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SMITH HOMOMORPHISMS AND Spin^h STRUCTURES

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Abstract. In this article, we answer two questions of Buchanan-McKean [BM23] about bordism for manifolds with spin^h structures: we establish a Smith isomorphism between the reduced spin^h bordism of $\mathbb{RP}^\infty$ and pin^{h-} bordism, and we provide a geometric explanation for the isomorphism $\Omega_{4k}^{\mathrm{Spin}^c}\otimes\mathbb{Z}[1/2]\cong\Omega_{4k}^{\mathrm{Spin}^h}\otimes\mathbb{Z}[1/2]$. Our proofs use the general theory of twisted spin structures and Smith homomorphisms that we developed in [DDK+24] joint with Devalapurkar, Liu, Pacheco-Tallaj, and Thorngren, specifically that the Smith homomorphism participates in a long exact sequence with explicit, computable terms.

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1. Introduction

At the start of the 1960s, C.T.C. Wall challenged the readers of [Wal60] to study the bordism groups of spin manifolds—and by the end of the decade, Anderson-Brown-Peterson [ABP67] had completely solved this problem, determining not just the spin bordism groups but also a convenient decomposition of the spectrum $MT\mathrm{Spin}$ itself, catalyzing computations of other, related bordism groups.

One such example is bordism for a complex analogue of spin structures, referred to as spin^c structures (see Example 2.2), which was solved almost immediately after Anderson-Brown-Peterson's work (see [Sto68, Chapter XI]). Similarly, one can replace the complex numbers with the quaternions, leading to the notion of a *spin^h structure*, i.e. a reduction of structure group to the group[1]

$$\mathrm{Spin}_n^h := \mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{SU}_2. \tag{1.1}$$

Spin^h structures have been studied in the mathematics and physics literature since the 1960s, with applications to quantum gravity [BFF78, Bec24], index theory, e.g. in [May65, Nag95, Bär99, FH21, Che17], Seiberg-Witten theory [OT96], immersion problems [Bär99, AM21], almost quaternionic geometry, e.g. in [Nag95, Bär99, AM21], and invertible field theories [FH21, BC18, WWW19,

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[1]Here and elsewhere in this article, the notation $G\times_{\{\pm 1\}} H$ indicates that there are central subgroups $\{\pm 1\}\subset G$, $\{\pm 1\}\subset H$ each isomorphic to the multiplicative group $\{\pm 1\}\subset\mathbb{R}^\times$; then $G\times_{\{\pm 1\}} H$ is the quotient of $G\times H$ by the diagonal $\{\pm 1\}$ subgroup. These subgroups of G and H will be clear from context.

WWZ20, WW20, DL21, DY22, Ste22, WWW22, BI23, DDK+23]. See [Law23] for a review of the mathematical aspects of spin^h structures.

However, spin^h *bordism* has attracted interest only in the last few years, beginning with Freed-Hopkins' work [FH21] applying low-degree spin^h bordism groups to condensed-matter physics; other important results include obstruction theory for spin^h structures [AM21], the construction of a quaternionic Atiyah-Bott-Shapiro map [FH21, Hu23] and an Anderson-Brown-Peterson-style splitting of the spin^h bordism spectrum at the prime 2 [Mil23].[2]

Recently, Buchanan-McKean [BM23] proved a number of key results on spin^h bordism, including describing the above splitting in terms of characteristic classes and showing that a collection of characteristic classes valued in quaternionic K-theory detect a manifold's spin^h bordism class. Using this splitting, they give an algorithm for computing $\Omega_n^{\mathrm{Spin}^h}$ for all n and analyze the asymptotics of the size of the n^{th} spin^h bordism group in n.

Buchanan-McKean also ask several questions on spin^h bordism [BM23, §10] coming from their work. The main goal of this article is to answer two of these questions, which we now describe.

Anderson-Brown-Peterson [ABP69] established a *Smith isomorphism* $\mathrm{sm}_\sigma\colon \widetilde{\Omega}_n^{\mathrm{Spin}}(\mathbb{RP}^\infty) \xrightarrow{\cong} \Omega_{n-1}^{\mathrm{Pin}^-}$,[3] described concretely by taking a spin manifold M with a map $M \to \mathbb{RP}^\infty$ to the zero set of a generic section of the pullback of the tautological line bundle to M. Then, Bahri-Gilkey [BG87a, BG87b] constructed a completely analogous isomorphism $\mathrm{sm}_\sigma^c\colon \widetilde{\Omega}_n^{\mathrm{Spin}^c}(\mathbb{RP}^\infty) \xrightarrow{\cong} \Omega_{n-1}^{\mathrm{Pin}^c}$.

Question 1.2 (Buchanan-McKean [BM23, Question 10.8])**.** Let $\mathrm{Pin}_n^{h-} \coloneqq \mathrm{Pin}_n^- \times_{\{\pm 1\}} \mathrm{SU}_2$.[4] Is there a Smith isomorphism for pin^{h-} bordism?

We affirmatively answer this question.

Theorem 3.1. *For all n, there is an isomorphism*

$$\mathrm{sm}_\sigma^h\colon \widetilde{\Omega}_n^{\mathrm{Spin}^h}(\mathbb{RP}^\infty) \xrightarrow{\cong} \Omega_{n-1}^{\mathrm{Pin}^{h-}} \tag{1.3}$$

given by sending a pair (M, f) of a spin^h manifold M with a generic map $f\colon M \to \mathbb{RP}^\infty$ to the zero set of a generic section of the pullback of the tautological line bundle $\sigma \to \mathbb{RP}^\infty$ by f.

Part of this theorem is the assertion that such a zero set is generically a closed $(n-1)$-manifold with pin^{h-} structure.

The technique we use to prove Theorem 3.1 also enables us to solve another one of Buchanan-McKean's questions.

Question 1.4 (Buchanan-McKean [BM23, Question 10.3])**.** For all $k \geq 0$, $\mathrm{rank}(\Omega_{4k}^{\mathrm{Spin}^c}) = \mathrm{rank}(\Omega_{4k}^{\mathrm{Spin}^h})$. Is there a geometric explanation for this fact? Is there a procedure to produce generators for the free summand of $\Omega_{4k}^{\mathrm{Spin}^h}$ from those of $\Omega_{4k}^{\mathrm{Spin}^c}$?

To answer this question, we exhibit a map $p_*\colon \Omega_n^{\mathrm{Spin}^c} \to \Omega_n^{\mathrm{Spin}^h}$ induced from an inclusion $\mathrm{Spin}_n^c \hookrightarrow \mathrm{Spin}_n^h$. We show that p_* is part of a long exact sequence of bordism groups whose third term is $\Omega_{n-3}^{\mathrm{Spin}}(B\mathrm{SO}_3)$ (4.10), and give geometric interpretations to the three maps of the long exact sequence in (LES-1)–(LES-3). Exactness yields a quick proof of the following theorem.

