K-module with the adjoint action. Let $\mathfrak{k}_Q \to X$ denote the adjoint bundle $\mathfrak{k}_Q = Q \times_K \mathfrak{k}$.

11.3.4 Lemma. *Let $p : Q \to X$ a principal K-bundle with connection. Let Ω be its curvature. Then Ω descends to a 2-form $\tilde{\Omega} \in \Omega^2(X; \mathfrak{g}_Q)$.*

11.3.5 Example. Take $K = \mathrm{GL}_n$ so that Q has an associated rank n vector bundle $V \to X$. The adjoint bundle can be identified with the endomorphism bundle $\mathrm{End}(V)$. Under this identification, a principal connection on Q corresponds to a connection on the vector bundle $V \to X$, and the curvature $\tilde{\Omega}$ from a principal connection on Q corresponds to the curvature of $V \to X$.

11.3.6 Theorem (Bianchi identities)**.** *We have* $\mathrm{d}\Omega + [\omega \wedge \Omega] = 0$ *and* $\mathrm{d}\Omega = 0$.

11.4 Invariant polynomials

11.4.a For vector bundles

In Gauss–Bonnet we used $\sqrt{\det}$ to turn the $R \in \Omega^2(\mathfrak{so}(\mathrm{T}\Sigma))$ into an $\mathbb{R}$-valued form to integrate. In general, since F_A isn't basis-invariant we want a map $P : \mathfrak{g} \to \mathbb{R}$ (for $G = \mathrm{SO}_k$ or SU_k) invariant under Ad. If P is a polynomial, we say it is an *invariant polynomial*. The space of Ad-invariant polynomials on $\mathfrak{g}$ is $\mathrm{Sym}^\bullet(\mathfrak{g}^\vee)^G$.

11.4.1 Example. Both tr and det are Ad-invariant.

Thus given P, A we get an $\mathbb{R}$-valued form $P(F_A) \in \Omega^*(M; \mathbb{R})$.

11.4.2 Proposition. *The form $P(F_A)$ is closed,* $\mathrm{d}P(F_A) = 0$. *Hence we get a homomorphism*

$$\mathrm{Sym}^\bullet(\mathfrak{g}^\vee)^G \to \mathrm{H}^*_{\mathrm{dR}}(M; \mathbb{R})\,.$$

The map above is called the *Chern–Weil homomorphism*, or sometimes just the Weil homomorphism.

Proof. Write $P(\xi) = \sum_I P_I \xi_{i_1}, \dots, \xi_{i_N}$. Since P is Ad-invariant, for $g_t = \exp(t\eta)$, we have

$$P(\xi) = P(\mathrm{Ad}_{g_t} \xi)$$

so

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t} P(\xi) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \sum_I P_I (\mathrm{Ad}_{g_t}\, \xi)_{i_1} \cdots (\mathrm{Ad}_{g_t}\, \xi)_{i_N} \\
&= \sum_{I,k} P_I \xi_{i_1} \cdots \xi_{i_{k-1}} [\eta, \xi]_{i_k} \cdots \xi_{i_N}\ .
\end{aligned}
$$

Writing $F_A = \sum F_A^i$, we have

$$
\begin{aligned}
\mathrm{d}P(F_A) &= \mathrm{d}\left(\sum_I P_I F^{i_1} \wedge \cdots \wedge F^{i_N} \right) \\
&= \sum_{I,k} P_I F^{i_1} \wedge \cdots \wedge dF^{i_k} \wedge \cdots \wedge F^{i_N} + \sum_{I,k} P_I F^{i_1} \wedge \cdots \wedge [A, F_A]_{i_k} \wedge \cdots \\
&= \sum_{I,k} P_I F^{i_1} \wedge \cdots \wedge (\mathrm{d}_A F_A)_{i_k} \wedge \cdots \wedge F_{i_N} \\
&= 0\ .
\end{aligned}
$$

□

11.4.3 Proposition (invariance). *The class $[P(F_A)]$ satisfies the following properties.*

(1) $[P(F_A)]$ *is independent of* A.

(2) $[P(F_A)]$ *is independent of* $\langle -, - \rangle$.

(3) *If* $E \simeq E'$ *then* $[P(F_A)] = [P(F_{A'})]$. *The* characteristic class *of* E *is* $[P(F_A)] \in \mathrm{H}^*$.

Proof Sketch. For (1), take A, A' and set $\nabla_A - \nabla_{A'} = B$. Define

$$
A_t : E \times I \to M \times I
$$

by $\nabla_A + tB$. Then $P(F_{A_t}) \in \Omega^\bullet(M \times I; \mathbb{R})$, and $i_0 : M \to M \times \{0\}$ has $i_0^* P(F_{A_t}) = P(F_A)$ for some i_1, A'. But i_0, i_1 are homotopic.

The proof of (2) is similar.

For (3), use the pullback connection plus the independence of A. □

11.4.b For principal bundles

We have an analogous story for principal bundles, using the corresponding notions of curvature and Bianchi identities.

11.4.4 Proposition. *Let $Q \to X$ be a principal K-bundle with curvature Ω. The assignment $P \mapsto P(\Omega)$ determines a map*

$$
\mathrm{Sym}(\mathfrak{k}^\vee)^K \to \Omega^\bullet_{\mathrm{dR}}(X)
$$

that descends to a map on cohomology.

11.5 Examples

Now the fun part: choose different P and see what we get.

11.5.a Chern classes

Consider the polynomial $P = \det(\lambda \,\mathrm{id} - \frac{1}{2\pi i}X) \colon \mathfrak{u}_k \to \mathbb{R}$. Then expanding out, we get

$$P = \lambda^k - c_1(X)\lambda^{k-1} + c_2(X)\lambda^{k-2} + \cdots$$

for c_k polynomials in X. Define the characteristic class c_k in H^{2k} obtained from P to be the k-th Chern class. Explicitly

$$\begin{aligned} c_k(F_A) &= \frac{\mathrm{tr}(F_A^{\wedge k})}{(2\pi i)^k} \\ &= 1 - \frac{1}{2\pi i}\,\mathrm{tr}(F_A) + \frac{\mathrm{tr}(F_A \wedge F_A) - \mathrm{tr}(F_A)^2}{8\pi^2} - \cdots . \end{aligned}$$

11.5.1 Remark. It's immediate that $c_1 = 0$ for an SU_n-bundle since $\mathfrak{su}_n$ is traceless. In fact, one can show c_k are a basis for Ad-invariant polynomials so this is a complete list.

11.5.b Pontryagin classes

Consider the polynomial P from

$$\det(\lambda \,\mathrm{id} - \frac{1}{2\pi}X) \colon \mathfrak{o}_k \to \mathbb{R} .$$

Expanding out, we get

$$P = \lambda^k - g_1(X)\lambda^{k-1} + \cdots .$$

Since $\mathfrak{o}_k$ is skew-symmetric $g_{\mathrm{odd}} = 0$ and $g_{2k} = p_k(E)$ is the k-th Pontryagin class. For example, we have

$$p_1 = -\frac{\mathrm{tr}(F_A \wedge F_A)}{8\pi^2}$$

and

$$p_2 = \frac{\mathrm{tr}(F_A \wedge F_A)^2 - 2\,\mathrm{tr}(F_A \wedge \cdots \wedge F_A)}{128\pi^4} .$$

11.5.c Euler class

If k is even, there is the Pfaffian $\mathrm{pf} \colon \mathfrak{o}(2k) \to \mathbb{R}$ with $\mathrm{pf}(X)^2 = \det(X)$. Then the *Euler class* is

$$e(E) = \mathrm{pf}(F_A) .$$

11.5.d Other classes

If $g(X) = a_0 + a_1X + a_2X^2 + \cdots$ is a power series, then $\det(g(X))$ is invariant. For example,

- we get the total Chern class from
$$g = 1 + \frac{z}{2\pi i}$$
- we get the L-genus from
$$g = \frac{z}{\tanh(z)}$$
- we get the Todd genus from
$$g = \frac{z^2}{1 - \exp(-z^2)} .$$

11.6 Axioms

There are a set of axioms that Chern classes satisfy. Moreover, these axioms uniquely determine the Chern classes. See, for example, [MS74, §4] for a discussion of this perspective. The axioms are

(1) $c_0(E) = 1$, $c_i(E) = 0$ for $i > \operatorname{rank}(E)$

$$c_i = \operatorname{tr}(\wedge^i F_A) \text{ and } \wedge^i = 0 \text{ for } i \geq \operatorname{rank}(E) + 1.$$

(2) Naturality with pullbacks

(3) Whitney sum, $c(E \oplus F) = c(E) \cup c(F)$

(4) Normalization $c_1(\mathcal{O}(1)) = -1$ on $\mathbb{C}\mathrm{P}^1$.

One can check that the Chern classes, as we have defined them above, satisfy these axioms, see [MS74, Appendix C]. Thus, the Chern–Weil definition gives the same Chern classes as other definitions.

11.6.1 Remark. Although, a priori, c_k has real coefficients $\mathrm{H}^{2k}_{\mathrm{dR}}(M;\mathbb{R})$, the normalization shows it is actually in the image of the map

$$\mathrm{H}^{2k}(M;\mathbb{Z}) \to \mathrm{H}^{2k}(M;\mathbb{R}) .$$

11.7 An application

Here's an application of Chern–Weil theory to something harder to see with other definitions of characteristic classes.

11.7.1 Lemma. *Let $E \to M$ be a complex vector bundle that admits a reduction of structure group to locally constant transition functions (i.e., E is a local system with group $\mathbb{C}^n$), then the Chern class $c_k(E) \in \mathrm{H}^{2k}(M;\mathbb{Z})$ is torsion.*

Proof. The vector bundle E admits a flat connection

$$A \mapsto gAg^{-1} + g^{-1}\mathrm{d}g$$

so we can take $\mathrm{d} + A$ with $A = 0$, and this is preserved by changing trivializations. □

12 Equivariant de Rham cohomology

by Greg Parker

12.1 Motivation

Let G be a Lie group and M be a smooth manifold with a G action. We want a cohomology theory that takes into account the G-action. If the action is free, then we can take

$$\mathrm{H}^\bullet_G(M) \coloneqq \mathrm{H}^\bullet(M/G)\,.$$

If the action is not free, we take the homotopy quotient $\mathrm{E}G \times_G M$ and set the *equivariant cohomology of M* to be

$$\mathrm{H}^\bullet_G(M) \coloneqq \mathrm{H}^\bullet(\mathrm{E}G \times_G M)\,.$$

Here $\mathrm{E}G \to \mathrm{B}G$ is the universal bundle, so that $\mathrm{E}G$ is a contractible space with a free G-action.

12.1.1 Question. How should one define equivariant cohomology using differential forms?

To answer this question, we will roughly follow [GS99, Chapter 1-4]. The reader is encouraged to read [GS99] for more details and applications.

As motivation, again consider a free action. That is, take $P \to X$ to be a principal G-bundle. We want to distinguish forms in $\Omega^\bullet(P)$ that pullback from $X = P/G$. Let $\mathfrak{g}$ be the Lie algebra of G.

For $\alpha \in \Omega^\bullet(P)$, we can locally write

$$\alpha = \sum_I \alpha_I \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_N}\,.$$

The form α is pulled back from M if, for all i the following conditions hold:

(1) The form $\mathrm{d}x_i$ is vertical: $i_\xi \alpha = 0$ for all $\xi \in \mathfrak{g}$.

(2) α does not depend on vertical coordinates: $i_\xi \mathrm{d}\alpha = 0$ for all $\xi \in \mathfrak{g}$.

Forms satisfying these two conditions are called *basic*. Let $\Omega^\bullet(P)_{\mathrm{basic}}$ denote the subcomplex of basic forms. Then, we have

$$\mathrm{H}^\bullet(\Omega(P)_{\mathrm{basic}}) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(P/G) = \mathrm{H}^\bullet_G(P)\,.$$

12.2 G^*-Algebras

Given an element $\xi \in \mathfrak{g}$, there are multiple maps on $\Omega^\bullet(M)$:

- a degree -1 map by contraction, $\xi \mapsto i_\xi$ and
- a degree 0 map by Lie derivative, $\xi \mapsto L_\xi$.

We can package these actions of $\mathfrak{g}$, together with the differential d, on $\Omega^\bullet(M)$ as a representation of a certain Lie superalgebra $\tilde{\mathfrak{g}}$. Take

$$\tilde{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}$$

where, for each element $\xi \in \mathfrak{g}$, we have corresponding elements of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ that we denote by their action on $\Omega^\bullet(M)$. That is, by i_ξ and L_ξ, respectively. The generator of $\mathbb{R}$ is denoted d. The bracket of the Lie superalgebra $\tilde{\mathfrak{g}}$ is defined by

$$\begin{aligned}
[i_\xi, i_\eta] &= 0 \\
[L_\xi, i_\eta] &= i_{[\xi,\eta]} \\
[L_\xi, L_\eta] &= L_{[\xi,\eta]} \\
[d, i_\xi] &= L_\xi \\
[d, L_\xi] &= 0 \\
[d, d] &= 2d^2 = 0
\end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}$.

The following is [GS99, Definition 2.3.1].

12.2.1 Definition. A *G^*-algebra* is a graded algebra A with an action $G \to \mathrm{Aut}(A)$ of G and an action $\tilde{\mathfrak{g}} \to \mathrm{End}(A)$ of $\tilde{\mathfrak{g}}$, so that the following hold:

(1) $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t_\xi) = L_\xi$.

(2) $gL_\xi g^{-1} = L_{\mathrm{Ad}_g \xi}$ and $gi_\xi g^{-1} = i_{\mathrm{Ad}_g \xi}$.

(3) $gd = dg$.

Note that the tensor product of two G^*-algebras is again a G^*-algebra.

12.2.2 Example. The complex $\Omega^\bullet(M)$ is a G^*-algebra with multiplication by the wedge product.

Considering a G^*-algebra A with its differential from the action of $d \in \tilde{\mathfrak{g}}$, we define $\mathrm{H}^\bullet(A) := \mathrm{H}_\bullet(A, d)$.

12.2.3 Definition. Let A be a G^*-algebra. A *basic form* in A is an element $\alpha \in A$ so that

$$i_\xi \alpha = L_\xi = 0$$

for all $\xi \in \mathfrak{g}$.

We will need to add an assumption on our G^*-algebra, referred to as *Condition C* in [GS99, §2.3.4]. Condition C will ensure the existence of a certain G-invariant subspace that acts like the vertical subbundle (Section 11.2.b) in the locally free case, see [GS99, Definition 2.3.3].

12.2.4 Definition. Let $\xi_1, \ldots, \xi_k$ be a basis for $\mathfrak{g}$. A G^*-algebra A is *satisfies Condition C* if there exists elements $\theta^1, \ldots, \theta^k \in A$ of degree 1 so that for all $i, j = 1, \ldots, k$,

$$\iota_{\xi_i} \theta^j = \delta_{ij}$$

and the subspace spanned by $\{\theta^i\}$ is invariant under G.

In particular, if the action of G on M is free, then $\Omega^\bullet(M)$ is a G^*-algebra satisfying Condition C.

We say a G^*-algebra A is *acyclic* if the chain complex (A, d) is.

12.2.5 Definition. Let M be a manifold with G action and let E be a G^*-algebra that is acyclic and satisfies Condition C. Define the equivariant de Rham cohomology by

$$\mathrm{H}^\bullet_{G,\mathrm{dR}}(M) \coloneqq \mathrm{H}^\bullet((\Omega(M) \otimes E)_{\mathrm{basic}}) .$$

The following is [GS99, Theorem 2.5.1]. In particular, by [GS99, Prop. 2.5.4], such G^*-algebras E as in Definition 12.2.5 exist in the context we care about.

12.2.6 Theorem (equivariant de Rham)**.** *There is an isomorphism*

$$\mathrm{H}^*_{G,\mathrm{dR}}(M) \cong \mathrm{H}^*_G(M) .$$

We discuss the idea of the proof here. For a full proof, see [GS99, §2.5].

Proof Idea. Approximate $\mathrm{E}G$ with a sequence of finite-dimensional manifolds E_k and take

$$E = \lim_k \Omega(E_k) .$$

By the free case,

$$\mathrm{H}^*(M \times E_k/G) = \mathrm{H}^*(\Omega(M \times E_k)_{\mathrm{basic}})$$

for $* \ll k$. To finish, one shows that

$$\Omega(M \times E_k)_{\mathrm{basic}} = \Omega(M) \otimes \Omega(E_k)_{\mathrm{basic}}$$

in the limit. □

12.2.7 Remark. By [GS99, §4.4], the definition of $\mathrm{H}^*_{G,\mathrm{dR}}$ is independent of E satisfying the assumptions (acyclic and Condition C).

12.3 Cartan model

Now we can look for a specific E that gives a nice algebraic structure, so it might be more computable.

For a vector space V, the *Koszul algebra* is $(\Lambda^\bullet(V) \otimes \mathrm{Sym}^\bullet(V), \mathrm{d})$ where

$$\mathrm{d}(\alpha \otimes 1) = 1 \otimes \alpha \qquad \text{and} \qquad \mathrm{d}(1 \otimes \alpha) = 0$$

extended as a derivation. The *Weil Algebra* is the Koszul algebra of $\mathfrak{g}^\vee$,

$$W = \Lambda^\bullet(\mathfrak{g}^\vee) \otimes \mathrm{Sym}^\bullet(\mathfrak{g}^\vee)$$

as a G^* algebra: For a basis (as an algebra) θ^i, z^j we have

$$\begin{aligned} i_a\theta_b &= \delta_{ab} \\ L_a\theta_b &= -[\theta_a, \theta_b] = -c^k_{ab}\theta_k \\ L_a z_b &= -c^k_{ab} z_k \\ i_a z_b &= -c^k_{ab}\theta_k \ . \end{aligned}$$

The following is [GS99, Theorem 3.2.1].

12.3.1 Proposition. *The Weil algebra W is acyclic and satisfies Condition C.*

Proof of Acyclicity. Define a chain homotopy Q from id to 0 by setting

$$Q(\alpha \otimes 1) \coloneqq 0 \qquad \text{and} \qquad Q(1 \otimes \alpha) \coloneqq \alpha \otimes 1 \ . \qquad \square$$

In particular, we can use W as a model for E.

The G^*-algebra W has a rather nice subalgebra of basic forms. By [GS99, Theorem 3.2.2], the basic cohomology ring of the Weil algebra W is $\mathrm{Sym}^\bullet(\mathfrak{g}^\vee)^G$. Thus

$$\mathrm{H}_*((W \otimes \Omega^*(M))_{\mathrm{basic}}, \mathrm{d}|_{\mathrm{basic}})$$

calculates $\mathrm{H}^*_G(M)$. One can use this description of the equivariant de Rham cohomology of the Weil algebra to obtain a description, called the *Cartan model*, of the equivariant de Rham cohomology of any G^*-algebra.

12.3.2 Theorem (Cartan model)**.** *For a G^*-algebra A, there is an isomorphism (the Mathai–Quillen isomorphism)*

$$\varphi \colon (W \otimes A)_{\mathrm{basic}} \xrightarrow{\sim} (\mathrm{Sym}^\bullet(\mathfrak{g}^\vee) \otimes A)^G$$

sending

$$\mathrm{d}|_{\mathrm{basic}} \mapsto \mathrm{d}_G = 1 \otimes \mathrm{d}_A - \mu^a \otimes i_a \ .$$

In particular, $\mathrm{H}^*_G(M)$ can be computed from $(\mathrm{Sym}^\bullet(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G, \mathrm{d}_G)$.

13 Classifying spaces for G-bundles

by Peter Haine

Let G be a Lie group. The purpose of this chapter is to study two differential cohomology variants of the classifying space $\mathrm{B}G$ of G. We first introduced these objects in Examples 3.3.6 and 3.3.7: they are the sheaves Bun_G and Bun_G^∇ that send a manifold M to the groupoid of principal G-bundles on M and principal G-bundles on M equipped with a connection, respectively. Our main goal is to show that Bun_G and Bun_G^∇ are differential *refinements* of the space $\mathrm{B}G$ in the sense that there are natural equivalences

$$\Gamma_\sharp(\mathrm{Bun}_G^\nabla) \simeq \Gamma_\sharp(\mathrm{Bun}_G) \simeq \mathrm{B}G$$

in the ∞-category Spc (Corollaries 13.2.6 and 13.3.29). We also show that Bun_G and Bun_G^∇ refine the classifying space of the underlying *discrete* group G^{disc} associated to G in the sense that there are natural equivalences

$$\Gamma_*(\mathrm{Bun}_G^\nabla) \simeq \Gamma_*(\mathrm{Bun}_G) \simeq \mathrm{B}G^{\mathrm{disc}}$$

in the ∞-category Spc (Proposition 13.3.33).

In order to prove that Bun_G and Bun_G^∇ refine the classifying spaces $\mathrm{B}G$ and $\mathrm{B}G^{\mathrm{disc}}$, we use explicit *presentations* of Bun_G and Bun_G^∇. The sheaf Bun_G can be realized as the geometric realization in $\mathsf{Sh}(\mathsf{Mfld})$ of the bar construction

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; G\times G \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; G \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; *$$

on the Lie group G. That is, Bun_G is the quotient $*/\!\!/G$ in $\mathsf{Sh}(\mathsf{Mfld})$ of the point by G (Proposition 13.2.5). Similarly, the sheaf Bun_G^∇ can be realized as the geometric realization in $\mathsf{Sh}(\mathsf{Mfld})$ of *action groupoid*

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; G\times G\times \Omega^1(-;\mathfrak{g}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; G\times\Omega^1(-;\mathfrak{g}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \Omega^1(-;\mathfrak{g})\,.$$

of the *adjoint action* of G on the sheaf $\Omega^1(-;\mathfrak{g})$ (see Definition 13.3.16 and Remark 13.3.17). That is, Bun_G^∇ is the quotient $\Omega^1(-;\mathfrak{g})/\!\!/G$ (Corollary 13.3.18). We then deduce that Bun_G and Bun_G^∇ refine $\mathrm{B}G$ and $\mathrm{B}G^{\mathrm{disc}}$ by using these presentations to compute the spaces $\Gamma_\sharp(\mathrm{Bun}_G)$ and $\Gamma_\sharp(\mathrm{Bun}_G^\nabla)$ as well as $\Gamma_*(\mathrm{Bun}_G)$ and $\Gamma_*(\mathrm{Bun}_G^\nabla)$.

Many texts take this colimit as the *definition* of Bun_G, and then try to argue that this colimit indeed classifies principal G-bundles. This is difficult for a few reasons, the main one being that colimits in ∞-categories of sheaves are not very explicit. Instead, we work in the other direction and use general categorical techniques to show that Bun_G admits such a presentation. After these categorical reductions, the result boils down to the straightforward claim that the group of automorphisms of the trivial G-bundle $M\times G \to M$ is naturally isomorphic to $\mathrm{C}^\infty(M,G)$; see Observation 13.2.4.

We also want to highlight two advantages of the sheaf Bun_G over the classifying space $\mathrm{B}G$:

(1) If G is not discrete, then $\mathrm{B}G$ generally has homotopy in all degrees; on the other hand, Bun_G is a sheaf of *groupoids*.

(2) Homotopy classes of maps $M \to \mathrm{B}G$ are in bijection with *isomorphism classes* of principal G-bundles on M; on the other hand, the space $\mathrm{Map}_{\mathsf{Sh}(\mathsf{Mfld})}(M, \mathrm{Bun}_G)$ is by definition the groupoid of principal G-bundles on M.

Thus, in many ways the sheaf Bun_G is more simple to work with than the classifying space $\mathrm{B}G$. The price we pay for this simplicity is that we have to work with a sheaf, rather than a single space.

Looking forward, there are also a number of applications of the work from this chapter:

(1) Chapter 14 uses the sheaves Bun_G and Bun_G^{∇} to lift characteristic classes to classes living in differential cohomology.

(2) Chapter 15 presents work of Freed–Hopkins that uses Chern–Weil theory to compute the de Rham complex of the sheaf Bun_G^{∇}.

(3) Chapter 17 computes differential cohomology groups of $\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{C})}$ and $\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})}$.

Section 13.1 sets up the general categorical framework that we need to provide presentations for Bun_G and Bun_G^{∇}. In particular, we discuss monoid and group objects in ∞-categories, actions of monoid objects, and effective epimorphisms. The reader comfortable with these ideas can safely skip this section. In § 13.2, we use these tools to provide a presentation for Bun_G. In § 13.3, we provide a presentation for Bun_G^{∇}.

13.1 Groupoid objects and effective epimorphisms

The purpose of this section is to develop a dictionary between sheaves of groupoids on Mfld and simplicial diagrams in $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Set})$. The point of developing this dictionary is that it will give us our desired presentations of the sheaves of groupoids Bun_G and Bun_G^{∇}.

13.1.a Motivation: monoid objects & group objects in ordinary categories

In order to introduce monoid, groupoid, and group objects in an ∞-category, we start with some motivation. Recall that in an ordinary category X with finite products, a *monoid object* in X is the data of an object $M \in \mathsf{X}$ along with unit and multiplication maps

$$u \colon * \to M \qquad \text{and} \qquad m \colon M \times M \to M$$

such that the unitality and associativity diagrams

$$\begin{tikzcd} M \cong M \times * \arrow[r, "\mathrm{id}_M \times u"] \arrow[dr, equal] & M \times M \arrow[d, "m"] & * \times M \cong M \arrow[l, "u \times \mathrm{id}_M"'] \arrow[dl, equal] \\ & M & \end{tikzcd}$$

and

$$\begin{tikzcd}
M \times M \times M \arrow[r, "m\times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times m"'] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

commute. To explain the definition of a monoid in an ∞-category, observe that this data can be neatly packaged as a truncated simplicial diagram

$$\begin{tikzcd}
M \times M \times M \arrow[r, shift left=6, "\mathrm{pr}_{2,3}"] \arrow[r, shift left=2, "m\times\mathrm{id}"] \arrow[r, shift right=2, "\mathrm{id}\times m"] \arrow[r, shift right=6, "\mathrm{pr}_{1,2}"'] & M \times M \arrow[l, shift left=4, "\mathrm{id}\times\mathrm{id}\times u"'] \arrow[l, "\mathrm{id}\times u\times\mathrm{id}"'] \arrow[l, shift right=4, "u\times\mathrm{id}\times\mathrm{id}"'] \arrow[r, shift left=4, "\mathrm{pr}_1"] \arrow[r, "m"] \arrow[r, shift right=4, "\mathrm{pr}_2"'] & M \arrow[l, shift left=2, "u\times\mathrm{id}"'] \arrow[l, shift right=2, "\mathrm{id}\times u"'] \arrow[r, shift left=2] \arrow[r, shift right=2] & \ast \arrow[l, "u"'] .
\end{tikzcd}$$

The simplicial identities encode the associativity and unitality of composition, as well as some information that is redundant in ordinary category theory.

Readers familiar with the bar construction may recognize this truncated simplicial diagram as the first few stages of the bar construction. To make a tighter connection between simplicial objects and monoids, let us recall the precise description of monoids via their bar constructions.

13.1.1 Notation. Write $\mathsf{Cat}_1^{\mathrm{alg}}$ for the $(1,1)$-category with objects 1-categories and morphisms functors.[4]

13.1.2 Notation. We write

$$\mathrm{N}\colon \mathsf{Cat}_1^{\mathrm{alg}} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

for the fully faithful *nerve* functor from 1-categories to simplicial sets. Write Mon for the 1-category of monoids. We regard Mon as a full subcategory of $\mathsf{Cat}_1^{\mathrm{alg}}$ via the functor sending a monoid M to the one-object category with endomorphisms the monoid M. The composite fully faithful functor

$$\mathsf{Mon} \hookrightarrow \mathsf{Cat}_1^{\mathrm{alg}} \xhookrightarrow{\mathrm{N}} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

is called the *bar construction*. We denote this composite by

$$\mathrm{Bar}\colon \mathsf{Mon} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set}) .$$

13.1.3 Lemma [Lan21, Theorem 1.1.52]**.** *The essential image of the nerve*

$$\mathrm{N}\colon \mathsf{Cat}_1^{\mathrm{alg}} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

consists of those simplicial sets $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ satisfying the Segal condition*:*

[4] We've used the somewhat strange notation $\mathsf{Cat}_1^{\mathrm{alg}}$ here to highlight that we are working with the ordinary category of categories, where we do not even keep track of natural isomorphisms. Thus this is the 'algebraic theory' of categories where we do not identify equivalent categories.

(13.1.3.1) *For each $n > 0$ and $t \in [n]$, the square*

$$\begin{tikzcd} X([n]) \arrow[r] \arrow[d] & X(\{t < \cdots < n\}) \arrow[d] \\ X(\{0 < \cdots < t\}) \arrow[r] & X(\{t\}) \end{tikzcd}$$

is a pullback square.

Consequently, the essential image of the bar construction

$$\mathrm{Bar} \colon \mathsf{Mon} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

consists of those simplicial sets $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ satisfying (13.1.3.1) and:

(13.1.3.2) *We have $X_0 \simeq *$.*

13.1.4 Remark. By induction, the Segal condition (13.1.3.1) is equivalent to the condition that for each $n > 0$, the natural map

$$X([n]) \to X(\{0 < 1\}) \underset{X(\{1\})}{\times} X(\{1 < 2\}) \underset{X(\{2\})}{\times} \cdots \underset{X(\{n-1\})}{\times} X(\{n-1 < n\})$$

is an isomorphism. Said differently, let $\mathrm{Spine}^n \subset \Delta^n$ be the simplicial subset given by the union of edges between successive vertices

$$\mathrm{Spine}^n \coloneqq \bigcup_{i=0}^{n-1} \Delta^{\{i<i+1\}} .$$

Lemma 13.1.3 says that a simplicial set X is in the essential image of the nerve if and only if every map $\mathrm{Spine}^n \to X$ admits a *unique* extension to an n-simplex $\Delta^n \to X$.

13.1.5 Remark. Note that the spine Spine^2 of Δ^2 is the horn Λ^2_1. Thus, the Segal condition (13.1.3.1) for $n = 2$ is equivalent to the statement that each horn $\Lambda^2_1 \to X$ admits a *unique* extension to a 2-simplex $\Delta^2 \to X$. More generally, the Segal condition (13.1.3.1) is equivalent to the condition that for each pair $n > 0$ and $0 < i < n$, every inner horn $\Lambda^n_i \to X$ admits a *unique* extension to an n-simplex $\Delta^n \to X$.

13.1.6 Observation (multiplication and unit via simplicial maps). If $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ is a simplicial set satisfying (13.1.3.1) and (13.1.3.2), the face map

$$d_1 \colon X_1 \times X_1 \simeq X_2 \to X_1$$

provides a multiplication on X_1 with unit given by the degeneracy map

$$s_0 \colon * \simeq X_0 \to X_1 .$$

The simplicial identities involving X_0, X_1, X_2, and X_3 encode the associativity and unitality of the multiplication. The simplicial identities involving the $(n+1)$-simplices for $n \geq 3$ encode higher-order associativity conditions for multiplying n elements. While these higher coherences are automatic for monoids in the category of sets, when working in higher category theory it is important to encode these coherences.

Since groups form a full subcategory of the category of monoids, the bar construction also identifies the category of groups with a full subcategory of the category of simplicial sets. For this it is better to use an alternative characterization of the existence of inverses: a monoid M is a group if and only if the *shear maps*

$$\begin{aligned} M \times M &\to M \times M \\ (x, y) &\mapsto (x, xy) \end{aligned} \qquad \text{and} \qquad \begin{aligned} M \times M &\to M \times M \\ (x, y) &\mapsto (xy, y) \end{aligned}$$

are bijections. Translating this into simplicial sets one sees that the category of groups is equivalent to the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$ spanned by the simplicial sets X satisfying (13.1.3.1), (13.1.3.2), and:

(3) The induced squares

$$\begin{tikzcd} X(\{0<1<2\}) \arrow[r, "d_1"] \arrow[d, "d_2"'] & X(\{0<2\}) \arrow[d] \\ X(\{0<1\}) \arrow[r] & X(\{0\}) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X(\{0<1<2\}) \arrow[r, "d_1"] \arrow[d, "d_0"'] & X(\{0<2\}) \arrow[d] \\ X(\{1<2\}) \arrow[r] & X(\{0\}) \end{tikzcd}$$

are pullback squares.

We emphasize that condition (3) is not implied by the Segal condition (13.1.3.1).

13.1.7 Notation. Write $\mathsf{Gpd}_1^{\mathrm{alg}} \subset \mathsf{Cat}_1^{\mathrm{alg}}$ for the full subcategory spanned by the groupoids.

13.1.8 Corollary. *The essential image of the nerve functor restricted to groupoids*

$$\mathrm{N} \colon \mathsf{Gpd}_1^{\mathrm{alg}} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

consists of those simplicial sets $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ *satisfying:*

(13.1.8.1) *For each object $S \in \mathbf{\Delta}^{\mathrm{op}}$ and partition $S = T \cup T'$ such that $T \cap T' = \{t\}$ consists of a single element, the induced square*

$$\begin{tikzcd} X(S) \arrow[r] \arrow[d] & X(T') \arrow[d] \\ X(T) \arrow[r] & X(\{t\}) \end{tikzcd}$$

is a pullback square.

Consequently, the essential image of the bar construction restricted to groups

$$\mathrm{Bar}\colon \mathsf{Grp} \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathsf{Set})$$

consists of those simplicial sets $X\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ *satisfying (13.1.8.1) and:*

(13.1.8.2) *We have* $X_0 \simeq *$.

13.1.b Monoid & group objects in ∞-categories

The conclusions of Lemma 13.1.3 and Corollary 13.1.8 give the correct definition of a monoid object in an arbitrary ∞-category.

13.1.9 Definition (monoid object)**.** Let X be an ∞-category with finite products. An *associative monoid* or $\mathbb{E}_1$*-monoid* in X is a simplicial object $M\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{X}$ such that

(13.1.9.1) The object M_0 is a terminal object of X.

(13.1.9.2) *Segal condition:* For each $n > 0$ and $t \in [n]$, the square

$$\begin{tikzcd} M([n]) \arrow[r] \arrow[d] & M(\{t < \cdots < n\}) \arrow[d] \\ M(\{0 < \cdots < t\}) \arrow[r] & M(\{t\}) \end{tikzcd}$$

is a pullback square in X.

In this case, we call $M_1 \in \mathsf{X}$ the *underlying object* of M. We often identify a monoid object by its underlying object. We write

$$\mathrm{Mon}(\mathsf{X}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathsf{X})$$

for the full subcategory spanned by the monoid objects.

13.1.10 Definition (groupoid object [HTT, Definitions 6.1.2.7 & 7.2.2.1])**.** Let X be an ∞-category with finite limits. A *groupoid object* in X is a simplicial object $G\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{X}$ such that for each object $S \in \boldsymbol{\Delta}^{\mathrm{op}}$ and partition $S = T \cup T'$ such that $T \cap T' = \{t\}$ consists of a single element, the induced square

$$\begin{tikzcd} G(S) \arrow[r] \arrow[d] & G(T') \arrow[d] \\ G(T) \arrow[r] & G(\{t\}) \end{tikzcd}$$

is a pullback square in X. We write

$$\mathrm{Gpd}(\mathsf{X}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathsf{X})$$

for the full subcategory spanned by the groupoid objects.

A groupoid object G is a *group object* if $G_0 \simeq *$. We write

$$\mathsf{Grp}(\mathsf{X}) \coloneqq \mathsf{Mon}(\mathsf{X}) \cap \mathsf{Gpd}(\mathsf{X})$$

for the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{X})$ spanned by the group objects.

Another way of phrasing Definitions 13.1.9 and 13.1.10 is that in the approach we take, we take the *definition* of a group object to be its delooping. The reader should consult [HA, §§4.1.2, 5.2.6, & 6.1.2] for a more complete treatment.

13.1.c Čech nerves & effective epimorphisms

In this subsection, we go in a slightly different direction and introduce a class of maps called *effective epimorphisms*. These maps generalize surjections between sets. They are important to us because specifying a groupoid object in $\mathsf{Sh}(\mathsf{Mfld})$ is equivalent to specifying an effective epimorphism (Theorem 13.1.22). In particular, we take this alternative perspective to give presentations of Bun_G and Bun_G^∇.

We begin with some notation.

13.1.11 Notation. Write $\mathbf{\Delta}_+$ for the *augmented simplex category*. That is $\mathbf{\Delta}_+$ is the category of (possibly empty) linearly ordered finite sets. We write $[-1] \in \mathbf{\Delta}_+$ for the empty linearly ordered set. The usual simplex category $\mathbf{\Delta}$ is the full subcategory of $\mathbf{\Delta}_+$ spanned by the *nonempty* linearly ordered sets.

Given an integer $n \geq -1$, write

$$\mathbf{\Delta}_{+,\leq n} \subset \mathbf{\Delta}_+$$

for the full subcategory containing $[i]$ for $-1 \leq i \leq n$ and closed under isomorphism.

13.1.12 Observation. Note that $\mathbf{\Delta}_{+,\leq 0}$ is equivalent to the category with two objects $[-1]$ and $[0]$ and a single non-identity arrow $[-1] \to [0]$.

13.1.13 Definition (Čech nerve)**.** Let X be an ∞-category with pullbacks, and let $e \colon W \to X$ be a morphism in X. The *Čech nerve* $\check{\mathrm{C}}_+(e)$ of e is the augmented simplicial object in X given by the right Kan extension of the functor

$$\mathbf{\Delta}^{\mathrm{op}}_{+,\leq 0} \to \mathsf{X}$$

that picks out the morphism $e \colon W \to X$ along the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{+,\leq 0} \subset \mathbf{\Delta}^{\mathrm{op}}_+$. Concretely, $\check{\mathrm{C}}_+(e)_n$ is the augmented simplical object

$$\cdots \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; W \underset{X}{\times} W \underset{X}{\times} W \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; W \underset{X}{\times} W \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \; W \xrightarrow{\;e\;} X$$

where $\check{\mathrm{C}}_+(e)_n$ is the $(n+1)$-fold fiber product of W over X, each degeneracy map is a diagonal morphism, and each face map is a projection.

We write $\check{C}(e)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ for the restriction of $\check{C}_+(e)$ to $\mathbf{\Delta}^{\mathrm{op}} \subset \mathbf{\Delta}_+^{\mathrm{op}}$. We also refer to $\check{C}(e)$ as the Čech nerve of e.

13.1.14 Observation. Let X be an ∞-category with pullbacks, and let $e\colon W \to X$ be a morphism in X. The Čech nerve $\check{C}(e)$ is always a groupoid object of X.

13.1.15 Definition (effective epimorphism)**.** Let X be an ∞-category with pullbacks. A morphism $e\colon W \to X$ in X is an *effective epimorphism* if the induced map $|\check{C}(e)| \to X$ is an equivalence. We often denote an effective epimorphism by the two-headed arrow '$\twoheadrightarrow$'.

13.1.16 Example. A morphism e in the 1-category of sets is an effective epimorphism if and only if e is a surjection.

13.1.17 Example [HTT, Corollary 7.2.1.15]**.** A morphism $e\colon W \to X$ in the ∞-category Spc is an effective epimorphism if and only if $\pi_0(e)\colon \pi_0(W) \to \pi_0(X)$ is a surjection.

The important fact that we need is that in $\mathsf{Sh}(\mathsf{Mfld})$ a map that is *locally* a π_0-surjection is an effective epimorphism:

13.1.18 Lemma. *Let $f\colon E \to E'$ be a morphism in $\mathsf{Sh}(\mathsf{Mfld})$. If for each $n \geq 0$, the induced morphism*

$$\pi_0 E(\mathbb{R}^n) \to \pi_0 E'(\mathbb{R}^n)$$

on connected components is a surjection, then f is an effective epimorphism.

Proof. Combine the equivalence

$$\mathsf{Sh}(\mathsf{Mfld}) \simeq \mathsf{Sh}(\mathsf{Euc})$$

of Lemma 3.5.3 with Example 13.1.17 and [HTT, Remark 6.5.1.15 & Proposition 7.2.1.14]. □

13.1.19 Warning (categorical epimorphisms vs. effective epimorphisms)**.** Recall that a morphism $e\colon W \to X$ in a 1-category X is called an *epimorphism* if the square

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, equal] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{13.1.20}$$

is a pushout square. In many 1-categories, the notions of an 'epimorphism' and an 'effective epimorphism' coincide. For example, a map of sets $e\colon W \to X$ is an effective epimorphism if and only if e is an epimorphism if and only if e is a surjection.

This is no longer the case in the setting of ∞-categories. For example, given a morphism $e\colon W \to X$ in Spc, the square (13.1.20) is a pushout if and only if e is *acyclic* [Rap19, Theorem 2.1]. That is, the reduced integral homology groups of all of the fibers of e vanish. Since $\pi_0\colon \mathsf{Spc} \to \mathsf{Set}$ preserves colimits and epimorphisms in Set are surjections, every acyclic map in Spc is an effective epimorphism. However, acyclicity is a much stronger condition.

The following results explain why the data of a groupoid object in the ∞-category $\mathsf{Sh}(\mathsf{Mfld})$ is *equivalent* to the data of an effective epimorphism.

13.1.21 Notation. Let X be an ∞-category with pullbacks. We write

$$\mathrm{Eff}(\mathsf{X}) \subset \mathrm{Fun}([1], \mathsf{X})$$

for the full subcategory of the arrow ∞-category of X spanned by those arrows $W \to X$ that are effective epimorphisms. Write $\mathrm{Eff}_*(\mathsf{X}) \subset \mathrm{Eff}(\mathsf{X})$ for the full subcategory spanned by those effective epimorphisms $* \twoheadrightarrow X$ with source the terminal object of X.

13.1.22 Theorem [HTT, p. 587]**.** *Let X be an ∞-topos. The formation of the Čech nerve defines an equivalence of ∞-categories*

$$\check{\mathrm{C}} \colon \mathrm{Eff}(\mathsf{X}) \xrightarrow{\sim} \mathsf{Gpd}(\mathsf{X})$$

between effective epimorphisms in X and groupoid objects in X. The inverse is given by sending a groupoid object G in X to the induced effective epimorphism $G_0 \twoheadrightarrow |G|$.

Moreover, the Čech nerve restricts to an equivalence

$$\check{\mathrm{C}} \colon \mathrm{Eff}_*(\mathsf{X}) \xrightarrow{\sim} \mathsf{Grp}(\mathsf{X})$$

between effective epimorphisms with source the terminal object of X and group objects of X.

13.1.23 Remark. Given a morphism $x \colon * \to X$, note that the 1-simplices of the Čech nerve of $x \colon * \to X$ are given by the loop object $\Omega_x X$, i.e., the pullback

$$\begin{tikzcd}
\Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "x"] \\
* \arrow[r, "x"'] & X .
\end{tikzcd}$$

In particular, under the equivalence of Theorem 13.1.22, an effective epimorphism $x \colon * \twoheadrightarrow X$ corresponds to the loop object $\Omega_x X$ equipped with a natural group structure.

13.1.24 Remark. The second part of Theorem 13.1.22 can be formulated in a slightly different manner. Write X_* for the ∞-category of pointed objects of X and $\mathsf{X}_*^{\mathrm{cn}} \subset \mathsf{X}_*$ for the full subcategory spanned by the *connected* objects. Then [HA, Theorem 5.2.6.15] asserts that the formation of the Čech nerve of the basepoint $* \to X$ lifts to an equivalence of ∞-categories

$$\Omega \colon \mathsf{X}_*^{\mathrm{cn}} \xrightarrow{\sim} \mathsf{Grp}(\mathsf{X}) .$$

13.1.d Actions of monoid objects

In this subsection, we explain how to use Definition 13.1.9 to give the correct definition of actions of monoid objects in ∞-categories. We begin with some motivation from ordinary category theory. Let X be an ordinary category with finite products, and let M be a monoid object in X with multiplication $m \colon M \times M \to M$ and unit $u \colon * \to M$. Recall that a *(left) action* of M on an object $X \in \mathsf{X}$ is a map $a\colon M \times X \to X$ such that the diagrams

$$
\begin{tikzcd}
M \cong * \times M \arrow[r, "u\times\mathrm{id}_M"] \arrow[dr, equal] & M \times X \arrow[d, "a"] \\
& X
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
M \times M \times X \arrow[r, "a\times\mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times a"'] & M \times X \arrow[d, "a"] \\
M \times X \arrow[r, "a"'] & X
\end{tikzcd}
$$

commute. Like with monoid objects themselves, this data can be neatly packaged as a truncated simplicial diagram

$$
\begin{tikzcd}
M \times M \times X \arrow[r, shift left=4, "\mathrm{id}\times a"] \arrow[r, shift right=4, "\mathrm{pr}_{2,3}"'] \arrow[r, "m\times\mathrm{id}\times\mathrm{id}" description] & M \times X \arrow[l, shift right=2, "u\times\mathrm{id}\times\mathrm{id}"'] \arrow[l, shift left=2, "\mathrm{id}\times u\times\mathrm{id}"] \arrow[r, shift left=3, "a"] \arrow[r, shift right=3, "\mathrm{pr}_2"'] & X \arrow[l, "u\times\mathrm{id}" description] .
\end{tikzcd}
$$

With this in mind, the following is the homotopy coherent definition of an action of a monoid object:

13.1.25 Definition (action of a monoid object)**.** Let X be an ∞-category with finite products, and let $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ be a monoid object in X. A *(left) action* of M on an object of X consists of:

(13.1.25.1) A simplicial object $A\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$.

(13.1.25.2) A map of simplicial objects $p\colon A \to M$ such that for each $[n] \in \mathbf{\Delta}$, the maps

$$f([n])\colon A([n]) \to M([n]) \qquad \text{and} \qquad A([n]) \to A(\{n\})$$

exhibit the n-simplices $A([n])$ as the product $M([n]) \times A(\{n\})$.

In this case, we say that M *acts* on the 0-simplices A_0. Given an object $X \in \mathsf{X}$, an *action of M on X* is an action $A\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ of M equipped with an identification $A_0 \simeq X$.

We write

$$\mathsf{LMod}_M(\mathsf{X}) \subset \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{X})_{/M}$$

for the full subcategory spanned by the (left) M-actions. (See [HA, Proposition 4.2.2.9].)

13.1.26 Example. The terminal object $* \in \mathsf{X}$ admits a unique M-action: this is just the simplicial object $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$.

13.1.27 Definition (quotient by an action)**.** Let X be an ∞-category with finite products and geometric realizations, let $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ be a monoid object in X, let $X \in \mathsf{X}$, and let $A\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ be an action of M on X. The *quotient $X /\!\!/ M$* of X by the action of M is the geometric realization

$$X /\!\!/ M \coloneqq |A| .$$

13.1.28 Example (classifying spaces of topological groups)**.** Let G be a topological group. Since the underlying homotopy type functor $\Pi_\infty \colon \mathsf{Top} \to \mathsf{Spc}$ preserves finite products, the underlying homotopy type $\Pi_\infty(G)$ is naturally a group object of Spc. The *classifying space* $\mathrm{B}G$ is the quotient $* /\!\!/ \Pi_\infty(G)$ of $*$ by the group object $\Pi_\infty(G) \in \mathsf{Grp}(\mathsf{Spc})$. That is, $\mathrm{B}G$ is the geometric realization of the simplicial space

$$\cdots \mathrel{\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}} \Pi_\infty(G) \times \Pi_\infty(G) \mathrel{\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}} \Pi_\infty(G) \mathrel{\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}} * \,.$$

13.1.29 Notation (classifying object)**.** Let X be an ∞-category with finite products and geometric realizations, and let $M \colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{X}$ be a monoid object in X. We write

$$\mathrm{B}_{\mathsf{X}} M \coloneqq * /\!\!/ M = |M|$$

and call $\mathrm{B}_{\mathsf{X}} M$ the *classifying object* of M.

Note that the unit defines a map $* \to M$ from the constant simplicial object at the terminal object of X to M. Passing to geometric realizations, we see that $\mathrm{B}_{\mathsf{X}} M$ admits a natural point $* \to \mathrm{B}_{\mathsf{X}} M$. Whenever we regard $\mathrm{B}_{\mathsf{X}} M$ as a pointed object, we use this natural point.

13.1.30 Remark (classifying spaces of Lie groups)**.** Let G be a Lie group. In most of this text, $\mathrm{B}G$ denotes the classifying space of a Lie group in the classical sense: in Notation 13.1.29, $\mathrm{B}G$ is the space $\mathrm{B}_{\mathsf{Spc}} \Pi_\infty(G)$. We are also interested in the classifying object of G regarded as an object of $\mathsf{Sh}(\mathsf{Mfld})$: Proposition 13.2.5 shows that the latter classifying object coincides with the sheaf Bun_G. We've included the subscript X in our notation for classifying objects in order to distinguish these two objects.

13.1.31 Observation. Let X be an ∞-category with finite products and geometric realizations, and let $M \colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{X}$ be a monoid object in X. By definition, M is the terminal in $\mathsf{LMod}_M(\mathsf{X})$, hence the quotient by M (i.e., geometric realization) naturally refines to a functor

$$(-) /\!\!/ M \colon \mathsf{LMod}_M(\mathsf{X}) \to \mathsf{X}_{/\mathrm{B}_{\mathsf{X}} M} \,.$$

13.1.32 Proposition [SAG, Proposition E.6.4.4]**.** *Let X be an ∞-category with finite limits and geometric realizations, and let G be a group object of X. Assume that geometric realizations preserve finite products in X. Then the functor*

$$(-) /\!\!/ G \colon \mathsf{LMod}_G(\mathsf{X}) \to \mathsf{X}_{/\mathrm{B}_{\mathsf{X}} M}$$

admits a right adjoint

$$\mathrm{F}_G \colon \mathsf{X}_{/\mathrm{B}_{\mathsf{X}} M} \to \mathsf{LMod}_G(\mathsf{X})$$

defined by

$$[f \colon Y \to \mathrm{B}_{\mathsf{X}} G] \mapsto \check{\mathrm{C}}(* \to \mathrm{B}_{\mathsf{X}} G) \underset{\mathrm{B}_{\mathsf{X}} G}{\times} Y \,.$$

In particular, the composite

$$\mathsf{X}_{/\mathrm{B}_{\mathsf{X}}M} \xrightarrow{\mathrm{F}_G} \mathsf{LMod}_G(\mathsf{X}) \xrightarrow{\text{forget}} \mathsf{X}$$

is given by

$$[f\colon Y \to \mathrm{B}_{\mathsf{X}}G] \mapsto \mathrm{fib}(f)\,.$$

13.1.33 Theorem (actions via classifying objects)**.** *Let* X *be an* ∞*-topos and let* $G\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ *be a group object in* X*. Then:*

(13.1.33.1) *The functor*

$$(-) /\!\!/ G\colon \mathsf{LMod}_G(\mathsf{X}) \to \mathsf{X}_{/\mathrm{B}_{\mathsf{X}}M}$$

is an equivalence of ∞*-categories with inverse* F_G*.*

(13.1.33.2) *Let* $X \in \mathsf{X}$*, and let* $A\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{X}$ *be an action of* G *on* X*. There is a natural equivalence*

$$X \simeq \mathrm{fib}(X /\!\!/ G \to \mathrm{B}_{\mathsf{X}}G)\,.$$

Proof. See the proof of [SAG, Theorem E.6.5.1].[5] □

13.2 The structure of Bun_G

In this section, we use the abstract material introduced in §13.1 to show that Bun_G is the quotient $* /\!\!/ G$ in $\mathsf{Sh}(\mathsf{Mfld})$ (Proposition 13.2.5). This presentation then lets us show that the space $\Gamma_\sharp(\mathrm{Bun}_G)$ recovers the classifying space $\mathrm{B}G$ (Corollary 13.2.6).

The first step is to define an effective epimorphism $* \twoheadrightarrow \mathrm{Bun}_G$.

13.2.1 Construction (triv_G)**.** Let G be a Lie group. Define a global section

$$\mathrm{triv}_G\colon * \to \mathrm{Bun}_G$$

of sheaves of groupoids on Mfld as follows. For each manifold M, the map

$$\mathrm{triv}_G(M)\colon * \to \mathrm{Bun}_G(M)$$

picks out the trivial G-bundle $\mathrm{pr}_M\colon M \times G \to M$.

13.2.2 Lemma. *Let* G *be a Lie group. Then the global section* $\mathrm{triv}_G\colon * \to \mathrm{Bun}_G$ *is an effective epimorphism in* $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$*.*

Proof. By Lemma 13.1.18, it suffices to check that for each $n \geq 0$, the groupoid $\mathrm{Bun}_G(\mathbb{R}^n)$ is connected. This is a consequence of the fact that every principal G-bundle on a contractible manifold is trivializable. □

[5] Lurie's result [SAG, Theorem E.6.5.1] is in the context of profinite homotopy theory (rather than ∞-topos theory). However, Lurie's proof only uses a few categorical properties that are true in both of these settings. Specifically, the conclusion of Theorem 13.1.33 holds in any ∞-category X with finite limits and geometric realizations in which geometric realizations are universal and geometric realizations of groupoid objects are effective.

In light of Theorem 13.1.22 and Lemma 13.2.2, the Čech nerve of triv_G provides a presentation of Bun_G. Our next goal is to identify this Čech nerve with the simplical object of $\mathrm{Sh}(\mathrm{Mfld})$ defined by the group Lie G.

13.2.3 Notation. Let G be a Lie group, that is, a group object in the category of manifolds. In order to distinguish G thought of as a manifold from G thought of as a group object, we write

$$\mathrm{Bar}(G) \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Sh}(\mathrm{Mfld}))$$

for the group object of $\mathrm{Sh}(\mathrm{Mfld})$ defined by G.

Computing the pullback $* \times_{\mathrm{Bun}_G} *$ amounts to computing the automorphisms of trivial bundles.

13.2.4 Observation (automorphism groups of trivial bundles)**.** Let M be a manifold and G a Lie group. Given a principal G-bundle $E \to M$, write $\mathrm{Aut}(E)$ for the group of automorphisms of E as a principal G-bundle over M.

The map

$$\begin{aligned} \mathrm{C}^{\infty}(M, G) &\to \mathrm{Aut}(M \times G) \\ f &\mapsto [(m, g) \mapsto (m, g \cdot f(m))] \end{aligned}$$

is an isomorphism of groups. The inverse is given by sending an isomorphism of principal G-bundles $\phi \colon M \times G \xrightarrow{\sim} M \times G$ to the composite

$$\begin{tikzcd} M \cong M \times \{e\} \arrow[r, hook] & M \times G \arrow[r, "\phi", "\sim"'] & M \times G \arrow[r, "\mathrm{pr}_G"] & G \\ m \arrow[rrr, mapsto] & & & \phi(m, e)\,. \end{tikzcd}$$

13.2.5 Proposition. *Let G be a Lie group. Then:*

(13.2.5.1) *There is a natural equivalence of simplicial objects*

$$\mathrm{Bar}(G) \xrightarrow{\sim} \check{\mathrm{C}}(\mathrm{triv}_G)$$

in $\mathrm{Sh}(\mathrm{Mfld})$.

(13.2.5.2) *There is a natural equivalence* $* /\!\!/ G \xrightarrow{\sim} \mathrm{Bun}_G$ *in* $\mathrm{Sh}(\mathrm{Mfld})$.

Proof. For (13.2.5.1), we begin by computing the n-fold pullbacks $* \times_{\mathrm{Bun}_G} \cdots \times_{\mathrm{Bun}_G} *$ along the effective epimorphism $\mathrm{triv}_G \colon * \twoheadrightarrow \mathrm{Bun}_G$. By definition, given a manifold M, an object of the groupoid

$$\Big(* \underset{\mathrm{Bun}_G}{\times} \cdots \underset{\mathrm{Bun}_G}{\times} * \Big)(M)$$

consists of a tuple

$$(M \times G, \ldots, M \times G, \phi_1 \colon M \times G \xrightarrow{\sim} M \times G, \ldots, \phi_{n-1} \colon M \times G \xrightarrow{\sim} M \times G)$$

of n trivial G-bundles along with $(n-1)$ isomorphisms of trivial G-bundles

$$\phi_1, \cdots, \phi_{n-1} \colon M \times G \xrightarrow{\sim} M \times G \ .$$

Moreover, the only morphisms in this groupoid are the identities; that is, $* \times_{\mathrm{Bun}_G} \cdots \times_{\mathrm{Bun}_G} *$ is a sheaf of sets. Observation 13.2.4 provides natural isomorphisms

$$\begin{aligned} * \underset{\mathrm{Bun}_G}{\times} \cdots \underset{\mathrm{Bun}_G}{\times} * &\cong \mathrm{C}^{\infty}(-, G) \times \cdots \times \mathrm{C}^{\infty}(-, G) \\ &= \text{よ}(G) \times \cdots \times \text{よ}(G) \ . \end{aligned}$$

Thus for each $n \geq 0$ we have provided natural isomorphisms

$$\check{\mathrm{C}}(\mathrm{triv}_G)_n \cong \mathrm{Bar}(G)_n \ .$$

It is immediate from the definitions that these isomorphisms are compatible with the simplicial identities, proving (13.2.5.1).

To conclude, note that (13.2.5.1) and Lemma 13.2.2 imply (13.2.5.2). □

Using this presentation, we compute the homotopification of Bun_G:

13.2.6 Corollary. *Let G be a Lie group. Then:*

(13.2.6.1) *There is a natural equivalence of spaces $\Gamma_\sharp(\mathrm{Bun}_G) \simeq \mathrm{B}G$.*

(13.2.6.2) *There is a natural equivalence $\mathrm{L}_{\mathrm{hi}}(\mathrm{Bun}_G) \simeq \Gamma^*(\mathrm{B}G)$ of sheaves on* Mfld.

Proof. First we show (13.2.6.1). Since $\mathrm{Bun}_G \simeq |\mathrm{Bar}(G)|$ and $\Gamma_\sharp$ preserves colimits and on Mfld agrees with the underlying homotopy type functor $\Pi_\infty \colon \mathsf{Mfld} \to \mathsf{Spc}$, we have natural equivalences

$$\begin{aligned} \Gamma_\sharp(\mathrm{Bun}_G) &\simeq \Gamma_\sharp|\mathrm{Bar}(G)| \\ &\simeq |\Gamma_\sharp \mathrm{Bar}(G)| \\ &\simeq |\mathrm{Bar}(\Pi_\infty(G))| \\ &\simeq \mathrm{B}G \ . \end{aligned}$$

To conclude note that (13.2.6.2) follows from (13.2.6.1) and the definition $\mathrm{L}_{\mathrm{hi}} = \Gamma^*\Gamma_\sharp$. □

13.2.7 Remark. Due to Proposition 13.2.5 and Corollary 13.2.6, Freed and Hopkins denoted the sheaf Bun_G by $\mathrm{B}_\bullet G$.

13.2.8 Warning. As Corollary 13.2.6 demonstrates, the sheaf Bun_G is *not* generally $\mathbb{R}$-invariant. This might seem surprising: one often quotes the classical result that 'principal G-bundles are homotopy-invariant'. However, what this classical result says is that the *set of isomorphism classses* of principal G-bundles is homotopy-invariant. On the other hand, the *groupoid* of principal G-bundles is not homotopy-invariant!

