Differential Cohomology

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Abstract

We give an overview of differential cohomology from the point of view of algebraic topology. This includes a survey of several different definitions of differential cohomology groups, a discussion of differential characteristic classes, an introduction to differential generalized cohomology theory, and some applications in physics.

Introduction

It is a truth universally acknowledged that a closed differential form, in possession of an interpretation as a gauge field in a quantum field theory, must be in want of an integral refinement. This refinement manifests the quantum nature of the physical theory: that quantities in the theory are "quantized," meaning that in some system of units they are integers, not arbitrary real numbers. The mathematical incarnation of this theory of closed forms with integrality data is called *differential cohomology*; the objective of this article is to survey this theory, including several different approaches to the basic definitions, some useful constructions in the theory, and some applications.

The basic data is as follows. For M a smooth manifold, there are differential cohomology groups $\check{H}^k(M;\mathbb{Z})$ equipped with a *characteristic class map* $\mathrm{cc} : \check{H}^k(M;\mathbb{Z}) \to H^k(M;\mathbb{Z})$ and a *curvature map* $\mathrm{curv} : \check{H}^k(M;\mathbb{Z}) \to \Omega^k(M)_{c\ell}$; there is a sense in which $\check{H}^k(M;\mathbb{Z})$ is the universal object classifying data of an integral cohomology class (its characteristic class), a closed differential form (its curvature), and an identification of the two induced de Rham cohomology classes. The first construction of $\check{H}^k(M;\mathbb{Z})$ was given by Cheeger and Simons (1985), and since then many constructions, concrete and abstract, have appeared; we will survey several in section "Definitions".

It is a general rule of thumb that ordinary cohomology is to topological objects as differential cohomology is to geometric ones. For example, $H^2(M;\mathbb{Z})$ classifies complex line bundles $L \to M$, and $\check{H}^2(M;\mathbb{Z})$ classifies complex line bundles with connection. The characteristic class and curvature maps capture the first Chern class of the line bundle, resp. the curvature of the connection.

As differential cohomology feels like ordinary cohomology, but upgraded, one can ask which facts about ordinary cohomology upgrade to differential cohomology. The answer is that quite a lot of them do, including integration along the fiber of a relatively oriented bundle of smooth manifolds. In addition, large parts of the theory of characteristic classes lift to differential cohomology, and even enhance: the differential cohomology refinement of the Chern-Weil map contains the information of Chern-Simons

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invariants, for example. In section "Differential Characteristic Classes" we discuss this and other aspects of characteristic classes in differential cohomology. Another avenue for analogy with ordinary cohomology is the prospect of differential refinements of generalized cohomology theories. These too exist, and we will discuss theory and examples in section "Differential Generalized Cohomology".

In section "Applications in Physics", we discuss applications of differential cohomology in theoretical physics: quantization of abelian gauge fields.

Finally, in section "Further Reading", we give some suggestions for further reading.

Definitions

Before the main content of this section, where we survey several different models for differential cohomology, let us begin with some basic key facts about these groups.

Differential cohomology is a theory assigning to each smooth manifold M a series of abelian Fréchet Lie groups $\check{H}^k(M;\mathbb{Z})$ which are *not* homotopy invariants of M

- $\check{H}^1(M;\mathbb{Z})$ is naturally isomorphic to the set of functions $M \to \mathbb{R}/\mathbb{Z}$.
- $\check{H}^2(M;\mathbb{Z})$ is naturally isomorphic to the isomorphism classes of complex line bundles on M with connection.
- $\check{H}^*(M;\mathbb{Z})$ comes with a cup product making it into a ring.
- There is a map $\mathrm{cc} : \check{H}^k(M;\mathbb{Z}) \to H^k(M;\mathbb{Z})$, called the *characteristic class map*. On $\check{H}^2$, this sends a line bundle with connection to its first Chern class.
- There is a map $\mathrm{curv} : \check{H}^k(M;\mathbb{Z}) \to \Omega^k(M)_{c\ell}$ (i.e. to closed k-forms), called the *curvature map*. On $\check{H}^2$, this sends a line bundle with connection to its curvature.

Cheeger-Simons' Differential Characters

Differential characters are for the reader who sees de Rham cohomology's philosophy of form over function and thinks, ''why can't I have both?'' They were the first definition of differential cohomology to appear, and have the feel of singular cohomology.

Definition 1.1 (Cheeger and Simons, 1985, section 1). Let M be a smooth manifold and write $C_k^{\mathrm{sm}}(M)$, resp. $Z_k^{\mathrm{sm}}(M)$, for the abelian groups of smooth k-chains, resp. k-cycles on M. A *differential character* of degree n on M is a homomorphism $\chi : Z_{n-1}^{\mathrm{sm}}(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists $\omega \in \Omega^n(M)$ such that for all $C \in C_{n-1}^{\mathrm{sm}}(M)$,

$$\chi(\partial_c) = \int_c \omega(\chi) \bmod \mathbb{Z} \tag{1.2}$$

The degree-n differential cohomology of M, denoted $\check{H}^n(M;\mathbb{Z})$, *is the group of degree-n differential characters.*

There is a unique ω satisfying this definition for a given χ, and ω is always a closed form. The curvature map sends $\chi \mapsto \omega$.

The characteristic class map has a slightly more elaborate definition. Since $Z_{n-1}^{\mathrm{sm}}(M)$ is a free abelian group and $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is an epimorphism, $\chi : Z_{n-1}^{\mathrm{sm}}(M) \to \mathbb{R}/\mathbb{Z}$ lifts to a homomorphism $\tilde{\chi} : Z_{n-1}^{\mathrm{sm}}(M) \to \mathbb{R}$. Now define $I(\tilde{\chi}) : C_{n-1}^{\mathrm{sm}}(M) \to \mathbb{Z}$ by

$$C \mapsto -\tilde{\chi}(\partial C) + \int_c \mathrm{curv}\,(\chi) \tag{1.3}$$

One can show this is indeed $\mathbb{Z}$-valued, and since $\mathrm{curv}(\chi)$ is a closed form, this is a cocycle. The characteristic class morphism sends $\chi \mapsto [I(\tilde{\chi})]$, which can be shown to not depend on the choice of lift $\tilde{\chi}$

Remark 1.4 Our indexing convention differs from Cheeger-Simons' original convention; we follow the standard convention in the field of differential cohomology, so that the characteristic class and curvature morphisms preserve the degree

Deligne Cohomology

Deligne cohomology refers to a sheaf cohomology model for differential cohomology. Deligne first studied this model in an algebro-geometric setting in Brylinski (1993) and Deligne (1971). was the first to consider this model on smooth manifolds.

Throughout this article, we make the category $\mathcal{M}an$ of manifolds and smooth functions into a site in which the coverings are surjective submersions, and we define a few sheaves on this site.

- Given a Lie group A, we let $\underline{A}$ denote the sheaf of smooth A-valued functions;A without underline denotes the sheaf of locally constant A-valued functions, i.e. smooth functions for the discrete Lie group structure on A.
- The *sheaf of differential k-forms*, denoted Ω^k, sends a manifold M to the real vector space $\Omega^k(M)$ of k-forms on M. Thus we have an isomorphism $\upsilon : \underline{\mathbb{R}} \xrightarrow{\cong} \Omega^0$.