[2]However, not everything carries over: just as the quaternions have less structure than $\mathbb{R}$ or $\mathbb{C}$, spin^h bordism has less structure than spin or spin^c bordism. For example, Ω_*^{Spin} and $\Omega_*^{\mathrm{Spin}^c}$ have ring structures induced from the direct product of manifolds, but $\Omega_*^{\mathrm{Spin}^h}$ does not. Thus the twisted Atiyah-Bott-Shapiro map mentioned above is not a ring homomorphism.

[3]Pin$^-$ structures are an unoriented generalization of spin structures that we discuss in Example 2.5.

[4]This group was first defined by Freed-Hopkins [FH21, (9.21)], who call it G^-.

Theorem 4.13. For all $k \geq 0$, the map

$$p_*\colon \Omega_{4k}^{\mathrm{Spin}^c} \otimes \mathbb{Z}[1/2] \longrightarrow \Omega_{4k}^{\mathrm{Spin}^h} \otimes \mathbb{Z}[1/2], \tag{1.5}$$

where p_* is as above, is an isomorphism.

This answers the first part of Question 1.4. Unfortunately, there is quite a bit of 2-torsion in $\Omega_*^{\mathrm{Spin}}(B\mathrm{SO}_3)$, preventing us from lifting to $\mathbb{Z}$. This also suggests that answering the second part of Buchanan-McKean's question, building manifold generators of free summands of spin^h bordism from manifold generators of free summands of spin^c bordism, would be very difficult.

We use the same technique to prove both Theorems 3.1 and 4.13: a method of easily producing geometrically-defined long exact sequences of bordism groups. The input is a virtual vector bundle V and a vector bundle W of ranks r_V, resp. r_W, both over a space X. From this data, there is a long exact sequence

$$\cdots \longrightarrow \Omega_n^{\mathrm{Spin}}(S(W)^{p^*V}) \xrightarrow{p_*} \Omega_n^{\mathrm{Spin}}(X^{V-r_V}) \xrightarrow{\mathrm{sm}_W} \Omega_{n-r_W}^{\mathrm{Spin}}(X^{V+W-r_V-r_W}) \longrightarrow \cdots \tag{1.6}$$

where p denotes the bundle map $S(W) \to X$ for the sphere bundle of W and sm_W is the *Smith homomorphism*, the map on bordism defined by taking a smooth representative of the Poincaré dual of the cobordism Euler class of W. This long exact sequence is natural in the data of X, V, and W. The spin bordism of the Thom spectrum X^{V-r_V} may be interpreted in terms of *twisted spin bordism*: the bordism of manifolds M equipped with a map $f\colon M \to X$ and a spin structure on $TM \oplus f^*(V)$ (see Definition 2.1 and Lemma 2.11). The exact sequence (1.6) is attributed to James and is well-known; its relationship to the Smith homomorphism is explained in our work [DDK+23, DDK+24] joint with Devalapurkar, Liu, Pacheco-Tallaj, and Thorngren. We call (1.6) the *Smith long exact sequence*. We prove Theorems 3.1 and 4.13 by making judicious choices for X, V, and W, then invoking exactness of the resulting instances of (1.6).

In §2, we go over the background we need to prove Theorems 3.1 and 4.13: twisted spin structures in §2.1 and the Smith long exact sequence in §2.2, including several examples of each. In §3, we prove Theorem 3.1, and in §4, we prove Theorem 4.13.

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2. Background

Here we review the Smith long exact sequence and the concepts needed to set it up.

2.1. **Twisted spin structures.** Recall that a *spin structure* on a vector bundle $W \to Y$ is defined to be a homotopy class of lift of the principal $\mathrm{GL}_r(\mathbb{R})$-bundle of frames of W to a principal Spin_r-bundle, where r is the rank of W. This data is equivalent to a trivialization of the Stiefel-Whitney classes $w_1(W)$ and $w_2(W)$ [BH59, §26.5]; i.e. data of nullhomotopies of the maps $Y \to K(\mathbb{Z}/2, 1)$ and $Y \to K(\mathbb{Z}/2, 2)$ representing $w_1(W)$, resp. $w_2(W)$.

Definition 2.1 ([HKT20, §4.1])**.** Let $V \to X$ be a virtual vector bundle. An (X, V)-*twisted spin structure* on a virtual vector bundle $W \to Y$ is data of a map $f\colon Y \to X$ and a spin structure on $W \oplus f^*(V)$.

This notion encompasses many commonly considered variations of spin structure.[5]

Example 2.2. A spin^c structure on a virtual vector bundle $W \to Y$ is a reduction of the structure group of W to the group [ABS64, §3]

$$\mathrm{Spin}_n^c \coloneqq \mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{U}_1, \tag{2.3}$$

where the map to O_n is the composition

$$\mathrm{Spin}_n^c \xrightarrow{\mathrm{proj}_1} \mathrm{SO}_n \hookrightarrow \mathrm{O}_n. \tag{2.4}$$

This amounts to the data of a trivialization of $w_1(W)$ and a class $c \in H^2(Y;\mathbb{Z})$ and an identification of $c \bmod 2 = w_2(W)$ (i.e. a trivialization of $c \bmod 2 + w_2(W)$). As $B\mathrm{U}_1$ is a $K(\mathbb{Z}, 2)$, there is a complex line bundle $L \to Y$ with $c_1(L) = c$, and L is unique up to isomorphism.

The condition "$c_1(L) \bmod 2 = w_2(W)$" is equivalent to "$W \oplus L$ is spin": the Whitney sum formula shows $w_2(W \oplus L) = w_2(W) \oplus w_2(L)$, because $w_1(W)$ and $w_1(L)$ both vanish. Then, $w_2(V) = c_1(V) \bmod 2$ for any complex vector bundle V.

Finally, since all complex line bundles are pullbacks of the tautological bundle $V_{\mathrm{U}_1} \to B\mathrm{U}_1$ in a unique way up to isomorphism, the data of a spin^c structure on W is equivalent to a map $f\colon Y \to B\mathrm{U}_1$ and a spin structure on $W \oplus f^*(V_{\mathrm{U}_1})$. That is, spin^c structures are $(B\mathrm{U}_1, V_{\mathrm{U}_1})$-twisted spin structures.

Example 2.5. The same argument as in Example 2.2 identifies several more kinds of twisted spin structures. The pin$^+$ and pin$^-$ groups are defined as central extensions

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_n^\pm \longrightarrow \mathrm{O}_n \longrightarrow 1. \tag{2.6}$$

Central extensions of this form are classified by $H^2(B\mathrm{O}_n; \{\pm 1\})$; Pin_n^+ is the extension corresponding to the class w_2, and Pin_n^- corresponds to $w_2 + w_1^2$.

Standard obstruction theory then implies a pin$^+$ structure on a vector bundle $W \to Y$ is equivalent to a trivialization of $w_2(W)$, while a pin$^-$ structure on W is equivalent to a trivialization of $w_2(W) + w_1(W)^2$. A similar characteristic-class argument as in Example 2.2 shows that pin$^+$ structures are equivalent to $(B\mathbb{Z}/2, -\sigma)$-twisted spin structures, where $\sigma \to B\mathbb{Z}/2$ is the tautological bundle; similarly, pin$^-$ structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin structures.

