Exploring Pintopia:
Reflection Branes, Bordisms, and U-Dualities

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Abstract

The U-dualities of maximally supersymmetric non-chiral supergravity (SUGRA) theories lead to strong constraints on the non-perturbative structure of quantum gravity. In this paper we determine Spin- and Pin-lifts of these dualities, which extend this action to fermionic degrees of freedom. Among other things, this allows us to access non-supersymmetric sectors of these low energy effective field theories in which bosonic and fermionic degrees of freedom are treated differently. We use this refinement of the duality groups, in tandem with the Swampland Cobordism Conjecture, to predict new codimension-two branes. These are a natural generalization of the recently discovered R7-branes of type II string theories. The first bordism groups for Spin-twisted duality bundles follow directly from the Abelianization of the duality groups. Viewing the SUGRA theory as the low energy limit of a toroidal compactification of M-theory, winding around these codimension-two defects enacts a reflection around one of the torus directions, which in the effective field theory appears as a charge conjugation symmetry. We establish some basic properties of such branes, including determining BPS objects which can end on it, as well as braiding rules and bound states realized by multiple reflection branes.

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1 Introduction

Dualities provide important insight into the non-perturbative structure of quantum field theory (QFT) and quantum gravity (QG). When combined with supersymmetry, this leads to powerful constraints on the spectrum and properties of BPS objects.

A celebrated example of this sort are the U-dualities of maximally supersymmetric non-chiral supergravity (SUGRA) theories in D spacetime dimensions.[1] From a top down perspective, such theories arise from M-theory compactified on a T^d with $D + d = 11$, unifying many string dualities [6, 7]. The resulting duality symmetries combine S-dualities and T-dualities, as inherited from 11D diffeomorphisms and T-dualities of type II string theories on a T^{d-1}.

What happens if we relax the assumption that our branes / defects preserve supersymmetry? In this case more care is needed both in defining what we mean by U-dualities, as well as in developing new tools to extract the spectrum of branes / defects.

Our aim in this work will be concentrated in two complementary directions. On the one hand, we revisit the structure of the U-duality groups in maximally supersymmetric theories, showing that there are natural Spin- and Pin$^+$-lifts which can act non-trivially on fermionic degrees of freedom. On the other hand, we shall use this structure to extract additional non-supersymmetric data on the spectrum of objects. The first goal will be achieved via a study of possible extensions of the bosonic U-duality groups. The second goal will be achieved by leveraging the recently proposed Swampland Cobordism Conjecture [8] to extract qualitatively new codimension-two objects.[2]

Recently, it was proposed that taking into account the fermionic degrees of freedom can lead to subtle extensions of such duality groups [33, 34, 14, 23].[3] The main reason we need to do this is to specify a choice of bundle assignment for fermions which transform under both Spin and duality transformations. In general, these can be globally correlated. The case that has been studied in the greatest detail is that of the IIB duality group $\mathrm{SL}(2, \mathbb{Z})$ and its corresponding Spin- (see [33]) and Pin$^+$- (see [34]) lifts. Much as SU(2) is the Spin-lift of SO(3), there is a non-trivial $\mathbb{Z}_2$ extension of the $\mathrm{SL}(2, \mathbb{Z})$ dualities generated by large diffeomorphisms of a T^2 to the metaplectic cover $\mathrm{Mp}(2, \mathbb{Z})$. In [33] some Spin-lifts of other U-duality groups were also given, but as far as we are aware, a systematic study of all possible Spin-lifts of U-duality groups was not undertaken. One of our goals will be to determine the Spin-lift of U-dualities:

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U} \to G_U \to 1\,. \tag{1.1}$$

Including reflections along one of the directions of the compactification torus T^d leads to a further generalization where one instead extends the bosonic duality group by allowing

[1] See [1–5].

[2] See e.g., [8–32] for recent work on the Swampland Cobordism Conjecture.

[3] See also [35] for additional discussion of U-duality bundles.

reflections, i.e., allowing orientation reversing transformations of the internal T^d as well. In the low energy effective field theory this reflection is a charge conjugation symmetry. This can be written in terms of a short exact sequence of groups:

$$1 \to G_U \to G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R \to 1\,, \tag{1.2}$$

where the R superscript in $\mathbb{Z}_2^R$ refers to reflections. For the subgroup $\mathrm{SL}(d,\mathbb{Z})$ of large diffeomorphisms acting on T^d, this extensions yields $\mathrm{GL}(d,\mathbb{Z})$. As found in [34], the full duality group of IIB string theory is then the Pin^+ cover of $\mathrm{GL}(2,\mathbb{Z})$. As far as we are aware, the Pin^+ cover of more general U-duality groups has not been considered. This is given by a non-trivial central extension:

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^{+} \to (G_U \rtimes \mathbb{Z}_2^R) \to 1\,. \tag{1.3}$$

Throughout, we shall refer to the Spin- and Pin^+-lifts of U-dualities as $\widetilde{G_U}$ and $\widetilde{G_U}^{+}$. Due to the correlation between the Spin structure and the duality bundle, the relevant structure group for spacetimes is then:

$$\text{Spin-}\widetilde{G_U}\ \text{structure:} \qquad \frac{\mathrm{Spin}\times\widetilde{G_U}}{\mathbb{Z}_2} \tag{1.4}$$

$$\text{Spin-}\widetilde{G_U}^{+}\ \text{structure:} \qquad \frac{\mathrm{Spin}\times\widetilde{G_U}^{+}}{\mathbb{Z}_2}, \tag{1.5}$$

where the $\mathbb{Z}_2$ in the quotient embeds as $(-1)^F$ in the Spin factor and the $\mathbb{Z}_2$ of the central extension in the U-duality group.

This extension of the duality groups provides additional access to the non-perturbative as well as non-supersymmetric sectors of these theories. In this work we use the Swampland Cobordism Conjecture [8] to extract some general predictions for new non-supersymmetric branes in D-dimensional supergravity theories. The main point is that while the Swampland Cobordism Conjecture asserts that the bordism groups of quantum gravity are trivial:

$$\Omega_k^{\mathrm{QG}} = 0\,, \tag{1.6}$$

in practice, actual bordism groups are often non-trivial! As such, the Cobordism Conjecture predicts that the low energy effective field theory must be supplemented by additional degrees of freedom. Note also that these new objects are necessarily singular in the low energy effective field theory, and are also stable against deformations to configurations describable in the supergravity theory at low energies [28, 32].

For our purposes, the relevant bordism groups are $\Omega_k^{\text{Spin-}\widetilde{G_U}}(\mathrm{pt})$ and $\Omega_k^{\text{Spin-}\widetilde{G_U}^{+}}(\mathrm{pt})$; namely, those in which the Spin structure and duality bundle are correlated, as in lines (1.4) and (1.5). While the computation of these bordism groups is in general quite difficult, in the case of low values of k much more can be said without much machinery. In particular, for the codimension-two objects of the theory, we prove that the bordism group in question is captured by just the Abelianization of the duality bundle structure group:

$$\Omega_1^{\text{Spin-}\widetilde{G_U}^{(+)}}(\text{pt}) = \text{Ab}\big[\widetilde{G_U}^{(+)}\big]\,. \tag{1.7}$$

In the case of $D = 8$, and 9, where G_U contains a simple $\text{SL}(2,\mathbb{Z})$ factor, the Abelianization of $\widetilde{G_U}^{+}$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where one of the factors is generated by a supersymmetric brane in codimension-two, and the other is a non-supersymmetric brane induced from a reflection on the internal torus. In the case of $D \leq 7$ we find that the only surviving element of the bordism group is a $\mathbb{Z}_2$, which geometrically descends from a reflection of a single direction in the internal T^d. Other reflections are obtained by conjugating with the duality group.

This codimension-two object is simply the D-dimensional version of the reflection 7-branes (R7-branes) found[4] in [16, 23] and further studied in [24, 37, 32]. Indeed, starting from the reflection 7-brane associated with a $(-1)^{F_L}$ monodromy transformation, we observe that compactification on additional circles leads to the corresponding objects in the lower-dimensional theory. This reflection brane is non-supersymmetric since it does not preserve any Killing spinors. Additionally, using the same reasoning presented in [28, 32], it is stable against deformations to any smooth field configurations in the low energy effective field theory.

Much as in earlier studies of the R7-brane, we can also use simple topological arguments to characterize some properties of reflection branes. For one, we immediately see that BPS branes can terminate on the reflection brane of the D-dimensional theory. Additionally, since the R7-brane serves as a collapsed bubble configuration for a IIA/IIB wall (with a $(-1)^{F_L}$ monodromy cut),[5] we also see that analogous statements hold for these branes as well, separating the same configurations wrapped on additional directions of an internal T^d.

Similar considerations hold for the dynamics of multiple reflection branes. We find that for reflections associated with an even number of distinct internal directions, the resulting branes form supersymmetric bound states characterized by a local geometry of the form $T^{d-2k} \times \left(\mathbb{C} \times T^{2k}\right)/\mathbb{Z}_2^k$, where the $\mathbb{Z}_2^k$ act on pairs of holomorphic coordinates. In the case of an odd number of distinct reflections, we instead arrive at a non-supersymmetric configuration, namely a supersymmetric background with an additional supersymmetry breaking reflection brane added in. This geometry can be written as $T^{d-(2k+1)} \times \text{Cone}(S^1_\infty \times T^{2k} \times S^1)/\Gamma \times \mathbb{Z}_2^{\text{KB}}$, where the quotient by $\mathbb{Z}_2^{\text{KB}}$ on the factor $S^1_\infty \times S^1$ is a Klein bottle (with Pin^+ structure).

The rest of this paper is organized as follows. In Section 2 we determine the Spin- and Pin^+-lifts of the bosonic duality groups. In Section 3 we compute the one-dimensional bordism groups associated to those Spin- and Pin^+-lifts, and in Section 4 we derive some

[4]See also hints of this brane in [36].

[5]See [32].

properties of the reflection branes predicted by the Cobordism Conjecture. We present our conclusions in Section 5. We discuss some additional features of reflections on the massive and massless spectrum of M-theory in Appendix A. This also provides a complementary perspective on why we must enlarge the U-duality group. Some additional technical details on the standard "bosonic" U-duality groups are reviewed in Appendix B. In Appendix C we discuss the Spin- and Pin$^+$-lifts of these duality groups to $\widetilde{G_U}$ and $\widetilde{G_U}^+$. Appendices D and E discuss some further aspects of the corresponding group (co)homology and bordism groups using spectral sequence techniques.

2 Spin- / Pin$^+$-Lifts of U-Duality Groups

In this paper we focus on the duality groups of maximally supersymmetric non-chiral supergravity theories. The D-dimensional effective supergravity theory can originate both from M-theory compactifications on $\mathbb{R}^{D-1,1} \times T^d$ or type II string theory on $\mathbb{R}^{D-1,1} \times T^{d-1}$. The symmetry / duality group of the theory is known as the U-duality group, and is composed of both strong / weak dualities as well as T-dualities. To be precise, the *bosonic* U-duality group takes the form:[6]

$$G_U = \mathrm{SL}(d, \mathbb{Z}) \ltimes \mathrm{Spin}(d-1, d-1, \mathbb{Z})\,, \tag{2.1}$$

where the components correspond to the group of large diffeomorphisms on the M-theory torus and the group of T-dualities in type II string theories respectively.[7] Here, it is important to note that the T-duality group must include spinor representations since the RR fields transform in bispinor representations of:

$$\frac{\mathrm{Spin}(d-1, \mathbb{Z})_L \times \mathrm{Spin}(d-1, \mathbb{Z})_R}{\mathbb{Z}_2} \subset \mathrm{Spin}(d-1, d-1, \mathbb{Z}), \tag{2.2}$$

namely the left- and right-moving spinors. That being said, observe that purely left-moving or right-moving spinors do not naturally lift to representations of $\mathrm{Spin}(d-1, d-1, \mathbb{Z})$, a fact we will need to handle with care.

In any case, from the perspective of low energy supergravity, the relevant U-duality group is $G_U(\mathbb{R})$, namely the split real form of a certain Lie group. The refinement to M-theory imposes a quantization condition to $G_U \equiv G_U(\mathbb{Z})$, in accord with a discrete spectrum of

[6] In [38] this is stated as $\mathrm{SL}(d, \mathbb{Z}) \ltimes \mathrm{SO}(d-1, d-1, \mathbb{Z})$, but taking into account the RR states and their transformation as spinors of the T-duality group requires this slight refinement.

[7] The classical U-duality group is defined over $\mathbb{R}$, but is broken to the discrete group over $\mathbb{Z}$ due to quantum effects [6].

branes / non-perturbative objects [6,7]. For $3 \leq D \leq 9$, the U-duality groups are:

D	**Bosonic U-duality Group** G_U
9	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{Z})$

(2.3)

See Appendix B for additional discussion of the U-duality groups.

The main subtlety we need to contend with is the behavior of fermions under toroidal compactification, and in particular, the interplay with U-dualities. Locally, one can characterize fermions as transforming in spinor representations, and this clearly descends from 11D. Globally, however, we need to also specify how such objects transform in duality bundles.

Consider, for example, an effective field theory compactified on a T^d. Now, in the case of supersymmetric compactifications to a spacetime with Spin structure, the spinors will transform in a section of a bundle with structure group $\mathrm{Spin}(D-1,1) \times \mathrm{SL}(d,\mathbb{Z})$. We can clearly consider more general U-duality groups Γ, and we characterize this by a connection of the schematic form:

$$\mathcal{D} = d + \omega_{\mathrm{Spin}} + A_{\Gamma}, \tag{2.4}$$

where ω_{Spin} denotes the Spin connection and A_{Γ} the connection for the relevant duality bundle with structure group Γ. In this case, it is natural to restrict to $\Gamma = G_U$, but in general Γ could be different from G_U.

Indeed, nothing really requires us to restrict ourselves to D-dimensional spacetimes with a Spin structure.[8] Rather, one can impose a milder condition in which the Spin connection and duality bundle are correlated in a Spin-Γ bundle with structure group:

$$\text{Spin-}\Gamma: \quad \frac{\mathrm{Spin} \times \Gamma}{\mathbb{Z}_2}, \tag{2.5}$$

where the $\mathbb{Z}_2$ is the diagonal of the $(-1)^F$ generator of $\mathrm{Spin}(D-1,1)$ (i.e., it characterizes the cover $\mathrm{Spin}/\mathbb{Z}_2 = \mathrm{SO}$), and some putative $\mathbb{Z}_2$ subgroup of Γ. From this perspective, we need to determine the Spin- and Pin^+-lifts of G_U. Again, we emphasize that if we restrict to supersymmetric backgrounds with Spin structure, the duality group will appear to be just G_U. If, however, we entertain more general backgrounds, more care will be needed.[9]

[8] F-theory [39–41] is a prototypical example of this sort, in which the base of an elliptically-fibered Calabi-Yau manifold typically may only have a Spin^c structure rather than a Spin structure.

[9] Another way to see the same issue is to work with a fixed T^d, but with tuned moduli. At these enhanced

Our aim in the remainder of this section will be to understand the Spin- and Pin$^+$-lifts of these duality groups, which we respectively denote by $\widetilde{G_U}$ and $\widetilde{G_U}^+$. See also Appendix C.

2.1 Warmup: Lifting $\mathrm{SL}(d, \mathbb{Z})$

Before examining the various extensions of the full U-duality groups in (2.3), our aim in this section will be to first motivate the general discussion with an analysis of Spin- and Pin$^+$-lifts of $\mathrm{SL}(d, \mathbb{Z})$ dualities. These are extensions of the duality groups specified by the large diffeomorphisms of an internal T^d.[10] After this, we will proceed to the more intricate case of U-dualities.

We first study the Spin-lift of $\mathrm{SL}(d, \mathbb{Z})$, the group of large diffeomorphisms of a T^d. The motivation for this case is a gravitational theory with fermionic degrees of freedom compactified on a T^d. In the presence of a non-trivial T^d fibration over the D-dimensional spacetime, the fermionic degrees of freedom will have correlated Spin and duality bundle transformations.

With this in mind, it is instructive to start with the continuous group $\mathrm{SL}(d, \mathbb{R})$, which has maximal compact subgroup $\mathrm{SO}(d)$. From this, we see that the first homotopy groups are given by

$$\pi_1\big(\mathrm{SL}(d, \mathbb{R})\big) = \pi_1\big(\mathrm{SO}(d)\big) = \begin{cases} \mathbb{Z} & \text{if } d = 2\,, \\ \mathbb{Z}_2 & \text{if } d \geq 3\,. \end{cases} \tag{2.6}$$

Thus, for $d \geq 3$ the Spin-lift / double cover of $\mathrm{SL}(d, \mathbb{R})$ is also the universal cover, and is given by a central extension of the form

$$1 \to \mathbb{Z}_2 \to \widetilde{\mathrm{SL}}(d, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R}) \to 1. \tag{2.7}$$

The main exception is $\mathrm{SL}(2, \mathbb{R})$, whose universal cover is a $\mathbb{Z}$-fold central extension. In this case, the metaplectic group $\mathrm{Mp}(2, \mathbb{R})$ is the only non-trivial double cover of $\mathrm{SL}(2, \mathbb{R})$. This singles out this extension as the appropriate one to act on fermions, which transform in Spin-$\widetilde{\mathrm{SL}}$ bundles. From this, we can get the desired Spin-lift / central extension of the discrete group $\mathrm{SL}(d, \mathbb{Z})$ via a pullback from the extension (2.7).

We can generalize even further by allowing for reflections, as necessary when considering the full Pin$^+$-lift of the duality group. In particular, we include a new generator R in $\mathrm{GL}(d, \mathbb{Z}) \setminus \mathrm{SL}(d, \mathbb{Z})$. For example, one such generator is

$$R = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{GL}(d, \mathbb{Z}) \setminus \mathrm{SL}(d, \mathbb{Z}). \tag{2.8}$$

The element R acts on the generators of $\mathrm{SL}(d, \mathbb{Z})$ via conjugation, and all other reflections

loci, additional symmetries act on fermionic degrees of freedom, leading to the same conclusion.

[10] This example is also studied in [33].

are also related to R by conjugation. Thus, the inclusion of this reflection enlarges the duality group to

$$\mathrm{GL}(d,\mathbb{Z}) = \mathrm{SL}(d,\mathbb{Z}) \rtimes \mathbb{Z}_2^R. \tag{2.9}$$

The full Pin$^+$-lift is then given by a $\mathbb{Z}_2$ central extension of the form

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^{+} \to \mathrm{SL}(d,\mathbb{Z}) \rtimes \mathbb{Z}_2^R \to 1, \tag{2.10}$$

where $\widetilde{R} \in \widetilde{G_U}^{+}$ (the lifted reflection generator) satisfies $\widetilde{R}^2 = 1$, as demanded by the Pin$^+$ structure.[11]

With this in hand, we now proceed to consider the more involved case of U-dualities in maximal supergravity theories.

2.2 Spin- and Pin$^+$-Lifts of U-Dualities

In this section we turn to the Spin- and Pin$^+$-lifts of U-dualities.