We will also take chain complexes of sheaves on $\mathcal{M}an$. The categories of chain complexes of sheaves on $\mathcal{M}an$ and sheaves of chain complexes on $\mathcal{M}an$ are isomorphic; given a sheaf of chain complexes $\mathcal{F}^\bullet$ on $\mathcal{M}an$ and a smooth manifold M, $H^*(M;\mathcal{F}^\bullet)$ refers to the *hypercohomology* of M valued in C; that is, form the double complex $C^p(M;\mathcal{F}^q)$ and take the cohomology with respect to the total differential.[1]

Definition 1.5 (Deligne, 1971, section 2.2). *The Deligne complex* $\mathbb{Z}(n)$ is the chain complex of sheaves

$$\mathbb{Z}(n) := (0 \to \mathbb{Z} \to \Omega^0 \to \ldots \to \Omega^{n-1} \to 0 \tag{1.6a}$$

Here the map $\mathbb{Z} \to \Omega^0$ is the inclusion of $\mathbb{Z}$-valued functions into $\mathbb{R}$-valued functions combined with the isomorphism $\upsilon : \underline{\mathbb{R}} \xrightarrow{\cong} \Omega^0$.

Remark 1.6b Etymologically, the Deligne complex $\mathbb{Z}(n)$ is related to the "Tate twist" that is also often denoted $\mathbb{Z}(n)$, but the two are not equivalent. For this reason, some authors denote the Deligne complex something like $\mathbb{Z}(n)_{\mathcal{D}}$

One also sees the complexes

$$\mathbb{R}(n) := (0 \to \mathbb{R} \xrightarrow{\tilde{\upsilon}} \Omega^0 \to \cdots \to \Omega^{n-1} \to 0), \tag{1.6c}$$

where $\tilde{\upsilon}$ is the inclusion of locally constant functions into all functions followed by υ, and

$$\mathbb{T}(n) := (0 \to \underline{\mathbb{T}} \xrightarrow{\varphi} \Omega^1 \to \ldots \to \Omega^n \to 0) \tag{1.6d}$$

where $\varphi(1/2\pi i)$dlog and dlog : $\mathbb{T} \to i\Omega^1$ is the morphism sending a $\mathbb{T}$-valued function f to the form $(1/f)\,\mathrm{d}f$

Proposition 1.7 (Brylinski, 1993, Proposition 1.5.7). *For any manifold* M, *there is a natural isomorphism* $H^n(M;\mathbb{Z}(n)) \cong \check{H}^n(M)$.

Thus the "diagonally graded" Deligne cohomology groups are differential cohomology groups. The "off-diagonal" groups $H^k(M;\mathbb{Z}(n))$ for $k \neq n$ are isomorphic to singular cohomology valued in $\mathbb{Z}$ (if $k > n$) or $\mathbb{R}/\mathbb{Z}$ (if $k < n$) (see Brylinski (1993, Theorem 1.5.3) and Hopkins and Singer (2005, section 3.2), so appear uninteresting at first glance, but they attain interesting values on certain stacks; see section "Off-Diagonal Characteristic Classes".

$\mathbb{R}(n)$ and $\mathbb{T}(n)$ are also familiar: $\mathbb{R}(n)$ is isomorphic to the sheaf of closed n-forms considered as a complex concentrated in degree n, and $\mathbb{T}(n) \simeq \mathbb{Z}(n+1)[-1]$ (see Brylinski and McLaughlin (1994, Remark 3.6)); the proof of the latter essentially amounts to the weak equivalence of the complexes $0 \to \mathbb{Z} \to \underline{\mathbb{R}} \to 0$ and $0 \to 0 \to \underline{\mathbb{T}} \to 0$.

In this model for differential cohomology, curv : $H^n(M;\mathbb{Z}(n)) \to \Omega^n_{c\ell}(M)$ is the map $\mathbb{Z}(n) \to \mathbb{R}(n)$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, together with the identification of $\mathbb{R}(n)$-cohomology with closed n-forms. The characteristic class map is the effect on cohomology of the *truncation map* $t : \mathbb{Z}(n) \to \mathbb{Z}$ defined by quotienting $\mathbb{Z}(n)$ by the subcomplex of sheaves in positive homological degrees.

Harvey-Lawson's model for differential cohomology in terms of "sparks" (see Harvey and Lawson (2006)) has a similar feel to Deligne cohomology, though with a cocycle model.

Hopkins-Singer's Homotopy Pullback Model

Hopkins-Singer's approach to differential cohomology (Hopkins and Singer, 2005, section 3) begins with the following observation.

Lemma 1.8 (Hopkins and Singer, 2005, section 3.2; Bunke *et al.*, 2016, section 4.1). *The truncation maps* $t : \mathbb{Z}(n) \to \mathbb{Z}$ *and* $\mathbb{R}(n) \to \mathbb{R}$ participate in a homotopy pullback square

$$\begin{tikzcd} \mathbb{Z}(n) \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\ \mathbb{R}(n) \arrow[r, "t"] & \mathbb{R}, \end{tikzcd} \tag{1.9}$$

where the vertical arrows are induced by the usual inclusion $\mathbb{Z} \to \mathbb{R}$.

This expresses the idea that differential cohomology is a homotopy pullback of closed differential forms and integral cohomology. Hopkins-Singer provide an explicit cocycle model for this homotopy pullback.

Definition 1.10 (Hopkins and Singer, 2005, section 3.2). Let $\check{C}(q)^\bullet(M)$ be the cochain complex given by

$$\check{C}(q)^n(M) := \begin{cases} C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M), & n \geq q \\ C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}), & n \geq q \end{cases} \tag{1.11}$$

[1]For an abelian group A, $H^*(M;A)$ is a priori ambiguous – do we mean singular A-cohomology of M or the sheaf cohomology of M with respect to the sheaf A? Fortunately, these two cohomology theories are naturally isomorphic.

with differential given by, when $n \geq q$,

$$d(c, h, w) := (\delta c, w - c - \delta h, dw \tag{1.12a}$$

and when $n < q$,

$$d(c, h) := \begin{cases} (\delta c, -c - \delta h, 0), & n = q - 1 \\ (\delta c, -c - \delta h), & n < q - 1 \end{cases} \tag{1.12b}$$

The degree-n *differential cohomology* of M is $\check{H}^n(M; \mathbb{Z}) H^n(\check{C}(n)^\bullet(M))$.

If (c, h, ω) is an n-cocycle, then c and ω are both closed; the characteristic class map sends (c, h, ω) to the class of c, and the curvature map sends $(c, h, \omega) \mapsto \omega$.

Simons-Sullivan's Hexagon

Simons and Sullivan (2008) produced a property of differential cohomology that uniquely characterizes it, in terms of a hexagon-shaped diagram.