Campbell [Cam17, §7.8] proves a related statement for 2σ: $(B\mathbb{Z}/2, 2\sigma)$-twisted spin structures are equivalent to G-structures for $G = \mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$.

Example 2.7. If one imitates the definition of Spin_n^c from (2.3) using the pin groups, the resulting group and its map down to O_n is the same whether one begins with Pin_n^+ or Pin_n^-. Thus using either, the group Pin_n^c is defined to be [ABS64, Corollary 3.19]

$$\mathrm{Pin}_n^c \coloneqq \mathrm{Pin}_n^\pm \times_{\{\pm 1\}} \mathrm{U}_1. \tag{2.8}$$

The map to O_n is analogous to that for Spin_n^c, and a pin^c structure on $W \to Y$ is the data of a class $c \in H^2(Y;\mathbb{Z})$ with $c \bmod 2 = w_2(W)$; i.e. the same as a spin^c structure with no condition on w_1. This is equivalent to a $(B\mathbb{Z}/2 \times B\mathrm{U}_1, \sigma \oplus V_{\mathrm{U}_1})$-twisted spin structure.

[5] However, see Stolz [Sto98, §2.6] for a different notion of twisted spin structure and [DY23, §3.1] for examples showing that Stolz' definition is strictly more general than Definition 2.1.

Example 2.9. A quaternionically-minded reader might expect analogues of Examples 2.2 and 2.7 with SU_2 in place of U_1. Indeed, there are groups

$$\mathrm{Spin}_n^h \coloneqq \mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{SU}_2 \tag{2.10a}$$

$$\mathrm{Pin}_n^{h\pm} \coloneqq \mathrm{Pin}_n^{\pm} \times_{\{\pm 1\}} \mathrm{SU}_2. \tag{2.10b}$$

Unlike for Pin_n^c, Pin_n^{h+} and Pin_n^{h-} do not define equivalent tangential structures. Freed-Hopkins [FH21, (10.20)] show that spin^h structures are equivalent to $(B\mathrm{SO}_3, V_{\mathrm{SO}_3})$-twisted spin structures and pin$^{h\pm}$ structures are equivalent to $(B\mathrm{O}_3, \pm V_{\mathrm{O}_3})$-twisted spin structures, where for $G = \mathrm{SO}_3$ or O_3, the bundle $V_G \to BG$ is the tautological bundle.

We will study bordism groups of manifolds with (X, V)-twisted spin structures. To do so, we express these twisted spin structures as untwisted tangential structures.

Lemma 2.11 (Shearing)**.** *Given $V \to X$ as above, (X, V)-twisted spin structures on a vector bundle $W \to Y$ are in natural bijection with homotopy classes of lifts in the diagram*

$$\begin{tikzcd} & B\mathrm{Spin} \times X \arrow[d, "V_{\mathrm{Spin}} - V + \mathrm{rank}(V)"] \\ Y \arrow[ur, dashed] \arrow[r, "W"'] & B\mathrm{O}. \end{tikzcd} \tag{2.12}$$

Here $V_{\mathrm{Spin}} \to B\mathrm{Spin}$ is the tautological virtual vector bundle.

The rank term appears so that the entire virtual bundle has rank zero.

Corollary 2.13. *The Thom spectrum whose homotopy groups correspond under the Pontrjagin-Thom theorem to the bordism groups of manifolds with (X, V)-twisted structures on the* tangent *bundle is homotopy equivalent as $MT\mathrm{Spin}$-modules to $MT\mathrm{Spin} \wedge X^{V - \mathrm{rank}(V)}$.*

Here the $-V$ becomes a $+V$ because we want tangential bordism, not normal bordism. For the same reason, we use the Madsen-Tillmann spectrum MTH, which is the Thom spectrum whose homotopy groups are the bordism groups of manifolds with H-structures on their stable tangent bundles. In homotopy theory, one more often encounters MH, which is the Thom spectrum for manifolds with H-structures on their stable *normal* bundles. For $H = \mathrm{Spin}$, Spin^c, and Spin^h, there is a canonical homotopy equivalence $MTH \simeq MH$, but this is not true for all H: for example, $MT\mathrm{Pin}^{h\pm} \simeq M\mathrm{Pin}^{h\mp}$.

2.2. **The Smith long exact sequence.** Next, we introduce our main tool: the Smith long exact sequence. We detailed this long exact sequence in [DDK+24] as part of a general framework for studying Smith homomorphisms. We begin by defining *Smith maps*: maps of Madsen-Tillman spectra induced by the inclusion of the zero section of a vector bundle.

Let $V \to X$ be as in the previous section, and let $W \to X$ be a real vector bundle. The inclusion of the zero section $0 \hookrightarrow W$ induces a map of Thom spaces $X_+ \to \mathrm{Th}(X; W)$, and upgrading this map to incorporate twisting by V yields the following.

Definition 2.14. Let $V \to X$ be a virtual vector bundle and $W \to X$ be a vector bundle. The *Smith map* associated to X, V, and W is the map of Thom spectra

$$\mathrm{sm}_W \colon X^V \to X^{V+W} \tag{2.15}$$

induced by the map $v \mapsto (v, 0) \colon V \to V \oplus W$.

Remark 2.16. In the case that V is a vector bundle, the formula $v \mapsto (v, 0)$ describes an actual function of spaces which descends to a map of Thom spaces

$$\mathrm{sm}_W\colon \mathrm{Th}(X; V) \to \mathrm{Th}(X; V \oplus W). \tag{2.17}$$

Then (2.15) is Σ^∞ applied to (2.17).

For more general V, one makes sense of (2.15) as follows: if X is homotopy equivalent to a finite-dimensional CW complex, we may replace V with an actual vector bundle up to a trivial summand, which only (de)suspends the map of Thom spectra. In general, one takes a colimit over n-skeleta. See [DDK+24, Definition 3.13] for more details.

We will consider the maps induced by Smith maps on spin bordism. There is a similar story for other generalized (co)homology theories; see [DDK+24, §7] for more examples.

Definition 2.18. With X, V, and W as above, let $r_V \coloneqq \mathrm{rank}(V)$ and $r_W \coloneqq \mathrm{rank}(W)$. The *Smith homomorphism* associated to X, V, and W is the homomorphism

$$\mathrm{sm}_W\colon \Omega_n^{\mathrm{Spin}}(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^{\mathrm{Spin}}(X^{V+W-r_V-r_W}) \tag{2.19}$$

induced by applying Ω_n^{Spin} to (2.15).

We describe the homomorphism (2.19) on the level of manifolds in (LES-2).