2.2.1 Spin-Lifts

Consider next the Spin-lift of the full U-duality group G_U, which contains $\mathrm{SL}(d,\mathbb{Z})$ as a subgroup. We denote this by $\widetilde{G_U}$ [33]. This is once again defined via a central extension of the form

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U} \to G_U \to 1\,. \tag{2.11}$$

In particular, we find that $\pi_1\big(G_U(\mathbb{R})\big) = \pi_1(K_U) = \mathbb{Z}_2$ for $3 \leq D \leq 7$, where $G_U(\mathbb{R})$ is the continuous bosonic U-duality group and $K_U \subset G_U(\mathbb{R})$ is the maximal compact subgroup. For $D = 8, 9$ the situation is a bit more delicate (since there is a torsion free factor), but one can still identify a universal $\mathbb{Z}_2$ extension.

As discussed in Appendix C.1, the Spin-lift for the discrete U-duality group G_U can be defined by pulling back from that for $G_U(\mathbb{R})$, much as was the case for $\mathrm{SL}(d,\mathbb{Z})$.[12] Once again, the main exception to this procedure is the group $\mathrm{SL}(2,\mathbb{Z})$, as the universal cover is infinite-sheeted. The metaplectic group $\mathrm{Mp}(2,\mathbb{Z})$ is still the unique nontrivial $\mathbb{Z}_2$ extension; it just arises from the unique nontrivial double cover of $\mathrm{SL}(2,\mathbb{R})$, which is not the universal cover.

2.2.2 Pin$^+$-Lifts

This story can be generalized even further by also allowing for compactifications of M-theory on non-orientable manifolds. In particular, 11D M-theory is well defined on manifolds with

[11] A Pin$^-$-lift would instead require $\widetilde{R}^2 = (-1)^F$.

[12] This argument essentially follows the discussion in [33].

Pin$^+$ structure [42,43], which in turn suggests a further lift of G_U to incorporate orientation reversing transformations such as reflections.[13]

The lift to include reflections is given by a short exact sequence of the form

$$1 \to G_U \to G_U^{(R)} \to \mathbb{Z}_2^R \to 1\,, \tag{2.12}$$

which describes a semi-direct product

$$G_U^{(R)} = G_U \rtimes \mathbb{Z}_2^R\,. \tag{2.13}$$

The generator R of the $\mathbb{Z}_2^R$ in this extension descends from a reflection element in the disconnected component of $G_U^{(R)}(\mathbb{R})$.[14] In practice, we determine R as a matrix in the disconnected component of the maximal compact subgroup $K_U^{(R)} \subset G_U^{(R)}(\mathbb{R})$. This in turn defines a short exact sequence of the form

$$1 \to K_U \to K_U^{(R)} \to \mathbb{Z}_2^R \to 1\,. \tag{2.14}$$

The relevant maximal compact subgroups are summarized as

D	**Maximal Compact Subgroup** K_U
9	SO(2)
8	SO(3) $\times$ SO(2)
7	SO(5)
6	(Spin(5) $\times$ Spin(5))/$\mathbb{Z}_2^{\text{diag}}$
5	USp(8)/$\mathbb{Z}_2$
4	SU(8)/$\mathbb{Z}_2$
3	SO(16)

. (2.15)

Finally, the full Pin$^+$-lift of interest is given by combining the two extensions (2.11) and (2.12) into an extension of the form

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^{+} \to G_U \rtimes \mathbb{Z}_2^R \to 1, \tag{2.16}$$

where we demand that the lifted reflection generator $\widetilde{R}$ satisfies $\widetilde{R}^2 = 1$. See Appendix C.2 for an explicit construction of the Pin$^+$-lifts in each dimension.

As a final comment, we note that we have now fixed that the fermions in the D-

[13] From the bottom up perspective, it may appear that, for example, 9D supergravity is well-defined on Pin$^-$-manifolds. However, in order to match with the top down construction, i.e., anomaly cancellation constraints in 11D M-theory, we must restrict to the Pin$^+$-lift.

[14] Once again, we only need to include a single new generator for the reflections, as all of the reflections are related by conjugation.

dimensional effective theory transform as sections of a

$$\frac{\text{Spin} \times \widetilde{G_U}^{+}}{\mathbb{Z}_2} \tag{2.17}$$

bundle over spacetime. With this in hand, we now move on to computing the relevant bordism groups, and analyze the predictions made by the Cobordism Conjecture in this setting.

3 Bordisms and Branes

In the previous section we discussed the Spin- and Pin$^+$-lifts of the bosonic U-duality groups. Our aim in this section will be to use the Swampland Cobordism Conjecture [8] to predict new objects in the low energy effective theory.

The general setup we consider involves a D-dimensional effective field theory which enjoys a duality symmetry Γ. We have in mind situations where Γ is realized as a $\mathbb{Z}_2$ extension, as appropriate for a Spin- and Pin$^+$-lift, which we characterize by the short exact sequence:

$$1 \to \mathbb{Z}_2 \to \Gamma \to \Gamma/\mathbb{Z}_2 \to 1\,. \tag{3.1}$$

In general, there can be a non-trivial correlation between the Spin structure of spacetime and the duality bundle, and so we refer to Spin-Γ twisted bundles as those which mix the tangential and internal symmetries:[15]

$$\text{Spin-}\Gamma \equiv \frac{\text{Spin} \times \Gamma}{\mathbb{Z}_2}\,, \tag{3.2}$$

where the $\mathbb{Z}_2$ embeds as $(-1)^F$ in Spin and the image of the $\mathbb{Z}_2$ in the central extension (3.1) in the duality group Γ.

The Swampland Cobordism Conjecture [8] asserts that when the effective field theory has a non-trivial bordism group $\Omega_k^{\mathcal{G}}$ (where we consider bordism of manifolds with respect to whatever structure $\mathcal{G}$ is needed to define the theory), one must enrich the low energy effective field theory by additional dynamical objects of codimension $(k+1)$. One way to understand this condition is that the presence of a non-trivial bordism group generator implies the existence of a p-form symmetry $G^{(p)} = \text{Hom}(\Omega_k^{\mathcal{G}}, \text{U}(1))$, where $p = D - (k+1)$; namely, one can construct an extended object filling p spacetime dimensions. Following the formulation of generalized global symmetries given in [44], there are corresponding topological symmetry operators which link with these defects. In particular, they fill out $q = k$ dimensions (so that $p + q = D - 1$).

The general lore from quantum gravity is that all such putative global symmetries are

[15] See, e.g., [23] for a more mathematical definition.

actually broken (i.e., absent) or gauged. Let us first discuss breaking and then turn to gauging. The topological linking can be destroyed if additional dynamical states are added to the spectrum; namely, if our original defect can terminate on an object which fills $p-1$ spatial dimensions, as well as time. This is then a codimension-$(k+1)$ object in the theory. Observe that in terms of the field content of the original effective field theory, this configuration is necessarily singular, i.e., there is no deformation to a smooth configuration. Supplementing the theory by these additional dynamical states allows any putative topological linking to be destroyed, thus removing the candidate global symmetry.

Gauging a finite p-form symmetry turns out to be somewhat subtle, and winds up introducing a magnetic dual global higher-form symmetry [44]. To get rid of these candidate symmetries we need to introduce additional degrees of freedom anyway. As such, we take the most conservative interpretation and include the expected defects right from the start.

Now, in the case at hand where we have fermions coupled to both the Spin connection and a duality bundle, the relevant connection is of the schematic form described in line (2.4), which we reproduce here:

$$\mathcal{D} = d + \omega_{\mathrm{Spin}} + A_{\Gamma}, \tag{3.3}$$

where ω_{Spin} denotes the Spin connection and A_Γ the connection for the relevant duality bundle, and there is a gauging by a diagonal $\mathbb{Z}_2$. This has the consequence that transition functions can be interpreted in multiple ways. For example, consider the transition function $(\theta, g) \in \mathrm{Spin} \times \Gamma$; due to the $\mathbb{Z}_2$ quotient, this is identified with

$$(\theta, g) \simeq ((-1)^F \theta, Fg)\,, \tag{3.4}$$

where we denote the image of $1 \in \mathbb{Z}_2$ in Γ, determined from (3.1), by F. This is precisely the reason why the twisted theory can be formulated on non-Spin manifolds, because the cocycle condition on triple overlaps involves the equivalence class of (θ, g) and not θ alone. Thus, the obstruction to Spin structure, captured by the cocycle condition for θ, is compensated by a non-trivial duality bundle, captured by the transition functions g. But even in the case the underlying manifold X allows for a Spin structure, the identification in (3.4) has the important consequence that the choice of Spin structure, classified by an element in $H^1(X;\mathbb{Z}_2)$, can be reinterpreted as a choice of duality bundle. The most important situation for us is the circle S^1, which we will discuss in more detail next.

The circle allows for two different Spin structures, because $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$. The two Spin structures are characterized by periodic, S^1_+, or anti-periodic, S^1_-, boundary conditions for fermions. The Γ bundles are characterized by $H^1(S^1;\Gamma)$,[16] which for discrete Γ can be identified with Γ itself. This means that Γ-bundles are classified by a transition function / monodromy g when going around the circle. Denoting a circle with boundary conditions $\pm$

[16] If G is a non-Abelian group, the standard definition of $H^k(X;A)$ with coefficients in an Abelian group A can be extended to G in a sensible way for $k \leq 1$, and like in the Abelian case, $H^1(X;G)$ classifies principal G-bundles through their monodromy around non-trivial 1-cycles.

and monodromy g as $(S^1_\pm)_g$, the gauging of the diagonal $\mathbb{Z}_2$ leads to the identification

$$(S^1_\pm)_g \simeq (S^1_\mp)_{Fg}\,, \tag{3.5}$$

i.e., one can trade a $(-1)^F$ for F. This leads to the reduction of inequivalent manifolds, which will be important for the determination of the bordism groups below.

We are interested in calculating the bordism group $\Omega_1^{\text{Spin-}\Gamma}(\text{pt})$. We present a direct calculation of this in Appendix E, where we establish that, when $D \leq 7$ and $\Gamma = \widetilde{G_U}$, this bordism group is trivial. Meanwhile, for $D \leq 7$ and $\Gamma = \widetilde{G_U}^+$, we instead have a single $\mathbb{Z}_2$ factor associated with monodromy by a reflection in the internal toroidal directions. The cases $D = 8, 9$ have additional (supersymmetry preserving) bordism generators.

Our aim here will be to establish the same result in more physical terms. The end result of our analysis is that for a non-trivial Spin-Γ bundle, the bordism group is:

$$\Omega_1^{\text{Spin-}\Gamma}(\text{pt}) = \text{Ab}[\Gamma]\,. \tag{3.6}$$

3.1 Spin-Lifts, Pin$^+$-Lifts, and Bordisms

In this section we turn to the Spin- and Pin$^+$-Lifts of U-duality groups, and the corresponding first bordism groups associated with Spin-twisted duality bundles.

3.1.1 Spin-Lift and Bordisms

Consider first the Spin-lift of the U-duality groups, $\widetilde{G_U}$. As discussed above, the only manifolds we need to consider are circles $(S^1_\pm)_g$ with $\widetilde{G_U}$ monodromy g. Using the identification (3.5), we always fix the fermionic boundary conditions to be bounding. This can modify the duality bundle. More concretely, for non-bounding boundary conditions we rewrite

$$(S^1_+)_g \simeq (S^1_-)_{Fg}\,. \tag{3.7}$$

However, since $\widetilde{G_U}$ is perfect (see Appendix D) for $D \leq 7$, the element Fg is part of the commutator subgroup of $\widetilde{G_U}$. Similar to the results in [11, 23, 37, 31], this provides a gravitational soliton configuration with a well-defined twisted Spin structure that bounds the one-dimensional manifold (see Figure 1). We therefore see that, for $D \leq 7$,

$$\Omega_1^{\text{Spin-}\widetilde{G_U}}(\text{pt}) = 0\,. \tag{3.8}$$

This is also proven by a spectral sequence argument in Appendix E. Since the bordism group vanishes there is no need to include any codimension-2 Spin-$\widetilde{G_U}$ defects in the theory to break global symmetries.

The story is different for duality groups with non-trivial Abelianization, i.e., for $D = 8, 9$,

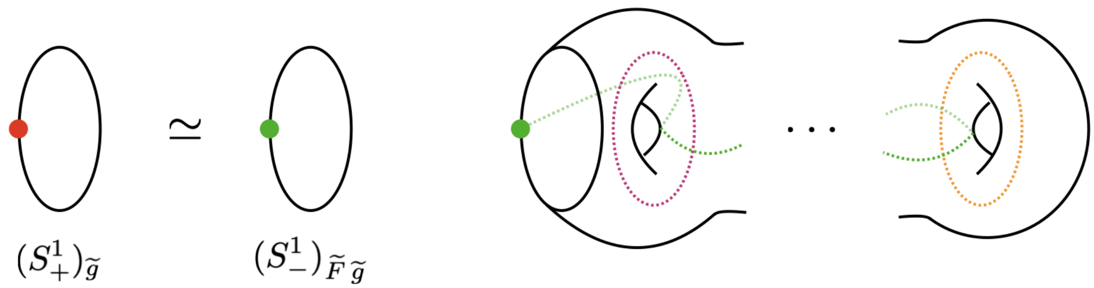

Figure 1: Bounding of every one-dimensional manifold with Spin-$\widetilde{G_U}$ structure (the duality bundle is depicted via transition functions on codimension-one sub-manifolds).

since G_U has a simple $\mathrm{SL}(2,\mathbb{Z})$ factor. There, the Spin-lift can lead to a non-trivial extension. For example, as found in [23]:

$$\Omega_1^{\text{Spin}}\big(B\mathrm{SL}(2,\mathbb{Z})\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_{12} \to \mathbb{Z}_{24} = \Omega_1^{\text{Spin-Mp}(2,\mathbb{Z})}(\mathrm{pt})\,. \tag{3.9}$$

Again, we can understand this pictorially by translating the Spin structure on the spacetime circle to a shift in the transition function of the duality bundle. This reasoning is compatible with the relation

$$\Omega_1^{\text{Spin-}\Gamma}(\mathrm{pt}) = \mathrm{Ab}[\Gamma]\,, \tag{3.10}$$

for more general Γ, which we prove in Appendix E when Γ is one of the U-duality groups.

3.1.2 Pin$^+$-Lift and Bordisms

Next, we turn to the Pin$^+$-lift of the duality group G_U after the inclusion of a reflection element R, which can be understood as extending G_U to the semi-direct product $G_U \rtimes \mathbb{Z}_2^R$. The fact that we do a Pin$^+$-lift means that R lifts to an element in $\widetilde{G_U}^+$, which we denote by $\widetilde{R}$, that still squares to the identity

$$\widetilde{R}^2 = 1 \in \widetilde{G_U}^+\,. \tag{3.11}$$

As before, the $\mathbb{Z}_2$ of the extension defining the lift is identified with the $(-1)^F$ coming from $\mathrm{Spin}(D-1,1)$, and we can translate the choice of Spin structure into a duality bundle.

The different backgrounds we need to consider are given by circles with a duality transition function. With the exact same argument as above, using that each element in G_U as well as F can be written in terms of a commutator, there are bounding configurations of the type depicted in Figure 1. The only generator left to discuss is $(S^1_-)_{\widetilde{R}}$, i.e., the circle with a transition function given by reflection. However, the reflection element can never be a commutator and thus cannot be bounded by manifolds of the type above.

For that let us recall that the Pin^+-lift can be described by the short exact sequence

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^+ \to G_U \rtimes \mathbb{Z}_2^R \to 1\,. \tag{3.12}$$

Now if there was a way to write $\widetilde{R}$ as a (product of) commutator(s), we could map this to $G_U \rtimes \mathbb{Z}_2^R$, where the equation should still hold and produce R as a commutator. Note, however that $G_U^{(R)}(\mathbb{R})$ has two disconnected components, and under the "determinant map" $\det : G_U^{(R)}(\mathbb{Z}) \to \mathbb{Z}_2^{(R)}$,[17] the image of G_U is $+1$ and the element R maps to -1. Then, it is clear that R cannot be a commutator, as it always contains an even number of elements with determinant -1 and hence can only produce elements with positive determinant. The same is true for $\widetilde{R}$ in $\widetilde{G_U}^+$.[18]

One might also ask whether there might be other manifolds that can bound the circle with $\widetilde{R}$ transition function which are not included in the class above. If that was true, one could glue two of them along their common boundary, given by $(S^1_-)_{\widetilde{R}}$, to obtain a compact, orientable, and smooth two-dimensional manifold. These are simply the Riemann surfaces, which are captured by the argument above. Any other such manifolds cannot exist.

With the arguments above, for G_U perfect (i.e., for $D \leq 7$), we find that,

$$\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt}) = \mathbb{Z}_2 \;\text{ for }\; 3 \leq D \leq 7, \tag{3.13}$$

with the generator given by $(S^1_-)_{\widetilde{R}}$. This is in accord with the general discussion given above, and the result:

$$\Omega_1^{\text{Spin-}\Gamma}(\text{pt}) = \text{Ab}[\Gamma]\,, \tag{3.14}$$

with proof in Appendix E in the case that Γ is one of the U-duality groups. This includes $D = 8, 9$ for which one has

$$\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt}) = \text{Ab}\big[\widetilde{G_U}^+\big] = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \;\text{ for }\; D = 8, 9, \tag{3.15}$$

with one $\mathbb{Z}_2$ associated to a circle with reflection and the other $\mathbb{Z}_2$ to a circle with non-trivial $\text{Mp}(2,\mathbb{Z})$ monodromy given by a Spin-lift of the S generator for $\text{SL}(2,\mathbb{Z})$.

We thus conclude that in the original D-dimensional effective theory, we get a non-trivial codimension-two defect, as obtained from the Pin^+-lift of reflections on the internal torus directions. We refer to this as a "reflection brane" since it arises from an internal reflection on the torus. Note also that nothing singles out a particular direction of reflection. Indeed, conjugating by the internal $\text{SL}(d,\mathbb{Z})$ transformations, or even G_U transformations, this reflection can instead act on any of the internal T^d torus coordinates and involve T-dualities. The main point is that under monodromy around the defect, the orientation of T^d

[17]For any representation it is just a determinant on the corresponding linear maps.

[18]Though not relevant for our present physical purposes, the same holds true for the Pin^- lift.

reverses:

$$T^d \to \overline{T^d}. \tag{3.16}$$

3.2 Extra Branes at $D = 8, 9$

Summarizing the above discussion, for $3 \leq D \leq 7$, the bordism group $\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt}) = \mathbb{Z}_2$, and this predicts the existence of a reflection brane which trivializes the corresponding bordism class in Ω_1^{QG}. For $D = 8, 9$, the Abelianization of $\widetilde{G_U}^+$ is actually $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, so there is an additional generator to contend with. This is essentially because in both of these cases, G_U contains a non-trivial $\text{SL}(2, \mathbb{Z})$ factor, and $\text{Ab}[G_U] = \mathbb{Z}_{12}$. The other brane predicted by these generators is essentially the same one already discussed in [23]: it is a supersymmetric codimension-two object.