Theorem 1.13 (Simons and Sullivan, 2008, Theorem 1.1). *Let $\hat{H}^*$ be a functor from manifolds to graded abelian groups, and suppose $\hat{H}$ is equipped with natural transformations*

(1) $i_1 : H^{n-1}(--; \mathbb{R}/\mathbb{Z}) \to \hat{H}^n$,
(2) $i_2 : \Omega^{n-1}(--)/\mathrm{Im}(\mathrm{d}) \to \hat{H}^k$,
(3) $\delta_1 : \hat{H}^n \to \Omega^n_{c\ell}$, and
(4) $\delta_2 : \hat{H}^n \to H^n(--; \mathbb{Z})$

such that the following diagram commutes:

$$\begin{tikzcd}
 & H^{n-1}(-;\mathbb{R}/\mathbb{Z}) \arrow[rr] \arrow[dr, "i_1"] & & H^n(-;\mathbb{Z}) \arrow[dr] & \\
H^{n-1}(-;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^n(-;\mathbb{R}), \\
 & \Omega^{n-1}(-)/\mathrm{Im}(\mathrm{d}) \arrow[rr, "\mathrm{d}"] \arrow[ur, "i_2"] & & \Omega^n_{c\ell} \arrow[ur] &
\end{tikzcd} \tag{1.14}$$

where the topmost arrows are the Bockstein long exact sequence associated to $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, and the bottommost arrows come from the de Rham theorem. Then $\hat{H}^* \cong \check{H}^*$, δ_1 and δ_2 are respectively the curvature and characteristic class maps, and i_1 and i_2 are their kernels.

So this diagram, the *differential cohomology hexagon*, contains quite a bit of information: the topmost arrows are a long exact sequence, the bottommost arrows are another long exact sequence, and the diagonals extend to a short exact sequence. Moreover, both squares, when lifted to the level of sheaves of complexes on $\mathcal{M}an$, are homotopy pullback squares.

See Stimpson (2011) for another axiomatic characterization of differential cohomology.

A Few Basic Properties of Differential Cohomology

We conclude this section with a few elementary properties of differential cohomology.

Higher gerbes with connection

Recall that $\check{H}^1(M; \mathbb{Z})$ is the group of functions $M \to \mathbb{R}/\mathbb{Z}$ – or equivalently, $M \to \mathbb{T}$, and that $\check{H}^2(M; \mathbb{Z})$ is the isomorphism classes of complex line bundles with connection. These can be thought of as categorifications of $\mathbb{T}$-valued functions, suggesting that higher differential cohomology groups ought to represent categorifications of the notion of line bundle with connection. This is correct: these higher-categorical objects are called *gerbes with connection*. See Brylinski (1993) and the references therein for more information.

Cup product

The differential cohomology groups $\check{H}^*(M; \mathbb{Z})$ jointly carry a ring structure, which the characteristic class map sends to the ordinary cup product and the curvature map sends to wedge product. There are various different ways to construct this cup product dating back to Cheeger-Simons' original construction (Cheeger and Simons, 1985) of differential cohomology.

Integration along the fiber

Suppose $E \to B$ is a submersion of smooth manifolds with fiber F and an oriented vertical tangent bundle – i.e. exactly the conditions needed to have integration along the fiber in ordinary cohomology. Then, there is also a differential-cohomology integration along the fiber:

$$\int_F : H^k(E;\mathbb{Z}(n)) \to H^{k-\dim(F)}(B\,;\,\mathbb{Z}(n-\dim(F))) \tag{1.15}$$

See Hopkins and Singer (2005, section 2.4) for a construction of this map.

Cohomology operations

Grady and Sati (2018a,b) have lifted primary and secondary cohomology operations to differential cohomology.

Differential Characteristic Classes

Characteristic classes are a place where differential cohomology shines: the analogy with ordinary cohomology is close enough to help both intuition and proofs, yet differential characteristic classes engender new phenomena, including geometric invariants such as Chern-Simons invariants. Chern-Weil theory is the key to the story, so we begin with that, and then explain how it manifests in differential cohomology.

Chern-Weil Theory

In this subsection only, we will let $\Omega_M^*(V)$ denote differential forms on the manifold M valued in the vector space V, i.e., sections of the bundle $\Lambda^*(T^*M\otimes V)$. For example, a connection $\boldsymbol{\Theta}$ on a principal G-bundle $P\to M$ is a form in $\Omega_P^1(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, and the curvature of $\boldsymbol{\Theta}$ is an element of $\Omega_P^2(\mathfrak{g})$.

Consider the algebra $\mathrm{Sym}^*(\mathfrak{g}^\vee)^G$, i.e. the G-invariants of the algebra of polynomial functions $\mathfrak{g}\to\mathbb{R}$. Here the G-action is induced from the adjoint action of G on $\mathfrak{g}$. Given $f\in\mathrm{Sym}^k(\mathfrak{g}^\vee)^G$, so that f is a degree-k polynomial, together with a principal G-bundle with connection, we will build a closed $2k$-form whose de Rham class is a characteristic class for G-bundles.

Let $\pi : P\to M$ be a principal G-bundle with connection $\boldsymbol{\Theta}$ and curvature $\Omega\in\Omega_P^2(\mathfrak{g})$. Wedge together k copies of Ω to produce $\Omega^{\wedge k}\in\Omega_P^{2k}(\mathfrak{g}^{\otimes k})$, then apply the polynomial f to obtain $f(\Omega^{\wedge k})\in\Omega_P^{2k}(\mathbb{R})$. Because f is Ad-invariant, this form descends to a form $CW(P,\boldsymbol{\Theta},f)\in\Omega_M^{2k}(\mathbb{R})$, called the *Chern-Weil form* of P, $\boldsymbol{\Theta}$, and f. The key properties of Chern-Weil forms are:

(1) $CW(P,\boldsymbol{\Theta},f)$ is a closed form,
(2) the de Rham class of $CW(P,\boldsymbol{\Theta},f)$ depends on P but not on the choice of $\boldsymbol{\Theta}$, and
(3) holding f fixed, $CW(P,\boldsymbol{\Theta},f)$ is natural in $(P,\boldsymbol{\Theta})$.

In fact, the Chern-Weil construction defines an isomorphism $\mathrm{Sym}^*(\mathfrak{g}^\vee)^G\to H^*(BG;\mathbb{R})$ for any compact Lie group G.

Differential Cohomology Lifts of Chern-Weil Forms

Suppose that the de Rham class of a Chern-Weil form $CW(P,\boldsymbol{\Theta},f)$ is in the image of the map $H^*(--;\mathbb{Z})\to H^*(--;\mathbb{R})$. Then we have, at least at a heuristic level, the data of a differential cohomology class: a closed form and an integral cohomology class with identified values in de Rham cohomology. Is there a lift to differential cohomology? Cheeger and Simons (1985) showed the answer is yes, and Bunke *et al.* (2016) showed the naturality properties of these classes allow one to work universally with the classifying stack $B_\nabla G$ of principal G-bundles with connection.[2]

Theorem 2.1 (Cheeger and Simons, 1985, Theorem 2.2; Bunke *et al.*, 2016, section 5.2). *Let G be a compact Lie group and $c\in H^{2k}(BG;\mathbb{Z})$. Let $f\in\mathrm{Sym}^k(\mathfrak{g}^\vee)^G$ be the polynomial uniquely characterized by asking for the de Rham class of $CW(P,\boldsymbol{\Theta},f)$ to equal $c(P)$ in $H^{2k}(BG;\mathbb{R})$. Then, there is a unique class $\check{c}\in\check{H}^{2k}(B_\nabla G;\mathbb{Z})$ whose characteristic class is c and whose curvature form is the Chern-Weil form.*

That is, Chern-Weil theory produces characteristic classes in differential cohomology depending on a principal bundle and a connection.