Example 2.20. Let $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma$, the tautological line bundle. Including the zero section into σ defines a map of spectra

$$\mathrm{sm}_\sigma\colon B\mathbb{Z}/2_+ \to (B\mathbb{Z}/2)^\sigma, \tag{2.21}$$

and taking spin bordism gives the Smith homomorphism

$$\mathrm{sm}_\sigma\colon \Omega_n^{\mathrm{Spin}}(B\mathbb{Z}/2) \to \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1}). \tag{2.22}$$

Using Example 2.5 and Lemma 2.11, we may recognize this as a map

$$\mathrm{sm}_\sigma\colon \Omega_n^{\mathrm{Spin}}(B\mathbb{Z}/2) \to \Omega_{n-1}^{\mathrm{Pin}^-} \tag{2.23}$$

between the bordism groups of spin manifolds equipped with a principal $\mathbb{Z}/2$-bundle and pin$^-$ manifolds. Restricted to reduced spin bordism, (2.23) is an isomorphism for all n [ABP69]; we will later establish this as a consequence of Example 2.34.

Example 2.24. Taking X and W as above but starting with $V = \sigma$, we obtain a map of spectra

$$\mathrm{sm}_\sigma\colon (B\mathbb{Z}/2)^\sigma \to (B\mathbb{Z}/2)^{2\sigma}, \tag{2.25}$$

which on spin bordism gives a Smith homomorphism

$$\mathrm{sm}_\sigma\colon \Omega_n^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1}) \to \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{2\sigma-2}). \tag{2.26}$$

Example 2.5 and Lemma 2.11 allow us to rewrite this as a map

$$\mathrm{sm}_\sigma\colon \Omega_n^{\mathrm{Pin}^-} \to \Omega_{n-1}^{\mathrm{Spin}\times_{\{\pm 1\}}\mathbb{Z}/4}. \tag{2.27}$$

This map is generally not an isomorphism. For example, when $n = 2$, $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [ABP69, Theorem 5.1][6] and $\Omega_1^{\mathrm{Spin}\times_{\{\pm 1\}}\mathbb{Z}/4} \cong \mathbb{Z}/4$ [Cam17, Theorem 7.9]. Unlike in Example 2.20, we cannot solve this problem by discarding a basepoint.

Example 2.28. Next, we again take $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma$, but instead take spin^c bordism. In other words, we smash with $MT\mathrm{Spin}^c$ instead of $MT\mathrm{Spin}$ and then take homotopy groups. We get

$$\mathrm{sm}_\sigma^c \colon \Omega_n^{\mathrm{Spin}^c}(B\mathbb{Z}/2) \to \Omega_{n-1}^{\mathrm{Spin}^c}((B\mathbb{Z}/2)^{\sigma-1}). \tag{2.29}$$

Using Examples 2.2 and 2.7 and Lemma 2.11, we can rewrite the codomain:

$$\Omega_{n-1}^{\mathrm{Spin}^c}((B\mathbb{Z}/2)^{\sigma-1}) \underset{(2.2)}{\cong} \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathrm{U}_1)^{V_{\mathrm{U}_1}-2}) \underset{(2.7)}{\cong} \Omega_{n-1}^{\mathrm{Pin}^c}, \tag{2.30}$$

allowing us to rephrase (2.29) as a map

$$\mathrm{sm}_\sigma^c \colon \Omega_n^{\mathrm{Spin}^c}(B\mathbb{Z}/2) \to \Omega_{n-1}^{\mathrm{Pin}^c}. \tag{2.31}$$

Like in Example 2.20, when restricted to reduced spin^c bordism of $B\mathbb{Z}/2$, (2.31) is an isomorphism for all n. This is a theorem of Bahri-Gilkey [BG87b]; we will prove it in Example 2.34.

We could equivalently describe this example using $X = B\mathbb{Z}/2 \times B\mathrm{U}_1$, $V = 0$, and $W = \sigma \oplus V_{\mathrm{U}_1}$ and taking spin bordism, applying Example 2.7.

Proving the quaternionic analog of Examples 2.20 and 2.28 is our objective in the next section.

Before then, we shall extend Smith homomorphisms to a long exact sequence, toward our second application. To do so, we identify the fiber of Equation (2.15). We write $S(W)$ for the sphere bundle of a vector bundle $W \to X$.

The following theorem is attributed to James (see, e.g., [KZ18, Remark 3.14]). See for example [DDK+24, Theorem 5.1] for a proof.[7]

Theorem 2.32. *Let V be a virtual bundle and let W be real vector bundle over X. Write $p\colon S(W) \to X$ for the projection map. Then there is a fiber sequence in spectra:*

$$S(W)^{p^*V} \to X^V \to X^{V\oplus W}. \tag{2.33}$$

Example 2.34. Return to the setup with $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma$ from Example 2.20. Since the sphere bundle of $\sigma \to B\mathbb{Z}/2$ gives the universal fibration $E\mathbb{Z}/2 \to B\mathbb{Z}/2$, we see that $S(\sigma)$ is contractible. Since $V = 0$, we simply we get a fiber sequence

$$\mathbb{S} \to B\mathbb{Z}/2_+ \xrightarrow{\mathrm{sm}_\sigma} (B\mathbb{Z}/2)^\sigma. \tag{2.35}$$

Lemma 2.36. *The sequence (2.35) splits. Explicitly, the crush map $c\colon (B\mathbb{Z}/2)_+ \to \mathbb{S}$ is a section and the restriction of sm_σ to the fiber of c is a homotopy equivalence $\widetilde{\mathrm{sm}}_\sigma \colon \Sigma^\infty B\mathbb{Z}/2 \simeq (B\mathbb{Z}/2)^\sigma$.*

This is a standard fact: see, e.g. [Koc96, Lemma 2.6.5]. In a sense, its proof is trivial: the crush map always splits $\mathbb{S}$ off of X_+ for any space X, and the rest follows from that and general properties of fiber sequences of spectra. The heart of the lemma is that the inclusion of a basepoint in $B\mathbb{Z}/2$

[6] Within the statement of [ABP69, Theorem 5.1], the piece relevant for $\Omega_2^{\mathrm{Pin}^-}$ is "The contribution to Ω_*^{Pin} of terms $\pi_*(RP^\infty \wedge \mathbf{B}O\langle 8n\rangle)$ is as follows... $Z_2{}^{4k+3}$ in dim $8n+8k+2$, $k \geq 0$." For us $n = k = 0$. There is a typo: $Z_2{}^{4k+3}$ should be $Z_{2^{4k+3}}$. See Giambalvo [Gia73, Theorem 3.4(b)] and Kirby-Taylor [KT90, Lemma 3.6] for additional calculations of $\Omega_2^{\mathrm{Pin}^-}$.

[7] This theorem can also be deduced from a theorem of Conner-Floyd [CF66, §16]; see [DDK+24, §5.1] for more information.

suspends to participate in a Smith fiber sequence, which is more nontrivial.[8] After smashing with $MT\mathrm{Spin}$, $MT\mathrm{Spin}^c$, or $MT\mathrm{Spin}^h$ and invoking the Pontrjagin-Thom theorem, (2.35) produces the Smith isomorphisms of Examples 2.20 and 2.28 and Theorem 3.1.[9]

There are many examples of Smith maps with more interesting fibers, including the Smith map (4.5) that we discuss in Section 4.