For example, in $D = 9$, this is characterized by a T^2 fibration over $\mathbb{C}$ in which the complex structure is fixed to $\tau = i$. There are different 6-brane configurations which realize this. For example, we can use a Kodaira fiber of type III* as well as a Kodaira fiber of type III. Collapsing this elliptic fiber to zero size would result in an $\mathfrak{e}_7$ (for type III*) and $\mathfrak{su}_2$ gauge symmetry (for type III). Indeed, essentially the same configuration was considered in the F-theory backgrounds of IIBordia in [23].

In the case of $D = 8$, we can realize the relevant generator by working with type IIB string theory compactified on a T^2_{spatial}. Allowing the elliptic fibration of F-theory to have precisely the form indicated above amounts to compactifying the relevant $\tau = i$ 7-brane configuration on a further T^2_{spatial}. This is essentially the same strategy also used in [31] to partially geometrize the U-duality groups in M- / F-theory.

4 Properties of Reflection Branes

In the previous section we used the Swampland Cobordism Conjecture to argue for the existence of codimension-two reflection branes, as captured by a $\mathbb{Z}_2$ factor of $\text{Ab}[\widetilde{G_U}^+]$. In this section we establish some basic properties of these objects. Some aspects of this analysis amount to generalizations of what was found in [16] for the R7-branes of type IIB / F-theory (see also [32]). Indeed, one can view our reflection branes as descending from R7-branes wrapped on internal cycles. We begin by establishing that these objects break supersymmetry, but are nevertheless stable.[19] We then turn to an analysis of the BPS objects which can terminate on our reflection branes. Following this, we consider some preliminary aspects of their dynamics, primarily focusing on properties well constrained by topological considerations. Along these lines, we study the braiding of branes constructed from different reflections, as well as the class of bound states these objects form.

[19]This is to be contrasted with the case of unstable non-supersymmetric orbifolds, see e.g., [45–52]. Stability in the present case follows from similar considerations to those presented in [28] (see also [32]).

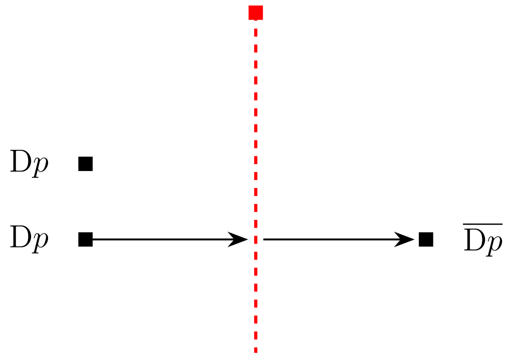

Figure 2: Depicition of a pair of BPS Dp-branes near a IIA R7-brane reflection brane. Passing one Dp-brane through the branch cut of the R7-brane (red) becomes a $\overline{\mathrm{D}p}$-brane when the $(p+1)$-form potential it is coupled to picks up a minus sign. Under toroidal compactification, similar considerations hold for all reflection branes and probes by BPS objects.

At a general level, we observe that the reflection branes considered here can be viewed as arising from the IIA reflection 7-brane of type IIA string theory (see [32]). The M-theory lift of this configuration is simply a cone over a Klein bottle, in which the Klein bottle is viewed as a circle fibration, where the fiber undergoes orientation reversal in winding around the base circle.[20] As such, many of the qualitative properties of a single reflection brane follow directly from compactification of such objects. With this in mind, many of the salient properties, such as supersymmetry breaking, stability, as well as the spectrum of objects which can terminate on these branes directly follow from toroidal compactification of the IIA R7-brane.

4.1 SUSY Breaking and Stability

In this section we argue that the reflection brane completely breaks supersymmetry in the D-dimensional effective supergravity theory.

As a warmup, we first consider type IIA string theory with a $(-1)^{F_L}$ R7-brane and some supersymmetric Dp-brane probes. Passing one such brane through the branch cut generates an anti-Dp-brane, so the combination of branes and anti-branes (and R7-brane) breaks all supersymmetries. Since we can compactify this configuration on a T^d, we conclude that the related reflection branes probed by BPS branes also break supersymmetry. See Figure 2 for a depiction of this configuration.

With this in hand, we will now show that the reflection brane itself breaks supersymmetry in the D-dimensional effective theory. To that end, recall that a background is only

[20]Indeed, the bordism class of the Klein bottle generates $\Omega_2^{\mathrm{Pin}^+}(\mathrm{pt})$ [53, Proposition 3.9], and the corresponding defect predicted by the Swampland Cobordism Conjecture [8] is the M-theory lift of the IIA R7-brane [32].

supersymmetric if there is a non-trivial solution to the Killing spinor equation:

$$\nabla_\mu Q = 0. \tag{4.1}$$

Here ∇_μ depends on background fields and Q is a complex supercharge in the D-dimensional theory. We claim that upon including the reflection brane there are no solutions to (4.1).[21]

We can see that supersymmetry is broken by studying the monodromy action on any candidate solution to (4.1). The reflection branes are codimension-two defects in the D-dimensional theory. As such, it is instructive to split the D-dimensional spacetime as $\mathbb{R}^{D-3,1} \times \mathbb{R}^2$. Then, the reflection brane breaks the Lorentz algebra in D-dimensions to

$$\mathfrak{so}(D-1,1) \to \mathfrak{so}(D-3,1) \times \mathfrak{so}(2), \tag{4.2}$$

where $\mathfrak{so}(D-3,1)$ is the Lorentz algebra in $(D-2)$-dimensions along the brane worldvolume and $\mathfrak{so}(2)$ are transformations in $\mathbb{R}^2$, which can be understood as rotations in the plane perpendicular to the reflection brane.

Under rotations by an angle θ associated to the $\mathfrak{so}(2)$ factor of (4.2), the supercharge Q transforms as[22]

$$Q \mapsto e^{i\theta/2} Q. \tag{4.3}$$

On the other hand, Q undergoes a monodromy under a full rotation around the reflection brane:

$$Q \mapsto \omega \overline{Q}, \tag{4.4}$$

where ω is a phase related to the net conical deficit angle obtained by circling around the reflection brane. Importantly, there are no solutions to (4.3) that can satisfy the condition (4.4), so the reflection brane must break supersymmetry in the D-dimensional theory. This can be seen by noting that any parity reversing operation, such as reflections, has determinant -1. On the other hand, any transformation under $\mathfrak{so}(2)$ must have determinant $+1$.

Stability While the reflection branes are charged, and thus cannot disappear entirely, the fact that these objects are non-supersymmetric might at first suggest that they are unstable and will eventually expand into an energetically favorable configuration. However, the branes in question are only charged under a discrete group which does not embed in a continuous group. Indeed, if the brane was actually unstable, then there would have to be a smooth field configuration in the low energy supergravity theory for the brane to expand into. The obstructions to such a configuration are characterized by the SUGRA bordism groups $\Omega_*^{\mathrm{SUGRA}}$ [28].[23] In other words, objects in $\Omega_*^{\mathrm{SUGRA}}$ cannot be realized as a smooth configuration in the effective theory and thus must be stable against deformations to EFT

[21] This was previously shown for the R7-brane in [16].

[22] Note that this is the standard transformation of spin-1/2 operators under rotations.

[23] See [32] for the case of the R7-brane.

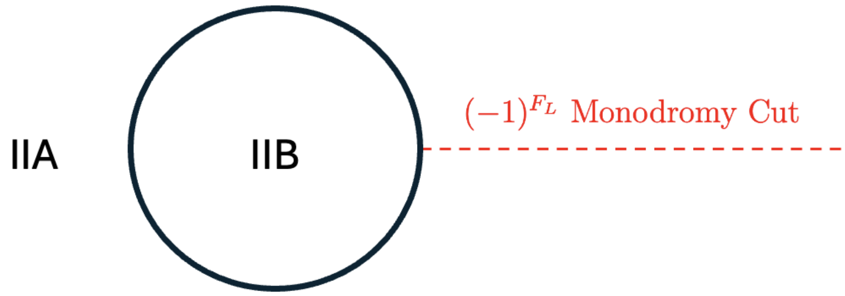

Figure 3: Top down view of the cylindrical configuration of the type IIA/IIB wall with topology $S^1 \times \mathbb{R}^{D-1} \times T^d$ with a $(-1)^{F_L}$ monodromy cut. IIB is on the inside of the wall while IIA is on the outside. The configuration collapses due to the tension of the wall, and the endpoint of the collapse is the reflection brane. This configuration lifts to a M- / F-theory wall.

configurations.[24] Indeed, the reflection branes in this paper are exactly objects in such bordism groups descending from M-theory compactifications.

4.2 Bubbles and Walls

Much as in the case of the reflection 7-branes of type IIA and IIB, we also expect these reflection branes to arise from codimension-one walls on collapsing cylindrical configurations with a monodromy cut. Similar to reference [32], we consider a IIA/IIB wall with an $(-1)^{F_L}$ cut emanating out of the wall. We put IIB in the interior of the cylindrical configuration, and IIA on the outside. This collapses to the R7-brane of type IIA, which in turn lifts in M-theory to a cone over a Klein bottle. We can take this same setup and compactify over an additional T^d. As such, we see that there is a bubble-like configuration which collapses to our reflection brane.[25] See Figure 3 for a depiction.

4.3 Lasso Configurations and Worldvolume Degrees of Freedom

We now determine some physical properties of the reflection branes by probing them with known supersymetric branes [16]. The probe analysis relies on determining how the various supergravity p-form potentials transform under a reflection, i.e., whether the reflection brane

[24]One could of course hypothesize some as yet unknown deformation to another singular configuration in the UV completion, but one might as well refer to this object as the same thing as the original one predicted by the Swampland Cobordism Conjecture.

[25]What about collapsing configurations involving multiple reflections? This is is considerably more subtle. The issue is that, as of this writing, the corresponding wall which can collapse to other reflection branes is not known (see [32] for further discussion on this point). Because of this complication, we defer an analysis of this case to future work.

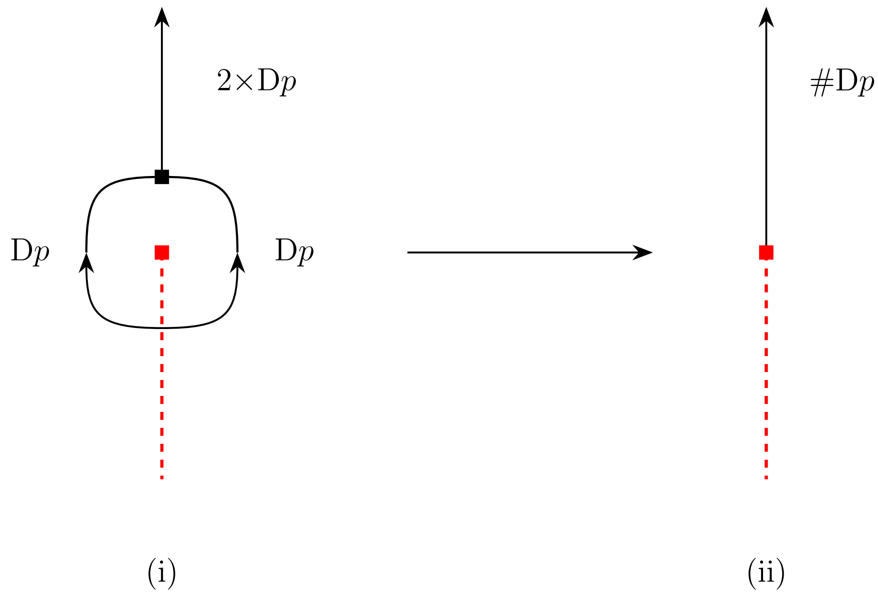

Figure 4: Type IIA $(-1)^{F_L}$ R7-brane in the presence of BPS brane probes. (i) Two Dp-branes (black lines) are joined by passing through the branch cut of the reflection brane (red). The branes combine at a junction (black square) and extend to infinity. (ii) If two Dp-branes can be lassoed in this way, then this implies that any number of the brane (not just an even number) can terminate on the R7-brane. Wrapping these branes on directions of an internal T^d results in similar configurations for reflection branes probed by BPS objects of the D-dimensional effective field theory.

acts via charge conjugation on the supersymmetric brane coupled to the p-form potential or not.

One of two things can happen: either the p-form potential picks up a minus sign under a reflection, in which case the reflection brane acts via charge conjugation on the probe supersymmetric brane, or the p-form potential is invariant under a reflection. In the former case, two copies of the probe supersymmetric brane, that extend out to infinity, can be joined by passing through the branch cut of the reflection brane. The pair of probe branes then combine at a junction (see e.g., [54–57]), completing a lasso around the reflection brane. The lasso then collapses to an energetically favored configuration, leaving an even number of probe supersymmetric branes that end on the reflection brane. See Figure 4 for an example.

Note also that since the reflection branes arise as a collapsing cylindrical configuration separating F-theory and M-theory backgrounds, we can also ask about the fate of such branes as they pass from one side of the M- / F-theory wall (the lift of the IIA/IIB wall) to the other side (see Figure 3). Following the discussion in [32], this in turn means that once we establish that a brane-lasso configuration can terminate on a reflection brane, we can actually strengthen the conclusion to argue that a single supersymmetric brane (instead of a pair) can actually terminate on it.

By determining the various types of supersymmetric branes that end on the reflection brane, we are also able to partially uncover the worldvolume degrees of freedom of the reflection branes themselves. It is important to note that our procedure is only sensitive to branes that can be "lassoed" in this way. Thus, the full worldvolume theory of the reflection branes may be more complicated. In the following, we review the R7-branes of type IIA and perform a detailed analysis for the reflection 6-brane, which is obtained by a circle compactification of the R7-brane. We then generalize the story to all the reflection branes, which are obtained via further circle compactifications.

4.3.1 Reflection 7-Branes of Type IIA

Our primary interest is in the reflection branes generated by $D \leq 9$ M-theory vacua. That being said, some of our considerations already follow from codimension-two defects of type IIA, namely the R7-brane associated with $(-1)^{F_L}$ monodromy (see [16, 32]). As such, we already know that all of the Dp-branes go to anti-Dp-branes under monodromy. We also know that there are lasso configurations where pairs of these branes can terminate on the reflection 7-brane. Furthermore, using the general arguments provided in [32], where we construct the 7-brane from a collapsed IIA / IIB wall wrapped on a cylindrical configuration with an $(-1)^{F_L}$ monodromy cut, one can actually conclude that a single Dp-brane can terminate on the R7-brane.[26]

In the lift to M-theory, the D0-brane descends from KK momenta on the M-theory circle, and the D6-brane is its magnetic dual counterpart. Likewise, the F1-string descends from a wrapped M2-brane, and the NS5-brane and D4-brane are the magnetic dual counterparts.

Our aim in the remainder of this section will be to characterize the relevant supersymmetric objects which can terminate on the reflection brane of the D-dimensional effective field theory. Since the objects associated with KK momenta behave universally across all spacetime dimensions, we primarily focus on those degrees of freedom which couple to the M-theory 3-form potential C_3 or its magnetic dual 6-form potential $\widetilde{C}_6$.

4.3.2 Reflection 6-Brane

We begin by analyzing all of the possible lasso configurations involving the reflection 6-brane, which is the codimension-two defect in the 9D effective supergravity theory predicted by the Cobordism Conjecture. The possible supersymmetric branes / p-form potentials in the 9D theory descend from the M2-branes coupled to the M-theory 3-form C_3, as well as the magnetic dual M5-branes coupled to the M-theory 6-form $\widetilde{C}_6$. It is important to note

[26] Another way to reach the same conclusion is to consider a pair of branes which end on the R7-brane, and to then let one of the branes move off to infinity. While this may not be energetically preferred, nothing obstructs this deformation at the level of realizing off-shell configurations.

that C_3 is a pseudo 3-form while $\widetilde{C}_6$ is a real 6-form.[27]

With this in hand, and to be concrete, we first partition the internal T^2 of the 11D supergravity compactification as

$$\mathbb{R}^{8,1} \times S^1_{(1)} \times S^1_{(2)}, \tag{4.7}$$

where the subscript denotes the two cycles of the T^2. Furthermore, and without loss of generality, let the reflection R_1 act on the first cycle $S^1_{(1)}$.

Let us begin by analyzing the descendants of C_3 that undergo a parity transformation under this setup. The possible candidates include 0-form, 1-form, 2-form, and 3-form potentials.

0-Form Potentials from C_3 There are no candidate 0-form potentials descending from C_3 in the 9D effective theory. This is because any candidate 0-form could only emerge if all of C_3 was compactified in the internal directions, which is not possible in this case.[28]

1-Form Potentials from C_3 Starting from C_3, there is a single candidate 1-form potential in the 9D effective theory that descends from compactifying C_3 on both internal directions. We denote the 1-form as $A_1^{[12]}$ where the superscript denotes the compactified directions. However, we see that

$$A_1^{[12]} \xrightarrow{R_1} A_1^{[12]}, \tag{4.8}$$

as the 1-form picks up two minus signs under the reflection (one from C_3 and one from the reflection of the coordinate itself). Thus, no branes coupled to $A_1^{[12]}$ can terminate on the reflection 6-brane via lasso configurations.

2-Form Potentials from C_3 Starting from C_3, there are two candidate 2-form potentials in the 9D effective theory that descend from compactifying C_3 on either $S^1_{(1)}$ or $S^1_{(2)}$. We

[27]We can see this by studying the 11D supergravity action, which should be invariant under parity transformations such as reflections. In particular, notice that the topological Chern-Simons term

$$\int C_3 \wedge G_4 \wedge G_4, \tag{4.5}$$

where locally $G_4 = dC_3$, is only reflection invariant if we take C_3 to be a pseudo 3-form. Likewise, the kinetic term

$$\int G_4 \wedge G_7, \tag{4.6}$$

where locally $G_7 = d\widetilde{C}_6$, is only invariant under reflections if G_7, and consequently $\widetilde{C}_6$, is a real 6-form.

[28]Note that there can still be scalars in the D-dimensional theory that do not originate from C_3 and undergo non-trivial monodromy. For example, the complex structure modulus has monodromy $\tau \to -\overline{\tau}$.

denote them as $B_2^{[1]}$ and $B_2^{[2]}$, respectively. Upon reflection, we see that

$$\begin{aligned} B_2^{[1]} &\xrightarrow{R_1} B_2^{[1]}, \\ B_2^{[2]} &\xrightarrow{R_1} -B_2^{[2]}. \end{aligned} \tag{4.9}$$

This implies that any number of branes coupled to $B_2^{[2]}$ can terminate on the reflection 6-brane. These are M2-branes compactified on $S^1_{(2)}$, i.e., effective strings obtained from wrapped D2-branes.[29]

3-Form Potentials from C_3 There is a single candidate 3-form potential in the 9D effective theory that descends from not compactifying C_3 on any of the internal directions. We already argued that

$$C_3 \xrightarrow{R_1} -C_3, \tag{4.10}$$

which implies that any number of M2-branes can terminate on the reflection 6-brane.