Example 2.2 For $G=\mathrm{O}_n$, SO_n, or U_n, we obtain characteristic classes of vector bundles with certain classes of connections.

(1) The Pontrjagin classes $p_i\in H^{4i}(B\mathrm{O}_n;\mathbb{Z})$ of a real vector bundle lift to *differential Pontrjagin classes* $\check{p}_i\in\check{H}^{4i}(B_\nabla\mathrm{O}_n;\mathbb{Z})$ of vector bundles equipped with a metric and a compatible connection. See Brylinski and McLaughlin (1996) and Grady and Sati (2021b, Proposition 3.6) for additional constructions of these classes.
(2) The Chern classes $c_i\in H^{2i}(B\mathrm{U}_n;\mathbb{Z})$ of a complex vector bundle lift to *differential Chern classes* $\check{c}_i\in\check{H}^{2i}(B_\nabla\mathrm{U}_n;\mathbb{Z})$ of complex vector bundles equipped with a Hermitian metric and a compatible connection. See Several authors construct differential Chern classes by other methods, including Brylinski and McLaughlin (1996), Berthomieu (2010), Bunke, (2010, 2013) and Ho (2015) for additional constructions of these classes.

[2]By a *stack* we mean a simplicial sheaf on $\mathcal{M}an$. See Freed and Hopkins (2013) for more information.

(3) The Euler class $e \in H^{2k}(B\mathrm{SO}_{2k}; \mathbb{Z})$ of an oriented real rank-$2k$ vector bundle lifts to a *differential Euler class* $\check{e} \in \check{H}^{2k}(B_\nabla \mathrm{SO}_{2k}; \mathbb{Z})$ of such vector bundles equipped with a metric and a compatible connection. See Brylinski and McLaughlin (1996) and Bunke (2013, Example 3.85) for additional constructions of $\check{e}$.

For differential Pontrjagin and Chern classes but *not* the differential Euler class, one can relax the condition of compatibility with the metric. See Amabel *et al.* (2021, Rmarkk 14.1.13). These classes also satisfy a Whitney sum formula, as described in Amabel *et al.* (2021, section 14.2).

Fiorenza *et al.* (2012) generalize this story to higher groups.

Chern-Simons Invariants

Choose a class $c \in H^{2k}(BG; \mathbb{Z})$ and let $\check{c} \in \check{H}^{2k}(B_\nabla G; \mathbb{Z})$ be its differential refinement as guaranteed by **Theorem 2.1** If M is a closed, oriented $2k$-manifold together with a principal G-bundle $P \to M$ with connection $\mathbf{\Theta}$, the data $(P, \mathbf{\Theta})$ pull $\check{c}$ back to a class $\check{c}(P, \mathbf{\Theta}) \in \check{H}^{2k}(M; \mathbb{Z})$. Since M is oriented, we can integrate:

$$\int_M \check{c}(P, \mathbf{\Theta}) \in \check{H}^0(pt; \mathbb{Z}) \cong \mathbb{Z} \tag{2.3}$$

This integral is not so interesting: we just recover $\int_M c(P)$, as if we had never entered the world of differential cohomology. But there is something better we can do: since $\check{H}^1(\mathrm{pt}; \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, we can integrate on a $(2k-1)$-manifold N with principal G-bundle P and connection $\mathbf{\Theta}$:

$$\int_N \check{c}(P, \mathbf{\Theta}) \in \check{H}^1(pt; \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \tag{2.4}$$

This number turns out to be more interesting – it recovers the Chern-Simons invariant.

Definition 2.5 (Chern and Simons, 1974). Choose $f \in \mathrm{Sym}^k(\mathfrak{g}^\vee)^G$ and let $P \to M$ be a principal G-bundle with two connections $\mathbf{\Theta}_0$ and $\mathbf{\Theta}_1$ on it. Since the space of connections is convex, we can let $\mathbf{\Theta}_t(1-t)\mathbf{\Theta}_0 + t\mathbf{\Theta}_1$ for $t \in [0,1]$; these connections stitch together to a connection $\overline{\mathbf{\Theta}}$ on $[0,1] \times M$, with curvature $\overline{\Omega}$.

The *Chern-Simons form* of P, $\mathbf{\Theta}_1$, and $\mathbf{\Theta}_2$ is

$$CS_f(\mathbf{\Theta}_1, \mathbf{\Theta}_2) = \int_0^1 f(\overline{\Omega}) \in \Omega^{2k-1}(M) \tag{2.6}$$

Given a bundle $p: P \to M$ with one connection $\mathbf{\Theta}$, the Chern-Simons form $\mathrm{CS}_f(\mathbf{\Theta})$ is defined by pulling $(P, \mathbf{\Theta})$ back along $P \to M$, then computing $\mathrm{CS}_f(\mathbf{\Theta}^{\mathrm{triv}}, p^*\mathbf{\Theta})$, where $\mathbf{\Theta}^{\mathrm{triv}}$ is the connection coming from the tautological trivialization of $p^*P \to P$.

Proposition 2.7 *With* c *and* $\check{c}$ *as above, let* f *be the invariant polynomial whose Chern-Weil form is* $\mathrm{curv}(\check{c})$, *and let* $i_2 : \Omega^{2k-1}(M)/\mathrm{Im}(\mathrm{d}) \to \check{H}^{2k}(M; \mathbb{Z})$ *be as in* (1.14) *(i.e. the kernel of the characteristic class map). Then, for any principal* G*-bundle* $\pi : P \to M$ *with connection* $\mathbf{\Theta}$

$$i_2(CS_f(\mathbf{\Theta})) = \pi^* \check{c}(P, \mathbf{\Theta}) \in \check{H}^{2k}(P; \mathbb{Z}). \tag{2.8}$$

This was known to Chern and Simons (1974) albeit not stated explicitly there; see Amabel *et al.* (2021, Proposition 19.1.9) for a proof.

To more explicitly relate Proposition 2.7 to the integration story we began with, fix $2k = 4$ and G to be a simple, simply connected Lie group, so that BG is 3-connected and any principal G-bundle $P \to N$ over a 3-manifold N admits a section. Choose a section, and call it $s: N \to P$. Then

$$\int_N s^* CS_f(\mathbf{\Theta}) = \int_N \check{c}(P, \mathbf{\Theta}) \in \mathbb{R}/\mathbb{Z}. \tag{2.9}$$

The left-hand side is a priori $\mathbb{R}$-valued, but depends on the section; the value in $\mathbb{R}/\mathbb{Z}$ is independent of the choice of s.