13.3 The structure of Bun_G^{∇}

The purpose of this section is to give a presentation of Bun_G^{∇} as the quotient of $\Omega^1(-;\mathfrak{g})$ by the 'adjoint action' of G (see Definition 13.3.16 and Remark 13.3.17). To do this, we use the equivalence

$$(-) /\!\!/ G \colon \mathsf{LMod}_G(\mathsf{Sh}(\mathsf{Mfld})) \to \mathsf{Sh}(\mathsf{Mfld})_{/\mathrm{Bun}_G}$$

provided by Theorem 13.1.33 and Proposition 13.2.5 and compute the fiber of the forgetful map as

$$\mathrm{fib}(\mathrm{Bun}_G^{\nabla} \to \mathrm{Bun}_G) \simeq \Omega^1(-;\mathfrak{g})$$

(Lemma 13.3.14).

In order to compute this fiber, we first need to define a map

$$\mathrm{triv}_G^{\nabla} \colon \Omega^1(-;\mathfrak{g}) \twoheadrightarrow \mathrm{Bun}_G^{\nabla} .$$

As the notation suggests, triv_G^{∇} sends a form $\omega \in \Omega^1(M;\mathfrak{g})$ to the trivial G-bundle $M \times G$ with a connection involving ω. In order to give a formula for this connection, we start in §13.3.a by explaining some background material on why every connection on a trivial G-bundle takes a particular form (Lemma 13.3.8). In §13.3.b we prove that

$$\mathrm{Bun}_G^{\nabla} \simeq \Omega^1(-;\mathfrak{g}) /\!\!/ G$$

(Corollary 13.3.18). In §13.3.c, we use this presentation to show that $\Gamma_\sharp(\mathrm{Bun}_G^{\nabla}) \simeq \mathrm{B}G$ (Corollary 13.3.29). Finally, in §13.3.d we show that the global sections of Bun_G and Bun_G^{∇} recover the classifying space of G equipped with the *discrete* topology (Proposition 13.3.33).

13.3.a Maurer–Cartan forms & connections on trivial bundles

Every Lie group admits a canonical 1-form valued in its Lie algebra called the *Maurer–Cartan form*. In this subsection, we introduce Maurer–Cartan forms and use them to explain why all connections on a trivial G-bundle have a very particular form (Lemma 13.3.8).

To define the Maurer–Cartan form, let us fix some notation.

13.3.1 Notation. Let G be a Lie group.

(13.3.1.1) We write $e \in G$ for the identity element, and $\mathfrak{g} \coloneqq \mathrm{T}_e G$ for the Lie algebra of G.

(13.3.1.2) For each $g \in G$, we write

$$\begin{aligned} \mathrm{L}_g \colon G &\to G \\ h &\mapsto gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{R}_g \colon G &\to G \\ h &\mapsto hg \end{aligned}$$

for the maps given by left and right multiplication by g, respectively.

(13.3.1.3) We write $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$ for the *adjoint action* of G on $\mathfrak{g}$. That is, Ad is the derivative of the conjugation action $G \to \mathrm{Aut}(G)$ at the identity element $e \in G$.

Geometrically, the Maurer–Cartan form is defined by using the fact that the tangent bundle of a Lie group naturally splits:

13.3.2 Observation. Let G be a Lie group. There is a natural splitting $\mathrm{T}G \simeq G \times \mathfrak{g}$ defined by

$$(g, v) \mapsto (g, (\mathrm{L}_{g^{-1}})_*(v)) .$$

Here note that $(\mathrm{L}_{g^{-1}})_*(v)$ is an element of $\mathrm{T}_{g^{-1}g}G = \mathfrak{g}$.

13.3.3 Definition (Maurer–Cartan form)**.** Let G be a Lie group. The *Maurer–Cartan form* of G is the the $\mathfrak{g}$-valued 1-form $\mathrm{MC} \in \Omega^1(G; \mathfrak{g})$ defined by the composite

$$\mathrm{T}G \xrightarrow{\sim} G \times \mathfrak{g} \xrightarrow{\mathrm{pr}_{\mathfrak{g}}} \mathfrak{g}$$

of the splitting of Observation 13.3.2 with the projection. Explicitly, the Maurer–Cartan form is defined by

$$\mathrm{MC}_g(v) \coloneqq (\mathrm{L}_{g^{-1}})_*(v) .$$

13.3.4 Remark. The Maurer–Cartan form of is often written as $g^{-1}\mathrm{d}g$.

It follows immediately from the definitions that the Maurer–Cartan form is the unique left-invariant $\mathfrak{g}$-valued 1-form on G that is the identity on $\mathrm{T}_e G = \mathfrak{g}$:

13.3.5 Proposition (characterization of the Maurer–Cartan form)**.** *Let G be a Lie group. The Maurer–Cartan form of G is the unique $\mathfrak{g}$-valued 1-form $\mathrm{MC} \in \Omega^1(G; \mathfrak{g})$ satisfying the following properties:*

(13.3.3.1) *The map $\mathrm{MC}_e \colon \mathrm{T}_e G \to \mathfrak{g}$ is the identity map.*

(13.3.3.2) *For each $g \in G$, we have*

$$\mathrm{MC}_g = \mathrm{Ad}_g(\mathrm{R}_g^* \mathrm{MC}_e) .$$

Here R_g^* denotes the pullback of forms under right translation by g.

13.3.6 Observation. Rephrasing Definition 13.3.3, the Maurer–Cartan form $\mathrm{MC} \in \Omega^1(G; \mathfrak{g})$ is the *unique* connection on the trivial G-bundle $G \to *$.

Using the Maurer–Cartan form, we can also see that every connection on a trivial G-bundle on a manifold has a very particular form. Namely, they are all obtained by pulling back the Maurer–Cartan form from G and adding a form that lives 'horizontally' over M.

13.3.7 Observation. Let M be a manifold and G a Lie group. Since the map $\mathrm{pr}_M \colon M \times G \to M$ admits a section, the pullback map $\mathrm{pr}_M^* \colon \Omega^1(M; \mathfrak{g}) \to \Omega^1(M \times G; \mathfrak{g})$ is injective. Hence the map

$$\begin{aligned} \Omega^1(M; \mathfrak{g}) &\to \Omega^1(M \times G; \mathfrak{g}) \\ \omega &\mapsto \mathrm{pr}_M^*(\omega) + \mathrm{pr}_G^*(\mathrm{MC}) \end{aligned}$$

is also injective.

13.3.8 Lemma (connections on trivial bundles)**.** *Let M be a manifold and G a Lie group. Write $i_e \colon M \hookrightarrow M \times G$ for the identity section $m \mapsto (m, e)$. Then:*

(13.3.8.1) *Given a 1-form $\omega \in \Omega^1(M; \mathfrak{g})$, the $\mathfrak{g}$-valued 1-form $\mathrm{pr}_M^*(\omega) + \mathrm{pr}_G^*(\mathrm{MC})$ is a connection 1-form on the trivial G-bundle $M \times G$.*

(13.3.8.2) *If $\theta \in \Omega^1(M \times G; \mathfrak{g})$ is a connection 1-form on the trivial G-bundle $M \times G$, then*

$$\theta = \mathrm{pr}_M^* \, i_e^*(\theta) + \mathrm{pr}_G^*(\mathrm{MC}) \, .$$

(13.3.8.3) *The image of the injection*

$$\begin{aligned} \Omega^1(M; \mathfrak{g}) &\hookrightarrow \Omega^1(M \times G; \mathfrak{g}) \\ \omega &\mapsto \mathrm{pr}_M^*(\omega) + \mathrm{pr}_G^*(\mathrm{MC}) \end{aligned}$$

is the subset of connection 1-forms on the trivial G-bundle.

13.3.b The presentation of Bun_G^∇

We now appeal to Theorem 13.1.33 to show Bun_G^∇ admits a presentation as $\Omega^1(-; \mathfrak{g}) /\!\!/ G$. To do this, we start by giving an explicit description of the fiber of the forgetful map $\mathrm{Bun}_G^\nabla \to \mathrm{Bun}_G$.

13.3.9 Notation ($\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$)**.** Let G be a Lie group. Write $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ for the pullback

$$\begin{tikzcd} \mathrm{Bun}_G^{\nabla,\mathrm{triv}} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Bun}_G^\nabla \arrow[d, "\mathrm{forget}"] \\ * \arrow[r, two heads, "\mathrm{triv}_G"'] & \mathrm{Bun}_G \, . \end{tikzcd}$$

13.3.10 Observation (explicit description of $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$)**.** Let G be a Lie group. By the explicit description of pullbacks of groupoids, for each manifold M, objects of $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}(M)$ consist of triples

$$(P, \theta \in \Omega^1(P; \mathfrak{g}), \phi \colon M \times G \xrightarrow{\sim} P) \, ,$$

where $P \to M$ is a G-bundle, θ is a connection on P, and ϕ is a trivialization of P. A morphism $(P_1, \theta_1, \phi_1) \to (P_1, \theta_1, \phi_1)$ is an isomorphism of G-bundles

$$f \colon P_1 \xrightarrow{\sim} P_2$$

such that $f^*(\omega_2) = \omega_1$ and the triangle of isomorphisms

$$\begin{tikzcd} & M \times G \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ P_1 \arrow[rr, "f"'] & & P_2 \end{tikzcd}$$

commutes.

13.3.11 Lemma. *Let G be a Lie group. Then the sheaf $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ is 0-truncated (i.e., equivalent to a sheaf of sets).*

Proof. Let M be a manifold and $f\colon (P_1, \theta_1, \phi_1) \to (P_1, \theta_1, \phi_1)$ a morphism in $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}(M)$. Note that the condition $\phi_2 = f\phi_1$ uniquely determines f: we necessarily have $f = \phi_2\phi_1^{-1}$. Hence the condition $f^*(\omega_2) = \omega_1$ is equivalent to the condition $\phi_1^*(\omega_1) = \phi_2^*(\omega_2)$. Thus we have

$$\mathrm{Map}_{\mathrm{Bun}_G^{\nabla,\mathrm{triv}}(M)}((P_1, \theta_1, \phi_1), (P_1, \theta_1, \phi_1)) = \begin{cases} \{\phi_2\phi_1^{-1}\} & \phi_1^*(\omega_1) = \phi_2^*(\omega_2) \\ \varnothing & \text{otherwise}\,. \end{cases} \qquad \square$$

Using this description of $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$, we now provide an equivalence $\Omega^1(-;\mathfrak{g}) \xrightarrow{\sim} \mathrm{Bun}_G^{\nabla,\mathrm{triv}}$.

13.3.12 Convention. Let V be an $\mathbb{R}$-vector space and $n \geq 0$ an integer. Throughout this chapter, we regard $\Omega^n(-;V)$ as a sheaf of *sets* (hence a sheaf of spaces) on Mfld.

13.3.13 Construction (triv_G^∇)**.** Let G be a Lie group. Define a morphism

$$\mathrm{triv}_G^\nabla\colon\ \Omega^1(-;\mathfrak{g}) \to \mathrm{Bun}_G^\nabla$$

of sheaves of groupoids on Mfld as follows. For each manifold M, the map $\mathrm{triv}_G^\nabla(M)$ is given by sending a 1-form $\omega \in \Omega^1(M;\mathfrak{g})$ to the trivial G-bundle $\mathrm{pr}_M\colon M \times G \to M$ equipped with the connection

$$\mathrm{pr}_M^*(\omega) + \mathrm{pr}_G^*(\mathrm{MC})\,.$$

13.3.14 Lemma. *Let G be a Lie group. Then the natural commutative square*

$$\begin{tikzcd} \Omega^1(-;\mathfrak{g}) \arrow[r, "\mathrm{triv}_G^\nabla"] \arrow[d] & \mathrm{Bun}_G^\nabla \arrow[d, "\mathrm{forget}"] \\ * \arrow[r, two heads, "\mathrm{triv}_G"'] & \mathrm{Bun}_G \end{tikzcd}$$

is a pullback square in $\mathsf{Sh}(\mathsf{Mfld})$. In particular, triv_G^∇ is an effective epimorphism.

Proof. Note that for each manifold M, the map of groupoids

$$t_M\colon\ \Omega^1(M;\mathfrak{g}) \to \mathrm{Bun}_G^{\nabla,\mathrm{triv}}(M)$$

induced by the universal property of the pullback is given by the assignment

$$\omega \mapsto (M\times G, \mathrm{pr}_M^*(\omega) + \mathrm{pr}_G^*(\mathrm{MC}), \mathrm{id}\colon\ M\times G \simeq M\times G)\ .$$

The map t_M is fully faithful because the groupoids $\Omega^1(M;\mathfrak{g})$ and $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}(M)$ are both equivalent to sets (Lemma 13.3.11). Lemma 13.3.8 implies that t is essentially surjective; hence t is an equivalence, as desired. □

We now explain why Lemma 13.3.14 gives rise to a presentation of Bun_G^{∇} as a quotient $\Omega^1(-;\mathfrak{g}) /\!\!/ G$.

13.3.15 Observation. Let G be a Lie group. By Proposition 13.2.5, $\mathrm{Bun}_G \simeq * /\!\!/ G$ in $\mathsf{Sh}(\mathsf{Mfld})$. Thus Theorem 13.1.33 provides an equivalence of ∞-categories

$$(-) /\!\!/ G\colon\ \mathsf{LMod}_G(\mathsf{Sh}(\mathsf{Mfld})) \simeq \mathsf{Sh}(\mathsf{Mfld})_{/\mathrm{Bun}_G}\ .$$

The inverse sends an object $f: E \to \mathrm{Bun}_G$ to the fiber $\mathrm{fib}(f)$ equipped with a G-action. By Lemma 13.3.14, applying this inverse equivalence to the forgetful map

$$\mathrm{Bun}_G^{\nabla} \to \mathrm{Bun}_G$$

defines a G-action on $\Omega^1(-;\mathfrak{g})$.

13.3.16 Definition (adjoint action)**.** Let G be a Lie group. We refer to the G-action on $\Omega^1(-;\mathfrak{g})$ described in Observation 13.3.15 as the *adjoint action*.

13.3.17 Remark (the adjoint action, explicitly)**.** Unwinding the definitions shows that the adjoint action admits the following explicit description. Given a manifold M, map $\phi\colon\ M \to G$, and 1-form $\omega \in \Omega^1(M;\mathfrak{g})$, write $\mathrm{Ad}_\phi\,\omega \in \Omega^1(M;\mathfrak{g})$ for the 1-form defined by

$$m \mapsto \mathrm{Ad}_{\phi(m)}\,\omega_m\ .$$

Then the adjoint action of G on $\Omega^1(-;\mathfrak{g})$ is given by

$$\begin{aligned} \mathrm{C}^\infty(M,G)\times\Omega^1(M;\mathfrak{g}) &\to \Omega^1(M;\mathfrak{g}) \\ (\phi,\omega) &\mapsto \mathrm{Ad}_\phi\,\omega\ . \end{aligned}$$

13.3.18 Corollary. *Let G be a Lie group. Then there is a natural equivalence*

$$\Omega^1(-;\mathfrak{g}) /\!\!/ G \simeq \mathrm{Bun}_G^{\nabla}\ .$$

from the quotient of $\Omega^1(-;\mathfrak{g})$ by the adjoint action to Bun_G^{∇}.

Proof. Immediate from Theorem 13.1.33 and the definition of the adjoint action. $\square$

13.3.19 Remark ($\mathrm{B}_\nabla G$)**.** Due to Corollary 13.3.18, Freed and Hopkins denoted the sheaf Bun_G^∇ by $\mathrm{B}_\nabla G$ [FH13, Example 5.11].

13.3.20 Remark ($\mathrm{E}_\nabla G$)**.** The sheaf $\mathrm{Bun}_G^{\nabla,\mathrm{triv}} \cong \Omega^1(-;\mathfrak{g})$ has the following alternative description. Write $\mathrm{E}_\nabla G$ for the sheaf on Mfld that assigns a manifold M the groupoid of triples

$$(P, s\colon M \to P, \theta \in \Omega^1(P;\mathfrak{g})) , \tag{13.3.21}$$

where $P \to M$ is a principal G-bundle on M, s is a global section of P, and θ is a connection on P. The morphisms in $\mathrm{E}_\nabla G(M)$ are isomorphisms of principal bundles preserving the specified sections and 1-forms. Since the data of a section of a principal G-bundle is equivalent to the a trivialization, there are isomorphisms

$$\mathrm{E}_\nabla G \cong \mathrm{Bun}_G^{\nabla,\mathrm{triv}} \cong \Omega^1(-;\mathfrak{g}) .$$

Freed and Hopkins use this alternative description in [FH13].

13.3.c The homotopification of Bun_G^∇

We now use the presentation $\mathrm{Bun}_G^\nabla \simeq \Omega^1(-;\mathfrak{g}) /\!\!/ G$ to compute the homotopification of Bun_G^∇. We begin by showing that the homotopification of $\Omega^1(-;\mathfrak{g})$ is trivial.

13.3.22 Notation. We write C^∞ for the sheaf of $\mathbb{R}$-algebras on Mfld given by the assignment

$$M \mapsto \mathrm{C}^\infty(M, \mathbb{R}) ,$$

with pointwise addition and multiplication. We regard C^∞ as an object of the ∞-category $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Vect}(\mathbb{R}))$.

13.3.23 Observation. As a sheaf of sets, C^∞ is just the sheaf represented by $\mathbb{R}$. We have introduced the notation Notation 13.3.22 to distinguish when we want to think of the sheaf of rings represented by $\mathbb{R}$ or the sheaf of sets represented by $\mathbb{R}$.

13.3.24 Lemma. *Let $E \in \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Vect}(\mathbb{R}))$. If E admits the structure of a C^∞-module, then the composite*

$$\mathsf{Sh}(\mathsf{Mfld}; \mathsf{Vect}(\mathbb{R})) \xrightarrow{\text{forget}} \mathsf{Sh}(\mathsf{Mfld}; \mathsf{Spc}) \xrightarrow{\Gamma_\sharp} \mathsf{Spc}$$

carries E to the terminal object.

Proof. To prove the claim, it suffices to show that the identity map $\Gamma_\sharp(E) \to \Gamma_\sharp(E)$ is nullhomotopic. Write $i_0, i_1 : * \to \mathrm{C}^\infty$ for the global sections specified by $0, 1 \in \mathbb{R}$, respectively. Since E admits the structure of a C^∞-module, there exist a point $0 : * \to E$ and a multiplication map

$$m\colon \mathrm{C}^\infty \times E \to E$$

such that the diagram

(13.3.25)
$$\begin{tikzcd} E \cong * \times E \arrow[r, "i_0\times \mathrm{id}_E"] \arrow[dr, "0_E"'] & \mathrm{C}^\infty \times E \arrow[d, "m"] & * \times E \cong E \arrow[l, "i_1\times \mathrm{id}_E"'] \arrow[dl, equal] \\ & E & \end{tikzcd}$$

commutes. Since the manifold $\mathbb{R}$ is contractible and $\Gamma_\sharp \colon \mathsf{Mfld} \to \mathsf{Spc}$ sends a manifold to its underlying homotopy type, we have $\Gamma_\sharp(\mathrm{C}^\infty) \simeq *$. Hence $\Gamma_\sharp(i_0)$ and $\Gamma_\sharp(i_1)$ are equivalences and

$$\Gamma_\sharp(i_0) \simeq \Gamma_\sharp(i_1) .$$

Since $\Gamma_\sharp$ preserves finite products (Corollary 5.1.7), the commutativity of the diagram (13.3.25) shows that there are equivalences

$$\begin{aligned} \mathrm{id}_{\Gamma_\sharp(E)} &\simeq \Gamma_\sharp(m) \circ (\Gamma_\sharp(i_1) \times \Gamma_\sharp(\mathrm{id}_E)) \\ &\simeq \Gamma_\sharp(m) \circ (\Gamma_\sharp(i_0) \times \Gamma_\sharp(\mathrm{id}_E)) \\ &\simeq \Gamma_\sharp(0_E) . \end{aligned}$$

Hence the map $\mathrm{id}_{\Gamma_\sharp(E)}$ is nullhomotopic, as desired. □

13.3.26 Remark (on the proof of Lemma 13.3.24)**.** One can strengthen Lemma 13.3.24 to the following claim: if $E \in \mathsf{Sh}(\mathsf{Mfld}; \mathsf{D}(\mathbb{R}))$ admits the structure of a C^∞-module, then $\Gamma_\sharp(E) \simeq 0$. The point is that from the colimit formula for $\Gamma_\sharp$ (Corollary 5.1.4), one can show that

$$\Gamma_\sharp \colon \mathsf{Sh}(\mathsf{Mfld}; \mathsf{D}(\mathbb{R})) \to \mathsf{D}(\mathbb{R})$$

admits a canonical *lax* monoidal structure with respect to the tensor products on both sides. In particular, $\Gamma_\sharp$ preserves algebras and modules over algebras. Hence $\Gamma_\sharp(E)$ is a module over the algebra $\Gamma_\sharp(\mathrm{C}^\infty)$ in $\mathsf{D}(\mathbb{R})$. The fact that the topological space $\mathbb{R}$ is contractible implies that $\Gamma_\sharp(\mathrm{C}^\infty)$ is zero; since every module over the zero algebra is also zero, $\Gamma_\sharp(E) \simeq 0$.

This sketch is the real idea behind the proof of Lemma 13.3.24. However, to spell out this argument in full detail requires a number of technical digressions that we do not need to use elsewhere. Thus we decided to give a more direct proof of the specific result we need in order to compute $\Gamma_\sharp(\mathrm{Bun}_G^\nabla)$.

13.3.27 Example. Let V be an $\mathbb{R}$-vector space and $n \geq 0$ an integer. The sheaf $\Omega^n(-; V)$ is a C^∞-module with multiplication defined by

$$\begin{aligned} \mathrm{C}^\infty(M, \mathbb{R}) \times \Omega^n(M; V) &\to \Omega^n(M; V) \\ (f, \omega) &\mapsto f\omega . \end{aligned}$$

Lemma 13.3.24 shows that $\Gamma_\sharp(\Omega^n(-; V)) \simeq *$.

13.3.28 Warning. The subsheaf $\Omega^n_{\mathrm{cl}} \subset \Omega^n$ of closed n-forms is not a C^∞-module. The reason is that multiplication by a function is not compatible with the de Rham differential: given a function $f \in \mathrm{C}^\infty(M, \mathbb{R})$ and form $\omega \in \Omega^n(M)$, we have

$$\mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega - f\mathrm{d}\omega .$$

Moreover, even if ω is closed, $\mathrm{d}f \wedge \omega$ need not be zero.

We now compute the homotopification of Bun_G^∇.

13.3.29 Corollary. *Let G be a Lie group. Then:*

(13.3.29.1) *The forgetful morphism* $\mathrm{Bun}_G^\nabla \to \mathrm{Bun}_G$ *induces an equivalence*

$$\Gamma_\sharp(\mathrm{Bun}_G^\nabla) \xrightarrow{\sim} \Gamma_\sharp(\mathrm{Bun}_G) .$$

(13.3.29.2) *There is a natural equivalence* $\Gamma_\sharp(\mathrm{Bun}_G^\nabla) \simeq \mathrm{B}G$.

Proof. First we prove (13.3.29.1). By definition, the map $\mathrm{Bun}_G^\nabla \to \mathrm{Bun}_G$ given by forgetting connection 1-forms is induced on geometric realizations by a map

$$G^{\times(\bullet-1)} \times \Omega^1(-;\mathfrak{g}) \to G^{\times(\bullet-1)}$$

from the simplicial object defining the adjoint action of G on $\Omega^1(-;\mathfrak{g})$ to the bar construction of G. Moreover, on each term this map is the projection. Hence using the fact that the left adjoint $\Gamma_\sharp$ preserves finite products (Corollary 5.1.7), we compute

$$\begin{aligned} \Gamma_\sharp(\mathrm{Bun}_G^\nabla) &\simeq \Gamma_\sharp|G^{\times(\bullet-1)} \times \Omega^1(-;\mathfrak{g})| \\ &\simeq |\Gamma_\sharp(G^{\times(\bullet-1)}) \times \Gamma_\sharp(\Omega^1(-;\mathfrak{g}))| \\ &\simeq |\Gamma_\sharp(G^{\times(\bullet-1)}) \times *| && \text{(Example 13.3.27)} \\ &\simeq \Gamma_\sharp|\operatorname{Bar}(G)| \\ &\simeq \Gamma_\sharp(\mathrm{Bun}_G) && \text{(Proposition 13.2.5)} . \end{aligned}$$

Item (13.3.29.2) now follows from Corollary 13.2.6. ☐

Later on, we make use of the following suspension spectra variant of Corollary 13.3.29:

13.3.30 Corollary. *Let G be a Lie group. There are natural equivalences of spectra*

$$\Gamma_\sharp(\Sigma^\infty_+ \mathrm{Bun}_G^\nabla) \simeq \Gamma_\sharp(\Sigma^\infty_+ \mathrm{Bun}_G) \simeq \Sigma^\infty_+ \mathrm{B}G .$$

Proof. Combine the compatibility of $\Gamma_\sharp$ and Σ^∞_+ (Example 4.4.21) with Corollary 13.3.29. ☐

13.3.d G-bundles with flat connection

In this final subsection, we prove that the global sections of Bun_G and Bun_G^∇ recover the classifying space of the group G equipped with the *discrete* topology.

13.3.31 Notation. Let G be a topological group. We write $G^{\mathrm{disc}} \in \mathsf{Grp}(\mathsf{Set})$ for the discrete group obtained by forgetting the topology on G. We write $\mathrm{B}G^{\mathrm{disc}} \in \mathsf{Spc}$ for the classifying space of G^{disc}.

13.3.32 Observation. Let G be a Lie group. Note that the global sections

$$\Gamma_*(G) = \mathrm{C}^\infty(*, G)$$

recover the discrete group G^{disc}.

13.3.33 Proposition. *Let G be a Lie group. Then:*

(13.3.33.1) *The forgetful map* $\mathrm{Bun}_G^\nabla \to \mathrm{Bun}_G$ *induces an equivalence*

$$\Gamma_*(\mathrm{Bun}_G^\nabla) \xrightarrow{\sim} \Gamma_*(\mathrm{Bun}_G)$$

on global sections.

(13.3.33.2) *There is a natural equivalence* $\Gamma_*(\mathrm{Bun}_G) \simeq \mathrm{B}G^{\mathrm{disc}}$.

Proof. To prove (13.3.33.1), note that since the Maurer–Cartan form is the *unique* connection form on the trivial G-bundle on the point (Observation 13.3.6), the map

$$\mathrm{Bun}_G^\nabla(*) \to \mathrm{Bun}_G(*)$$

forgetting the connection form is an equivalence. For (13.3.33.2), using the fact that Γ_* preserves limits and colimits, we compute

$$\begin{aligned}
\Gamma_*(\mathrm{Bun}_G) &\simeq \Gamma_* |\operatorname{Bar}(G)| && \text{(Proposition 13.2.5)} \\
&\simeq |\Gamma_* \operatorname{Bar}(G)| \\
&\simeq |\operatorname{Bar}(\Gamma_*(G))| \\
&\simeq |\operatorname{Bar}(G^{\mathrm{disc}})| && \text{(Observation 13.3.32)} \\
&= \mathrm{B}G^{\mathrm{disc}} .
\end{aligned}$$

□

14 On-diagonal differential characteristic classes

by Arun Debray

In Chapter 11, we constructed Chern, Pontryagin, and Euler classes of vector bundles in the de Rham cohomology of manifolds M. The catalyst for this chapter is the observation that these classes are always in the image of the map $\mathrm{H}^*(M;\mathbb{Z}) \to \mathrm{H}^*_{\mathrm{dR}}(M)$. That is, we have the diagram

(14.0.1)
$$\begin{tikzcd}
\mathrm{H}^k(M;\mathbb{Z}) \arrow[dr, "c^{\mathbb{Z}}(V)\mapsto c(V)"] & \\
& \mathrm{H}^k_{\mathrm{dR}}(M), \\
\Omega^k_{\mathrm{cl}}(M) \arrow[ur, "P(F_A)\mapsto c(V)"'] &
\end{tikzcd}$$

which looks suspiciously like two sides of the differential cohomology hexagon. We therefore ask whether it is possible to fill in the middle: can one choose a class $\hat{c} \in \hat{\mathrm{H}}^*(M;\mathbb{Z})$ whose image under the curvature map is the Chern–Weil form, and whose image under the characteristic class map is the lift of the characteristic class to $\mathbb{Z}$-valued cohomology?

The answer is yes, and in fact this was one of Cheeger–Simons' original applications of their theory of differential characters [CS85, §2]. In this section, we will follow the proof of Bunke–Nikolaus–Völkl [BNV16, §5.2], who work universally on the classifying stack Bun_G^∇ from Example 3.3.7. After that, we review our examples, constructing differential lifts of Chern, Pontryagin, and Euler classes, and discuss how the Whitney sum formula behaves in the differential context. Finally, we use the differential refinement of Chern–Weil theory to give a clean general description of secondary invariants. These invariants in particular include Chern–Simons invariants, which we will use again and again in Part III.

14.1 Lifting the Chern–Weil map to differential cohomology

Begin with a Lie group G and an invariant polynomial $P \in \mathrm{Sym}^n(\mathfrak{g}^\vee)^G$. From P, the Chern–Weil machine constructs a closed form $P(\Omega) \in \Omega^\bullet_{\mathrm{cl}}(\mathrm{Bun}_G^\nabla)$.[6]

We next need to choose an integer lift $c^{\mathbb{Z}}$ of c. There is both an existence and a uniqueness question: an arbitrary cohomology class need not be in the lattice

$$\mathrm{im}(\mathrm{H}^k(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^k(\mathrm{B}G;\mathbb{R}))\ ,$$

and if there is torsion in $\mathrm{H}^k(\mathrm{B}G;\mathbb{Z})$, the lift is not unique.[7]

14.1.1 Theorem (Cheeger–Simons [CS85, Theorem 2.2], Bunke–Nikolaus–Völkl [BNV16, §5.2])**.**

[6] We end up with a form on the universal object Bun_G^∇ because Chern–Weil forms are natural in the connection. For more information, see [FH13, (7.21)] and the surrounding text.

[7] For example, there is torsion in $\mathrm{H}^*(\mathrm{BO}_n;\mathbb{Z})$ and $\mathrm{H}^*(\mathrm{BSO}_n;\mathbb{Z})$ [Bro82].

Given this data, there is a unique natural class $\hat{c} \in \hat{\mathrm{H}}^k(\mathrm{Bun}_G^\nabla; \mathbb{Z})$ whose image under the characteristic class map is $c^{\mathbb{Z}}$ and whose image under the curvature map is $P(\Omega)$.

Naturality is with respect to G, keeping track of the data $c^{\mathbb{Z}}$.

Proof. The invariant polynomial P gives us a map of sheaves of sets on Mfld:

$$\Omega^1(-;\mathfrak{g}) \xrightarrow{\omega \mapsto d\omega + [\omega,\omega]} \Omega^2(-;\mathfrak{g}) \xrightarrow{P} \mathrm{Cyc}^{2p}(\Omega)\,, \tag{14.1.2}$$

where Cyc is the sheaf of differential cycles from Definition 6.2.4. If $よ(G)$ denotes the sheaf of groups associated to G by the Yoneda embedding, then the maps in (14.1.2) are $よ(G)$-equivariant, where $\mathrm{Cyc}^{2p}(\Omega)$ is given the trivial $よ(G)$-action. Take the groupoid quotient

$$(\Omega^1(-;\mathfrak{g})) /\!\!/ よ(G) \longrightarrow (\Omega^2(-;\mathfrak{g})) /\!\!/ よ(G) \longrightarrow \mathrm{Cyc}^{2p}(\Omega) /\!\!/ よ(G)\,, \tag{14.1.3}$$

then take the nerve and sheafify, giving Bun_G^∇ and Bun_G as we discussed in Proposition 13.2.5 and Corollary 13.3.18:

$$\mathrm{Bun}_G^\nabla \longrightarrow \mathrm{Bun}_G \times i(\mathrm{Cyc}^{2p}(\Omega))\,, \tag{14.1.4}$$

where $i\colon \mathsf{Set} \to \mathsf{sSet}$ builds the constant simplicial set out of a set. There is an equivalence of simplicial sheaves

$$i(\mathrm{Cyc}^{2p}(\Omega)) \xrightarrow{\sim} \Omega^\infty \mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0])\,, \tag{14.1.5}$$

where $\mathrm{H}\colon \mathsf{D}(\mathbb{Z}) \to \mathsf{Spt}$ is the Eilenberg–MacLane functor and $[0]$ means we regard the sheaf $\mathrm{Cyc}^{2p}(\Omega)$ of abelian groups as a sheaf of complexes concentrated in degree zero.

Take (14.1.4), compose with the projection onto $i(\mathrm{Cyc}^{2p}(\Omega))$, and apply (14.1.5) to obtain a map

$$\varphi_P\colon \mathrm{Bun}_G^\nabla \to \Omega^\infty \mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0])\,.$$

Let

$$\psi_P\colon \Sigma^\infty_+ \mathrm{Bun}_G^\nabla \to \mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0])$$

be the adjoint of φ_P under the $(\Sigma^\infty, \Omega^\infty)$ adjunction.

Now apply the homotopification functor $\mathrm{L_{hi}}\colon \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) \to \mathsf{Sh}_{\mathbb{R}}(\mathsf{Mfld};\mathsf{Spc})$ from Definition 4.4.5 to ψ_P. We claim this produces a map

$$\Gamma^*(\Sigma^\infty_+ BG) \xrightarrow{\chi_P} \Gamma^*(\mathrm{H}\mathbb{R}[2p])\,. \tag{14.1.6}$$

To see this, use the identifications $\mathrm{L_{hi}} \simeq \Gamma^*\Gamma_\sharp$ (Definition 4.4.5) and $\Gamma_\sharp(E) \simeq |E(\Delta^\bullet_{\mathrm{alg}})|$ (Corollary 5.1.4). The identification

$$\Gamma_\sharp(\mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0])) \simeq \mathrm{H}\mathbb{R}[2p]$$

is a dressed-up version of the de Rham theorem, and the equivalence

$$\Gamma_\sharp(\Sigma^\infty_+ \mathrm{Bun}^\nabla_G) \simeq \Sigma^\infty_+ \mathrm{B}G$$

is Corollary 13.3.30.

Next we have to identify χ_P. On cohomology, the Chern–Weil construction uses P to naturally assign a degree-$2p$ real cohomology class to a principal G-bundle; this soups up to a map $\xi_P \colon \Sigma^\infty_+ \mathrm{B}G \to \mathrm{H}\mathbb{R}[2p]$. Looking back at the definition of φ_P, we see that $\Gamma_\sharp(\psi_P) = \xi_P$; therefore $\chi_P = \Gamma^*(\xi_P)$.

Here is where $c^\mathbb{Z}$ comes in. It is data of a lift

$$\begin{tikzcd} & \mathrm{H}\mathbb{Z}[2p] \arrow[d] \\ \Sigma^\infty_+ \mathrm{B}G \arrow[ur, "c^\mathbb{Z}"] \arrow[r, "\xi_P"'] & \mathrm{H}\mathbb{R}[2p], \end{tikzcd} \tag{14.1.7}$$

giving us a diagram

$$\begin{tikzcd} \Sigma^\infty_+ \mathrm{Bun}^\nabla_G \arrow[rr, bend left, "\psi_P"] \arrow[d] & & \mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0]) \arrow[d] \\ \Gamma^*(\Sigma^\infty_+ \mathrm{B}G) \arrow[r, "\Gamma^*(c^\mathbb{Z})"] \arrow[rr, bend right, "\Gamma^*(\xi_P)"'] & \Gamma^*(\mathrm{H}\mathbb{Z}[2p]) \arrow[r] & \Gamma^*(\mathrm{H}\mathbb{R}[2p])\,. \end{tikzcd}$$

The vertical arrows are both of the form $F \to \mathrm{L_{hi}}(F)$, and are unit maps for the adjunction $(\mathrm{L_{hi}}, \text{inclusion})$ from (4.4.2).

The map from the upper left to the lower right factors through the pullback

$$\begin{tikzcd} \Sigma^\infty_+ \mathrm{Bun}^\nabla_G \arrow[rr, bend left, "\psi_P"] \arrow[r, "\hat{c}"'] \arrow[d] & \widehat{\mathrm{H}\mathbb{Z}}(2p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{H}(\mathrm{Cyc}^{2p}(\Omega)[0]) \arrow[d] \\ \Gamma^*(\Sigma^\infty_+ \mathrm{B}G) \arrow[r, "\Gamma^*(c^\mathbb{Z})"] \arrow[rr, bend right, "\Gamma^*(\xi_P)"'] & \Gamma^*(\mathrm{H}\mathbb{Z}[2p]) \arrow[r] & \Gamma^*(\mathrm{H}\mathbb{R}[2p])\,. \end{tikzcd}$$

and $\hat{c}$ is the desired differential refinement. $\square$

14.1.8 Remark. Cheeger–Simons' original proof did not use this language: they did not have Bun_G^∇ available. Instead, they use *n-classifying spaces* $\beta_\nabla^{(n)}G$. These are spaces such that all connections on principal G-bundles $P \to M$ pull back from $\beta_\nabla^{(n)}G$, provided $\dim M < n$, and the pullback need not be unique. Narasimhan–Ramanan [NR61; NR63] proved n-classifying spaces exist for all n and G, provided $\pi_0(G)$ is finite.

14.1.9 Example (differential Chern classes)**.** Borel [Bor53, §29] shows that

$$\mathrm{H}^*(\mathrm{BU}_n;\mathbb{Z}) \cong \mathbb{Z}[c_1,\dots,c_n]\,,$$

so integer lifts are unique, and using Grothendieck's axioms, one can show that the images of these Chern classes in de Rham cohomology are equal to the Chern classes we constructed in §11.5.a. Therefore we obtain *on-diagonal differential Chern classes* $\hat c_k(P,A) \in \hat{\mathrm{H}}^{2k}(M;\mathbb{Z})$ associated to principal U_n-bundles $P \to M$ with connection A. See Remark II.6 for a note on terminology.

Several authors construct differential Chern classes by other methods, including Brylinski–McLaughlin [BM96], Berthomieu [Ber10], Bunke [Bun10; Bun13], and Ho [Ho15]. Schreiber [Sch13b] constructs $\hat c_1$.

14.1.10 Example (differential Pontryagin classes)**.** Brown [Bro82, Theorem 1.6] shows there is torsion in $\mathrm{H}^*(\mathrm{BO}_n;\mathbb{Z})$, so choosing $p_k^{\mathbb{Z}}$ is not automatic. Let $c\colon \mathrm{BO}_n \to \mathrm{BU}_n$ be the complexification map, and for a principal O_n-bundle $P \to M$ define

$$p_k(P) \coloneqq (-1)^k c_{2k}(c(P)) \in \mathrm{H}^{2k}(M;\mathbb{Z})\,. \tag{14.1.11}$$

The images of these classes in de Rham cohomology are equal to the Pontryagin classes we defined in §11.5.b, so Theorem 14.1.1 produces for us *on-diagonal differential Pontryagin classes* $\hat p_k(P,A) \in \hat{\mathrm{H}}^{4k}(M;\mathbb{Z})$ associated to principal O_n-bundles with connection A.

Brylinski–McLaughlin [BM96] and Grady–Sati [GS21, Proposition 3.6] construct $\hat p_k$ in a different way.

14.1.12 Example (differential Euler classes)**.** Brown [Bro82, Theorem 1.6] shows that there is also torsion in $\mathrm{H}^*(\mathrm{BSO}_n;\mathbb{Z})$, so we must choose a lift $e^{\mathbb{Z}}$ of the Euler class we constructed in §11.5.c. There, we defined e only for n even; for odd n, we set $e \coloneqq 0$.

Let $V \to \mathrm{BSO}_n$ denote the tautological bundle. Since V is oriented, it has a $\mathbb{Z}$-cohomology Thom class $\tau(E) \in \widetilde{\mathrm{H}}^n(V, V\smallsetminus 0;\mathbb{Z})$. We let $e^{\mathbb{Z}}$ be the pullback of $\tau(E)$ by the zero section of V. The image of this class is e, so the class defined by the Pfaffian when n is even, and 0 when n is odd. For all n, however, $e^{\mathbb{Z}} \neq 0$; it is 2-torsion when n is odd.

Therefore we obtain a *on-diagonal differential Euler class* $\hat e(P,A) \in \hat{\mathrm{H}}^n(M;\mathbb{Z})$ associated to a principal SO_n-bundle with connection A, and it can be nonzero for all n, not just even n.

Brylinski–McLaughlin [BM96] and Bunke [Bun13, Example 3.85] construct $\hat e$ in a different way.

14.1.13 Remark (from principal bundles to vector bundles: an important nuance)**.** We would like to use the characteristic classes we just constructed to define differential lifts of characteristic classes of vector bundles with connection. The way this usually works for characteristic classes is that a vector bundle has an associated principal G-bundle, and we consider characteristic classes for G. For example, a rank-n complex vector bundle has a principal $\mathrm{GL}_n(\mathbb{C})$-bundle of frames. The maximal compact of $\mathrm{GL}_n(\mathbb{C})$ is U_n, so inclusion $\mathrm{U}_n \to \mathrm{GL}_n(\mathbb{C})$ induces a homotopy equivalence of classifying spaces, which means characteristic classes of principal U_n-bundles give you characteristic classes of principal $\mathrm{GL}_n(\mathbb{C})$-bundles give you characteristic classes of complex vector bundles. Both of these steps are necessary: the Chern–Weil map is only guaranteed to be an isomorphism for compact groups, and without additional structure such as a metric, the structure group of a vector bundle is noncompact.

In differential cohomology, this becomes a stumbling block: homotopy equivalences do not always induce isomorphisms on differential cohomology, so what we learn about principal U_n-bundles does not necessarily help us with complex vector bundles. Therefore a priori, the differential characteristic classes we defined above only make sense for vector bundles with a metric and a compatible connection as in §11.2.a, because these correspond to connections on principal U_n-bundles, rather than principal $\mathrm{GL}_n(\mathbb{C})$-bundles.

In addition to complex vector bundles and U_n versus $\mathrm{GL}_n(\mathbb{C})$, which is about differential Chern classes, there are two more cases to worry about.

(1) Real vector bundles and O_n and $\mathrm{GL}_n(\mathbb{R})$, and differential Pontryagin classes.

(2) Oriented real vector bundles, SO_n, and $\mathrm{GL}_n(\mathbb{R})_0$ (i.e. the connected component of $\mathrm{GL}_n(\mathbb{R})$ containing the identity), for the differential Euler class.

First, Chern classes. For $\mathrm{GL}_n(\mathbb{C})$ the Chern–Weil map is not an isomorphism, but it is surjective [CS85, §4; Pro07, §11.8.1], so differential Chern classes can be defined in the absence of a metric.

Next, Pontryagin classes. The construction in Example 14.1.10 implies differential Pontryagin classes of V, A are equal to differential Chern classes of $V \otimes \mathbb{C}$ with connection induced from A, so differential Pontryagin classes can be defined in the absence of a metric.

But Euler classes are different! If $A \in \mathrm{GL}_n(\mathbb{R})$ and $X \in \mathfrak{so}_n$, then

$$\mathrm{pf}(AXA^{-1}) = \det(A)\mathrm{pf}(X)\,, \tag{14.1.14}$$

so the Pfaffian is not $\mathrm{GL}_n(\mathbb{R})_0$-invariant. Therefore the differential Euler class requires an oriented vector bundle, a Euclidean metric, and a compatible connection.

We will use these classes in a few different ways in Part III, including obstructing conformal immersions in Chapter 20 and constructing non-topological invertible field theories in Chapter 22. Cheeger–Simons [CS85] discuss some additional applications, including characteristic classes associated to foliations and a geometric refinement of the Atiyah–Singer index theorem. There are also differential refinements of the Todd genus, $\widehat{A}$-genus [GS21, Definition 3.9], and so forth.

14.2 The Whitney sum formula for on-diagonal differential characteristic classes

The Whitney sum formula expresses the Chern, Pontryagin, and Euler classes of a direct sum $E \oplus F$ of vector bundles in terms of the respective characteristic classes of E and of F. Let $c \coloneqq 1 + c_1 + c_2 + \cdots$ denote the total Chern class and $p \coloneqq 1 + p_1 + p_2 + \cdots$ denote the total Pontryagin class.[8] For complex vector bundles $E, F \to X$, we have

$$\begin{aligned} c(E \oplus F) &= c(E)c(F) \\ c_k(E \oplus F) &= \sum_{i+j=k} c_i(E)c_j(F)\,. \end{aligned} \tag{14.2.1a}$$

For oriented real vector bundles $E, F \to X$,

$$e(E \oplus F) = e(E)e(F)\,. \tag{14.2.1b}$$

Both of these equations take place in the ring $\mathrm{H}^*(X;\mathbb{Z})$. However, for Pontryagin classes, the corresponding formula only holds modulo 2-torsion. That is, in the ring $\mathrm{H}^*(X;\mathbb{Z}[1/2])$,

$$\begin{aligned} p(E \oplus F) &= p(E)p(F) \\ p_k(E \oplus F) &= \sum_{i+j=k} p_i(E)p_j(F)\,. \end{aligned} \tag{14.2.1c}$$

The formula for the Pontryagin classes of a direct sum with $\mathbb{Z}$ coefficients is known by work of Thomas [Tho62] and Brown [Bro82, Theorem 1.6], but it is a little more complicated.

On to differential cohomology. Given vector bundles with connection (E, A^E) and (F, A^F) over a space X, the direct sum $E \oplus F$ has an induced connection $A^E \oplus A^F$. One can prove the Whitney sum formulas (14.2.1) by studying the effect of the maps

$$B(\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})) \to B\mathrm{GL}_{n_1+n_2}(\mathbb{C})$$

(resp. $B\mathrm{SO}_{n_i}$, $B\mathrm{GL}_{n_i}(\mathbb{R})$) on cohomology. Naturality of Theorem 14.1.1 then implies

$$\hat{c}(E \oplus F, A^E \oplus A^F) = \hat{c}(E, A^E)\hat{c}(F, A^F) \tag{14.2.2a}$$

$$\hat{e}(E \oplus F, A^E \oplus A^F) = \hat{e}(E, A^E)\hat{e}(F, A^F) \tag{14.2.2b}$$

$$\hat{p}(E \oplus F, A^E \oplus A^F) = \hat{p}(E, A^E)\hat{p}(F, A^F), \tag{14.2.2c}$$

where E and F are complex or oriented where needed. For (14.2.2b) we must assume E and F come with Euclidean metrics which A^E and A^F are compatible with, because of Remark 14.1.13; and as usual (14.2.2c) takes place in $\hat{\mathrm{H}}^*(X;\mathbb{Z}[1/2])$.

The formulas (14.2.2) are less useful than they might seem: in some places you might want to use it, the connection you care about on $E \oplus F$ is not a direct sum connection. This happens,

[8] Though these appear to be infinite sums, they are finite when evaluated on any vector bundle, because $c_k(E)$ and $p_k(E)$ vanish when $k > \mathrm{rank}(E)$.

for example, in the proof of Theorem 20.2.5 in Part III. Fortunately, the differential Whitney sum formula is true in more generality.

14.2.3 Definition. Choose connections A^E on E, A^F on F, and $\overline{A}$ on $E \oplus F$. The projections $E \oplus F \rightrightarrows E, F$ induce connections $\overline{A}^E$, resp. $\overline{A}^F$ on E, resp. F from $\overline{A}$. Let $F_{\overline{A}} \in \Omega^2_X(\mathrm{End}(E \oplus F))$ be the curvature of $\overline{A}$. We say A is *compatible* with $A^E \oplus A^F$ if

(1) $\overline{A}^E = A^E$ and $\overline{A}^F = A^F$, and

(2) given vector fields v, w on X, $F_{\overline{A}}(v, w) \in \Gamma(\mathrm{End}(E \oplus F))$ is block diagonal.

There are two notions of compatibility floating around: compatibility with a metric, and compatibility with the direct-sum connection. They are different.

14.2.4 Theorem (Cheeger–Simons [CS85, Theorem 4.7])**.**

(1) *If $\overline{A}$ is compatible with $A^E \oplus A^F$, then*

$$\hat{p}(E \oplus F, \overline{A}) = \hat{p}(E \oplus F, A^E \oplus A^F)\,.$$

(2) *If E and F are oriented and Euclidean, then*

$$\hat{e}(E \oplus F, \overline{A}) = \hat{e}(E \oplus F, A^E \oplus A^F)\,.$$

(3) *If E and F are complex, then*

$$\hat{c}(E \oplus F, \overline{A}) = \hat{c}(E \oplus F, A^E \oplus A^F)\,.$$

Therefore analogues of (14.2.2) *hold with $\overline{A}$ in place of $A^E \oplus A^F$.*

The proof uses a variation formula for the Chern–Simons form similar to Lemma 20.1.2.

Recall that the Whitney sum formula can be used to show that the Euler class obstructs the existence of a section of an oriented vector bundle. In the same way, the differential Euler class obstructs flat sections.

14.2.5 Lemma. *Let $V \to M$ be an oriented Euclidean vector bundle with compatible connection A admitting a flat section. Then $\hat{e}(V, A) = 0$.*

Proof. The flat section splits $V = V' \oplus \underline{\mathbb{R}}$ such that A is compatible with the direct sum connection, where $\underline{\mathbb{R}}$ carries the standard connection d. Because $\hat{e}(\underline{\mathbb{R}}, \mathrm{d}) = 0$, the Whitney sum formula finishes the proof for us. □

14.3 Secondary invariants and Chern–Simons forms

Degree-n characteristic classes provide invariants of closed, oriented n-manifolds by integration, and these invariants provide useful topological information: integrating the Euler class produces the Euler characteristic, and integrating products of Pontryagin classes produces oriented bordism invariants. In this section we discuss the analogous invariants defined by integrating on-diagonal differential characteristic classes; since the differential cohomology of a point is not concentrated in degree zero, we do not have to stick to n-manifolds.

Let G be a compact Lie group and $c^{\mathbb{Z}} \in \mathrm{H}^n(BG;\mathbb{Z})$. Theorem 14.1.1 gives us an on-diagonal differential lift $\hat{c} \in \hat{\mathrm{H}}^n(\mathrm{Bun}_G^{\nabla};\mathbb{Z})$ of $c^{\mathbb{Z}}$. Let M be a closed, oriented $(n-1)$-manifold, and let $P \to M$ be a principal G-bundle with connection A. In Chapter 9, we constructed an integration map on differential cohomology. Integration has degree $-(n-1)$, so if $\alpha_c(P,A)$ denotes the integral of $\hat{c}(P,A)$, then $\alpha_c(P,A)$ is an element of $\mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} \int_M \colon \hat{\mathrm{H}}^n(M;\mathbb{Z}) &\longrightarrow \hat{\mathrm{H}}^1(\mathrm{pt};\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \\ \hat{c}(P,A) &\longmapsto \alpha_c(P,A)\,. \end{aligned} \tag{14.3.1}$$

The quantity $\alpha_c(P,A)$, as an $\mathbb{R}/\mathbb{Z}$-valued invariant of principal bundles with connection, is called the *secondary invariant* associated to c. In this context, the $\mathbb{Z}$-valued purely topological invariant $\int_M c^{\mathbb{Z}}(P)$ on n-manifolds is called the *primary invariant*.

In examples, secondary invariants tend to be very geometric, despite our general abstract definition.

14.3.2 Example (holonomy of a connection on a principal U_1-bundle)**.** Let $P \to M$ be a principal U_1-bundle with connection A and consider the differential first Chern class $\hat{c}_1(P,A)$, built from the curvature form of A. Given an embedded, oriented loop $i\colon \mathrm{S}^1 \hookrightarrow M$, we can pull back $\hat{c}_1(P,A)$ to S^1 and integrate, defining an element of $\mathbb{R}/\mathbb{Z}$. Cheeger–Simons [CS85, Example 1.5] show that this $\mathbb{R}/\mathbb{Z}$-valued quantity is the log of the holonomy of P around S^1. That is, holonomy is the secondary invariant associated to the first Chern class or the curvature for principal U_1-bundles.

14.3.3 Example (Chern–Simons invariants)**.** Chern–Simons invariants are important examples of secondary invariants: they will appear several times in several different ways in Part III. In some settings, any secondary invariant constructed via Chern–Weil theory is called a Chern–Simons invariant, but by far the most commonly considered example is in dimension 3.

Choose a compact Lie group G and an element $\lambda \in \mathrm{H}^4(BG;\mathbb{Z})$, which we call the *level*. Given a closed 3-manifold Y, a principal G-bundle $P \to Y$, and a connection A on P, the *Chern–Simons invariant* $\mathrm{CS}_\lambda(P,A) \in \mathbb{R}/\mathbb{Z}$ [CS74] is defined to be value of the secondary invariant associated to λ on (P,A).

The standard construction of $\mathrm{CS}_\lambda(P,A)$, which is the construction Chern–Simons gave, is more geometric. We will discuss this in Chapter 19. The approach here, using differential cohomology, is due to Cheeger–Simons [CS85].

Chern [Che44] defines a differential form in a sphere bundle related to the secondary invariant built from the Euler class.

14.3.4 Remark (secondary invariants and differential generalized cohomology)**.** We can try to run the same story with a generalized cohomology theory E. To do so, we need a differential refinement $\hat{E}$ of E, an integration map for $\hat{E}$-cohomology (possibly on manifolds with some additional structure) and an on-diagonal differential characteristic class $\hat{c} \in \hat{E}^*(\mathrm{Bun}_G^\nabla)$. Together these data are a lot to ask for, but everything goes through in K-theory, for example.

Definitions of differential refinements of K and KO were first sketched by Freed [Fre00, Examples 1.12 and 1.13] and Freed–Hopkins [FH00]. Hopkins–Singer [HS05, §4.4] first constructed differential K-theory, and Grady–Sati [GS21] first systematically study differential KO-theory. There are differential lifts of the Atiyah–Bott–Shapiro integration maps in K- and KO-theory on closed spin^c, resp. spin manifolds.

We can therefore study secondary invariants for K- and KO-theories. The final piece of data we need is a differential characteristic class, and we choose $1 \in \mathrm{K}^0(X)$ or $\mathrm{KO}^0(X)$. The primary invariant associated with this data on a spin or spin^c manifold admits a geometric interpretation as the index of the spinor Dirac operator [AS68]. The secondary invariant has a related description [Lot94], as the *η-invariant* of the Dirac operator, defined and studied by Atiyah–Patodi–Singer [APS75a; APS75b; APS76]. See Freed [Fre21, §7.4] for more information.

There are several additional models for differential K-theory constructed by Klonoff [Klo08], Bunke–Schick [BS09, §2], Simons–Sullivan [SS10], Bunke–Nikolaus–Völkl [BNV16, §6], Schlegel [Sch13a, §4.2], Tradler–Wilson–Zenalian [TWZ13; TWZ16], Hekmati–Murray–Schlegel–Vozzo [HMSV15], Park [Par17], Gorokhovski–Lott [GL18], Schlarmann [Sch19], Park–Parzygnat–Redden–Stoffel [PPRS21], Cushman [Cus21], and Gomi–Yamashita [GY21]; Cushman and Gomi-Yamashita also construct models for differential KO-theory.

See Bunke–Schick [BS12] for a survey.

15 Chern–Weil forms after Freed–Hopkins

by Dexter Chua

Let G be a Lie group. The main theorem of the Freed–Hopkins paper *Chern–Weil forms and abstract homotopy theory* [FH13] is that Chern–Weil forms are the only natural way to get a differential form from a principal G-bundle. That is, Freed and Hopkins computed the de Rham complex of the sheaf Bun_G^∇ in terms of Chern–Weil forms.

Theorems along these lines are of interest historically. It is an important ingredient in the heat kernel proof of the Atiyah–Singer index theorem. Essentially, the idea of the proof is to use the heat equation to show that there is *some* formula for the index of a vector bundle in terms of the derivatives of the metric, and then by invariant theory, this must be given by the Chern–Weil forms we know and love. One then computes this for sufficiently many examples to figure out exactly which characteristic class it is, as Hirzebruch originally did for his signature formula.

15.1 The statement

To state the theorem, we work in the category $\mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$.

15.1.1 Recollection. There are a number of important sheave on Mfld that we have already encountered:

(15.1.1.1) Any $M \in \mathsf{Mfld}$ defines a representable (discrete) sheaf, which we denote by M again (Example 3.1.7).

(15.1.1.2) For $p \geq 0$, we have a discrete sheaf

$$\Omega^p \in \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc}) .$$

This is in fact a sheaf of vector spaces (Example 3.3.3). Moreover, there are linear natural transformations $\mathrm{d} : \Omega^p \to \Omega^{p+1}$. Thus, we get a sheaf of chain complexes $\Omega^\bullet$ whose value on a manifold M is the de Rham complex of M.

(15.1.1.3) Fix G a Lie group. We write $\mathrm{Bun}_G^\nabla : \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{Spc}_{\leq 1}$ for the sheaf sending a manifold M to be the groupoid of principal G-bundles on M with connection and isomorphisms (Example 3.3.7).

15.1.2 Notation. We also write

$$\Omega^\bullet : \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc})^{\mathrm{op}} \to \mathrm{D}(\mathbb{R})$$

for the right Kan extension of $\Omega^\bullet : \mathsf{Mfld}^{\mathrm{op}} \to \mathrm{D}(\mathbb{R})$ along the Yoneda embedding. Thus, for any presheaf $\mathcal{F} \in \mathsf{PSh}(\mathsf{Mfld};\mathsf{Spc})$, we can think of

$$\Omega^\bullet(\mathcal{F}) := \lim_{M \to \mathcal{F}} \Omega^\bullet(M)$$

as the de Rham complex of $\mathcal{F}$.

The main theorem is:

15.1.3 Theorem [FH13, Theorem 7.20]**.** *Let G be a Lie group. The Chern–Weil homomorphism induces an isomorphism:*

$$(\mathrm{Sym}^\bullet \mathfrak{g}^\vee)^G \xrightarrow{\sim} \Omega^\bullet(\mathrm{Bun}_G^\nabla)\,.$$

Here $(\mathrm{Sym}^\bullet \mathfrak{g}^\vee)^G$ is regarded as a complex with zero differential and $(\mathrm{Sym}^i \mathfrak{g}^\vee)^G$ is in degree $2i$. In particular, the de Rham differential on Bun_G^∇ is zero.

This implies that the Chern–Weil construction is the only natural way of obtaining differential forms from a principal G-bundle.

To prove the theorem, we use the universal principal G-bundle

$$\mathrm{triv}_G^\nabla \colon \Omega^1(-;\mathfrak{g}) \twoheadrightarrow \mathrm{Bun}_G^\nabla$$

introduced in Construction 13.3.13, as well as the presentation of Bun_G^∇ as the quotient

$$\mathrm{Bun}_G^\nabla \simeq \Omega^1(-;\mathfrak{g}) /\!\!/ G$$

of $\Omega^1(-;\mathfrak{g})$ by the adjoint action of G (Corollary 13.3.18).