Magnetic Dual Branes In addition to the branes found above, the magnetic dual branes can also terminate on the reflection 6-brane. These branes couple to p-form potentials descending from the M-theory 6-form $\widetilde{C}_6$. In other words, the branes are M5-branes wrapping some number of internal direction. The only difference is that since $\widetilde{C}_6$ is a real 6-form, the M5-brane has to wrap the cycle acted on by the reflection, i.e., $S^1_{(1)}$, in order to pick up a minus sign.

For the case of the reflection 6-brane, the relevant magnetic dual branes couple to $\widetilde{B}_5$, which is the dual of B_2, and $\widetilde{C}_4$, which is the dual of C_3. From this, we see that any number of 4-branes and 3-branes (coming from M5-branes wrapping either a single or both internal cycles respectively) can terminate on the reflection 6-brane.

Worldvolume Degrees of Freedom We can use lasso arguments to partially determine the worldvolume degrees of freedom of the reflection 6-brane. To be concrete, take the case of 4-branes terminating on the reflection 6-brane. The worldvolume coupling of the 4-brane is:

$$\int_{\Sigma_5} B_5. \tag{4.11}$$

This is not gauge invariant under the gauge transformation $B_5 \to B_5 + d\lambda_4$ when the 4-brane worldvolume has a boundary. The problem comes from the following boundary term

$$\int_{\partial\Sigma_5} \lambda_4, \tag{4.12}$$

[29] Here we take $S^1_{(1)}$ to be the M-theory / IIA circle.

and is important in this case because the boundary of the 4-brane is contained in the reflection 6-brane. We can cancel this boundary term via another coupling term, this time in the reflection 6-brane, that also transforms as $B_5 \to B_5 + d\lambda_4$:

$$-\int_{\text{6-brane}} \lambda_4 \wedge (\partial\Sigma_5)_{\text{PD}}, \tag{4.13}$$

where the subscript denotes the Poincaré dual. This can come from a term such as

$$\int_{\text{6-brane}} B_5 \wedge f_2, \tag{4.14}$$

where f_2 is the field strength for a 1-form gauge field. Upon variation, this term gives

$$\int_{\text{6-brane}} \lambda_4 \wedge df_2. \tag{4.15}$$

From this, we see that if we identify df_2 with $(\partial\Sigma_5)_{\text{PD}}$, then the configuration will be gauge invariant. Furthermore, this suggests that there is a 1-form gauge field on the worldvolume of the 6-brane. Applying this reasoning to the other relevant p-form potentials, we see that there must also be 2-form, 3-form, and 4-form gauge fields on the worldvolume of the reflection 6-brane.

4.3.3 Generalizations

We now generalize the previous analysis for the reflection 6-brane to the reflection $(D-3)$-brane, which is the codimension-two defect in the D-dimensional effective supergravity theory predicted by the Cobordism Conjecture. Let d denote the number of internal directions. I.e., the 11D spacetime splits as $\mathbb{R}^{D-1,1} \times T^d$. We begin by enumerating all of the p-form potentials descending from C_3 that undergo a reflection. Any number of supersymmetric branes coupled to these p-form potentials can terminate on the reflection branes.

0-Form Potentials from C_3 For $d > 3$, there are $\binom{d-1}{3}$ 0-form potentials that undergo a reflection. These couple to M2-branes wrapping three internal directions, namely pointlike instantons. Strictly speaking, these instantons do not end on the reflection brane, but they become anti-instantons after winding around the reflection brane.

1-Form Potentials from C_3 For $d > 2$, there are $\binom{d-1}{2}$ 1-form potentials that undergo a reflection. These couple to M2-branes wrapping two internal directions.

2-Form Potentials from C_3 There are $(d-1)$ 2-form potentials that undergo a reflection. These couple to M2-branes wrapping a single internal direction.

3-Form Potentials from C_3 There is a single 3-form potential that undergoes a reflection. This is C_3 itself, which suggests that any number of M2-branes can terminate on all of the reflection branes.

There are an equal number of magnetic dual branes, descending from $\widetilde{C}_6$, that terminate on the reflection branes. The worldvolume degrees of freedom for each of the reflection branes are determined in exactly the same manner as was done for the reflection 6-brane above. They can also be deduced by compactifiying the reflection 6-brane down to the relevant dimension.

4.4 Braiding and Binding

We now consider the interplay of multiple reflection branes. Returning to our geometric perspective given by M-theory compactified on a square T^d:

$$T^d = \underbrace{S^1 \times ... \times S^1}_{d \text{ times}}, \tag{4.16}$$

we label the reflection brane associated with the i^{th} factor as $R_i \in G_U^{(R)}$, and its Pin$^+$-lift by $\widetilde{R}_i$.

To begin, let us consider what happens when we have a pair of such branes. If it is the same sort of brane, then the fact that we have a Pin$^+$-lift means that the corresponding element of $\widetilde{G_U}^+$ squares to 1, namely $\widetilde{R}_i^2 = 1$. Physically, we take this to mean that a pair of such branes annihilate to pure radiation.

Next, suppose we have a pair of such branes $\widetilde{R}_i$ and $\widetilde{R}_j$ with $i \neq j$. Since we have singled out two distinguished directions, we can focus our attention on this T^2. The general $\widetilde{G_U}^+$ transformations descend to the Pin$^+$-cover, $\mathrm{GL}^+(2,\mathbb{Z})$, a situation that was analyzed in detail in [16]. In this case, the group theory relations tell us that:

$$\widetilde{R}_i \widetilde{R}_j = (-1)^F \widetilde{R}_j \widetilde{R}_i, \tag{4.17}$$

where $(-1)^F$ is spacetime fermion parity. One way to establish this is to return to the case of type IIB reflection 7-branes. In that setting, one has an F-theory torus $T^2 = S^1 \times S^1$ and the two reflections amount to worldsheet $\mathbb{Z}_2$ actions given by left-moving fermion parity $(-1)^{F_L}$ and worldsheet orientation reversal Ω. Since worldsheet orientation reversal sends left-movers to right-movers, one has $(-1)^{F_L}\Omega = \Omega(-1)^{F_R} = (-1)^F \Omega (-1)^{F_L}$, where we used the fact that $(-1)^F = (-1)^{F_L}(-1)^{F_R}$ commutes with Ω (see also [34]).

What is the geometry of this $(-1)^F$ factor? As explained in [16], this amounts to a non-compact elliptically-fibered 1/2 K3 surface, i.e., a dP_9 geometry. Viewed as an elliptic fibration over a compact $\mathbb{P}^1$, the Weierstrass model for dP_9 is:

$$y^2 = x^3 + f_4 x + g_6, \tag{4.18}$$

with f_4 and g_6 degree 4 and 6 polynomials of the homogeneous coordinates $[z_1, z_2]$ of the base $\mathbb{P}^1$.

Observe also that as elements of $\mathrm{GL}(2,\mathbb{Z})$, the combined product of $R_i R_j = \mathrm{diag}(-1,-1)$. As such, we see that the resulting geometry produced by a pair of coincident reflection branes is just the orbifold:

$$T^{d-2} \times \left(\mathbb{C} \times T^2\right)/\mathbb{Z}_2, \tag{4.19}$$

where the $\mathbb{Z}_2$ acts on $\mathbb{C} \times T^2$ factors with local coordinates (z,w) as $(z,w) \mapsto (-z,-w)$. Now, in contrast to the case of F-theory models, the T^2 in this case is of finite size. This means there is no "further enhancement" in the singularity, and we instead have four A_1 singularities with local presentation $\mathbb{C}^2/\mathbb{Z}_2$, i.e., the brane supports an $\mathfrak{su}(2)^4$ gauge symmetry, one gauge algebra for each factor. Observe that this object is supersymmetric because the orbifold group action preserves the holomorphic 2-form $dz \wedge dw$. We take this to mean that there is an attractive potential between a pair of non-commuting[30] reflection branes which has this supersymmetric bound state as its end product (accompanied by radiation).

While we have phrased our discussion in terms of M-theory backgrounds, it is also natural to consider the corresponding F-theory models associated with these reflection branes. This corresponds to shrinking the T^2 factor in line (4.19) to zero size, in which case we reach a non-compact elliptically-fibered K3 with a singular I_0^* fiber, i.e., we get an 8D $\mathfrak{so}(8)$ gauge theory which is wrapped on a further $S^1 \times T^{d-2}$, where the S^1 factor decompactifies under M- / F-theory duality. This is essentially the same background as that studied in [16].

4.4.1 $M > 2$ Reflection Branes: Supersymmetric Case

Let us now turn to supersymmetric bound states with more than two reflection branes. An even number of distinct reflections will produce such examples. To illustrate, let us consider the case of four reflection branes, grouped according to their action on the four-torus $T^2_{(12)} \times T^2_{(34)}$, where the first factor is associated with monodromy generated by the pair $R_1 R_2$ (as indicated by the subscript) and similar conventions for the second factor. Restricting to $\mathrm{GL}(4,\mathbb{Z})$, the monodromy in this case involves the reflection $\mathrm{diag}(-1,-1,-1,-1)$ on a T^4 factor.

We now give a geometric characterization of the M-theory background which realizes this configuration. Observe that if we had not included the additional $R_3 R_4$ branes, the resulting geometry would be captured by the quotient:

$$T^{d-4} \times \left(\mathbb{C} \times T^2_{(12)}\right)/\mathbb{Z}_2^{(12)} \times T^2_{(34)}. \tag{4.20}$$

Including this extra set of reflection branes amounts to introducing a quotient by another

[30] See line (4.17).

$\mathbb{Z}_2$.[31] In other words, the whole singular geometry is of the form:

$$T^{d-4} \times \left(\mathbb{C} \times T^2_{(12)} \times T^2_{(34)}\right) / \mathbb{Z}_2^{(12)} \times \mathbb{Z}_2^{(34)}, \tag{4.21}$$

where the two $\mathbb{Z}_2$'s act on the local holomorphic coordinates as:

$$\mathbb{Z}_2^{(12)} : (z, w_{(12)}, w_{(34)}) \mapsto (-z, -w_{(12)}, w_{(34)}) \tag{4.22}$$

$$\mathbb{Z}_2^{(34)} : (z, w_{(12)}, w_{(34)}) \mapsto (-z, w_{(12)}, -w_{(34)}). \tag{4.23}$$

This results in sixteen fixed points, all of the same type, locally being given by $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., we locally get the 5D T_2 theory, a hypermultiplet in the trifundamental of the flavor symmetry $\mathfrak{su}(2)^3$ (see [58]). The further compactification (from working on a compact $T^2_{(12)} \times T^2_{(34)}$) means that these flavor symmetries are all gauged, and we are considering the further dimensional reduction of this theory on a T^{d-4}. This sort of orbifold geometry, including the global form of the gauge group (including additional Abelian factors) was treated in [59] (see also [60]).

Consider next the F-theory background obtained by shrinking one of these T^2 factors, namely we treat $T^2_{(12)}$ as the F-theory elliptic fiber. Observe that in this case, the local collision of singularities involves I_0^* collisions at the four orbifold fixed points in the $\mathbb{C} \times T^2_{(34)}$ directions, namely we get an $\mathfrak{so}(8)^4$ global symmetry with pairwise collisions resulting in (D_4, D_4) conformal matter, as in references [61–64] (see [65, 66] for reviews).

Similar considerations hold for additional supersymmetric combinations of reflection branes. For an even number of reflections on an internal torus T^{2k}, the resulting monodromy for the codimension-two defect in $\mathrm{GL}(2k, \mathbb{Z})$ is generated by the element:

$$\mathrm{diag}(\underbrace{-1, ..., -1}_{2k \text{ times}}) \in \mathrm{GL}(2k, \mathbb{Z}) \,. \tag{4.24}$$

In this more general case, the supersymmetric M-theory background is:

$$T^{d-2k} \times \left(\mathbb{C} \times T^{2k}\right) / \Gamma, \tag{4.25}$$

where the group Γ is:

$$\Gamma = \underbrace{\mathbb{Z}_2 \times ...\mathbb{Z}_2}_{k \text{ times}}, \tag{4.26}$$

where each $\mathbb{Z}_2$ acts via a sign flip on the z coordinate of $\mathbb{C}$ and one of the holomorphic T^2 factors of:

$$T^{2k} = \underbrace{T^2 \times ...T^2}_{k \text{ times}}. \tag{4.27}$$

[31] One way to see the presence of an additional quotient is to consider a background where we have separated the two codimension-two defects $R_1 R_2$ and $R_3 R_4$ in the spacetime directions. These are specified by locally independent $\mathbb{Z}_2$ actions.

As an example along these lines, consider $k = 3$. This results in an M-theory background with local $\mathbb{C}^4/\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ singularities. This local singularity structure results in a 3D $\mathcal{N} = 2$ theory with an $\mathfrak{su}(2)^6$ flavor symmetry (fixed divisors coming from pairs of local equations of the form $z_i = z_j = 0$ for $i \neq j$) and four matter fields in tri-fundamental representations (fixed curves coming from triples of local equations of the form $z_i = z_j = z_k = 0$ for i, j, k distinct) and a localized interaction term at $z_i = 0$ for all i with superpotential of the schematic form $W = X_1X_2X_3X_4$ (which is classically marginal in 3D). We summarize the matter content for this local model below, where subscripts denote directions in which the associated singular loci are localized:

	$\mathfrak{su}(2)_{(12)}$	$\mathfrak{su}(2)_{(13)}$	$\mathfrak{su}(2)_{(14)}$	$\mathfrak{su}(2)_{(23)}$	$\mathfrak{su}(2)_{(24)}$	$\mathfrak{su}(2)_{(34)}$
$X_{(123)}$	**2**	**2**	·	**2**	·	·
$X_{(124)}$	**2**	·	**2**	·	**2**	·
$X_{(134)}$	·	**2**	**2**	·	·	**2**
$X_{(234)}$	·	·	·	**2**	**2**	**2**

. (4.28)

On compact tori we simply get additional copies of this same system with gauged combinations of the original flavor symmetries.

Likewise, the F-theory lift of the local model $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ involves a triple intersection of $\mathfrak{so}(8)$ flavor symmetries, with local Weierstrass model of the form:

$$y^2 = x^3 + \alpha x(u_1u_2u_3)^2 + \beta(u_1u_2u_3)^3, \tag{4.29}$$

i.e., it is an example of an $\mathfrak{so}(8)^3$ conformal Yukawa in the sense of reference [67].[32]

Compactifying all the way to 3D is the lowest we can go while still retaining a sensible notion of a codimension-two object.

4.4.2 $M > 2$ Reflection Branes: Non-Supersymmetric Case

Consider next the case of configurations involving an odd number of reflection branes, say $2k + 1$. In this case, we do not preserve any supersymmetries. Topologically, the geometry in this case involves a combination of the supersymmetry preserving quotient by the group $\Gamma = \mathbb{Z}_2^k$, as well as a $\mathbb{Z}_2^{\mathrm{KB}}$ group action which produces the Klein bottle (KB) from a quotient of T^2, namely $\mathrm{KB} = T^2/\mathbb{Z}_2^{\mathrm{KB}}$. In this case, the full geometry takes the form:

$$T^{d-(2k+1)} \times \mathrm{Cone}(S^1_\infty \times T^{2k} \times S^1)/(\Gamma \times \mathbb{Z}_2^{\mathrm{KB}}), \tag{4.30}$$

[32]Note in particular that these further collisions of singularities do not result in matter in even bigger representations of the flavor symmetry, but rather additional interactions. Thus, this is in accord with the conjectured absence of an isolated 5-plet of $\mathfrak{su}(2)$ discussed in [68].

where $S^1_\infty = \partial\mathbb{C}$, the circle at infinity which winds around the codimension-two defect. One way to understand these examples is to start with one of our supersymmetric backgrounds and simply add an additional non-supersymmetric reflection brane. We expect that this engineers an interacting quantum field theory, and it would be interesting to extract many of its properties.

5 Conclusions

Dualities provide important constraints on the non-perturbative structure of quantum theories. In this work we have determined the Spin- and Pin$^+$-lifts of the U-dualities of maximally supersymmetric non-chiral supergravity theories. Using this, we applied the Swampland Cobordism Conjecture to predict the existence of codimension-two reflection branes. These branes are lower-dimensional analogs of the reflection 7-branes found in type II string theory. Indeed, the reflection branes found in this paper arise from wrapping R7-branes on higher-dimensional cycles of the internal torus of an M-theory compactification. We have also argued that these branes support non-trivial degrees of freedom since BPS branes can terminate on them, and moreover, have also established some basic features such as braiding and bound state formation. In the remainder of this section we discuss some potential avenues of future investigation.

One of the motivations for the present work was to determine the spectrum of objects predicted by the Swampland Cobordism Conjecture. Now that we have determined the full U-duality group, it is natural to return to the question of the corresponding bordism groups $\Omega_k^{\text{Spin-}\widetilde{G_U}}(\text{pt})$ and $\Omega_k^{\text{Spin-}\widetilde{G_U}^+}(\text{pt})$ and extract predictions for non-perturbative objects of these gravitational theories. Especially in the case of Spin-$\widetilde{G_U}^+$ bordisms, we expect that some of these defects will be non-supersymmetric.

Another natural extension would be to consider even more general tangential structures on our D-dimensional effective field theories. While we still required a Spin-Γ structure for our spacetime, it would be interesting to also consider further refinements, as motivated by M-theory, such as suitable twistings of Pin, String and other related structures.

It would also be interesting to determine the corresponding Spin- and Pin$^+$-lifts for systems with reduced supersymmetry. For example Calabi-Yau compactifications often have non-trivial duality groups inherited from the automorphisms of the Calabi-Yau manifold [69]. One could thus carry out an analysis of (possiby non-supersymmetric) objects predicted by the Cobordism Conjecture.

We have primarily used group-theoretic and topological considerations to argue for the existence of these reflection branes and to determine their properties. It would be interesting to work out the corresponding supergravity solutions which include these defects. Among other things, this would allow us to extract their tension.

Wrapping branes on "cycles at infinity" has been a fruitful way to engineer a wide class of topological symmetry operators in stringy QFTs as well as holographic systems.[33] It would be interesting to study how these branes realize discrete symmetry operators, perhaps along the lines of [24].

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A Reflections on the States of M-Theory

We now explain in greater detail why the reflections of M-theory require us to enlarge the U-duality groups. To this end, let us begin in type IIA string theory compactified on a T^{d-1}. In this case, the T-duality symmetry group is $\mathrm{Spin}(d-1, d-1, \mathbb{Z})$, with possible discrete quotients. The T-duality group contains $\mathrm{SL}(d-1, \mathbb{Z})$, the group of large diffeomorphisms on the T^{d-1}. From the perspective of the low energy supergravity theory, there is little difference between IIA and IIB, and so one can also entertain the $\det = -1$ elements which amount to inverting the length of a circle, namely $L \mapsto 1/L$. Additionally, it is worth noting that the RR states transform in spinor representations of the T-duality group, and so we have written $\mathrm{Spin}(d-1, d-1, \mathbb{Z})$. We emphasize that this is still a statement purely connected with the bosonic sector of the theory, and is not directly associated with the fermionic degrees of freedom (which require a Spin-lift of the U-duality group).