Corollary 2.37. *Taking the spin bordism of* (2.33) *yields a homology long exact sequence:*
(2.38)
$$\cdots \longrightarrow \Omega_n^{\mathrm{Spin}}(S(W)^{p^*V}) \xrightarrow{p_*} \Omega_n^{\mathrm{Spin}}(X^V) \xrightarrow{\mathrm{sm}_W} \Omega_{n-r_W}^{\mathrm{Spin}}(X^{V+W-r_W}) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}}(S(W)^{p^*V}) \longrightarrow \cdots$$

The central map is the Smith homomorphism, and the other two are the pullback and the connecting homomorphism. This long exact sequence is remarkably useful for bordism computations, specifically for resolving extension problems that arise in spectral sequence calculations. Moreover, we understand this sequence on the level of manifolds:

(LES-1) Let $[M,h]$ be the bordism class of an n-manifold M equipped with a map $h\colon M\to S(W)$ such that $TM\oplus h^*p^*V$ is spin, so that $[M,h]\in\Omega_n^{\mathrm{Spin}}(S(W)^{p^*V})$. Its image under p^* is the class $[M, M\xrightarrow{h} S(W)\xrightarrow{p} X]$ of the same underlying manifold equipped with a map to X given by composing with the projection.

(LES-2) Let $[M,f]\in\Omega_n^{\mathrm{Spin}}(X^V)$, so that M is an n-manifold such that $TM\oplus f^*V$ is spin. Consider the pullback of W to M. The intersection of the zero section of W with a generic section is, by transversality, a submanifold N of codimension r_W. The image of M under the Smith homomorphism is the class $[N, N\hookrightarrow M\xrightarrow{f} X]$. In this setting, the normal bundle $\nu\to N$ of the embedding $i\colon N\hookrightarrow M$ is isomorphic to $W|_N$, so $TN\oplus(i\circ f)^*W\oplus(i\circ f)^*V\cong i^*(TM\oplus f^*V)$ is spin, and therefore N has a $(X, V\oplus W)$-twisted spin structure.

(LES-3) Let $[N,g]$ be a class in $\Omega_n^{\mathrm{Spin}}(X^{V+W-r_W})$, so $g\colon N\to X$ is such that $TN+g^*(V+W)$ is spin. The image of $[N,g]$ under the connecting homomorphism is the class $[S(W)|_N, S(W)|_N\hookrightarrow S(W)]\in\Omega_{n-1}^{\mathrm{Spin}}(S(W))$ given by restricting the sphere bundle of W to N.

For a justification of these descriptions, see [DDK+24, Appendix A].

3. A Pin^{h-} Smith Isomorphism

In this section we answer [BM23, Question 10.8] (Question 1.2 in this article): is there a Smith isomorphism for pin^{h-} bordism, generalizing Examples 2.20 and 2.28?

Theorem 3.1. *For all n, there is an isomorphism $\mathrm{sm}_\sigma^h\colon \widetilde\Omega_{n+1}^{\mathrm{Spin}^h}(\mathbb{RP}^\infty)\xrightarrow{\cong}\Omega_n^{\mathrm{Pin}^{h-}}$ given by sending a pair (M,f) of a spin^h manifold M with a map $f\colon M\to\mathbb{RP}^\infty$ (which may without loss of generality be assumed to be transverse to $\mathbb{RP}^{\infty-1}\subset\mathbb{RP}^\infty$) to the pin^{h-} manifold $f^{-1}(\mathbb{RP}^{\infty-1})$.*

The idea of our proof is this: using Example 2.9 and Lemma 2.11, pin^{h-} bordism is isomorphic to the spin bordism of the Thom spectrum $(B\mathrm{O}_3)^{3-V_{\mathrm{O}_3}}$, and spin^h bordism is isomorphic to the spin bordism of $(B\mathrm{SO}_3)^{3-V_{\mathrm{SO}_3}}$, where again $V_G\to BG$ denotes a tautological bundle. The isomorphism $\mathrm{O}_3\cong\mathrm{SO}_3\times\mathbb{Z}/2$ allows one to factor $(B\mathrm{O}_3)^{3-V_{\mathrm{O}_3}}$ as a smash product of $(B\mathrm{SO}_3)^{3-V_{\mathrm{SO}_3}}$ and a piece that corresponds to $\mathbb{RP}^\infty$, leading to the isomorphism in the theorem statement.

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[8] Another way to approach Lemma 2.36 is to directly observe that the Thom space of $\sigma\to\mathbb{RP}^n$ is homeomorphic to $\mathbb{RP}^{n+1}$ and that the zero section inside the Thom space can be homotoped into the standard inclusion $\mathbb{RP}^n\hookrightarrow\mathbb{RP}^{n+1}$ coming from the equatorial $S^n\hookrightarrow S^{n+1}$. Then check compatibility as $n\to\infty$ and conclude.

[9] The Smith homomorphism interpretation of this equivalence is well-known, but we are not sure who was the first to discuss it in general: see [DDK+24, §7.1] and the references therein.

Now we give the details, beginning with some lemmas. Recall that $H^*(B\mathrm{O}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3]$ with $|w_i| = i$ [Bor53a], $H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[a]$ with $|a| = 1$, and $H^*(B\mathrm{SO}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{w}_2, \overline{w}_3]$ with $|\overline{w}_i| = i$ [Bor53a, Proposition 8.1]. (The classes $\overline{w}_i$ are the usual Stiefel-Whitney classes, but we write $\overline{w}$ so that the classes on $B\mathrm{O}_3$ and $B\mathrm{SO}_3$ have different names.)

Lemma 3.2. *Write $\varphi\colon \mathrm{SO}_3\times\mathbb{Z}/2 \xrightarrow{\cong} \mathrm{O}_3$ for the isomorphism. Then the map $\varphi^*\colon H^*(B\mathrm{O}_3;\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2 \times B\mathrm{SO}_3;\mathbb{Z}/2)$ on cohomology is such that $\varphi^*(w_1) = a$ and $\varphi^*(w_2) = a^2 + \overline{w}_2$.*

Proof. Since φ is an isomorphism, so is φ^*. Therefore, since $w_1 \neq 0$, $\varphi^*(w_1)$ must also be nonzero. Since $H^1(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2 \cdot a$ and $H^1(B\mathrm{SO}_3;\mathbb{Z}/2) \cong 0$, the Künneth formula tells us that the only nonzero class, which must be $\varphi^*(w_1)$, is a.