Our proof naturally breaks into two steps. First, we compute $\Omega^\bullet(\Omega^1(-;\mathfrak{g}))$, and then we need to know how to compute $\Omega^\bullet(\mathcal{F} /\!\!/ G)$ from $\Omega^\bullet(\mathcal{F})$ for any discrete sheaf $\mathcal{F}$.

We first do the second part.

15.1.4 Lemma. *Let $\mathcal{F} \in \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$ be a discrete sheaf with a G-action $\alpha \colon G \times \mathcal{F} \to \mathcal{F}$. Then $\Omega^\bullet(\mathcal{F} /\!\!/ G)$ is the subcomplex of $\Omega^\bullet(\mathcal{F})$ consisting of the ω satisfying the following properites:*

(15.1.4.1) *For all $g \in G$, we have $\alpha^*\omega|_{\{g\}\times\mathcal{F}} = \omega$.*

(15.1.4.2) *For all $\xi \in \mathfrak{g}$, we have $\iota_\xi\omega = 0$.*

The first condition says ω should be G-invariant, and the second condition says ω is suitably "horizontal".

15.1.5 Remark. Let us explain what we mean by $\iota_\xi\omega$. In general, for M a manifold and X a vector field on M, for each manifold N, contraction with X on M defines a map

$$\iota_X \colon \Omega^p(M \times N) \to \Omega^{p-1}(M \times N)\,.$$

Then by left Kan extension, this induces a map

$$\iota_X \colon \Omega^p(M \times \mathcal{F}) \to \Omega^{p-1}(M \times \mathcal{F})$$

for all $\mathcal{F} \in \mathsf{Sh}(\mathsf{Mfld};\mathsf{Spc})$.

Now if $\mathcal{F}$ has a G-action and $\xi \in \mathfrak{g}$, then ξ induces an invariant vector field on G, which we also call ξ. We then define $\iota_\xi \colon \Omega^p(\mathcal{F}) \to \Omega^{p-1}(\mathcal{F})$ by the following composition

$$\Omega^p(\mathcal{F}) \xrightarrow{\alpha^*} \Omega^p(G\times\mathcal{F}) \xrightarrow{\iota_\xi} \Omega^{p-1}(G\times\mathcal{F}) \longrightarrow \Omega^{p-1}(\{e\}\times\mathcal{F}) = \Omega^{p-1}(\mathcal{F})\,,$$

where the last map is induced by the inclusion.

This gives us a very explicit method to compute the natural transformation $\iota_\xi\omega$ for $\omega \in \Omega^p(\mathcal{F})$ and $\xi \in \mathfrak{g}$. Given a test manifold M and $\phi \in \mathcal{F}(M)$, which we think of as a natural transformation $\phi\colon M \to \mathcal{F}$, we form the composite

$$G \times M \xrightarrow{1\times\phi} G \times \mathcal{F} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\omega} \Omega^p$$

This defines a differential form $\eta \in \Omega^p(G \times M)$. Then we have

$$(\iota_\xi\omega)_M(\phi) = \iota_\xi\eta|_{\{e\}\times M} .$$

Proof. We have

$$\Omega^p(\mathcal{F} /\!\!/ G) = \Omega^p(|(\mathcal{F} /\!\!/ G)_\bullet|) = \mathrm{Tot}(\Omega^p((\mathcal{F} /\!\!/ G)_\bullet)) .$$

Since $(\mathcal{F} /\!\!/ G)_\bullet$ is a simplicial discrete sheaf, its totalization can be computed by

$$\Omega^p(\mathcal{F} /\!\!/ G) = \ker\left(\Omega^p(\mathcal{F}) \xrightarrow{\mathrm{pr}^* - \alpha^*} \Omega^p(G \times \mathcal{F}) \right),$$

where $\mathrm{pr}\colon G \times \mathcal{F} \to \mathcal{F}$ is the projection.

To prove the lemma, we have to show that $\mathrm{pr}^*\omega = \alpha^*\omega$ if and only if the conditions in the lemma are satisfied. This follows from the more general claim below with $\eta = \alpha^*\omega - \mathrm{pr}^*\omega$.

15.1.6 Claim. *Let M be a manifold and $\mathcal{F}$ a sheaf. Then $\eta \in \Omega^p(M \times \mathcal{F})$ is zero if and only if:*

(15.1.6.1) *For all $x \in M$, we have $\eta|_{\{x\}\times\mathcal{F}} = 0$.*

(15.1.6.2) *For any vector field X on M, we have $\iota_X\eta = 0$.*

The conditions (15.1.4.1) and (15.1.6.1) match up exactly. Unwrapping the definition of ι_ξ and noting that $\iota_X \mathrm{pr}^*\omega = 0$ always, the only difference between (15.1.4.2) and (15.1.6.2) is that in (15.1.4.2), we only test on invariant vector fields on G, instead of all vector fields, and we only check the result is zero after restricting to a fiber $\{e\}\times\mathcal{F}$. The former is not an issue because the condition $\mathrm{C}^\infty(G)$-linear and the invariant vector fields span as a $\mathrm{C}^\infty(G)$-module. The latter also doesn't matter because we have assumed that $\alpha^*\omega$ is invariant.

To prove the claim, if $\mathcal{F}$ were a manifold, this is automatic, since the first condition says η vanishes on vectors in the N direction while the second says it vanishes on vectors in the M direction.

If $\mathcal{F}$ were an arbitrary sheaf, we know η is zero when pulled back along any map

$$(1 \times \phi)\colon M \times N \to M \times \mathcal{F}$$

where N is a manifold, by naturality of the conditions. But since $M \times \mathcal{F}$ is a colimit of such maps, η must already be zero on $M \times \mathcal{F}$. $\square$

Now it remains to describe $\Omega^\bullet(\Omega^1(-;\mathfrak{g}))$. More generally, for any vector space V, we can calculate $\Omega^\bullet(\Omega^1(-;V))$. We first state the result in the special case where $V = \mathbb{R}$.

15.1.7 Theorem. *For each $p \geq 0$ there is an equivalence*

$$\Omega^p(\Omega^1) \cong \mathbb{R}\,.$$

For $p = 2q$, it sends ω to $(\mathrm{d}\omega)^q$. For $p = 2q+1$, it sends ω to $\omega \wedge (\mathrm{d}\omega)^q$.

The general case is no harder to prove, and the result is described in terms of the Koszul complex.

15.1.8 Definition. Let V be a vector space. The *Koszul complex* $\mathrm{Kos}^\bullet V$ is a differential graded algebra whose underlying algebra is

$$\mathrm{Kos}^\bullet V = \Lambda^\bullet(V) \otimes \mathrm{Sym}^\bullet(V)\,.$$

For $v \in V$, we write v for the corresponding element in $\Lambda^1 V$, and $\tilde{v}$ for the corresponding element in $\mathrm{Sym}^1 V$. We set $|v| = 1$ and $|\tilde{v}| = 2$. The differential is then defined by

$$\mathrm{d}(v) = \tilde{v} \qquad \text{and} \qquad \mathrm{d}(\tilde{v}) = 0\,.$$

15.1.9 Theorem. *For any vector space V, we have an isomorphism of differential graded algebras*

$$\eta\colon\ \mathrm{Kos}^\bullet(V^\vee) \xrightarrow{\sim} \Omega^\bullet(\Omega^1(-;V))\,.$$

In particular,

$$\Omega^\bullet(\Omega^1(-;\mathfrak{g})) = \mathrm{Kos}^\bullet(\mathfrak{g}^\vee)\,.$$

Explicitly, for $\ell \in V^\vee = \Lambda^1(V^\vee)$, the element $\eta(\ell) \in \Omega^1(\Omega^1(-;V))$ is defined by

$$\eta(\ell)(\alpha \otimes v) = \langle v, \ell\rangle\,\alpha$$

for $\alpha \in \Omega^1$ and $v \in V$. This is then extended to a map of differential graded algebras.

In other words, the theorem says every natural transformation

$$\omega_M\colon\ \Omega^1(M;V) \to \Omega^p(M)$$

is (uniquely) a linear combination of transformations of the form

$$\sum \alpha_i \otimes v_i \mapsto \sum_{I,J} M_{I,J}(v_{i_1},\dots,v_{i_k},v_{j_1},\dots,v_{j_\ell})\,\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}\wedge\mathrm{d}\alpha_{j_1}\wedge\cdots\wedge\mathrm{d}\alpha_{j_\ell}$$

where $M_{I,J}$ is anti-symmetric in the first k variables and symmetric in the last ℓ.

Using this, we conclude:

15.1.10 Theorem. *The Chern–Weil homomorphism gives an isomorphism*

$$(\mathrm{Sym}^\bullet \mathfrak{g}^\vee)^G \xrightarrow{\sim} \Omega^\bullet(\mathrm{Bun}_G^\nabla)\,,$$

and the differential on $\Omega^\bullet(\mathrm{Bun}_G^\nabla)$ *is zero.*

Note that this $\mathrm{Sym}^\bullet \mathfrak{g}^\vee$ is different from that appearing in the Koszul complex.

Proof. We apply the criteria in Lemma 15.1.4. The first condition is the G-invariance condition, and translates to the $(-)^G$ part of the statement. So we have to check that the forms satisfying the second condition are isomorphic to $\mathrm{Sym}^\bullet \mathfrak{g}^\vee$.

To do so, we have to compute the action of ι_ξ on $\Omega^1(-;\mathfrak{g})$ following the recipe in Remark 15.1.5. Fix $\omega \in \Omega^p(\Omega^1(-;\mathfrak{g}))$ and $\xi \in \mathfrak{g}$.

Let $\phi\colon M \to \mathrm{E}_\nabla G$ be a trivial principal G-bundle with connection $A \in \Omega^1(M;\mathfrak{g})$. The induced principal G-bundle on $G \times M$ under the action then has connection $\theta + \mathrm{Ad}_{g^{-1}} A$. So by definition,

$$(\iota_\xi\omega)_M(A) = \iota_\xi\big(\omega(\theta + \mathrm{Ad}_{g^{-1}} A)\big)\big|_{\{e\}\times M}\,.$$

To compute the action on $\mathrm{Kos}^\bullet \mathfrak{g}^\vee$, it suffices to compute it on $\Lambda^1\mathfrak{g}^\vee$ and $\mathrm{Sym}^1 \mathfrak{g}^\vee$.

(1) If $\lambda \in \mathfrak{g}^\vee = \Lambda^1\mathfrak{g}^\vee$, then $\lambda(A) = \langle A, \lambda\rangle$, and

$$\iota_\xi\langle\theta + \mathrm{Ad}_{g^{-1}} A, \lambda\rangle = \langle\iota_\xi\theta + \iota_\xi \mathrm{Ad}_{g^{-1}} A, \lambda\rangle.$$

We know $\iota_\xi\theta = \xi$, and $\iota_\xi \mathrm{Ad}_{g^{-1}} A = 0$ since $\mathrm{Ad}_{g^{-1}} A$ vanishes on all vectors in the G direction. So we know

$$\iota_\xi\lambda = \langle\xi,\lambda\rangle \in \Lambda^0\mathfrak{g}^\vee.$$

(2) Next, $\tilde\lambda(A) = \langle \mathrm{d}A, \lambda\rangle$. We compute

$$\begin{aligned}
\iota_\xi\langle \mathrm{d}(\theta + \mathrm{Ad}_{g^{-1}} A), \lambda\rangle|_{\{e\}\times M} &= \iota_\xi\Big\langle -\frac{1}{2}[\theta,\theta] + \mathrm{Ad}_{\mathrm{d}g^{-1}} \wedge A + \mathrm{Ad}_{g^{-1}} \mathrm{d}A, \lambda\Big\rangle\Big|_{\{e\}\times M}\\
&= \langle -\mathrm{Ad}_\xi A, \lambda\rangle\\
&= \langle A, -\mathrm{Ad}_\xi^* \lambda\rangle\,.
\end{aligned}$$

So

$$\iota_\xi\tilde\lambda = -\mathrm{Ad}_\xi^*\lambda \in \Lambda^1\mathfrak{g}^\vee\,.$$

First observe that in $\Lambda^\bullet\mathfrak{g}^\vee$, the only elements killed by ι_ξ are those in $\Lambda^0\mathfrak{g}^\vee \cong \mathbb{R}$. To take care of the Sym part, set

$$\Omega_\lambda = \tilde\lambda + \frac{1}{2}[\lambda,\lambda]\,.$$

Since $\tilde\lambda(A) = \langle \mathrm{d}A, \lambda\rangle$, we see that $\Omega_\lambda(A) = \langle\Omega_A, \lambda\rangle$, where Ω_A is the curvature, and one calculates $\iota_\xi\Omega_\lambda = 0$. By a change of basis, we can identify

$$\mathrm{Kos}^\bullet(\mathfrak{g}^\vee) \cong \Lambda^\bullet(\mathfrak{g}^\vee) \otimes \mathrm{Sym}^\bullet\langle\Omega_\lambda : \lambda \in \mathfrak{g}^\vee\rangle\,,$$

and ι_ξ vanishes on the second factor entirely. So we are done. □

More generally, the same proof shows that:

15.1.11 Theorem. *If M is a smooth manifold, the de Rham complex of $M \times (\Omega^1(-;V))$ is the total complex of $\Omega^\bullet(M; \mathrm{Kos}^\bullet(V^\vee))$.*

In particular, if M has a G-action, then $(M \times (\Omega^1(-;\mathfrak{g}))) /\!\!/ G$ is exactly the Cartan model for equivariant de Rham cohomology.

See Theorem 12.3.2 for more on the Cartan model.

This would follow immediately if we had a result that said that

$$\Omega^\bullet(M \times \mathcal{F}) \cong \Omega^\bullet(M) \hat{\otimes} \Omega^\bullet(\mathcal{F})\,.$$

Here $\hat{\otimes}$ denotes the *completed* tensor product. Since $\Omega^\bullet(\Omega^1(-;\mathfrak{g}))$ is finite dimensional, in the case of interest the completed tensor product is the usual tensor product.

15.2 The proof

We now prove of Theorem 15.1.9. The $p = 0$ case is trivial, so assume $p > 0$.

Recall that we have to show that any natural transformation

$$\omega_M : \Omega^1(M;V) \to \Omega^p(M)$$

is (uniquely) a linear combination of transformations of the form

$$\sum \alpha_i \otimes v_i \mapsto \sum_{I,J} M_{I,J}(v_{i_1}, \dots, v_{i_k}, v_{j_1}, \dots, v_{j_\ell})\, \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \wedge \mathrm{d}\alpha_{j_1} \wedge \cdots \wedge \mathrm{d}\alpha_{j_\ell}\,.$$

The uniqueness part is easy to see since we can extract $M_{I,J}$ by evaluating $\omega_M(\alpha)$ for M of dimension large enough. So we have to show every ω_M is of this form.

The idea of the proof is to first use naturality to show that for $x \in M$, the form $\omega_M(\alpha)_x$ depends only on the N-jet of α at x for some large but finite number N (of course, *a posteriori*, $N = 1$ suffices). Once we know this, the problem is reduced to one of finite dimensional linear algebra and invariant theory.

15.2.1 Lemma. *For $\omega \in \Omega^p(\Omega^1(-;V))$ and $\alpha \in \Omega^1(M;V)$, the value of $\omega_M(\alpha)$ at $x \in M$ depends only on the N-jet of α at p for some N. In fact, $N = p$ suffices.*

We elect to introduce the constant N, despite it being equal to p, because the precise value does not matter.

Proof. Suppose α and α' have identical p-jets at x. Then there are functions $f_0, f_1, \dots, f_p$ vanishing at p and $\beta \in \Omega^1(M;V)$ such that

$$\alpha' = \alpha + f_0 f_1 \cdots f_p \beta\,.$$

The first step is to replace the f_i with more easily understood coordinate functions. Consider the maps

$$M \xrightarrow{1_M \times (f_0,\dots,f_p)} M \times \mathbb{R}^{p+1} \xrightarrow{\mathrm{pr}_1} M.$$

Let $\tilde{\alpha}, \tilde{\beta}$ be the pullbacks of the corresponding forms under pr_1, and $t_0, \dots, t_p$ the standard coordinates on $\mathbb{R}^{p+1}$. Then $\alpha, f_0 f_1 \cdots f_p \beta$ are the pullbacks of $\tilde{\alpha}, t_0 t_1 \cdots t_p \tilde{\beta}$ under the first map.

So it suffices to show that $\omega_{M\times\mathbb{R}^{p+1}}(\tilde{\alpha})$ and $\omega_{M\times\mathbb{R}^{p+1}}(\tilde{\alpha}+t_0 t_1 \cdots t_p \tilde{\beta})$ agree as p-forms at $(x, 0)$.

The point now is that by multilinearity of a p-form, it suffices to evaluate these p-forms on p-tuples of standard basis vectors (after choosing a chart for M), and there is at least one i for which the ∂_{t_i} is not in the list. So by naturality we can perform this evaluation in the submanifold defined by $t_i = 0$, in which these two p-forms agree. □

By naturality, we may assume $M = W$ is a vector space and x is the origin. The value of $\omega_W(\alpha)$ at the origin is given by a map

$$\tilde{\omega}_W : \ \mathrm{Jet}^N(W; W^\vee \otimes V) \to \Lambda^p(W^\vee)\,,$$

where $\mathrm{Jet}^N(W; W^\vee \otimes V)$ is the space of N-jets of elements of $\Omega^1(W; V)$. This is a finite dimensional vector space, given explicitly by

$$\mathrm{Jet}^N(W; W^\vee \otimes V) = \bigoplus_{j=0}^{N} \mathrm{Sym}^j(W^\vee) \otimes W^\vee \otimes V\,.$$

Under this decomposition, the j-th piece captures the j-th derivatives of α. Throughout the proof, we view $\mathrm{Sym}^j(W^\vee)$ as a *quotient* of $(W^\vee)^{\otimes j}$, hence every function on $\mathrm{Sym}^j(W^\vee)$ is in particular a function on $(W^\vee)^{\otimes j}$.

At this point, everything else follows from the fact that $\tilde{\omega}_W$ is functorial in W, and in particular $\mathrm{GL}(W)$-invariant.

15.2.2 Lemma. *The map $\tilde{\omega}_W : \ \mathrm{Jet}^N(W; W^\vee \otimes V) \to \Lambda^p(W^\vee)$ is a polynomial function.*

This lemma is true in much greater generality — it holds for any set-theoretic natural transformation between "polynomial functors" $\mathsf{Vect} \to \mathsf{Vect}$. Here a set-theoretic natural transformation is a natural transformations of the underlying set-valued functors. This is a polynomial version of the fact that a natural transformation between additive functors is necessarily additive, because being additive is a *property* and not a structure.

Proof. Write

$$F(W) := \bigoplus_{j=0}^{N} \mathrm{Sym}^j W^\vee \otimes W^\vee \otimes V \qquad \text{and} \qquad G(W) := \Lambda^p W\,.$$

We think of these as functors $\mathsf{Vect} \to \mathsf{Vect}$ (with V fixed). The point is that for $f \in \mathrm{Hom}_{\mathsf{Vect}}(W, W')$, the functions $F(f), G(f)$ are polynomial in f. This together with naturality will force $\tilde{\omega}_W$ to be polynomial as well.

To show that $\tilde{\omega}_W$ is polynomial, we have to show that if $v_1, \dots, v_n \in F(W)$, then $\tilde{\omega}_W(\sum \lambda_i v_i)$ is a polynomial function in $\lambda_1, \dots, \lambda_n$. Without loss of generality, we may assume each v_i lives in the $(j_i - 1)$-th summand (so that the summand has j_i tensor powers of $W^\vee$).

Fix a number j such that $j_i \mid j$ for all i. We first show that $\tilde{\omega}_W(\sum \lambda_i^j v_i)$ is a polynomial function in the λ_i's.

Let $f : W^{\oplus n} \to W^{\oplus n}$ be the map that multiplies by λ_i^{j/j_i} on the i-th factor, and let $\Sigma : W^{\oplus n} \to W$ be the sum map. Consider the commutative diagram

$$\begin{tikzcd}
F(W^{\oplus n}) \arrow[r, "F(f)"] \arrow[d, "\tilde{\omega}_{W^{\oplus n}}"'] & F(W^{\oplus n}) \arrow[r, "F(\Sigma)"] \arrow[d, "\tilde{\omega}_{W^{\oplus n}}"] & F(W) \arrow[d, "\tilde{\omega}_W"] \\
G(W^{\oplus n}) \arrow[r, "G(f)"'] & G(W^{\oplus n}) \arrow[r, "G(\Sigma)"'] & G(W)
\end{tikzcd}$$

Let $\tilde{v}_i \in F(W^{\oplus n})$ be the image of v_i under the inclusion of the i-th summand. Then $x = \sum \tilde{v}_i$ gets sent along the top row to $\sum \lambda_i^j v_i$. On the other hand, $\tilde{\omega}_{W^{\oplus n}}(x)$ is some element in $G(W^{\oplus n})$, and whatever it might be, the image along the bottom row gives a polynomial function in the λ_i^{j/j_i}, hence in the λ_i. So we are done.

We now know that for any finite set $v_1, \dots, v_n$, we can write

$$\tilde{\omega}_W(\lambda_1^j v_1 + \cdots + \lambda_n^j v_n) = \sum_{r_1,\dots,r_m} a_R \lambda_1^{r_1} \cdots \lambda_n^{r_n} .$$

We claim each r_i is a multiple of j (if the corresponding a_R is non-zero). Indeed, if we set

$$\lambda_i := (\mu_i^j - \nu_i^j)^{1/j} ,$$

then the result must be a polynomial in the μ_i and ν_i as well, since it is of the form

$$\tilde{\omega}_W(\sum \mu_i^j v_i - \nu_i^j v_i) .$$

But

$$\sum a_R(\mu_1^j - \nu_1^j)^{r_1/j} \cdots (\mu_n^j - \nu_n^j)^{r_n/j}$$

is polynomial in μ_i, ν_i if and only if $j \mid r_i$.

Now by taking j-th roots, we know $\tilde{\omega}_W(\sum \lambda_i v_i)$ is polynomial in the λ_i when $\lambda_i \geq 0$. That is, it is polynomial when restricted to the cone spanned by the v_i's. But since the v_i's are arbitrary, this implies it is polynomial everywhere. $\square$

15.2.3 Lemma. *Any non-zero* $\mathrm{GL}(W)$*-invariant linear map*

$$(W^\vee)^{\otimes M} \to \Lambda^p(W^\vee)$$

has $M = p$ *and is a multiple of the anti-symmetrization map. In particular, any such map is anti-symmetric.*

Proof. For convenience of notation, replace $W^\vee$ with W. Since the map is in particular invariant under $\mathbb{R}^\times \subseteq \mathrm{GL}(W)$, we must have $M = p$. By Schur's lemma, the second part of the lemma is equivalent to claiming that if we decompose $W^{\otimes p}$ as a direct sum of irreducible $\mathrm{GL}(W)$ representations, then $\Lambda^p W$ appears exactly once. In fact, we know the complete decomposition of $W^{\otimes p}$ by Schur–Weyl duality.

Let $\{V_\lambda\}$ be the set of irreducible representations of S_p. Then as an $S_p \times \mathrm{GL}(W)$-representation, we have

$$W^{\otimes p} = \bigoplus_\lambda V_\lambda \otimes W_\lambda\,,$$

where $W_\lambda = \mathrm{Hom}_{S_p}(V_\lambda, W^{\otimes p})$ is either zero or irreducible, and are distinct for different λ. Under this decomposition, $\Lambda^p W$ corresponds to the sign representation of S_p. □

So we know $\tilde{\omega}_W$ is a polynomial in $\bigoplus_j \mathrm{Sym}^j(W^\vee) \otimes W^\vee \otimes V$, and is anti-symmetric in the $W^\vee$. So the only terms that can contribute are when $j = 0$ or $j = 1$. In the $j = 1$ case, it has to factor through $\Lambda^2 W^\vee \otimes V$. So $\tilde{\omega}_W$ is polynomial in $(W^\vee \otimes V) \oplus (\Lambda^2 W^\vee \otimes V)$. This exactly says $\omega_W(\alpha)$ is given by wedging together α and $\mathrm{d}\alpha$ (and pairing with elements of $V^\vee$).

16 Bott's method

by Araminta Amabel

For G a Lie group, recall the sheaf of groupoids Bun_G from Example 3.3.6. The goal of this section is to prove Bott's theorem [Bot73, Theorem 1]:

16.0.1 Theorem. *There is an isomorphism*

$$\mathrm{H}^p(\mathrm{Bun}_G; \Omega^q) = \mathrm{H}^{p-q}_{\mathrm{cont}}(G; \mathrm{Sym}^q(\mathfrak{g}^\vee)),$$

where the right-hand side is the continuous cohomology group.

16.1 Motivation and set up

Let G be a Lie group. Recall the Chern–Weil homomorphism

$$\phi\colon\ \mathrm{Sym}(\mathfrak{g}^\vee)^G \to \mathrm{H}^*(\mathrm{B}G; \mathbb{R}).$$

Here, $\mathfrak{g}^\vee$ denotes the linear dual of $\mathfrak{g}$. We view $\mathfrak{g}^\vee$ as a G-module under the adjoint action. If G is compact, then this map φ is an isomorphism.

Given any principal G-bundle on X with connection, we get an induced map

$$\mathrm{Sym}(\mathfrak{g}^\vee)^G \to \Omega^*(X).$$

Taking $X = \mathrm{B}G$ with principal G-bundle $\mathrm{E}G \to \mathrm{B}G$, recovers the universal case, φ. Note that this construction depends on a choice of connection, but this dependence no longer matters once we descend to cohomology. Bott's method will allow us to construct a similar map with no mention of a connection.

16.2 Continuous cohomology

The following definition can be found in [Sta78, §2].

16.2.1 Definition. Let G be a topological group. Let W be a G-space. Then the *continuous cohomology* of G with coefficients in W is the cohomology $\mathrm{H}^p_{\mathrm{cont}}(G; W)$ of the cochain complex

$$\mathrm{Map}_{\mathrm{cont}}(G^{\times p}, W)$$

of continuous maps, with differential

$$\partial\colon\ \mathrm{Map}_{\mathrm{cont}}(G^{\times p}, W) \to \mathrm{Map}_{\mathrm{cont}}(G^{\times p+1}, W)$$

sending a map $f\colon G^{\times p} \to W$ to the map $(\partial f)\colon G^{\times p+1} \to W$ by

$$(\partial f)(g_1, \dots, g_{p+1}) \coloneqq f(g_2, \dots, g_{p+1}) + \left(\sum_{i=1}^{p} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1})\right) \\ + (-1)^{p+1} f(g_1, \dots, g_p) \cdot g_{p+1} .$$

Note that on the third term in (∂f), we are using the action of G on W.

16.2.2 Example. Let G be a topological group and W a G-module. The zeroeth continuous cohomology of G with values in W is the fixed points,

$$\mathrm{H}^0_{\mathrm{cont}}(G; W) \simeq W^G .$$

The following theorem of van Est can be found in [vEst53].

16.2.3 Theorem (van Est)**.** *Let G be a connected Lie group and $K \subset G$ a maximal compact subgroup. Then there is an equivalence*

$$\mathrm{H}^\bullet_{\mathrm{cont}}(G; A) \simeq \mathrm{H}^\bullet_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{k}; A)$$

for any G-space A.

See [Sta78, §5] for a discussion of this result, and [HM62] for generalizations.

16.2.4 Corollary. *Let G be a compact, connected Lie group. For $i > 0$,*

$$\mathrm{H}^i_{\mathrm{cont}}(G; A) = 0 .$$

16.3 Relating continuous cohomology to ordinary cohomology

We would like to produce a map

$$\mathrm{H}^\bullet(\mathrm{B}G; \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cont}}(G; \mathbb{R})$$

when G is a connected Lie group. We will produce this map as the edge map of a spectral sequence.

For K a Lie group, let $\mathfrak{k} \coloneqq \mathrm{Lie}(K)$.

16.3.1 Lemma. *Let G be a connected Lie group with maximal compact subgroup K. There is a spectral sequence whose E_1 term is*

$$E_1^{p,q} = \left(\Lambda^p((\mathfrak{g}/\mathfrak{k})^\vee) \otimes \mathrm{Sym}^q(\mathfrak{g}^\vee)\right)^{\mathfrak{k}}$$

converging to

$$E_\infty^{p,q} = \mathrm{Sym}^{q-p}(\mathfrak{k}^\vee) .$$

Proof. Note that $\mathfrak{g}$ splits as
$$\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{k} \oplus \mathfrak{k} .$$
Thus we can rewrite the E_1 page as

$$E_1^{p,q} = \left(\Lambda^p((\mathfrak{g}/\mathfrak{k})^\vee) \otimes \bigoplus_{a+b=q} \mathrm{Sym}^a((\mathfrak{g}/\mathfrak{k}))^\vee \otimes \mathrm{Sym}^b(\mathfrak{k})^\vee) \right)^{\mathfrak{k}} .$$

Note that the terms $\Lambda^p((\mathfrak{g}/\mathfrak{k})^\vee)$ and $\mathrm{Sym}^p((\mathfrak{g}/\mathfrak{k})^\vee)$ are Koszul dual. During the course of the spectral sequence, these Koszul dual terms cancel each other. The E_∞ page is thus

$$E_\infty^{p,q} = \mathrm{Sym}^{q-p}(\mathfrak{k}^\vee) . \qquad \square$$

We can compute the E_2 term of this spectral sequence directly. The E_1 page comes from the relative Chevalley–Eilenberg complex,

$$E_1^{p,q} = \mathrm{H}^p_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{k}; \mathrm{Sym}^q(\mathfrak{g}^\vee)) .$$

The d_1 differential is the Chevalley–Eilenberg differential. Thus the E_2 page is just relative Lie algebra cohomology,
$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{k}; \mathrm{Sym}^q(\mathfrak{g}^\vee)) .$$
By the van Est theorem, this relative Lie algebra cohomology can be recognized in terms of continuous cohomology,

$$\mathrm{H}^p_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{k}; \mathrm{Sym}^q(\mathfrak{g}^\vee)) \simeq \mathrm{H}^p_{\mathrm{cont}}(G; \mathrm{Sym}^q(\mathfrak{g}^\vee)) .$$

16.3.2 Corollary. *Let G be a connected Lie group with maximal compact subgroup K. There is a map* $\mathrm{H}^\bullet(\mathrm{B}G; \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cont}}(G; \mathbb{R})$.

Proof. One of the edge maps of the spectral sequence from Lemma 16.3.1 goes from the E_∞ term to the $E_2^{p,0}$ column. Since K is compact, the E_∞ term can be identified with $\mathrm{H}^\bullet(\mathrm{B}K; \mathbb{R})$ be the Chern–Weil homomorphism. The $E_2^{p,0}$ column is

$$\mathrm{H}^p_{\mathrm{cont}}(G; \mathrm{Sym}^0(\mathfrak{g}^\vee)) \simeq \mathrm{H}^p_{\mathrm{cont}}(G; \mathbb{R}) . \qquad \square$$

17 Lifts of Chern classes

Talk by Mike Hopkins
Notes by Araminta Amabel

17.1 Introduction

Let $\mathbb{Z}(n)$ be the Deligne complex

$$\mathbb{Z} \to \Omega^0 \to \cdots \to \Omega^{n-1}$$

We'll also let $\mathbb{Z}(\infty)$ denote the untruncated complex,

$$\mathbb{Z} \to \Omega^0 \to \cdots$$

Similarly, we define $\mathbb{R}(n)$ where $n = 1, \dots, \infty$ to be the complex

$$\mathbb{R} \to \Omega^0 \to \cdots \to \Omega^{n-1}$$

and $\mathbb{Z}_{\mathbb{C}}(n)$ to be the complex

$$\mathbb{Z} \to \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \cdots \to \Omega^{n-1}_{\mathbb{C}} .$$

One can also think of $\mathbb{Z}(n)$ as the homotopy pullback

$$\begin{tikzcd} \mathbb{Z}(n) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Sigma^{-n}\Omega^n_{\mathrm{cl}} \arrow[r] & \mathbb{R} . \end{tikzcd}$$

One take away is that there are a lot more characteristic classes in differential cohomology than you would expect.

17.1.a Virasoro group motivation

The Virasoro group is a certain central extension of $\mathrm{Diff}^+(S^1)$ by U_1,

$$U_1 \to \widetilde{\mathrm{Diff}}^+(S^1) \to \mathrm{Diff}^+(S^1) .$$

Let $\Gamma = \mathrm{Diff}^+(S^1)$ be the group of orientation preserving diffeomorphisms of S^1. Central extensions of Γ are classified by elements of $H^3(B\Gamma; \mathbb{Z}(1))$; i.e., by homotopy classes of maps $B\Gamma \to K(\mathbb{Z}(1), 3)$. We have a fiber sequence

$$K(\mathbb{Z}(1), 2) \to E\Gamma \to B\Gamma \to K(\mathbb{Z}(1), 3) .$$

Consider the fibration $E\Gamma \times_\Gamma S^1 \to B\Gamma$ with fiber S^1. Integration along the fibers gives a map

$$H^4(E\Gamma \times_\Gamma S^1; \mathbb{Z}(2)) \to H^3(B\Gamma; \mathbb{Z}(1)) .$$

There is a map $E\Gamma \times_\Gamma S^1 \to BSL(\mathbb{R})$. Thus given a class $\tilde{p}_1 \in H^4(BSL(\mathbb{R}); \mathbb{Z}(2))$, we can pull it back to get a class in $H^4(E\Gamma \times_\Gamma S^1; \mathbb{Z}(2))$. Integrating along the fiber produces a class in $H^3(B\Gamma; \mathbb{Z}(1))$. Thus, classes in $H^4(BSL(\mathbb{R}); \mathbb{Z}(2))$ produce central extensions of $\mathrm{Diff}^+(S^1)$.

17.1.b Hopes

Let G be a Lie group. Recall the sheaf of groupoids Bun_G from Example 3.3.6.

(1) If $V \to X$ is a real vector bundle, we want lifted Pontryagin classes $\tilde{p}_n(V) \in H^{4n}(X; \mathbb{Z}(2n))$.

To obtain such lifts, it suffices to construct $\tilde{p}_n \in H^{4n}(\mathrm{Bun}_{GL_n(\mathbb{R})}; \mathbb{Z}(2n))$ such that $\tilde{p}_n$ maps to p_n under the map

$$H^{4n}(\mathrm{Bun}_{GL_n(\mathbb{C})}; \mathbb{Z}(2n)) \to H^{4n}(BGL_n(\mathbb{C}); \mathbb{Z})$$

(2) If $W \to X$ is a complex vector bundle, we want (off-diagonal) Chern classes

$$\tilde{c}_m(W) \in H^{2m}(X; \mathbb{Z}_{\mathbb{C}}(m)) .$$

To obtain such lifts, it suffices to construct $\tilde{c}_n \in H^{2n}(\mathrm{Bun}_{GL_n(\mathbb{C})}; \mathbb{Z}_{\mathbb{C}}(n))$ such that $\tilde{c}_n$ maps to c_n under the map

$$H^{2n}(\mathrm{Bun}_{GL_n(\mathbb{C})}; \mathbb{Z}_{\mathbb{C}}(n)) \to H^{2n}(BGL_n(\mathbb{C}); \mathbb{Z}) .$$

(3) *Cartan formula:* Given a short exact sequence of vector bundles

$$0 \to V \to W \to U \to 0$$

an expression of the differential characteristic classes of W in terms of the differential characteristic classes for U and V. Every short exact sequence of vector bundles is split, but this splitting might not be smooth. Thus it's possible that such a formula exists for split short exact sequences, $V \oplus U$.

(4) *Projective bundle formula:* More generally, higher characteristic classes being determined by those for line bundles.

17.1.c Statement of results

The following are things Hopkins has worked out and attributes to ideas found in papers of Bott, [Bot73; BMP73]

17.1.1 Theorem. *There is a pullback square*

$$\begin{tikzcd}
\mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{C})};\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{Z}) \arrow[d] \\
\mathrm{H}^{n}(\mathrm{BU}_m \times \mathrm{BU}_m;\mathbb{C}) \arrow[r, "\text{diagonal}^*"'] & \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{C}) .
\end{tikzcd}$$

This is Corollary 17.2.9 below.

So if we wanted to lift the first Chern class c_1, we could take

$$\frac{1}{2}(c_1 \otimes 1 + 1 \otimes c_1) \in \mathrm{H}^2(\mathrm{BU}_1 \times \mathrm{BU}_1;\mathbb{C}) .$$

But, could also add to this any terms that are in the kernel of the diagonal map. So there are many possible off-diagonal lifts of c_1 to something with $\mathbb{Z}_{\mathbb{C}}(1)$ coefficients.

Using the $e^{2\pi i}$ induced isomorphism $\mathrm{K}(\mathbb{Z}_{\mathbb{C}}(1);2) \xrightarrow{\sim} \mathrm{BGL}_1(\mathbb{C})$ produces the lift of c_1 corresponding to $\frac{1}{2}(c_1 \otimes 1 + 1 \otimes c_1)$.

17.1.2 Remark. This also works for products of copies of $\mathrm{GL}_n(\mathbb{C})$. For example, let

$$G = \mathrm{GL}_n(\mathbb{C}) \times \cdots \times \mathrm{GL}_n(\mathbb{C}) .$$

Then we have a pullback square

$$\begin{tikzcd}
\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{Z}) \arrow[d] \\
\mathrm{H}^{n}(\mathrm{B}G \times \mathrm{B}G;\mathbb{C}) \arrow[r, "\text{diagonal}^*"'] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{C}) .
\end{tikzcd}$$

Let $P_{a|b} \subset \mathrm{GL}_{a+b}(\mathbb{C})$ be the subset of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is an $(a \times a)$-matrix and B is a $(b \times b)$-matrix. Note that there is a map

$$\mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C}) \to P_{a|b}$$

sending (A, B) to the block matrix with A and B on the diagonal.

17.1.3 Conjecture. *The induced map*

$$\mathrm{H}^{2n}(P_{a|b};\mathbb{Z}_{\mathbb{C}}(n)) \to \mathrm{H}^{2n}(\mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C});\mathbb{Z}_{\mathbb{C}}(n))$$

is an isomorphism.

Proof Outline. Completing Exercise 17.2.12 below, one should find that $\mathrm{H}^{2n}(P_{a|b};\mathbb{Z}_{\mathbb{C}}(n))$ fits into a pullback diagram

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_{P_{a|b}};\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{B}(P_{a|b}\cap U_{a+b});\mathbb{Z}) \arrow[d] \\ \mathrm{H}^{n}(\mathrm{B}(P_{a|b}\cap U_{a+b})\times \mathrm{B}(P_{a|b}\cap U_{a+b});\mathbb{C}) \arrow[r, "\mathrm{diagonal}^*"'] & \mathrm{H}^{2n}(\mathrm{B}(P_{a|b}\cap U_{a+b});\mathbb{C}) \end{tikzcd}$$

and $\mathrm{H}^{2n}(\mathrm{GL}_a(\mathbb{C})\times\mathrm{GL}_b(\mathbb{C}));\mathbb{Z}_{\mathbb{C}}(n))$ fits into a pullback diagram

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_a(\mathbb{C})\times\mathrm{GL}_b(\mathbb{C})};\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{BU}_{a+b};\mathbb{Z}) \arrow[d] \\ \mathrm{H}^{n}(\mathrm{BU}_{a+b}\times\mathrm{BU}_{a+b};\mathbb{C}) \arrow[r, "\mathrm{diagonal}^*"'] & \mathrm{H}^{2n}(\mathrm{BU}_{a+b};\mathbb{C}) \end{tikzcd}$$

Since every short exact sequence of vector bundles splits, the inclusion $\mathrm{BU}_{a+b}\hookrightarrow \mathrm{B}P_{a|b}$ is a homotopy equivalence. Thus so is the inclusion $\mathrm{BU}_{a+b}\hookrightarrow \mathrm{B}(P_{a|b}\cap U_{a+b})$. Hence the lower left corners of the above two pullback diagrams are isomorphic. □

Thus if we have a Cartan-like formula for split short exact sequences, we can get a Cartan-like formula for any short exact sequence.

The following is an example of Corollary 17.2.5 below.

17.1.4 Theorem. *There is a pullback square*

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})};\mathbb{Z}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{BO}_m;\mathbb{Z}) \arrow[d] \\ \mathrm{H}^{2n}(\mathrm{BGL}_m(\mathbb{C});\mathbb{R}) \arrow[r] & \mathrm{H}^{n}(\mathrm{BO}_m;\mathbb{R}) . \end{tikzcd}$$

17.1.5 Example. Take $n = 1$ and choose m large. The first Pontryagin class p_1 lives in $\mathrm{H}^4(\mathrm{BO}_m;\mathbb{Z})$. By Theorem 17.1.4, off-diagonal differential lifts of p_1 are given by a choice of class in

$$\mathrm{H}^4(\mathrm{BGL}_m(\mathbb{C});\mathbb{R}) \simeq \mathbb{R}\oplus\mathbb{R}$$

that agrees with the image of p_1 in

$$\mathrm{H}^4(\mathrm{BO}_m;\mathbb{R}) \simeq \mathbb{R} .$$

Pictorially, there is a pullback diagram

$$\begin{tikzcd} \mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{R})};\mathbb{Z}(2)) \arrow[r, "f"] \arrow[d] & \mathrm{H}^4(\mathrm{BO}_m;\mathbb{Z}) \arrow[d] \\ \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R}\,. \end{tikzcd}$$

Since this is pullback diagram, the kernel of f is the same as the kernel of the bottom horizontal map. That is, $\ker(f) = \mathbb{R}$. Thus there is a 1-parameter family of differential lifts of p_1.

One way to choose such a lift $\tilde{p}_1$ is to ask for $\tilde{p}_1$ to be primitive; i.e.,

$$\tilde{p}_1(V \oplus U) = \tilde{p}_1(V) + \tilde{p}_1(U)$$

Up to a scalar λ, there is only one choice of primitive element of $\mathrm{H}^4(\mathrm{BGL}_m;\mathbb{R})$ that agrees with p_1 in $\mathrm{H}^4(\mathrm{BO}_m;\mathbb{R})$. That class is

$$\frac{1}{2}(\lambda c_1^2 - 2c_2)\,.$$

17.2 Computations

Suppose that G is a finite-dimensional Lie group. We are interested in computing

$$\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{Z}(n))\,.$$

We start with $\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n))$.

17.2.1 Proposition. *For all k one has* $\mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(\infty)) = 0$.

Proof. By definition, $\mathbb{R}(\infty)$ is the complex

$$\mathbb{R} \to \Omega^0 \to \Omega^1 \to \cdots$$

which is acyclic by the Poincaré Lemma. □

17.2.2 Corollary. *For $k < 2n$ one has* $\mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(n)) = 0$.

Proof. We will show that for $k < 2n$ the map

$$\mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(n+1)) \to \mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(n))$$

is surjective. For this we have the long exact sequence associated to the short exact sequence

$$0 \to \Sigma^{-(n+1)}\Omega^n \to \mathbb{R}(n+1) \to \mathbb{R}(n) \to 0\,.$$

It gives us an exact sequence

$$\mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(n+1)) \to \mathrm{H}^k(\mathrm{Bun}_G;\mathbb{R}(n)) \to \mathrm{H}^{k-n}(\mathrm{Bun}_G;\Omega^n)\,.$$

By Bott's theorem [Bot73, Theorem 1], we have

$$\mathrm{H}^{k-n}(\mathrm{Bun}_G;\Omega^n) = \mathrm{H}^{k-2n}_{\mathrm{cont}}(G;\mathrm{Sym}^n(\mathfrak{g}^\vee)) ,$$

where the right-hand side is the continuous cohomology group. Since $k-2n<0$, this group is zero. □

17.2.3 Corollary. *The map*

$$\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n)) \to \mathrm{H}^n(\mathrm{Bun}_G;\Omega^n)$$

is an isomorphism.

Proof. This map is part of the long exact sequence

$$\cdots \to \mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n{+}1)) \to \mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n)) \to \mathrm{H}^n(\mathrm{Bun}_G;\Omega^n) \to \mathrm{H}^{2n+1}(\mathrm{Bun}_G;\mathbb{R}(n{+}1)) \to \cdots$$

and the two end terms are zero by Corollary 17.2.2. □

17.2.4 Corollary. *We have an isomorphism*

$$\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n)) \cong \mathrm{Sym}^n(\mathfrak{g}^\vee)^G .$$

Proof. By Corollary 17.2.3, we have an isomorphism

$$\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n)) \xrightarrow{\sim} \mathrm{H}^n(\mathrm{Bun}_G;\Omega^n) .$$

Bott's theorem gives an isomorphism

$$\mathrm{H}^n(\mathrm{Bun}_G;\Omega^n) \cong \mathrm{H}^{n-n}_{\mathrm{cont}}(G;\mathrm{Sym}^n(\mathfrak{g}^\vee)) .$$

One has

$$\mathrm{H}^0_{\mathrm{cont}}(G;\mathrm{Sym}^n(\mathfrak{g}^\vee)) \cong (\mathrm{Sym}^n(\mathfrak{g}^\vee))^G .$$ □

17.2.5 Corollary. *For every n there is a pullback square*

$$\begin{tikzcd}
\mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{Z}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{Z}) \arrow[d] \\
\mathrm{Sym}^n(\mathfrak{g}^\vee)^G \arrow[r] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{R}) .
\end{tikzcd}$$

Proof. For this consider the pullback square

$$\begin{tikzcd}
\mathbb{Z}(n) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{R}(n) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

The associated Mayer–Vietoris sequence shows that the kernel of the map from the upper left corner of

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{Z}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{Z}) \arrow[d] \\ \mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{R}(n)) \arrow[r] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{R}) \end{tikzcd}$$

to the pullback is $\mathrm{H}^{2n-1}(\mathrm{Bun}_G;\mathbb{R})$, which is zero by Chern–Weil. $\square$

Tensoring with $\mathbb{C}$ gives:

17.2.6 Corollary. *For every n there is a pullback square*

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_G;\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{Z}) \arrow[d] \\ \mathrm{Sym}^n_{\mathbb{C}}(\mathfrak{g}^\vee \otimes \mathbb{C})^{G_{\mathbb{C}}} \arrow[r] & \mathrm{H}^{2n}(\mathrm{B}G;\mathbb{C}) \end{tikzcd}$$

where $G_{\mathbb{C}}$ is the complexification of the Lie group G.

17.2.7 Remark. When G is connected, the map

$$\mathrm{Sym}^n(\mathfrak{g}^\vee)^G \to \mathrm{Sym}^n(\mathfrak{g}^\vee)^{\mathfrak{g}}$$

is an isomorphism. Otherwise, there is a residual action of $\pi_0 G$ and one has an isomorphism

$$\mathrm{Sym}^n(\mathfrak{g}^\vee)^G \to \left(\mathrm{Sym}^n(\mathfrak{g}^\vee)^{\mathfrak{g}}\right)^{\pi_0 G}.$$

We now turn to evaluating these groups.

17.2.8 Example. Let's take $G = \mathrm{GL}_n(\mathbb{C})$. Then since $\mathrm{GL}_n(\mathbb{C})$ is connected, we have

$$\mathrm{Sym}^n(\mathfrak{g}^\vee)^G = \mathrm{Sym}^n(\mathfrak{g}^\vee)^{\mathfrak{g}}$$

which depends only on $\mathfrak{g}$. Since $\mathfrak{g}$ is complex, we have

$$\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus \mathfrak{g}$$

and so

$$\mathbb{C} \otimes \mathrm{Sym}^n(\mathfrak{g}^\vee)^G = \mathrm{Sym}^n_{\mathbb{C}}(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee)^{\mathfrak{g}\oplus\mathfrak{g}}$$

Now $\mathfrak{g}$ is also the complexification of the Lie algebra $\mathfrak{u}_n$ of the unitary group U_n. Thus the above is isomorphic to

$$\mathbb{C} \otimes (\mathrm{Sym}^n(\mathfrak{u}_n \oplus \mathfrak{u}_n))^{\mathrm{U}_n \times \mathrm{U}_n}$$

which, by Chern–Weil, is

$$\mathrm{H}^{2n}(\mathrm{BU}_n \times \mathrm{BU}_n;\mathbb{C}).$$

17.2.9 Corollary. *There is a pullback diagram*

$$\begin{tikzcd} \mathrm{H}^{2n}(\mathrm{Bun}_{\mathrm{GL}_m(\mathbb{C})};\mathbb{Z}_{\mathbb{C}}(n)) \arrow[r] \arrow[d] & \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{Z}) \arrow[d] \\ \mathrm{H}^{2n}(\mathrm{BU}_m \times \mathrm{BU}_m;\mathbb{C}) \arrow[r] & \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{C}) \end{tikzcd}$$

17.2.10 Example. Let's now take the case $G = \mathrm{GL}_n(\mathbb{R})$. The main thing now is to compute

$$\mathrm{Sym}^n(\mathfrak{g}^\vee)^G = \left(\mathrm{Sym}^n(\mathfrak{g}^\vee)^{\mathfrak{g}}\right)^{\mathbb{Z}/2}$$

Using Weyl's unitary trick again, we can complexify and recognize

$$\mathfrak{g}_{\mathbb{C}} \cong (\mathfrak{u}_n) \otimes \mathbb{C}$$

and we find by Chern–Weil that

$$\left(\mathrm{Sym}^n(\mathfrak{g}^\vee)^{\mathfrak{g}}\right)_{\mathbb{C}} \cong \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{C})\,.$$

The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation on both $\mathbb{C}$ and on U_m so

$$\mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \cong \begin{cases} \mathrm{H}^{2n}(\mathrm{BU}_m;i\mathbb{R}), & n \text{ odd} \\ \mathrm{H}^{2n}(\mathrm{BU}_m;\mathbb{R}), & n \text{ even}\,. \end{cases}$$

In this case, the action of $\pi_0\,\mathrm{GL}_m$ is trivial.

17.2.11 Remark. Maybe the easiest way to be convinced of the action of complex conjugation and of $\pi_0\,\mathrm{GL}_m$ is to remember the formula for the Chern classes in terms of $\mathrm{Sym}^\bullet(\mathfrak{g}^\vee)$. For $x \in \mathfrak{gl}_n(\mathbb{C})$, the total Chern class

$$1 + c_1 t + \cdots + c_n t^n$$

is given by the homogeneous terms in the characteristic polynomial

$$\det \frac{t}{2\pi i}\begin{pmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & \ddots & \vdots \\ e_{n,1} & \cdots & e_{n,n} \end{pmatrix} - 1$$

where $e_{i,j} \in \mathfrak{gl}_n\mathbb{C}^*$ is the function associating to a matrix its (i,j) entry. If we apply this to a matrix with real entries, we see that the k-th chern class lies in $\frac{1}{(2\pi i)^k}\mathbb{R}$ and that it is invariant under conjugation by any matrix.

17.2.12 Exercise. Let $P_{a,b} \subset \mathrm{GL}_{a+b}(\mathbb{C})$ be the subgroup which sends vectors whose last b coordinates are zero to vectors whose last b coordinates are zero, as above. One may compute $\mathrm{Sym}^\bullet(\mathfrak{p}_{a,b}^\vee)^{P_{a,b}}$ by first computing $\mathrm{Sym}^\bullet(\mathfrak{p}_{a,b}^\vee)^{G_{a,b}}$ and appealing to the unitary trick. This is the relevant computation for working out a Cartan formula for an exact sequence which does not necessarily split.

18 The Virasoro algebra

by Arun Debray

The contents of this section can be summarized as follows:

- The Virasoro group is a particular central extension of $\mathrm{Diff}^+(S^1)$ by $\mathbb{T}$.
- A theorem of Segal [Seg81, Corollary 7.5] proves that

  (18.0.1) $$\mathrm{Cent}_{\mathbb{T}}(\mathrm{Diff}^+(S^1)) \xrightarrow{\sim} \mathrm{Cent}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R})) \times \mathrm{Cent}_{\mathbb{R}}(\mathrm{Witt}_{\mathbb{R}}),$$

  where $\mathrm{Witt}_{\mathbb{R}} = \mathrm{Lie}(\mathrm{Diff}^+(S^1))$ is the Witt algebra. The map is: restrict the central extension to $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Diff}^+(S^1)$ for the first component, and differentiate for the second component.
- It is possible to construct this central extension using an off-diagonal differential lift of p_1 and a transgression map.

18.1 Review of central extensions

18.1.1 Definition. Let G be a group and A be an abelian group. A *central extension* of G by A is a short exact sequence of groups

(18.1.2) $$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1,$$

such that $A \subset Z(\widetilde{G})$. An equivalence of central extensions is a map of short exact sequences which is the identity on G and on A. These form an abelian group we denote $\mathrm{Cent}_A(G)$.

When G and A have additional structure, we will ask that central extensions respect that structure: for example, when both are Lie groups (possibly infinite-dimensional), we want (18.1.2) to be a short exact sequence of Lie groups.

For discrete G and A, central extensions are classified by $\mathrm{H}^2(G; A)$. Explicitly, given a cocycle $b\colon G \times G \to A$, we build the central extension by setting $\widetilde{G} = G \times A$ as sets, with the twisted multiplication

(18.1.3) $$(g_1, a_1) \cdot_b (g_2, a_2) := (g_1 g_2, a_1 + a_2 + b(g_1, g_2)).$$

Associativity follows from the cocycle condition; if two cocycles are related by a coboundary, their induced central extensions are equivalent.

Generalizing this to Lie groups is not straightforward — you can't just use smooth cochains unless A is a topological vector space. We are interested in central extensions by $\mathbb{T}$, so we'll have to be craftier. The fix is due to Segal [Seg70], and was later rediscovered by Brylinski [Bry00], following Blanc [Bla85]. We rephrase it in language familiar to this seminar.

Let A be an abelian Lie group. Throughout today's talk, $\underline{A}$ denotes the simplicial sheaf on Man whose value on a test manifold M is the space of *smooth* maps $M \to A$.[9]

18.1.4 Theorem (Segal [Seg70], Brylinski [Bry00])**.** *Let G and A be abelian Lie groups. Then, equivalences classes of central extensions in which $\widetilde{G} \to G$ is a principal A-bundle are classified by* $\mathrm{H}^2(\mathrm{Bun}_G; \underline{A})$.

The idea of the characterization is that Bun_G admits a simplicial resolution

$$\mathrm{Bun}_G \simeq \left(\cdots \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} G \times G \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} G \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} * \right)$$

which is the content of the bar construction (see Proposition 13.2.5), and we want to compute π_0 of the simplicial set of maps

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G \times G \arrow[d, blue] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & * \arrow[d] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & * \arrow[r, shift left] \arrow[r, shift right] & *
\end{tikzcd} \tag{18.1.5}$$

The blue map corresponds to the 2-cocycle for the extension in ordinary group cohomology.

18.1.6 Remark. Differentiating a central extension of Lie groups produces a *central extension of Lie algebras*

$$0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0\,,$$

which is what you would expect ($\mathfrak{a}$ is an abelian Lie algebra contained in the center of $\tilde{\mathfrak{g}}$).

Central extensions of Lie algebras are classified by second *Lie algebra cohomology* $\mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}; \mathfrak{a})$. Cocycles are alternating bilinear maps $\omega\colon \Lambda^2\mathfrak{g} \to \mathfrak{a}$ satisfying a version of the Jacobi identity:

$$\omega(X, [Y, Z]) + \omega(Y, [Z, X]) + \omega(Z, [X, Y]) = 0\,. \tag{18.1.7}$$

From such an ω, we build a central extension which, as a vector space, is $\mathfrak{g} \oplus \mathfrak{a}$, but with Lie bracket

$$[(X_1, A_1), (X_2, A_2)] \coloneqq [X_1, X_2] + \omega(X_1, X_2)\,. \tag{18.1.8}$$

A 1-cochain is a map $\lambda\colon \mathfrak{g} \to \mathfrak{a}$, and its differential is $d\lambda(X, Y) \coloneqq \lambda([X, Y])$.

So we have a map

$$\mathrm{H}^2(\mathrm{Bun}_G, \underline{A}) \to \mathrm{H}^2_{\mathrm{Lie}}(\mathfrak{g}; \mathfrak{a})\,.$$

The van Est theorem says this is an equivalence in certain nice situations (not ours, unfortunately).

[9] By contrast, the simplicial sheaf just denoted "A" treats A as having the discrete topology. This is a little bit counterintuitive but is standard notation.

18.2 The Virasoro algebra and the Virasoro group

Let $\Gamma \coloneqq \mathrm{Diff}^+(S^1)$, the group of orientation-preserving diffeomorphisms of the circle. This is an infinite-dimensional *Fréchet Lie group*, meaning it is locally modeled on a Fréchet space and has a group structure in which multiplication and inversion are smooth.

18.2.1 Definition. The *Witt algebra* $\mathrm{Witt}_{\mathbb{R}}$ is the infinite-dimensional real Lie algebra of polynomial vector fields on S^1. Explicitly, it is generated by $\xi_n \coloneqq -x^{n+1}\frac{\partial}{\partial x}$ for $n \in \mathbb{Z}$, with bracket

$$[\xi_m, \xi_n] \coloneqq (m-n)\xi_{m+n} \ . \tag{18.2.2}$$

Skating over issues of regularity, the Witt algebra is the Lie algebra of Γ.[10]

The Virasoro algebra $\mathrm{Vir}_{\mathbb{R}}$ is a central extension of $\mathrm{Witt}_{\mathbb{R}}$ by $\mathbb{R}$. There is also a Virasoro group $\widetilde{\Gamma}$, a central extension of Γ; the Virasoro algebra is its Lie algebra, and is easier to define (since Lie algebra H^2_{Lie} just works to produce central extensions, whereas we had to modify group cohomology). Specifically, consider the 2-cocycle $c\colon \Lambda^2\,\mathrm{Witt}_{\mathbb{R}} \to \mathbb{R}$ given by

$$c(\xi_m, \xi_n) \coloneqq \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \ , \tag{18.2.3}$$

where c is a chosen basis for $\mathbb{R}$. The $1/12$ is not there for any deep reason, just as a normalization constant. Anyways, as in (18.1.8) this defines for us an extension

$$1 \to \mathbb{R} \to \mathrm{Vir}_{\mathbb{R}} \to \mathrm{Witt}_{\mathbb{R}} \to 1 \ ,$$

called the *Virasoro algebra*. The element c inside $\mathrm{Vir}_{\mathbb{R}}$ is called the *central charge*.