Now, in addition to these T-duality symmetries, we also have $(-1)^{F_L}$, namely left-moving fermion parity of the IIA theory. This is an additional $\mathbb{Z}_2$ symmetry and acts by sending RR fields to minus themselves. In terms of M-theory on $S^1 \times T^{d-1}$, with reduction on S^1 taking us to type IIA, this $\mathbb{Z}_2$ corresponds to the reflection $\theta \mapsto -\theta$ on the local coordinate.

It is instructive to see how this reflection acts on the spectrum of massless and massive states of the theory. To illustrate, we focus on a toroidal compactification in which we impose

[33] See e.g., [70–75].

periodic boundary conditions for bosonic fields. As a representative example, we consider the spectrum of two-index anti-symmetric tensor fields, as obtained from dimensional reduction of the pseudo 3-form potential C_3 and its magnetic dual real 6-form potential $\widetilde{C}_6$.[34] We also retain the explicit internal dependence on θ by splitting up our potentials into reflection even Fourier modes and reflection odd Fourier modes, namely $\cos n\theta$ and $\sin n\theta$, which we denote by $m \in \{0, 1\}$, with $m = 0$ for the parity even modes (cosines) and $m = 1$ for the parity odd modes (sines). We introduce the notation $C_3(m)$ and $\widetilde{C}_6(m)$ to capture these modes. Note that the massless sector is in $m = 0$, but that $m = 0$ also includes massive excitations. For $m = 1$ all modes are massive.

Let us explore the consequences of this in an explicit example, with M-theory compactified on a T^d. We mostly keep the discussion general for arbitrary d, but specialize to $d = 4$ when convenient to illustrate the main ideas since similar considerations hold for more general cases.

Consider first the reduction of $C_3(m)$ on our T^d. Observe that in the D-dimensional spacetime, we get two-index anti-symmetric tensor fields by keeping one leg internal. This results in d such fields which we write as $C_{\mu\nu i}(m)$. Reflections on the M-theory circle further partition up these degrees of freedom. Since we are dealing with a pseudo 3-form, we have:

$$R_1 : C_{\mu\nu 1}(m) \mapsto (-1)^m \, C_{\mu\nu 1}(m) \tag{A.1}$$

$$R_1 : C_{\mu\nu i}(m) \mapsto (-1)^{m+1} \, C_{\mu\nu i}(m) \quad \text{for} \quad i \neq 1. \tag{A.2}$$

Consider next the reduction of $\widetilde{C}_6(m)$ on our T^d. To get a 2-index anti-symmetric tensor field, four indices must be kept internal, so this only contributes when $d > 3$. When this holds, we have $\binom{d}{4} = \frac{d(d-1)(d-2)(d-3)}{12}$ such fields. Of these, there is a further refinement depending on their sign under an internal reflection. In particular, we have, for i, j, k, l all distinct:

$$R_1 : \widetilde{C}_{\mu\nu 1ijk}(m) \mapsto (-1)^{m+1} \, \widetilde{C}_{\mu\nu 1ijk}(m) \quad \text{for} \quad i, j, k \neq 1 \tag{A.3}$$

$$R_1 : \widetilde{C}_{\mu\nu ijkl}(m) \mapsto (-1)^m \, \widetilde{C}_{\mu\nu ijkl}(m) \qquad \text{for} \quad i, j, k, l \neq 1. \tag{A.4}$$

Totaling everything up, we now see that our states organize according to their sign under reflections. In particular, the same sign under reflection is fixed for:

$$(-1)^m \text{ parity:} \quad C_{\mu\nu 1}(m) \text{ and } \widetilde{C}_{\mu\nu ijkl}(m) \quad \text{for} \quad i, j, k, l \neq 1 \tag{A.5}$$

$$(-1)^{m+1} \text{ parity:} \quad C_{\mu\nu i}(m) \text{ and } \widetilde{C}_{\mu\nu 1ijk}(m) \quad \text{for} \quad i, j, k \neq 1. \tag{A.6}$$

Let us now specialize further to $d = 4$. Fixing the overall mass, we see that for each mass

[34] Recall that a pseudo-form transforms with an extra minus sign under a reflection.

we have the following number of tensor fields:

$$(-1)^m \text{ parity:} \qquad C_{\mu\nu 1}(m) \Rightarrow 1 \quad \text{tensor field} \tag{A.7}$$

$$(-1)^{m+1} \text{ parity:} \qquad C_{\mu\nu i}(m) \text{ and } \widetilde{C}_{\mu\nu 1234}(m) \Rightarrow 4 \quad \text{tensor fields,} \tag{A.8}$$

which have opposite signs under an internal reflection. Of course, the massless sector simply reproduces the expected spectrum of IIA, where $C_{\mu\nu 1}$ descends to the familiar NSNS 2-form potential (inert under $(-1)^{F_L}$), while the remaining massless potentials $C_{\mu\nu i}$ and $\widetilde{C}_{\mu\nu 1234}$ are the descendants of RR potentials and transform in the four-dimensional spinor representation of $\mathrm{Spin}(3,3,\mathbb{Z})$.

By inspection, we see that as expected, the massless modes respect the breaking pattern in reducing from M-theory to IIA, namely $\mathrm{SL}(5,\mathbb{Z}) \supset \mathrm{Spin}(3,3,\mathbb{Z}) \simeq \mathrm{SL}(4,\mathbb{Z})$ where the vector representation decomposes as $\mathbf{5} \to \mathbf{4} \oplus \mathbf{1}$, as expected. Observe also that on these massless states, $(-1)^{F_L}$ embeds as $\mathrm{diag}(+1,-1,-1,-1,-1)$, namely it appears as an element of $\mathrm{SL}(5,\mathbb{Z})$. On the other hand, for the first massive excitations which are odd under reflections, we again have a decomposition into representations of $\mathbf{5} \to \mathbf{4} \oplus \mathbf{1}$, but $(-1)^{F_L}$ embeds as $\mathrm{diag}(-1,+1,+1,+1,+1)$, namely an element of $\mathrm{GL}(5,\mathbb{Z})$ but not $\mathrm{SL}(5,\mathbb{Z})$.

Similar considerations hold for other toroidal compactifications, and is in line with the general expectation that we cannot generate $(-1)^{F_L}$ from large diffeomorphisms of M-theory in tandem with T-dualities. As such, we must extend the U-duality group to include these symmetries.

B Split Real Form versus Compact Real Form

U-duality groups naturally arise from toroidal compactifications of M-theory. At the level of supergravity, this results in $G_U(\mathbb{R})$, the split real form of a complex Lie group. As proposed in [6], the refinement to a quantized spectrum of objects results in $G_U(\mathbb{Z}) \equiv G_U$, with a natural embedding $i : G_U(\mathbb{Z}) \to G_U(\mathbb{R})$. A simple example is the inclusion $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{R})$. At the other extreme, we have the U-duality group in three dimensions $G_U^{\mathrm{3D}} = E_{8(8)}(\mathbb{Z}) \to E_{8(8)}(\mathbb{R})$.

We are interested in possible Spin- and Pin$^+$-lifts of our U-dualities, as required by the fermionic degrees of freedom of our theory. We use the same strategy deployed in [33] and [34], namely we first study extensions of $G_U(\mathbb{R})$ and then show that this induces an extension of $G_U = G_U(\mathbb{Z})$.

The first important comment is that as opposed to the compact real forms, the split real form of Lie groups do have non-trivial $\mathbb{Z}_2$ extensions [77].[35]

For ease of exposition, we focus on the case of 3D supergravity with U-duality group

[35]Indeed, the only $\mathbb{Z}_2$ extension of E_8^{cpct} is the trivial one to $E_8^{\mathrm{cpct}} \times \mathbb{Z}_2$. For the split real form $E_{8(8)}(\mathbb{R})$, a non-trivial extension is possible.

$G_U(\mathbb{R}) = E_{8(8)}(\mathbb{R})$. Similar considerations hold for the $D > 3$ U-duality groups, and can also be deduced by taking suitable decompactification limits.

The existence of a non-trivial central extension, such as a Spin-lift, is governed by the topology of $G_U(\mathbb{R})$. More precisely, central extensions of the U-duality group $G_U(\mathbb{R})$, which is a connected group, by a discrete Abelian group A (such as $\mathbb{Z}_2$) are classified by the second group cohomology $H^2(G_U(\mathbb{R}); A)$, which is in turn encoded by the fundamental group:

$$H^2(G_U(\mathbb{R}); A) \cong \mathrm{Hom}(\pi_1(G_U(\mathbb{R})), A). \tag{B.1}$$

Thus, a non-trivial fundamental group can give rise to non-trivial central extensions. Since we are interested in the double cover / Spin-lift of $G_U(\mathbb{R})$, this simplifies to

$$H^2(G_U(\mathbb{R}); \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(G_U(\mathbb{R})), \mathbb{Z}_2). \tag{B.2}$$

For the split real form of an exceptional Lie group, such as $E_{8(8)}(\mathbb{R})$, we have that $\pi_1(G_U(\mathbb{R})) = \pi_1(K_U)$ (see Appendix C for more details). In this case, $K_U^{3\mathrm{D}} = \mathrm{SO}(16)$:

$$\pi_1(K_U^{3\mathrm{D}}) = \pi_1(G_U^{3\mathrm{D}}(\mathbb{R})) = \mathbb{Z}_2. \tag{B.3}$$

As a result, there exists a non-trivial double cover $\mathrm{Spin}(16) \to \mathrm{SO}(16)$, and one is led to consider a non-trivial $\mathbb{Z}_2$ central extension $\widetilde{E}_{8(8)}(\mathbb{R})$ in theories that include fermions.

By contrast, the compact real form of E_8 is simply connected:

$$\pi_1(E_8^{\mathrm{cpct}}) = 0. \tag{B.4}$$

This implies that all central extensions of E_8^{cpct} are trivial. That is, any central extension of the form

$$1 \to \mathbb{Z}_2 \to \widetilde{E}_8 \to E_8^{\mathrm{cpct}} \to 1 \tag{B.5}$$

splits as a direct product:

$$\widetilde{E}_8 \cong E_8^{\mathrm{cpct}} \times \mathbb{Z}_2. \tag{B.6}$$

In summary, the split real form $E_{8(8)}$ admits a non-trivial Spin-lift due to the non-trivial topology of the maximal compact subgroups. On the other hand, the compact real form E_8^{cpct} is simply connected and admits no non-trivial central extensions.

This analysis extends to the U-duality groups in 4D, given by $G_U^{4\mathrm{D}}(\mathbb{R}) = E_{7(7)}(\mathbb{R})$, and 5D, given by $G_U^{5\mathrm{D}}(\mathbb{R}) = E_{6(6)}(\mathbb{R})$. The maximal compact subgroups can be found in Table 1 and are given by

$$K_U^{4\mathrm{D}} = \mathrm{USp}(8)/\mathbb{Z}_2 \qquad \text{and} \qquad K_U^{5\mathrm{D}} = \mathrm{SU}(8)/\mathbb{Z}_2. \tag{B.7}$$

From this we see that $\pi_1(G_U^{4\mathrm{D}}) = \pi_1(K_U^{4\mathrm{D}}) = \mathbb{Z}_2$ and $\pi_1(G_U^{5\mathrm{D}}) = \pi_1(K_U^{5\mathrm{D}}) = \mathbb{Z}_2$, which indicates that $E_{6(6)}$ and $E_{7(7)}$ admit non-trivial Spin-lifts. In contrast, the compact real forms E_7^{cpct} and E_6^{cpct} are simply connected.[36] Thus, the fundamental groups are trivial in each case, and any candidate central extension splits as a direct product.

C Explicit Spin- / Pin$^+$-Lifts

In this Appendix we explicitly construct the Spin- and Pin$^+$-lifts of the bosonic U-duality groups. The Spin-lifts are given by a non-trivial $\mathbb{Z}_2$ extension of the original U-duality groups, while the Pin$^+$-lift is given by including an additional reflection generator in the disconnected component of the U-duality group. The reflection generator acts on the other generators of the U-duality group via conjugation. We also comment on the decompactification limits of the extended U-duality groups.

C.1 Spin-Lifts

The discrete bosonic U-duality group $G_U(\mathbb{Z}) \equiv G_U$ of the D-dimensional effective theory arising in toroidal compactifications of maximal supergravity does not act linearly on fermionic fields. To define a consistent duality action on spinors, one must replace $G_U(\mathbb{Z})$ with its double cover. This is given by a non-trivial central extension of $G_U(\mathbb{Z})$ by $\mathbb{Z}_2$ known as the Spin-lift.

The Spin-lift $\widetilde{G_U}(\mathbb{Z})$ acts on fermionic states via linear representations. When $D = 9$, this construction recovers the metaplectic group $\mathrm{Mp}(2, \mathbb{Z})$ as the Spin-lift of $\mathrm{SL}(2, \mathbb{Z})$. At higher rank, it defines unique $\mathbb{Z}_2$ central extensions of groups such as $E_{6(6)}(\mathbb{Z})$, $E_{7(7)}(\mathbb{Z})$, and $E_{8(8)}(\mathbb{Z})$.

Following the construction of Pantev and Sharpe [33], the double cover $\widetilde{G_U}(\mathbb{Z})$ is defined as the pullback of the universal cover $\widetilde{G_U}(\mathbb{R}) \to G_U(\mathbb{R})$ along the inclusion $G_U(\mathbb{Z}) \hookrightarrow G_U(\mathbb{R})$. Explicitly, this gives:

$$\widetilde{G_U}(\mathbb{Z}) := \left\{ (a, g) \in \widetilde{G_U}(\mathbb{R}) \times G_U(\mathbb{Z}) \;\middle|\; p(a) = i(g) \right\}, \tag{C.1}$$

where p is the covering map and i is the inclusion. Since $\pi_1(G_U(\mathbb{R})) \supset \mathbb{Z}_2$ in almost all cases, we see that

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}(\mathbb{R}) \to G_U(\mathbb{R}) \to 1. \tag{C.2}$$

This can be seen by computing $\pi_1(K_U)$, where K_U is the maximal compact subgroup of $G_U(\mathbb{R})$. In particular, in all cases $G_U(\mathbb{R})$ is a connected, real group and has finite center. Furthermore, $G_U(\mathbb{R})/K_U$ is contractible in all cases. Thus, the inclusion $K_U \hookrightarrow G_U$ induces an isomorphism

$$\pi_1(K_U) = \pi_1(G_U(\mathbb{R})). \tag{C.3}$$

[36]We use the simply connected form of the exceptional Lie group to construct the real forms.

From this, we can define the desired Spin-lift / double cover of $G_U(\mathbb{Z})$ via pullback from line (C.2):

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}(\mathbb{Z}) \to G_U(\mathbb{Z}) \to 1. \tag{C.4}$$

For the 9D U-duality group $G_U^{9\text{D}}(\mathbb{Z}) = \text{SL}(2, \mathbb{Z})$, the story is slightly different: $\pi_1(K_U^{9\text{D}} = \text{SO}(2)) = \mathbb{Z}$, so we use not the universal cover, but the unique double cover of $K_U^{9\text{D}}$ to obtain the metaplectic double cover $\text{Mp}(2, \mathbb{Z})$.

The Spin-lift of the bosonic U-duality groups in 8D, $G_U^{8\text{D}}(\mathbb{Z}) = \text{SL}(3, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$, while still given by (C.4), has some additional subtleties. $G_U^{8\text{D}}$ is a product of two groups, both of which have a universal cover. Thus, the correct Spin-lift / double cover is found by extending both groups and then quotienting by a diagonal $\mathbb{Z}_2^{\text{diag}}$, as can be verified by comparing the Spin-lifts of the U-duality groups across different dimensions.

See Table 1 for a summary of the continuous and discrete bosonic U-duality groups, the corresponding maximal compact subgroups, and the Spin-lifts of each. This universal construction provides a systematic and dimension-independent method for determining the correct U-duality symmetry group acting on all fields, including fermions, in maximal supergravity.

D	**Classical U-duality Group** $G_U(\mathbb{R})$	**Discrete U-duality Group** $G_U(\mathbb{Z})$	**Maximal Compact Subgroup** K_U	**Spin Lift** $\widetilde{K_U}$	**Spin Lift** $\widetilde{G_U}(\mathbb{Z})$
9	$\text{SL}(2,\mathbb{R})$	$\text{SL}(2,\mathbb{Z})$	$\text{SO}(2)$	$\text{Spin}(2)$	$\text{Mp}(2,\mathbb{Z})$
8	$\text{SL}(3,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$	$\text{SL}(3,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$	$\text{SO}(3) \times \text{SO}(2)$	$\text{Spin}(3) \times \text{Spin}(2)/\mathbb{Z}_2^{\text{diag}}$	$\widetilde{\text{SL}}(3,\mathbb{Z}) \times \text{Mp}(2,\mathbb{Z})/\mathbb{Z}_2^{\text{diag}}$
7	$\text{SL}(5,\mathbb{R})$	$\text{SL}(5,\mathbb{Z})$	$\text{SO}(5)$	$\text{Spin}(5)$	$\widetilde{\text{SL}}(5,\mathbb{Z})$
6	$\text{Spin}(5,5,\mathbb{R})$	$\text{Spin}(5,5,\mathbb{Z})$	$\text{Spin}(5) \times \text{Spin}(5)/\mathbb{Z}_2^{\text{diag}}$	$\text{Spin}(5) \times \text{Spin}(5)$	$\widetilde{\text{Spin}}(5,5,\mathbb{Z})$
5	$E_{6(6)}$	$E_{6(6)}(\mathbb{Z})$	$\text{USp}(8)/\mathbb{Z}_2$	$\text{USp}(8)$	$\widetilde{E}_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}$	$E_{7(7)}(\mathbb{Z})$	$\text{SU}(8)/\mathbb{Z}_2$	$\text{SU}(8)$	$\widetilde{E}_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}$	$E_{8(8)}(\mathbb{Z})$	$\text{SO}(16)$	$\text{Spin}(16)$	$\widetilde{E}_{8(8)}(\mathbb{Z})$

Table 1: The classical U-duality groups $G_U(\mathbb{R})$, the discrete U-duality groups $G_U(\mathbb{Z})$, the corresponding maximal compact subgroups K_U, and the respective Spin-lifts $\widetilde{K_U}$ and $\widetilde{G_U}$ appearing in D-dimensional supergravity theories for $3 \leq D \leq 9$.