Remark 2.10 For $\check{c}_1$, $\exp\left(2\pi i \int_{S^1} \check{c}_1(P, \mathbf{\Theta})\right)$ computes the holonomy of the connection $\mathbf{\Theta}$ around S^1.

Off-Diagonal Characteristic Classes

Let G be a Lie group and $B_\bullet G$ be the classifying stack of principal G-bundles. Paralleling work of Beĭlinson (1984, section 1.7), Bloch (1978), Soulé (1989), Brylinski (1999a, 1999b) and Dupont *et al.* (2000) in algebraic geometry, people have lifted characteristic classes of principal G-bundles to the "off-diagonal" Deligne cohomology groups $H^{2q}(B_\bullet G; \mathbb{Z}(q))$, beginning with work of Bott (1973) calculating $H^*(B_\bullet G; \Omega^q)$ and of Shulman (1972) and Bott and Shulman (1976) on $H^*(B_\bullet G; \Omega^{\geq q})$; these calculations were interpreted in differential cohomology by Waldorf (2010) and Amabel *et al.* (2021, Chapter 15-17). One key result is a lift of the Chern-Weil map.

Theorem 2.11 (Bott, 1973, Hopkins). *Let G be a Lie group with $\pi_0(G)$ finite, let $i: K \hookrightarrow G$ be the inclusion of the maximal compact subgroup of G, and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then there is a commutative diagram*

$$\begin{tikzcd} H^{2n}(B_\bullet G;\mathbb{Z}(n)) \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & H^{2n}(BG;\mathbb{Z}) \arrow[r, "i^*", "\cong"'] \arrow[d] & H^{2n}(BK;\mathbb{Z}) \arrow[d] \\ \mathrm{Sym}^n(\mathfrak{g}^\vee)^G \arrow[r, "i^*"] & \mathrm{Sym}^n(\mathfrak{k}^\vee)^K \arrow[r, "CW", "\cong"'] & H^{2n}(BK;\mathbb{R}) \end{tikzcd} \tag{2.12}$$

where the left-hand square is a pullback square and CW is the usual Chern-Weil isomorphism.

Bott (1973) proved a related result; this reinterpretation is due to Hopkins, and a proof can be found in Amabel *et al.* (2021, Corollaries 16.2.4, 16.2.5).

Hence if G is compact, the truncation map $t: H^{2n}(B_\bullet G;\mathbb{Z}(n)) \to H^{2n}(BG;\mathbb{Z})$ is an isomorphism. For noncompact G,**Theorem 2.11** allows one to use information on $\mathrm{Sym}^*(\mathfrak{g}^\vee)^G$ to gain leverage on characteristic classes in $H^{2n}(B_\bullet G;\mathbb{Z}(n))$; see Amabel *et al.* (2021, Chapter 17) and Debray *et al.* (2023, section 3) for examples of this technique.

If H is a Fréchet Lie group, there is a natural isomorphism from $H^3(B_\bullet H;\mathbb{Z}(1))$ to the abelian group of Fréchet Lie group central extensions of H by $\mathbb{T}$ (Amabel *et al.*, 2021), Corollary 18.3.2]). Thus one can construct such extensions by using big_off_diagonal_thm to construct an off-diagonal characteristic class, then move it into $H^3(--;\mathbb{Z}(1))$ using some sort of transgression map. Work of Brylinski and McLaughlin (1994, section 5)) shows how to use this to construct the Kac-Moody central extensions of loop groups of compact simple Lie groups,[3] and (Debray *et al.*, 2023) use this approach to construct the Virasoro central extensions of $\mathrm{Diff}^+(S^1)$.

Differential Generalized Cohomology

Generalized cohomology theories such as K-theory and cobordism have long been an important ingredient in the algebraic topologist's toolbox. In differential cohomology, analogous theories were motivated by ideas in string theory, before more recent work studying all such "differential generalized cohomology theories"[4]; from a homotopical point of view. In this section, we will begin with the general theory in section "Differential Generalized Cohomology Theories and Sheaves On Manifolds", then turn to examples in section "Examples Of Differential Generalized Cohomology Theories".

Differential generalized cohomology theories were first proposed by Freed (2000, section 1), who sketched a definition. Hopkins and Singer (2005, section 4) provided the first comprehensive treatment of differential generalized cohomology. Bunke *et al.* (2016) and Schreiber (2013) provide additional, more homotopical treatments; in this section, we will follow Bunke-Nikolaus-Völkl's account.

Differential Generalized Cohomology Theories and Sheaves on Manifolds

Let Sp denote the ∞-category of spectra, and for any presentable ∞-category C, such as Sp,let $Sh(\mathcal{M}an C)$ denote the ∞-category of C-valued sheaves on $\mathcal{M}an$. These are the functors $\mathcal{F}: Man^{op} \to C$ whose restriction to each manifold is a sheaf in the usual sense.

Definition 3.1 (Bunke *et al.*, 2016). A *differential generalized cohomology theory* is a cohomology theory on $\mathcal{M}an$ given by the sheaf cohomology of some sheaf in $Sh(\mathcal{M}an, Sp)$.

That is, generalized cohomology theories are to Sp as differential generalized cohomology theories are to $Sh(\mathcal{M}an, Sp)$. Much of the theory in this section works with target an arbitrary presentable ∞-category C in place of Sp; see Amabel *et al.* (2021) and Schreiber (2013) for more information.

Generalized differential cohomology theories are in general not homotopy-invariant. One easy example is $H\Omega^k$, given by composing the sheaf of differential k-forms with the Eilenberg-Mac Lane functor. $H\Omega^k$-cohomology is nontrivial on $\mathbb{R}^k$.

Definition 3.2 A sheaf $\mathcal{F} \in Sh(\mathcal{M}an, Sp)$ *is homotopy invariant*, or *concordance-invariant*, or $\mathbb{R}$-*invariant*, if for every map of manifolds $f: M \to N$ that is a homotopy equivalence, $\mathcal{F}(f)$ is an isomorphism. The full subcategory of homotopy invariant sheaves of spectra is denoted $Sh_{\mathbb{R}}(\mathcal{M}an, Sp)$.

[3]Brylinski-McLaughlin did not have **Theorem 2.11** available, so constructed their off-diagonal differential characteristic classes a different way, using objects called *multiplicative bundle gerbes*. **Theorem 2.11** gives an alternative to that part of their proof.

[4]One hears both "differential generalized cohomology theory and generalized differential cohomology theory." In this article, we favor the former: the way these theories have been studied in the literature generally treats them as differential analogs of generalized cohomology theories, rather than generalizations of ordinary differential cohomology. For example, one does not often see Eilenberg-Steenrod-type axioms for differential generalized cohomology theories.

Constant sheaves provide good examples of homotopy invariant sheaves.

The following lemma is essentially due to Dugger (2001) and Morel and Voevodsky (1999), though they considered space-valued sheaves. See Bunk (2022) for a general, model-categorical version.