The Virasoro group $\widetilde{\Gamma}$ is the extension of Γ by $\mathbb{T}$ which is, as a space, $\mathbb{T}\times\Gamma$, with multiplication

$$(z_1, f)\cdot(z_2, g) \coloneqq (z_1 + z_2 + B(f,g), f\circ g) \ , \tag{18.2.4}$$

where $B\colon \Gamma\times\Gamma \to \mathbb{T}$ is the *Bott cocycle*

$$B(f,g) \coloneqq \oint_{S^1} \log(f\circ g)' \mathrm{d}(\log g)' \ . \tag{18.2.5}$$

18.2.6 Remark. The identification $S^1 \cong \mathbb{RP}^1$ embeds $\mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{PSL}_2(\mathbb{R}) \subset \Gamma$ as the real fractional linear transformations; hence also

$$\mathfrak{psl}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R}) \subset \mathrm{Witt}_{\mathbb{R}} \ ,$$

as the Lie algebra generated by ξ_{-1}, ξ_0, and ξ_1. Restricted to $\mathrm{PSL}_2(\mathbb{R})$, the Virasoro central extension is trivializable, which will be useful later.

18.2.7 Remark. Some authors' definitions will differ. For example, defining the Witt and Virasoro algebras as complex Lie algebras, or defining the Virasoro group as the universal cover of

[10] If we were to treat regularity more carefully, we would allow some infinite linear combinations of the ξ_n, corresponding to the Fourier series of a smooth vector field.

ours.

18.2.8 Remark (applications)**.** The Virasoro group and algebra appear in two-dimensional conformal field theory (CFT). Usually, in quantum field theory, one specifies a (Riemannian or Lorentzian) metric on spacetime, and the information in the theory depends on the metric. A *conformal field theory* is a quantum field theory in which all information only depends on the conformal class of the metric. Two-dimensional CFTs in particular connect to many areas of mathematics and physics.

- The mathematical formalization of 2d CFT, using vertex algebras, has connections to representation theory, and, famously, to monstrous moonshine.
- One way to think of string theory is as a 2d CFT on the worldsheet, one of whose fields is a map into (10- or 26-dimensional) spacetime.
- In condensed-matter physics, Wess–Zumino–Witten models (particular 2d CFTs) are used in modeling the quantum Hall effect. See also Example 22.2.8.
- Maybe closest to the hearts of the attendees of this seminar: the Stolz–Teichner conjecture suggests that cocycles for TMF on a space X are given by families of 2d supersymmetric quantum field theories parametrized by X. Superconformal field theories are particularly nice examples of these, and have been used to shine light on this conjecture.[11]

So how does the Virasoro appear in CFT? Let's suppose we're on a Riemann surface Σ in a local holomorphic coordinate z. If you write out commutators for the Lie algebra $\mathfrak{c}$ of infinitesimal conformal transformations, you might notice they look like those for the Witt algebra — in fact, if you complexify it, you obtain precisely $\mathrm{Witt}_{\mathbb{C}} \oplus \mathrm{Witt}_{\mathbb{C}}$. So this acts on the system as a symmetry; you can think of it as two different Witt group symmetries.

The fact that we obtain a central extension is standard lore from quantum mechanics. The state space in a quantum system is a complex Hilbert space, but if $\lambda \in \mathbb{C}^\times$, the states $|\psi\rangle$ and $\lambda|\psi\rangle$ are thought of as the same, in that measurements cannot distinguish them. Nonetheless, the formalism of quantum mechanics uses the Hilbert space structure.

The takeaway, though, is that a symmetry of the system, as in acting on the states and all that, only has to be a projective representation on the state space! So to describe an honest Lie group or Lie algebra acting on the state space, we need to take a central extension of the symmetry group or Lie algebra. This leads us to the (complexified) Virasoro algebra and Virasoro group. Thus, the symmetry algebra of conformal field theory is (at least) a product of two copies of the Virasoro algebra, and the space of states is a representation of the Virasoro algebra.

18.3 Constructing the central extension with differential cohomology

The key fact bridging differential cohomology and central extensions is:

[11] The appearances of SCFTs, rather than just CFTs, in superstring theory and in the Stolz–Teichner conjecture aren't as related to the Virasoro group and algebra; they have a larger symmetry algebra, though it's closely related.

18.3.1 Lemma. *There is an equivalence of simplicial sheaves* $\mathbb{Z}(1) \simeq \Sigma^{-1}\underline{\mathbb{T}}$.

Proof. By definition, $\mathbb{Z}(1)$ is the sheaf $0 \to \mathbb{Z} \to \Omega^0 \to 0$, and $\Omega^0 = \underline{\mathbb{R}}$. The chain map

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \underline{\mathbb{R}} \arrow[r] \arrow[d, "\mathrm{mod}\ \mathbb{Z}"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \underline{\mathbb{T}} \arrow[r] & 0 \end{tikzcd}$$

is a quasi-isomorphism. □

18.3.2 Corollary. *For any Lie group G, possibly infinite-dimensional, we have an isomorphism*

$$\mathrm{H}^2(\mathrm{Bun}_G; \underline{\mathbb{T}}) \cong \mathrm{H}^3(\mathrm{Bun}_G; \mathbb{Z}(1)) .$$

In particular, the group $\mathrm{H}^3(\mathrm{Bun}_G; \mathbb{Z}(1))$ *classifies central extensions of G by $\mathbb{T}$ which are principal $\mathbb{T}$-bundles over G.*

Thus, we would like to construct the Virasoro central extension via a differential cohomology class in $\mathrm{H}^3(\mathrm{Bun}_\Gamma; \mathbb{Z}(1))$. This builds on the hard work of the previous few talks. Let $\mathrm{GL}_n^+(\mathbb{R})$ denote the group of orientation-preserving, invertible $n \times n$ matrices. The restriction map

$$\mathrm{H}^*(B\mathrm{GL}_n(\mathbb{R}); \mathbb{R}) \to \mathrm{H}^*(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{R})$$

is not an isomorphism, but is close enough to one that we still have Pontryagin classes for $\mathrm{GL}_n^+(\mathbb{R})$-bundles, or for oriented vector bundles.

In Corollary 17.2.5, Hopkins described how $\mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_n^+(\mathbb{R})}; \mathbb{Z}(2))$ fits into a pullback square

(18.3.3)
$$\begin{tikzcd} \mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_n^+(\mathbb{R})}; \mathbb{Z}(2)) \arrow[r] \arrow[d] & \mathrm{H}^4(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z}) \arrow[d] \\ \mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_n^+(\mathbb{R})}; \mathbb{R}(2)) \arrow[r] & \mathrm{H}^4(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{R}) . \end{tikzcd}$$

18.3.4 Definition. An *off-diagonal differential lift of p_1* is a class

$$\tilde{p}_1 \in \mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_n^+(\mathbb{R})}; \mathbb{Z}(2))$$

whose image under the blue map is the usual $p_1 \in \mathrm{H}^4(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z})$.

By Corollary 17.2.4, we have an isomorphism

$$\mathrm{H}^4(\mathrm{Bun}_G; \mathbb{R}(2)) \cong \mathrm{Sym}^2(\mathfrak{g}^\vee)^G .$$

For $\mathrm{GL}_n^+(\mathbb{R})$, this is an $\mathbb{R}^2$, spanned by the invariant polynomials $\mathrm{tr}(A)^2$ and $\mathrm{tr}(A)^2$, which we call c_1^2 and c_2, respectively. The group $\mathrm{H}^4(BG; \mathbb{R})$ can be dispatched with ordinary Chern–Weil theory: we repeat the same story, but retracting G onto its maximal compact. Here, we get

$$\mathrm{H}^4(\mathrm{BSO}(n);\mathbb{R}) \cong \mathbb{R} ,$$

spanned by $\mathrm{tr}(A^2)$, as $\mathrm{tr}(A)^2 = 0$. Accordingly, the red map in (18.3.3) is a rank-1 map $\mathbb{R}^2 \to \mathbb{R}$. Since (18.3.3) is a pullback square, there is an $\mathbb{R}$ worth of differential lifts of p_1: explicitly, $\lambda \in \mathbb{R}$ gives you the lift of p_1 which maps in the lower left to $(1/2)(\lambda c_1^2 - 2c_2)$. However, if you want

$$\tilde{p}_1(E_1 \oplus E_2) = \tilde{p}_1(E_1) + \tilde{p}_1(E_2) ,$$

you force $\lambda = 1$, which is a quick calculation with the Whitney formula. (All this was in Chapter 8.)[12]

In Chapter 9, we also discussed the fiber integration map for an $\hat{\mathrm{H}}$-oriented fiber bundle

$$F \to E \to B ,$$

which has the form

$$\mathrm{H}^k(E;\mathbb{Z}(\ell)) \to \mathrm{H}^{k-\dim(F)}(B;\mathbb{Z}(\ell - \dim(F))) .$$

Combining all this, consider the universal oriented sphere bundle $\mathrm{E}_\bullet\Gamma \times_\Gamma \mathrm{S}^1 \to \mathrm{Bun}_\Gamma$, which is an $\hat{\mathrm{H}}$-oriented fiber bundle with fiber S^1. Therefore, given a differential lift of p_1, we can apply it to the vertical tangent bundle $V \to \mathrm{E}_\bullet\Gamma \times_\Gamma \mathrm{S}^1$, and get a class

$$\tilde{p}_1(V) \in \mathrm{H}^4(\mathrm{E}_\bullet\Gamma \times_\Gamma \mathrm{S}^1;\mathbb{Z}(2)) .$$

Then we can push it forward to a class in $\mathrm{H}^3(\mathrm{Bun}_\Gamma;\mathbb{Z}(1))$, which determines an isomorphism class of central extensions of Γ as above. The goal is to determine the choice of λ such that this central extension gives the Virasoro group. I'll suggest some ways forward.

The first thing we need is a way to get a handle on the group of extensions of Γ. Recall that $\mathrm{PSL}_2(\mathbb{R}) \subset \Gamma$ as the real fractional linear transformations of $\mathbb{RP}^1 = \mathrm{S}^1$; hence a central extension of Γ restricts to a central extension of $\mathrm{PSL}_2(\mathbb{R})$.

18.3.5 Theorem (Segal [Seg81, Corollary 7.5]). *A central extension of* Γ *by* $\mathbb{T}$ *is determined by the pair of (1) its restriction to* $\mathrm{PSL}_2(\mathbb{R})$ *and (2) the induced Lie algebra central extension of* $\mathrm{Witt}_\mathbb{R}$ *by* $\mathbb{R}$*. Said differently, there is an isomorphism of abelian groups*

$$\mathrm{Cent}_\mathbb{T}(\Gamma) \xrightarrow{\sim} \mathrm{Cent}_\mathbb{T}(\mathrm{PSL}_2(\mathbb{R})) \times \mathrm{Cent}_\mathbb{R}(\mathrm{Witt}_\mathbb{R}) .$$

We can identify both of these groups. First, $\pi_1 \mathrm{PSL}_2(\mathbb{R}) \cong \mathbb{Z}$, and the universal cover $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$[13] is the *universal central extension* of $\mathrm{PSL}_2(\mathbb{R})$: for any abelian group A,

[12] As we have mentioned a few times already, the Whitney sum formula is not true for Pontryagin classes in $\mathbb{Z}$-valued cohomology. So what's going on here? Brown [Bro82, Theorem 1.6] identified the obstructions to the Whitney sum formula holding for the Pontryagin classes of a pair of vector bundles, and for p_1 the obstructions vanish for oriented vector bundles. This is one reason to use $\mathrm{GL}_n^+(\mathbb{R})$ instead of $\mathrm{GL}_n(\mathbb{R})$.

[13] The notation is because it's also the universal cover of $\mathrm{SL}_2(\mathbb{R})$, which is the connected double cover of $\mathrm{PSL}_2(\mathbb{R})$.

central extensions of $\mathrm{PSL}_2(\mathbb{R})$ by A are in bijection with maps $\varphi\colon \mathbb{Z} \to A$, given by

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\varphi"'] & \widetilde{\mathrm{SL}}_2(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{PSL}_2(\mathbb{R}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & (\widetilde{\mathrm{SL}}_2(\mathbb{R}))_\varphi \arrow[r] & \mathrm{PSL}_2(\mathbb{R}) \arrow[r] & 0\,. \end{tikzcd} \tag{18.3.6}$$

So

$$\mathrm{Cent}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R})) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) = \mathbb{T}\,.$$

The computation that $\mathrm{H}^2(\mathrm{Witt}_{\mathbb{R}}; \mathbb{R}) \cong \mathbb{R}$ is standard, e.g. [Obl17, §6.2.1].

Thus the map from off-diagonal differential lifts of p_1 to central extensions of Γ is a map $\mathbb{R} \to \mathbb{R} \times \mathbb{T}$.

One can then ask the following question, which was posed to us by Dan Freed and Mike Hopkins.

18.3.7 Question. Does there exist an off-diagonal differential lift $\tilde{p}_1$ of the first Pontryagin class that hits the Virasoro algebra central extension in $\mathbb{R} \times \mathbb{T}$?

Note that the Virasoro central extension is in the first factor of $\mathbb{R} \times \mathbb{T}$. Indeed, it induces the Virasoro algebra central extension, and hence is nontrivial on the first factor and trivial when restricted to $\mathrm{PSL}_2(\mathbb{R})$.

The answer to Question 18.3.7 is: yes!

18.3.8 Theorem [DLW21]**.** *Let $\tilde{p}_1 \in \mathrm{H}^4(\mathrm{Bun}_{\mathrm{GL}_n^+(\mathbb{R})}; \mathbb{Z}(2))$ be the unique off-diagonal differential lift of p_1 which satisfies the Whitney sum formula. The central extension of Γ defined by transgressing $\tilde{p}_1$ according to the procedure above is the Virasoro group with central charge -12.*

"Central charge -12" means that this group extension is classified by -12 times the Bott cocycle (18.2.5).

Part III

Applications

In Part III, we survey some applications of differential cohomology to questions in geometry and physics. Some of these applications belong to the pattern that what ordinary cohomology tells us about topological objects, differential cohomology tells us about their geometric analogues: this includes both the use of differential cohomology to obstruct conformal immersions as well as the classification of invertible field theories, both of which we say more about below. For other applications, the analogy with ordinary cohomology is subtler; some use the differential characteristic classes we built in Part II, such as the study of loop groups and the Virasoro group.

Chern–Simons invariants

Chern–Simons invariants, which we define and study in Chapter 19, are the key to many of these applications. Let G be a compact Lie group; choose a class $\lambda \in \mathrm{H}^4(BG;\mathbb{Z})$ and let $\langle -,-\rangle$ be the degree-2 G-invariant symmetric polynomial on $\mathfrak{g}$ associated to the image of λ in de Rham cohomology. The Chern–Simons invariant associated to λ is defined for a 3-manifold Y, a principal G-bundle $\pi\colon P\to Y$, and a connection A on P with curvature F_A. If we assume that π has a section, so that we can descend F_A to a form on Y, the Chern–Simons invariant is

$$\mathrm{CS}_\lambda(P,A) = \int_Y \langle A\wedge F_A\rangle - \frac{1}{6}\langle A\wedge [A,A]\rangle \in \mathbb{R}/\mathbb{Z}\,. \tag{III.1}$$

We first met this invariant in a different guise in Example 14.3.3. In Theorem 14.1.1 we showed λ and $\langle -,-\rangle$ determine a differential refinement $\hat\lambda \in \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z})$, and said but did not prove that the Chern–Simons invariant is the secondary invariant associated to $\hat\lambda$. We will prove the latter fact in Chapter 19.

Chern–Simons invariants and their generalizations play a central role in most of the applications of differential cohomology which we survey: they bridge the geometry of connections with the algebraic topology of (differential) characteristic classes, and therefore have something to say about both worlds.

For example, in Chapter 19 we follow Evans-Lee–Saveliev [ES16] and use Chern–Simons invariants as a tool to determine when two homotopy-equivalent lens spaces are not diffeomorphic; to do so, we also spend time developing a little of the theory of Chern–Simons invariants. The classification of lens spaces up to diffeomorphism or homotopy equivalence is classical [Rei35; Whi41, §5; Bro60], which makes it a good testing ground to determine how powerful manifold invariants are. For example, Longoni–Salvatore [LS05] proved the surprising result that the homotopy type of the two-point configuration space of a lens space can distinguish homotopy-equivalent lens spaces. Evans-Lee–Saveliev build on Longoni–Salvatore's work, providing more comprehensive tools for understanding when the homotopy type of the two-point configuration space of $L(p,q)$ is a stronger invariant than the homotopy type of $L(p,q)$. They extend Chern–Simons invariants to two-point configuration spaces and use them to give a numerical criterion (Proposition 19.3.4) for a map of configuration spaces to be a homotopy equivalence. They combine this criterion with a few other tools, including Massey products, to provide many pairs of homotopy-equivalent lens spaces whose two-point configuration spaces are not homotopy equivalent.

In Chapter 20, we use on-diagonal differential characteristic classes to obstruct conformal immersions, following Chern–Simons [CS74]. Recall that characteristic classes in ordinary cohomology can obstruct immersions into $\mathbb{R}^n$ as follows: if M is a smooth m-manifold that immerses into $\mathbb{R}^n$ with normal bundle ν, then $TM \oplus \nu \cong T\mathbb{R}^n|_M \cong \underline{\mathbb{R}}^n$, and ν is rank $n - m$, so all of its characteristic classes in degree greater than $n - m$ vanish. This places constraints on the characteristic classes of M. For example, let w_i denote the i-th Stiefel–Whitney class; if $\mathbb{C}\mathrm{P}^2$ immersed in $\mathbb{R}^5$, then the normal bundle ν would be one-dimensional, so

$$w_2(T\mathbb{C}\mathrm{P}^2 \oplus \nu) = w_2(T\mathbb{C}\mathrm{P}^2) + \underbrace{w_1(T\mathbb{C}\mathrm{P}^2)w_1(\nu) + w_2(\nu)}_{=0} = w_2(\underline{\mathbb{R}}^5) = 0\,, \tag{III.2}$$

but $w_2(T\mathbb{C}\mathrm{P}^2) \neq 0$, which prevents such an immersion. One can run the same argument using Cheeger–Simons' differential characteristic classes, which we discussed in Chapter 14: since these characteristic classes are defined for vector bundles with connection, they can obstruct isometric embeddings of a Riemannian manifold M by placing constraints on TM with its Levi-Civita connection. Chern–Simons [CS74] improve on this argument in two ways, giving it considerably more power: they prove that the on-diagonal differential Pontryagin classes of the Levi-Civita connection only depend on the conformal class of the metric (Theorem 20.1.1), so can be used to obstruct conformal immersions. They then use the Chern–Simons form to obtain additional obstructions: in some cases, the Chern–Simons form is closed, and conformal immersions restrict what its de Rham class can be. The proofs of these obstructions make use of the close relationship between differential characteristic classes and Chern–Simons forms. Chern–Simons' obstructions are strong enough to prove that $\mathbb{R}\mathrm{P}^3$ with the round metric cannot conformally immerse in $\mathbb{R}^4$ (Theorem 20.3.9).

Our third application of Chern–Simons invariants is to physics: there is a classical field theory whose Lagrangian is the Chern–Simons invariant (III.1). We discuss this theory in Example 22.2.3, focusing on how various pieces of the theory can be described using differential cohomology. Schwarz [Sch77] and Witten [Wit89] quantized this theory, producing a topological field theory called Chern–Simons theory which has been a major object of study in both mathematics and physics. See Remark 22.2.6 for references and more information on the quantum theory.

Quantum physics

Speaking of physics, several of the applications of differential cohomology that we survey are in physics or are closely related to it. In these applications, differential cohomology tends to appear because quantization imposes integrality conditions on objects in field theories; in many cases these can be lifted to integrality data, allowing differential cohomology to enter the picture.

Chapter 21 is dedicated to this idea, working with the example of electromagnetism. We first discuss classical Maxwell theory, describing how information in this theory can be expressed with differential forms. Then we walk through Dirac's argument [Dir31] that the presence of magnetic monopoles forces electric and magnetic charges to be quantized, i.e. valued in a discrete subgroup of $\mathbb{R}$. As a consequence, the fields in the quantum theory are cocycles for differential cohomology, and the action can be rewritten using the differential-cohomological cup product and integration. For electromagnetism, the appearance of differential cohomology is relatively explicit and simple, making it a good example, but the concept of quantization of abelian gauge fields leading to differential cohomology appears in numerous other places in quantum physics, and can involve fancier objects such as differential K-theory.

The next chapter, Chapter 22, is about a different application of differential (generalized) cohomology to physics: the classification of invertible field theories. This is one of the applications which is a geometric analogue of a use of ordinary (generalized) cohomology for something topological. Following Atiyah and Segal, a topological field theory (TFT) is a symmetric monoidal functor

$$Z\colon \mathsf{Bord}_n \to \mathsf{C}\,,$$

where Bord_n is a bordism (higher) category and C is some symmetric monoidal (higher) category, often $\mathsf{Vect}_{\mathbb{C}}$. The simplest nontrivial TFTs are the invertible TFTs, which are the TFTs whose values on all objects and morphisms in Bord_n are invertible in C, meaning that objects are invertible under the tensor product, and morphisms are invertible under composition. We are interested in *reflection-positive* invertible TFTs; this extra requirement is a physically motivated version of unitarity. The classification of reflection-positive invertible TFTs is due to Freed–Hopkins [FH21b], who show that, up to isomorphism, reflection-positive invertible TFTs are classified by the torsion subgroup of $[\mathrm{MTH}, \Sigma^n \mathrm{I}_{\mathbb{Z}}]$ (see § 22.1 for definitions of these spectra). In typical examples, the partition functions of these theories are bordism invariants defined by integrating characteristic classes in (generalized) cohomology. Freed–Hopkins (*ibid.*) go further and conjecture that the entirety of $[\mathrm{MTH}, \Sigma^n \mathrm{I}_{\mathbb{Z}}]$ classifies invertible field theories that need not be topological, which would be defined on some yet-to-be-constructed geometric bordism category. Again, partition functions can often be described by integrating characteristic classes, but this time in differential (generalized) cohomology, and typically in one dimension lower, so as to obtain a secondary invariant. We discuss this conjecture and several examples: classical Chern–Simons theory as mentioned above, the classical Wess–Zumino–Witten model, and an example using differential KO-theory.

Representations of loop groups

In Chapter 23, we turn to the representation theory of loop groups. These are infinite-dimensional Lie groups, but unusually nice ones: as long as you are careful about what you mean by a representation, their representation theory closely resembles that of compact Lie groups! The representations we care about are projective representations, so genuine representations of a central extension by the circle group

$$1 \to \mathbb{T} \to \widetilde{\mathrm{L}G} \to \mathrm{L}G \to 1\,, \tag{III.3}$$

satisfying a "positive energy" condition: restricting the representation to $\mathbb{T}$, its weight subspaces for negative weights are trivial. The reader may wonder how invariant this definition is, and is right to be concerned: it is a significant theorem of Pressley–Segal [PS86, Theorem 13.4.2] that when G is simply connected and compact, every positive energy representation of $\mathrm{L}G$ admits an intertwining projective $\mathrm{Diff}^+(\mathbb{T})$-action, meaning that the notion of positive energy is preserved under reparametrizations of $\mathbb{T}$. One of the major goals of Chapter 23 is to discuss the key ideas in this theorem and its proof: we introduce and motivate the positive energy condition, we discuss the nice properties of positive energy representations, and we sketch the proof of Pressley–Segal's theorem. Along the way, we discuss some connections with physics. In §23.4, we discuss two different connections to differential cohomology: first, the central extensions of the sort we consider are principal $\mathbb{T}$-bundles over $\mathrm{L}G$, hence determine classes in $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$. It turns out that every element of this cohomology group comes from a central extension, and moreover, as principal $\mathbb{T}$-bundles they carry canonical connections, allowing for a lift to $\hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$. This class is related to the "level" that one starts with via transgression maps

$$\hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z}) \to \hat{\mathrm{H}}^3(G;\mathbb{Z}) \to \hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})\,.$$

Central extensions that are principal $\mathbb{T}$-bundles correspond to off-diagonal classes in the differential cohomology group $\mathrm{H}^3(\mathrm{Bun}_{\mathrm{L}G};\mathbb{Z}(1))$, as in Chapter 18, and we say a little about this perspective too.

Our final chapter, Chapter 24, takes the above story and makes it explicit, albeit at the level of Lie algebras. The Lie algebra of a central extension $\widetilde{\mathrm{L}G}$, denoted $\widetilde{\mathrm{L}\mathfrak{g}}$, is an example of a *Kac–Moody algebra*, and is a central extension of the loop algebra of the Lie algebra of $\mathfrak{g}$. The Pressley–Segal theorem cooks up an intertwining projective $\mathrm{Diff}^+(\mathbb{T})$-action on the representations of $\widetilde{\mathrm{L}G}$, so at the level of Lie algebras we might expect a compatible Virasoro algebra action on the representations of Kac–Moody algebras. This is true, and Segal–Sugawara show we can do better, explicitly identifying how the central $\mathbb{C}$ in the Virasoro algebra acts in terms of the level of the central extension (III.3). Both this chapter and the previous chapter on loop groups are closely related to two-dimensional conformal field theory: the data of the category of positive-energy representations of $\widetilde{\mathrm{L}G}$ can be used to build a two-dimensional conformal field theory called the Wess–Zumino–Witten model. This CFT is further related to Chern–Simons theory, a 3d TFT. All of this data — the central extension of $\mathrm{L}G$, the specific Wess–Zumino–Witten model, the specific Chern–Simons theory — is indexed by groups such as $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$, $\mathrm{H}^3(G;\mathbb{Z})$, and $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$, which when G is simple and simply connected are all canonically isomorphic to $\mathbb{Z}$. These groups are related to each other by transgression maps, and this corresponds to the relationship between, e.g. loop groups and the WZW model, or the WZW model and Chern–Simons theory. These cohomology classes have differential refinements, as do the transgression maps relating them.

These are not the only applications of differential cohomology to topology, geometry, or physics, but we hope they illustrate the diversity of things that can be done with differential cohomology, and that they make for an interesting and enjoyable read.

19 Chern–Simons invariants

by YiYu (Adela) Zhang

Our first application is to the theory of Chern–Simons forms and invariants, tools in geometry which are closely tied to differential cohomology. We first mentioned these in Example 14.3.3, where we said that Chern–Simons invariants are defined to be the secondary invariants associated to the on-diagonal differential characteristic classes constructed in Chern–Weil theory. But they also have a much more geometric description, given by integrating a specific form built from the connection and curvature forms. These two descriptions are part of the reason Chern–Simons invariants are so useful: one can use homotopy-theoretic methods in differential cohomology to learn facts about geometry, and vice versa. This will be a common theme throughout this part of the book, and Chern–Simons forms will appear several times.

We begin in § 19.1, defining and discussing Chern–Simons forms associated to a principal bundle $\pi : P \to M$ with connection, and relating them to the differential lifts of Chern–Weil characteristic classes from Chapter 14. In § 19.2, we focus on the case when π is a principal SU_2-bundle over a 3-manifold, where we can descend the Chern–Simons invariant from an integral on P to an integral on M. Finally, in § 19.3, we show an application of Chern–Simons invariants, as a tool to determine when two-point configuration spaces of lens spaces are homotopy equivalent.

19.1 Chern–Simons forms

Let G be a compact Lie group and $\pi : P \to M$ a principal G-bundle. Fix a degree-k invariant polynomial $f \in \mathrm{Sym}^k(\mathfrak{g}^\vee)^G$. Given a connection A on P with curvature F_A, we will write $f(F_A) \in \mathrm{H}^{2k}_{\mathrm{dR}}(M)$ for the associated Chern–Weil form.

19.1.1 Recall. A *connection* A on the principal G-bundle $\pi : P \to M$ is a $\mathfrak{g}$-valued 1-form on P which is G-equivariant in the sense that $(R_g)^*A = \mathrm{Ad}_{g^{-1}} A$, and it is "the identity" on tangent vectors along the fiber, i.e. $A(X_\xi) = \xi$ for $\xi \in \mathfrak{g}$ and X_ξ its fundamental vector field.

The *curvature* of A, which we usually denote F_A, is the form $\mathrm{d}\omega + [\omega, \omega] \in \Omega^2(M; \mathfrak{g})$.

Analogous to connections on vector bundles, a G-connection corresponds to a splitting

$$\mathrm{T}P \cong H \oplus V\ ,$$

where V is the vertical tangent bundle (the kernel of $\pi_* : \mathrm{T}P \to \mathrm{T}M$), and H is the horizontal tangent bundle. A priori, there is only a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathrm{T}P \longrightarrow H \longrightarrow 0\ ; \tag{19.1.2}$$

a connection is a G-equivariant splitting. Because the fibers of a principal G-bundle are G-torsors, there is an isomorphism $V \cong \underline{\mathfrak{g}}$, and the G-action is the fiberwise adjoint action, leading to the definition of connection given in 19.1.1.

Recall from § 11.3.b that the *adjoint bundle* to a principal G-bundle $P \to M$, denoted $\mathfrak{g}_P$, is the associated vector bundle to the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$. The affine space of connections on P can be identified with $\mathcal{A}_P = \Omega^1(M; \mathfrak{g}_P)$, i.e., 1-forms on M with values in the adjoint bundle. Given two connections A_0, $A_1 \in \mathcal{A}_P$, the straight-line path $A_t : I \to \mathcal{A}_P$ determines a connection $\overline{A}$ on the G-bundle $P \times [0,1]$ over $M \times [0,1]$. Let $F_{\overline{A}}$ be the curvature of $\overline{A}$.

19.1.3 Definition. The *Chern–Simons form* associated to $A_0, A_1 \in \mathcal{A}_P$ and f is given by

$$\mathrm{CS}_f(A_1, A_0) = \int_{[0,1]} f(F_{\overline{A}}) \in \Omega^{2k-1}(M)\,.$$

Let F_{A_i} denote the curvature of A_i; then, by Stokes' theorem,

$$\mathrm{dCS}_f(A_1, A_0) = f(F_{A_1}) - f(F_{A_0})\,. \tag{19.1.4}$$

That is, the de Rham class $[f(F_{A_1})]$ is independent of the choice of connection, a fact that we first saw in Chapter 11.

19.1.5 Remark. The path from A_0 to A_1 matters — if we choose a different path, the Chern–Simons form will differ by an exact term. This is beyond the scope of this chapter.

Suppose instead we take the G-bundle $\pi^*P \to P$, which has a tautological section and hence a tautological (flat) connection A_0. Then we can define a Chern–Simons form on P (not on M!) for a single connection A:

$$\mathrm{CS}_f(A) = \mathrm{CS}_f(\pi^*A, A_0) \in \Omega^{2k-1}(P)\,. \tag{19.1.6}$$

Since A_0 is flat, (19.1.4) implies

$$\mathrm{dCS}_f(A) = f(\pi^*F_A) = \pi^* f(F_A)\,. \tag{19.1.7}$$

At this point, we want you to recall the differential cohomology hexagon from Theorem 2.3.2.

(19.1.8)
$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{H}^{*-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\,\mathrm{Bock}"] \arrow[dr] & & \mathrm{H}^{*}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
\mathrm{H}^{*-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^{*}(M;\mathbb{Z}) \arrow[ur, "\mathrm{ch}"] \arrow[dr, "\mathrm{curv}"] & & \mathrm{H}^{*}_{\mathrm{dR}}(M) \\
& \dfrac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \arrow[ur, "\iota"] \arrow[rr, "\mathrm{d}"'] & & \Omega^{*}_{\mathrm{cl}}(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

The squares and triangles are commutative, and the diagonals are short exact sequences.

19.1.9 Proposition. *Suppose $c^{\mathbb{Z}} \in \mathrm{H}^{2k}(\mathrm{B}G;\mathbb{Z})$ is an integral lift of the Chern–Weil characteristic class of f and $\hat{c} \in \hat{\mathrm{H}}^{2k}(\mathrm{Bun}_G^{\nabla};\mathbb{Z})$ is the differential refinement of $c^{\mathbb{Z}}$ and f guaranteed by Theorem 14.1.1. Then for any principal G-bundle $\pi\colon P \to M$ with connection A,*

(19.1.10)
$$\iota(\mathrm{CS}_f(A)) = \pi^*\hat{c}(P,A) \in \hat{\mathrm{H}}^{2k-1}(P;\mathbb{Z}).$$

Proof. As usual, we can prove this for all principal bundles with connection at once by working universally on $\pi\colon (\mathrm{Bun}_G^{\nabla,\mathrm{triv}}, A) \to \mathrm{Bun}_G^{\nabla}$; here $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ denotes the classifying stack of trivialized principal G-bundles with connection (see Notation 13.3.9) and π is the universal principal G-bundle with connection A in the setting of stacks on Mfld. By construction, if F_A denotes the curvature of A,
$$\mathrm{curv}(\hat{c}) = f(F_A) \in \Omega^{2k}(\mathrm{Bun}_G^{\nabla}),$$
so by (19.1.7),
$$\mathrm{d}\mathrm{CS}_f(A) = \pi^*\mathrm{curv}(\hat{c}) \in \hat{\mathrm{H}}^{2k}(\mathrm{Bun}_G^{\nabla,\mathrm{triv}};\mathbb{Z}).$$
The hexagon does all the hard work for us: the homotopification of $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ is contractible (Example 13.3.27), so
$$\mathrm{H}^{2k-1}(\mathrm{Bun}_G^{\nabla,\mathrm{triv}};\mathbb{R}/\mathbb{Z}) = 0$$
and thus the curvature map is injective. Since $\mathrm{d} = \mathrm{curv}\circ\iota$, we can conclude. □

Now suppose that $\pi : P \to M$ admits a section $\sigma : M \to P$. Then we further deduce that

$$\hat{c}(P, A) = \sigma^* \pi^* \hat{c}(P, A) = \iota(\sigma^* \mathrm{CS}_f(A)),$$

meaning that

$$\int_M \hat{c}(P, A) = \int_M \sigma^*(\mathrm{CS}_f(A)) \in \mathbb{R}/\mathbb{Z}. \tag{19.1.11}$$

That is, as promised in Example 14.3.3, this Chern–Simons invariant is the secondary invariant associated to $\hat{c}$. This is conceptually nice, but how do we obtain computable topological invariants from this formula?

19.2 Chern–Simons invariants for 3-manifolds

As an example, we examine the case where P is a principal SU_2-bundle over a path-connected 3-manifold M,

$$f(A) = \frac{1}{8\pi^2} \operatorname{tr}(F_A \wedge F_A),$$

and $c^{\mathbb{Z}} \in \mathrm{H}^4(\mathrm{BSU}_2; \mathbb{Z})$ is the second Chern class. We mostly follow the exposition in [KK90].

The quaternionic projective space $\mathbb{HP}^\infty$ is a BSU_2, so BSU_2 is 3-connected; hence every principal SU_2-bundle over a 3-manifold is trivializable. Fix a trivialization; then there is a trivial (flat) connection A_0, which allows us to identify $\mathcal{A}_P$ with $F^1_A(M; \mathfrak{su}_2)$. Recall that SU_2 acts on $\mathcal{A}_P$ by

$$g \cdot A = gAg^{-1} - dg\, g^{-1}.$$

This action preserves flatness: if F_A is the curvature of A, then the curvature of $g \cdot A$ is gF_Ag^{-1}. The *gauge group* of P is the group of bundle automorphisms of P which cover the identity on M. In this case, the gauge group is $\mathcal{G} \cong \mathrm{Map}_{\mathrm{sm}}(M, \mathrm{SU}_2)$ and it acts on $P \cong M \times \mathrm{SU}_2$ by left multiplication, so the $\mathcal{G}$-action preserves flat connections.

On the other hand, each flat connection A gives rise to a *holonomy representation* $\pi_1(M) \to G$: parallel transport along a loop γ at m_0 gives an automorphism of the fiber SU_2 at m_0, which depends only on the homotopy class $[\gamma] \in \pi_1(M, m_0)$. With a bit of work, one can recover the well-known fact that

$$\{\text{flat connections on } P\}/\mathcal{G} \hookrightarrow R(M) := \mathrm{Hom}(\pi_1(M), \mathrm{SU}_2)/\text{conjugation}. \tag{19.2.1}$$

Since P is trivial, this injection becomes a bijection. In fact, this can be upgraded to a homeomorphism, with the right-hand side the character variety of M.

Now look at the 3-form

$$\mathrm{CS}_f(A) = \mathrm{CS}_f(A, A_0) = \int_{[0,1]} \frac{1}{8\pi^2} \operatorname{tr}(F_A \wedge F_A),$$

where as usual F_A is the curvature of A. Integrating over M gives us the *Chern–Simons func-*

tional on $\mathcal{A}_P$:

$$\tilde{\mathrm{cs}}(A) = \int_{M\times[0,1]} \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\int_M \operatorname{tr}\Big(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge[A\wedge A]\Big)\,. \tag{19.2.2}$$

This map is smooth and functorial in $P \to M$, and up to $\mathbb{Z}$ factors, it is independent of the trivialization of P. Therefore $\tilde{\mathrm{cs}}$ descends to a functional

$$\mathrm{cs}\colon R(M) \cong \mathcal{A}_P/\mathcal{G} \to \mathbb{R}/\mathbb{Z}\,. \tag{19.2.3}$$

The reason is that if $\sigma \in \mathcal{G}$, there is a straight-line path in $\mathcal{A}_P$ from A to $\sigma\cdot A$, which we can interpret as a connection $\overline{A}$ on $[0,1]\times P \to [0,1]\times M$ with curvature $F_{\overline{A}}$. When we quotient by $\mathcal{G}$, we obtain a loop in $\mathcal{A}_P/\mathcal{G}$, or a connection on $P\times \mathrm{S}^1 \to M\times\mathrm{S}^1$. Then

$$\mathrm{cs}(\sigma\cdot A) - \mathrm{cs}(A) = \int_{M\times\mathrm{S}^1}\frac{1}{8\pi^2}\operatorname{tr}(F_{\overline{A}}\wedge F_{\overline{A}}) = \int_{M\times\mathrm{S}^1} c_2(P\times\mathrm{S}^1)\,,$$

which is an integer because c_2 is an integer-valued characteristic class.

The function $\mathrm{cs}\colon R(M) \to \mathbb{R}/\mathbb{Z}$ is a homotopy invariant of M. In practice, it is relatively computable, as we will see for lens spaces.

19.2.a Chern–Simons invariants of lens spaces

Let p and q be coprime positive integers and ζ be a primitive p-th root of unity. Then $\mathbb{Z}/p$ acts on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_2)\,. \tag{19.2.4}$$

Restricting to the unit $\mathrm{S}^3 \subset \mathbb{C}^2$, this is a free action, and the quotient is called a *lens space* and denoted $L(p,q)$ [Tie08, §20].

Lens spaces form a nice collection of examples of 3-manifolds, and given an invariant of 3-manifolds, one can test how powerful it is by checking how well it distinguishes inequivalent lens spaces. For example, $L(5,1)$ and $L(5,2)$ have the same homology and fundamental group, but are not homotopy equivalent [Ale19]; and there are homotopy-equivalent lens spaces which are not homeomorphic [Rei35; Whi41, §5; Bro60]. The full classifications of lens spaces up to homotopy equivalence and homeomorphism are known, due to work of Whitehead [Whi41, §5], resp. Reidemeister [Rei35] and Brody [Bro60].

Let's test the power of Chern–Simons invariants on lens spaces.

19.2.5 Theorem [KK90, Theorem 5.1]**.** *The image of* $\mathrm{cs}\colon R(L(p,q)) \to \mathbb{R}/\mathbb{Z}$ *is the set*

$$\left\{-\frac{n^2 r}{p}\;\middle|\; n = 0, 1, \dots, \left\lfloor\frac{p}{2}\right\rfloor\right\},$$

where r is an integer satisfying $qr \equiv -1 \bmod p$.

You can think of im(cs) as the set of Chern–Simons invariants of a 3-manifold.

19.2.6 Remark. Two lens spaces $L(p,q)$ and $L(p',q')$ have the same set of Chern–Simons invariants if and only if $p = p'$ and $q'q^{-1} \equiv a^2 \mod p$ for some $a \in \mathbb{Z}$, i.e., there is an orientation preserving homotopy equivalence between the two [Whi41, §5]. Hence Chern–Simons invariants detect the homotopy type of lens spaces.

Proof sketch of Theorem 19.2.5. The lens space $L(p,q)$ can be obtained by gluing the boundary of two solid tori X, K together via an element

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Let $x = \mathrm{S}^1 \times \{1\}$ represent a generator of $\pi_1(X)$ and y a meridian of ∂X. Let μ, λ be the corresponding generators of ∂K, so $\mu = px + qy$ and $\lambda = rx + sy$.

Now we utilize some general results about 3-manifolds with a single torus boundary in [KK90]. Suppose we have a path f_t in $\mathrm{Hom}(\pi_1(X), \mathrm{SU}_2)$ with

$$f_t(\mu) = \begin{pmatrix} e^{2\pi i\alpha(t)} & \\ & e^{-2\pi i\alpha(t)} \end{pmatrix} \qquad \text{and} \qquad f_t(\lambda) = \begin{pmatrix} e^{2\pi i\beta(t)} & \\ & e^{-2\pi i\beta(t)} \end{pmatrix},$$

where $\alpha, \beta \colon [0,1] \to \mathbb{R}$. The corresponding path of flat connections takes the form

$$A_t = \begin{pmatrix} i\alpha(t) & \\ & -i\alpha(t) \end{pmatrix} \mathrm{d}x + \begin{pmatrix} i\beta(t) & \\ & -i\beta(t) \end{pmatrix} \mathrm{d}y$$

near the torus boundary. If f_0 and f_1 send μ to 1, then [KK90, Theorem 4.2]

$$\mathrm{cs}(f_1) - \mathrm{cs}(f_0) = -2\int_0^1 \beta\alpha'\, dt \bmod \mathbb{Z}. \tag{19.2.7}$$

On the other hand, a holonomy representation on X extends to one on the Dehn filling M (in our case, the lens space itself) if and only if it sends μ to 1 (*ibid.*, proof of Theorem 4.2).

Back to the sketch. We take γ_t to be a path sending x to $e^{2\pi i\theta}$ with $\theta \in [0, 1/2]$. (Every representation of $\pi_1(X)$ is conjugate to a representation in the image of the path.) Then γ_{t_1} extends to a representation f_t of $\pi_1(L(p,q)) = \mathbb{Z}/p$ if and only if $pt_1 \in \mathbb{Z}$, so we can obtain $\lfloor p/2 \rfloor + 1$ conjugacy classes of representations of $\mathbb{Z}/p$, which correspond to $t_1 = n/p$ for $0 \leq n \leq \lfloor p/2 \rfloor$.

On the other hand, $\alpha(t) = pt$ and $\beta(t) = rt$, so

$$\mathrm{cs}(f_{t_1}) = -2\int_0^{t_1} \beta\alpha' dt = -rpt_1^2\,.$$

Plug in $t_1 = n/p$ and conclude. □

19.3 Application: configuration spaces of lens spaces

To strengthen our Chern–Simons invariants, let's use them to study a related invariant of lens spaces: the homotopy type of $F_2(L(p,q))$, the space of two-point subsets of $L(p,q)$. Longoni–Salvatore [LS05] showed that this distinguishes $L(7,1)$ and $L(7,2)$, which are homotopy equivalent; the fact that the homotopy type of $F_2(X)$ knows more than the homotopy type of X was a surprising result. Differential cohomology enters the story with work of Evans-Lee–Saveliev [ES16] using Chern–Simons invariants to provide a more comprehensive way to test whether the two-point configuration spaces of two homotopy-equivalent lens spaces are homotopy equivalent.

Choose a lens space $L = L(p,q)$ and a CW structure on it with a single i-cell e_i for $0 \le i \le 3$. Let $X = L \times L$. The *two-point configuration space* of L is

$$X_0 \coloneqq \mathrm{Conf}_2(L) \cong X \setminus \Delta\,, \tag{19.3.1}$$

where $\Delta \subset X$ is the diagonal, i.e. the subspace of elements (x,x) with $x \in L$. Taking the product CW structure on X, $X_0 \subset L$ is a subcomplex, and the inclusion $X_0 \hookrightarrow L \times L$ induces an isomorphism of fundamental groups.

Using this CW structure, one can compute that

$$\mathrm{H}_3(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/p\,;$$

the classes $[e_0 \times e_3] = [e_0 \times L]$ and $[e_3 \times e_0] = [L \times e_0]$ generate the two $\mathbb{Z}$ summands and $[e_1 \times e_2 + e_2 \times e_1]$ generates the $\mathbb{Z}/p$ summand.

19.3.2 Lemma. *There is a closed, oriented* 3*-manifold S with a map $f\colon M \to X$ such that $f_*[M] = [e_1 \times e_2 + e_2 \times e_1]$.*

Proof. This is a special case of the *Steenrod realization problem* asking when a given degree-n homology class can be represented as a map from a closed, oriented n-manifold. This can be reformulated as a question about oriented bordism $\Omega_n^{\mathrm{SO}}(X)$, a generalized homology theory, and the natural transformation

$$\Omega_n^{\mathrm{SO}} \to \mathrm{H}_n$$

sending

$$(M, f\colon M \to X) \mapsto f_*[M]\,.$$

In this form, the question was answered negatively in general by Thom [Tho54, Théorème III.9], but when X is a manifold, $\Omega_3^{\mathrm{SO}}(X) \to \mathrm{H}_3(X)$ is surjective (*ibid.*, Théorème III.3). □

Evans-Lee–Saveliev [ES16, §3] give an explicit example of such a representative manifold S.

With this choice of generators of $\mathrm{H}_3(X)$, the inclusion $\mathrm{H}_3(X_0) = \mathbb{Z} \oplus \mathbb{Z}/p \hookrightarrow \mathrm{H}_3(X)$ sends a generator of the free summand to $(1,1,0)$ and a generator of the torsion summand to $[S] = (0,0,1)$.

Given a representation

$$\alpha \colon \pi_1(X) = \mathbb{Z}/p \times \mathbb{Z}/p \to \mathrm{SU}_2$$

and a closed, oriented 3-manifold M with a map $f \colon M \to X$, we get a representation $f^*\alpha$ of $\pi_1(M)$. Hence we can define an extension of the Chern–Simons invariants

$$\mathrm{cs}_X \colon R(X_0) \to \mathrm{Hom}(\mathrm{H}_3(X_0), \mathbb{R}/\mathbb{Z}) \tag{19.3.3a}$$

by

$$\mathrm{cs}_X(\alpha) = \mathrm{cs}_M(f^*\alpha) = \frac{1}{8\pi^2}\int_M \mathrm{tr}\Big(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge [A \wedge A]\Big)\,. \tag{19.3.3b}$$

A priori this depends on our choice of (M, f), but it is actually independent of this choice, and is also functorial in X. Thus we obtain a homotopy invariant for each pair of conjugacy class of representation and third homology class.

Now we compute. Fix an SU_2-representation α, which is conjugate to one sending the generators of $\pi_1(X)$ to $e^{2\pi i k/p}$ and $e^{2\pi i \ell/p}$; we will call this representation $\alpha(k, \ell)$. Under the two maps $L \rightrightarrows X$ realizing our two nontorsion generators of $\mathrm{H}_3(X)$, $\alpha(k,\ell)$ pulls back to the representations sending a generator of $\pi_1(L)$ to $e^{2\pi i k/p}$ and $e^{2\pi i \ell/p}$. By Theorem 19.2.5, the Chern–Simons invariants of these representations are $-k^2 r/p$ and $-\ell^2 r/p$, where r can be any integer such that $qr \equiv -1 \bmod p$.

Evaluating the Chern–Simons invariant for $S \to X$ is harder. Evans-Lee-Saveliev show that the choice of S they constructed is Seifert fibered over S^2 (*ibid.*, Lemma 4.4), allowing them to use a theorem of Auckly [Auc94, §2] computing the Chern–Simons invariants of such 3-manifolds. The upshot is that the Chern–Simons invariant of $f^*\alpha$ on S is $2k\ell/p$. Pulling back along $X_0 \hookrightarrow X$, our nontorsion generator of $\mathrm{H}_3(X_0)$ has Chern–Simons invariant $r(k^2 + \ell^2)/p$, and our torsion generator has invariant $2k\ell/p$.

Let

$$X_0 = \mathrm{Conf}_2(L(p,q)) \qquad \text{and} \qquad X_0' = \mathrm{Conf}_2(L(p,q'))$$

and suppose that $f \colon X_0 \to X_0'$ is a homotopy equivalence. Then the induced isomorphism $\mathbb{Z}/p \times \mathbb{Z}/p \to \mathbb{Z}/p \times \mathbb{Z}/p$ on fundamental groups corresponds to a matrix

$$f_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p)\,.$$

The induced isomorphism on $\mathrm{H}_3 = \mathbb{Z} \oplus \mathbb{Z}/p$ has the form

$$h_3 = \begin{pmatrix} \epsilon & 0 \\ a & b \end{pmatrix},$$

where $\epsilon = \pm 1$ and $b \in (\mathbb{Z}/p)^\times$. Using naturality of Chern–Simons invariants, we can deduce the following numerical constraints:

19.3.4 Proposition [ES16, Proposition 5.2]. *If f is a homotopy equivalence, then $\epsilon q' \equiv qa^2 \bmod p$ and*

$$f_1 = \begin{pmatrix} a & 0 \\ 0 & \pm a \end{pmatrix}, \begin{pmatrix} 0 & a \\ \pm a & 0 \end{pmatrix}; \ h_3 = \begin{pmatrix} \epsilon & 0 \\ 0 & \pm a^2 \end{pmatrix}.$$

Composing with the swap map $(x, y) \mapsto (y, x)$ if necessary, we can and do make f diagonal, rather than antidiagonal.

To learn more information about lens spaces, we have to combine Proposition 19.3.4 with other invariants. These invariants are further away from differential cohomology, so we will be terser and point the reader towards references with more information. Specifically, we will combine the Chern–Simons invariants results from above with information about Massey products in the cohomology of the universal cover $\tilde{X}_0$ of X_0.

19.3.5 Proposition [ES16, Lemma 6.1]. *There is an isomorphism*

$$\mathrm{H}^*(\tilde{X}_0) \cong \mathbb{Z}[a_1, \dots, a_{p-1}, b]/(a_i^2, b^2)\,,$$

where $|a_i| = 2$ and $|b| = 3$.

Proof sketch. The universal cover of X is $\mathrm{S}^3 \times \mathrm{S}^3$; therefore the universal cover of X_0 is a subspace of $\mathrm{S}^3 \times \mathrm{S}^3$, specifically the complement of the orbit of the diagonal of $\mathrm{S}^3 \times \mathrm{S}^3$ under the $\pi_1(X)$-action. Therefore there is a map $\pi\colon \tilde{X}_0 \hookrightarrow \mathrm{S}^3 \times \mathrm{S}^3 \to \mathrm{S}^3$ given by inclusion followed by projection onto the first factor; it is a surjective submersion, and the fiber is a $(p-1)$-punctured S^3. Set up the Serre spectral sequence; there are only a few differentials not zeroed out by degree considerations, and they vanish because π has a section. Thus the spectral sequence collapses. There are no nontrivial extension questions, so the cohomology ring of $\tilde{X}_0$ is the tensor product of

$$\mathrm{H}^*(\mathrm{S}^3; \mathbb{Z}/2) \cong \mathbb{Z}/2[b]/(b^2) \qquad \text{and} \qquad \mathrm{H}^*(\mathrm{S}^3 \setminus \{x_1, \dots, x_{p-1}\}) \cong \mathbb{Z}/2[a_1, \dots, a_{p-1}]/(a_i^2)\,. \ \square$$

Let $a_0 = -a_1 - \cdots - a_{p-1}$. Miller [Mil11, §2.1] calculates the $\pi_1(X_0)$-action on $\mathrm{H}^2(\tilde{X}_0)$. Specifically, for $k, \ell \in \mathbb{Z}/p$, let $\tau_{k,\ell}$ denote the element corresponding to (k, ℓ) under the identification

$$\pi_1(X_0) \cong \mathbb{Z}/p \times \mathbb{Z}/p$$

above. Then,

$$\tau_{k,\ell} \cdot a_i = a_{i+k-\ell}\,.$$

This puts an additional constraint on a homotopy equivalence $f\colon X_0 \to X_0'$: f must intertwine the action map $\pi_1(X_0) \to \mathrm{Aut}(\mathrm{H}^2(\tilde{X}_0))$. With α, ϵ as above, this implies $f = \alpha \cdot \mathrm{id}$ and that the following diagram commutes [ES16, Proposition 6.3]:

$$\begin{tikzcd} \mathrm{H}^2(\tilde{X}_0') \arrow[r, "\tilde{f}^*"] \arrow[d, "\tau_{\alpha k,\alpha\ell}"'] & \mathrm{H}^2(\tilde{X}_0) \arrow[d, "\tau_{k,\ell}"] \\ \mathrm{H}^2(\tilde{X}_0') \arrow[r, "\tilde{f}^*"'] & \mathrm{H}^2(\tilde{X}_0)\,. \end{tikzcd}$$

This provides an additional constraint on f.

Next we need information about Massey products in $\mathrm{H}^*(\tilde{X}_0;\mathbb{Z}/2)$. The Massey product is a secondary cohomology operation; the corresponding primary operation is the cup product. As a quick review, a Massey product [UM57, §2; Mas58] is defined for $x, y, z \in \mathrm{H}^*(X; A)$ when A is a ring, $x \smile y = 0$, and $y \smile z = 0$: one chooses cocycles $\overline{x}$, $\overline{y}$, and $\overline{z}$ representing x, y, and z respectively, and chooses cochains A and B such that

$$\delta A = \overline{x} \smile \overline{y} \qquad \text{and} \qquad \delta B = \overline{y} \smile \overline{z}\,.$$

The Massey product $\langle x, y, z\rangle$ is defined to be the set of cohomology classes $[A \smile \overline{z} - \overline{x} \smile B]$ for all possible choices of A and B. Massey products are functorial, which follows directly from their definition.

Assume p is odd and $0 < q < p/2$. It follows from Proposition 19.3.5 that there are identifications of abelian groups

$$\mathbb{F}_2(\zeta_p) \coloneqq \mathbb{F}_2[t]/(1 + t + \cdots + t^{p-1}) \xrightarrow{\sim} \mathrm{H}^m(\tilde{X}_0; \mathbb{Z}/2),\ m = 2, 5\,; \tag{19.3.6}$$

for $m = 2$, this map sends $t^k \mapsto a_k \bmod 2$, and for $m = 5$, $t^k \mapsto a_k b \bmod 2$.[14] If $x, y, z \in \mathrm{H}^2(\tilde{X}_0; \mathbb{Z}/2)$ satisfy $xy = yz = 0$ (so that their Massey product is defined), then

$$\langle x, y, z\rangle \subset \mathrm{H}^5(\tilde{X}_0; \mathbb{Z}/2)\,,$$

so we may describe these Massey products as (possibly multivalued) maps

$$\langle -,-,-\rangle\colon \mathbb{F}_2(\zeta_p) \times \mathbb{F}_2(\zeta_p) \times \mathbb{F}_2(\zeta_p) \to \mathbb{F}_2(\zeta_p)\,. \tag{19.3.7}$$

Miller [Mil11, Theorem 3.33] calculates these Massey products. For example,

$$t^n \cdot \langle t^k, t^\ell, t^j\rangle = \langle t^{k+n}, t^{\ell+n}, t^{j+n}\rangle \qquad \text{and} \qquad \langle t^k, t^\ell, t^j\rangle = \langle t^j, t^\ell, t^k\rangle\,.$$

These two relations allow us to inductively reduce to the case when at least one of j, k, or ℓ is 0; the description of the Massey products in that case is a little more complicated, and can be found in [ES16, Theorem 7.1].

[14] We chose the notation $\mathbb{F}_2(\zeta_p)$ because this is the cyclotomic field associated to a primitive p-th root of unity ζ_p over $\mathbb{F}_2$.

This leads us to our last obstruction. The two different maps

$$\tilde{f}^* : \mathrm{H}^m(\tilde{X}_0'; \mathbb{Z}_2) \to \mathrm{H}^m(\tilde{X}_0; \mathbb{Z}/2)$$

for $m = 2, 5$, become the same map $\tilde{f}^* : \mathbb{F}_2(\zeta_p) \to \mathbb{F}_2(\zeta_p)$ under the identification (19.3.6). Therefore we obtain the constraint that this $\tilde{f}^*$ must intertwine the Massey product map (19.3.7).

Our three constraints (coming from Chern–Simons invariants, cohomology of $\tilde{X}_0$, and Massey products) each boil down to numerical constraints on p and q, and these are amenable to computer calculation. This is how Evans-Lee–Saveliev showed that these constraints can detect some homotopy-equivalent but not homeomorphic lens spaces that Longoni–Salvatore's techniques miss. These pairs include $L(11, 2)$ and $L(11, 3)$; $L(13, 2)$ and $L(13, 5)$; and $L(17, 3)$ and $L(17, 5)$.

20 Conformal immersions

by Charlie Reid

Let M be a smooth, m-dimensional manifold and suppose M immerses in $\mathbb{R}^n$ with normal bundle $\mathrm{N}M$. Then there is a short exact sequence

$$0 \longrightarrow \mathrm{T}M \longrightarrow \mathrm{T}\mathbb{R}^n|_M \longrightarrow \mathrm{N}M \longrightarrow 0, \tag{20.0.1}$$

so the Pontryagin classes of $\mathrm{T}M$ and $\mathrm{N}M$ satisfy[15]

$$p(\mathrm{T}M)p(\mathrm{N}M) = p(\mathrm{T}\mathbb{R}^n|_M) = 1\,. \tag{20.0.2}$$

The total Pontryagin class is the sum of 1 and a nilpotent element $(p_1(M)+p_2(M)+\cdots)$, hence is invertible. This means $p(\mathrm{N}M)$ is uniquely determined if it exists: there is a formula for $p_k(\mathrm{N}M)$ in terms of $p(\mathrm{T}M)$. If M immerses in $\mathbb{R}^n$, then $\mathrm{N}M$ is rank $n-m$, so $p_k(\mathrm{N}M) = 0$ for $k > n-m$, and because of the formula, this is actually a constraint on the Pontryagin classes of $\mathrm{T}M$. Thus Pontryagin classes can be used to prove nonimmersion results for smooth manifolds by showing this constraint is not met.

In Chapter 14, we saw that given a connection on the tangent bundle, Pontryagin classes lift to differential cohomology. It therefore seems worthwhile to imitate the above argument and use on-diagonal differential Pontryagin classes given by the Levi-Civita connection to obstruct isometric immersions of Riemannian manifolds. Chern and Simons [CS74] did this, though with a few key differences.

(1) Chern and Simons were able to show (*ibid.*, Theorem 4.5) that if g and g' are two conformally equivalent metrics on a manifold M, with Levi-Civita connections A, resp. A', then $\hat{p}(M,A) = \hat{p}(M,A')$. Therefore the differential Pontryagin classes of M are conformal invariants, and can be used to study conformal immersions.

(2) There is an additional integrality result which has no analogue in the purely topological case (*ibid.*, Theorem 5.14): when a Pontryagin class' Chern–Weil form vanishes, the corresponding Chern–Simons form is closed, and one-half of its de Rham class is contained within the lattice $\mathrm{im}(\mathrm{H}^*(-;\mathbb{Z}) \to \mathrm{H}^*(-;\mathbb{R}))$. After some more work, this leads to another necessary condition for the existence of a conformal immersion.