To match w_2, we have three nonzero classes: $\overline{w}_2$, a^2, and $\overline{w}_2 + a^2$. To tell them apart, first consider the map $i_1\colon B\mathrm{SO}_3 \to B\mathrm{O}_3$ induced by the inclusion $\mathrm{SO}_3 \hookrightarrow \mathrm{O}_3$; this factors through φ and $i_1^*(w_2) = \overline{w}_2$ by definition, so $\varphi^*(w_2)$ must be either $\overline{w}_2$ or $\overline{w}_2 + a^2$. Likewise, take the map $i_2\colon B\mathbb{Z}/2 \to B\mathrm{O}_3$ induced by φ; since this map is defined by sending $1 \in \mathbb{Z}/2$ to an inversion in O_3, $\mathbb{Z}/2$ acts on the pullback representation $i_2^* V_{\mathrm{O}_3}$ as $\{\pm 1\}$, i.e. as the representation 3σ. Thus $i_2^*(w_2) = w_2(3\sigma)$, which by the Whitney sum formula is $w_1(\sigma)^2$, i.e. a^2. Thus $\varphi^*(w_2)$ must have an a^2 term, and we already saw it must have an $\overline{w}_2$ term, so $\varphi^*(w_2) = \overline{w}_2 + a^2$. $\square$

If $V_1 \to X_1$ and $V_2 \to X_2$ are virtual vector bundles, then there is a homotopy equivalence $(X_1 \times X_2)^{V_1 \boxplus V_2} \simeq (X_1)^{V_1} \wedge (X_2)^{V_2}$. Since $B\mathrm{O}_3$ splits as a direct product, one might hope that $3 - V_{\mathrm{O}_3} \to B\mathrm{O}_3$ is an external direct sum, leading to a splitting of $(B\mathrm{O}_3)^{3-V_{\mathrm{O}_3}}$. This is not true, but we will be able to replace $3 - V_{\mathrm{O}_3}$ with a different vector bundle that is an external direct sum using the following lemma.

Lemma 3.3 (Relative Thom isomorphism, c.f. [Deb21, Theorem 1.39] or [DY23])**.** *Let $V_1, V_2 \to X$ be rank-zero vector bundles. A spin structure on V_2 determines a homotopy equivalence of $MT\mathrm{Spin}$-module spectra $MT\mathrm{Spin} \wedge X^{V_1} \xrightarrow{\simeq} MT\mathrm{Spin} \wedge X^{V_1+V_2}$.*

We will replace $\pm(V_{\mathrm{O}_3} - 3)$ with $\pm((3\sigma - 3) \boxplus (V_{\mathrm{SO}_3} - 3))$, so we must check the hypothesis of Lemma 3.3.

Lemma 3.4. *The virtual vector bundles $\varphi^*(\pm(V_{\mathrm{O}_3} - 3)) - \pm((3\sigma - 3) \boxplus (V_{\mathrm{SO}_3} - 3))$ are spin.*

Proof. Directly compute with the Whitney sum formulas that $w_1(V_1 \oplus V_2) = w_1(V_1) + w_1(V_2)$ and $w_2(V_1 \oplus V_2) = w_2(V_1) + w_1(V_1)w_1(V_2) + w_2(V_2)$. For any vector bundle E, setting $V_1 = E$ and $V_2 = -E$ (so that $V_1 \oplus V_2 = 0$) gives that $w_1(-E) = w_1(E)$ and $w_2(-E) = w_2(E) + w_1(E)^2$. Stability of the Stiefel-Whitney classes implies we may add or subtract trivial bundles without affecting their characteristic classes.

Thus, for $E_+ \coloneqq \varphi^*(V_{\mathrm{O}_3} - 3) - ((3\sigma - 3) \boxplus V_{\mathrm{SO}_3} - 3)$, we have using Lemma 3.2 that

$$\begin{aligned} w_1(E_+) &= w_1(\varphi^*(V_{\mathrm{O}_3} - 3)) + w_1(-((3\sigma - 3) \boxplus (V_{\mathrm{SO}_3} - 3))) \\ &= \varphi^*(w_1(V_{\mathrm{O}_3})) + w_1(-3\sigma) + w_1(-V_{\mathrm{SO}_3}) \\ &= \varphi^*(w_1) + w_1(3\sigma) + w_1(V_{\mathrm{SO}_3}) \\ &= a + a + 0 = 0, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}w_2(E_+) &= w_2(\varphi^*(V_{\mathrm{O}_3}-3)) + w_1(\varphi^*(V_{\mathrm{O}_3}-3))w_1(-(3(\sigma-3)\boxplus(V_{\mathrm{SO}_3}-3))) + w_2(-((3(\sigma-3)\boxplus(V_{\mathrm{SO}_3}-3)))\\ &= \varphi^*(w_2(V_{\mathrm{O}_3})) + \varphi^*(w_1(V_{\mathrm{O}_3}))(w_1(-3\sigma)+w_1(-V_{\mathrm{SO}_3})) + w_2(-3\sigma) + w_1(-3\sigma)w_1(-V_{\mathrm{SO}_3}) + w_2(-V_{\mathrm{SO}_3})\\ &= \overline{w}_2 + a^2 + a^2 + w_2(3\sigma) + w_1(3\sigma)^2 + w_1(3\sigma)w_1(W) + w_2(V_{\mathrm{SO}_3}) + w_1(V_{\mathrm{SO}_3})^2\\ &= \overline{w}_2 + a^2 + a^2 + a^2 + a^2 + 0 + \overline{w}_2 + 0 = 0.\end{aligned} \tag{3.6}$$

Since the first and second Stiefel-Whitney classes of E_+ vanish, E_+ is spin. This also implies w_1 and w_2 of $-E_+$ vanish, so we have proven the claim for both bundles in the lemma statement. ☐

Corollary 3.7. *There are equivalences of spectra (in fact, of $MT\mathrm{Spin}$-module spectra)*

$$MT\mathrm{Spin}\wedge(B\mathrm{SO}_3)^{V_{\mathrm{SO}_3}-3}\wedge(B\mathbb{Z}/2)^{\pm(3\sigma-3)}\simeq MT\mathrm{Spin}\wedge(B\mathrm{O}_3)^{\pm(V_{\mathrm{O}_3}-3)}. \tag{3.8}$$

Proof. We prove the $+$ case; the $-$ case is analogous. Lemma 3.3 tells us that, since $\varphi^*(V_{\mathrm{O}_3}-3)-((3\sigma-3)\boxplus(V_{\mathrm{SO}_3}-3))$ is spin and rank-zero, there are equivalences of $MT\mathrm{Spin}$-module spectra

$$\begin{aligned}MT\mathrm{Spin}\wedge(B\mathrm{O}_3)^{V_{\mathrm{O}_3}-3} &\simeq MT\mathrm{Spin}\wedge(B\mathbb{Z}/2\times B\mathrm{SO}_3)^{\varphi^*V_{\mathrm{O}_3}-3}\\ &\simeq MT\mathrm{Spin}\wedge(B\mathbb{Z}/2\times B\mathrm{SO}_3)^{(3\sigma-3)\boxplus(V_{\mathrm{SO}_3}-3)}.\end{aligned} \tag{3.9}$$

As we noted above, the Thom spectrum functor sends external direct sums to smash products, so the Thom spectrum in (3.9) factors as $MT\mathrm{Spin}\wedge(B\mathbb{Z}/2)^{3\sigma-3}\wedge(B\mathrm{SO}_3)^{V_{\mathrm{SO}_3}-3}$, as we wanted to prove. ☐

Now we are ready to prove the main theorem of this section.