Lemma 3.3 *The assignment $\mathcal{F} \mapsto \mathcal{F}(\mathrm{pt})$ defines an equivalence $Sh_{\mathbb{R}}(\mathcal{M}an, Sp) \to Sp$.*

The inclusion $\iota_{\mathbb{R}} : Sh_{\mathbb{R}}(\mathcal{M}an, Sp) \to Sh(\mathcal{M}an, Sp)$ admits both a left adjoint L_{hi} and a right adjoint R_{hi}. R_{hi} is the composition of the global sections functor $\Gamma_* : Sh(\mathcal{M}an, Sp) \to Sp$ followed by the constant sheaf functor $\Gamma^* : Sp \to Sh(\mathcal{M}an, Sp)$; for a formula for L_{hi}, see Amabel *et al.* (2021, Chapter 5).

Definition 3.4 A sheaf $\mathcal{F} \in Sh(\mathcal{M}an, Sp)$ is pure if $\Gamma_*(\mathcal{F}) = 0$. The full subcategory of pure sheaves of spectra is denoted $Sh_{\mathrm{pu}}(\mathcal{M}an, Sp)$.

For example, $H\Omega^k$ is a pure sheaf. Pure sheaves tend to look like sheaves of differential forms, and contain the "infinitesimal" information in a differential generalized cohomology theory.

Definition 3.5 Let $\varepsilon : R_{\mathrm{hi}} \Rightarrow \mathrm{id}$ be the counit of the adjunction $\iota_{\mathbb{R}} \dashv R_{\mathrm{hi}}$. Define a functor $\mathrm{Cyc} : Sh(\mathcal{M}an, Sp) \to Sh(\mathcal{M}an, Sp)$ and a natural transformation $\mathrm{curv} : \mathrm{id} \Rightarrow \mathrm{Cyc}$ by asking that $\mathrm{curv} : \mathrm{id} \Rightarrow \mathrm{Cyc}$ is the cofiber of ε. We call curv the curvature map and $\mathrm{Cyc}(\mathcal{F})$ for a sheaf $\mathcal{F}$ the sheaf of differential cycles of $\mathcal{F}$.

Cycfactors through $Sh_{\mathrm{pu}}(\mathcal{M}an, Sp)$, and is in fact left adjoint to the inclusion $\iota_{\mathrm{pu}} : Sh_{\mathrm{pu}}(\mathcal{M}an, Sp) \rightarrowtail Sh(\mathcal{M}an, Sp)$.

Definition 3.6 In a similar way, let $\eta : \mathrm{id} \Rightarrow L_{\mathrm{hi}}$ be the unit of the adjunction $L_{\mathrm{hi}} \dashv \iota_{\mathbb{R}}$ and let $\psi : \mathrm{Def} \Rightarrow \mathrm{id}$ be the fiber of η. Given a sheaf $\mathcal{F}$, $\mathrm{Def}(\mathcal{F})$ is called the *sheaf of differential deformations* of $\mathcal{F}$.

Def is left adjoint to Cyc.

The data of Def, Cyc, L_{hi}, and R_{hi} assemble to generalizations of (1.9) and (1.14).

Theorem 3.7 (Bunke *et al.*, 2016, section 3). *$Sh(\mathcal{M}an, Sp)$ is a recollement of its subcategories $Sh_{\mathbb{R}}(\mathcal{M}an, sp)$ and $Sh_{\mathrm{pu}}(\mathcal{M}an, Sp)$. That is,*

(1) *both ι_{pu} and $\iota_{\mathbb{R}}$ admit left adjoints, namely* Cyc *and* L_{hi}*;*
(2) $\mathrm{Cyc} \circ \iota_{\mathbb{R}} \simeq 0$*; and*
(3) *a morphism of sheaves is an equivalence if and only if both* Cyc *and* L_{hi} *map it to an equivalence.*

Corollary 3.8 (Fracture square, (Bunke *et al.*, 2016,[Proposition 3.3)). *There is a pullback square of natural transformations*

$$\begin{tikzcd} \mathrm{id} \arrow[r, "\eta"] \arrow[d, "\mathrm{curv}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & L_{\mathrm{hi}} \arrow[d, "\mathrm{curv}"'] \\ \mathrm{Cyc} \arrow[r, "\eta"] & L_{\mathrm{hi}}\mathrm{Cyc}. \end{tikzcd} \tag{3.9}$$

This is the analog of Lemma 1.8: it factors an arbitrary differential generalized cohomology theory as a pullback of something like closed forms (the pure part, in the lower left corner) and a non-differential generalized cohomology theory (something homotopy-invariant, in the upper right corner).

Corollary 3.10 (Differential cohomology hexagon, (Bunke *et al.*, 2016, (9))). *There is a commutative diagram of natural transformations*

$$\begin{tikzcd} & R_{\mathrm{hi}} \arrow[rr] \arrow[dr, "\varepsilon"'] & & L_{\mathrm{hi}} \arrow[dr] & \\ \Sigma^{-1}L_{\mathrm{hi}}\mathrm{Cyc} \arrow[ur] \arrow[dr] & & \mathrm{id} \arrow[ur, "\eta"'] \arrow[dr, "\mathrm{curv}"] & & L_{\mathrm{hi}}\mathrm{Cyc} \\ & \mathrm{Def} \arrow[rr] \arrow[ur, "\psi"] & & \mathrm{Cyc} \arrow[ur] & \end{tikzcd} \tag{3.11}$$

with the following properties.

(1) *The diagonals* $(\varepsilon, \mathrm{curv})$ *and* (ψ, η) *are cofiber sequences.*
(2) *The top and bottom rows are once-extended cofiber sequences.*
(3) *Both squares are pullback squares.*

Plug in a sheaf $\mathcal{F}$ to obtain the differential cohomology hexagon for the differential generalized cohomology theory associated to $\mathcal{F}$.

Remark 3.12 This flurry of adjoints suggests that it is the presence of so many adjoints that makes the whole theory of the differential cohomology hexagon possible (Schreiber, 2013). takes this attitude, which he names *cohesion*, and uses it to study differential cohomology in a very general setting.

Examples of Differential Generalized Cohomology Theories

Example 3.13 (Ordinary differential cohomology) For ordinary differential cohomology, apply the Eilenberg-Mac Lane functor H to the Deligne complexes $\mathbb{Z}(n)$. The resulting hexagon coincides with the differential cohomology hexagon from **Theorem 1.13**for example, $\mathrm{Cyc}(H\mathbb{Z}(n)) \simeq H\mathbb{R}(n)$, recovering the sheaf of closed forms.

Example 3.14 (Differential K-theory) Differential K-theory was first studied by Freed and Hopkins (2000) and Freed (2000) for applications in string theory, with related objects considered earlier by Gillet and Soulé (1990) and Lott (2000). Hopkins and Singer (2005, section 4.4) gave the first comprehensive construction of differential K-theory, and additional constructions have been given by Bunke and Schick (2009), Klonoff (2008, section 2), Bunke *et al.* (2016), Simons and Sullivan (2010, section 6), Schlegel (2013, section 4.2), Tradler *et al.* (2013), Tradler *et al.* (2016), Hekmati *et al.* (2015), Park (2017), Gorokhovsky and Lott (2018), Schlarmann (2019), Cushman (2021), Park *et al.* (2022), Gomi and Yamashita (2023) and Lee and Park (2023). See Bunke and Schick (2012) for a survey.