As an example, $\mathbb{RP}^3$ smoothly immerses in $\mathbb{R}^4$ [Boy03], and given the round metric, $\mathbb{RP}^3$ locally conformally immerses in $\mathbb{R}^4$. But Chern–Simons show (*ibid.*, §6) that there is no conformal immersion $\mathbb{RP}^3 \hookrightarrow \mathbb{R}^4$.

In § 20.1, we prove that the on-diagonal differential Pontryagin classes of the Levi-Civita connection are conformal invariants of the Riemannian metric. Then, in § 20.2, we use on-diagonal differential Pontryagin classes to obstruct conformal immersions. Finally, in §20.3, we

[15] Because the Whitney sum formula for Pontryagin classes only holds up to 2-torsion, this formula should be thought of as taking place in cohomology with $\mathbb{Z}[1/2]$ or $\mathbb{R}$ coefficients.

produce the integrality obstruction using the Chern–Simons form and use it to show $\mathbb{RP}^3$ with the round metric cannot conformally immerse in $\mathbb{R}^4$.

20.0.3 Remark. The story we just told is a little anachronistic: Chern–Simons' work came before Cheeger–Simons' paper on differential characters, and was not stated in this language. But Chern and Simons were aware that their ideas could be rephrased as calculations in the ring of differential characters, as they write in the introduction to their paper. In any case, the paper [CS74] is best known for an entirely different reason: for introducing the Chern–Simons form of a connection!

20.1 Conformal invariance of differential Pontryagin classes

Let G be a compact Lie group. Recall that given a degree-k invariant polynomial f on the Lie algebra $\mathfrak{g}$ and a characteristic class $c^{\mathbb{Z}} \in \mathrm{H}^{2k}(BG;\mathbb{Z})$, we obtain a differential characteristic class $\hat{c} \in \hat{\mathrm{H}}^{2k}(\mathrm{Bun}_G^{\nabla};\mathbb{Z})$ (as proven in Theorem 14.1.1) and a Chern–Simons form $\mathrm{CS}_f(A) \in \Omega^{2k-1}(P)$ given a principal G-bundle $\pi : P \to M$ and a connection A on P (as defined in (19.1.6)). We are specifically interested in the Pontryagin polynomials P_k from § 11.5.b, which we lifted to on-diagonal differential Pontryagin classes $\hat{p}_k$ in Example 14.1.10.

Our aim in this section is to prove:

20.1.1 Theorem [CS74, Theorem 4.5]**.** *Let M be a manifold and g_0, g_1 be conformally equivalent Riemannian metrics on M. If A_0 and A_1 denote the Levi-Civita connections for g_0 and g_1, then for all k, we have*

$$\hat{p}_k(M, A_0) = \hat{p}_k(M, A_1)$$

and $\mathrm{CS}_{P_k}(A_0) - \mathrm{CS}_{P_k}(A_1)$ is exact.

The first ingredient in the proof is a variation formula.

20.1.2 Lemma (variation formula [CS74, Proposition 3.8])**.** *Suppose A_t is a smooth path of connections on a principal G-bundle $P \to M$ and F_{A_t} is the curvature of A_t. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{CS}_f(A_t)\bigg|_{t=0} = k \cdot f(A' \wedge F_{A_0}^{k-1}) + \omega, \tag{20.1.3}$$

where ω is exact and $A' = \frac{\mathrm{d}}{\mathrm{d}t}(A_t)\Big|_{t=0}$.

Proof. It suffices to work universally in $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$, the stack of trivial principal G-bundles with connection: the forgetful map

$$\mathrm{Bun}_G^{\nabla,\mathrm{triv}} \to \mathrm{Bun}_G^{\nabla}$$

is the universal principal G-bundle with connection in the world of stacks on Mfld (see §13.3.b). The de Rham complex of $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ is acyclic [FH13, Theorem 7.19], so it suffices to apply the de Rham differential to (20.1.3) and then show both sides are equal.[16]

[16] One can avoid the use of the abstract object $\mathrm{Bun}_G^{\nabla,\mathrm{triv}}$ by using Narasimhan–Ramanan's n-classifying spaces [NR61; NR63].

For the left-hand side, we know

$$\begin{aligned}
\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{CS}_f(A_t)\Big|_{t=0}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{d}(\mathrm{CS}_f(A_t))\big)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(f((F_{A_t})^k)\big)\Big|_{t=0}\\
&= k\cdot f(F'_{A_0}\wedge F_{A_0}^{k-1}),
\end{aligned}$$

where $F'_{A_0} = \frac{\mathrm{d}}{\mathrm{d}t}(F_{A_t})\big|_{t=0}$.

For the right-hand side,

$$\begin{aligned}
\mathrm{d}(k\cdot f(A'\wedge F_{A_0}^{k-1})) &= k\cdot f(\mathrm{d}A'\wedge F_{A_0}^{k-1}) - k(k-1)f(A'\wedge \mathrm{d}F_A\wedge F_A^{k-2})\\
&= kf(\mathrm{d}A'\wedge F_{A_0}^{k-1}) - k(k-1)f(A'\wedge [F_{A_0},A_0]\wedge F_A^{k-2})\\
&= kf(\mathrm{d}A'\wedge F_{A_0}^{k-1}) + kf([A',A_0]\wedge F_{A_0}^{k-1}).
\end{aligned}$$

This uses two important facts from Chern–Weil theory: that $\mathrm{d}F_{A_0} = [F_{A_0}, A_0]$ together with the value of the invariant polynomial for a commutator [CS74, (2.9)]. Now

$$\begin{aligned}
\mathrm{d}A' &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}A_t)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(F_{A_t} - \frac{1}{2}[A_t,A_t]\right)\Big|_{t=0}\\
&= F'_{A_0} - [A',A_0],
\end{aligned}$$

so $\mathrm{d}(k\cdot f(A'\wedge F_{A_t}^{k-1})) = k\cdot(F'_{A_0}\wedge F_{A_t}^{k-1})$ and we are done. $\square$

Proof of Theorem 20.1.1. Now for f we take P_k, the invariant polynomial that we used in §11.5.b to define the k-th Pontryagin class. This is the pullback of the $2k$-th Chern polynomial under the complexification map $\mathfrak{o}(n)\to\mathfrak{u}(n)$; we tend not to use the pullback of the $(2k+1)^{\text{st}}$ Chern polynomial as much because it is 2-torsion and its Chern–Simons form is exact [CS74, Proposition 4.3].

It suffices to show that $\delta \coloneqq \mathrm{CS}_{P_k}(A_0) - \mathrm{CS}_{P_k}(A_1)$ is exact; this implies it is a closed form with integral periods, so the image $\overline{\delta}$ of δ in $\Omega^{4k-1}(M)/\Omega^{4k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}$ vanishes. This is the lower-left corner of the differential cohomology hexagon, and as we saw in Proposition 19.1.9, applying

$$\iota\colon\ \Omega^{4k-1}(M)/\Omega^{4k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}} \to \check{\mathrm{H}}^{4k}(M;\mathbb{Z})$$

sends

$$\overline{\delta} \mapsto \hat{p}_k(P,A_0) - \hat{p}_k(P,A_1)\,,$$

so showing $\overline{\delta}=0$ is good enough.

Now to show δ is exact. It is always possible to connect g_0 and g_1 by a path g_t, $t\in(-\varepsilon, 1+\varepsilon)$

of conformally equivalent metrics. Moreover, this path may be chosen to satisfy

$$g_t = e^{2th} g_0$$

for some real-valued smooth function h. Choose such a path and let A_t be the Levi-Civita connection of g_t. Differentiating in t commutes with the de Rham differential, so is suffices to show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{CS}_{P_k}(A_t)$ is exact; without loss of generality, we prove this for $t = 0$. Lemma 20.1.2 means we only have to show

$$P_k(A_0' \wedge F_{A_0}^{2k-1}) = 0.$$

For a little while we work locally on the bundle $\pi \colon B(M) \to M$ of frames: the fiber at $x \in M$ is the $\mathrm{GL}_n(\mathbb{R})$-torsor of orthonormal bases $(e_1, \dots, e_n)$ of T_xM. There are canonical one-forms $\omega_i \in \Omega^1(B(M))$ defined at a point $(x, (e_1, \dots, e_n))$ so that

$$\mathrm{d}\pi = \sum_{i=1}^n \omega_i \cdot e_i.$$

Let E_i be the horizontal vector field dual to ω_i; here "horizontal" is with respect to the connection A_0. Then on frames orthogonal to g_0, there is a decomposition [CS74, Lemma 4.4]

$$A_{ij}' = \underbrace{\delta_{ij}\mathrm{d}(h\circ\pi)}_{\alpha} + \underbrace{E_i(h\circ\pi)\omega_j - E_j(h\circ\pi)\omega_i}_{\beta}.$$

We will address each piece separately. First, one directly checks that for $\varphi = (\varphi_{ij}) \in \Omega^k(F(M))$,

$$P_k(\varphi \wedge F_A^{k-1}) = \sum_{i_1,\dots,i_k=1}^n \varphi_{i_1 i_2} \wedge (F_A)_{i_2 i_3} \wedge \cdots \wedge (F_A)_{i_n i_1}. \tag{20.1.4}$$

Plugging in $\varphi = \alpha$, we obtain

$$P_k(\alpha \wedge F_A^{2k-1}) = \mathrm{d}(f\circ\pi) \wedge P_{2k-1}(F_A^{2k-1}) = 0,$$

because A is compatible with the metric. Now plugging β into (20.1.4),

$$P_k(\beta \wedge F_A^{2k-1}) = \sum_{i_1,\dots,i_{2k}=1}^n (E_{i_1}(f\circ\pi)\omega_{i_2} - E_{i_2}(f\circ\pi)\omega_{i_1}) \wedge (F_A)_{i_2 i_3} \wedge \cdots \wedge (F_A)_{i_{2k} i_1}. \tag{20.1.5}$$

The Jacobi identity implies $\sum \omega_i \wedge (F_A)_{ij} = 0$, so (20.1.5) vanishes as well. Lastly, we need to descend from $B(M)$ to M, and 0 descends to 0. $\square$

20.2 Obstructing conformal immersions with differential Pontryagin classes

Recall the on-diagonal differential lifts of Chern classes we constructed in Chapter 14, specifically Example 14.1.9, defined as follows: define the *Chern polynomials* $C_k \in I^k(\mathrm{U}_n)$ by

$$\det(\lambda I - \frac{1}{2\pi i}A) = \sum_{k=0}^{n} C_k(A)\lambda^{n-k}; \tag{20.2.1}$$

apply Chern–Weil theory to C_k, producing a characteristic class c_k. The integer cohomology of U_n is torsion-free [Bor53, §29] and its image in de Rham cohomology contains c_k, so there is a unique lift to $\hat{c}_k$ to degree-$2k$ differential cohomology.

We will also need the *inverse Chern polynomials* $C_k^\perp$, which are defined to satisfy

$$(1 + C_1 + \cdots + C_n)(1 + C_1^\perp + C_2^\perp + \cdots) = 1\,. \tag{20.2.2}$$

For example, $C_1^\perp = -C_1$, $C_2^\perp = -C_2 - C_1C_1^\perp$, $C_3^\perp = -C_3 - C_2C_1^\perp - C_1C_2^\perp$, and so on. Chern–Weil theory associates de Rham characteristic classes $c_k^\perp \in \mathrm{H}^{2k}_{\mathrm{dR}}$ to these, and like ordinary Chern classes, these classes lift uniquely to differential cohomology classes $\hat{c}_k^\perp \in \hat{\mathrm{H}}^{2k}(\mathrm{Bun}^\nabla_{\mathrm{U}_n};\mathbb{Z})$. They satisfy analogous formulas to the inverse Chern polynomials: for example

$$\hat{c}_2^\perp = -\hat{c}_2 - \hat{c}_1\hat{c}_1^\perp\,. \tag{20.2.3}$$

In Example 14.1.10, we defined on-diagonal differential Pontryagin classes $\hat{p}_k$ in much the same way as we defined differential Chern classes. Using the *inverse Pontryagin polynomials* $P_k^\perp$, defined to satisfy

$$(1 + P_1 + \cdots + P_n)(1 + P_1^\perp + P_2^\perp + \cdots) = 1\,, \tag{20.2.4}$$

we define on-diagonal inverse differential Pontryagin classes $\hat{p}_k^\perp \in \hat{\mathrm{H}}^{4k}(\mathrm{Bun}^\nabla_{\mathrm{O}_n};\mathbb{Z})$. Because there is torsion in $\mathrm{H}^*(\mathrm{BO}_n;\mathbb{Z})$, a priori the lift to differential cohomology requires a choice, but there is a canonical way to do this: complexify to pass to on-diagonal inverse differential Chern classes. This means that analogues of (20.2.3) and its higher-rank generalizations hold for on-diagonal inverse Pontryagin classes. For example, $\hat{p}_2^\perp = -\hat{p}_2 - \hat{p}_1\hat{p}_1^\perp$.

20.2.5 Theorem. *Let M be a Riemannian manifold and $\phi\colon M^n \to \mathbb{R}^{n+k}$ be a conformal immersion of M into Euclidean space. Then the image of $\hat{p}_i^\perp(M, A^{\mathrm{LC}})$ in $\hat{\mathrm{H}}^{4i}(M;\mathbb{Z}[1/2])$ vanishes for all $i > k/2$.*

Proof. Since the classes $\hat{p}_k$ are conformally invariant (Theorem 20.1.1), so too are the classes $\hat{p}_k^\perp$. Therefore, without loss of generality, we can assume ϕ is isometric. Let $\mathrm{N}M$ denote the orthogonal normal bundle: there is an orthogonal direct sum

$$\mathrm{T}M \oplus \mathrm{N}M = \mathrm{T}\mathbb{R}^{n+k} = \underline{\mathbb{R}}^{n+k}.$$

The Levi-Civita connection $A^{\mathrm{LC}}_{\mathrm{T}\mathbb{R}^{n+k}}$ on $\mathbb{R}^{n+k}$ compresses to the Levi-Civita connection $A^{\mathrm{LC}}_{\mathrm{T}M}$ on M, and to a connection $A_{\mathrm{N}M}$ on $\mathrm{N}M$. Since $A^{\mathrm{LC}}_{\mathrm{T}\mathbb{R}^{n+k}}$ is flat, it is compatible with $A^{\mathrm{LC}}_{\mathrm{T}M} \oplus A_{\mathrm{N}M}$ (Definition 14.2.3). Hence

$$\hat{p}(\mathrm{T}M, A^{\mathrm{LC}}_{\mathrm{T}M}) * \hat{p}(\mathrm{N}M, A_{\mathrm{N}M}) = \hat{p}(\mathrm{T}\mathbb{R}^{n+k}, A^{\mathrm{LC}}_{\mathrm{T}\mathbb{R}^{n+k}}) = 1\,,$$

implying

$$\hat{p}^{\perp}(\mathrm{T}M, A^{\mathrm{LC}}_{\mathrm{T}M}) = \hat{p}(\mathrm{N}M, A_{\mathrm{N}M})\,.$$

Since $\mathrm{N}M$ has rank k, $\hat{p}_i(\mathrm{N}M, A^{\mathrm{LC}}_{\mathrm{T}M})$ vanishes for $i > k/2$. □

20.2.6 Remark. As always, we use $\mathbb{Z}[1/2]$ coefficients because the Whitney sum for Pontryagin classes is more complicated over the integers. See Thomas [Tho62] and Brown [Bro82, Theorem 1.6]. The extra factors ultimately come from Chern classes, so they too admit differential refinements, and a $\mathbb{Z}$-valued differential Whitney sum formula exists. Using this, it is possible to upgrade Theorem 20.2.5 to take place in $\hat{\mathrm{H}}^*(M;\mathbb{Z})$.

20.3 Dividing by 2

We foreshadowed that Chern–Simons theory will allow us to prove that $\mathbb{RP}^3$ with the round metric does not conformally immerse in $\mathbb{R}^4$, but to actually prove this we need another obstruction. This one is an evenness result: we will use the Chern–Simons form to define a de Rham cohomology class of on the frame bundle of $\mathbb{RP}^3$, and prove that a conformal immersion would imply this class is in the image of the map induced by the inclusion $2\mathbb{Z} \to \mathbb{R}$. A direct calculation shows this is not the case, and we conclude.

20.3.1 Lemma [CS74, Proposition 3.15]**.** *If $\pi : P \to M$ is a principal G-bundle with connection A, there is a cochain $u \in C^{2k-1}(M;\mathbb{R}/\mathbb{Z})$ such that $\delta(u) = f(F_A) \bmod \mathbb{Z}$ and in $C^*(P;\mathbb{R}/\mathbb{Z})$, $\mathrm{CS}_f(A) \bmod \mathbb{Z}\pi^*(u)$ is a coboundary.*

Proof. Since $[f(F_A)]$ is in the image of the map from integer cohomology to de Rham cohomology, $f(F_A) \bmod \mathbb{Z}$ is a coboundary, so choose $u \in C^{2k-1}(M;\mathbb{R}/\mathbb{Z})$ with $\delta u = f(F_A) \bmod \mathbb{Z}$. Then

$$\begin{aligned}\delta(\pi^*(u)) = \pi^*(\delta u) &= \pi^*(f(F_A)) \bmod \mathbb{Z}\\ &= \delta(\mathrm{CS}_f(A)) \bmod \mathbb{Z} = \delta(\mathrm{CS}_f(A) \bmod \mathbb{Z})\,.\end{aligned}$$

That is, $\delta(\pi^*(u) - \mathrm{CS}_f(A) \bmod \mathbb{Z})$ vanishes. □

Let $\pi : P \to M$ be a principal G-bundle with connection A. In the previous chapter, specifically (19.1.7), we showed that $\mathrm{dCS}_f(A) = \pi^* f(F_A)$. Therefore if $f(F_A) = 0$, $\mathrm{CS}_f(A)$ is closed and defines a class $[\mathrm{CS}_f(A)] \in \mathrm{H}^{2k-1}(P;\mathbb{R})$.

20.3.2 Corollary [CS74, Theorem 3.16]**.** *Assume that $f(F_A) = 0$. Then there is a class*

$$\overline{u} \in \mathrm{H}^{2k-1}(M;\mathbb{R}/\mathbb{Z})$$

such that in $\mathrm{H}^{2k-1}(P;\mathbb{R}/\mathbb{Z})$*, we have*

$$[\mathrm{CS}_f(A)] \bmod \mathbb{Z} = \pi^*(\overline{u}).$$

Proof. By hypothesis of Lemma 20.3.1, $\delta(u) = f(F_A) = 0$, so we can choose $\overline{u}$ to be the class of u in cohomology. ☐

20.3.3 Example. Let $\mathrm{St}_n(\mathbb{C}^{n+k})$ denote the *Stiefel manifold* of isometric immersions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+k}$. Sending an immersion to its image defines a map π to the *Grassmannian manifold* $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ parametrizing codimension-k subspaces of $\mathbb{C}^{n+k}$, and this map is a principal U_n-bundle. This bundle has a natural connection. It is equivalent to describe the connection on the associated rank-n complex vector bundle $\pi'\colon S \to \mathrm{Gr}_n(\mathbb{C}^{n+k})$, which is the tautological bundle. If $\rho\colon (-\varepsilon,\varepsilon) \to S$ is a smooth curve, $\rho(t)$ is an element of the vector space $\pi(\rho(t)) \in \mathrm{Gr}_n(\mathbb{C}^{n+k})$; we specify the connection by declaring the covariant derivative of $\rho(t)$ along $\pi\circ\rho$ to be the orthogonal projection of $\rho'(t)$ into the subspace $\pi(\rho(t))$. Call this connection A^{can}.

There is a canonically defined rank-k complex vector bundle $Q \to \mathrm{Gr}_n(\mathbb{C}^{n+k})$, whose fiber at an n-dimensional subspace $V \subset \mathbb{C}^{n+k}$ is $V^\perp \subset \mathbb{C}^{n+k}$. Thus $S \oplus Q = \underline{\mathbb{C}}^{n+k}$, so in a similar manner as in the proof of Theorem 20.2.5, $[C_i^\perp(A^{\mathrm{can}})] = 0$, i.e. $C_i^\perp(A^{\mathrm{can}})$ is exact. The Grassmannian is a compact, irreducible Riemannian symmetric space, so since $C_i^\perp(A^{\mathrm{can}})$ is an invariant, exact differential form, it must vanish. Therefore Corollary 20.3.2 tells us $[\mathrm{CS}_{C_i^\perp}(A^{\mathrm{can}})] \bmod \mathbb{Z}$ pulls back from $\overline{u} \in \mathrm{H}^{2k-1}(\mathrm{Gr}_n(\mathbb{C}^{n+k});\mathbb{R}/\mathbb{Z})$. Because the cohomology of complex Grassmannians is concentrated in even degrees, $\overline{u} = 0$, meaning $[\mathrm{CS}_{C_i^\perp}(A^{\mathrm{can}})]$ is in the image of the map

$$\mathrm{H}^*(\mathrm{St}_n(\mathbb{C}^{n+k});\mathbb{Z}) \to \mathrm{H}^*(\mathrm{St}_n(\mathbb{C}^{n+k});\mathbb{R}).$$

By passing to real vector bundles, we will gain an additional factor of 2. We will say a real-valued cohomology class is *contained in the even integer lattice* if it is in the image of the composite

$$\mathrm{H}^*(-;\mathbb{Z}) \xrightarrow{\cdot 2} \mathrm{H}^*(-;\mathbb{Z}) \longrightarrow \mathrm{H}^*(-;\mathbb{R}). \tag{20.3.4}$$

20.3.5 Lemma [CS74, Lemma 5.12]**.** *Let* $c\colon \mathrm{St}_n(\mathbb{R}^{n+k}) \to \mathrm{St}_n(\mathbb{C}^{n+k})$ *be the complexification map. For* $\ell > 0$*, the image of*

$$c^*\colon \mathrm{H}^\ell(\mathrm{St}_n(\mathbb{C}^{n+k});\mathbb{Z}) \to \mathrm{H}^\ell(\mathrm{St}_n(\mathbb{R}^{n+k});\mathbb{Z})$$

is contained in the even integer lattice.

Proof. First suppose $k = 0$, for which $\mathrm{St}_n(\mathbb{C}^n) \cong \mathrm{U}_n$ and $\mathrm{St}_n(\mathbb{R}^n) \cong \mathrm{O}_n$; c is the usual complexification map. It suffices to show that the mod 2 reductions of all positive-degree classes in the image of c^* vanish.

At this point we need a tool called the *inverse transgression map*. We will say more about this map in Remark 20.3.11 at the end of this chapter; for this proof, we need only that inverse transgression is a map $\tau\colon \mathrm{H}^\ell(BG;\mathbb{Z}) \to \mathrm{H}^{\ell-1}(G;\mathbb{Z})$ satisfying two key properties:

(1) τ is natural in G, and

(2) for $A = \mathbb{Z}$ or $\mathbb{Z}/2$ and $x \in \mathrm{H}^*(\mathrm{B}G; A)$, $\tau(x^2) = 0$.

Let $\mathrm{B}c\colon\ \mathrm{BO}_n \to \mathrm{BU}(n)$ be the map induced from complexification on classifying spaces. We know $(\mathrm{B}c)^*(c_i) \bmod 2 = w_i^2$ [Bro82, Theorem 1.5], so

$$c^*(\tau(c_i)) \bmod 2 = \tau((\mathrm{B}c)^*(c_i) \bmod 2) = 0\,. \tag{20.3.6}$$

This suffices because $\{\tau(c_i)\}$ generates $\mathrm{H}^*(\mathrm{U}_n; \mathbb{Z})$ [Bor54, Théorèmes 8.2 et 8.3].

For more general k, recall that

$$\mathrm{St}_n(\mathbb{R}^{n+k}) \cong \mathrm{O}_{n+k} \,/\, \mathrm{O}_k \qquad \text{and} \qquad \mathrm{St}_n(\mathbb{C}^{n+k}) \cong \mathrm{U}_{n+k} \,/\, \mathrm{U}_k\ .$$

Let π denote the quotient $\mathrm{O}_{n+k} \to \mathrm{St}_n(\mathbb{R}^{n+k})$ as well as its complex analogue. Then π commutes with complexification, so it suffices to show that

$$\pi^*\colon\ \mathrm{H}^*(\mathrm{St}_n(\mathbb{R}^{n+k}); \mathbb{Z}/2) \to \mathrm{H}^*(\mathrm{O}_{n+k}; \mathbb{Z}/2)$$

is injective, and this is due to Borel [Bor53, §10]. ☐

This extra factor of two provides an additional obstruction to the existence of a conformal immersion, and this is what we will use to show $\mathbb{RP}^3$ cannot conformally immerse in $\mathbb{R}^4$.

20.3.7 Theorem [CS74, Theorem 5.14]**.** *Let M be an n-dimensional Riemannian manifold,*

$$B(M) \to M$$

be the principal O_n-bundle of frames, and A be the Levi-Civita connection on $B(M)$. Suppose M conformally immerses in $\mathbb{R}^{n+k}$; then, for $i \geq \lfloor k/2 \rfloor$, $\mathrm{CS}_{P_i^\perp}(A)$ is contained in the even integer lattice.

Proof. Let $\varphi\colon\ M \to \mathbb{R}^{n+k}$ be a conformal immersion. By Theorem 20.1.1, we can assume φ is an isometric immersion. We then have a Gauss map $\Phi\colon\ M \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ sending $x \mapsto T_xM \subset T_x\mathbb{R}^{n+k} = \mathbb{R}^{n+k}$, as well as its analogue on total spaces $\Phi\colon\ B(M) \to \mathrm{St}_n(\mathbb{R}^{n+k})$ defined analogously.

For $i > \lfloor k/2 \rfloor$, we know by Example 20.3.3 and Lemma 20.3.5 that

$$[\mathrm{CS}_{P_i^\perp}(A^{\mathrm{can}})] \in \mathrm{H}^{2i-1}(\mathrm{St}_n(\mathbb{R}^{n+k}); \mathbb{R})$$

is contained in the even integer lattice. This property is natural in principal bundles with a connection, and $A = \Phi^*(A^{\mathrm{can}})$, so this is also true for $\mathrm{CS}_{P_i^\perp}(A)$. ☐

We use this to define an $\mathbb{R}/\mathbb{Z}$-valued invariant which obstructs conformal immersions of an orientable Riemannian 3-manifold Y into $\mathbb{R}^4$. The frame bundle $B(Y) \to Y$ admits a section χ;

define

$$\Phi(Y) := \int_Y \frac{1}{2}\chi^*\mathrm{CS}_{P_1}(A) \in \mathbb{R}/\mathbb{Z}\,, \tag{20.3.8}$$

where A is the Levi-Civita connection. A priori this depends on the section, but one can calculate (e.g. [CS74, §6]) that if χ and χ' are two sections, the difference of their pullbacks of the Chern–Simons invariant consists of torsion and an integer number of copies of an integral cohomology class; the torsion disappears when we integrate, and the integer-valued cohomology class does not affect the answer mod $\mathbb{Z}$. Theorem 20.3.7 (and the fact that $P_1^\perp = -P_1$) implies that if Y conformally immerses in $\mathbb{R}^4$, then $\Phi(Y) = 0$.

And now the moment we've all been waiting for.

20.3.9 Theorem [CS74, §6, Example 1]**.** *The manifold* $\mathbb{RP}^3$ *with the round metric does not conformally immerse into* $\mathbb{R}^4$.

Proof. We will calculate $\mathrm{CS}_{P_1}(A)$ for A the Levi-Civita connection on $\mathbb{RP}^3$. The identification $\mathbb{RP}^3 = \mathrm{SO}(3)$ gives us an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{so}(3)$, the space of left-invariant vector fields; in the Levi-Civita connection, $\nabla_{v_1} v_2 = v_3$, $\nabla_{v_2} v_3 = v_1$, and $\nabla_{v_1} v_3 = -v_2$. If $\pi : B_{\mathrm{O}}(\mathbb{RP}^3) \to \mathbb{RP}^3$ denotes the bundle of orthonormal frames, the above basis gives us a section χ of π. We have a formula for $\chi^*\mathrm{CS}_{P_1}(A)$ (19.2.2); expanding in coordinates and using the covariant derivatives of the v_is, and we obtain

$$\chi^*\left(\frac{1}{2}\mathrm{CS}_{P_1}(A)\right) = -\frac{1}{2\pi^2}\,\mathrm{vol}\,, \tag{20.3.10}$$

where vol is the volume form on $\mathbb{RP}^3$. As a Riemannian manifold, $\mathbb{RP}^3$ with the round metric is the quotient of S^3 with the round metric under the antipodal map, so the volume of $\mathbb{RP}^3$ is one-half that of S^3, i.e. $\mathrm{Vol}(\mathbb{RP}^3) = \pi^2$. Thus $\Phi(\mathbb{RP}^3) = 1/2$. □

There are numerous examples in the literature of calculations of this sort to obtain conformal nonimmersion results: see [HL74; APS75b; Mil75; Don77; Tsu81; Bac82; Tsu84; Ouy94; MM01; MZ10; PT10; Li15] for some examples.

20.3.11 Remark (transgression and inverse transgression)**.** Here we go into a little more detail about the transgression and inverse transgression maps, the latter of which appeared in the proof of Lemma 20.3.5. We follow [Bor55, §9; CS74].

20.3.12 Definition. Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fiber bundle, $x \in \mathrm{H}^k(F; A)$, and $y \in \mathrm{H}^{k+1}(B; A)$. We say that x *transgresses to* y when there is a cochain $c \in Z^k(F; A)$ such that $[i^*(c)] = x$ and $\delta c = \pi^* b$ for some cocycle b in the cohomology class of y.

Given x, y may not exist, and may not be unique if it exists. Transgression is natural under pullback of fiber bundles, so when studying transgression in principal G-bundles, it makes sense to work universally in $G \to \mathrm{E}G \to \mathrm{B}G$.

Transgression has something to say about the Serre spectral sequence for the fiber bundle $F \to E \to B$. We can identify x and y with their images on the E_2-page, in $E_2^{0,k}$ and $E_2^{k+1,0}$ respectively. Transgression as defined above is equivalent to asking that

(1) no differential d_r for $r < k+1$ kills x or y, so that their images in the E_{k+1}-page are nonzero; and

(2) $d_{k+1}(x) = y$.

The Serre spectral sequence is first-quadrant, so d_{k+1} is the last differential that could kill x or y. In the bundle $G \to \mathrm{E}G \to \mathrm{B}G$, all positive-degree elements must be killed by differentials, because $\mathrm{E}G$ is contractible; this is another indication that transgression is important here.[17] When G is a connected Lie group, transgression is often as nice as it can be: $\mathrm{H}^*(G;A)$ is an exterior algebra on odd-degree generators $x_1, \dots, x_n$, $\mathrm{H}^*(\mathrm{B}G;A)$ is a polynomial algebra on even-degree generators $y_1, \dots, y_n$, and x_i transgresses to y_i. Here A may be $\mathbb{Q}$, $\mathbb{Z}/p$, or $\mathbb{Z}$ depending on G; for example, when $G = \mathrm{U}_n$, we can use $\mathbb{Z}$ coefficients. In these settings we can begin to see how to define the inverse transgression map: ignoring gradings, the only differences between the rings $\mathrm{H}^*(\mathrm{B}G;A)$ and $\mathrm{H}^*(G;A)$ are the relations $x_i^2 = 0$, so we can think of transgression as a map $\mathrm{H}^*(G;A) \to \mathrm{H}^{*+1}(\mathrm{B}G;A)$ whose image is everything not containing terms of the form y_i^m for $m > 1$. Thus we can define an inverse transgression map τ by sending $y_i \mapsto x_i$ and $y_i^2 = 0$.

Chern–Simons [CS74, §5] define τ differently, and more directly: given $y \in \mathrm{H}^{k+1}(\mathrm{B}G;A)$, let b be a cocycle representative for y which vanishes when pulled back to any point of $\mathrm{B}G$; since $\mathrm{E}G$ is contractible, $\pi^*(b) = \delta c$ for some $c \in Z^k(\mathrm{E}G;A)$. Then $\tau(y)$ is defined to be the cohomology class of the restriction of c to a fiber; one has to check this is well-defined, but it is. When $\mathrm{H}^*(G;A)$ is an exterior algebra on odd-degree generators, this definition recovers the definition from the previous paragraph, but this definition is more general. It is natural in G, and $\tau(y^2) = 0$ follows because if we choose b, c as above, then

$$\delta(b \smile c) = \pi^*(b \smile b)\,,$$

and restricted to a fiber, $b \smile c$ vanishes.

From here it is natural to wonder whether the inverse transgression map admits a differential refinement $\hat\tau\colon \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z}) \to \hat{\mathrm{H}}^3(G;\mathbb{Z})$. This is true, and there are constructions of this map due to Carey–Johnson–Murray–Stevenson–Wang [CJM+05, §3] and Schreiber [Sch13b, 1.4.1.2].

Chern–Simons (*ibid.*, §3) also discuss transgression in the context of the Chern–Simons form and when the fiber bundle is a principal G-bundle $P \to M$ with connection A. Fixing an invariant polynomial f, they use the Maurer–Cartan form on G to define a class in $\mathrm{H}^*_{\mathrm{dR}}(G)$ which transgresses to $[f(F_A)] \in \mathrm{H}^*_{\mathrm{dR}}(M)$.

[17] Similarly, when A is an abelian group, there is a fibration $\mathrm{K}(A,n) \to E \to \mathrm{K}(A,n+1)$, where E is contractible, and a theorem of Borel [Bor53, Theorem 13.1] on transgression is a crucial part of Serre's calculation [Ser53] of the cohomology of Eilenberg–MacLane spaces.

21 Charge quantization

Talk by Dan Freed
Notes by Arun Debray

There are a few different applications of differential cohomology to quantum physics; today, we'll focus on charge quantization, using Maxwell theory as an example. First, in §21.1, we introduce classical Maxwell theory, formulated in the language of differential forms. Then, in §21.2, we pass to the quantum theory. This imposes integrality conditions on differential forms, leading to the appearance of differential cohomology. This lecture is based on [Fre02a, Part 3].

The history of the use of differential cohomology to implement charge quantization is closely tied to the development of the theory of differential cohomology itself. Alvarez [Alv85] was the first to use differential cohomology in this context, though he does not use the words "differential cohomology."[18] Gawędzki [Gaw88] then explicitly brings in differential cohomology in the form of Deligne cohomology.

The original motivation to consider generalized differential cohomology came from charge quantization in string theory: work of Minasian–Moore [MM97], Sen [Sen98], and Witten [Wit98] argued that D-brane charges and Ramond–Ramond field strengths are valued in K-theory,[19] leading to a search for a K-theoretic analogue of differential cohomology. Freed–Hopkins [FH00] first provided a definition of differential K-theory for this purpose, and Freed [Fre00] considers more general differential generalized cohomology theories. Hopkins–Singer [HS05], who comprehensively studied differential generalized cohomology theories, write that they originally began their project to investigate string-theoretic phenomena.[20]

21.1 Classical Maxwell theory

Let (N, g_N) be a Riemannian 3-manifold without boundary and $M = \mathbb{R} \times N$. Let t be the $\mathbb{R}$ coordinate, so we give M the Lorentz metric

$$g_M = \mathrm{d}t^2 - g_N\,. \tag{21.1.1}$$

Choose differential forms $E \in \Omega^1(N)$ and $B \in \Omega^2(N)$, respectively the electric and magnetic fields; also choose the *charge density* $\rho_E \in \Omega^3_c(N)$, and the *current* $J_E \in \Omega^2_c(N)$.[21] If $\star_N$ denotes

[18] Alvarez also uses differential cohomology to characterize quantized topological terms. This is a related but different application of differential cohomology to physics, and is more closely related to the discussion of invertible field theories in the next chapter. See Deligne–Freed [DF99, Chapter 6] for a mathematical exposition of topological terms and their relationship to differential cohomology, along with more recent work of Davighi, Gripaios, and Randal-Williams [DG18a; DGR20] and Córdova–Freed–Lam–Seiberg [CFLS20a; CFLS20b] from a more physics-based perspective.

[19] See [FW99; MW00; DMW02] for some related work.

[20] Similarly, twisted differential cohomology was first motivated by the appearance of examples of twisted differential K-theory in string theory [Wit98, §5.3; BM00; Fre00], and has since become an object of study in its own right [Sch13b, §4.1.2; GS18; BN19; GS19a; GS19d; GS19c; FSS20a].

[21] Here $\Omega^k_c(X)$ denotes the space of compactly supported k-forms on X.

the Hodge star on N, then *Maxwell's equations*, as you might see them on a t-shirt, are

$$\begin{aligned}
\mathrm{d}B &= 0\\
\frac{\partial B}{\partial t} + \mathrm{d}E &= 0\\
\mathrm{d}\star_N E &= \rho_E\\
\star_N \frac{\partial E}{\partial t} - \mathrm{d}\star_N B &= J_E\,.
\end{aligned}$$

Writing $F = B - \mathrm{d}t \wedge E \in \Omega^2(M)$ and $j_E = \rho_E + \mathrm{d}t \wedge J_E \in \Omega^3(M)$, we obtain a more concise form of Maxwell's equations:

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d}\star_M F = j_E\,. \tag{21.1.2}$$

Now we include topology. We just saw that j_E is exact, so it cannot define an interesting de Rham cohomology class, but F is closed, so may be interesting. Define the *charge* at time t to be the de Rham class

$$Q_E = [j_E|_{\{t\}\times N}] \in \mathrm{H}^3_{\mathrm{c}}(N;\mathbb{R})\,. \tag{21.1.3}$$

This is in the kernel of the map $\mathrm{H}^3_{\mathrm{c}}(N;\mathbb{R}) \to \mathrm{H}^3(N;\mathbb{R})$; hence, on a compact manifold, $Q_E = 0$.

Let W be the worldline of a charged particle with electric charge $q_E \in \mathbb{R}$. Then $j_E = q_E \cdot \delta_W$, where δ_W is the "current sitting at W." We have two ways of making sense of this.

- First, we could take δ_W to be a current in the de Rham sense, akin to a differential form but built with distributions instead of smooth functions. Amusingly, this is a current in both the Maxwell and de Rham senses. This is a typical example of a current in electromagnetism.

- Alternatively, we could take δ_W to be an honest 3-form Poincaré dual to W. In this case we can choose δ_W to be supported in an arbitrary neighborhood of W.

One more ingredient in Maxwell theory, though not strictly necessary, is an action principle. This follows the Lagrangian formulation of physics: we aim to find a variational problem whose solutions are the Maxwell equations. We add an assumption from classical physics: that $[F] = 0$ in $\mathrm{H}^2_{\mathrm{dR}}(M)$; this means there are no magnetic monopoles.

This assumption also implies $F = \mathrm{d}A$ for some 1-form A called the *electromagnetic potential*. This is not unique, but its class in $\Omega^1(M)/\Omega^1_{\mathrm{cl}}(M)$ (i.e. up to closed 1-forms) is unique. Then, the *classical action* of Maxwell theory is

$$S = \int_M -\frac{1}{2}\mathrm{d}A \wedge \star \mathrm{d}A + A \wedge j_E\,. \tag{21.1.4}$$

Since M is noncompact, this could be infinite, but we're just interested in its first variation anyways, which is well-behaved.

21.1.5 Exercise. Show that the Euler–Lagrange equation for (21.1.4) is $\mathrm{d}{\star}F = j_E$. (We already assumed $\mathrm{d}F = 0$, the other half of Maxwell's equations.)

One caveat: defining the action requires A to be in $\Omega^1(M)$, not $\Omega^1(M)/\Omega^1_{\mathrm{cl}}(M)$. This ends up not a problem; adding a closed form to A does not change the Euler–Lagrange equation.

21.2 Quantum Maxwell theory

In the quantum theory, we allow magnetic monopoles. Dirac [Dir31] argues that this forces electric and magnetic charges to be *quantized*, i.e. taking values in a discrete subgroup of $\mathbb{R}$. This is how differential cohomology enters the picture.

So assume $N = \mathbb{R}^3$ with the usual Euclidean metric, and introduce a magnetic monopole of charge $q_B \in \mathbb{R}$ at the origin. Then we have a *magnetic current* $j_B = q_B \cdot \delta_0$. The condition that $\mathrm{d}F = 0$ is modified to

$$\mathrm{d}F = q_B \cdot \delta_0\,. \tag{21.2.1}$$

The input to the path integral is the exponentiated action $\exp(iS/\hbar)$ (where S is as in (21.1.4)). However, this is not quite consistent with (21.2.1) — there is a problem at the origin. On $\mathbb{R} \times (\mathbb{R}^3 \setminus 0)$, we can write $F = \mathrm{d}A$, and therefore realize F as the curvature of a connection A on a principal $\mathbb{R}/q_B\mathbb{Z}$-bundle P. The characteristic class of P is

$$[P] \in \mathrm{H}^2(\mathbb{R} \times (\mathbb{R}^3 \setminus 0); q_B\mathbb{Z}) \cong \mathrm{H}^2(S^2; q_B\mathbb{Z}) = q_B\mathbb{Z}\,, \tag{21.2.2}$$

and $[P]$ is a generator of this abelian group.

The space of fields in the quantum theory is the groupoid of principal $\mathbb{R}/q_B\mathbb{Z}$-bundles with connection. Now we can revisit the action (21.1.4) — it doesn't have to make sense as is (e.g. A isn't exactly a 1-form), but we do want $\exp(iS/\hbar)$ to make sense.

Let's work on a general 4-manifold X. To avoid causality issues, let's make X a Riemannian manifold, rather than a Lorentz one. Assume j_E is Poincaré dual to some loop $\gamma \subset X$. If there is a q_E charge moving along this loop, then

$$\int_M A \wedge j_E = \oint_\gamma q_E A = q_E \operatorname{Hol}_\gamma(A)\,. \tag{21.2.3}$$

Now $\operatorname{Hol}_\gamma(A) \in \mathbb{R}/q_B\mathbb{Z}$, so the quantity

$$\exp\!\left(\frac{i}{\hbar} q_E \operatorname{Hol}_\gamma(A)\right) \tag{21.2.4}$$

is well-defined if and only if

$$\frac{1}{\hbar} q_E q_B \in 2\pi\mathbb{Z}\,. \tag{21.2.5}$$

This is Dirac's quantization condition. Thus integrality enters a story told with differential forms; this is already suggestive of differential cohomology!

To say it more explicitly, the space of quantum fields is the stack $\mathrm{Bun}^{\nabla}_{\mathbb{R}/q_B\mathbb{Z}}(X)$; the set of isomorphism classes of objects is $\hat{\mathrm{H}}^2(X;q_B\mathbb{Z})$. The curvature map lands in those 2-forms with periods in $q_B\mathbb{Z}$, giving us a short exact sequence we've seen before:

$$0 \longrightarrow \mathrm{H}^1(X;\mathbb{R}/q_B\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2(X;q_B\mathbb{Z}) \xrightarrow{\text{curv}} \Omega^2_{\mathrm{cl}}(X)_{q_B\mathbb{Z}} \longrightarrow 0\,.$$

The classical fields $\Omega^1(X)/\Omega^1_{\mathrm{cl}}(X)$ sit as a subspace in $\hat{\mathrm{H}}^2(X;q_B\mathbb{Z})$; the cokernel is $\mathrm{H}^2(X;q_B\mathbb{Z})$ modulo torsion, indicating the new information in the quantum theory.

Another interesting upshot is that since the kernel of the curvature map corresponds to the flat connections, i.e. those on which F is boring, the electric flux really lives in $\hat{\mathrm{H}}^2(X;q_B\mathbb{Z})$. This is new. The flat connections are new, too — even if you don't usually get to observe them, they manifest in the physics, e.g. through the Aharonov–Bohm effect. And all of this is still "semiclassical," i.e. about the input to the path integral, before we try to evaluate said path integral.

21.2.6 Remark. One important clarification: F is not a differential cohomology class; it's the curvature of an actual bundle with connection, not an equivalence class. So really we need a cochain model: bundles and connections glue, but equivalence classes don't. Cheeger–Simons characters aren't built in this way, so for physics applications one must do something different.

Now we revisit the electric charge, a closed 3-form. Because $(i/\hbar)j_E j_B \in 2\pi\mathbb{Z}$, we'd like to impose that $[j_E] \in \mathrm{H}^3_{\mathrm{dR}}(X)$ is also in the image of the map $\mathrm{H}^3(X;q_E\mathbb{Z}) \to \mathrm{H}^3(X;\mathbb{R})$, i.e. that we're in the homotopy pullback, which is $\hat{\mathrm{H}}^3(X;q_E\mathbb{Z})$. Again, though, we want a local object in the end, not just its isomorphism class.

We can also rewrite one term in the exponentiated action in terms of differential cohomology, as

$$\exp\left(\frac{i}{\hbar}\int_X \hat{F}\cdot\hat{\jmath}\right). \tag{21.2.7}$$

Here $\hat{F}$ and $\hat{\jmath}$ are the differential cohomology refinements of F and j_E, respectively. The product $\cdot$ is the cup product from Chapter 8, which is a map

$$\hat{\mathrm{H}}^2(X;q_B\mathbb{Z})\otimes\hat{\mathrm{H}}^3(X;q_E\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^5(X;q_Eq_B\mathbb{Z})\,. \tag{21.2.8}$$

Since X is a 4-manifold, the integration map has degree -4, so is of the form

$$\int_X : \hat{\mathrm{H}}^5(X;q_Eq_B\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^1(\mathrm{pt};q_Eq_B\mathbb{Z}) \cong \mathbb{R}/q_Eq_B\mathbb{Z}\,. \tag{21.2.9}$$

21.2.10 Exercise. Show that if $\hat{F}$ is topologically trivial, meaning that it comes from a connection on a trivial vector bundle, or equivalently that its image under the characteristic class map vanishes, then $\hat{F}\cdot\hat{\jmath}$ is also topologically trivial.

21.2.11 Remark. There are many variations of this story in field theory and string theory, generally for abelian gauge fields. For example, F might have some other degree, or even be inhomogeneous. Dirac charge quantization still applies, and will refine F to an appropriate differential cohomology group.

More recently, people realized that this story sometimes yields generalized differential cohomology theories. Understanding which cohomology theory one obtains is a bit of an art — physics tells you some constraints, but not an algorithm. For example, this happens in superstring theory: the Ramond–Ramond field is realized in differential K-theory [FH00; MW00], and the B-field in a differential refinement of (a truncation of) ko [DFM11a; DFM11b]. These and other refinements of Dirac quantization to generalized differential cohomology are also studied in [BM06a; BM06b; DFM07; Fre08; Sat10; SV10; Sat11; SSS12; KM13a; KV14; DMR14; FSS15c; Fer16; GS19b; Sat19; FR20]. The choice of generalized cohomology theory is not always an exact science: for example, there are different proposals for the C-field in M-theory. Witten [Wit97, §2.3] argues that the C-field should be quantized in w_1-twisted degree-4 ordinary differential cohomology, which passes consistency checks for various possible anomalies [Wit97, §4; Wit16, §4; FH21a]; there is also the ambitious "hypothesis H" of Fiorenza–Sati–Schreiber [Sat18; FSS20b; FSS21a] proposing that the C-field in M-theory is quantized using a differential refinement of $\mathrm{im}(J)$-twisted stable cohomotopy instead. Work of Fiorenza, Sati, Schreiber, and their collaborators [SS19; FSS20b; GS20; SS20a; SS20b; BSS21; FSS21a; FSS21b; SS21a; SS21b; FSS22] and Roberts [Rob20] recovers as consequences of hypothesis H several things physicists predicted to be true about M-theory.

If we consider Maxwell theory with both electric and a magnetic currents, the theory has an "anomaly," meaning that some quantity that we'd like to obtain as a complex number is actually an element of a complex line that's not trivialized (and in some cases cannot be trivialized canonically for all manifolds of a given dimension). Differential cohomology also provides a perspective on the anomaly. The expression $\hat F \cdot \hat\jmath_E$ in (21.2.8) is valid if there's electric current but not magnetic current; if $\hat\jmath_B \neq 0$, then F isn't closed, hence isn't the curvature of a line bundle. But $\hat\jmath_B$ is also quantized, hence represents a differential cohomology class, and we can ask for $\hat F$ to trivialize $\hat\jmath_B$. Now the action is

$$\exp\left(\frac{i}{\hbar}\int_X \hat F \cdot \hat\jmath_E \hat\jmath_B\right). \tag{21.2.12}$$

Since $\hat F \cdot \hat\jmath_E\hat\jmath_B \in \hat{\mathrm{H}}^6$, integrating brings us to $\hat{\mathrm{H}}^2(\mathrm{pt}; q_E q_B\mathbb{Z})$, yielding the complex line which signals the anomaly. More on this anomaly can be found in Freed–Moore–Segal [FMS07a; FMS07b].

22 Invertible field theories

by Arun Debray

Freed–Hopkins [FH21b, §5.4] conjecture a different application of generalized differential cohomology to field theory, describing reflection-positive invertible field theories which are not necessarily topological. In this chapter we go over this conjecture. This story is similar to an established theorem, Freed–Hopkins' classification of reflection-positive invertible *topological* field theories [FH21b], so we begin in §22.1 by going over that classification; then in §22.2 we generalize to the nontopological setting.

22.1 Topological invertible field theories

22.1.1 Definition. Let $\rho(n)\colon H_n \to \mathrm{O}_n$ be a Lie group homomorphism. An *H_n-structure* on a smooth manifold M is a principal H_n-bundle $P \to M$ together with an isomorphism of principal O_n-bundles

$$\theta\colon P \times_{H_n} \mathrm{O}_n \xrightarrow{\sim} \mathcal{B}_{\mathrm{O}}(M),$$

where $\mathcal{B}_{\mathrm{O}}(M)$ is the frame bundle of M.

For example, an SO_n-structure is equivalent data to an orientation, a Spin_n-structure is equivalent to a spin structure, and so forth.

An H_n-structure on a manifold M induces an H_n-structure on ∂M, and we may therefore consider bordism groups Ω_n^H of H_n-manifolds, as Lashof [Las63] did, and their categorified analogues: bordism (∞, n)-categories Bord_n^H of n-manifolds with H_n-structure, such as the bordism categories constructed by Lurie [Lur09b], Schommer-Pries [Sch17], and Calaque–Scheimbauer [CS19a].

Recall that a topological field theory (TFT) is a symmetric monoidal functor

$$Z\colon \mathsf{Bord}_n^H \to \mathsf{C}\,, \tag{22.1.2}$$

where C is some symmetric monoidal (∞, n)-category. The ∞-category of TFTs is symmetric monoidal under "pointwise tensor product:"

$$(Z_1 \otimes Z_2)(M) \coloneqq Z_1(M) \otimes Z_2(M)\,.$$

22.1.3 Definition (Freed–Moore [FM06])**.** A TFT

$$Z\colon \mathsf{Bord}_n^H \to \mathsf{C}$$

is *invertible* if there is some other TFT Z^{-1} such that $Z \otimes Z^{-1}$ is isomorphic to the trivial theory (i.e. the constant functor valued in $\mathbf{1}_{\mathsf{C}}$).

Equivalently, Z carries objects of M to $\otimes$-invertible objects in C and k-morphisms to composition-invertible k-morphisms in C for all k. In many cases it suffices to check invertibility on a subset of objects, such as certain spheres [Fre12a] or tori [Sch18].

22.1.4 Example (Euler theories)**.** Let $\lambda \in \mathbb{C}^\times$. The *Euler theory*

$$Z_\lambda : \mathsf{Bord}^{\mathrm{O}}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$$

is an invertible TFT which to every object assigns the vector space $\mathbb{C}$, and to every bordism $X : M_1 \to M_2$ assigns multiplication by $\lambda^{\chi(X,M_1)}$. These compose properly because the Euler characteristic satisfies a gluing formula.

Freed–Hopkins–Teleman [FHT10] classified invertible TFTs using work of Galatius–Madsen–Tillmann–Weiss [GTMW09] and Nguyen [Ngu17]. Freed–Hopkins [FH21b] went further: they studied *reflection-positive invertible TFTs*, which have additional structure. This structure is related to the notion of unitarity in quantum field theory, so invertible TFTs appearing in the study of unitary QFTs should have reflection-positive structures.

Let MTH denote the Thom spectrum of $-\mathrm{B}\rho : \mathrm{B}H \to \mathrm{BO}$.[22] Thom's collapse map identifies the homotopy groups of MTH with the bordism groups of manifolds with H_n-structure [Tho54, Théorème IV.8; Pon59; Las63, Theorem C].[23] Let $\mathrm{I}_{\mathbb{Z}}$ denote the *Anderson dual of the sphere spectrum* [And69; Yos75], which satisfies the universal property that there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{n-1}(X), \mathbb{Z}) \longrightarrow [X, \Sigma^n \mathrm{I}_{\mathbb{Z}}] \longrightarrow \mathrm{Hom}(\pi_n(X), \mathbb{Z}) \longrightarrow 0\,, \tag{22.1.5}$$

which noncanonically splits.

22.1.6 Theorem (Freed–Hopkins [FH21b])**.** *There is an isomorphism of abelian groups from π_0 of the space reflection-positive, invertible, n-dimensional, topological field theories to the torsion subgroup of* $[\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}}]$.

22.1.7 Remark. Any classification of TFTs $Z : \mathsf{Bord}^H_n \to \mathsf{C}$ depends on what we take C to be. For this theorem, Freed–Hopkins make an ansatz about the choice of C. Example C meeting this ansatz are known in category number 2 and below: see [Fre12b, Theorem 1.52] and [DG18b, Proposition 4.21].

If B admits a CW structure with finitely many cells in each dimension, so that the homotopy groups of MTH are finitely generated, then

$$\mathrm{Tors}([\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}}]) \cong \mathrm{Tors}(\mathrm{Hom}(\pi_n(\mathrm{MTH}), \mathbb{C}^\times))\,.$$

Thus we have identified $\mathrm{Tors}([\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}}])$ with the group of torsion $\mathbb{C}^\times$-valued bordism invariants for n-dimensional H-manifolds. Given such a bordism invariant φ, it is possible to

[22] There is an important subtlety here: we started with $\rho(n) : H_n \to \mathrm{O}_n$, not the stabilized version $\rho : H \to \mathrm{O}$. Freed–Hopkins [FH21b, Theorem 2.19] show that the additional data associated to reflection positivity allows one to define ρ and H such that $\rho(n) : H_n \to \mathrm{O}_n$ is the pullback of $\rho : H \to \mathrm{O}$ along the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{O}$.

[23] The use of $-\rho$ ensures that we obtain an H-structure on the stable tangent bundle. Homotopy theorists more traditionally study the Thom spectrum of ρ, denoted MH, which corresponds to bordism of manifolds with an H-structure on the stable *normal* bundle. Often $\mathrm{MTH} \simeq \mathrm{MH}$, as is the case for MTO, MTSO, MTSpin, MTSpin^c, MTString, and MTU, but not always: $\mathrm{MTPin}^+ \not\simeq \mathrm{MPin}^+$.

choose a reflection-positive invertible TFT Z in the component of $\pi_0(\mathrm{ITFT}s)$ corresponding to φ such that the partition function of Z is equal to φ.

22.1.8 Example (classical Dijkgraaf–Witten theory [DW90; FQ93])**.** Let G be a group and $\lambda \in H^n(BG; \mathbb{Q}/\mathbb{Z})$. Then λ defines a bordism invariant of oriented n-manifolds M with a principal G-bundle P by integrating, then exponentiating:

$$(M, P) \longmapsto \exp\left(2\pi i \int_M \lambda(P)\right) \in \mathbb{C}^\times, \tag{22.1.9}$$

where $\lambda(P)$ denotes the pullback of λ along the map $M \to BG$ defined by P. Stokes' theorem implies this is a bordism invariant, and it is torsion; therefore (22.1.8) is the partition function of a unique (up to isomorphism) reflection-positive invertible TFT. This TFT is called *classical Dijkgraaf–Witten theory*. The state space assigned to any codimension-1 manifold is noncanonically isomorphic to $\mathbb{C}$; see Freed–Quinn [FQ93, §1] for a fuller description and Yonekura [Yon19, §4] for another construction.

22.1.10 Example (Arf theory)**.** We have $\Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2$, and the *Arf invariant* is a complete invariant

$$\mathrm{Arf} : \Omega_2^{\mathrm{Spin}} \to \{\pm 1\}$$

[Ati71, Proposition (4.1)]. Using Freed–Hopkins' classification, there is a reflection-positive invertible TFT $Z_A : \mathsf{Bord}_2^{\mathrm{Spin}} \to \mathsf{C}$, called the *Arf theory*, whose partition function is the Arf invariant, and Z_A is unique up to isomorphism. Gunningham [Gun16, Example 2.19] showed that we can take C to be $\mathsf{sAlg}_{\mathbb{C}}$, the Morita bicategory of complex superalgebras.

As in Example 22.1.8, we can recast this example as integration, this time in generalized cohomology. Atiyah–Bott–Shapiro [ABS64] showed that spin manifolds admit pushforward maps for KO-theory. On a spin surface, the partition function of the Arf theory (i.e. the Arf invariant) is the pushforward

$$\begin{aligned} \exp 2\pi i \int_\Sigma^{\mathrm{KO}} \; : \; \mathrm{KO}^0(\Sigma) &\longrightarrow \mathrm{KO}^{-2}(\mathrm{pt}) \cong \{\pm 1\} \\ 1 &\longmapsto Z_A(\Sigma) \end{aligned}$$

That is, the KO-theoretic pushforward lands in $\mathbb{Z}/2$, and exponentiation brings us to $\{\pm 1\} \subset \mathbb{C}^\times$.

Something similar also works in positive codimension! Let C be a closed spin 1-manifold.

$$\int_C^{\mathrm{KO}} \; : \; \mathrm{KO}^0(C) \longrightarrow \mathrm{KO}^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2 \tag{22.1.11}$$

$$1 \longmapsto Z_A(C)\,. \tag{22.1.12}$$

This $\mathbb{Z}/2$ is different — we interpret it as the group of isomorphism classes of complex super lines $\{\mathbb{C}, \Pi\mathbb{C}\}$ under tensor product. That is, an invertible field theory valued in $\mathsf{sAlg}_{\mathbb{C}}$ assigns to a codimension-1 manifold a $\otimes$-invertible complex super vector space; up to isomorphism this is either the even line or the odd line, and (22.1.11) tells us which one the Arf theory assigns to C. For example, the bounding spin circle is assigned an even line, and the nonbounding spin circle is assigned an odd line.

When we turn to non-topological invertible field theories, these integrals will use differential (generalized) cohomology.

22.2 Non-topological invertible field theories

Using reflection-positive invertible TFTs, we saw the torsion subgroup of $[\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}}]$. Freed–Hopkins [FH21b, §5.4] go further and conjecture that the entire group classifies reflection-positive invertible field theories that are not necessarily topological. At present, it is not clear how to define these field theories. But Freed–Hopkins predict what the partition functions of these theories should be, which is a differential-cohomological lift of the topological story, where we had bordism invariants. We follow Freed [Fre19, Lecture 9] and Freed–Hopkins [FH21b, §5.4] in this section.