C.2 Pin$^+$-Lifts

In addition to orientation-preserving symmetries, physical duality groups often include elements that reverse orientation, such as spacetime or internal reflections. These generate an extension of the bosonic U-duality group by a discrete reflection symmetry, resulting in the semi-direct product

$$G_U \rtimes \mathbb{Z}_2^R, \tag{C.5}$$

where the $\mathbb{Z}_2$ factor corresponds to a chosen reflection representative. Such reflections lie outside the identity component of $K_U^{(R)} \supset G_U^{(R)}(\mathbb{R})$, the maximal compact subgroup. More precisely, including such a reflection element extends the maximal compact subgroup K_U as determined by the short exact sequence:

$$1 \to K_U \to K_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R \to 1. \tag{C.6}$$

Then, via the inclusion $K_U \hookrightarrow G_U(\mathbb{R})$, this also induces an extension of $G_U(\mathbb{R})$ as determined by a similar short exact sequence:

$$1 \to G_U(\mathbb{R}) \to G_U(\mathbb{R}) \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R \to 1. \tag{C.7}$$

From this, we again get the desired extension on the discrete group via a pullback:

$$1 \to G_U \to G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R \to 1. \tag{C.8}$$

To accommodate fermions in the presence of such orientation-reversing symmetries, one must lift this group to a central extension that incorporates both the Spin and reflection structure, i.e., a Pin^+-lift. This is given by combining the extensions (C.4) and (C.8):

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^+ \to G_U \rtimes \mathbb{Z}_2^R \to 1. \tag{C.9}$$

This lift is governed by the structure of the Pin^+ group, which is the double cover of the full orthogonal group $\mathrm{O}(n)$, just as $\mathrm{Spin}(n)$ is the double cover of $\mathrm{SO}(n)$. In particular, a reflection element $R \in \mathrm{O}(n) \setminus \mathrm{SO}(n)$, when lifted to an element $\widetilde{R}$ in the Pin^+ group, satisfies $\widetilde{R}^2 = 1$ on spinors.

In each dimension, we include only a single reflection generator to extend the U-duality group because the relevant outer automorphism group is $\mathbb{Z}_2^R$, corresponding to the disconnected component of the full U-duality group (e.g., $\mathrm{GL}(n, \mathbb{Z})$ versus $\mathrm{SL}(n, \mathbb{Z})$). Although there are many reflection-like elements in the full group, they are all conjugate to each other, so their effect on the U-duality group is captured by a single non-trivial automorphism. Hence, adjoining one reflection generator that implements this outer action suffices to generate the full semi-direct product structure. This also ensures the minimal and correct extension when considering spinor representations.

In practice, this minimal extension is achieved by performing a Pin^+-lift of the maximal compact subgroup K_U of $G_U(\mathbb{R})$, which in turn gives the correct extension of $G_U(\mathbb{R})$ via embedding $\widetilde{K_U}^+$, and from this the correct extension on G_U itself via a pullback. All of the relevant extensions are summarized in Table 2. Note that the case of $D = 8$ is more subtle, and is treated with more care in the next section. The case of $D = 6$ is also subtle, as the maximal compact subgroup is the product of two groups: $K_U^{5D} = (\mathrm{Spin}(5) \times \mathrm{Spin}(5))/\mathbb{Z}_2$. Since the two $\mathrm{Spin}(5)$ factors embed into $G_U(\mathbb{R}) = \mathrm{Spin}(5, 5)$ block diagonally, it is important to only extend one of the Spin(5) factors to have an overall element with determinant -1. It does not matter which factor gets extended, as they can be related via conjugation.

D	**Maximal Compact Subgroup** $K_U \subset G_U(\mathbb{R})$	$G_U^{(R)} = G_U \rtimes \mathbb{Z}_2^R$
9	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^R$
8	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$(\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \rtimes \mathbb{Z}_2^R$
7	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z}) \rtimes \mathbb{Z}_2^R$
6	$\mathrm{Spin}(5) \times \mathrm{Spin}(5)/\mathbb{Z}_2^{\mathrm{diag}}$	$\mathrm{Spin}(5,5,\mathbb{Z}) \rtimes \mathbb{Z}_2^R$
5	$\mathrm{USp}(8)/\mathbb{Z}_2$	$E_{6(6)}(\mathbb{Z}) \rtimes \mathbb{Z}_2^R$
4	$\mathrm{SU}(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z}) \rtimes \mathbb{Z}_2^R$
3	$\mathrm{SO}(16)$	$E_{8(8)}(\mathbb{Z}) \rtimes \mathbb{Z}_2^R$

Table 2: The maximal compact subgroup K_U of the classical U-duality group, and the lift of the bosonic U-duality group $G_U^{(R)}$ for $3 \leq D \leq 9$. Extending K_U to include reflections induces a lift of G_U to $G_U^{(R)}$. The full Pin$^+$-lift of G_U is given by a $\mathbb{Z}_2$ central extension of $G_U^{(R)}$.

C.3 Decompactification Limits

In this section we study the decompactification limit from the $D = 7$ U-duality group to the $D = 8$ U-duality group. The bosonic U-duality groups are

$$G_U^{7\mathrm{D}} = \mathrm{SL}(5,\mathbb{Z}) \quad \text{and} \quad G_U^{8\mathrm{D}} = \mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}), \tag{C.10}$$

respectively. In the bosonic case, the decompactification limit is simple, as $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ embeds block diagonally in $\mathrm{SL}(5,\mathbb{Z})$. However, additional subtleties arise when considering the Pin$^+$-lift. In particular, there is an $\mathrm{SL}(2,\mathbb{Z})$ simple factor of $G_U^{8\mathrm{D}}$ which we can geometrically interpret as the group of large diffeomorphisms of a T^2 arising from type II string theory on T^2. Type IIB string theory is not well defined on non-orientable manifolds, so the lifted duality group should never have orientation-reversing elements belonging *solely* to $\mathrm{GL}(2,\mathbb{Z})$. We will show that this is indeed the case by studying the decompactification limit from 7D.

To be precise, start with the block diagonal embedding[37]

$$\iota: \ \mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \hookrightarrow \mathrm{SL}(5,\mathbb{Z}), \qquad \iota(A,B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{C.11}$$

realizing $\mathbb{Z}^5 = \mathbb{Z}^3 \oplus \mathbb{Z}^2$. At the purely bosonic level one can move a diagonal sign among

[37]We argue for this particular embedding by noting that $\mathrm{SL}(3,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ are each subgroups of $\mathrm{SL}(5,\mathbb{Z})$ that need to commute with each other.

coordinates by conjugation in the full group. For instance, with

$$R = \mathrm{diag}(-1,1,1,1,1), \qquad P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}(5,\mathbb{Z}), \tag{C.12}$$

one has $P^4RP^{-4} = \mathrm{diag}(1,1,1,1,-1)$. Crucially, P is not block-diagonal with respect to the chosen decomposition, so this conjugation does not stay inside the block-diagonal subgroup.

Now include the reflection / Pin^+-lift as

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^+ \to SL(5,\mathbb{Z}) \rtimes \mathbb{Z}_2^R \to 1, \tag{C.13}$$

and let $H := \big(\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})\big) \rtimes \mathbb{Z}_2$ be the restricted semi-direct subgroup. Pulling back the extension gives $\iota^*\widetilde{G_U}^+$, the unique Pin^+-lift of the subgroup as it sits inside the full lattice automorphism group.

The obstruction is now immediate and unavoidable: any conjugation that would relocate the -1 into the $\mathrm{SL}(2,\mathbb{Z})$-block requires a conjugator P outside the block-diagonal normalizer, and lifting that conjugation to $\widetilde{G_U}^+$ will insert a central sign. In short, the bosonic group admits coordinate moves by non-block conjugation. Thus, once the reflection and Pin^+ central sign are included, the block-diagonal subgroup cannot contain a pure reflection on the $\mathrm{SL}(2,\mathbb{Z})$ factor, as the semi-direct structure prevents it.[38]

D Lyndon–Hochschild–Serre Spectral Sequence

In our analysis we examine various extensions of groups, by the addition of a reflection as well as the introduction of Spin- and Pin^+-lifts. The group cohomology of these extensions, which we use for the analysis of Abelianizations and bordism groups, can be accessed via the Lyndon-Hochschild-Serre (LHS) spectral sequence. In particular, for an extension

$$1 \to N \to \widetilde{G} \to G \to 1\,, \tag{D.1}$$

the second page of the homological LHS spectral sequence is given by

$$E^2_{p,q} = H_p\big(B\widetilde{G}; H_q(BN;A)\big) \Longrightarrow H_{p+q}(B\widetilde{G};A)\,, \tag{D.2}$$

[38]One may wonder if the issue may arise if we had instead started with a reflection element in the $\mathrm{SL}(2,\mathbb{Z})$ sub-block of $\mathrm{SL}(5,\mathbb{Z})$ in (C.11). However, we see that this element will not only extend the $\mathrm{SL}(2,\mathbb{Z})$ block, but a combination of the the $\mathrm{SL}(3,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ blocks due to the semi-direct product.

where in general A is a G-module, and in all cases $H_q(BN;A)$ has a $\widetilde{G}$-module structure arising from the G-action on N induced by (D.1). In our case it will suffice to consider $A=\mathbb{Z}$ or $A=\mathbb{Z}_2$ with trivial G-action. Thus, the group homology of $\widetilde{G}$ can be related to the group homology of G and N.

D.1 Inclusion of Reflections

Let us first include the reflection in the bosonic U-duality group, describing a semi-direct product with $\mathbb{Z}_2^R$. In this subsection, we make the following computations.

Proposition D.3. *Suppose the dimension $D \leq 7$.*

1. *Let A be either $\mathbb{Z}$ or $\mathbb{Z}_2$. Then $H_1(B(G_U \rtimes \mathbb{Z}_2^R);A) \cong \mathbb{Z}_2$.*
2. *The map $H_2(BG_U;\mathbb{Z}) \to H_2(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z})$ is an isomorphism.*

Proof. We have a short exact sequence

$$1 \to G_U \to G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R \to 1\,, \tag{D.4}$$

so we can apply the LHS spectral sequence (D.2):

$$E^2_{p,q} = H_p\big(B\mathbb{Z}_2^R; H_q(BG_U;\mathbb{Z})\big) \Longrightarrow H_{p+q}\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}\big)\,. \tag{D.5}$$

Since $D \leq 7$, G_U is perfect, i.e.,

$$\mathrm{Ab}\big[G_U\big] = H_1(BG_U;\mathbb{Z}) = 0\,. \tag{D.6}$$

We also have

$$H_k(B\mathbb{Z}_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}_2 & k>0,\ k \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{D.7a}$$

and

$$H_k(B\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2\,, \quad k \geq 0\,. \tag{D.7b}$$

Thus the E^2-page of the LHS spectral sequence (D.5) for p,q small is

2	$H_2(BG_U;\mathbb{Z})$	$H_2(BG_U;\mathbb{Z}_2)$	0	$H_2(BG_U;\mathbb{Z}_2)$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
q/p	0	1	2	3

(D.8)

Thus in degrees 1 and below, degree considerations mean there are no nonzero differentials nor extension problems and this spectral sequence collapses at the E^2-page. Thus

$$\mathrm{Ab}\big[G_U \rtimes \mathbb{Z}_2^R\big] = H_1\big(B(G_U \rtimes \mathbb{Z}_2^R); \mathbb{Z}\big) \cong \mathbb{Z}_2. \tag{D.9}$$

This takes care of part (1) for $A = \mathbb{Z}$; the proof for $\mathbb{Z}_2$ coefficients is essentially the same. For part (2), looking at the E^2-page (D.8), since total degree 2 otherwise vanishes, it suffices to prove that $E^2_{0,2} \cong H_2(BG_U; \mathbb{Z})$ survives to the E^∞-page. The only differential to or from it that does not vanish for degree reasons is $d_3\colon E^2_{3,0} \to E^2_{0,2}$. However, in the LHS spectral sequence for a semidirect product, all differentials which cross the line $q = 0$ must vanish. This is because the quotient map $q\colon G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R$ has a section given by a choice of reflection, so the pushforward map $q_*\colon H_*(B(G_U \rtimes \mathbb{Z}_2^R); \mathbb{Z}) \to H_*(B\mathbb{Z}_2^R)$ also has a section, hence must be surjective. This pushforward map is the edge homomorphism in the spectral sequence, meaning that it is realized as the quotient by all elements with $q > 0$; for this to be surjective, differentials cannot kill any classes on the line $q = 0$. Therefore the d_3 of interest vanishes and the inclusion $G_U \to G_U \rtimes \mathbb{Z}_2^R$ is indeed an isomorphism on H_2. ☐

D.2 Spin- and Pin$^+$-Lifts

In this section we turn to the analogous statements incorporating Spin- and Pin$^+$-Lifts.

D.2.1 Spin-Lift

Next, we discuss the non-trivial $\mathbb{Z}_2$ central extension associated to the Spin-lift of G_U and described by the short exact sequence

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U} \to G_U \to 1\,. \tag{D.10}$$

Proposition D.11. *For $D \leq 7$, $H_1(B\widetilde{G_U}; \mathbb{Z}) = \mathrm{Ab}\big[\widetilde{G_U}\big] = 0$.*

We will once again use the LHS spectral sequence, but this time we will need to know the value of a differential. To do so, we will want to work in a more general setting. Let

$$B\mathbb{Z}_2 \to Y \to X \tag{D.12}$$

be a principal $B\mathbb{Z}_2$-bundle, which is classified by a map $f\colon X \to B^2\mathbb{Z}_2$. Since $B^2\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 2)$, the homotopy class of f is equivalent data to a class $w \in H^2(X; \mathbb{Z}_2)$. This generalizes the case of a central extension

$$1 \to \mathbb{Z}_2 \to \widetilde{G} \to G \to 1; \tag{D.13}$$

set $X = BG$ and $Y = B\widetilde{G}$. Then the classifying space functor turns the extension (D.13) into the fibration (D.12), and identifies the LHS spectral sequence for this extension with the

Serre spectral sequence for the fibration. The class $w \in H^2(BG;\mathbb{Z}_2)$ equals the cohomology class classifying the central extension (D.13).

Lemma D.14. *Given X, Y, f, and w as above, such that X is connected, then in the homological Serre spectral sequence for* (D.12)*, which has signature*

$$E^2_{p,q} = H_p(X; H_q(B\mathbb{Z}_2;\mathbb{Z})) \Longrightarrow H_{p+q}(Y;\mathbb{Z}), \tag{D.15}$$

the differential[39] $d_2\colon E^2_{2,0} \to E^2_{0,1}$ *is the map*

$$\begin{aligned} H_2(X;\mathbb{Z}) &\longrightarrow H_0(X;\mathbb{Z}_2) \cong \mathbb{Z}_2 \\ x &\longmapsto w \frown (x \bmod 2). \end{aligned} \tag{D.16}$$

Here, "$\frown$" denotes the cap product, which evaluates a cohomology class on a homology class. Sometimes this can be calculated using the universal coefficient theorem, which gives us a short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),\mathbb{Z}_2) \to H^n(X;\mathbb{Z}_2) \xrightarrow{h} \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z}_2) \to 0 \tag{D.17}$$

such that the map h sends a cohomology class y to the homomorphism $x \mapsto y \frown (x \bmod 2)$.

Proof. The classifying map f induces a map of E^2-pages of spectral sequences from (D.15) to the Serre spectral sequence for the universal principal $B\mathbb{Z}_2$-bundle

$$1 \to B\mathbb{Z}_2 \to E(B\mathbb{Z}_2) \to B^2\mathbb{Z}_2 \to 1. \tag{D.18}$$

This map of spectral sequences commutes with differentials, and under the identification of the E^2-page with homology, this map is the pushforward map on homology. Let ${}'E^r_{p,q}$ denote the Serre spectral sequence associated to (D.18). Then we obtain a commutative diagram

$$\begin{array}{ccc} E^2_{2,0} = H_2(X;\mathbb{Z}) & \xrightarrow{\quad f_* \quad} & {}'E^2_{2,0} = H_2(B^2\mathbb{Z}_2;\mathbb{Z}) \\ \Big\downarrow d_2 & & \Big\downarrow d_2 \\ E^2_{0,1} = H_0(X;\mathbb{Z}_2) & \xrightarrow{\quad f_* \quad} & {}'E^2_{0,1} = H_0(B^2\mathbb{Z}_2;\mathbb{Z}_2) \\ & \searrow^{\cong} \quad {}^{\cong}\swarrow & \\ & \mathbb{Z}_2. & \end{array} \tag{D.19}$$

The claimed formula for d_2 in (D.16) also commutes with these maps, so it suffices to prove the lemma in the case of the universal principal $B\mathbb{Z}_2$-bundle (D.18), for which $f = \mathrm{id}$ and w is the tautological class.

[39] Typically the differential is denoted by d^2 in the homological version of the spectral sequence, but we stick to d_2 here in order to avoid confusion with the square of the differential.

The Hurewicz theorem implies $H_2(B^2\mathbb{Z}_2;\mathbb{Z}) \cong \mathbb{Z}_2$; let y be the nonzero element. Since $H_1(B^2\mathbb{Z}_2;\mathbb{Z}) = 0$ because $B^2\mathbb{Z}_2$ is simply connected, the universal coefficient long exact sequence (D.17) collapses to an isomorphism $h\colon H^2(B^2\mathbb{Z}_2;\mathbb{Z}_2) \to \mathrm{Hom}(H_2(B^2\mathbb{Z}_2;\mathbb{Z}),\mathbb{Z}_2)$. This map sends w to the homomorphism "cap product with w," so $w \frown (y \bmod 2) = 1 \in \mathbb{Z}_2$, since $y \bmod 2$ is the only class on which that cap product can be nonzero. Since $B^2\mathbb{Z}_2$ is connected, $H_0(B^2\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Therefore $d_2(y) = w \frown (y \bmod 2)$ if and only if this d_2 is nonzero. But this d_2 must be nonzero: since $E(B\mathbb{Z}_2)$ is contractible, $E^\infty_{p,q}$ must vanish if $p+q>0$, and since $E^2_{0,1}$ can only be killed by this d_2, this d_2 is nonzero. ☐

Proof of Proposition D.11. We study the homological LHS spectral sequence with $\mathbb{Z}$ coefficients associated to (D.10), which has E^2-page

$$\begin{array}{c|cccc} 2 & 0 & 0 & 0 & 0 \\ 1 & H_0(BG_U;\mathbb{Z}_2) & H_1(BG_U;\mathbb{Z}_2) & H_2(BG_U;\mathbb{Z}_2) & H_3(BG_U;\mathbb{Z}_2) \\ 0 & H_0(BG_U;\mathbb{Z}) & H_1(BG_U;\mathbb{Z}) & H_2(BG_U;\mathbb{Z}) & H_3(BG_U;\mathbb{Z}) \\ \hline q/p & 0 & 1 & 2 & 3 \end{array} \tag{D.20}$$

where we already plugged in the homology groups of $B\mathbb{Z}_2$ from (D.7). Since we have restricted to $D \le 7$, so that G_U is perfect, (D.20) simplifies to

$$\begin{array}{c|cccc} 2 & 0 & 0 & 0 & 0 \\ 1 & \mathbb{Z}_2 & 0 & H_2(BG_U;\mathbb{Z}_2) & H_3(BG_U;\mathbb{Z}_2) \\ 0 & \mathbb{Z} & 0 & H_2(BG_U;\mathbb{Z}) & H_3(BG_U;\mathbb{Z}) \\ \hline q/p & 0 & 1 & 2 & 3 \end{array} \tag{D.21}$$

From this we can see that $H_0(B\widetilde{G}_U;\mathbb{Z}) = \mathbb{Z}$, and $H_1(B\widetilde{G}_U;\mathbb{Z})$ is either 0 or $\mathbb{Z}_2$ if $d_2\colon E^2_{2,0} \to E^2_{0,1}$ is nonzero, resp. 0. By Lemma D.14, for any class $x \in H_2(BG_U;\mathbb{Z})$, $d_2(x) = w \frown (x \bmod 2)$.