The idea of differential K-theory is to use the Chern character as the source of differential form information refining a K-theory class. Let $A := KU^*(\mathrm{pt}) \cong \mathbb{Z}[t, t^{-1}]$, with $t = 2$. The *Chern character* is the map of spectra sending KU to its tensor product with $\mathbb{R}$:

$$\mathrm{ch}: \quad KU \to KU \wedge H\mathbb{R} \simeq H(\mathbb{R} \otimes A) \tag{3.15}$$

Fix $n \in \mathbb{Z}$, though only the value of $n \bmod 2$ will matter in the end, due to Bott periodicity. Then define a K-theoretic analog of the Deligne complex $KU(n)$ as the homotopy pullback

$$\begin{tikzcd} KU(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & KU \arrow[d, "\mathrm{ch}"] \\ H\tau_{\geq 0}(\mathbb{R}(n) \otimes A) \arrow[r, "t"] & H(\mathbb{R} \otimes A). \end{tikzcd} \tag{3.16}$$

In the lower left corner, $\tau_{\geq 0}$ means taking the nonnegatively graded parts only (since A can contribute negative grading). This sheaf consists of closed $\mathbb{R} \otimes A$-valued forms whose degrees, possibly shifted by multiplication by a power of t, are nonnegative and of the same parity as n. The reason for this complicated object is that the Chern character associated to a connection on a vector bundle is a form of this type.

The *differential K-theory* groups $\check{K}^n(M)$ are the hypercohomology groups $H^n(M; KU(n))$. They are 2-periodic, like for ordinary K-theory. $\check{K}^0(M)$ is naturally isomorphic to the group completion of the commutative monoid of vector bundles with connection on M.

We can then fill in the rest of the hexagon for differential K-theory. This diagram was first constructed by (Simons and Sullivan, 2010):

$$\begin{tikzcd}[column sep=small] & K^n(M;\mathbb{R}/\mathbb{Z}) \arrow[rr] \arrow[dr] & & K^n(M) \arrow[dr, "\mathrm{ch}"] & \\ H^{n-1}(M;\mathbb{R}\otimes A) \arrow[ur] \arrow[dr] \arrow[rr, phantom, ">"] & & \check{K}^n(M) \arrow[ur, "\mathrm{cc}"] \arrow[dr, "\mathrm{ch}^\nabla"] \arrow[rr, phantom, ">"] & & H^n(M;\mathbb{R}\otimes A) \\ & \Omega^{n-1}(M;\mathbb{R}\otimes A)/\mathrm{Im(d)} \arrow[rr, "\mathrm{d}"] \arrow[ur] & & \Omega^n(M;\mathbb{R}\otimes A)_{c\ell} \arrow[ur] & \end{tikzcd} \tag{3.17}$$

The story is still roughly similar to the hexagon for ordinary differential cohomology, but there is some new notation.

- $K^*(-; \mathbb{R}/\mathbb{Z})$ is *K-theory with* $\mathbb{R}/\mathbb{Z}$ *coefficients*, the generalized cohomology theory represented by the spectrum which is the cofiber of $\mathrm{ch}: KU \to KU \wedge H\mathbb{R} \simeq H(\mathbb{R} \otimes A)$. This theory first appears in Atiyah *et al.* (1976, section 5), who attribute it to Segal. The long exact sequence in cohomology induced by the cofiber sequence $KU \to KU \wedge H\mathbb{R} \to KU(--; \mathbb{R}/\mathbb{Z})$, which is a K-theoretic analog of the $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ Bockstein long exact sequence, is the upper long exact sequence in (3.17).
- cc is the characteristic class map, which is the topmost map in (3.16).
- ch^∇ is the version of the Chern character which takes in a vector bundle with connection and produces a closed form. This is the curvature map for differential K-theory.

Example 3.18 (Differential *KO*-theory). Like differential *K*-theory, differential *KO*-theory was first studied by Freed and Hopkins (2000) and Freed (2000); Grady and Sati (2019, 2021b) were the first to comprehensively study differential *KO*-theory, and Cushman (2021) and Gomi and Yamashita (2023) provide additional constructions.

The real version of Example 3.14 is completely analogous. Instead of using the Chern character, one uses its real analog (sometimes called the *Pontrjagin character*)

$$\mathrm{ph} : KO \to KO \wedge H\mathbb{R} \simeq H(\mathbb{R} \otimes B) \tag{3.19}$$

where $B \cong \mathbb{Z}[t, t^{-1}]$ with $t = 4$.[5] One succinct way to define both ph and its form-level version ph^{∇} for a real vector bundle with connection is: first complexify, then take the Chern character. The result a priori lands in $\mathbb{R} \otimes A$-valued forms (resp. cohomology), but in fact factors through $\mathbb{R} \otimes B$-valued forms (resp. cohomology).

Thus we have Deligne-type complexes, now depending on $n \bmod 8$:

$$\begin{tikzcd}
KO(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & KO \arrow[d, "\mathrm{ph}"'] \\
H\tau_{\geq 0}(\mathbb{R}(n) \otimes B) \arrow[r, "t"] & H(\mathbb{R} \otimes B),
\end{tikzcd} \tag{3.20}$$

and the differential *KO*-cohomology hexagon:

$$\begin{tikzcd}
 & KO^n(M; \mathbb{R}/\mathbb{Z}) \arrow[rr] \arrow[dr] & & KO^n(M) \arrow[dr, "\mathrm{ph}"] & \\
H^{n-1}(M; \mathbb{R} \otimes B) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & \check{KO}^n(M) \arrow[ur, "\mathrm{cc}"] \arrow[dr, "\mathrm{ph}^{\nabla}"'] \arrow[rr, phantom, "\circlearrowright"] & & H^n(M; \mathbb{R} \otimes B), \\
 & \Omega^{n-1}(M; \mathbb{R} \otimes B)/\mathrm{Im}(\mathrm{d}) \arrow[rr] \arrow[ur] & & \Omega^n(M; \mathbb{R} \otimes B)_{c\ell} \arrow[ur] &
\end{tikzcd} \tag{3.21}$$

where as usual $\check{K}O^n(M)$ is the n^{th} cohomology of M valued in $KO(n)$.

Remark 3.22 (Some more examples) Though differential *K*- and *KO*-theory are the most commonly studied differential generalized cohomology theories, several others appear in the literature.

(1) *Supercohomology SH*, defined by Freed (2008, section 1)) and Gu and Wen (2014) is the spectrum with $\pi_0(SH) \cong \mathbb{Z}$, $\pi_2(SH) \cong \mathbb{Z}/2$, and the unique nontrivial Postnikov invariant connecting them.[6] Freed and Neitzke (2022, 2023) introduce a differential refinement of this theory for the purpose of studying classical spin Chern-Simons theory.
(2) Differential refinements of algebraic *K*-theory spectra appear in work of Bunke and Gepner (2021), Bunke and Tamme (2015, 2016), Bunke (2018a, 2018b), Park *et al.* (2022); and Schrade (2018) where among other things they are applied to construct a topological version of Beilinson's regulator homomorphisms.
(3) A complex-analytic differential refinement of *MU*, the spectrum representing complex cobordism, appears in work of Haus and Quick (2023a, 2023b) and Hopkins and Quick (2015), Kaspersen and Quick (2023) and Quick (2016, 2019); another differential cobordism theory appears in work of Bunke *et al.* (2009). See also Grady and Sati (2017) for a closely related construction.