22.2.1 Definition. A *differential H_n-structure* on a smooth manifold M is

(1) a Riemannian metric on M,

(2) an H-structure in the sense above, i.e. a principal H_n-bundle $P \to M$ with an isomorphism $\theta\colon P \times_{H_n} \mathrm{O}_n \xrightarrow{\sim} \mathcal{B}_{\mathrm{O}}(M)$, and

(3) a connection A on P whose induced connection under θ is the Levi-Civita connection for the metric.

A differential H_n-structure on M induces a differential H_n-structure on a collar neighborhood of ∂M, so analogously to Bord_n^H, there should be a "geometric bordism category" $\mathsf{Bord}_n^{H_n,\nabla}$. Then one should be able to define field theories as symmetric monoidal functors from $\mathsf{Bord}_n^{H_n,\nabla}$ to something like a category of topological vector spaces, and define invertibility as above. Following ideas of Atiyah, Kontsevich, and Segal [Seg11], various geometric versions of bordism categories have been constructed or sketched by Cheung [Che07], Ayala [Aya09], Hohnhold–Stolz–Teichner [HST10, §6.2], Hohnhold–Kreck–Stolz–Teichner [HKST11, §5.2], Stolz–Teichner [ST11], Tachikawa [Tac13, §1], Schommer-Pries–Stapleton [SS14, §7], Kandel [Kan16], Grady–Sati [GS17, §5.2], Ulrickson [Ulr17, §2.1.2], Müller–Szabo [MS18, §2.1], Grady–Pavlov [GP20, §4.2], Ludewig–Stoffel [LS21, §3], and Kontsevich–Segal [KS21]; Müller–Szabo use their model to study examples of invertible, non-topological field theories.

22.2.2 Conjecture (Freed–Hopkins [FH21b, Conjecture 8.37])**.** *There is an isomorphism of abelian groups from π_0 of the space reflection-positive, invertible, n-dimensional field theories to* $[\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}}]$.

Key to this conjecture is formulating a good definition of invertible, non-topological field theory. In the rest of this section, we assume the conjecture is true, which in particular means finding a definition.

This conjecture includes a prediction for the value of the partition function of an invertible field theory given by $\varphi \in \operatorname{Map}(\mathrm{MTH}, \Sigma^{n+1}\mathrm{I}_{\mathbb{Z}})$. An H-manifold M gives a point in MTH, i.e. a map $M\colon \Sigma^n\mathbb{S} \to \mathrm{MTH}$. Composing with φ and desuspending, we have a map $\mathbb{S} \to \Sigma\mathrm{I}_{\mathbb{Z}}$; its homotopy class is an element of $\mathrm{I}_{\mathbb{Z}}^1(\mathrm{pt}) = \pi_{-1}\mathrm{I}_{\mathbb{Z}} = 0$, so this construction is not very interesting. But conjecturally, a differential refinement of this procedure takes a manifold M with a differential H_n-structure and obtains an element $\varphi(M) \in \hat{\mathrm{I}}_{\mathbb{Z}}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$; then the partition function of the corresponding invertible field theory is predicted to be $\exp(2\pi i\varphi(M))$. See Hopkins–Singer [HS05, §5.1] for a construction which adopts this perspective; they in particular construct the differential refinement $\hat{\mathrm{I}}_{\mathbb{Z}}$ of $\mathrm{I}_{\mathbb{Z}}$, by using that $\mathrm{H}\mathbb{Z} \to \mathrm{I}_{\mathbb{Z}}$ is a rational equivalence. Yamashita–Yonekura [YY21; Yam21] take another approach, directly constructing a differential refinement of $\operatorname{Map}(\mathrm{MTH}, \Sigma^2\mathrm{I}_{\mathbb{Z}})$ and using it to access the partition functions of these conjectured field theories.

Often there is a simpler description. Assume φ can be identified with the element of the group $\operatorname{Hom}(\Omega_{n+1}^H, \mathbb{Z})$ given by integrating a (generalized) cohomology class c. Then the partition function of the theory associated to φ is the secondary invariant associated to c, as defined in §14.3.

22.2.3 Example (classical Chern–Simons theory)**.** The Chern–Simons invariants we discussed above in §19.2 fit together into an invertible, non-topological field theory which is a differential analogue of Example 22.1.8. Fix a compact Lie group and a *level* $\lambda \in \mathrm{H}^4(BG;\mathbb{Z})$. Assume λ is not torsion. Since G is compact, the Chern–Weil map is an isomorphism, so as in Chapter 14, λ refines to a class $\hat{\lambda} \in \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z})$.

The level λ defines an element of $\operatorname{Hom}(\Omega_4^{\mathrm{SO}}(BG);\mathbb{Z})$: send an oriented 4-manifold X with principal G-bundle $P \to M$ to the integer $\int_M \lambda(P)$, where $\lambda(P)$ denotes the pullback of λ along the homotopy class of maps $M \to BG$ defined by P. Again, Stokes' theorem is why this is a bordism invariant. According to Conjecture 22.2.2, this bordism invariant determines (up to isomorphism) an invertible field theory for 3-manifolds with a differential $\mathrm{SO}_3 \times G$-structure. This field theory is classical Chern–Simons theory [Fre95; Fre02b; Gom01b]

$$\alpha_{(G,\lambda)}\colon \mathsf{Bord}_3^{\mathrm{SO}\times G,\nabla} \longrightarrow \mathsf{Line}_{\mathbb{C}}\,. \tag{22.2.4}$$

Let Y be a closed 3-manifold with a differential $\mathrm{SO}\times G$-structure, which means an orientation, a Riemannian metric, a principal G-bundle $P \to Y$, and a connection A for P. The data of (P, A) gives us a map $Y \to \mathrm{Bun}_G^\nabla$, allowing us to pull $\hat{\lambda}$ back to Y, and the orientation allows us to integrate differential cohomology classes, as in Chapter 9. The partition function of $\alpha_{(G,\lambda)}$ is $\exp\bigl(2\pi i \int_Y \hat{\lambda}(P,A)\bigr)$, which is exactly the exponentiated Chern–Simons invariant of (P, A), as we established in (19.1.11):

$$\begin{aligned} \exp 2\pi i \int_Y \colon \hat{\mathrm{H}}^4(Y) &\longrightarrow \hat{\mathrm{H}}^1(\mathrm{pt}) \to \mathbb{C}^\times \\ \hat{\lambda}(P,A) &\longmapsto \exp(2\pi i \mathrm{CS}_\lambda(P,A))\,, \end{aligned}$$

That is, $\hat{\mathrm{H}}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$, and exponentiating gets us to $\mathbb{C}^\times$.

On a closed, oriented surface Σ with a Riemannian metric, principal G-bundle $P \to \Sigma$, and connection A, $\alpha_{(G,\lambda)}$ again assigns the pushforward of $\hat{\lambda}(P, A)$, but this time the pushforward map has signature

$$\int_\Sigma : \hat{\mathrm{H}}^4(\Sigma) \longrightarrow \hat{\mathrm{H}}^2(\mathrm{pt}) \cong \mathsf{Line}_{\mathbb{C}}\,, \tag{22.2.5}$$

which sends $\hat{\lambda}(P, A)$ to the Chern–Simons line constructed in, e.g., [Fre95, §4]. This story continues in extended TFT, assigning higher-categorical objects to lower-dimensional manifolds, such as in [Gom01a].

See also Fiorenza–Sati–Schreiber [FSS15a], Yamashita–Yonekura [YY21, Example 4.56], and Yamashita [Yam21, §3.4.2] for additional constructions of classical Chern–Simons theory as an invertible field theory, and Freed–Neitzke [FN20] for an application to special functions.

22.2.6 Remark (quantizing Chern–Simons theory)**.** One of the interesting things you can do with the classical Chern–Simons theory is to quantize it. This amounts to summing $\alpha_{(G,\lambda)}$ over the space of all principal G-bundles with connection on a given closed, oriented 3-manifold. This procedure, known as taking the path integral, is still only heuristically defined,[24] but enough is known about it in the physics literature that we can ask mathematical questions about the quantized theory. In physics, this quantum Chern–Simons theory was first studied by Schwarz [Sch77] and Witten [Wit89].

Something strange happens in this quantization procedure, though: Witten (*ibid.*) gives a physical argument that quantum Chern–Simons theory is in fact a topological field theory! Therefore it should be possible to formalize it mathematically as a symmetric monoidal functor

$$Z_{G,k} : \mathsf{Bord}_3^{\mathrm{SO}} \longrightarrow \mathsf{C}\,, \tag{22.2.7}$$

where C is some symmetric monoidal $(\infty, 3)$-category. It is not known how to do this in general,[25] but it is known how to extend it to a theory of 1-, 2-, and 3-manifolds, valued in the 2-category of $\mathbb{C}$-linear categories, by work of Reshetikhin–Turaev [RT90; RT91], Walker [Wal91], Bakalov–Kirillov [BK01], Kerler–Lyubashenko [KL01], and Bartlett–Douglas–Schommer-Pries–Vicary [BDSV15].[26] Much more can be said about this TFT and its connections to various parts of geometry, topology, representation theory, and physics; see Freed [Fre09] for a general survey on Chern–Simons theory and the references therein for more information.

22.2.8 Example (classical Wess–Zumino–Witten theory)**.** This example is related to the previous example, but with a slightly different flavor. Let G be a compact Lie group and $\hat{h} \in \hat{\mathrm{H}}^3(G; \mathbb{Z})$.

[24] When G is finite, Freed–Quinn [FQ93] define a path integral of *topological* field theories whose fields include a principal G-bundle. Applied to classical Dijkgraaf–Witten theory from Example 22.1.8, the resulting TFT, called *(quantum) Dijkgraaf–Witten theory*, is a commonly studied model organism in topological field theory.

[25] There are a few different perspectives on what $Z_{G,k}(\mathrm{pt}_+)$ should be. For G finite, the answer is known by work of Freed–Hopkins–Lurie–Teleman [FHLT10, §4.2] and Wray [Wra10, §9]; for G a torus, the answer is due to Freed–Hopkins–Lurie–Teleman (*ibid.*). For general G, two different approaches are provided by Freed–Teleman (see [Fre12a]) and Henriques [Hen17a; Hen17b]. See also [FT21].

[26] These constructions require some additional structure on our manifolds, such as a choice of trivialization of the first Pontryagin class. As theories of merely oriented manifolds, Chern–Simons theories are *anomalous*. See [FHLT10, §9.3; Fre12a] for more information.

If $h \coloneqq \mathrm{cc}(\hat{h})$ (the image of $\hat{h}$ under the characteristic class map of Construction 2.2.5), then h defines a bordism invariant of oriented 3-manifolds M with a map $\psi\colon M \to G$:

$$\begin{aligned} \Omega_3^{\mathrm{SO}}(G) &\longrightarrow \mathbb{Z} \\ (M, \psi) &\longmapsto \int_M \psi^*(h)\,. \end{aligned}$$

Conjecture 22.2.2 therefore says there is a two-dimensional invertible field theory $\beta_{G,h}$ whose partition function is the secondary invariant associated to $\hat{h}$. This theory is called *classical Wess–Zumino–Witten (WZW) theory*; it was originally studied by Witten [Wit83], following Wess–Zumino [WZ71]. See Freed [Fre95, Appendix A] for a discussion of the classical theory specifically.[27]

As part of a trend you may have noticed by now, the original description of the classical WZW partition function $\int_M \psi^*(\hat{h})$ was not phrased in this way; the connection with differential cohomology is due to Gawędzki [Gaw88]. For a moment assume that G is connected, simple, and simply connected, so that $\mathrm{H}^3(G;\mathbb{Z}) \cong \mathbb{Z}$. Let $\mathrm{MC} \in \Omega^1(G;\mathfrak{g})$ denote the Maurer–Cartan form (see Definition 13.3.3)., As mentioned in Remark 20.3.11, the transgression map

$$\tau^{-1}\colon \mathrm{H}^3(G;\mathbb{Z}) \to \mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$$

is an isomorphism; since G is compact, the Chern–Weil machine associates to $\tau^{-1}(h)$ (or rather, its image in $\mathbb{R}$-valued cohomology) a degree-two invariant polynomial f. In this case, the Wess–Zumino–Witten action is

$$\beta_{G,h}(M,\psi) = \int_M -\frac{1}{6}\psi^*(f(\mathrm{MC} \wedge [\mathrm{MC},\mathrm{MC}])). \tag{22.2.9}$$

The differential refinement of $\tau\colon \mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^3(G;\mathbb{Z})$ constructed by Carey–Johnson–Murray–Stevenson–Wang [CJM+05, §3] and Schreiber [Sch13b, 1.4.1.2] can be thought of as starting with a classical Chern–Simons theory and obtaining a classical Wess–Zumino–Witten theory in one dimension lower.

22.2.10 Remark (quantizing the Wess–Zumino–Witten model)**.** Just as in Remark 22.2.6, it is possible to quantize the classical WZW model, at least at a physical level of rigor: one sums over the space of maps to G. The result is called the quantum Wess–Zumino–Witten model, or just the Wess–Zumino–Witten or WZW model. This theory is a conformal field theory, meaning its value on a manifold depends only on the conformal class of the Riemannian metric. Some of what we do in the next two chapters, involving the representation theory of loop groups, is related to the WZW model.

Given a level $h \in \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z})$, there is a (quantum) Chern–Simons theory and a quantum WZW model (obtained by transgressing h to $\hat{\mathrm{H}}^3(G;\mathbb{Z})$), and the two are related: the WZW model

[27] There are considerably more general objects studied in quantum physics under the name "Wess–Zumino–Witten theory" or "Wess–Zumino–Witten term." See [DF99, §6; Fre08; Sch13b, §5.6; FSS15b; LOT21; Yon21] for some examples taking an algebro-topological viewpoint.

is a boundary theory for the Chern–Simons theory. There are different ways of formulating this precisely: one uses *relative field theory* [FT14]. In this formalism, the bulk theory α is a symmetric monoidal functor out of a bordism category, and its boundary theory Z is a natural transformation from (a truncation of) α to the trivial field theory. Among other things, this implies that the partition function of Z on an $(n-1)$-manifold M is not a number, but an element of the state space $\alpha(M)$; when α is Chern–Simons theory and Z is the WZW model, this fact was first noticed by Witten [Wit89]. See Gwilliam–Rabinovich–Williams [GRW22] for another approach to this bulk-boundary correspondence, in the language of factorization algebras.

22.2.11 Example (exponentiated η-invariants)**.** We now give a differential analogue of Example 22.1.10: in that example, we used the Atiyah–Bott–Shapiro pushforward [ABS64] in KO-theory to produce a torsion bordism invariant, hence an invertible topological field theory. Here we will use the same pushforward to produce a nontorsion bordism invariant, hence an invertible, non-topological field theory. This theory is discussed by Freed [Fre19, Example 9.24].

The bordism invariant in question is the $\widehat{A}$-*genus* $\widehat{A}\colon \Omega_4^{\mathrm{Spin}} \to \mathbb{Z}$,[28] which, like the Arf invariant, is a pushforward in KO-theory: for a closed spin 4-manifold X, we have

$$\begin{aligned} \int_X^{\mathrm{KO}} : \mathrm{KO}^0(X) &\longrightarrow \mathrm{KO}^{-4}(\mathrm{pt}) \cong \mathbb{Z} \\ 1 &\longmapsto \widehat{A}(X)\,. \end{aligned}$$

This is nonvanishing on the K3 surface, hence nontorsion. By Freed–Hopkins' conjecture, this bordism invariant corresponds to some invertible, non-topological field theory on 3-dimensional differential spin manifolds (i.e. 3-manifolds with a spin structure and a Riemannian metric):

$$\alpha'\colon \mathsf{Bord}_3^{\mathrm{Spin},\nabla} \longrightarrow \mathsf{sLine}_{\mathbb{C}}\,.$$

And analogously to the Arf theory, we can describe the value of α' on closed 2- and 3-manifolds with differential spin structure using the pushforward in differential KO-theory. Grady–Sati [GS21, §4.3] construct this pushforward for a closed spin manifold; using this, the partition function of α' on a closed spin Riemannian 3-manifold Y is

$$\begin{aligned} \exp 2\pi i \int_Y^{\widehat{\mathrm{KO}}} : \widehat{\mathrm{KO}}^0(Y) &\longrightarrow \widehat{\mathrm{KO}}^{-3}(\mathrm{pt}) \to \mathbb{C}^\times \\ 1 &\longmapsto \alpha'(Y)\,, \end{aligned}$$

[28] $\widehat{A}$ is pronounced "A-hat" or "A-roof." This gives rise to the following joke: A man walks into a bar with a dog and says to the bartender, "This is a talking dog. I'll bet you a drink he can answer a question."

The bartender says, "Sure. Ok dog, what's your favorite spin bordism invariant?"

"Arf!"

"…"

"Clifford, how about a different one?"

"A-roof!"

(they get thrown out)

The dog looks at the man and says, "Ok fine, next time I'll say 'index of the Dirac operator.' "

where as usual $\widehat{\mathrm{KO}}^{-3}(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$, and we exponentiate to obtain the partition function in $\mathbb{C}^\times$. The isomorphism type of the state space assigned to a closed spin Riemannian 2-manifold Σ is in a similar way the image of 1 under the pushforward $\widehat{\mathrm{KO}}^0(\Sigma) \to \widehat{\mathrm{KO}}^{-2}(\mathrm{pt}) \cong \mathbb{Z}/2$, corresponding to the two isomorphism classes of complex super lines, $\mathbb{C}$ and $\Pi\mathbb{C}$.

Like in Example 22.2.3, the partition function of Y also has a more geometric description. A differential spin structure is the data needed to define the Dirac operator on the spinor bundle of Y, and index-theoretic methods allow one to extract an *exponentiated η-invariant* from this Dirac operator, as constructed by Atiyah–Patodi–Singer [APS75a; APS75b; APS76]. The Dai–Freed theorem [DF94] proves this exponentiated η-invariant satisfies a gluing law which can be interpreted as implying that α' is symmetric monoidal.

For more examples of invertible, non-topological field theories and their relationship to differential cohomology, see Monnier [Mon15, §4; Mon17, §5; Mon18], Monnier–Moore [MM19], Córdova–Freed–Lam–Seiberg [CFLS20a, §§6.2 & 7], Yamashita–Yonekura [YY21, §§4.2, 6], and Yamashita [Yam21, §3.4].

23 Loop groups and intertwining of positive-energy representations

by Sanath Devalapurkar

We will give an introduction to the representation theory of loop groups of compact Lie groups: we will discuss what positive energy representations are, why they exist, how to construct them (via a Schur–Weyl style construction and a Borel–Weil style construction), and how to show that they don't depend on choices. Motivation will come from both mathematics and quantum mechanics.

The theory of positive-energy representations of loop groups is modeled on the representation theory of compact Lie groups. Some parts of the talk will make more sense if you are familiar with the compact Lie group story, but this is not a requirement: in this section, we try to emphasize the "big picture" over details, and we hope that this choice makes it readable for you. Likewise, we will not assume any familiarity with loop groups or infinite-dimensional topology, nor will we dig into those details.

In §23.1, we state the main theorem (Theorem 23.1.1) and discuss some motivation for caring about representations of loop groups. In §23.2, we begin thinking about projective representations of loop groups and the corresponding central extensions. In §23.3, we provide an extended proof sketch of Theorem 23.1.1, and discuss some connections to physics. Finally, in §23.4, we discuss how this relates to differential cohomology. There are two ways to lift the construction of central extensions of loop groups to differential cohomology; one follows the Chern–Weil story we've used several times already in this part, and the other more closely resembles the story we told about off-diagonal Deligne cohomology and the Virasoro algebra in Chapter 18.

23.1 Overview

The objective of this chapter is to explain the following theorem of Pressley–Segal [PS86, Theorem 13.4.2]:

23.1.1 Theorem. *Let G be a simply connected compact Lie group. Then any positive energy representation E of the loop group* $\mathrm{L}G$ *admits a projective intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$.

If this means nothing to you, that's okay: the goal of this talk is to explain all the components of this theorem (§ 23.2) and sketch a proof (§ 23.3). Then, in § 23.4, we discuss how the representation theory of loop groups is related to differential cohomology.

Here's a rough sketch of what Theorem 23.1.1 is about. The representation theory of a semisimple compact Lie group G is very well-behaved: the Peter–Weyl theorem [PW27] allows one to provide any finite-dimensional G-representation with a G-invariant Hermitian inner product, and this inner product decomposes the representation into a direct sum of irreducibles. Moreover, the irreducibles are in bijection with dominant weights, where by the Borel–Weil theorem (see [Ser54]), the representation associated to a dominant weight is given as the global sections of a line bundle associated to a homogeneous space of G (a particular flag variety).

Most representations of loop groups will not satisfy analogues of this property, so we'd like to hone down on the ones which do. These are the "positive energy representations"; these essentially satisfy properties necessary to be able to write down highest/lowest weight vectors. Theorem 23.1.1 then states that positive energy representations are preserved under reparametrizations of the circle (which give automorphisms of the loop group $\mathrm{L}G$). One can therefore think of Theorem 23.1.1 as a consistency result.

Before proceeding, I'd like to give some motivation for caring about the representation theory of loop groups.

(1) One motivation comes from the connection between representation theory and homotopy theory. The Atiyah–Segal completion theorem [Ati61, Theorem 7.2; AH61, §4.8; AS69, Theorem 2.1] relates representations of a compact Lie group G to G-equivariant K-theory, and likewise the representation theory of the loop group $\mathrm{L}G$ is related to (twisted) G-equivariant elliptic cohomology. This has been explored in [Bry90; Dev96; Liu96; And00; And03; Gro07; Lur09a; Gan14; Lau16; Kit19; Rez20; BT21].

(2) Another motivation comes from the hope that geometry on the free loop space $\mathrm{L}M$ of a manifold M is supposed to correspond to correspond to "higher-dimensional geometry" over M. For instance, if M has a Riemannian metric, one can think of the scalar curvature of $\mathrm{L}M$ at a loop as the integral of the Ricci curvature of g over the loop. Similarly, spin structures on M are closely related to orientations on $\mathrm{L}M$ [Wit85; Ati85, §3; Wit88; McL92, §2; ST05, Theorem 9; Wal16b, Corollary E, §1.2], and string structures on M are closely related to spin structures on $\mathrm{L}M$ [Kil87; NW13, Theorem 6.9].[29]

In light of this hope, it is rather pacifying to have a strong analogy between representation theory of compact Lie groups and of loop groups. In fact, all of these motivations are related by a story that still seems to be mysterious at the moment.

There's also motivation from physics for studying the representation theory of loop groups. The wavefunction of a free particle on the circle S^1 must be an L^2-function on S^1 (because the probability of finding the particle somewhere on the circle is 1). There is an action of the loop group LU_1 on $\mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$ given by pointwise multiplication (a pair $\gamma : \mathrm{S}^1 \to \mathrm{U}_1$ and $f \in \mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$ is sent to the L^2-function $f_\gamma(z) = \gamma(z)f(z)$). In particular, LU_1 gives a lot of automorphisms of the Hilbert space $\mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$; this is relevant to quantum mechanics, where observables are (Hermitian) operators on the Hilbert space of states. Having a particularly (mathematically) natural source of symmetries is useful. In [Seg85], Segal in fact says: "In fact it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups".

23.2 Representations of loop groups

23.2.1 Definition. Let G be a compact connected Lie group. The loop group $\mathrm{L}G \coloneqq \mathrm{C}^\infty(\mathrm{S}^1, G)$ is the group of smooth unbased loops in G.

[29] There are a number of other works providing additional proofs of this fact or pointing out subtleties in the definitions, including [PW88; CP89; McL92, §3; KY98; ST05; KM13b; Wal15; Cap16; Wal16a; Kri20].

If G is positive-dimensional, LG is not finite-dimensional. A fair amount of the theory of finite-dimensional manifolds generalizes to infinite-dimensional spaces locally modeled by nice classes of topological vector spaces, and in this sense LG is an infinite-dimensional Lie group, in fact quite a nice one. Reading this chapter does not require any additional familiarity with infinite-dimensional topology, but if you're interested, you can learn more in [Ham82b; Mil84; PS86, §3.1]

There will be a lot of circles floating around, and so we will distinguish these by subscripts. Some of these will be denoted by $\mathbb{T}$, for "torus".

23.2.2 Remark (classification of compact Lie groups)**.** We quickly review the classification of compact Lie groups. This may clarify the generality in which some of the results in this section hold.

- Let G be a compact Lie group and $G_0 \subset G$ denote the connected component containing the identity. Then there is a short exact sequence

$$1 \to G_0 \to G \to \pi_0(G) \to 1 .$$

- Let G be a compact, connected Lie group. Then there is a short exact sequence

$$1 \to F \to \tilde{G} \to G \to 1 ,$$

  where F is finite and $\tilde{G}$ is a product of a torus $\mathbb{T}^n$ and a simply connected group.

- Let G be a compact, connected, simply connected Lie group. Then G is a product of *simple* simply connected Lie groups.

- Let G be a compact, simply connected, simple Lie group. Then G is isomorphic to one of SU_n, Spin_n, Sp_n, G_2, F_4, E_6, E_7, or E_8.

Most of the results in this section require G to be connected and simply connected; a few will also require G to be simple. In particular, when G is simple, $\mathrm{H}^4(BG;\mathbb{Z}) \cong \mathbb{Z}$.[30]

23.2.3 Remark. The loop group LG is an infinite-dimensional Lie group, and it has an action of S^1 by rotation. We will denote this "rotation" circle by $\mathbb{T}_{\mathrm{rot}}$. This action will turn out to be very useful shortly.

The action of $\mathbb{T}_{\mathrm{rot}}$ allows one to consider the semidirect product $LG \rtimes \mathbb{T}_{\mathrm{rot}}$. The following proposition is then an exercise in manipulating symbols:

23.2.4 Proposition. *An action of* $LG \rtimes \mathbb{T}_{\mathrm{rot}}$ *on a vector space* V *is the same data as an action* R *of* $\mathbb{T}_{\mathrm{rot}}$ *on* V *and an action* U *of* LG *on* V *satisfying*

$$R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma} .$$

[30] This isomorphism can be made canonical by specifying that under the Chern–Weil map, the Killing form $B\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defines a positive element of $\mathrm{H}^4_{\mathrm{dR}}(BG) \cong \mathbb{R}$.

Most interesting representations U of $\mathrm{L}G$ on a vector space V are not, strictly speaking, representations: instead of $U_\gamma U_{\gamma'} = U_{\gamma\gamma'}$, they satisfy the weaker condition that

$$U_\gamma U_{\gamma'} = c(\gamma, \gamma')U_{\gamma\gamma'}\,, \tag{23.2.5}$$

where $c(\gamma, \gamma') \in \mathbb{C}^\times$. This is precisely:

23.2.6 Definition. A *projective representation* of $\mathrm{L}G$ on a Hilbert space V is a continuous homomorphism $\mathrm{L}G \to \mathrm{PU}(V)$.

23.2.7 Remark. Why Hilbert spaces? From a mathematical perspective, this is because Hilbert spaces are well-behaved infinite-dimensional vector spaces. From a physical perspective, this is because Hilbert spaces are spaces of states. In fact, this also explains why most interesting representations are projective: the state of a quantum system is not a vector in the Hilbert space, but rather a vector in the projectivization of the Hilbert space. This corresponds to the statement that shifting the wavefunction by a phase does not affect physical observations.

Assume V is an infinite-dimensional, separable Hilbert space. Then $\mathrm{PU}(V)$ is a $\mathrm{K}(\mathbb{Z}, 2)$, so projective representations determine cohomology classes in $\mathrm{H}^2(\mathrm{L}G; \mathbb{Z})$.

23.2.8 Lemma. *When G is compact and simply connected,* $\mathrm{H}^2(\mathrm{L}G; \mathbb{Z}) \cong \mathrm{H}^3(G; \mathbb{Z})$*.*

Proof. Since G is simply connected, $\pi_1(G) = 0$, and π_2 vanishes for any Lie group. Therefore the Hurewicz theorem identifies $\pi_3(G)$ and $\mathrm{H}_3(G; \mathbb{Z})$. Let ΩG denote the based loop space of G, i.e. the subspace of $\mathrm{L}G$ consisting of loops beginning and ending at the identity. Essentially by definition, there is an isomorphism $\pi_k(G) \to \pi_{k-1}(\Omega G)$ for $k > 1$, so we learn $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) \cong \pi_3(G)$.

To get to $\mathrm{L}G$, we use that as topological spaces, $\mathrm{L}G \cong G \times \Omega G$ [PS86, §4.4]. Thus $\pi_1(\mathrm{L}G) = 0$ and $\pi_2(\mathrm{L}G) \cong \pi_3(G)$, and the Hurewicz and universal coefficient theorems allow us to conclude. ☐

Another way to construct this isomorphism is as follows: there is an evaluation map

$$\begin{aligned} \mathrm{ev}\colon \mathrm{S}^1 \times \mathrm{L}G &\to G \\ (x, \ell) &\mapsto \ell(x)\,; \end{aligned}$$

then the isomorphism in Lemma 23.2.8 is: pull back by ev, then integrate in the S^1 direction.

It turns out that when G is compact and simply connected, every class in $\mathrm{H}^2(\mathrm{L}G; \mathbb{Z})$ arises from a projective representation as above [PS86, Theorem 4.4.1]. There is a central extension[31]

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \mathrm{U}(V) \to \mathrm{PU}(V) \to 1\,, \tag{23.2.9}$$

[31] This central extension is also a fiber bundle, and by Kuiper's theorem [Kui65], the total space $\mathrm{U}(V)$ is contractible (see also [DD63, Lemme 3; AS04, Proposition A2.1]). This fiber bundle is homotopy equivalent to two other interesting fiber bundles: the universal principal U_1-bundle $\mathrm{U}_1 \to \mathrm{EU}_1 \to \mathrm{BU}_1$, and the loop space-path space bundle $\Omega\mathrm{K}(\mathbb{Z}, 2) \to P\mathrm{K}(\mathbb{Z}, 2) \to \mathrm{K}(\mathbb{Z}, 2)$.

and so any projective representation ρ of $\mathrm{L}G$ determines a central extension by pulling (23.2.9) back:

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \widetilde{\mathrm{L}G}_{\rho} \to \mathrm{L}G \to 1\,. \tag{23.2.10}$$

Conversely, any central extension of $\mathrm{L}G$ gives rise to a projective representation of $\mathrm{L}G$. In particular:

23.2.11 Definition. Let G be a simple and simply connected compact Lie group. The *universal central extension* $\widetilde{\mathrm{L}G}$ of $\mathrm{L}G$ is the central extension corresponding to the generator of $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z}) \cong \mathbb{Z}$.

We first met universal central extensions in a different context, in §18.3.

The following result is key.

23.2.12 Theorem [PS86, Theorem 4.4.1]**.** *Let G be simply connected. Then there is a unique action of* $\mathrm{Diff}^+(\mathbb{T}_{\mathrm{rot}})$ *on $\widetilde{\mathrm{L}G}$ which covers the action on* $\mathrm{L}G$*. Moreover, $\widetilde{\mathrm{L}G}$ deserves to be called "universal", because there is a unique map of extensions from $\widetilde{\mathrm{L}G}$ to any other central extension of* $\mathrm{L}G$.

23.2.13 Remark. As a consequence, the action of $\mathbb{T}_{\mathrm{rot}}$ on $\mathrm{L}G$ lifts canonically to $\widetilde{\mathrm{L}G}$. Every projective unitary representation of $\mathrm{L}G$ with an intertwining action of $\mathbb{T}_{\mathrm{rot}}$ is equivalently a unitary representation of $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$. For the remainder of this talk, we will assume G is simply connected and abusively say write "representation of $\mathrm{L}G$" to mean a representation of $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$.

23.2.14 Notation. It is a little inconvenient to constantly keep writing $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$, so we will henceforth denote it by $\widetilde{\mathrm{L}G}^+$. The subgroup $\mathbb{T}_{\mathrm{rot}}$ of $\widetilde{\mathrm{L}G}^+$ is also known as the "energy circle" (for reasons to be explained below).

One of the nice properties of tori is that their representations take on a particularly simple form, thanks to the magic of Fourier series. The action of S^1 on a finite-dimensional vector space is the same data as a $\mathbb{Z}$-grading. The case of topological vector spaces is slightly more subtle: if S^1 acts on a topological vector space V, then one can consider the closed "weight" subspace V_n of V where the action of S^1 is by the character[32] $z \mapsto z^{-n}$. Then the direct sum $\bigoplus_{n\in\mathbb{Z}} V_n$ is a dense subspace of V; it is known as the subspace of *finite energy* vectors in V. This is simply the usual weight decomposition adapted to the topological setting.

23.2.15 Definition. The action of S^1 on a topological vector space V is said to satisfy the *positive energy condition* if the weight subspace $V_n = 0$ for $n < 0$. Equivalently, the action of S^1 is represented by $e^{-iA\theta}$, where A is an operator with positive spectrum.

23.2.16 Remark. The motivation for this definition comes from quantum mechanics: the wavefunction of a free particle on a circle is e^{inx} (up to normalization), and requiring that the energy (which is essentially the weight n) to be positive is mandated by physics.

[32] Some conventions are different: the action might be by $z \mapsto z^n$. We're following [PS86].

23.2.17 Definition. A representation of $\mathrm{L}G$ (which, recall, means a representation of $\widetilde{\mathrm{L}G}^+$) is said to satisfy the *positive energy condition* if it satisfies the positive energy condition when viewed as a representation of the energy/central circle $\mathbb{T}_{\mathrm{rot}}$.

23.2.18 Remark. It doesn't make sense for a representation of $\mathrm{L}G$ to be positive energy if you take "representation of $\mathrm{L}G$" to mean a literal representation of $\mathrm{L}G$; one needs to interpret that phrase as meaning a representation of $\widetilde{\mathrm{L}G}^+$.

We can now see the utility of Theorem 23.1.1: the positive energy condition involves the canonical parametrization of the circle, and to ensure that our definition would agree with that of an alien civilization's, we should ensure that the pullback f^*V of any positive energy representation V of $\mathrm{L}G$ along an orientation-preserving diffeomorphism $f \in \mathrm{Diff}^+(\mathbb{T}_{\mathrm{rot}})$ is another positive energy representation. That is precisely the content of Theorem 23.1.1.

At the beginning of this chapter, we said that positive energy representations of loop groups satisfy analogues of many properties of representations of compact Lie groups. To make that statement precise, we need to introduce some definitions that impose sanity conditions on the representations we want to study.

23.2.19 Definition. Let V be a representation of a topological group G (possibly infinite-dimensional). Then V is said to be:

- *irreducible* if it has no closed G-invariant subspace;
- *smooth* if the following condition is satisfied: let V_{sm} denote the subspace of vectors $v \in V$ such that the orbit map $G \to V$ sending g to gv is continuous; then V_{sm} is dense in V.

Two G-representations V and W are *essentially equivalent* if there is a continuous G-equivariant map $V \to W$ which is injective and has dense image.

23.2.20 Warning. Essential equivalence is *not* an equivalence relation!

The representation theory of compact Lie groups is really nice: every finite-dimensional complex representation of a compact Lie group G is semisimple (i.e. it is a direct sum of irreducible representations), and unitary, and extends to a representation of the complexification $G_{\mathbb{C}}$ of G.[33] These properties have analogues for positive energy representations of loop groups.

23.2.21 Theorem [PS86, Theorem 9.3.1]**.** *Let V be a smooth positive energy representation of* $\mathrm{L}G$*. Then up to essential equivalence:*

- *V is completely reducible into a discrete direct sum of irreducible representations,*
- *V is unitary,*
- *V extends to a holomorphic projective representation of* $\mathrm{L}(G_{\mathbb{C}})$*, and*

[33] A complexification of a real Lie group G is a complex Lie group, generally noncompact, whose Lie algebra is isomorphic to $\mathfrak{g} \otimes \mathbb{C}$. When G is compact, $G_{\mathbb{C}}$ is unique up to isomorphism.

- *V admits a projective intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*, where this* S^1 *is the energy/rotation circle. (This is Theorem 23.1.1.)*

The proof of this result takes up the bulk of the second part of Pressley–Segal.

23.2.22 Remark. The group G includes into $\mathrm{L}G$ as the subgroup of constant loops. Let G be simple and simply connected. If T is a maximal torus of G, then one has

$$\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}} \subseteq \widetilde{\mathrm{L}}G^+ .$$

Consequently, if V is a representation of $\widetilde{\mathrm{L}}G^+$, then V can be decomposed (up to essential equivalence) as a $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$-representation:

$$V = \bigoplus_{(n,\lambda,h)\in\mathbb{T}_{\mathrm{rot}}^\vee\times T^\vee\times\mathbb{T}_{\mathrm{cent}}^\vee} V_{(n,\lambda,h)} . \tag{23.2.23}$$

Here, n is the energy of V; λ is a weight of V (regarded as a representation of T); and h is a character of $\mathbb{T}_{\mathrm{cent}}$. The notation $(-)^\vee \coloneqq \mathrm{Hom}(-,\mathbb{C}^\times)$ denotes the character dual: because $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$ is a compact abelian group, its unitary representations are direct sums of one-dimensional representations. Therefore as a $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$-representation, V splits as a direct sum of one-dimensional representations, which are indexed by the character dual

$$(\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}})^\vee = \mathbb{T}_{\mathrm{rot}}^\vee \times T^\vee \times \mathbb{T}_{\mathrm{cent}}^\vee .$$

If V is irreducible, then $\mathbb{T}_{\mathrm{cent}}$ must act by scalars by Schur's lemma, and so only one value of h can occur; this is called the *level* of V. It turns out that if V is a smooth positive energy representation, then each weight space $V_{n,\lambda,h}$ is finite-dimensional. In fact, a representation of $\mathrm{L}G$ of level h is the same as a representation of $\widetilde{\mathrm{L}}G_h \rtimes \mathbb{T}_{\mathrm{rot}}$, where $\widetilde{\mathrm{L}}G_h$ is the central extension of $\mathrm{L}G$ corresponding to $h \in \mathbb{Z} \cong \mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$.

23.2.24 Remark. By Remark 23.2.22, an irreducible positive energy representation V of $\mathrm{L}G$ is uniquely determined by the level h and its lowest energy subspace V_0: the representation V is generated as a $\widetilde{\mathrm{L}}G^+$-representation by V_0.

23.2.25 Remark. Since G is simply connected, there are transgression isomorphisms

$$\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^3(G;\mathbb{Z}) \to \mathrm{H}^2(\mathrm{L}G;\mathbb{Z}) ,$$

meaning we can understand the level as (up to homotopy) a map $\mathrm{B}G \to \mathrm{K}(\mathbb{Z},4)$. This $\mathrm{K}(\mathbb{Z},4)$ is closely tied to the twisting $\mathrm{K}(\mathbb{Z},4) \to \mathrm{BGL}_1(\mathrm{tmf})$ of tmf constructed in [ABG10, Theorem 1.1]: see [And00; Gro07; BT21].

As a side note, we observe the following:

23.2.26 Proposition. *Let V be a smooth positive energy representation of* $\mathrm{L}G$*. Then V is irreducible as a representation of* $\widetilde{\mathrm{L}}G$*.*

Proof. Assume V is not irreducible as a $\widetilde{\mathrm{L}G}$-representation. Projection onto a proper $\widetilde{\mathrm{L}G}$-invariant summand defines a bounded self-adjoint operator $T : V \to V$ which commutes with $\widetilde{\mathrm{L}G}$, but (by hypothesis) not with the action of $\mathbb{T}_{\mathrm{rot}}$. Choose $R \in \mathbb{T}_{\mathrm{rot}}$; then define for each $n \in \mathbb{Z}$ the bounded operator

$$T_n = \int_{\mathbb{T}_{\mathrm{rot}}} z^n R_z T R_z^{-1} \,\mathrm{d}z \,. \tag{23.2.27}$$

T_n commutes with the action of $\widetilde{\mathrm{L}G}$, and T_n sends the weight space V_m to V_{m+n}. Because T does not commute with $\mathbb{T}_{\mathrm{rot}}$, the operator T_n must be nontrivial for at least one $n < 0$. Suppose that m is the lowest energy of V (i.e., the smallest m such that the weight space $V_m \neq 0$).[34] Then $T_n(V_m) = 0$ if $n < 0$. Since V is irreducible as a representation of $\widetilde{\mathrm{L}G}^+$, it is generated as a representation by V_m. But then $T_n(V) = 0$ for all $n < 0$. The adjoint to T_n is T_{-n}, and so $T_n(V) = 0$ for all $n \neq 0$.

This implies that T commutes with the action of $\mathbb{T}_{\mathrm{rot}}$, which is a contradiction: the T_n are the Fourier coefficients of the loop $\mathrm{S}^1 \to \mathrm{End}(V)$ sending z to $R_z T R_z^{-1}$, so we find that this loop must be constant. Consequently, T must commute with the action of $\mathbb{T}_{\mathrm{rot}}$, as desired. $\square$

23.3 A proof sketch of Theorem 23.1.1

The goal of this section is to go through the proof of Theorem 23.1.1. As with all proofs in representation theory, we may first reduce to the irreducible case, thanks to the first part of Theorem 23.2.21.

23.3.1 Observation. Recall that Schur–Weyl duality sets up a one-to-one correspondence between representations of SU_n and representations of the symmetric groups, by studying the decomposition of the tensor power $V^{\otimes d}$ of the standard representation V under the action of Σ_d.

One may hope that some analogue of Observation 23.3.1 is true for representations of loop groups: suppose we could construct a giant representation of LSU_n whose h-fold tensor product contains all the irreducible positive energy representations of level h, such that this big representation admits an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$. Then (with a little bit of work), we would obtain an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$ on all irreducible positive representations of LSU_n, which would prove Theorem 23.1.1 in this particular case. We would like to then reduce from the case of a general G to the case of SU_n. The Peter–Weyl theorem says that a simply connected G is a closed subgroup of SU_n for some n, suggesting that a technique like this might work.

Pressley–Segal's approach is similar, but not the same.

- Their base case consists not just of LSU_n, but the loop groups of all simply connected, simply laced compact Lie groups.[35] In [PS86, Lemma 13.4.4], they extend from simply

[34] Because V is positive energy, $m \geq 0$ — but that doesn't matter for now.

[35] Recall that G is simply laced if all its nonzero roots have the same length; in other words, if the Dynkin diagram of G does not have multiple edges (so the Dynkin diagram is of ADE type). The simple, simply connected, simply laced Lie groups are SU_n for all n, Spin_n for n even, E_6, E_7, and E_8.

laced groups to all simply connected Lie groups; the reason they cannot just use an embedding $j\colon G \hookrightarrow \mathrm{SU}_n$ is that, given a representation V of $\widetilde{\mathrm{L}}G$, Pressley–Segal need not just the embedding j, but also the condition that there is an irreducible representation V' of the bigger group with V a summand in j^*V'.

- Now assume G is simply connected and simply laced. Instead of constructing a huge tensor product, Pressley–Segal reduce to the case of level 1 representations in a different way. Let $m_n\colon \mathrm{L}G \to \mathrm{L}G$ be the map precomposing a loop $\mathrm{S}^1 \to G$ with the n-th-power map $\mathrm{S}^1 \to \mathrm{S}^1$. Then [PS86, Proposition 9.3.9] every irreducible representation V of $\widetilde{\mathrm{L}}G$ is contained in m_h^*F for some level 1 representation F. This allows Pressley–Segal to carry the $\mathrm{Diff}^+(\mathrm{S}^1)$-action from F to V.
- Finally, when G is simply laced and F is level 1, Pressley–Segal construct the $\mathrm{Diff}^+(\mathrm{S}^1)$-action directly using the "blip construction" [PS86, §13.2, §13.3].

23.3.2 Remark. Pressley–Segal write that "one hopes that a more satisfactory proof of Theorem 23.1.1 can be found," [PS86, p. 271], so perhaps there's a proof out there that more closely resembles the Schur–Weyl-style argument.

Now we will see how the story goes for LSU_n.

23.3.3 Construction. Let $G = \mathrm{SU}_n$. Define $H \coloneqq \mathrm{L}^2(\mathrm{S}^1, V)$, where V is the standard representation. Let $\mathrm{Har}^2(\mathrm{S}^1, V) \subseteq H$ denote the *Hardy space* of L^2-functions on S^1 with only nonnegative Fourier coefficients, and let P denote orthogonal projection of H onto $\mathrm{Har}^2(\mathrm{S}^1, V)$. Then $H = PH \oplus P^\perp H$. The *Fock space* Fock_P is the Hilbert space completion of the alternating algebra:

$$\mathrm{Fock}_P = \widehat{\Lambda}(PH \oplus \overline{P^\perp H}) \cong \widehat{\bigoplus_{i,j\geq 0}} \Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H})\,. \tag{23.3.4}$$

Here $\overline{V}$ denotes the complex conjugate vector space to V, and $\widehat{\Lambda}$ and $\widehat{\bigoplus}$ denote Hilbert space completions. The Fock space turns out to be the "giant representation" we were after: it's the fundamental representation of LSU_n.

23.3.5 Remark (the Fock space in physics)**.** The process of building a Fock space out of a Hilbert space H, as in (23.3.4), has a quantum-mechanical interpretation. Suppose that H is the space of states describing the mechanics of a particle: for example, $\mathrm{L}^2(\mathrm{S}^1, \mathbb{C})$ corresponds to a particle moving on a circle. The corresponding Fock space is the space of states for systems with any number of particles. In Construction 23.3.3, we used the alternating algebra, which means that the particles are fermions: the relation $f \wedge f = 0$ is the Pauli exclusion principle, imposing that two fermions cannot be in the same state. For a bosonic many-body system, one would use the (Hilbert space completion of the) symmetric algebra. The process of building a Fock space from a single-particle Hilbert space is called second quantization.

In our setting, $\mathrm{L}^2(\mathrm{S}^1, V)$ corresponds to a system with a fermion moving on a circle, together with some kind of G-symmetry. The subspace $\Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H})$ consists of i fermionic particles and j fermionic antiparticles. This explains why we take the conjugate space to $P^\perp H$: it is so that the antiparticles have positive energy.

A loop on G acts on H by pointwise multiplication, and $f \in \mathrm{Diff}^+(\mathrm{S}^1)$ acts on H by sending $\xi : \mathrm{S}^1 \to V$ to $\xi(f^{-1}(z)) \cdot |(f^{-1})'(z)|^{1/2}$. (The square root factor is a normalization factor to ensure unitarity of the action.) In fact, this gives an action of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on H, and one can ask when this descends to a projective representation of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on the Fock space Fock_P. Segal wrote down a *quantization condition* for when a unitary operator on H descends to a projective transformation of Fock_P: namely, u descends to Fock_P if and only if the commutator $[u, P]$ is Hilbert–Schmidt.[36] One checks that the action of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on H satisfies Segal's quantization criterion, and so descends to a projective representation of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on the Fock space Fock_P.

Almost by definition, the action of $\mathrm{S}^1 = \mathbb{T}_{\mathrm{rot}}$ on Fock_P is of positive energy, and so Fock_P is a representation of positive energy. It turns out that:

23.3.6 Theorem [PS86, Section 10.6; Was98, Chapter I.5]**.** *The irreducible summands of* $\mathrm{Fock}_P^{\otimes h}$ *give all the irreducible positive energy representations of* LSU_n *of level* h*.*

We will expand on this construction of the irreducible level h representations of LSU_n in Chapter 24, when we discuss the Segal–Sugawara construction.

The first reduction comes from:

23.3.7 Lemma [PS86, Lemma 13.4.3]**.** *Let V and W be positive energy representations of $\widetilde{\mathrm{L}}G$. Suppose that V is irreducible, and that $V \oplus W$ admits an intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*. Then V admits an intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*.*

We will prove this shortly; first, we will indicate how to use this to prove the general case.

23.3.8 Remark. It suffices to prove by Lemma 23.3.7 that for every irreducible positive energy representation V of $\mathrm{L}G$, there is some G' and an embedding $i : \mathrm{L}G \to \mathrm{L}G'$ where Theorem 23.1.1 is true for G', and an irreducible representation V' of $\mathrm{L}G'$ such that V is a summand of i^*V'.

To use this reduction, we first need to establish that Theorem 23.1.1 is true for a class of Lie groups G. In fact:

23.3.9 Theorem. *Theorem 23.1.1 is true if G is simple, simply connected, and simply laced.*

The proof of this result is quite similar to that of Theorem 23.3.6: one constructs the analogue of the Fock space for $\mathrm{L}G$ (which, like in the SU_n case, has an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$), and then shows that every irreducible positive energy representation is a summand of some twist of this representation of $\mathrm{L}G$. See [PS86, §13.4] for more details.

23.3.10 Construction. Let ΩG denote the *based* loop space of G, regarded as the homogeneous quotient $\mathrm{L}G/G \simeq \mathrm{L}G_{\mathbb{C}}/\mathrm{L}^+G_{\mathbb{C}}$. Since G is simple any simply connected,

$$\mathrm{H}^2(\Omega G; \mathbb{Z}) \cong \mathrm{H}^3(G; \mathbb{Z}) \cong \mathbb{Z} ,$$

so every integer gives rise to a complex line bundle on ΩG. The holomorphic sections Γ of the line bundle corresponding to the generator is called the *basic representation* of $\mathrm{L}G$.[37]

[36] Recall that a bounded operator A on a Hilbert space is *Hilbert–Schmidt* if $\mathrm{tr}(A^*A)$ is finite.

[37] Of course, the abelian group $\mathbb{Z}$ has two generators. Here we have a canonical one: as discussed above, we have a canonical generator for $\mathrm{H}^4(\mathrm{B}G; \mathbb{Z})$, hence $\mathrm{H}^3(G; \mathbb{Z})$ via transgression, and therefore also for $\mathrm{H}^2(\Omega G; \mathbb{Z})$.

23.3.11 Example. If $G = \mathrm{SU}_n$, then Γ is the Fock space described above.

Then:

23.3.12 Proposition [PS86, Proposition 9.3.9]**.** *Let G be a simple, simply connected, and simply laced Lie group. Then any irreducible positive energy representation of level h of $\mathrm{L}G$ is a summand in $i_h^*\Gamma$, where $i_h : \mathrm{L}G \to \mathrm{L}G$ is the map induced by the degree h map $\mathrm{S}^1 \to \mathrm{S}^1$.*

The level 1 representation Γ admits an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$ via the "blip construction." We will not go into the details here; see [PS86, §13.3]. Assuming this, combining Proposition 23.3.12 with Lemma 23.3.7 shows that Theorem 23.1.1 is true for $\mathrm{L}G$ when G is simply laced (and simple and simply connected).

According to Remark 23.3.8, it now suffices to show:

23.3.13 Proposition. *For every irreducible positive energy representation V of $\mathrm{L}G$, there is a simply laced G' and an embedding $i : \mathrm{L}G \to \mathrm{L}G'$, as well as an irreducible representation V' of $\mathrm{L}G'$ such that V is a summand of i^*V'.*

This is proved in [PS86, Lemma 13.4.4] in the following manner.

One first classifies all the irreducible representations of $\mathrm{L}G$. Using the loop group analogue of Schur–Weyl duality worked well when $G = \mathrm{SU}_n$, but that won't do in the general case. Instead, one utilizes a loop group analogue of Borel–Weil (see [Seg85, Section 4.2]). Recall how this works for finite-dimensional, compact Lie groups: fix a maximal torus T of G, and then, for every antidominant weight λ of T (i.e., $\langle h_\alpha, \lambda\rangle \leq 0$ for every positive root α), there is an associated line bundle $\mathcal{L}_\lambda$ on $G/T \cong G_\mathbb{C}/B^+$. The space of holomorphic sections of $\mathcal{L}_\lambda$ is an irreducible representation of G of lowest weight λ, and all irreducible representations of G arise this way.

In the loop group case, one again begins by fixing a maximal torus T of G (one should think of $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$ as a maximal torus of $\mathrm{L}G$). Consider the homogeneous space $\mathrm{L}G/T$. There is a fiber sequence

$$G/T \to \mathrm{L}G/T \to \Omega G\,, \tag{23.3.14}$$

and the set of isomorphism classes of complex line bundles on $\mathrm{L}G/T$ is

$$\mathrm{H}^2(\mathrm{L}G/T;\mathbb{Z}) \cong \mathrm{H}^2(\Omega G;\mathbb{Z}) \oplus \mathrm{H}^2(G/T;\mathbb{Z}) = \mathbb{Z} \oplus \widehat{T}\,, \tag{23.3.15}$$

where $\widehat{T}$ is the character group of T. You can prove this using the Serre spectral sequence, which as usual is easier because G is simple and simply connected. Anyways, we learn that line bundles on $\mathrm{L}G/T$ are indexed by $(h,\lambda) \in \mathbb{Z} \oplus \widehat{T}$.

23.3.16 Theorem (Borel–Weil for loop groups [PS86, Theorem 9.3.5])**.** *One has:*

- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ of holomorphic sections is zero or irreducible of positive energy of level h; moreover, every projective irreducible representation of $\mathrm{L}G$ arises this way.*

- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ is nonzero if and only if (h,λ) is antidominant,*[38] *i.e.,*

$$0 \geq \lambda(h_\alpha) \geq -\frac{h}{2}\langle h_\alpha, h_\alpha\rangle$$

for each positive coroot h_α of G. (In particular, λ is antidominant as a weight of $T \subseteq G$.)

The upshot is that irreducible representations correspond to antidominant weights. To prove Proposition 23.3.13, it suffices to show that all antidominant weights of $\mathrm{L}G$ are restrictions of antidominant weights of $\mathrm{L}G'$ for some simply laced G'. The argument now proceeds case-by-case, as G ranges over all simple simply connected simply laced compact Lie groups. The proof is not very enlightening, so we will not go into more detail here.

23.3.17 Remark (relationship with Wess–Zumino–Witten theory)**.** Segal [Seg04] studies the theory of positive energy representations of $\mathrm{L}G$ from a different perspective, that of conformal field theory. Specifically, the category of level h positive energy representations of $\mathrm{L}G$ has the structure of a *modular tensor category* Given a modular tensor category C, one can build

(1) a 3-dimensional topological field theory Z_{C} [RT90; RT91; Wal91; BK01; KL01; BDSV15], and

(2) a 2-dimensional conformal field theory [MS89].

These two theories are related: the 2d CFT is a boundary theory for the 3d TFT [Wit89; FT14]. When C is the category of level h representations of $\mathrm{L}G$, the TFT is Chern–Simons theory (see Remark 22.2.6) and the CFT is the Wess–Zumino–Witten model (see Remark 22.2.10).[39]

You do not need Theorem 23.1.1 to construct the modular tensor category structure on $\mathsf{Rep}_k(\mathrm{L}G)$, and the TFT and CFT provide a very large amount of data associated to that structure. It may be possible to coax Theorem 23.1.1 out of that extra structure. For example, Segal [Seg04, §12] discusses this for abelian Lie groups.

23.4 OK, but what does this have to do with differential cohomology?

There is differential cohomology hiding in the background of the story of central extensions of loop groups. There are two ways in which it appears: one which is related to the story of on-diagonal differential characteristic classes built from Chern–Weil theory, and another which relates central extensions to off-diagonal Deligne cohomology similarly to the discussion of the Virasoro group in Chapter 18. This, together with the appearance of $\mathrm{Diff}^+(S^1)$ in the representation theory of loop groups, suggests that loop groups and the Virasoro group should interact somehow, as we will see in the next chapter.

[38] Recall that if G is the simply laced group SU_n, then the weight lattice is $\bigoplus_{1\leq i\leq n+1}\mathbb{Z}\chi_i/\mathbb{Z}\sum_i \chi_i$, and the roots are $\chi_i-\chi_j$ with $i \neq j$. The positive roots, corresponding to the usual Borel subgroup of upper-triangular matrices, are $\chi_i-\chi_j$ for $i < j$. Therefore, $(h, \lambda = \lambda_1, \cdots, \lambda_n)$ is antidominant if λ is antidominant, i.e., $\lambda_1 \leq \cdots \leq \lambda_n$, and if $\lambda_n - \lambda_1 \leq h$.

[39] One might wonder if every modular tensor category arises in this way, as a category of positive-energy representations of a loop group. This is the Moore–Seiberg conjecture, and is open at the time of writing. See, e.g., [HRW08].

23.4.a The on-diagonal story

Suppose G is simple and simply connected, so that $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$, $\mathrm{H}^3(G;\mathbb{Z})$, and $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$ are all isomorphic to $\mathbb{Z}$, and the transgression maps

$$\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^3(G;\mathbb{Z}) \to \mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$$

are isomorphisms. The level h canonically refines to $\hat{h} \in \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z})$ (Theorem 14.1.1), and the transgression map refines to a map

$$\hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z}) \to \hat{\mathrm{H}}^3(G;\mathbb{Z})$$

[CJM+05, §3], as we discussed in Remark 20.3.11. Does the story continue to a differential refinement $\hat{\mathrm{H}}^3(G;\mathbb{Z}) \to \hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$? That is, a projective representation $LG \to \mathrm{PU}(V)$ determines a central extension $\widetilde{\mathrm{L}}G$ of $\mathrm{L}G$, which is a principal $\mathbb{T}$-bundle over $\mathrm{L}G$. Does this $\mathbb{T}$-bundle come with a canonical connection?