Now use the universal coefficient theorem again. Since G_U is perfect, $H_1(BG_U;\mathbb{Z}) = 0$, so just as in the proof of Lemma D.14, the universal coefficient short exact sequence (D.17) collapses to an isomorphism $h\colon H^2(BG_U;\mathbb{Z}_2) \to \mathrm{Hom}(H_2(B^2\mathbb{Z}_2;\mathbb{Z}),\mathbb{Z}_2)$, where $h(y)$ is the function $x \mapsto y \frown (x \bmod 2)$. The class $w \in H^2(BG_U;\mathbb{Z}_2)$ representing the extension $\widetilde{G}_U \to G_U$ is nonzero, because in all examples of interest, this extension is nonsplit. Therefore the function $d_2(x) = x \mapsto w \frown (x \bmod 2)$ is also nonzero, which implies $H_1(B\widetilde{G}_U;\mathbb{Z}) = 0$. ☐

As a consistency check, this matches what we observed in bordism in §3.1.1. The argument in the proof of Proposition D.11 also fixes the second homology group to be

$$H_2(B\widetilde{G}_U;\mathbb{Z}) = \ker(d_2) \subset H_2(BG_U;\mathbb{Z}), \tag{D.22}$$

which in the case of $\mathrm{SL}(n,\mathbb{Z})$ vanishes.

D.2.2 Pin$^+$-Lift

Finally, we analyze the homology for the full Pin$^+$-lift of U-duality groups in $D \leq 7$. This lift is described by a short exact sequence

$$1 \to \mathbb{Z}_2 \to \widetilde{G_U}^+ \to G_U \rtimes \mathbb{Z}_2^R \to 1\,. \tag{D.23}$$

Proposition D.24. *For $D \leq 7$, $H_1(B\widetilde{G_U}^+;\mathbb{Z}) \cong \mathrm{Ab}\big[\widetilde{G_U}^+\big] \cong \mathbb{Z}_2$.*

Proof. Set up the homological LHS spectral sequence associated to (D.23) with $\mathbb{Z}$ coefficients. The E^2-page is

$$\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 1 & \mathbb{Z}_2 & H_1\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2\big) & H_2\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2\big) \\ 0 & \mathbb{Z} & \mathbb{Z}_2 & H_2\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}\big). \\ \hline q/p & 0 & 1 & 2 \end{array} \tag{D.25}$$

The $\mathbb{Z}_2$ summand in $E^2_{1,0}$ survives to the E^∞-page for degree reasons, so we are done if we can show that $d_2\colon E^2_{2,0} \to E^2_{0,1}$ is nonzero, to remove the other $\mathbb{Z}_2$ summand in total degree 1. For this, consider the map of short exact sequences induced by the homomorphism $j\colon \widetilde{G_U} \hookrightarrow \widetilde{G_U}^+$ including the Spin-lift of G_U into the Pin$^+$-lift:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \widetilde{G_U} & \longrightarrow & G_U & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow j & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \widetilde{G_U}^+ & \longrightarrow & G_U \rtimes \mathbb{Z}_2^R & \longrightarrow & 1 \end{array} \tag{D.26}$$

This induces a map of LHS spectral sequences which on the E^2-page is the pushforward map j_* on homology, and which commutes with all differentials. For the time being, let $E^r_{p,q}$ denote the LHS for the Spin-lift and ${}^+E^r_{p,q}$ denote the LHS for the Pin$^+$-lift. Thus we have a commutative diagram

$$\begin{array}{ccc} E^2_{2,0} = H_2(BG_U;\mathbb{Z}) & \xrightarrow{\;j_*\;} & {}^+E^2_{2,0} = H_2(B(G_U \rtimes \mathbb{Z}_2);\mathbb{Z}) \\ \downarrow d_2 & & \downarrow {}^+d_2 \\ E^2_{0,1} = H_0(BG_U;\mathbb{Z}_2) & \xrightarrow{\;j_*\;} & {}^+E^2_{0,1} = H_0(B(G_U \rtimes \mathbb{Z}_2);\mathbb{Z}_2) \\ & \searrow^{\cong} \quad \mathbb{Z}_2. \quad {}^{\cong}\swarrow & \end{array} \tag{D.27}$$

In the proof of Proposition D.11 we saw that the leftmost d_2 is surjective; since j_* is an isomorphism in degree 0, then ${}^+d_2 \circ j_*$ (i.e. traveling along the upper right of (D.27)) is also surjective. Therefore ${}^+d_2$ must also be surjective, so as we mentioned above, we have finished showing $H_1(B\widetilde{G_U}^+;\mathbb{Z}) \cong \mathbb{Z}_2$. □

Like for the Spin-lift, Proposition D.24 matches the results we obtained in bordism in §3.1.2.

E Calculation of $\Omega_1^{\text{Spin-}\widetilde{G_U}}(\text{pt})$ and $\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt})$

In this Appendix we calculate $\Omega_1^{\text{Spin-}\widetilde{G_U}}(\text{pt})$ and $\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt})$ for the U-duality groups G_U in dimensions $3 \le D \le 9$. We summarize the answers in the following theorems.

Theorem E.1. *For $3 \le D \le 9$, there is an isomorphism*

$$\Omega_1^{\text{Spin-}\widetilde{G_U}}(\text{pt}) \cong \text{Ab}\big[\widetilde{G_U}\big] \cong \begin{cases} 0, & 3 \le D \le 7 \\ \mathbb{Z}_{24}, & D = 8, 9. \end{cases} \tag{E.2}$$

Theorem E.3. *For $3 \le D \le 9$, there is an isomorphism*

$$\Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\text{pt}) \cong \text{Ab}\big[\widetilde{G_U}^+\big] \cong \begin{cases} \mathbb{Z}_2, & 3 \le D \le 7 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & D = 8, 9. \end{cases} \tag{E.4}$$

In all cases, one $\mathbb{Z}_2$ summand is generated by the bordism class of a circle whose duality bundle has monodromy given by the nontrivial element of $\mathbb{Z}_2^R$; in $D = 8, 9$ the other $\mathbb{Z}_2$ summand is represented by a circle with monodromy given by a Spin-lift of $S \in \mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{Mp}(2,\mathbb{Z})$.

We also discuss the bosonic version of the spin result; this is not a new result, but offers a nice parallel to Theorem E.1.

Proposition E.5. *For $3 \le D \le 10$, there is an isomorphism*

$$\Omega_1^{\text{Spin}}(BG_U) \cong \mathbb{Z}_2 \oplus \text{Ab}\big[G_U\big] \cong \begin{cases} \mathbb{Z}_2, & 3 \le D \le 7 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{12}, & D = 8, 9. \end{cases} \tag{E.6}$$

In all cases, one $\mathbb{Z}_2$ summand is generated by the circle with periodic spin structure and trivial duality bundle; in $D = 8, 9$, the other $\mathbb{Z}_{12}$ summand is generated by a circle with either spin structure and a duality bundle whose monodromy is $T \in \mathrm{SL}(2,\mathbb{Z})$.

We will prove these theorems by two different methods: an Atiyah-Hirzebruch spectral sequence calculation in §E.1, and an Adams spectral sequence calculation in §E.2. We focus on the case $D \le 7$ to simplify the arguments. Adams and Atiyah-Hirzebruch spectral sequence calculations for the $D = 9$ cases of Theorems E.1 and E.3 and Proposition E.5 appear in [10, §A] and [23, §§12–14]. This leaves $D = 8$, which we briefly discuss at the end of §E.1.

E.1 Atiyah-Hirzebruch Spectral Sequence Calculations

The Atiyah-Hirzebruch spectral sequence for the reduced Spin bordism of a space X has signature

$$E^2_{p,q} = \widetilde{H}_p(X; \Omega^{\mathrm{Spin}}_q(\mathrm{pt})) \Longrightarrow \widetilde{\Omega}^{\mathrm{Spin}}_{p+q}(X). \tag{E.7}$$

Proof of Proposition E.5 using the Atiyah-Hirzebruch spectral sequence. Apply the Atiyah-Hirzebruch spectral sequence (E.7) with $X = BG_U$, and recall $\Omega_0^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}$ and $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}_2$. On the E^2-page, in total degree 1, we have

$$E^2_{1,0} \cong \widetilde{H}_1(BG_U; \Omega^{\mathrm{Spin}}_0(\mathrm{pt})) \cong \widetilde{H}_1(BG_U; \mathbb{Z}) = \mathrm{Ab}\big[G_U\big] \tag{E.8a}$$

$$E^2_{0,1} \cong \widetilde{H}_0(BG_U; \Omega^{\mathrm{Spin}}_1(\mathrm{pt})) \cong \widetilde{H}_0(BG_U; \mathbb{Z}_2) \cong 0. \tag{E.8b}$$

For degree reasons, all differentials into or out of $E^2_{1,0}$ vanish, so it survives intact to the E^∞-page. There is no extension problem, so $\widetilde{\Omega}^{\mathrm{Spin}}_1(BG_U) \cong \mathrm{Ab}\big[G_U\big]$. For unreduced Spin bordism, $\Omega_1^{\mathrm{Spin}}(BG_U) \cong \widetilde{\Omega}_1^{\mathrm{Spin}}(BG_U) \oplus \Omega_*^{\mathrm{Spin}}(\mathrm{pt})$, so we direct-sum on $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}_2$. $\square$

For the Spin- and Pin-lifts, we must use a twisted variant.

Definition E.9 (Wang [78, Definition 8.2])**.** Let X be a space and $w \in H^2(X; \mathbb{Z}_2)$. An (X, w)-twisted spin structure on an oriented vector bundle $E \to M$ is the data of a map $f\colon M \to X$ and a trivialization of $w_2(E) + f^*(w)$.[40]

We will let $\Omega_k^{\mathrm{Spin}}(X, w)$ denote the group of bordism classes of k-dimensional manifolds with (X, w)-twisted spin structures.

Lemma E.10 ([80, §3.1])**.** *Let $w \in H^2(BG; \mathbb{Z}_2)$ be the cohomology class of the extension $1 \to \mathbb{Z}_2 \to \widetilde{G} \to G \to 1$. Then the notions of a* Spin-$\widetilde{G}$ *structure and a (BG, w)-twisted spin structure on a vector bundle are canonically equivalent.*

Thus, $\Omega_*^{\mathrm{Spin}\text{-}\widetilde{G_U}} \cong \Omega_*^{\mathrm{Spin}}(BG_U, w)$ and $\Omega_*^{\mathrm{Spin}\text{-}\widetilde{G_U}^+} \cong \Omega_*^{\mathrm{Spin}}(B(G_U \rtimes \mathbb{Z}_2^R), w)$. Here we are implicitly using that the class $w \in H^2(BG_U; \mathbb{Z}_2)$ is invariant under the $\mathbb{Z}_2^R$-action, therefore passes through the LHS spectral sequence to define a class in $H^2(B(G_U \rtimes \mathbb{Z}_2^R); \mathbb{Z}_2)$, which we also call w. Alternatively, the Spin double cover $\widetilde{G_U} \to G_U$ extends to the Pin^+ double cover $\widetilde{G_U}^+ \to G_U \rtimes \mathbb{Z}_2^R$, so the cohomology class w classifying it is the restriction of a class $w \in H^2(B(G_U \rtimes \mathbb{Z}_2^R); \mathbb{Z}_2)$. When we write "$w$," it will always be clear from context which of these two classes we mean.[41]

[40] There is a more general notion of twisted spin structure allowing modifications of both w_1 and w_2: see Hebestreit-Joachim [79] as well as [80, Definition 1.24]. The James spectral sequence, and the formulas for its low-degree d_2s, both exist in this generality, as is proven in Kasprowski-Powell [81, Proposition 3.9] following Teichner [82]; see also [83, §5]. The Adams spectral sequence we use in §E.2 for twisted Spin bordism also generalizes to this setting; see [80].

[41] In addition to the Pin^+ lift of $G_U \rtimes \mathbb{Z}_2^R$, there is also a Pin^- lift $\widetilde{G_U}^-$, classified by $w + r^2 \in H^2(B(G_U \rtimes \mathbb{Z}_2); \mathbb{Z}_2)$. Here r is the pullback of the unique nonzero class in $H^1(B\mathbb{Z}_2^R; \mathbb{Z}_2)$ by the quotient map $G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R$.

Theorem E.11 (Teichner [84, Proposition 1]). *With (X, w) as in Definition E.9, there is a spectral sequence with signature*

$$E^2_{p,q} = H_p(X; \Omega_q^{\mathrm{Spin}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{Spin}}(X, w). \tag{E.12a}$$

such that, at least for $2 \le p \le 4$, the differential $d_2\colon E^2_{p,0} \to E^2_{p-2,1}$, as a map $H_p(X;\mathbb{Z}) \to H_{p-2}(X;\mathbb{Z}_2)$, is reduction modulo 2 followed by the dual of the map

$$\begin{aligned} \mathrm{Sq}^2_w\colon H^{p-2}(X;\mathbb{Z}_2) &\to H^p(X;\mathbb{Z}_2) \\ x &\mapsto \mathrm{Sq}^2 x + wx. \end{aligned} \tag{E.12b}$$

"Dual" in Theorem E.11 means: the universal coefficient theorem shows that the cap product pairing canonically identifies $H_k(X;\mathbb{Z}_2)$ and $H^k(X;\mathbb{Z}_2)$ as dual $\mathbb{Z}_2$-vector spaces, and we take the dual of the linear map Sq^2_w.

Teichner calls the spectral sequence (E.12) the James spectral sequence. See [84–87] for some example computations with this spectral sequence similar to those in this paper.

Teichner shows that if there is an orientable vector bundle $V \to X$ with $w_2(V) = w$, then the James spectral sequence is isomorphic to the Atiyah-Hirzebruch spectral sequence for the Spin bordism of the Thom spectrum $X^{V-\mathrm{rank}(V)}$ (that this is $\Omega_*^{\mathrm{Spin}}(X, w_2(V))$-twisted Spin bordism is a folk theorem, with one proof given in [23, Corollary 10.19]). It is not always possible to realize every degree-2 cohomology class w as w_2 of a vector bundle [88, §2], and indeed this issue can occur when w is the extension class for the Spin-lift of a real U-duality group $G_U(\mathbb{R})$ [89, Theorem 4.2], so the extra generality of the James spectral sequence is necessary.[42]

Proof of Theorem E.1 for $D \le 7$ using the James spectral sequence. We begin by drawing the E^2-page of this spectral sequence (E.12a) for (BG_b, w)-twisted Spin bordism, which by Lemma E.10 also computed Spin-$\widetilde{G_U}$ bordism. Since $D \le 7$, G_U is perfect and so $H_1(BG_U;\mathbb{Z})$ and $H_1(BG_U;\mathbb{Z}_2)$ both vanish.

$$\begin{array}{c|ccc} 1 & \mathbb{Z}_2 & 0 & H_2\big(BG_U;\mathbb{Z}_2\big) \\ 0 & \mathbb{Z} & 0 & H_2\big(BG_U;\mathbb{Z}\big). \\ \hline q/p & 0 & 1 & 2 \end{array} \tag{E.13}$$

The only nonzero group in total degree 1 is $E^2_{0,1} \cong \mathbb{Z}_2$, and this is the target of $d_2\colon E^2_{2,0} \to E^2_{0,1}$. So if we can show that this d_2 is nonzero, we are done. In Theorem E.11, we learned that $d_2 = (\mathrm{Sq}^2_w)^\vee \circ r$, where r is reduction mod 2; we will show $(\mathrm{Sq}^2_w)^\vee$ and r are both surjective, which implies their composition is nonzero.

[42]More general twisted Atiyah-Hirzebruch spectral sequences have been constructed and discussed in [90–96], but the differentials we need in the case of twisted Spin bordism have not been computed, so we use the James spectral sequence.

For r, surjectivity follows from the Bockstein long exact sequence

$$\cdots \to H_2(BG_U;\mathbb{Z}) \xrightarrow{r} H_2(BG_U;\mathbb{Z}_2) \to H_1(BG_U;\mathbb{Z}) = 0 \to \cdots \tag{E.14}$$

The codomain of $(\mathrm{Sq}^2_w)^\vee$ is isomorphic to $\mathbb{Z}_2$, so this map is surjective if and only if it is nonzero, which is true if and only if its dual $\mathrm{Sq}^2_w\colon H^0(BG_U;\mathbb{Z}_2) \to H^2(BG_U;\mathbb{Z}_2)$ is nonzero. And indeed:

$$\mathrm{Sq}^2_w(1) = \mathrm{Sq}^2(1) + w\cdot 1 = 0 + w = w, \tag{E.15}$$

as Sq^2 vanishes in degree 0 of any space. Since $\widetilde{G_U} \to G_U$ is a nonsplit extension, $w \neq 0$ and so we are done. □

Proof of Theorem E.3 for $D \le 7$ using the James spectral sequence. The proof for the Pin^+-lift is similar. By Lemma E.10, we want to calculate $(B(G_U \rtimes \mathbb{Z}_2^R), w)$-twisted Spin bordism. To draw the E^2-page, we recall from Proposition D.3 that, since $D \le 7$, $H_1(BG_U;\mathbb{Z})$ and $H_1(BG_U;\mathbb{Z}_2)$ are both isomorphic to $\mathbb{Z}_2$.

$$\begin{array}{c|ccc} 1 & \mathbb{Z}_2 & \mathbb{Z}_2 & H_2\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2\big) \\ 0 & \mathbb{Z} & \mathbb{Z}_2 & H_2\big(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}\big). \\ \hline q/p & 0 & 1 & 2 \end{array} \tag{E.16}$$

Once again we are done if we can show that $d_2\colon E^2_{2,0} \to E^2_{0,1}$ is nonzero; this kills $E^2_{0,1}$, and the $\mathbb{Z}_2$ in $E^2_{1,0}$ survives to the E^∞-page for degree reasons, so this would imply that in total degree 1 on the E^∞-page, there is a single $\mathbb{Z}_2$ summand and no other nonzero classes, which would finish the proof.

We will compute this differential by comparing it with the differential for (BG_U, w)-twisted Spin bordism. Specifically, because the inclusion map $j\colon G_U \to G_U \rtimes \mathbb{Z}_2^R$ pulls $w \in H^2(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$ back to $w \in H^2(BG_U;\mathbb{Z}_2)$, there is a map of James spectral sequences commuting with differentials, analogous to the map of LHS spectral sequences that we used in the proof of Proposition D.24.