Applications in Physics

Closed differential forms are commonplace in the classical theory of electromagnetism, encoding quantities such as the field strength. Passing to the quantum theory amounts to choosing integrality data for the de Rham classes of these forms — in other words, lifting them to differential cocycles. We will discuss this story in this section, where it also leads to the original motivation for differential generalized cohomology theories (section "Quantizing in More General Cohomology Theories").

Dirac Quantization in Electromagnetism

For the first part of this section, we follow Freed (2000); see Amabel *et al.* (2021, Chapter 21).

Let us go over the basic objects of nonrelativistic classical electromagnetic theory in three-dimensional space, which for us will be an oriented Riemannian 3-manifold Y with empty boundary. Let $X\mathbb{R} \times Y$, with the Lorentzian metric $\mathrm{d}t^2 - g$, where t is the $\mathbb{R}$-coordinate.

One may be used to thinking of an electric field as a vector field, representing at each point the magnitude and direction of force exerted on a unit test charge. We will use the metric to pass between TY and T^*Y and describe the electric field as a 1-form $E \in \Omega^1(Y)$. For the magnetic field, it is helpful to instead pass through the Hodge star and obtain a 2-form $B \in \Omega^2(Y)$. The *charge density* ρ_c is a compactly supported differential 3-form, and the *(electric) current* J_E is a compactly supported 2-form.

[5] B is not isomorphic to $KO^*(\mathrm{pt})$. When we tensor with $\mathbb{R}$, this discrepancy goes away.

[6] Sometimes SH is called *restricted supercohomology* to contrast with *extended supercohomology*, a different spectrum studied by Kapustin and Thorngren (2017) and Qing-Rui Wang and Gu (2020). See Gaiotto and Johnson-Freyd (2019, sections 5.3, 5.4).

The *field strength* is $FB - \mathrm{d}t \wedge E \in \Omega^2(X)$, and let $j_E \rho_E - \mathrm{d}t \wedge J_E \in \Omega_c^3(X)$. Maxwell's equations can be concisely expressed in terms of F and j_E:

$$\begin{aligned} dF &= 0 \\ d{\star}F &= jE \end{aligned} \tag{4.1}$$

If there is a *magnetic current* $j_B \in \Omega^3(X)$, we modify the first equation to $\mathrm{d}F = j_B$.

One can then use these forms to write down a Lagrangian action, compute quantities such as the total charge, and so on. The total charge $\overline{Q}$ is a cohomological object, in fact —it is the de Rham class of j_E in $H_c^3(Y;\mathbb{R})$. There is an analogous total magnetic charge.

Quantization tells us that the total charge ought to be discrete — for example, if $Y = \mathbb{R}^3$, $H_c^3(\mathbb{R}^3;\mathbb{R}) \cong \mathbb{R}$, and we assume the total electric charge is some integer multiple of a unit charge q_E. In general, we postulate that the charge must be in the image of the map $H_c^3(Y; q_E\mathbb{Z}) \to H_c^3(Y;\mathbb{R})$, and likewise for a unit magnetic charge q_B.

So the electric charge is a closed form with what looks like data of a lift of its de Rham class to $q_E\mathbb{Z}$-cohomology. This suggests:

Ansatz 4.2 Objects represented by closed differential forms in a classical theory of physics should be represented by cocycles for ordinary differential cohomology in the corresponding quantum theory.

We use cocycles, rather than cohomology classes, in order to obtain something which sheafifies, part of the principle of locality of quantum field theory.

In general, differential forms represent plenty of objects in field theories. Notably, they are gauge fields for abelian gauge groups, including for "higher gauge theory" where the gauge group is a categorification of the circle group and one uses (higher) gerbes instead of principal bundles with connection.

Quantizing in More General Cohomology Theories

String theory teaches us a striking lesson: that for some differential forms, the natural home for the fields in the quantized theory is a differential generalized cohomology theory. Typically this is differential K- or KO-theory, but choosing the correct theory is more of an art than a science and there are different proposals using different differential generalized cohomology theories.

For example, consider type IIB string theory on a 10-manifold X. There is a 3-form field B, which as above should be upgraded to a cocycle for $\check{B} \in \check{H}^3(X;\mathbb{Z})$. For now, assume this field is zero[7]; then there are several forms called *Ramond-Ramond field strengths* $G_i \in \Omega^i(X)$, where $i = 1, 3, 5, 7, 9$. These field strengths satisfy related integrality conditions implying that they are the Chern character of a cocycle for $\check{K}^1(X)$, so we postulate that the Ramond-Ramond field *is* a cocycle for differential K-theory. See Freed (2002, section 3) for further discussion. Other examples include $\check{K}^0$ appearing in type IIA string theory, $\check{K}O^*$ in type I string theory, and the type II B-field lifting to a differential refinement of a Postnikov truncation of $\mathrm{Pic}(KU)$-cohomology, as described by Distler *et al.* (2011a,b).

See also Belov and Moore (2006a,b), Diaconescu *et al.* (2007), Doran *et al.* (2014), Fiorenza *et al.* (2015), Freed *et al.* (2007a,b), Freed (2008), Grady and Sati (2019), Kahle and Minasian (2013), Kahle and Valentino (2014), Ruffino and Barriga (2020), Ruffino (2016); Sati and Schreiber (2023a,b), Sati *et al.* (2012); Sati (2010, 2011, 2019) and Szabo and Valentino (2010) for more examples of quantization in differential generalized cohomology theories. Of particular note is "hypothesis H" of Fiorenza *et al.* (2020), Fiorenza *et al.* (2021a), Sati (2018) proposing that the C-field in M-theory is quantized using twisted differential cohomotopy; work of Burton *et al.* (2021), Fiorenza *et al.* (2020), Fiorenza *et al.* (2021a,b), Fiorenza *et al.* (2022),Grady and Sati (2021a), Roberts (2020); Sati and Schreiber (2020, 2021, 2022, 2023b,c) and Sati (2020), explores this hypothesis and its consequences.

Further Reading

The book Amabel *et al.* (2021) is an introduction to differential cohomology with much the same attitude as the current article; we also recommend the other book-length introductions (Bär and Becker, 2014; Bunke, 2013; Schreiber, 2013). Hopkins and Singer (2005) is a research article that we also recommend as a book-length introduction.

One application of differential generalized cohomology in physics that we did not get into is the classification of reflection-positive invertible field theories, conjectured by Freed and Hopkins (2021) and proven by Grady (2023). See Freed and Hopkins (2021) and Freed (2019) for more on this conjecture, and Amabel *et al.* (2021, Chapter 220) for a review, and see Davighi *et al.* (2020), Yamashita and Yonekura (2023) and Yamashita (2023a, 2023b) for some related work.

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[7]If this field is nonzero, one should repeat this discussion with *twisted differential K-theory*.

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