Of course, this is a loaded question, and we'll see that the answer is yes. But first, a (relatively) down-to-Earth plausibility argument. Given a central extension

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \widetilde{\mathrm{L}}G \to \mathrm{L}G \to 1\,, \tag{23.4.1a}$$

we can differentiate it to obtain a central extension of Lie algebras

$$0 \to \mathbb{R} \to \widetilde{\mathrm{L}\mathfrak{g}} \to \mathrm{L}\mathfrak{g} \to 0\,. \tag{23.4.1b}$$

Recall from Remark 18.1.6 that the central extension (23.4.1b) can be described by a cocycle for the Lie algebra cohomology group $\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{L}\mathfrak{g};\mathbb{R})$. Cocycles are alternating maps $\omega\colon \mathrm{L}\mathfrak{g}\times\mathrm{L}\mathfrak{g} \to \mathbb{R}$ satisfying the cocycle condition (18.1.7). Choose a cocycle ω; then, $\widetilde{\mathrm{L}\mathfrak{g}}$ is the vector space $\mathrm{L}\mathfrak{g}\oplus\mathbb{R}$ with the Lie bracket

$$[(\xi,a),(\eta,b)] := ([\xi,\eta],\omega(\xi,\eta))\,. \tag{23.4.2}$$

For example, an element of $\mathrm{H}^4(\mathrm{B}G;\mathbb{R})$ corresponds via the Chern–Weil machine to an invariant symmetric bilinear form $\langle -,-\rangle\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, and it defines a degree-2 Lie algebra cocycle for $\mathrm{L}\mathfrak{g}$ by [PS86, §4.2]

$$\omega(\xi,\eta) := \frac{1}{2\pi}\int_{S^1} \langle \xi(\theta), \eta'(\theta)\rangle\, \mathrm{d}\theta\,. \tag{23.4.3}$$

Suppose that ω comes from a central extension of $\mathrm{L}G$ which is a principal $\mathbb{T}$-bundle $\pi\colon \widetilde{\mathrm{L}}G \to \mathrm{L}G$. Then $\mathrm{T}\widetilde{\mathrm{L}}G$ fits into a short exact sequence

$$0 \to \mathrm{T}\mathbb{T} \to \mathrm{T}\widetilde{\mathrm{L}}G \to \pi^*\mathrm{T}\mathrm{L}G \to 0\,. \tag{23.4.4}$$

At the identity of $\widetilde{\mathrm{L}}G$ this is (23.4.1b), and left translation carries this identification to every tangent space. The data of ω includes a splitting of (23.4.1b), and left translation turns this into a splitting of (23.4.4). A connection on $\pi\colon \widetilde{\mathrm{L}G} \to \mathrm{L}G$ is a $\mathbb{T}$-invariant splitting, and since $\mathbb{T}$ acts trivially on its Lie algebra, we have just built a connection with curvature ω. Thus the class of (23.4.1b) in $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$ refines to a class in $\hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$. Pressley–Segal [PS86, Theorem 4.4.1] show that this is a necessary and sufficient condition on ω for any compact, simply connected Lie group G, and that ω determines the extension.[40]

23.4.5 Remark. It may be possible to do this "all at once" by finding a canonical connection A on the principal $\mathbb{T}$-bundle $\pi\colon\ \mathrm{U}(V) \to \mathrm{PU}(V)$ where V is an infinite-dimensional separable Hilbert space; this would lift the tautological class

$$c_1(\mathrm{U}(V)) \in \mathrm{H}^2(\mathrm{PU}(V);\mathbb{Z}) = \mathrm{H}^2(\mathrm{K}(\mathbb{Z},2);\mathbb{Z})$$

to

$$\hat{c}_1(\mathrm{U}(V),A) \in \hat{\mathrm{H}}^2(\mathrm{PU}(V);\mathbb{Z})\,.$$

Then a projective representation would pull back $\hat{c}_1(\mathrm{U}(V),A)$ (and A) to $\mathrm{L}G$.

To summarize a little differently, given $\hat{h} \in \hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z})$, we can obtain a Chern–Weil form $\langle -,-\rangle$, hence a cocycle $\omega \in \mathrm{H}^2_{\mathrm{Lie}}(\mathrm{L}\mathfrak{g};\mathbb{R})$. Because $\mathrm{curv}(\hat{h})$ satisfies an integrality condition, so does ω, which turns out to be the same condition needed to define a central extension $\widetilde{\mathrm{L}G} \to \mathrm{L}G$ with a connection. That is, we built a map $\hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z}) \to \hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$. We would like to describe it more directly.

The first step is the transgression map $\hat{\mathrm{H}}^4(\mathrm{Bun}_G^\nabla;\mathbb{Z}) \to \hat{\mathrm{H}}^3(\mathrm{B}G;\mathbb{Z})$ constructed by [CJM+05, §3]. To get from 3 to 2, Gawędzki [Gaw88, §3] constructs for any closed manifold M a transgression map

$$\hat{\mathrm{H}}^3(M;\mathbb{Z}) \to \hat{\mathrm{H}}^2(\mathrm{L}M;\mathbb{Z}) \tag{23.4.6}$$

from the perspective that differential cohomology is isomorphic to the hypercohomology of the Deligne complex[41]

$$0 \to \mathbb{Z} \to \Omega^0 \to \cdots \to \Omega^{n-1} \to 0\,.$$

Another option is to construct the transgression as follows: first pull back by the evaluation map $\mathrm{S}^1 \times \mathrm{L}M \to M$, then integrate over the S^1 factor using the map we constructed in Chapter 9.

23.4.b The off-diagonal story

In Chapter 18, we saw in Corollary 18.3.2 that central extensions of a Lie group Γ (possibly infinite-dimensional) which are principal $\mathbb{T}$-bundles are classified by $\mathrm{H}^3(\mathrm{Bun}_\Gamma;\mathbb{Z}(1))$. The central extensions of loop groups we constructed in this chapter are principal $\mathbb{T}$-bundles. There-

[40] When G is not simply connected, the theorem is not quite as nice: see [PS86, Theorem 4.6.9] and [Wal17].

[41] Gawędzki actually works with a different complex, namely $0 \to \mathbb{T} \to i\Omega^1 \to \cdots \to i\Omega^{n-1} \to 0$, where the map $\mathbb{T} \to i\Omega^1$ is $\mathrm{d}\circ\log$. This is equivalent to $\Sigma\mathbb{Z}(n)$ [BM94, Remark 3.6], and the proof is a straightforward generalization of Lemma 18.3.1.

fore there is in principle a way to start with a class $h \in \mathrm{H}^4(BG;\mathbb{Z})$ and obtain a class $\phi(h) \in \mathrm{H}^3(\mathrm{Bun}_{LG};\mathbb{Z}(1))$, and that is what we are going to do next.

Recall that truncating defines a map of complexes of sheaves of abelian groups $\mathbb{Z}(n) \to \mathbb{Z}$, inducing for us a map

$$\mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}(2)) \to \mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(BG;\mathbb{Z})\,. \tag{23.4.7}$$

23.4.8 Lemma. *For G a compact Lie group,* (23.4.7) *is an isomorphism.*

Proof. Recall from Corollary 17.2.5 that (23.4.7) is part of the pullback square

$$\begin{tikzcd} \mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}(2)) \arrow[r, "(23.4.7)"] \arrow[d] & \mathrm{H}^4(BG;\mathbb{Z}) \arrow[d] \\ \mathrm{Sym}^2(\mathfrak{g}^\vee)^G \arrow[r] & \mathrm{H}^4(BG;\mathbb{R}), \end{tikzcd} \tag{23.4.9}$$

where the bottom map is the Chern–Weil map. Since G is compact, the Chern–Weil map is an isomorphism, so (23.4.7) is as well. ☐

Therefore our level $h \in \mathrm{H}^4(BG;\mathbb{Z})$ is equivalent data to an off-diagonal characteristic class $\tilde{h} \in \mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}(2))$. The next step is the construction of yet another transgression map, this time due to Brylinski–McLaughlin [BM94, §5, on p. 618]:

$$\mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3(\mathrm{Bun}_{LG};\mathbb{Z}(1))\,. \tag{23.4.10}$$

Their construction models elements of these two differential cohomology groups simplicially: they identify $\mathrm{H}^4(\mathrm{Bun}_G;\mathbb{Z}(2))$ as the abelian group of equivalence classes of gerbes with a connective structure over a simplicial manifold model for Bun_G, and $\mathrm{H}^3(\mathrm{Bun}_{LG};\mathbb{Z}(1))$ as equivalence classes of line bundles over a simplicial model for Bun_{LG} (*ibid.*, Theorem 5.7).

We have obtained some class in $\mathrm{H}^3(\mathrm{Bun}_{LG};\mathbb{Z}(1))$ from a level $h \in \mathrm{H}^4(BG;\mathbb{Z})$, hence some central extension. That this coincides with the central extension obtained from h by the other methods in this chapter is due to Brylinski–McLaughlin (*ibid.*, §5). See also Brylinski [Bry08, §6.5] for related discussion and Waldorf [Wal10, §3.1] for another construction of this transgression map.

24 The Segal–Sugawara construction

by Peter Haine

Let G be a simply connected, simple, compact Lie group with Lie algebra $\mathfrak{g}$. In Chapter 23, we looked at central extensions

$$1 \longrightarrow \mathrm{S}^1 \longrightarrow \widetilde{\mathrm{L}}G \longrightarrow \mathrm{L}G \longrightarrow 1$$

of the *loop group* $\mathrm{L}G \coloneqq \mathrm{C}^\infty(\mathrm{S}^1, G)$. The group $\mathrm{Diff}^+(\mathrm{S}^1)$ of orientation-preserving diffeomorphisms of the circle acts on $\mathrm{L}G$ by precomposition. So we might expect an action of the Virasoro group $\widetilde{\mathrm{Diff}}^+(\mathrm{S}^1)$ on $\widetilde{\mathrm{L}}G$. We saw that even though there is not an action of $\widetilde{\mathrm{Diff}}^+(\mathrm{S}^1)$ on $\widetilde{\mathrm{L}}G$, roughly, the Virasoro group acts on any *positive energy* representation of $\widetilde{\mathrm{L}}G$. However, the Virasoro action on positive energy representations of $\widetilde{\mathrm{L}}G$ is very inexplicit, and we can only guarantee the existence of the Virasoro action up to "essential equivalence," which is not actually an equivalence relation. In particular, the Pressley–Segal Theorem [PS86, Theorem 13.4.3] (Theorem 23.1.1) does not explicitly explain how the central circle $\mathrm{S}^1 \subset \widetilde{\mathrm{L}}G$ acts.

The goal of this chapter is to explain the Lie algebra version of the Pressley–Segal Theorem, which gives an explicit representation of the Virasoro algebra on any positive energy representation of the *Kac–Moody algebra* $\widetilde{\mathrm{L}}\mathfrak{g}$ associated to a simple Lie algebra $\mathfrak{g}$ (over the complex numbers). We'll be able to do this by writing down explicit universal formulas for "elements" of the universal enveloping algebra $\mathcal{U}(\widetilde{\mathrm{L}}\mathfrak{g})$ that satisfy the Virasoro relations. The catch is that these universal formulas involve infinite sums, so they do not actually make sense as elements of $\mathcal{U}(\widetilde{\mathrm{L}}\mathfrak{g})$, but they do make sense whenever we act on a representation where only finitely many of the terms don't act by zero; this is what the positive energy condition guarantees.

Like in the previous chapter, we are not assuming you're familiar with all of these words. In § 24.1, we review some important definitions from Chapter 18. In § 24.2, we define the loop algebra of a Lie algebra, which up to regularity issues is the Lie-algebraic analogue of the loop group of a Lie group. We also introduce *Kac–Moody algebras*, the analogues of the central extensions of loop groups we constructed in § 23.2. In § 24.3, we introduce the Segal–Sugawara construction, first at a high level, then digging into the details.

24.1 Reminders on Virasoro & Witt algebras

24.1.1 Definition. The (complex) *Witt algebra* is the complex Lie algebra $\mathrm{Witt}_\mathbb{C}$ of polynomial vector fields on S^1. Explicitly, $\mathrm{Witt}_\mathbb{C}$ has generators $L_m \coloneqq ie^{im\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}$ for $m \in \mathbb{Z}$ with Lie bracket

$$[L_m, L_n] \coloneqq (m-n)L_{m+n}$$

for all $m, n \in \mathbb{Z}$.

This is the complexification of the Witt algebra we discussed in Definition 18.2.1.

24.1.2. Ignoring regularity issues, the Witt algebra is the complexification of the Lie algebra of the group $\mathrm{Diff}^+(\mathrm{S}^1)$ of orientation-preserving diffeomorphisms of the circle.[42]

24.1.3. Recall from Remark 18.1.6 that central extensions of Lie algebras are classified by Lie algebra cohomology. We have that

$$\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_{\mathbb{C}};\mathbb{C}) \cong \mathbb{C} ,$$

so there is a 1-dimensional space of central extensions of the Witt algebra.

24.1.4 Definition. The (complex) *Virasoro algebra* $\mathrm{Vir}_{\mathbb{C}}$ is the central extension

$$1 \longrightarrow \mathbb{C}\,\mathrm{chg} \longrightarrow \mathrm{Vir}_{\mathbb{C}} \longrightarrow \mathrm{Witt}_{\mathbb{C}} \longrightarrow 1 \tag{24.1.5}$$

of $\mathrm{Witt}_{\mathbb{C}}$ with generators L_m for $m \in \mathbb{Z}$ and a central element chg, and nontrivial Lie bracket given by

$$[L_m, L_n] := (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}\,\mathrm{chg} \tag{24.1.6}$$

for all $m, n \in \mathbb{Z}$.

We call the central element $\mathrm{chg} \in \mathrm{Vir}_{\mathbb{C}}$ the *central charge*.

Said a little differently, (24.1.6) spells out a cocycle for $\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{Witt}_{\mathbb{C}};\mathbb{C})$, which determines the central extension (24.1.5).

24.1.7. Again, ignoring regularity issues, the Virasoro algebra is the complexification of the Lie algebra of the Virasoro group $\widetilde{\mathrm{Diff}}^+(\mathrm{S}^1)$.

24.2 Loop algebras and Kac–Moody algebras

The first thing we need to explain in order to state the Segal–Sugawara construction is what the Kac–Moody algebra $\widetilde{\mathrm{L}}\mathfrak{g}$ is. As the notation suggests, $\widetilde{\mathrm{L}}\mathfrak{g}$ is the Lie algebra analog of the central extension $\widetilde{\mathrm{L}}G$ of the loop group $\mathrm{L}G$ (with suitable finiteness hypotheses). Before talking about Kac–Moody algebras, we need to talk about loop algebras.

24.2.a Loop algebras

24.2.1 Recollection. Let $\mathfrak{g}$ be a Lie algebra over a ring R, and let S be an R-algebra. The basechange $\mathfrak{g} \otimes_R S$ of $\mathfrak{g}$ to S is the Lie algebra over S with underlying S-module the basechange $\mathfrak{g} \otimes_R S$ of the underlying R-module of $\mathfrak{g}$ to S with Lie bracket extended from pure tensors from the formula

$$[X_1 \otimes s_1, X_2 \otimes s_2]_{\mathfrak{g}\otimes_R S} := [X_1, X_2]_{\mathfrak{g}} \otimes s_1 s_2 .$$

[42]For the readers who care about regularity: the Lie algebra of $\mathrm{Diff}^+(\mathrm{S}^1)$ is the Lie algebra of all smooth vector fields on S^1, and $\mathrm{Witt}_{\mathbb{C}}$ is a dense subset of the complexification. See [PS86, §3.3; Ano20].

24.2.2 Definition. Let $\mathfrak{g}$ be a complex Lie algebra. The *loop algebra* $\mathrm{L}\mathfrak{g}$ of $\mathfrak{g}$ is the Lie algebra

$$\mathrm{L}\mathfrak{g} \coloneqq \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \, ,$$

regarded as a Lie algebra over $\mathbb{C}$ (rather than $\mathbb{C}[t^{\pm 1}]$).

24.2.3 Notation. Let $\mathfrak{g}$ be a complex Lie algebra, $X \in \mathfrak{g}$, and m an integer. We write

$$X\langle m \rangle \coloneqq X \otimes t^m \in \mathrm{L}\mathfrak{g} \ .$$

24.2.4. If $\{u_i\}_{i \in I}$ is a Lie algebra basis for $\mathfrak{g}$, then $\{u_i\langle m\rangle\}_{(i,m)\in I\times\mathbb{Z}}$ is a basis for $\mathrm{L}\mathfrak{g}$.

24.2.5 Remark. The loop algebra functor $\mathrm{L} \colon \mathsf{Lie}_{\mathbb{C}} \to \mathsf{Lie}_{\mathbb{C}}$ preserves finite products.

24.2.6 Recollection. A finite dimensional Lie algebra $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and 0.

24.2.7 Theorem (Garland [Gar80, §§1 & 2])**.** *If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$, then*

$$\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{L}\mathfrak{g};\mathbb{C}) \cong \mathbb{C} \, .$$

In particular, if $\mathfrak{g}$ is simple there is a 1-dimensional space of central extensions of $\mathrm{L}\mathfrak{g}$.

24.2.b Recollection on bilinear forms & semisimplicity

24.2.8 Notation. Let $\mathfrak{g}$ be a complex Lie algebra. We write $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ for the adjoint representation, defined by

$$\mathrm{ad}(X) \coloneqq [X, -] \, .$$

24.2.9 Example. A Lie algebra $\mathfrak{g}$ is abelian if and only if the adjoint representation of $\mathfrak{g}$ is trivial.

24.2.10 Recollection (Killing form)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The *Killing form* on $\mathfrak{g}$ is the bilinear form

$$\begin{aligned} \mathrm{Kil}_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{C} \\ (X, Y) &\mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \, . \end{aligned}$$

The Killing form is symmetric and *invariant* in the sense that

$$\mathrm{Kil}_{\mathfrak{g}}([X,Y],Z) = \mathrm{Kil}_{\mathfrak{g}}(X,[Y,Z])$$

for all $X, Y, Z \in \mathfrak{g}$.

24.2.11 Example. If $\mathfrak{g}$ is a simple Lie algebra, then every invariant symmetric bilinear form on $\mathfrak{g}$ is a $\mathbb{C}$-multiple of the Killing form $\mathrm{Kil}_{\mathfrak{g}}$. See [Cés13] for a nice exposition of this fact. It is also related to Chern–Weil theory, which tells us that the space of invariant symmetric bilinear forms is isomorphic to $\mathrm{H}^4(\mathrm{B}G;\mathbb{R})$, and when G is a compact, simple, simply connected Lie group,

$\mathrm{H}^4(\mathrm{B}G;\mathbb{R}) \cong \mathbb{R}$. This is because $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \cong \mathbb{Z}$, which we have discussed and used in previous chapters.

24.2.12 Example. Let $\mathfrak{a}$ be a finite-dimensional abelian Lie algebra over $\mathbb{C}$. Since the adjoint representation of $\mathfrak{a}$ is trivial, the Killing form of $\mathfrak{a}$ is identically zero. Also note that every bilinear form on the underlying vector space of $\mathfrak{a}$ is an invariant bilinear form on $\mathfrak{a}$.

24.2.13 Proposition [Ser01, Chapter II, Theorems 2 & 4]**.** *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. The following conditions are equivalent:*

(24.2.13.1) *The center of $\mathfrak{g}$ is trivial.*

(24.2.13.2) *The only abelian ideal in $\mathfrak{g}$ is* 0*.*

(24.2.13.3) *The Lie algebra $\mathfrak{g}$ is isomorphic to a product of simple Lie algebras.*

(24.2.13.4) Cartan–Killing criterion: *the Killing form of $\mathfrak{g}$ is nondegenerate.*

24.2.14 Definition. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. If the equivalent conditions (24.2.13.1)–(24.2.13.4) are satisfied, we say that $\mathfrak{g}$ is *semisimple*.

24.2.c Kac–Moody algebras

Now we define the Lie algebra analogue of the central extensions $\widetilde{\mathrm{L}}G$ of the loop group $\mathrm{L}G$ that we studied in Chapter 23. Those central extensions were parametrized by an element of $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$, and these similarly require the additional data of an invariant symmetric bilinear form on $\mathfrak{g}$, i.e. an element of $\mathrm{H}^4(\mathrm{B}G;\mathbb{R})$. The Killing form provides a canonical choice. The forms not in the image of $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^4(\mathrm{B}G;\mathbb{R})$ correspond to loop algebra central extensions which do not lift to loop groups.

24.2.15 Definition [Kac68; Moo68]**.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with invariant symmetric bilinear form $B\colon\ \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$. The *Kac–Moody algebra* of $\mathfrak{g}$ with respect to the form B is the central extension

$$1 \longrightarrow \mathbb{C}c \longrightarrow \widetilde{\mathrm{L}}_B\mathfrak{g} \longrightarrow \mathrm{L}\mathfrak{g} \longrightarrow 1$$

with central element c and with Lie bracket extended from the relation

$$\begin{aligned}[][X\langle m\rangle, Y\langle n\rangle]_{\widetilde{\mathrm{L}}_B\mathfrak{g}} &\coloneqq [X\langle m\rangle, Y\langle n\rangle]_{\mathrm{L}\mathfrak{g}} + \delta_{m,-n} mB(X,Y)c\\ &= [X,Y]_{\mathfrak{g}}\langle m+n\rangle + \delta_{m,-n} mB(X,Y)c\end{aligned}$$

for all $X, Y \in \mathfrak{g}$.

24.2.16. If $\{u_i\}_{i\in I}$ is a Lie algebra basis for $\mathfrak{g}$, then $\{u_i\langle m\rangle\}_{(i,m)\in I\times\mathbb{Z}} \cup \{c\}$ is a basis for $\widetilde{\mathrm{L}}_B\mathfrak{g}$.

24.2.17 Remark. The Kac–Moody algebra $\widetilde{\mathrm{L}}\mathfrak{g}$ is usually denoted by $\hat{\mathfrak{g}}$ and is also known as the *affine Lie algebra* of $\mathfrak{g}$.

24.2.18 Remark. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be complex Lie algebras equipped with invariant symmetric bilinear forms

$$B_1 \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C} \qquad \text{and} \qquad B_2 \colon \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathbb{C} .$$

Write B for the bilinear form on the product Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$ defined by

$$B((x_1, x_2), (y_1, y_2)) \coloneqq B_1(x_1, y_1) + B(x_2, y_2) .$$

Then we have a canonical isomorphism

$$\widetilde{\mathrm{L}}_B(\mathfrak{g}_1 \times \mathfrak{g}_2) \cong \widetilde{\mathrm{L}}_{B_1}\mathfrak{g}_1 \times \widetilde{\mathrm{L}}_{B_2}\mathfrak{g}_2 .$$

24.3 The Segal–Sugawara construction

We now have enough of of the background on Lie algebras to give a vague statement of the Segal–Sugawara construction.

24.3.1 Definition. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and B an invariant symmetric bilinear form on $\mathfrak{g}$. A representation $\rho \colon \widetilde{\mathrm{L}}_B\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ has *positive energy* if for all $v \in V$ and $X \in \mathfrak{g}$ there exists an integer $m > 0$ such that

$$\rho(X\langle m\rangle)v = 0 .$$

24.3.2 Remark. In the theory of Kac–Moody algebras, positive energy representations are more often called *admissible*. We have chosen the term “positive energy” to align with the loop group terminology (see Definition 23.2.17).

24.3.3 Theorem (Segal–Sugawara construction, vague formulation)**.** *Let $\mathfrak{g}$ be an abelian or simple Lie algebra over $\mathbb{C}$ and let $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Write $\mathrm{Cas}_B(\mathfrak{g}) \in \mathcal{U}(\mathfrak{g})$ for the Casimir element of $\mathfrak{g}$ with respect to the bilinear form B. Let*

$$\rho \colon \widetilde{\mathrm{L}}_B\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$$

be a positive energy representation of $\widetilde{\mathrm{L}}_B\mathfrak{g}$ such that

(24.3.3.1) *the central element $c \in \widetilde{\mathrm{L}}_B\mathfrak{g}$ acts by multiplication by a complex number ℓ,*

(24.3.3.2) *and the complex number $-\ell$ is not equal to*

$$\lambda_B(G) \coloneqq \frac{\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g})))}{2\dim(\mathfrak{g})} .$$

Then there is an explicit action of the Virasoro algebra on V where the central charge $\mathrm{chg} \in \mathrm{Vir}_{\mathbb{C}}$ acts by multiplication by

$$\frac{\ell \dim(\mathfrak{g})}{\ell + \lambda_B(\mathfrak{g})} .$$

As special cases:

(24.3.3.3) *If $\mathfrak{g}$ is abelian, then $\lambda_B(\mathfrak{g}) = 0$ for any nondegenerate invariant symmetric bilinear form B, and the central charge* $\mathrm{chg} \in \mathrm{Vir}_{\mathbb{C}}$ *acts by multiplication by* $\dim(\mathfrak{g})$.

(24.3.3.4) *If $\mathfrak{g}$ is simple and B is the normalization of the Killing form such that the long roots of $\mathfrak{g}$ have square length 2, then $\lambda_B(\mathfrak{g})$ is a positive integer known as the* dual Coxeter number *of $\mathfrak{g}$.*

24.3.4. The complex number ℓ in Theorem 24.3.3 is known as the *level* of the positive energy representation ρ.

24.3.5 Goal. The goal for the rest of the talk is to explain this construction, the Casimir element $\mathrm{Cas}_B(\mathfrak{g})$, and give a better description of the normalized trace $\lambda_B(\mathfrak{g})$ as an eigenvalue of $\mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g}))$.

24.3.a Motivating case: the Heisenberg algebra

As motivation for the Segal–Sugawara construction, we start with the most simple case, where $\mathfrak{g}$ is the 1-dimensional abelian Lie algebra. Since the constant $\lambda_B(\mathfrak{g})$ will be zero in this case, we can do this without yet introducing the Casimir element.

24.3.6 Definition. The *Heisenberg algebra* is the Kac–Moody algebra

$$\mathrm{Heis} \coloneqq \widetilde{\mathrm{L}}\mathbb{C}$$

of the 1-dimensional abelian Lie algebra $\mathbb{C}$ with respect to the bilinear form $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by multiplication.

24.3.7. Write $u \in \mathbb{C}$ for the element 1, which we regard as a basis for $\mathbb{C}$ as a 1-dimensional abelian Lie algebra. Then the Heisenberg algebra has generators $\{c\} \cup \{u\langle m\rangle\}_{m\in\mathbb{Z}}$, where c is central and the nontrivial bracket relation is given by

$$[u\langle m\rangle, u\langle n\rangle] \coloneqq \delta_{m,-n} mc\ .$$

24.3.8 Definition. Let $\mu, \hbar \in \mathbb{C}$. Write $u \in \mathbb{C}$ for the element 1, which we regard as a basis for $\mathbb{C}$ as a 1-dimensional abelian Lie algebra. The *Fock representation* $\mathrm{Fock}(\mu, \hbar)$ is the representation of the Heisenberg algebra on the polynomial ring

$$\mathrm{Fock}(\mu, \hbar) \coloneqq \mathbb{C}[x_1, x_2, \ldots]$$

in infinitely many variables, where

$$c \mapsto \hbar \,\mathrm{id}$$

$$u\langle n\rangle \mapsto \begin{cases} \frac{\partial}{\partial x_n}\,, & n > 0 \\ -\hbar x_{-n}\,, & n < 0 \\ \mu \,\mathrm{id}\,, & n = 0\,. \end{cases}$$

The following fact about the irreducibility of Fock representations is easy:

24.3.9 Lemma [KR87, Lemma 2.1]**.** *Let $\mu, \hbar \in \mathbb{C}$. If $\hbar \neq 0$, then the* Heis*-representation* Fock$(\mu, \hbar)$ *is irreducible.*

24.3.10. If $\hbar = 0$, then the constants $\mathbb{C} \subset \mathrm{Fock}(\mu, 0)$ are invariant.

24.3.11 Properties. The following are some important properties of the Fock representations of the Heisenberg algebra.

(24.3.11.1) The elements $u(0)$ and c of Heis act by multiplication.

(24.3.11.2) For every polynomial $p \in \mathrm{Fock}(\mu, \hbar)$, there exists an integer $n \gg 0$ such that $u\langle n\rangle p = 0$: let n be any positive such that the variable x_n does not appear in p. That is, the Fock representation Fock$(\mu, \hbar)$ is “positive energy” in the sense of Definition 24.3.1.

(24.3.11.3) For each integer $n > 0$, the element $u\langle n\rangle \in$ Heis acts locally nilpotently on Fock$(\mu, \hbar)$.

Now we can give the Segal–Sugawara construction for the Fock representations of the Heisenberg algebra.

24.3.12 Construction (Virasoro action of Fock representations)**.** For each integer $m \in \mathbb{Z}$, define an infinite sum of elements of $\mathcal{U}(\mathrm{Heis})$ by

$$L_m^S \coloneqq \frac{1}{2} \sum_{j \in \mathbb{Z}} {:}u\langle -j\rangle u\langle j+m\rangle{:}\ \ .$$

Here, ${:}u\langle -j\rangle u\langle j+m\rangle{:}$ denotes the *normal ordering* on $u\langle -j\rangle u\langle j+m\rangle$, defined by

$${:}u\langle -j\rangle u\langle j+m\rangle{:} \coloneqq \begin{cases} u\langle -j\rangle u\langle j+m\rangle\,, & -j \leq j+m \\ u\langle j+m\rangle u\langle -j\rangle\,, & -j \geq j+m\,. \end{cases}$$

Explicitly,

$$L_m^S = \begin{cases} \frac{1}{2}u\langle n\rangle^2 + \sum_{j>0} u\langle n-j\rangle u\langle n+j\rangle\,, & m = 2n \\ \sum_{j>0} u\langle n+1-j\rangle u\langle n+j\rangle\,, & m = 2n+1\,. \end{cases}$$

The operators L_m^S are not well-defined elements of $\mathcal{U}(\mathrm{Heis})$, but since the Fock representations of Heis are positive energy (24.3.11.2), the operators L_m^S make sense as operators on $\mathrm{Fock}(\mu, \hbar)$.

24.3.13 Theorem (Segal–Sugawara for $\mathrm{Fock}(\mu, 1)$ [KR87, Proposition 2.3])**.** *Under the representation of* Heis *on the Fock space* $\mathrm{Fock}(\mu, 1)$*, the operators* L_m^S *on* $\mathrm{Fock}(\mu, 1)$ *satisfy the commutation relation*

$$[L_m^S, L_n^S] = (m-n)L_{m+n}^S + \delta_{m,-n}\frac{m^3-m}{12}\,.$$

Hence the assignment

$$\begin{aligned} \mathrm{Vir}_{\mathbb{C}} &\to \mathrm{End}_{\mathbb{C}}(\mathrm{Fock}(\mu, 1)) \\ L_m &\mapsto L_m^S \\ \mathrm{chg} &\mapsto \mathrm{id} \end{aligned}$$

is a $\mathrm{Vir}_{\mathbb{C}}$*-representation with central charge* 1.

24.3.14 Remark. To derive the Segal–Sugawara action on $\mathrm{Fock}(\mu, \hbar)$ for $\hbar \neq 0$, let L_m act by $\frac{1}{\hbar}L_m^S$.

24.3.15 Remark. Gordon's notes [Gor09] give a nice exposition of the Segal–Sugawara construction for Fock representations and the representation theory of the Virasoro algebra.

24.3.b The Casimir element

In the general case, the idea is to try to mimic the formulas that we wrote down defining the operators on the Fock representations that satisfy the Virasoro relations. First, we need to explain the "Casimir element" and normalized trace $\lambda_B(\mathfrak{g})$ appearing in Theorem 24.3.3.

24.3.16 Definition. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$ and let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. The *Casimir element* $\mathrm{Cas}_B(\mathfrak{g})$ of $\mathfrak{g}$ with respect to the form B is the element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ given by the image of $\mathrm{id}_{\mathfrak{g}}$ under the composite

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}^{\vee} \xrightarrow{\sim} \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g} \hookrightarrow \mathrm{T}_{\mathbb{C}}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g})\,.$$

Here the isomorphism $\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}^{\vee} \xrightarrow{\sim} \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}$ is the identity on the first factor and the isomorphism $\mathfrak{g}^{\vee} \xrightarrow{\sim} \mathfrak{g}$ induced by the form B on the second factor, and $\mathrm{T}_{\mathbb{C}}(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ over $\mathbb{C}$.

The following are some key properties that we need to know about the Casimir element:

24.3.17 Properties.

(24.3.17.1) The Casimir element $\mathrm{Cas}_B(\mathfrak{g})$ is a central element of $\mathcal{U}(\mathfrak{g})$.

(24.3.17.2) If $\{u_1, \dots, u_d\}$ and $\{u^1, \dots, u^d\}$ are bases of $\mathfrak{g}$ that are dual with respect to the bilinear form B in the sense that $B(u_i, u^j) = \delta_{i,j}$, then

$$\mathrm{Cas}_B(\mathfrak{g}) = \sum_{i=1}^{d} u_i u^i .$$

(24.3.17.3) Assume that $\mathfrak{g}$ is simple. Then the Casimir element of the Killing form of $\mathfrak{g}$ acts by the identity in the adjoint representation. Hence for any nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$, the Casimir element $\mathrm{Cas}_B(\mathfrak{g})$ acts by scalar multiplication in the adjoint representation of $\mathfrak{g}$. If B is the normalization of the Killing form on $\mathfrak{g}$ such that long roots have square length 2, then in the adjoint representation $\mathrm{Cas}_B(\mathfrak{g})$ acts by multiplication by an even positive integer.

(24.3.17.4) If $\mathfrak{g}$ is abelian, then since the adjoint representation of $\mathfrak{g}$ is trivial, for any nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$ we have $\mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g})) = 0$. In particular, in the adjoint representation $\mathrm{Cas}_B(\mathfrak{g})$ acts by scalar multiplication.

Even though there are no Lie algebras that are both abelian and simple, it is important for us that both types of Lie algebras have the property that the Casimir element associated to any nondegenerate invariant symmetric bilinear form acts by scalar multiplication in the adjoint representation. In particular, if $\mathfrak{g}$ is abelian or simple, then $\mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g}))$ only has exactly one eigenvalue.

24.3.18 Definition. Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Define a complex number $\lambda_B(\mathfrak{g})$ by

$$\lambda_B(\mathfrak{g}) \coloneqq \frac{1}{2}\Big(\text{eigenvalue of } \mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g}))\Big) .$$

24.3.19. If $\dim(\mathfrak{g}) > 0$, then

$$\lambda_B(\mathfrak{g}) = \frac{\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_B(\mathfrak{g})))}{2\dim(\mathfrak{g})} ,$$

which aligns with the vague formulation of the Segal–Sugawara construction (Theorem 24.3.3).

24.3.20 Example. If $\mathfrak{g}$ is simple and B is the normalization of the Killing form on $\mathfrak{g}$ such that long roots have square length 2, then $\lambda_B(\mathfrak{g})$ is a positive integer (24.3.17.3) known as the *dual Coxeter number* of $\mathfrak{g}$.

24.3.21 Example. If $\mathfrak{a}$ is an abelian Lie algebra, then for any nondegenerate invariant symmetric bilinear form B on $\mathfrak{a}$, we have $\lambda_B(\mathfrak{a}) = 0$.

24.3.c The general case

Now let us try using "the same" formula to write down a Virasoro action on positive energy representations of $\widetilde{\mathrm{L}}\mathfrak{g}$ as we did for the Heisenberg algebra. The first modification is that we need to sum over a basis of $\mathfrak{g}$.

24.3.22 Construction. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$ and let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Given a positive energy representation $\rho \colon \widetilde{\mathrm{L}}_B\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$, for each integer $m \in \mathbb{Z}$ define

$$T_m^{\rho} \coloneqq \frac{1}{2}\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}} :\rho(u_i\langle -j\rangle)\rho(u^i\langle j+m\rangle): \in \mathrm{End}_{\mathbb{C}}(V)\,.$$

Note that even though the formula defining T_m^{ρ} involves an infinite sum, since ρ is a positive energy representation, for each $v \in V$, all but finitely many terms in the sum defining T_m^{ρ} annihilate v. Hence T_m^{ρ} is well-defined as an element of $\mathrm{End}_{\mathbb{C}}(V)$.

We used the letter "T" instead of "L" because the commutation relation is not quite right:

24.3.23 Lemma [KR87, Theorem 10.1]**.** *Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. For every positive energy representation $\rho \colon \widetilde{\mathrm{L}}_B\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$, we have the following commutation relation in* $\mathrm{End}_{\mathbb{C}}(V)$*:*

$$\begin{aligned}[T_m^{\rho}, T_n^{\rho}] = (\rho(c) + \lambda_B(\mathfrak{g}))(m-n)T_{m+n}^{\rho} \\ + \delta_{m,-n}\dim(\mathfrak{g})\frac{m^3-m}{12}\rho(c)(\rho(c)+\lambda_B(\mathfrak{g}))\,.\end{aligned}$$

24.3.24 Idea. The naive guess that the operators T_m^{ρ} satisfy the Virasoro relations is not correct. However, if we could invert $\rho(c) + \lambda_B(\mathfrak{g})$, then the operators

$$\frac{1}{\rho(c)+\lambda_B(\mathfrak{g})}T_m^{\rho}$$

would satisfy the Virasoro relations. We can do this provided that the central element $c \in \widetilde{\mathrm{L}}_B\mathfrak{g}$ acts by a scalar ℓ on V, and $\ell \neq -\lambda_B(\mathfrak{g})$.

24.3.25 Theorem (Segal–Sugawara construction [KR87, Corollary 10.1])**.** *Let $\mathfrak{g}$ be a finite dimensional abelian or simple Lie algebra over $\mathbb{C}$ and let $B \colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ be a nondegenerate invariant symmetric bilinear form. Let*

$$\rho \colon \widetilde{\mathrm{L}}_B\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$$

be a positive energy representation of $\widetilde{\mathrm{L}}_B\mathfrak{g}$ such that

(24.3.25.1) *the central element $c \in \widetilde{\mathrm{L}}_B\mathfrak{g}$ acts by multiplication by a complex number ℓ,*

(24.3.25.2) *and $\ell \neq -\lambda_B(\mathfrak{g})$.*

Choose bases $\{u_1, \dots, u_d\}$ and $\{u^1, \dots, u^d\}$ of $\mathfrak{g}$ that are dual with respect to the bilinear form B.

Then the assignment

$$L_m \mapsto L_m^\rho := \frac{1}{2(\ell + \lambda_B(\mathfrak{g}))} \sum_{i=1}^{d} \sum_{j \in \mathbb{Z}} :\rho(u_i\langle -j \rangle)\rho(u^i\langle j+m \rangle):$$

extends to a $\mathrm{Vir}_{\mathbb{C}}$*-representation on V with central charge*

$$\frac{\ell \dim(\mathfrak{g})}{\ell + \lambda_B(\mathfrak{g})} .$$

That is, in $\mathrm{End}_{\mathbb{C}}(V)$*, the operators L_m^ρ satisfy the commutation relation*

$$[L_m^\rho, L_n^\rho] = (m-n)L_{m+n}^\rho + \delta_{m,-n} \frac{m^3 - m}{12} \frac{\ell \dim(\mathfrak{g})}{\ell + \lambda_B(\mathfrak{g})} .$$

24.3.26 Remark. For $a, b \in \mathbb{Z}$, the sum $\sum_{i=1}^d u_i(a)u^i(b)$ is independent of the choice of basis $\{u_1, \dots, u_d\}$ of $\mathfrak{g}$. In particular, the operators L_m^ρ are independent of the choice of basis.

24.3.27 Remark. If $\ell = -\lambda_B(\mathfrak{g})$, then the formulas we wrote down for the Segal–Sugawara operators L_m^ρ do not make sense, and there is a fundamental difficulty in dealing with the "critical level" $\ell = -\lambda_B(\mathfrak{g})$. At the critical level, the theory seems to resemble the positive characteristic situation rather than the classical one; see [Hum10] for some discussion of this point.

24.3.28 Remark. In light of Remark 24.2.18, the Segal–Sugawara construction can be extended to the case where $\mathfrak{g}$ is *reductive*, i.e., $\mathfrak{g}$ decomposes as a product

$$\mathfrak{g} \cong \mathfrak{a} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r ,$$

where $\mathfrak{a}$ is an abelian Lie algebra and $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ are simple Lie algebras. In this case, the central charge of the resulting $\mathrm{Vir}_{\mathbb{C}}$-representation is

$$\dim(\mathfrak{a}) + \sum_{i=1}^{r} \frac{\ell_i \dim(\mathfrak{g}_i)}{\ell_i + \lambda_{B_i}(\mathfrak{g}_i)} .$$

Here the central element of $\widetilde{\mathrm{L}}\mathfrak{a}$ acts by multiplication by a nonzero complex number and the central element of each $\widetilde{\mathrm{L}}\mathfrak{g}_i$ acts by multiplication by $\ell_i \in \mathbb{C} \smallsetminus \{-\lambda_{B_i}(\mathfrak{g}_i)\}$. This is rather useful as all of the classical Lie algebras are reductive [Kir08, Theorem 5.49]; see [KR87, Remark 10.3] for details.

24.3.29 Remark. The Segal–Sugawara construction is usually stated with the assumptions that $\mathfrak{g}$ is simple and B is the normalization of the Killing form such that the long roots of $\mathfrak{g}$ have square length 2 (so that $\lambda_B(\mathfrak{g})$ is the *dual Coxeter number*, often denoted by $h^\vee$). This is somewhat unfortunate; because the Killing form of an abelian Lie algebra is trivial, to include the abelian case (and the reductive extension) the "usual" statement needs to be modified to include arbitrary nondegenerate invariant symmetric bilinear forms as in Theorem 24.3.3.

24.3.30 Remark. One of the motivations for the formula for the Segal–Sugawara operators L_m^ρ comes from the theory of vertex algebras. See [BF04, §3], in particular [BF04, Proposition 3.3.1], for more details on the relation to vertex algebras.

References

Ale19 J. W. Alexander, *Note on two three-dimensional manifolds with the same group*, Trans. Amer. Math. Soc., vol. 20, no. 4, pp. 339–342, 1919. DOI: 10.2307/1988806 (cit. on p. 189).

Alv85 O. Alvarez, *Topological quantization and cohomology*, Comm. Math. Phys., vol. 100, no. 2, pp. 279–309, 1985 (cit. on p. 206).

And69 D. Anderson, *Universal coefficient theorems for* K*-theory*, 1969 (cit. on p. 212).

And00 M. Ando, *Power operations in elliptic cohomology and representations of loop groups*, Trans. Amer. Math. Soc., vol. 352, no. 12, pp. 5619–5666, 2000. DOI: 10.1090/S0002-9947-00-02412-0 (cit. on pp. 221, 226).

And03 ______, *The sigma orientation for analytic circle-equivariant elliptic cohomology*, Geom. Topol., vol. 7, pp. 91–153, 2003. DOI: 10.2140/gt.2003.7.91 (cit. on p. 221).

ABG10 M. Ando, A. J. Blumberg, and D. Gepner, *Twists of* K*-theory and TMF*, in *Superstrings, geometry, topology, and C^*-algebras*, Proc. Sympos. Pure Math. Vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 27–63. DOI: 10.1090/pspum/081/2681757 (cit. on p. 226).

Ano20 Anonymous, *Answer to Math Overflow Question 267249: Relationship between the Witt algebra and vector fields on the circle*, MO:267249, 2020 (cit. on p. 236).

AE17 B. Antieau and E. Elmanto, *A primer for unstable motivic homotopy theory*, in *Surveys on recent developments in algebraic geometry*, Proc. Sympos. Pure Math. Vol. 95, Amer. Math. Soc., Providence, RI, 2017, pp. 305–370 (cit. on p. 53).

AE21 ______, *Descent for semiorthogonal decompositions*, Adv. Math., vol. 380, Paper No. 107600, 2021. DOI: 10.1016/j.aim.2021.107600 (cit. on p. 69).

Aok20 K. Aoki, *Tensor triangular geometry of filtered objects and sheaves*, Preprint available at arXiv:2001.00319, Jan. 2020 (cit. on pp. 30, 103).

AHW17 A. Asok, M. Hoyois, and M. Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles*, Duke Math. J., vol. 166, no. 10, pp. 1923–1953, 2017. DOI: 10.1215/00127094-0000014X (cit. on pp. 30, 53).

Ati61 M. F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math., no. 9, pp. 23–64, 1961 (cit. on p. 221).

Ati85 ______, *Circular symmetry and stationary-phase approximation*, in, 131, Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983), 1985, pp. 43–59 (cit. on p. 221).

ABS64 M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology, vol. 3, no. suppl, suppl. 1, pp. 3–38, 1964. DOI: 10.1016/0040-9383(64)90003-5 (cit. on pp. 213, 218).

AH61 M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, in *Proc. Sympos. Pure Math., Vol. III*, American Mathematical Society, Providence, R.I., 1961, pp. 7–38 (cit. on p. 221).

APS75a M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc., vol. 77, pp. 43–69, 1975. DOI: 10.1017/S0305004100049410 (cit. on pp. 151, 219).

APS75b ______, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc., vol. 78, no. 3, pp. 405–432, 1975. DOI: 10.1017/S0305004100051872 (cit. on pp. 151, 204, 219).

APS76 ______, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc., vol. 79, no. 1, pp. 71–99, 1976. DOI: 10.1017/S0305004100052105 (cit. on pp. 151, 219).

AS69 M. F. Atiyah and G. B. Segal, *Equivariant* K*-theory and completion*, J. Differential Geometry, vol. 3, pp. 1–18, 1969 (cit. on p. 221).

AS68 M. F. Atiyah and I. M. Singer, *The index of elliptic operators. I*, Ann. of Math. (2), vol. 87, pp. 484–530, 1968. DOI: 10.2307/1970715 (cit. on p. 151).

Ati71 M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4), vol. 4, pp. 47–62, 1971 (cit. on p. 213).

AS04 M. Atiyah and G. Segal, *Twisted* K*-theory*, Ukr. Mat. Visn., vol. 1, no. 3, pp. 287–330, 2004 (cit. on p. 223).

Auc94 D. R. Auckly, *Topological methods to compute Chern–Simons invariants*, Math. Proc. Cambridge Philos. Soc., vol. 115, no. 2, pp. 229–251, 1994. DOI: 10.1017/S0305004100072066 (cit. on p. 192).

Aya09 D. Ayala, *Geometric cobordism categories*. ProQuest LLC, Ann Arbor, MI, 2009, p. 166, Thesis (Ph.D.)–Stanford University, ISBN: 978-1109-24281-2 (cit. on p. 214).

Ayo10 J. Ayoub, *Note sur les opérations de Grothendieck et la réalisation de Betti*, J. Inst. Math. Jussieu, vol. 9, no. 2, pp. 225–263, 2010. DOI: 10.1017/S1474748009000127 (cit. on p. 42).

BH21 T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, Astérisque, no. 425, p. 207, 2021. DOI: 10.24033/ast (cit. on pp. 9, 15, 48, 89, 90, 93–95).

Bac82 A. Back, *Pontryagin forms on homogeneous spaces*, Comment. Math. Helv., vol. 57, no. 3, pp. 349–355, 1982. DOI: 10.1007/BF02565864 (cit. on p. 204).

BK01 B. Bakalov and A. Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series. American Mathematical Society, Providence, RI, 2001, vol. 21, pp. x+221, ISBN: 0-8218-2686-7. DOI: 10.1090/ulect/021 (cit. on pp. 216, 231).

BB14 C. Bär and C. Becker, *Differential characters*, Lecture Notes in Mathematics. Springer, Cham, 2014, vol. 2112, pp. viii+187, ISBN: 978-3-319-07033-9; 978-3-319-07034-6. DOI: 10.1007/978-3-319-07034-6 (cit. on p. 19).

BA14 T. Barthel and O. Antolín-Camarena, *Chromatic fracture cubes*, Preprint available at arXiv:1410.7271, Oct. 2014 (cit. on p. 68).

BDSV15 B. Bartlett, C. L. Douglas, C. J. Schommer-Pries, and J. Vicary, *Modular categories as representations of the 3-dimensional bordism 2-category*, 2015, Preprint available at arXiv:1509.06811 (cit. on pp. 216, 231).

Bar17 C. Barwick, *Spectral Mackey functors and equivariant algebraic* K*-theory (I)*, Adv. Math., vol. 304, pp. 646–727, 2017. DOI: 10.1016/j.aim.2016.08.043 (cit. on pp. 90, 93).

BG16 C. Barwick and S. Glasman, *A note on stable recollements*, Preprint available at arXiv:1607.02064, Jul. 2016 (cit. on pp. 64, 68).

BGH20 C. Barwick, S. Glasman, and P. J. Haine, *Exodromy*, Preprint available at arXiv:1807.03281, Aug. 2020 (cit. on pp. 30, 103).

Beĭ84a A. A. Beĭlinson, *The derived category of coherent sheaves on* $\mathbb{P}^n$, in, 3, vol. 3, Selected translations, 1983/84, pp. 233–237 (cit. on p. 69).

Beĭ78 ______, *Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra*, Funktsional. Anal. i Prilozhen., vol. 12, no. 3, pp. 68–69, 1978 (cit. on p. 69).

Beĭ84b ______, *Higher regulators and values of L-functions*, in *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238 (cit. on p. 85).

BBD82 A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171 (cit. on p. 62).

BM06a D. Belov and G. W. Moore, *Holographic action for the self-dual field*, 2006, Preprint available at `arXiv:0605038` (cit. on p. 210).

BM06b ______, *Type II actions from 11-dimensional Chern–Simons theories*, 2006, Preprint available at `arXiv:0611020` (cit. on p. 210).

BF04 D. Ben-Zvi and E. Frenkel, *Geometric realization of the Segal–Sugawara construction*, in *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser. Vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 46–97. DOI: `10.1017/CBO9780511526398.006` (cit. on p. 246).

Ber10 A. Berthomieu, *A version of smooth* K*-theory adapted to the total Chern class*, J. K-Theory, vol. 6, no. 2, pp. 197–230, 2010. DOI: `10.1017/is010009026jkt104` (cit. on p. 146).

BBP19 D. Berwick-Evans, P. Boavida de Brito, and D. Pavlov, *Classifying spaces of infinity-sheaves*, Preprint available at `arXiv:1912.10544`, Dec. 2019 (cit. on pp. 31, 48).

BT21 D. Berwick-Evans and A. Tripathy, *A de Rham model for complex analytic equivariant elliptic cohomology*, Adv. Math., vol. 380, Paper No. 107575, 2021. DOI: `10.1016/j.aim.2021.107575` (cit. on pp. 221, 226).

Bla85 P. Blanc, *Cohomologie différentiable et changement de groupes*, Astérisque, vol. 174–175, pp. 113–130, 1985 (cit. on p. 173).

BK89 A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat., vol. 53, no. 6, pp. 1183–1205, 1337, 1989. DOI: `10.1070/IM1990v035n03ABEH000716` (cit. on p. 69).

Bor53 A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), vol. 57, pp. 115–207, 1953. DOI: `10.2307/1969728` (cit. on pp. 146, 200, 203, 205).

Bor54 ______, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math., vol. 76, pp. 273–342, 1954. DOI: `10.2307/2372574` (cit. on p. 203).

Bor55 ______, *Topology of Lie groups and characteristic classes*, Bull. Amer. Math. Soc., vol. 61, pp. 397–432, 1955. DOI: `10.1090/S0002-9904-1955-09936-1` (cit. on p. 204).

Bot73 R. Bott, *On the Chern–Weil homomorphism and the continuous cohomology of Lie-groups*, Advances in Math., vol. 11, pp. 289–303, 1973. DOI: `10.1016/0001-8708(73)90012-1` (cit. on pp. 9, 104, 105, 161, 165, 169).

BMP73 R. Bott, M. Mostow, and J. Perchik, *Gelfand-fuks cohomology and foliations*, in *Proceedings of the eleventh annual Symposium at New Mexico State University*, 1973 (cit. on p. 165).

BM00 P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted* K*-theory*, J. High Energy Phys., no. 3, Paper 7, 11, 2000. DOI: 10.1088/1126-6708/2000/03/007 (cit. on p. 206).

Boy03 W. Boy, *Über die curvatura integra und die topologie geschlossener flächen*, Math. Ann., vol. 57, pp. 151–184, 1903 (cit. on p. 196).

Bra18 T. Brazelton, *$\mathbb{A}^1$-homotopy theory from an $(\infty, 1)$-categorical viewpoint*, Notes available at www.math.upenn.edu/~tbraz/BMS2018.pdf, Jun. 2018 (cit. on p. 53).

Bro60 E. J. Brody, *The topological classification of the lens spaces*, Ann. of Math. (2), vol. 71, pp. 163–184, 1960. DOI: 10.2307/1969884 (cit. on pp. 180, 189).

Bro82 E. H. Brown Jr., *The cohomology of* BSO_n *and* BO_n *with integer coefficients*, Proc. Amer. Math. Soc., vol. 85, no. 2, pp. 283–288, 1982. DOI: 10.2307/2044298 (cit. on pp. 143, 146, 148, 178, 201, 203).

Bro73 K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc., vol. 186, pp. 419–458, 1973. DOI: 10.2307/1996573 (cit. on p. 100).

BM94 J.-L. Brylinski and D. A. McLaughlin, *The geometry of degree-four characteristic classes and of line bundles on loop spaces. I*, Duke Math. J., vol. 75, no. 3, pp. 603–638, 1994. DOI: 10.1215/S0012-7094-94-07518-2 (cit. on pp. 233, 234).

BM96 ______, *Čech cocycles for characteristic classes*, Comm. Math. Phys., vol. 178, no. 1, pp. 225–236, 1996 (cit. on p. 146).

Bry90 J.-L. Brylinski, *Representations of loop groups, Dirac operators on loop space, and modular forms*, Topology, vol. 29, no. 4, pp. 461–480, 1990. DOI: 10.1016/0040-9383(90)90016-D (cit. on p. 221).

Bry00 ______, *Differentiable cohomology of gauge groups*, 2000, Preprint available at arXiv:0011069 (cit. on pp. 173, 174).

Bry08 ______, *Loop spaces, characteristic classes and geometric quantization*, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. xvi+300, Reprint of the 1993 edition, ISBN: 978-0-8176-4730-8. DOI: 10.1007/978-0-8176-4731-5 (cit. on p. 234).

Bun22a S. Bunk, *Principal ∞-bundles and smooth string group models*, Sep. 2022 (cit. on p. 53).

Bun22b ______, *The $\mathbb{R}$-local homotopy theory of smooth spaces*, J. Homotopy Relat. Struct., vol. 17, no. 4, pp. 593–650, 2022. DOI: 10.1007/s40062-022-00318-7 (cit. on p. 41).

Bun10 U. Bunke, *Chern classes on differential* K*-theory*, Pacific J. Math., vol. 247, no. 2, pp. 313–322, 2010. DOI: 10.2140/pjm.2010.247.313 (cit. on p. 146).

Bun13 ______, *Differential cohomology*, Preprint available at arXiv:1208.3961, Aug. 2013 (cit. on p. 146).

BN19 U. Bunke and T. Nikolaus, *Twisted differential cohomology*, Algebr. Geom. Topol., vol. 19, no. 4, pp. 1631–1710, 2019. DOI: 10.2140/agt.2019.19.1631 (cit. on p. 206).

BNV16 U. Bunke, T. Nikolaus, and M. Völkl, *Differential cohomology theories as sheaves of spectra*, J. Homotopy Relat. Struct., vol. 11, no. 1, pp. 1–66, 2016. DOI: 10.1007/s40062-014-0092-5 (cit. on pp. 25, 76, 143, 151).

BS09 U. Bunke and T. Schick, *Smooth* K*-theory*, Astérisque, no. 328, 45–135 (2010), 2009 (cit. on p. 151).

BS12	______, *Differential K-theory: A survey*, in *Global differential geometry*, Springer Proc. Math. Vol. 17, Springer, Heidelberg, 2012, pp. 303–357. DOI: 10.1007/978-3-642-22842-1_11 (cit. on p. 151).
BSS21	S. Burton, H. Sati, and U. Schreiber, *Lift of fractional D-brane charge to equivariant cohomotopy theory*, J. Geom. Phys., vol. 161, pp. 104034, 20, 2021. DOI: 10.1016/j.geomphys.2020.104034 (cit. on p. 210).
CS19a	D. Calaque and C. Scheimbauer, *A note on the* (∞, n)*-category of cobordisms*, Algebr. Geom. Topol., vol. 19, no. 2, pp. 533–655, 2019. DOI: 10.2140/agt.2019.19.533 (cit. on p. 211).
Cap16	A. Capotosti, *From string structures to spin structures on loop spaces*, PhD thesis, Università degli Studi Roma Tre, 2016 (cit. on p. 221).
CS19b	D. Carchedi and P. Steffens, *On the universal property of derived manifolds*, Preprint available at arXiv:1905.06195, May 2019 (cit. on p. 90).
CJM+05	A. L. Carey, S. Johnson, M. K. Murray, D. Stevenson, and B.-L. Wang, *Bundle gerbes for Chern–Simons and Wess–Zumino–Witten theories*, Comm. Math. Phys., vol. 259, no. 3, pp. 577–613, 2005. DOI: 10.1007/s00220-005-1376-8 (cit. on pp. 205, 217, 232, 233).
CSY21	S. Carmeli, T. M. Schlank, and L. Yanovski, *Ambidexterity and height*, Adv. Math., vol. 385, Paper No. 107763, 90, 2021. DOI: 10.1016/j.aim.2021.107763 (cit. on p. 37).
Cés13	J. César, *Answer to Math Stack Exchange Question 298287: Lie algebra and Killing form*, MSE:298287, Feb. 2013 (cit. on p. 237).
CS85	J. Cheeger and J. Simons, *Differential characters and geometric invariants*, in *Geometry and topology (College Park, Md., 1983/84)*, Lecture Notes in Math. Vol. 1167, Springer, Berlin, 1985, pp. 50–80. DOI: 10.1007/BFb0075216 (cit. on pp. 8, 9, 12, 16, 19, 104, 143, 147, 149, 150).
CS74	S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2), vol. 99, pp. 48–69, 1974. DOI: 10.2307/1971013 (cit. on pp. 10, 150, 181, 196–199, 201–205).
Che44	S.-S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. (2), vol. 45, pp. 747–752, 1944. DOI: 10.2307/1969302 (cit. on p. 150).
Che07	P. M. Cheung, *Supersymmetric field theories and generalized cohomology*. ProQuest LLC, Ann Arbor, MI, 2007, p. 68, Thesis (Ph.D.)–Stanford University, ISBN: 978-0542-98306-1 (cit. on p. 214).
CM21	D. Clausen and A. Mathew, *Hyperdescent and étale* K*-theory*, Invent. Math., vol. 225, no. 3, pp. 981–1076, 2021. DOI: 10.1007/s00222-021-01043-3 (cit. on p. 99).
CP89	R. Coquereaux and K. Pilch, *String structures on loop bundles*, Comm. Math. Phys., vol. 120, no. 3, pp. 353–378, 1989 (cit. on p. 221).
CFLS20a	C. Córdova, D. S. Freed, H. T. Lam, and N. Seiberg, *Anomalies in the space of coupling constants and their dynamical applications I*, SciPost Phys., vol. 8, no. 1, pp. 001, 57, 2020. DOI: 10.21468/SciPostPhys.8.1.001 (cit. on pp. 206, 219).
CFLS20b	______, *Anomalies in the space of coupling constants and their dynamical applications II*, SciPost Phys., vol. 8, no. 1, pp. 002, 32, 2020. DOI: 10.21468/SciPostPhys.8.1.002 (cit. on p. 206).

Cra10 J. Cranch, *Algebraic theories and $(\infty, 1)$-categories*, Preprint available at arXiv:11011.3243, Nov. 2010 (cit. on p. 93).

Cus21 M. T. Cushman, *A geometric model for real and complex differential K-theory*, PhD thesis, City University of New York, 2021 (cit. on p. 151).

DF94 X. Dai and D. S. Freed, *η-invariants and determinant lines*, Journal of Mathematical Physics, vol. 35, no. 10, pp. 5155–5194, 1994. DOI: 10.1063/1.530747 (cit. on p. 219).

DG18a J. Davighi and B. Gripaios, *Homological classification of topological terms in sigma models on homogeneous spaces*, J. High Energy Phys., no. 9, 155, front matter+40, 2018. DOI: 10.1007/jhep09(2018)155 (cit. on p. 206).

DGR20 J. Davighi, B. Gripaios, and O. Randal-Williams, *Differential cohomology and topological actions in physics*, 2020, Preprint available at arXiv:2011.05768 (cit. on p. 206).