Specifically, if we let $E^r_{p,q}$ denote the James spectral sequence for (BG_U, w) and ${}^+E^r_{p,q}$ denote the James spectral sequence for $(B(G_U \rtimes \mathbb{Z}_2^R), w)$, then we have a commutative diagram

$$\begin{array}{ccc} E^2_{2,0} = H_2(BG_U;\mathbb{Z}) & \xrightarrow{\quad j_* \quad} & {}^+E^2_{2,0} = H_2(BG_U;\mathbb{Z}) \\ \Big\downarrow d_2 & & \Big\downarrow d_2 \\ E^2_{0,1} = H_0(BG_U;\mathbb{Z}_2) & \xrightarrow{\quad j_* \quad} & {}^+E^2_{0,1} = H_0(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2) \\ & \searrow^{\cong} \quad \mathbb{Z}_2 \quad {}^{\cong}\swarrow & \end{array} \tag{E.17}$$

Since both classifying spaces are connected, the pushforward map j_* on H_0 is an isomorphism. We proved in Proposition D.3 that j_* is also an isomorphism on H_2. In the proof of Theorem E.1 by the James spectral sequence earlier in this subsection, we showed that the left-hand vertical map, the d_2 for (BG_U, w), is surjective. Commutativity thus implies the right-hand vertical map, which is the d_2 of interest, is also surjective. As noted above, this finishes the proof. □

Thus only dimensions $D = 8, 9$ are left. These fall to the James spectral sequence in a similar manner; we highlight a few differences and leave the details to the interested reader. For the Spin lifts:

- In these examples, $d_2 \colon E^2_{2,0} \to E^2_{0,1}$ is 0, so in total degree 1, the E^∞ page consists of a $\mathbb{Z}_2$ in $E^2_{0,1}$ and a $\mathbb{Z}_{12} = \mathrm{Ab}\big[G_U\big]$ in $E^\infty_{1,0}$.
- There is a hidden extension joining $E^\infty_{1,0}$ and $E^\infty_{0,1}$. This can be seen by embedding $i \colon \mathbb{Z}_4 \hookrightarrow \mathrm{SL}(2,\mathbb{Z})$ (or into $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$) and using the map of spectral sequences. The presence of a nontrivial hidden extension in this degree in $(B\mathbb{Z}_4, i^*w)$-twisted Spin bordism follows from [23, §13.4].

For the Pin$^+$-lifts, the story is similar: $E^2_{1,0} \cong \mathbb{Z}_2$, and again the differential vanishes. This time, the extension splits, as can be seen by embedding $D_{16} \hookrightarrow \mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^R$ or $(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})) \rtimes \mathbb{Z}_2^R$ and comparing with [23, Theorem 14.18].

E.2 Adams Spectral Sequence Calculations

Lastly, we use the Adams spectral sequence to compute the one-dimensional Spin-$\widetilde{G_U}$ and Spin-$\widetilde{G_U}^+$ bordism groups. This tool is more abstract than the Atiyah-Hirzebruch spectral sequence, but in the last several years has become more prominent in the theoretical physics literature as a powerful yet tractable way to compute twisted Spin bordism groups. See [97, 23] for introductions to this technique aimed at a mathematical physics audience as well as several example computations.

The references cited apply the Adams spectral sequence to (X, w)-twisted Spin bordism under the assumption that there is an oriented vector bundle $V \to X$ with $w_2(V) = w$. As we noted above in §E.1, not only is this not true in general, there are specific counterexamples for the Spin-lifts of the real versions of U-duality groups. Thus, following [80], we use Baker-Lazarev's relative Adams spectral sequence [98] in this section. See [80, 99] for example computations with this version of the Adams spectral sequence.

Let $\mathcal{A}(1)$ be the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. Since Sq^1 and Sq^2 act naturally on mod 2 cohomology groups, the mod 2 cohomology of any space is naturally an $\mathcal{A}(1)$-module.

Lemma E.18 ([80, Lemma 2.27(3)]). *Let X be a space, $w \in H^2(X;\mathbb{Z}_2)$, and Sq^2_w be the operator defined in* (E.12b). *Then the actions of Sq^1 and Sq^2_w on $H^*(X;\mathbb{Z}_2)$ satisfy the Adem* *relations for* Sq^1 *and* Sq^2*, and therefore define another* $\mathcal{A}(1)$*-module structure on* $H^*(X;\mathbb{Z}_2)$ *which we call* $H^*_w(X;\mathbb{Z}_2)$*.*

That is: if we modify the action of Sq^2 on $H^*(X;\mathbb{Z}_2)$ by having it act by Sq^2_w instead, the result is still a well-defined $\mathcal{A}(1)$-module, and we call that $\mathcal{A}(1)$-module $H^*_w(X;\mathbb{Z}_2)$.

Theorem E.19 ([80, §2.2])**.** *Let* X *be a space and* $w \in H^2(X;\mathbb{Z}_2)$*. Then there is a graded* $\mathcal{A}(1)$*-module* M *and a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*_w(X;\mathbb{Z}_2) \otimes M, \mathbb{Z}_2) \Longrightarrow \Omega^{\mathrm{Spin}}_{t-s}(X,w)^\wedge_2. \tag{E.20}$$

There is an $\mathcal{A}(1)$*-module map* $M \to \mathbb{Z}_2$ *which is an isomorphism in degrees* 7 *and below.*

Here Ext is the derived functor of Hom (see [23, §11.2]) and $(-)^\wedge_2$ denotes "2-completion." For the spaces we consider in this paper, whose Spin bordism groups are finitely generated Abelian groups, 2-completion can heuristically be thought of as keeping the free and 2-torsion summands and throwing out the odd-torsion summands. Thus in this subsection we will do something else to account for odd-primary torsion.

For any space X and class $w \in H^2(X;\mathbb{Z}_2)$, the map $\phi\colon \Omega^{\mathrm{Spin}}_*(X,w) \to \Omega^{\mathrm{SO}}_*(X)$ is an isomorphism after localizing at any odd prime. This means that ϕ is an isomorphism on odd-torsion subgroups. When $w = 0$, this is a standard fact (see, e.g., [23, §10.5]); for general w it is less well-known but still a folklore theorem. See [100, Proof of Lemma 3.23] for a proof.

There is a map $\psi\colon \Omega^{\mathrm{SO}}_*(X) \to H_*(X;\mathbb{Z})$ obtained by sending an oriented manifold M with map $f\colon M \to X$ to $f_*([M])$, and Thom's computation of Ω^{SO}_* in low degrees [101, Théorème IV.13] implies ψ is an isomorphism in degrees 3 and below. Therefore to compute the odd-primary torsion in $\Omega_1^{\mathrm{Spin}\text{-}\widetilde{G_U}}$ and $\Omega_1^{\mathrm{Spin}\text{-}\widetilde{G_U}^+}$, it suffices to know $H_1(BG_U;\mathbb{Z})$ and $H_1(B(G_U \rtimes \mathbb{Z}_2^R),\mathbb{Z})$, i.e. the Abelianizations of these groups, which we discussed above, e.g. in Proposition D.3. Thus in what follows we will ignore odd torsion.

Before we get into the proofs of Theorems E.1 and E.3 and Proposition E.5, we have one more simplification to discuss. To run the Adams spectral sequence, we need to compute the Ext groups of $H^*_w(BG_U;\mathbb{Z}_2)$ and $H^*_w(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$, but we know very little about these cohomology groups in degrees 2 and above. Fortunately, graded Ext is "local" in the sense that low-degree information in cohomology completely determines Ext in low topological degrees. We have two points of view on this phenomenon.

1. Suppose you want to compute $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N,\mathbb{Z}_2)$ with a minimal $\mathcal{A}(1)$-module resolution $P_\bullet \to N$ (see [97, §4.4]). In practice, if one only knows the structure of N in degrees d and below, a minimal resolution can be constructed explicitly in degrees $t - s < d$, simply by trying to work out the minimal resolution and stopping once higher-degree information on N is necessary. This technique works because of a theoretical guarantee that, with one exception we discuss in a moment, if N is concentrated

in nonnegative degrees, the lowest-degree class in P_s, the s^{th} step of the minimal resolution, is approximately $3s$ [102, 103]. The exception is h_0-towers, which are not hard to recognize in a minimal resolution and so can be accounted for (see [97, Example 4.4.2]).

2. Alternatively, we can use a long exact sequence in Ext associated to a short exact sequence of $\mathcal{A}(1)$ modules, as in [97, §4.6]. Let N be an $\mathcal{A}(1)$ module and $N_{>\ell}$ the submodule of N consisting of elements in degrees $\ell+1$ and above. Then there is a short exact sequence
$$0 \to N_{>\ell} \to N \to N/N_{>\ell} \to 0\,, \tag{E.21}$$
which induces a long exact sequence in Ext. Since $N_{>\ell}$ is only concentrated in degree $\ell+1$ and above, $\mathrm{Ext}_{\mathcal{A}(1)}(N_{>\ell})$ is concentrated in degree $t-s \geq \ell+1$. Exactness then implies that
$$\mathrm{Ext}_{\mathcal{A}(1)}(N) \to \mathrm{Ext}_{\mathcal{A}(1)}(N/N_{>\ell}) \tag{E.22}$$
is an isomorphism for $t-s \leq \ell$. In our approach $\ell = 1$ due to our limited knowledge of the group homology of the U-duality groups.

Thus in particular, we may completely ignore the $\mathcal{A}(1)$-module M appearing in Lemma E.18, as it cannot affect the behavior of the spectral sequence in degrees 6 and below.[43]

Proof of Proposition E.5 for $D \leq 7$ using the Adams spectral sequence. We want to compute untwisted Spin bordism of BG_U, i.e. twisted Spin bordism for the class $0 \in H^2(BG_U;\mathbb{Z}_2)$. Therefore the input to Ext is simply $H^*(BG_U;\mathbb{Z}_2)$ as an $\mathcal{A}(1)$-module, as Sq^2_w for $w=0$ equals Sq^2.

Because $\widetilde{G_U} \to G_U$ is a non-trivial extension, its class in H^2 is nonzero, so there is at least one factor of $\mathbb{Z}_2$ in $H^2(BG_U;\mathbb{Z}_2)$. Therefore we can completely determine $H^*(BG_U;\mathbb{Z}_2)$ as an $\mathcal{A}(1)$-module in degrees 2 and below: we have the class $1 \in H^0$ and some number of classes in H^2 (and nothing in H^1 because G_U is perfect), and there is no action of the Steenrod algebra that connects $H^0(BG_U;\mathbb{Z}_2)$ with any element in $H^2(BG_U;\mathbb{Z}_2)$ (simply because $\mathrm{Sq}^1(1) = \mathrm{Sq}^2(1) = 0$). This means that when we apply Ext to $H^0(BG_U;\mathbb{Z})$, it simply produces the usual Spin bordism contributions of $\Omega_k^{\mathrm{Spin}}(\mathrm{pt})$ and we can focus on the positive-degree part of $H^*(BG_U;\mathbb{Z}_2)$, which we call X. The first step in a minimal resolution is
$$X \longleftarrow \Sigma^2\mathcal{A}(1)^{\oplus n} \oplus \Sigma^3 R\,, \tag{E.23}$$
where n is given in terms of
$$H^2(BG_U;\mathbb{Z}_2) = \mathbb{Z}_2^{\oplus n}\,, \tag{E.24}$$
and R denotes an unknown remainder. For the next step we only obtain new copies of $\mathcal{A}(1)$

[43] In fact, because the Anderson-Brown-Peterson splitting of Spin bordism [104] generalizes to (X,w)-twisted Spin bordism [79, §5.2], this extends to degree 7.

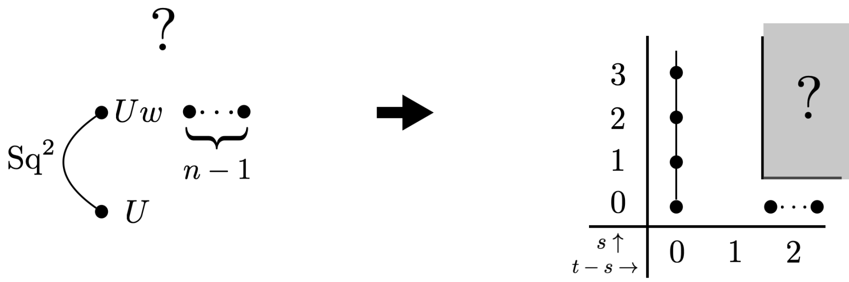

Figure 5: Steenrod structure of twisted Spin structure for the Spin-lift of perfect U-duality groups (left) and associated Adams chart (right).

in suspension Σ^3 and higher, in the next only in suspension Σ^4 and so on:

$$X \longleftarrow \Sigma^2\mathcal{A}(1)^{\oplus n} \oplus \Sigma^3 R \longleftarrow \Sigma^3 \tilde{R} \longleftarrow \Sigma^4 \tilde{R}' \longleftarrow \dots\,, \tag{E.25}$$

where the particular form of $\tilde{R}$ and $\tilde{R}'$ depends on the details we do not know. However, this is enough to draw the Adams chart for the first two columns, which are completely empty for X. From this we deduce the well-known fact that for perfect groups G,

$$\Omega_1^{\mathrm{Spin}}(BG_U) = \Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \oplus \mathrm{Ab}[G_U] = \Omega_1^{\mathrm{Spin}}(\mathrm{pt}) = \mathbb{Z}_2\,. \tag{E.26}$$

Thus $\Omega_*^{\mathrm{Spin}}(BG_U)$ only receives contributions from the Spin bordism of a point. □

Proof of Theorem E.1 for $D \leq 7$ using the Adams spectral sequence. This time around, we must use Sq^2_w with $w \neq 0$. This acts nontrivially from degree 0 to degree 2: as we saw in (E.15), $\mathrm{Sq}^2_w(1) = w$, which modifies the minimal resolution:

$$H^*_w(BG_U;\mathbb{Z}_2) \longleftarrow \mathcal{A}(1) \oplus \mathcal{A}(1)^{\oplus n-1} \oplus \Sigma^3 Q \longleftarrow \Sigma^1\mathcal{A}(1) \oplus \Sigma^3\widetilde{Q} \longleftarrow \Sigma^2\mathcal{A}(1) \oplus \Sigma^4\widetilde{Q}' \longleftarrow \dots\,, \tag{E.27}$$

where we denote the unknown contributions by Q, $\widetilde{Q}$, and $\widetilde{Q}'$, respectively. This leads to an Adams chart with the first column empty (see Figure 5), from which we read off

$$\Omega_1^{\mathrm{Spin}\text{-}\widetilde{G_U}}(\mathrm{pt}) = 0\,, \tag{E.28}$$

irrespective of the details. □

Proof of Theorem E.3 for $D \leq 7$ using the Adams spectral sequence. Let $r \in H^1(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$ be the pullback of the unique nonzero class in $H^1(B\mathbb{Z}_2^R;\mathbb{Z}_2)$ by the quotient map $q\colon G_U \rtimes \mathbb{Z}_2^R \to \mathbb{Z}_2^R$, and let w be the H^2 class of the extension $\widetilde{G_U}^+ \to G_U \rtimes \mathbb{Z}_2^R$. Then we have

$$\mathrm{Sq}^1(r) = r^2\,, \quad \mathrm{Sq}^2(r) = 0\,, \quad \mathrm{Sq}^2(w) = w^2\,, \quad \mathrm{Sq}^1(w) = rw + w_3(V)\,, \tag{E.29}$$

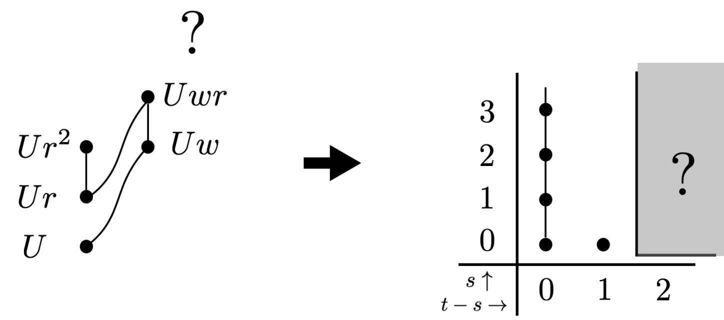

Figure 6: Steenrod structure of Pin$^+$-lift of perfect U-duality groups (left) and associated Adams chart (right).

which follow from the axioms and the Wu formula. Further note that r^2 is non-trivial: because the quotient q has a section given by a choice of reflection, $H^*(B\mathbb{Z}_2^R;\mathbb{Z}_2)$ is a direct summand of $H^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$. Since $r^2 \neq 0$ in $H^*(B\mathbb{Z}_2^R;\mathbb{Z}_2)$, the same is true pulled back to $B(G_U \rtimes \mathbb{Z}_2^R)$. Thus, we get a picture for the $\mathcal{A}(1)$-module structure on $H_w^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$ in degrees two and lower:

$$\mathrm{Sq}^1(1) = 0\,,\ \ \mathrm{Sq}_w^2(1) = w\,,\ \ \mathrm{Sq}^1(r) = r^2\,,\ \ \mathrm{Sq}_w^2(r) = wr\,, \tag{E.30}$$

If R_2 denotes the $\mathcal{A}(1)$-module which is the kernel of the unique nontrivial $\mathcal{A}(1)$-module map $\Sigma^{-1}\mathcal{A}(1) \to \Sigma^{-1}\mathbb{Z}_2$ (see [97, Figure 28, left] for a picture), then in degrees 2 and below, $H_w^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$ is isomorphic to the quotient of R_2 by its degree-≥ 3 elements.[44] We can therefore write down the first few steps in a minimal resolution for $H_w^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$:

$$H_w^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2) \longleftarrow \mathcal{A}(1) \oplus \Sigma\mathcal{A}(1) \oplus \Sigma^2 P \longleftarrow \Sigma^3 \widetilde{P} \longleftarrow \Sigma^4 \widetilde{P}' \longleftarrow \ldots\,, \tag{E.31}$$

with unknowns P, $\widetilde{P}$, $\widetilde{P}'$. This is enough to determine the the first two columns of the Adams chart as summarized in Figure 6. From this we read off

$$\Omega_0^{\text{Spin-}\widetilde{G_U}^+}(\mathrm{pt}) = \mathbb{Z}\,, \quad \Omega_1^{\text{Spin-}\widetilde{G_U}^+}(\mathrm{pt}) = \mathbb{Z}_2\,. \tag{E.32}$$

The $\mathbb{Z}_2$ is associated to the reflection element r, hence in twisted Spin bordism is represented by a circle with a duality bundle whose monodromy is a reflection. □

[44] For the nine-dimensional U-duality group, $H_w^*(B(G_U \rtimes \mathbb{Z}_2^R);\mathbb{Z}_2)$ is computed in degrees 12 and below in [23, Proposition 14.21], and the entire R_2 summand is visible. There is also another element in degree 1, which was associated to rotations; this part is absent here, since the U-duality groups for $D \leq 7$ are perfect and the Abelianization of $(G_U \rtimes \mathbb{Z}_2^R)$ is given by $\mathbb{Z}_2^R$ only.

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