Proof of Theorem 3.1. Example 2.9 and Lemma 2.11 combine to produce homotopy equivalences

$$MT\mathrm{Spin}^h\simeq MT\mathrm{Spin}\wedge(B\mathrm{SO}_3)^{V_{\mathrm{SO}_3}-3} \tag{3.10a}$$

$$MT\mathrm{Pin}^{h\pm}\simeq MT\mathrm{Spin}\wedge(B\mathrm{O}_3)^{\pm(V_{\mathrm{O}_3}-3)}, \tag{3.10b}$$

which are originally due to Freed-Hopkins [FH21, (10.2)]. Combining (3.10) with Corollary 3.7, we have produced equivalences

$$MT\mathrm{Pin}^{h\pm}\simeq MT\mathrm{Spin}^h\wedge(B\mathbb{Z}/2)^{\pm(3\sigma-3)}. \tag{3.11}$$

One can check using the Whitney sum formula that the bundle $4\sigma\to B\mathbb{Z}/2$ has a spin structure. Thus we may once again invoke Lemma 3.3 to obtain an equivalence $MT\mathrm{Spin}\wedge(B\mathbb{Z}/2)^{-(3\sigma-3)}\simeq MT\mathrm{Spin}\wedge(B\mathbb{Z}/2)^{\sigma-1}$: the difference between the two vector bundles is $4\sigma-4$, which is spin, so adding $4\sigma-4$ to $-(3\sigma-3)$ does not change the homotopy type.

The only remaining task is to get from $(B\mathbb{Z}/2)^{\sigma-1}$ to $\mathbb{RP}^\infty$ and interpret the resulting equivalence as a Smith isomorphism. This is done in Example 2.34. ☐

Remark 3.12. Buchanan-McKean's original question asked about a Smith isomorphism between pin^{h-} bordism and the spin^h bordism of $\mathbb{HP}^\infty$. These bordism groups are not isomorphic: to see this, run the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q}=\widetilde{H}_p(\mathbb{HP}^\infty;\Omega_q^{\mathrm{Spin}^h}(\mathrm{pt})\otimes\mathbb{Q})\Longrightarrow\widetilde{\Omega}_{p+q}^{\mathrm{Spin}^h}(\mathbb{HP}^\infty)\otimes\mathbb{Q}. \tag{3.13}$$

All Atiyah-Hirzebruch spectral sequences with $\mathbb{Q}$ coefficients collapse, so $E^2_{4,0}\cong\widetilde{H}_4(\mathbb{HP}^\infty;\mathbb{Q})\cong\mathbb{Q}$ [BH58, §15.5] implies $\widetilde{\Omega}_4^{\mathrm{Spin}^h}(\mathbb{HP}^\infty)\otimes\mathbb{Q}\neq 0$, but $\Omega_3^{\mathrm{Pin}^{h-}}$ is zero [FH21, Theorem 9.97].

The pin$^-$ and pin^c Smith isomorphisms of Examples 2.20 and 2.28 both use $B\mathbb{Z}/2 \simeq \mathbb{RP}^\infty$, ultimately because Pin_n^-, resp. Pin_n^c are extensions of $\mathbb{Z}/2$ by Spin_n, resp. Spin_n^c. That Pin_n^{h-} is also an extension of $\mathbb{Z}/2$, this time by Spin_n^h, suggested to us that the Smith isomorphism should also use $B\mathbb{Z}/2$. Smith isomorphisms involving $B\mathrm{SU}_2 \simeq \mathbb{HP}^\infty$ do exist [DDK+24, Example 7.42], in a setting where one group is an extension of SU_2 by another.

4. Rational Generators for Spin^h Bordism from Spin^c Bordism

Next, we address the Question 10.3 asked by Buchanan and McKean in [BM23] comparing spin^c and spin^h bordism in dimensions 0 mod 4.

Theorem 4.1 (Buchanan-McKean [BM23, Corollary 8.6])**.** *For all $k \geq 0$, $\mathrm{rank}(\Omega_{4k}^{\mathrm{Spin}^c}) = \mathrm{rank}(\Omega_{4k}^{\mathrm{Spin}^h})$.*

Question 4.2 ([BM23, Question 10.3])**.** Is there a geometric explanation for the equality of ranks in Theorem 4.1 between degree-$4k$ spin^c and spin^h bordism? Specifically, is there a procedure for producing generators for the free part of $\Omega_{4k}^{\mathrm{Spin}^h}$ from that of $\Omega_{4k}^{\mathrm{Spin}^c}$?

We use the Smith long exact sequence to mostly answer this question: it provides a geometric explanation for the equality of ranks and allows one to produce *rational* generators for spin^h bordism from generators of spin^c bordism. In the course of the proof, we will lift from $\mathbb{Q}$ to $\mathbb{Z}[1/2]$, but we will also see why it is hard to lift to a result over $\mathbb{Z}$.

Construction 4.3. The inclusion $\{\pm 1\} \hookrightarrow \mathrm{SU}_2$ used in the definition of Spin_n^h (Example 2.9) factors as the composition of the usual inclusion $\{\pm 1\} \hookrightarrow \mathrm{U}_1$ and the standard inclusion $\mathrm{U}_1 \hookrightarrow \mathrm{SU}_2$. Taking the product with Spin_n and quotienting by the diagonal central $\{\pm 1\}$ subgroup, we obtain an inclusion $\iota\colon \mathrm{Spin}_n^c \hookrightarrow \mathrm{Spin}_n^h$ commuting with the structure maps to O_n.

Given a vector bundle $V \to X$ with spin^c structure $\mathfrak{s}$, the spin^h structure $\iota(\mathfrak{s})$ is called the *induced spin^h structure* of $\mathfrak{s}$.

Theorem 4.4.

(1) Taking the induced spin^h structure of a spin^c structure defines a map of bordism groups $\Omega_n^{\mathrm{Spin}^c} \to \Omega_n^{\mathrm{Spin}^h}$ that participates in a Smith long exact sequence.
(2) The induced map $\Omega_{4k}^{\mathrm{Spin}^c} \otimes \mathbb{Z}[1/2] \to \Omega_{4k}^{\mathrm{Spin}^h} \otimes \mathbb{Z}[1/2]$ is an isomorphism.

In particular, part (2) follows from (1) and a few computations in the literature. In light of the explicit interpretations of the maps in a Smith long exact sequence in (LES-1)–(LES-3), we believe Theorem 4.4 provides a geometric answer to the first part of Question 4.2.

Consider the Smith map of spectra (2.15) for $X = B\mathrm{SO}_3$ and $V = W = V_{\mathrm{SO}_3}$, where V_{SO_3} is the tautological rank-three oriented bundle:

$$\mathrm{sm}_{V_{\mathrm{SO}_3}}\colon (B\mathrm{SO}_3)^{V_{\mathrm{SO}_3}} \to (B\mathrm{SO}_3)^{2V_{\mathrm{SO}_3}} \tag{4.5}$$

By Theorem 2.32, the fiber is given by the Thom spectrum over the sphere bundle: $S(V_{\mathrm{SO}_3})^{p^*V_{\mathrm{SO}_3}}$.