DG18b A. Debray and S. Gunningham, *The Arf-Brown TQFT of pin$^-$ surfaces*, in *Topology and quantum theory in interaction*, Contemp. Math. Vol. 718, Amer. Math. Soc., Providence, RI, 2018, pp. 49–87. DOI: 10.1090/conm/718/14478 (cit. on p. 212).

DLW21 A. Debray, Y. L. Liu, and C. Weis, *Constructing the Virasoro groups using differential cohomology*, 2021, Preprint available at arXiv:2112.10837 (cit. on p. 179).

Del71 P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math., no. 40, pp. 5–57, 1971 (cit. on pp. 16, 22, 77).

DF99 P. Deligne and D. S. Freed, *Classical field theory*, in *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, Amer. Math. Soc., Providence, RI, 1999, pp. 137–225 (cit. on pp. 206, 217).

Dev96 J. A. Devoto, *Equivariant elliptic homology and finite groups*, Michigan Math. J., vol. 43, no. 1, pp. 3–32, 1996. DOI: 10.1307/mmj/1029005387 (cit. on p. 221).

DMW02 D.-E. Diaconescu, G. Moore, and E. Witten, *E_8 gauge theory, and a derivation of* K*-theory from M-theory*, Adv. Theor. Math. Phys., vol. 6, no. 6, 1031–1134 (2003), 2002. DOI: 10.4310/ATMP.2002.v6.n6.a2 (cit. on p. 206).

DFM07 E. Diaconescu, D. S. Freed, and G. M. Moore, *The M-theory 3-form and E_8 gauge theory*, in *Elliptic cohomology*, London Math. Soc. Lecture Note Ser. Vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 44–88. DOI: 10.1017/CBO9780511721489.005 (cit. on p. 210).

DW90 R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys., vol. 129, no. 2, pp. 393–429, 1990 (cit. on p. 213).

Dir31 P. Dirac, *Quantised singularities in the electromagnetic field*, Proc. R. Soc. Lond. A, vol. 133, no. 821, pp. 60–72, 1931 (cit. on pp. 182, 208).

DFM11a J. Distler, D. S. Freed, and G. W. Moore, *Orientifold précis*, in *Mathematical foundations of quantum field theory and perturbative string theory*, Proc. Sympos. Pure Math. Vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 159–172. DOI: 10.1090/pspum/083/2742428 (cit. on p. 210).

DFM11b ———, *Spin structures and superstrings*, in *Surveys in differential geometry. Volume XV. Perspectives in mathematics and physics*, Surv. Differ. Geom. Vol. 15, Int. Press, Somerville, MA, 2011, pp. 99–130. DOI: 10.4310/SDG.2010.v15.n1.a4 (cit. on p. 210).

DD63 J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France, vol. 91, pp. 227–284, 1963 (cit. on p. 223).

DI85 P. Donato and P. Iglesias-Zemmour, *Exemples de groupes difféologiques: Flots irrationnels sur le tore*, C. R. Acad. Sci. Paris Sér. I Math., vol. 301, no. 4, pp. 127–130, 1985 (cit. on p. 35).

Don77 H. Donnelly, *Chern–Simons invariants of reductive homogeneous spaces*, Trans. Amer. Math. Soc., vol. 227, pp. 141–164, 1977. DOI: 10.2307/1997455 (cit. on p. 204).

DMR14 C. Doran, S. Méndez-Diez, and J. Rosenberg, *T-duality for orientifolds and twisted KR-theory*, Lett. Math. Phys., vol. 104, no. 11, pp. 1333–1364, 2014. DOI: 10.1007/s11005-014-0715-0 (cit. on p. 210).

Dug98 D. Dugger, *Sheaves and homotopy theory*, Preprint available at pages.uoregon.edu/ddugger/cech.html, Apr. 1998 (cit. on p. 41).

Dug01 ______, *Universal homotopy theories*, Adv. Math., vol. 164, no. 1, pp. 144–176, 2001. DOI: 10.1006/aima.2001.2014 (cit. on p. 41).

ES16 K. Evans-Lee and N. Saveliev, *On the deleted squares of lens spaces*, Topology Appl., vol. 209, pp. 134–152, 2016. DOI: 10.1016/j.topol.2016.06.001 (cit. on pp. 10, 180, 191, 193, 194).

Fer16 F. Ferrari Ruffino, *Differential* K*-characters and D-branes*, in *String-Math 2014*, Proc. Sympos. Pure Math. Vol. 93, Amer. Math. Soc., Providence, RI, 2016, pp. 151–165 (cit. on p. 210).

FR20 F. Ferrari Ruffino and J. C. Rocha Barriga, *Twisted differential* K*-characters and D-branes*, Nuclear Phys. B, vol. 960, pp. 115169, 72, 2020. DOI: 10.1016/j.nuclphysb.2020.115169 (cit. on p. 210).

FSS15a D. Fiorenza, H. Sati, and U. Schreiber, *A higher stacky perspective on Chern–Simons theory*, in *Mathematical aspects of quantum field theories*, Math. Phys. Stud. Springer, Cham, 2015, pp. 153–211 (cit. on p. 216).

FSS15b ______, *Super-Lie n-algebra extensions, higher WZW models and super-p-branes with tensor multiplet fields*, Int. J. Geom. Methods Mod. Phys., vol. 12, no. 2, pp. 1550018, 35, 2015. DOI: 10.1142/S0219887815500188 (cit. on p. 217).

FSS15c ______, *The Wess–Zumino–Witten term of the M5-brane and differential cohomotopy*, J. Math. Phys., vol. 56, no. 10, pp. 102301, 10, 2015. DOI: 10.1063/1.4932618 (cit. on p. 210).

FSS20a ______, *The character map in (twisted differential) non-abelian cohomology*, 2020, Preprint available at arXiv:2009.11909 (cit. on p. 206).

FSS20b ______, *Twisted cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Comm. Math. Phys., vol. 377, no. 3, pp. 1961–2025, 2020. DOI: 10.1007/s00220-020-03707-2 (cit. on p. 210).

FSS21a ______, *Twisted cohomotopy implies level quantization of the full 6d Wess–Zumino term of the M5-brane*, Comm. Math. Phys., vol. 384, no. 1, pp. 403–432, 2021. DOI: 10.1007/s00220-021-03951-0 (cit. on p. 210).

FSS21b ______, *Twisted cohomotopy implies twisted string structure on M5-branes*, J. Math. Phys., vol. 62, no. 4, Paper No. 042301, 16, 2021. DOI: 10.1063/5.0037786 (cit. on p. 210).

FSS22 ______, *Twistorial cohomotopy implies Green-Schwarz anomaly cancellation*, Rev. Math. Phys., vol. 34, no. 5, Paper No. 2250013, 42, 2022. DOI: 10.1142/S0129055X22500131 (cit. on p. 210).

Fre95 D. S. Freed, *Classical Chern–Simons theory. I*, Adv. Math., vol. 113, no. 2, pp. 237–303, 1995. DOI: 10.1006/aima.1995.1039 (cit. on pp. 215–217).

Fre00 ______, *Dirac charge quantization and generalized differential cohomology*, in *Surveys in differential geometry*, Surv. Differ. Geom. Vol. 7, Int. Press, Somerville, MA, 2000, pp. 129–194. DOI: 10.4310/SDG.2002.v7.n1.a6 (cit. on pp. 151, 206).

Fre02a ______, K*-theory in quantum field theory*, in *Current developments in mathematics, 2001*, Int. Press, Somerville, MA, 2002, pp. 41–87 (cit. on p. 206).

Fre02b ______, *Classical Chern–Simons theory. II*, in, 2, vol. 28, Special issue for S. S. Chern, 2002, pp. 293–310 (cit. on p. 215).

Fre08 ______, *Pions and generalized cohomology*, J. Differential Geom., vol. 80, no. 1, pp. 45–77, 2008 (cit. on pp. 210, 217).

Fre09 ______, *Remarks on Chern–Simons theory*, Bull. Amer. Math. Soc. (N.S.), vol. 46, no. 2, pp. 221–254, 2009. DOI: 10.1090/S0273-0979-09-01243-9 (cit. on p. 216).

Fre12a ______, *4-3-2-8-7-6*, Conference talk at *ASPECTS of Topology* available at people.maths.ox.ac.uk/tillmann/ASPECTSfreed.pdf, 2012 (cit. on pp. 211, 216).

Fre12b ______, *Lectures on twisted* K*-theory and orientifolds*, 2012, Notes available at web.ma.utexas.edu/users/dafr/vienna.pdf (cit. on p. 212).

Fre19 ______, *Lectures on field theory and topology*, CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI, 2019, vol. 133, pp. xi+186, Published for the Conference Board of the Mathematical Sciences, ISBN: 978-1-4704-5206-3. DOI: 10.1090/cbms/133 (cit. on pp. 214, 218).

Fre21 ______, *The Atiyah-Singer index theorem*, Bull. Amer. Math. Soc. (N.S.), vol. 58, no. 4, pp. 517–566, 2021. DOI: 10.1090/bull/1747 (cit. on p. 151).

FH00 D. S. Freed and M. J. Hopkins, *On Ramond–Ramond fields and* K*-theory*, J. High Energy Phys., no. 5, Paper 44, 14, 2000. DOI: 10.1088/1126-6708/2000/05/044 (cit. on pp. 151, 206, 210).

FH13 ______, *Chern–Weil forms and abstract homotopy theory*, Bull. Amer. Math. Soc. (N.S.), vol. 50, no. 3, pp. 431–468, 2013. DOI: 10.1090/S0273-0979-2013-01415-0 (cit. on pp. 9, 25, 29, 100, 101, 104, 139, 143, 152, 153, 197).

FH21a ______, *Consistency of M-theory on Non-Orientable Manifolds*, Q. J. Math., vol. 72, no. 1-2, pp. 603–671, 2021. DOI: 10.1093/qmath/haab007 (cit. on p. 210).

FH21b ______, *Reflection positivity and invertible topological phases*, Geom. Topol., vol. 25, no. 3, pp. 1165–1330, 2021. DOI: 10.2140/gt.2021.25.1165 (cit. on pp. 10, 182, 211, 212, 214).

FHLT10 D. S. Freed, M. J. Hopkins, J. Lurie, and C. Teleman, *Topological quantum field theories from compact Lie groups*, in *A celebration of the mathematical legacy of Raoul Bott*, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403. DOI: 10.1090/crmp/050/26 (cit. on p. 216).

FHT10 D. S. Freed, M. J. Hopkins, and C. Teleman, *Consistent orientation of moduli spaces*, in *The many facets of geometry*, Oxford Univ. Press, Oxford, 2010, pp. 395–419. DOI: 10.1093/acprof:oso/9780199534920.003.0019 (cit. on p. 212).

FM06 D. S. Freed and G. W. Moore, *Setting the quantum integrand of M-theory*, Comm. Math. Phys., vol. 263, no. 1, pp. 89–132, 2006. DOI: 10.1007/s00220-005-1482-7 (cit. on p. 211).

FMS07a D. S. Freed, G. W. Moore, and G. Segal, *Heisenberg groups and noncommutative fluxes*, Ann. Physics, vol. 322, no. 1, pp. 236–285, 2007. DOI: 10.1016/j.aop.2006.07.014 (cit. on p. 210).

FMS07b ______, *The uncertainty of fluxes*, Comm. Math. Phys., vol. 271, no. 1, pp. 247–274, 2007. DOI: 10.1007/s00220-006-0181-3 (cit. on p. 210).

FN20 D. S. Freed and A. Neitzke, *The dilogarithm and abelian Chern–Simons*, 2020, Preprint available at arXiv:2006.12565 (cit. on p. 216).

FQ93 D. S. Freed and F. Quinn, *Chern–Simons theory with finite gauge group*, Comm. Math. Phys., vol. 156, no. 3, pp. 435–472, 1993 (cit. on pp. 213, 216).

FT14 D. S. Freed and C. Teleman, *Relative quantum field theory*, Comm. Math. Phys., vol. 326, no. 2, pp. 459–476, 2014. DOI: 10.1007/s00220-013-1880-1 (cit. on pp. 218, 231).

FT21 ______, *Gapped boundary theories in three dimensions*, Comm. Math. Phys., vol. 388, no. 2, pp. 845–892, 2021. DOI: 10.1007/s00220-021-04192-x (cit. on p. 216).

FW99 D. S. Freed and E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math., vol. 3, no. 4, pp. 819–851, 1999. DOI: 10.4310/AJM.1999.v3.n4.a6 (cit. on p. 206).

GTMW09 S. Galatius, U. Tillmann, I. Madsen, and M. Weiss, *The homotopy type of the cobordism category*, Acta Math., vol. 202, no. 2, pp. 195–239, 2009. DOI: 10.1007/s11511-009-0036-9 (cit. on p. 212).

Gan14 N. Ganter, *The elliptic Weyl character formula*, Compos. Math., vol. 150, no. 7, pp. 1196–1234, 2014. DOI: 10.1112/S0010437X1300777X (cit. on p. 221).

Gar80 H. Garland, *The arithmetic theory of loop groups*, Inst. Hautes Études Sci. Publ. Math., no. 52, pp. 5–136, 1980 (cit. on p. 237).

Gaw88 K. Gawędzki, *Topological actions in two-dimensional quantum field theories*, in *Nonperturbative quantum field theory (Cargèse, 1987)*, NATO Adv. Sci. Inst. Ser. B Phys. Vol. 185, Plenum, New York, 1988, pp. 101–141 (cit. on pp. 206, 217, 233).

GG73 M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*. Springer-Verlag, New York-Heidelberg, 1973, pp. x+209, Graduate Texts in Mathematics, Vol. 14 (cit. on p. 34).

Gom01a K. Gomi, *Gerbes in classical Chern–Simons theory*, 2001, Preprint available at arXiv:0105072 (cit. on p. 216).

Gom01b ______, *The formulation of the Chern–Simons action for general compact Lie groups using Deligne cohomology*, J. Math. Sci. Univ. Tokyo, vol. 8, no. 2, pp. 223–242, 2001 (cit. on p. 215).

GY21 K. Gomi and M. Yamashita, *Differential KO-theory via gradations and mass terms*, 2021, Preprint available at arXiv:2111.01377 (cit. on p. 151).

Gor09 I. Gordon, *Infinite-dimensional Lie algebras*, Course notes available at www.maths.ed.ac.uk/~igordon/LA1.pdf, 2009 (cit. on p. 242).

GL18 A. Gorokhovsky and J. Lott, *A Hilbert bundle description of differential* K*-theory*, Adv. Math., vol. 328, pp. 661–712, 2018. DOI: 10.1016/j.aim.2018.02.002 (cit. on p. 151).

GP20 D. Grady and D. Pavlov, *Extended field theories are local and have classifying spaces*, 2020, Preprint available at arXiv:2011.01208 (cit. on p. 214).

GS17 D. Grady and H. Sati, *Parametrized geometric cobordism and smooth Thom stacks*, 2017, Preprint available at arXiv:1709.00686 (cit. on p. 214).

GS18 ———, *Twisted smooth Deligne cohomology*, Ann. Global Anal. Geom., vol. 53, no. 3, pp. 445–466, 2018. DOI: 10.1007/s10455-017-9583-z (cit. on p. 206).

GS19a ———, *Higher-twisted periodic smooth Deligne cohomology*, Homology Homotopy Appl., vol. 21, no. 1, pp. 129–159, 2019. DOI: 10.4310/HHA.2019.v21.n1.a7 (cit. on p. 206).

GS19b ———, *Ramond-Ramond fields and twisted differential K-theory*, 2019, Preprint available at arXiv:1903.08843 (cit. on p. 210).

GS19c ———, *Twisted differential* KO*-theory*, 2019, Preprint available at arXiv:1905.09085 (cit. on p. 206).

GS19d ———, *Twisted differential generalized cohomology theories and their Atiyah-Hirzebruch spectral sequence*, Algebr. Geom. Topol., vol. 19, no. 6, pp. 2899–2960, 2019. DOI: 10.2140/agt.2019.19.2899 (cit. on p. 206).

GS20 ———, *Differential cohomotopy versus differential cohomology for M-theory and differential lifts of Postnikov towers*, 2020, Preprint available at arXiv:2001.07640 (cit. on p. 210).

GS21 ———, *Differential* KO*-theory: Constructions, computations, and applications*, Adv. Math., vol. 384, Paper No. 107671, 117, 2021. DOI: 10.1016/j.aim.2021.107671 (cit. on pp. 146, 147, 151, 218).

Gro07 I. Grojnowski, *Delocalised equivariant elliptic cohomology*, in *Elliptic cohomology*, London Math. Soc. Lecture Note Ser. Vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 114–121. DOI: 10.1017/CBO9780511721489.007 (cit. on pp. 221, 226).

GS99 V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present. Springer-Verlag, Berlin, 1999, pp. xxiv+228, With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)], ISBN: 3-540-64797-X. DOI: 10.1007/978-3-662-03992-2 (cit. on pp. 116–119).

Gun16 S. Gunningham, *Spin Hurwitz numbers and topological quantum field theory*, Geom. Topol., vol. 20, no. 4, pp. 1859–1907, 2016. DOI: 10.2140/gt.2016.20.1859 (cit. on p. 213).

GRW22 O. Gwilliam, E. Rabinovich, and B. R. Williams, *Factorization algebras and abelian CS/WZW-type correspondences*, Pure Appl. Math. Q., vol. 18, no. 4, pp. 1485–1553, 2022. DOI: 10.4310/pamq.2022.v18.n4.a7 (cit. on p. 218).

Hai79 R. M. Hain, *A characterization of smooth functions defined on a Banach space*, Proc. Amer. Math. Soc., vol. 77, no. 1, pp. 63–67, 1979. DOI: 10.2307/2042717 (cit. on p. 34).

Hai21 P. J. Haine, *From nonabelian basechange to basechange with coefficients*, Preprint available at arXiv:2108.03545, Aug. 2021 (cit. on pp. 50, 65, 97, 99, 102).

HPT22 P. J. Haine, M. Porta, and J.-B. Teyssier, *The homotopy-invariance of constructible sheaves*, Preprint available at arXiv:2010.06473, Feb. 2022 (cit. on p. 42).

Ham82a R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.), vol. 7, no. 1, pp. 65–222, 1982. DOI: 10.1090/S0273-0979-1982-15004-2 (cit. on p. 34).

Ham82b ______, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.), vol. 7, no. 1, pp. 65–222, 1982. DOI: 10.1090/S0273-0979-1982-15004-2 (cit. on p. 222).

Har20 Y. Harpaz, *Ambidexterity and the universality of finite spans*, 2020 (cit. on p. 94).

Hau13 R. Haugseng, *The Becker–Gottlieb transfer is functorial*, Preprint available at arXiv:1310.6321v1, Oct. 2013 (cit. on p. 89).

HL74 J. L. Heitsch and H. B. Lawson Jr., *Transgressions, Chern–Simons invariants and the classical groups*, J. Differential Geometry, vol. 9, pp. 423–434, 1974 (cit. on p. 204).

HMSV15 P. Hekmati, M. K. Murray, V. S. Schlegel, and R. F. Vozzo, *A geometric model for odd differential* K*-theory*, Differential Geom. Appl., vol. 40, pp. 123–158, 2015. DOI: 10.1016/j.difgeo.2015.02.001 (cit. on p. 151).

Hen17a A. Henriques, *Bicommutant categories from conformal nets*, 2017, Preprint available at arXiv:1701.02052 (cit. on p. 216).

Hen17b ______, *What Chern–Simons theory assigns to a point*, Proc. Natl. Acad. Sci. USA, vol. 114, no. 51, pp. 13 418–13 423, 2017. DOI: 10.1073/pnas.1711591114 (cit. on p. 216).

Ho15 M.-H. Ho, *On differential characteristic classes*, J. Aust. Math. Soc., vol. 99, no. 1, pp. 30–47, 2015. DOI: 10.1017/S1446788714000627 (cit. on p. 146).

HM62 G. Hochschild and G. D. Mostow, *Cohomology of Lie groups*, Illinois J. Math., vol. 6, pp. 367–401, 1962 (cit. on p. 162).

HKST11 H. Hohnhold, M. Kreck, S. Stolz, and P. Teichner, *Differential forms and 0-dimensional supersymmetric field theories*, Quantum Topol., vol. 2, no. 1, pp. 1–41, 2011. DOI: 10.4171/QT/12 (cit. on p. 214).

HST10 H. Hohnhold, S. Stolz, and P. Teichner, *From minimal geodesics to supersymmetric field theories*, in *A celebration of the mathematical legacy of Raoul Bott*, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 207–274. DOI: 10.1090/crmp/050/20 (cit. on p. 214).

HRW08 S.-M. Hong, E. Rowell, and Z. Wang, *On exotic modular tensor categories*, Commun. Contemp. Math., vol. 10, no. suppl. 1, pp. 1049–1074, 2008. DOI: 10.1142/S0219199708003162 (cit. on p. 231).

HS05 M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom., vol. 70, no. 3, pp. 329–452, 2005 (cit. on pp. 10, 12, 22, 25, 77, 80, 87, 88, 151, 206, 215).

Hoy13 M. Hoyois, *Answer to Math Overflow Question 130999: Is the site of (smooth) manifolds hypercomplete?* MO:130999, May 2013 (cit. on p. 102).

Hoy17 ______, *The six operations in equivariant motivic homotopy theory*, Adv. Math., vol. 305, pp. 197–279, 2017. DOI: 10.1016/j.aim.2016.09.031 (cit. on p. 54).

Hum10 J. Humphreys, *Math Overflow Question 25592: What role does the "dual Coxeter number" play in Lie theory (and should it be called the "Kac number")?* MO:25592, May 2010 (cit. on p. 245).

Igl86 P. Iglesias-Zemmour, *Difféologie d'espace singulier et petits diviseurs*, C. R. Acad. Sci. Paris Sér. I Math., vol. 302, no. 14, pp. 519–522, 1986 (cit. on p. 35).

Igl87 ______, *Connexions et difféologie*, in *Aspects dynamiques et topologiques des groupes infinis de transformation de la mécanique (Lyon, 1986)*, Travaux en Cours, vol. 25, Hermann, Paris, 1987, pp. 61–78 (cit. on p. 35).

Igl07a ______, *Diffeology of the infinite Hopf fibration*, in *Geometry and topology of manifolds*, Banach Center Publ. Vol. 76, Polish Acad. Sci. Inst. Math., Warsaw, 2007, pp. 349–393. DOI: 10.4064/bc76-0-17 (cit. on p. 35).

Igl07b ______, *Dimension in diffeology*, Indag. Math. (N.S.), vol. 18, no. 4, pp. 555–560, 2007. DOI: 10.1016/S0019-3577(07)80062-0 (cit. on p. 35).

Igl13 ______, *Diffeology*, Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2013, vol. 185, pp. xxiv+439, ISBN: 978-0-8218-9131-5. DOI: 10.1090/surv/185 (cit. on p. 35).

IK12 P. Iglesias-Zemmour and Y. Karshon, *Smooth Lie group actions are parametrized diffeological subgroups*, Proc. Amer. Math. Soc., vol. 140, no. 2, pp. 731–739, 2012. DOI: 10.1090/S0002-9939-2011-11301-7 (cit. on p. 35).

IKZ10 P. Iglesias-Zemmour, Y. Karshon, and M. Zadka, *Orbifolds as diffeologies*, Trans. Amer. Math. Soc., vol. 362, no. 6, pp. 2811–2831, 2010. DOI: 10.1090/S0002-9947-10-05006-3 (cit. on p. 35).

Jar87 J. F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra, vol. 47, no. 1, pp. 35–87, 1987. DOI: 10.1016/0022-4049(87)90100-9 (cit. on p. 100).

Kac68 V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Izv. Akad. Nauk SSSR Ser. Mat., vol. 32, pp. 1323–1367, 1968 (cit. on p. 238).

KR87 V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987, vol. 2, pp. xii+145, ISBN: 9971-50-395-6; 9971-50-396-4 (cit. on pp. 241, 242, 244, 245).

KM13a A. Kahle and R. Minasian, *D-brane couplings and generalised geometry*, 2013, Preprint available at arXiv:1301.7238 (cit. on p. 210).

KV14 A. Kahle and A. Valentino, *T-duality and differential* K*-theory*, Commun. Contemp. Math., vol. 16, no. 2, pp. 1350014, 27, 2014. DOI: 10.1142/S0219199713500144 (cit. on p. 210).

Kan16 S. Kandel, *Functorial quantum field theory in the Riemannian setting*, Adv. Theor. Math. Phys., vol. 20, no. 6, pp. 1443–1471, 2016. DOI: 10.4310/ATMP.2016.v20.n6.a5 (cit. on p. 214).

KL01 T. Kerler and V. V. Lyubashenko, *Non-semisimple topological quantum field theories for 3-manifolds with corners*, Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001, vol. 1765, pp. vi+379, ISBN: 3-540-42416-4 (cit. on pp. 216, 231).

Kil87 T. P. Killingback, *World-sheet anomalies and loop geometry*, Nuclear Phys. B, vol. 288, no. 3-4, pp. 578–588, 1987. DOI: 10.1016/0550-3213(87)90229-X (cit. on p. 221).

Kir08 A. Kirillov Jr., *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008, vol. 113, pp. xii+222, ISBN: 978-0-521-88969-8. DOI: 10.1017/CBO9780511755156 (cit. on p. 245).

KK90 P. A. Kirk and E. P. Klassen, *Chern–Simons invariants of 3-manifolds and representation spaces of knot groups*, Math. Ann., vol. 287, no. 2, pp. 343–367, 1990. DOI: 10.1007/BF01446898 (cit. on pp. 188–190).

Kit19 N. Kitchloo, *Quantization of the modular functor and equivariant elliptic cohomology*, in *Homotopy theory: tools and applications*, Contemp. Math. Vol. 729, Amer. Math. Soc., [Providence], RI, 2019, pp. 157–177. DOI: 10.1090/conm/729/14694 (cit. on p. 221).

Klo08 K. R. Klonoff, *An index theorem in differential K-theory*. ProQuest LLC, Ann Arbor, MI, 2008, p. 119, Thesis (Ph.D.)–The University of Texas at Austin, ISBN: 978-0549-70973-2 (cit. on p. 151).

KN96a S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996, pp. xii+329, Reprint of the 1963 original, A Wiley-Interscience Publication, ISBN: 0-471-15733-3 (cit. on p. 106).

KN96b ———, *Foundations of differential geometry. Vol. II*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996, pp. xvi+468, Reprint of the 1969 original, A Wiley-Interscience Publication, ISBN: 0-471-15732-5 (cit. on p. 106).

KS21 M. Kontsevich and G. Segal, *Wick rotation and the positivity of energy in quantum field theory*, Q. J. Math., vol. 72, no. 1-2, pp. 673–699, 2021. DOI: 10.1093/qmath/haab027 (cit. on p. 214).

KM13b C. Kottke and R. Melrose, *Equivalence of string and fusion loop-spin structures*, 2013, Preprint available at arXiv:1309.0210 (cit. on p. 221).

KL79 D. Kraines and T. Lada, *A counterexample to the transfer conjecture*, in *Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978)*, Lecture Notes in Math. Vol. 741, Springer, Berlin, 1979, pp. 588–624 (cit. on p. 89).

Kri20 P. Kristel, *The spinor bundle on loop space and its fusion product*, PhD thesis available at epub.ub.uni-greifswald.de/frontdoor/index/index/docId/3574, PhD thesis, Universität Greifswald, 2020 (cit. on p. 221).

Kui65 N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology, vol. 3, pp. 19–30, 1965. DOI: 10.1016/0040-9383(65)90067-4 (cit. on p. 223).

Kum18 N. Kumar, *Notes on differential cohomology and gerbes*, 2018 (cit. on p. 78).

KY98 K. Kuribayashi and T. Yamaguchi, *The vanishing problem of the string class with degree 3*, J. Austral. Math. Soc. Ser. A, vol. 65, no. 1, pp. 129–142, 1998 (cit. on p. 221).

Lan21 M. Land, *Introduction to infinity-categories*, Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, ©2021, pp. ix+296, ISBN: 978-3-030-61523-9; 978-3-030-61524-6. DOI: 10.1007/978-3-030-61524-6 (cit. on p. 122).

Las63 R. Lashof, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc., vol. 109, pp. 257–277, 1963. DOI: 10.2307/1993906 (cit. on pp. 211, 212).

Lau16 G. Laures, *Characteristic classes in TMF of level $\Gamma_1(3)$*, Trans. Amer. Math. Soc., vol. 368, no. 10, pp. 7339–7357, 2016. DOI: 10.1090/tran/6575 (cit. on p. 221).

Law94 F. W. Lawvere, *Cohesive toposes and Cantor's "lauter Einsen"*, in, 1, vol. 2, Categories in the foundations of mathematics and language, 1994, pp. 5–15. DOI: 10.1093/philmat/2.1.5 (cit. on p. 47).

Law05 F. W. Lawvere, *Categories of spaces may not be generalized spaces as exemplified by directed graphs*, Repr. Theory Appl. Categ., no. 9, pp. 1–7, 2005, Reprinted from Rev. Colombiana Mat. **20** (1986), no. 3-4, 179–185 (cit. on p. 47).

Law07 ______, *Axiomatic cohesion*, Theory Appl. Categ., vol. 19, No. 3, 41–49, 2007 (cit. on p. 47).

LR03 F. W. Lawvere and R. Rosebrugh, *Sets for mathematics*. Cambridge University Press, Cambridge, 2003, pp. xiv+261, ISBN: 0-521-80444-2; 0-521-01060-8. DOI: 10.1017/CBO9780511755460 (cit. on p. 47).

LOT21 Y. Lee, K. Ohmori, and Y. Tachikawa, *Revisiting Wess–Zumino–Witten terms*, SciPost Phys., vol. 10, no. 3, Paper No. 061, 51, 2021. DOI: 10.21468/scipostphys.10.3.061 (cit. on p. 217).

Li15 Q. Li, *Isometric immersions of generalized Berger spheres in* $\mathbb{S}^4(1)$ *and* $\mathbb{C}P^2(4)$, J. Geom. Phys., vol. 98, pp. 21–27, 2015. DOI: 10.1016/j.geomphys.2015.07.022 (cit. on p. 204).

Liu96 K. Liu, *On elliptic genera and theta-functions*, Topology, vol. 35, no. 3, pp. 617–640, 1996. DOI: 10.1016/0040-9383(95)00042-9 (cit. on p. 221).

LS05 R. Longoni and P. Salvatore, *Configuration spaces are not homotopy invariant*, Topology, vol. 44, no. 2, pp. 375–380, 2005. DOI: 10.1016/j.top.2004.11.002 (cit. on pp. 180, 191).

Los92 M. V. Losik, *Fréchet manifolds as diffeological spaces*, Izv. Vyssh. Uchebn. Zaved. Mat., no. 5, pp. 36–42, 1992 (cit. on p. 34).

Los94 ______, *Categorical differential geometry*, Cahiers Topologie Géom. Différentielle Catég., vol. 35, no. 4, pp. 274–290, 1994 (cit. on p. 34).

Lot94 J. Lott, $\mathbb{R}/\mathbb{Z}$ *index theory*, Comm. Anal. Geom., vol. 2, no. 2, pp. 279–311, 1994. DOI: 10.4310/CAG.1994.v2.n2.a6 (cit. on p. 151).

LS21 M. Ludewig and A. Stoffel, *A framework for geometric field theories and their classification in dimension one*, SIGMA Symmetry Integrability Geom. Methods Appl., vol. 17, Paper No. 072, 58, 2021. DOI: 10.3842/SIGMA.2021.072 (cit. on p. 214).

Lur09a J. Lurie, *A survey of elliptic cohomology*, in *Algebraic topology*, Abel Symp. Vol. 4, Springer, Berlin, 2009, pp. 219–277. DOI: 10.1007/978-3-642-01200-6_9 (cit. on p. 221).

HTT J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009, vol. 170, pp. xviii+925, ISBN: 978-0-691-14049-0; 0-691-14049-9 (cit. on pp. 28, 29, 42, 57, 58, 100–102, 125, 127, 128).

Lur09b ______, *On the classification of topological field theories*, in *Current developments in mathematics, 2008*, Int. Press, Somerville, MA, 2009, pp. 129–280 (cit. on p. 211).

HA ______, *Higher algebra*, Preprint available at math.harvard.edu/~lurie/papers/HA.pdf, Sep. 2017 (cit. on pp. 13, 26, 43, 56, 64–66, 90, 97, 99, 126, 128, 129).

SAG ______, *Spectral algebraic geometry*, Preprint available at math.harvard.edu/~lurie/papers/SAG-rootfile.pdf, Feb. 2018 (cit. on pp. 30, 48, 64, 65, 68, 69, 96, 100, 130, 131).

Lur18 ______, *Ultracategories*, Preprint available at www.math.ias.edu/~lurie/papers/Conceptual.pdf, 2018 (cit. on p. 103).

Lur19 ______, *Elliptic cohomology III: Tempered cohomology*, Preprint available at math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf, Jul. 2019 (cit. on p. 46).

MZ10 L. Ma and J. Zhou, *Chern–Simons invariants and isometric immersions of warped products*, Monatsh. Math., vol. 159, no. 4, pp. 361–378, 2010. DOI: 10.1007/s00605-009-0166-3 (cit. on p. 204).

MM94 S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Universitext. Springer-Verlag, New York, 1994, pp. xii+629, A first introduction to topos theory, Corrected reprint of the 1992 edition, ISBN: 0-387-97710-4 (cit. on p. 37).

MW07 I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2), vol. 165, no. 3, pp. 843–941, 2007. DOI: 10.4007/annals.2007.165.843 (cit. on p. 31).

Mas58 W. S. Massey, *Some higher order cohomology operations*, in *Symposium internacional de topología algebraica International symposium on algebraic topology*, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 145–154 (cit. on p. 194).

MVW06 C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006, vol. 2, pp. xiv+216, ISBN: 978-0-8218-3847-1; 0-8218-3847-4 (cit. on pp. 39, 48).

McL92 D. A. McLaughlin, *Orientation and string structures on loop space*, Pacific J. Math., vol. 155, no. 1, pp. 143–156, 1992 (cit. on p. 221).

Mil11 M. S. Miller, *Rational homotopy models for two-point configuration spaces of lens spaces*, Homology Homotopy Appl., vol. 13, no. 2, pp. 43–62, 2011. DOI: 10.4310/HHA.2011.v13.n2.a4 (cit. on pp. 193, 194).

Mil75 J. J. Millson, *Examples of nonvanishing Chern–Simons invariants*, J. Differential Geometry, vol. 10, no. 4, pp. 589–600, 1975 (cit. on p. 204).

Mil84 J. Milnor, *Remarks on infinite-dimensional Lie groups*, in *Relativity, groups and topology, II (Les Houches, 1983)*, North-Holland, Amsterdam, 1984, pp. 1007–1057 (cit. on p. 222).

MS74 J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331 (cit. on pp. 106, 114).

MM97 R. Minasian and G. Moore, K*-theory and Ramond-Ramond charge*, J. High Energy Phys., no. 11, Paper 2, 7, 1997. DOI: 10.1088/1126-6708/1997/11/002 (cit. on p. 206).

Mon15 S. Monnier, *Hamiltonian anomalies from extended field theories*, Comm. Math. Phys., vol. 338, no. 3, pp. 1327–1361, 2015. DOI: 10.1007/s00220-015-2369-x (cit. on p. 219).

Mon17 ______, *Topological field theories on manifolds with Wu structures*, Rev. Math. Phys., vol. 29, no. 5, pp. 1750015, 103, 2017. DOI: 10.1142/S0129055X17500155 (cit. on p. 219).

Mon18 ______, *The anomaly field theories of six-dimensional* $(2,0)$ *superconformal theories*, Adv. Theor. Math. Phys., vol. 22, no. 8, pp. 2035–2089, 2018. DOI: 10.4310/ATMP.2018.v22.n8.a6 (cit. on p. 219).

MM19 S. Monnier and G. W. Moore, *Remarks on the Green-Schwarz terms of six-dimensional supergravity theories*, Comm. Math. Phys., vol. 372, no. 3, pp. 963–1025, 2019. DOI: 10.1007/s00220-019-03341-7 (cit. on p. 219).

Moo68 R. V. Moody, *A new class of Lie algebras*, J. Algebra, vol. 10, pp. 211–230, 1968. DOI: 10.1016/0021-8693(68)90096-3 (cit. on p. 238).

MS89 G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys., vol. 123, no. 2, pp. 177–254, 1989 (cit. on p. 231).

MW00 G. Moore and E. Witten, *Self-duality, Ramond–Ramond fields and* K*-theory*, J. High Energy Phys., no. 5, Paper 32, 32, 2000. DOI: 10.1088/1126-6708/2000/05/032 (cit. on pp. 206, 210).

MM01 J. D. Moore and J.-M. Morvan, *On isometric Lagrangian immersions*, Illinois J. Math., vol. 45, no. 3, pp. 833–849, 2001 (cit. on p. 204).

Mor06 F. Morel, $\mathbb{A}^1$*-algebraic topology*, in *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, pp. 1035–1059 (cit. on p. 48).

Mor12 ______, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics. Springer, Heidelberg, 2012, vol. 2052, pp. x+259, ISBN: 978-3-642-29513-3. DOI: 10.1007/978-3-642-29514-0 (cit. on p. 48).

MV99 F. Morel and V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math., no. 90, 45–143 (2001), 1999 (cit. on pp. 30, 41, 48, 53).

MS18 L. Müller and R. J. Szabo, *Extended quantum field theory, index theory, and the parity anomaly*, Comm. Math. Phys., vol. 362, no. 3, pp. 1049–1109, 2018. DOI: 10.1007/s00220-018-3169-x (cit. on p. 214).

NR61 M. S. Narasimhan and S. Ramanan, *Existence of universal connections*, Amer. J. Math., vol. 83, pp. 563–572, 1961. DOI: 10.2307/2372896 (cit. on pp. 146, 197).

NR63 ______, *Existence of universal connections. II*, Amer. J. Math., vol. 85, pp. 223–231, 1963. DOI: 10.2307/2373211 (cit. on pp. 146, 197).

Nar19 D. Nardin, *Answer to Math Overflow Question 333239: Describing fiber products in stable* ∞*-categories*, MO:333239, Jun. 2019 (cit. on p. 68).

Ngu17 H. K. Nguyen, *On the infinite loop space structure of the cobordism category*, Algebraic & Geometric Topology, vol. 17, no. 2, pp. 1021–1040, 2017, Preprint available at arXiv:1505.03490 (cit. on p. 212).

NW13 T. Nikolaus and K. Waldorf, *Lifting problems and transgression for non-abelian gerbes*, Adv. Math., vol. 242, pp. 50–79, 2013. DOI: 10.1016/j.aim.2013.03.022 (cit. on p. 221).

Noe14 J. Noel, $H_\infty \neq E_\infty$, in *An alpine expedition through algebraic topology*, Contemp. Math. Vol. 617, Amer. Math. Soc., Providence, RI, 2014, pp. 237–240. DOI: 10.1090/conm/617/12285 (cit. on p. 89).

Obl17 B. Oblak, *The Virasoro group*, in *BMS Particles in Three Dimensions*. Cham: Springer International Publishing, 2017, pp. 163–199, ISBN: 978-3-319-61878-4. DOI: 10.1007/978-3-319-61878-4_6 (cit. on p. 179).

Ouy94 M. Q. Ouyang, *Geometric invariants for Seifert fibred* 3*-manifolds*, Trans. Amer. Math. Soc., vol. 346, no. 2, pp. 641–659, 1994. DOI: 10.2307/2154864 (cit. on p. 204).

Par17 B. Park, *A smooth variant of Hopkins-Singer differential* K*-theory*, New York J. Math., vol. 23, pp. 655–670, 2017 (cit. on p. 151).

PPRS21 B. Park, A. J. Parzygnat, C. Redden, and A. Stoffel, *Noncommutative differential K-theory*, 2021, Preprint available at arXiv:2106.12073 (cit. on p. 151).

Pav22 D. Pavlov, *Numerable open covers and representability of topological stacks*, Topology Appl., vol. 318, Paper No. 108203, 28, 2022. DOI: 10.1016/j.topol.2022.108203 (cit. on p. 31).

PT10 C. Peng and Z. Tang, *Chern–Simons invariant and conformal embedding of a 3-manifold*, Acta Math. Sin. (Engl. Ser.), vol. 26, no. 1, pp. 25–28, 2010. DOI: 10.1007/s10114-010-8559-8 (cit. on p. 204).

PW27 F. Peter and H. Weyl, *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe*, Math. Ann., vol. 97, no. 1, pp. 737–755, 1927. DOI: 10.1007/BF01447892 (cit. on p. 220).

PW88 K. Pilch and N. P. Warner, *String structures and the index of the Dirac-Ramond operator on orbifolds*, Comm. Math. Phys., vol. 115, no. 2, pp. 191–212, 1988 (cit. on p. 221).

Pon59 L. S. Pontryagin, *Smooth manifolds and their applications in homotopy theory*, in *American Mathematical Society Translations, Ser. 2, Vol. 11*, American Mathematical Society, Providence, R.I., 1959, pp. 1–114 (cit. on p. 212).

PS86 A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1986, pp. viii+318, Oxford Science Publications (cit. on pp. 10, 183, 220, 222–225, 227–230, 232, 233, 235, 236).

Pro07 C. Procesi, *Lie groups: An approach through invariants and representations*, Universitext. 2007, pp. xxiv+596, ISBN: 978-0-387-26040-2; 0-387-26040-4 (cit. on p. 147).

Pst18 P. Pstrągowski, *Synthetic spectra and the cellular motivic category*, Preprint available at arXiv:1803.01804, Mar. 2018 (cit. on pp. 98, 99).

QS19 J. Quigley and J. Shah, *On the parametrized Tate construction and two theories of real p-cyclotomic spectra*, Preprint available at arXiv:1909.03920, Sep. 2019 (cit. on pp. 65, 66).

Rap19 G. Raptis, *Some characterizations of acyclic maps*, J. Homotopy Relat. Struct., vol. 14, no. 3, pp. 773–785, 2019. DOI: 10.1007/s40062-019-00231-6 (cit. on p. 127).

Red17 C. Redden, *Differential Borel equivariant cohomology via connections*, New York J. Math., vol. 23, pp. 441–487, 2017 (cit. on p. 104).

Rei35 K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg, vol. 11, no. 1, pp. 102–109, 1935. DOI: 10.1007/BF02940717 (cit. on pp. 180, 189).

RT90 N. Y. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys., vol. 127, no. 1, pp. 1–26, 1990 (cit. on pp. 216, 231).

RT91 N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math., vol. 103, no. 3, pp. 547–597, 1991. DOI: 10.1007/BF01239527 (cit. on pp. 216, 231).

Rez14 C. Rezk, *Cohesion and global homotopy theory*, Preprint available at faculty.math.illinois.edu/~rezk/global-cohesion.pdf, Oct. 2014 (cit. on p. 46).

Rez20 ———, *Looijenga line bundles in complex analytic elliptic cohomology*, Tunis. J. Math., vol. 2, no. 1, pp. 1–42, 2020. DOI: 10.2140/tunis.2020.2.1 (cit. on p. 221).

Rob20 D. M. Roberts, *Topological sectors for heterotic M5-brane charges under Hypothesis H*, J. High Energy Phys., no. 6, pp. 052, 16, 2020. DOI: 10.1007/jhep06(2020)052 (cit. on p. 210).

Roe98 J. Roe, *Elliptic operators, topology and asymptotic methods*, Second, Pitman Research Notes in Mathematics Series. Longman, Harlow, 1998, vol. 395, pp. ii+209, ISBN: 0-582-32502-1 (cit. on p. 106).

Sat10 H. Sati, *Geometric and topological structures related to M-branes*, in *Superstrings, geometry, topology, and C^*-algebras*, Proc. Sympos. Pure Math. Vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 181–236. DOI: 10.1090/pspum/081/2681765 (cit. on p. 210).

Sat11 ———, *Geometric and topological structures related to M-branes II: Twisted string and string^c structures*, J. Aust. Math. Soc., vol. 90, no. 1, pp. 93–108, 2011. DOI: 10.1017/S1446788711001261 (cit. on p. 210).

Sat18 ———, *Framed M-branes, corners, and topological invariants*, J. Math. Phys., vol. 59, no. 6, pp. 062304, 25, 2018. DOI: 10.1063/1.5007185 (cit. on p. 210).

Sat19 ———, *Six-dimensional gauge theories and (twisted) generalized cohomology*, 2019, Preprint available at arXiv:1908.08517 (cit. on p. 210).

SS19 H. Sati and U. Schreiber, *Differential cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams*, 2019, Preprint available at arXiv:1912.10425 (cit. on p. 210).

SS20a ———, *Equivariant cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys., vol. 156, pp. 103775, 40, 2020. DOI: 10.1016/j.geomphys.2020.103775 (cit. on p. 210).

SS20b ———, *The character map in equivariant twistorial cohomotopy implies the Green-Schwarz mechanism with heterotic M5-branes*, 2020, Preprint available at arXiv:2011.06533 (cit. on p. 210).

SS21a ———, *M/F-theory as Mf-theory*, 2021, Preprint available at arXiv:2103.01877 (cit. on p. 210).

SS21b ———, *Twisted cohomotopy implies M5-brane anomaly cancellation*, Lett. Math. Phys., vol. 111, no. 5, Paper No. 120, 25, 2021. DOI: 10.1007/s11005-021-01452-8 (cit. on p. 210).

SSS12 H. Sati, U. Schreiber, and J. Stasheff, *Twisted differential string and fivebrane structures*, Comm. Math. Phys., vol. 315, no. 1, pp. 169–213, 2012. DOI: 10.1007/s00220-012-1510-3 (cit. on p. 210).

Sch19 E. Schlarmann, *Smooth classifying spaces for differential* K*-theory*, 2019, Preprint available at arXiv:1905.03059 (cit. on p. 151).

Sch13a V. S. Schlegel, *The caloron correspondence and odd differential K-theory*, 2013, Preprint available at arXiv:1309.2601 (cit. on p. 151).

Sch17 C. Schommer-Pries, *Invertible topological field theories*, 2017, Preprint available at arXiv:1712.08029 (cit. on p. 211).

Sch18 C. J. Schommer-Pries, *Tori detect invertibility of topological field theories*, Geom. Topol., vol. 22, no. 5, pp. 2713–2756, 2018. DOI: 10.2140/gt.2018.22.2713 (cit. on p. 211).

SS14 C. Schommer-Pries and N. Stapleton, *Singular cohomology from supersymmetric field theories*, 2014, Preprint available at arXiv:1403.1303 (cit. on p. 214).

Sch13b U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, Preprint available at ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos, 2013 (cit. on pp. 11, 46, 146, 205, 206, 217).

Sch77 A. S. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants*, Lett. Math. Phys., vol. 2, no. 3, pp. 247–252, 1977. DOI: 10.1007/BF00406412 (cit. on pp. 181, 216).

Seg85 G. Segal, *Loop groups*, Lecture Notes in Math. Vol. 1111, pp. 155–168, 1985 (cit. on pp. 221, 230).

Seg04 ______, *The definition of conformal field theory*, in *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser. Vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577 (cit. on p. 231).

Seg70 G. Segal, *Cohomology of topological groups*, in *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, Academic Press, London, 1970, pp. 377–387 (cit. on pp. 173, 174).

Seg81 ______, *Unitary representations of some infinite dimensional groups*, Communications in Mathematical Physics, vol. 80, no. 3, pp. 301–342, Sep. 1981. DOI: 10.1007/BF01208274 (cit. on pp. 173, 178).

Seg11 ______, *Three roles of quantum field theory*, Felix Klein Lectures available at mpim-bonn.mpg.de/node/3372., 2011 (cit. on p. 214).

Sen98 A. Sen, SO(32) *spinors of type* I *and other solitons on brane-antibrane pair*, J. High Energy Phys., no. 9, Paper 23, 27, 1998. DOI: 10.1088/1126-6708/1998/09/023 (cit. on p. 206).

Ser53 J.-P. Serre, *Cohomologie modulo* 2 *des complexes d'Eilenberg-Mac Lane*, Comment. Math. Helv., vol. 27, pp. 198–232, 1953. DOI: 10.1007/BF02564562 (cit. on p. 205).

Ser54 ______, *Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d'aprèes Armand Borel et André Weil)*, in *Séminaire Bourbaki, Vol. 2*, Soc. Math. France, Paris, 1954, Exp. No. 100, 447–454 (cit. on p. 220).

Ser01 ______, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001, pp. x+74, Translated from the French by G. A. Jones, Reprint of the 1987 edition, ISBN: 3-540-67827-1. DOI: 10.1007/978-3-642-56884-8 (cit. on p. 238).

SGA 4_{I} *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois Marie 1963–64 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint–Donat. Lecture Notes in Mathematics, Vol. 269. Berlin: Springer-Verlag, 1963–64, pp. xix+525 (cit. on p. 62).

SS08 J. Simons and D. Sullivan, *Axiomatic characterization of ordinary differential cohomology*, J. Topol., vol. 1, no. 1, pp. 45–56, 2008. DOI: 10.1112/jtopol/jtm006 (cit. on pp. 8, 12, 16, 19–21).

SS10 ______, *Structured vector bundles define differential* K*-theory*, in *Quanta of maths*, Clay Math. Proc. Vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 579–599 (cit. on pp. 22, 151).

SS12 ______, *Differential characters for* K*-theory*, in *Metric and differential geometry*, Progr. Math. Vol. 297, Birkhäuser/Springer, Basel, 2012, pp. 353–361. DOI: 10.1007/978-3-0348-0257-4_12 (cit. on p. 22).

ST97 C. Simpson and C. Teleman, *De Rham's Theorem for* ∞*-stacks*, Preprint available at math.berkeley.edu/~teleman/math/simpson.pdf, 1997 (cit. on p. 47).

Sou80 J.-M. Souriau, *Groupes différentiels*, in *Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, Lecture Notes in Math. Vol. 836, Springer, Berlin-New York, 1980, pp. 91–128 (cit. on p. 35).

Spi10 D. I. Spivak, *Derived smooth manifolds*, Duke Math. J., vol. 153, no. 1, pp. 55–128, 2010. DOI: 10.1215/00127094-2010-021 (cit. on p. 90).

Sta78 J. D. Stasheff, *Continuous cohomology of groups and classifying spaces*, Bull. Amer. Math. Soc., vol. 84, no. 4, pp. 513–530, 1978. DOI: 10.1090/S0002-9904-1978-14488-7 (cit. on pp. 161, 162).

ST05 S. Stolz and P. Teichner, *The spinor bundle on loop space*, 2005, Preprint available at people.mpim-bonn.mpg.de/teichner/Math/ewExternalFiles/MPI.pdf (cit. on p. 221).

ST11 ______, *Supersymmetric field theories and generalized cohomology*, in *Mathematical foundations of quantum field theory and perturbative string theory*, Proc. Sympos. Pure Math. Vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 279–340. DOI: 10.1090/pspum/083/2742432 (cit. on p. 214).

SV10 R. J. Szabo and A. Valentino, *Ramond-Ramond fields, fractional branes and orbifold differential* K*-theory*, Comm. Math. Phys., vol. 294, no. 3, pp. 647–702, 2010. DOI: 10.1007/s00220-009-0975-1 (cit. on p. 210).

Tac13 Y. Tachikawa, *A pseudo-mathematical pseudo-review on 4d $N = 2$ supersymmetric quantum field theories*, Preprint available at member.ipmu.jp/yuji.tachikawa/tmp/review-rebooted7.pdf, 2013 (cit. on p. 214).

Tho54 R. Thom, *Quelques propriétés globales des variétés differentiables*, PhD thesis, University of Paris, 1954, pp. 17–86 (cit. on pp. 191, 212).

Tho62 E. Thomas, *The torsion Pontryagin classes*, Proc. Amer. Math. Soc., vol. 13, pp. 485–488, 1962. DOI: 10.2307/2034967 (cit. on pp. 148, 201).

Tie08 H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys., vol. 19, no. 1, pp. 1–118, 1908. DOI: 10.1007/BF01736688 (cit. on p. 189).

TWZ13 T. Tradler, S. O. Wilson, and M. Zeinalian, *An elementary differential extension of odd K-theory*, J. K-Theory, vol. 12, no. 2, pp. 331–361, 2013. DOI: 10.1017/is013002018jkt218 (cit. on p. 151).

TWZ16 ______, *Differential* K*-theory as equivalence classes of maps to Grassmannians and unitary groups*, New York J. Math., vol. 22, pp. 527–581, 2016 (cit. on p. 151).

Tsu81 K. Tsuboi, *Eta invariants and conformal immersions*, Publ. Res. Inst. Math. Sci., vol. 17, no. 1, pp. 201–214, 1981. DOI: 10.2977/prims/1195186712 (cit. on p. 204).

Tsu84 ______, *On Chern–Simons invariants of spherical space forms*, Japan. J. Math. (N.S.), vol. 10, no. 1, pp. 9–28, 1984. DOI: 10.4099/math1924.10.9 (cit. on p. 204).

UM57 H. Uehara and W. S. Massey, *The Jacobi identity for Whitehead products*, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, Princeton University Press, Princeton, N. J., 1957, pp. 361–377 (cit. on p. 194).

Ulr17 P. W. Ulrickson, *Oriented one-dimensional supersymmetric Euclidean field theories and K-theory*. ProQuest LLC, Ann Arbor, MI, 2017, p. 128, Thesis (Ph.D.)–University of Notre Dame, ISBN: 978-1369-54184-7 (cit. on p. 214).

vEst53 W. T. van Est, *Group cohomology and Lie algebra cohomology in Lie groups. I, II*, Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math., vol. 15, pp. 484–492, 493–504, 1953 (cit. on p. 162).

Voe98 V. Voevodsky, $\mathbb{A}^1$*-homotopy theory*, in *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, 1998, pp. 579–604 (cit. on p. 48).

Voi07 C. Voisin, *Hodge theory and complex algebraic geometry. I*, English, Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, vol. 76, pp. x+322, Translated from the French by Leila Schneps, ISBN: 978-0-521-71801-1 (cit. on pp. 16, 22, 77).

Wal10 K. Waldorf, *Multiplicative bundle gerbes with connection*, Differential Geom. Appl., vol. 28, no. 3, pp. 313–340, 2010. DOI: 10.1016/j.difgeo.2009.10.006 (cit. on p. 234).

Wal12 ______, *Transgression to loop spaces and its inverse, I: Diffeological bundles and fusion maps*, Cah. Topol. Géom. Différ. Catég., vol. 53, no. 3, pp. 162–210, 2012 (cit. on p. 34).

Wal15 ______, *String geometry vs. spin geometry on loop spaces*, J. Geom. Phys., vol. 97, pp. 190–226, 2015. DOI: 10.1016/j.geomphys.2015.07.003 (cit. on p. 221).

Wal16a ______, *Spin structures on loop spaces that characterize string manifolds*, Algebr. Geom. Topol., vol. 16, no. 2, pp. 675–709, 2016. DOI: 10.2140/agt.2016.16.675 (cit. on p. 221).

Wal16b ______, *Transgression to loop spaces and its inverse, II: Gerbes and fusion bundles with connection*, Asian J. Math., vol. 20, no. 1, pp. 59–115, 2016. DOI: 10.4310/AJM.2016.v20.n1.a4 (cit. on p. 221).

Wal17 ______, *Transgressive loop group extensions*, Math. Z., vol. 286, no. 1-2, pp. 325–360, 2017. DOI: 10.1007/s00209-016-1764-0 (cit. on p. 233).

Wal91 K. Walker, *On Witten's 3-manifold invariants*, 1991, Notes available at canyon23.net/math/1991TQFTNotes.pdf (cit. on pp. 216, 231).

Was98 A. Wassermann, *Operator algebras and conformal field theory. III. Fusion of positive energy representations of* $\mathrm{LSU}(N)$ *using bounded operators*, Invent. Math., vol. 133, no. 3, pp. 467–538, 1998 (cit. on p. 229).

WZ71 J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, Phys. Lett., vol. 37B, pp. 95–97, 1971. DOI: 10.1016/0370-2693(71)90582-x (cit. on p. 217).

Whi41 J. H. C. Whitehead, *On incidence matrices, nuclei and homotopy types*, Ann. of Math. (2), vol. 42, pp. 1197–1239, 1941. DOI: 10.2307/1970465 (cit. on pp. 180, 189, 190).

Wit83 E. Witten, *Global aspects of current algebra*, Nuclear Phys. B, vol. 223, no. 2, pp. 422–432, 1983. DOI: 10.1016/0550-3213(83)90063-9 (cit. on p. 217).

Wit85 ______, *Global anomalies in string theory*, in *Symposium on anomalies, geometry, topology (Chicago, Ill., 1985)*, World Sci. Publishing, Singapore, 1985, pp. 61–99 (cit. on p. 221).

Wit88 ______, *The index of the Dirac operator in loop space*, in *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, Lecture Notes in Math. Vol. 1326, Springer, Berlin, 1988, pp. 161–181. DOI: 10.1007/BFb0078045 (cit. on p. 221).

Wit89 ______, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys., vol. 121, no. 3, pp. 351–399, 1989 (cit. on pp. 181, 216, 218, 231).

Wit97 ______, *On flux quantization in M-theory and the effective action*, J. Geom. Phys., vol. 22, no. 1, pp. 1–13, 1997. DOI: 10.1016/S0393-0440(96)00042-3 (cit. on p. 210).

Wit98 ______, *D-branes and* K*-theory*, J. High Energy Phys., no. 12, Paper 19, 41, 1998. DOI: 10.1088/1126-6708/1998/12/019 (cit. on p. 206).

Wit16 ______, *The "parity" anomaly on an unorientable manifold*, Phys. Rev. B, vol. 94, p. 195 150, 19 Nov. 2016. DOI: 10.1103/PhysRevB.94.195150 (cit. on p. 210).

Wra10 K. Wray, *Extended topological gauge theories in codimension zero and higher*, Master's thesis, Universiteit van Amsterdam, 2010 (cit. on p. 216).

Yam21 M. Yamashita, *Differential models for the Anderson dual to bordism theories and invertible QFT's, II*, 2021, Preprint available at arXiv:2110.14828 (cit. on pp. 215, 216, 219).

YY21 M. Yamashita and K. Yonekura, *Differential models for the Anderson dual to bordism theories and invertible QFT's, I*, 2021, Preprint available at arXiv:2106.09270 (cit. on pp. 215, 216, 219).

Yon19 K. Yonekura, *On the cobordism classification of symmetry protected topological phases*, Comm. Math. Phys., vol. 368, no. 3, pp. 1121–1173, 2019. DOI: 10.1007/s00220-019-03439-y (cit. on p. 213).

Yon21 ______, *General anomaly matching by Goldstone bosons*, J. High Energy Phys., no. 3, Paper No. 057, 18, 2021. DOI: 10.1007/jhep03(2021)057 (cit. on p. 217).

Yos75 Z.-i. Yosimura, *Universal coefficient sequences for cohomology theories of CW-spectra*, Osaka J. Math., vol. 12, no. 2, pp. 305–323, 1975 (cit. on p. 212).

Index of Notation

Index of Terminology