Lemma 4.6. *For all $n \geq 1$, there is a homotopy equivalence $\varphi\colon S(V|_{\mathrm{SO}_n}) \xrightarrow{\simeq} B\mathrm{SO}_{n-1}$, and φ identifies the bundle map $p\colon S(V|_{\mathrm{SO}_n}) \to B\mathrm{SO}_n$ with the map $B\mathrm{SO}_{n-1} \to B\mathrm{SO}_n$ induced by the standard inclusion $\mathrm{SO}_{n-1} \hookrightarrow \mathrm{SO}_n$, up to homotopy.*

This is well-known; see [DDK+24, Example 7.57] for a proof. Because of Lemma 4.6, we will also write p for the map $B\mathrm{U}_1 \to B\mathrm{SO}_3$ induced by the standard inclusion $\mathrm{U}_1 \cong \mathrm{SO}_2 \hookrightarrow \mathrm{SO}_3$. Then, the pullback $p^*V_{\mathrm{SO}_3}$ to $B\mathrm{SO}_2$ is the rank-two tautological bundle over $B\mathrm{SO}_2$ plus a trivial real line bundle, and under the equivalence $\mathrm{SO}_2 \cong \mathrm{U}_1$, the tautological rank-two oriented bundle $V_{\mathrm{SO}_2} \to B\mathrm{SO}_2$ is identified with the tautological complex line bundle $V_{\mathrm{U}_1} \to B\mathrm{U}_1$.[10] Overall, we have argued an equivalence

$$S(V_{\mathrm{SO}_3})^{p^* V_{\mathrm{SO}_3}} \simeq (B\mathrm{U}_1)^{V_{\mathrm{U}_1} \oplus \underline{\mathbb{R}}} \simeq \Sigma (B\mathrm{U}_1)^{V_{\mathrm{U}_1}}. \tag{4.7}$$

To study spin bordism, we smash the fiber sequence for $\mathrm{sm}_{V_{\mathrm{SO}_3}}$ with $MT\mathrm{Spin}$:

$$MT\mathrm{Spin} \wedge \Sigma(B\mathrm{U}_1)^{V_{\mathrm{U}_1}} \to MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)^{V_{\mathrm{SO}_3}} \xrightarrow{\mathrm{sm}_{V_{\mathrm{SO}_3}}} MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)^{2V_{\mathrm{SO}_3}}. \tag{4.8}$$

Under shearing, this sequence becomes more familiar. Using Example 2.2 and Corollary 2.13, we may recast the first spectrum as $\Sigma^3 MT\mathrm{Spin}^c$. By Example 2.9, the second spectrum becomes $\Sigma^3 MT\mathrm{Spin}^h$. Finally, the third spectrum represents $(B\mathrm{SO}_3, 2V_{\mathrm{SO}_3})$-twisted spin bordism, but this twist is actually this is no twist at all: since $2V_{\mathrm{SO}_3}$ is spin, this spectrum reduces to $\Sigma^6 MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)_+$ by Lemma 3.3.

Altogether, after desuspending thrice, we have a fiber sequence of spectra[11]

$$MT\mathrm{Spin}^c \xrightarrow{p} MT\mathrm{Spin}^h \xrightarrow{\mathrm{sm}_{V_{\mathrm{SO}_3}}} \Sigma^3 MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)_+. \tag{4.9}$$

The associated Smith long exact sequence is

$$\cdots \longrightarrow \Omega_n^{\mathrm{Spin}^c} \xrightarrow{p_*} \Omega_n^{\mathrm{Spin}^h} \xrightarrow{\mathrm{sm}_{V_{\mathrm{SO}_3}}} \Omega_{n-3}^{\mathrm{Spin}}(B\mathrm{SO}_3) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}^c} \longrightarrow \cdots \tag{4.10}$$

Lemma 4.6 and (LES-1) imply that p_* is the map taking the induced spin^h structure of a spin^c manifold, proving the first part of Theorem 4.4.

We are interested in (4.10) in degrees $i = 4k$ after inverting 2.

Lemma 4.11.
(1) $\Omega_*^{\mathrm{Spin}^c} \otimes \mathbb{Z}[1/2]$ *is concentrated in even degrees.*
(2) $\Omega_*^{\mathrm{Spin}^h} \otimes \mathbb{Z}[1/2]$ *is concentrated in degrees* $0 \bmod 4$.
(3) $\Omega_*^{\mathrm{Spin}}(B\mathrm{SO}_3) \otimes \mathbb{Z}[1/2]$ *is concentrated in degrees* $0 \bmod 4$.

Proof. (1) is in Stong [Sto68, Chapter XI, p. 349]. For (2), use the equivalence $\Omega_*^{\mathrm{Spin}^h} \otimes \mathbb{Z}[1/2] \cong \Omega_*^{\mathrm{Spin}} \otimes H_*(B\mathrm{SU}_2; \mathbb{Z}[1/2])$ [Hu23, Remark A.2] together with the fact that both $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}[1/2]$ and $H_*(B\mathrm{SU}_2; \mathbb{Z})$ are concentrated in degrees 0 mod 4 ([ABP67], resp. [Bor53b, §29]); use the universal coefficient theorem to get to $H_*(B\mathrm{SU}_2; \mathbb{Z}[1/2])$ and thus to $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}[1/2]$.

For (3), use the Atiyah-Hirzebruch spectral sequence of signature

$$E^2_{p,q} = H_p(B\mathrm{SO}_3; \Omega_q^{\mathrm{Spin}} \otimes \mathbb{Z}[1/2]) \implies \Omega_{p+q}^{\mathrm{Spin}}(B\mathrm{SO}_3) \otimes \mathbb{Z}[1/2] \tag{4.12}$$

to compute. As noted above, spin bordism tensored with $\mathbb{Z}[1/2]$ is concentrated in degrees 0 mod 4, and $H_*(B\mathrm{SO}_3; \mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2, p_1]$ is concentrated in degrees divisible by 4 as well [Bor53b, §29], so the spectral sequence collapses on E_2 and the result is also concentrated in degrees 0 mod 4. $\square$

Theorem 4.13. *For all $k \geq 0$, the map*

$$p_* \colon \Omega_{4k}^{\mathrm{Spin}^c} \longrightarrow \Omega_{4k}^{\mathrm{Spin}^h} \tag{4.14}$$

[10] One way to see this is that these two vector bundles are induced from the defining representations $\mathrm{SO}_2 \to \mathrm{GL}_2(\mathbb{R})$, resp. $\mathrm{U}_1 \to \mathrm{GL}_1(\mathbb{C})$, and that the standard isomorphism $\mathbb{C} \cong \mathbb{R}^2$ induces an isomorphism of these two representations, hence also of their associated bundles.

[11] This fiber sequence and its corresponding Smith homomorphism also appears in [DDK+24, Example 7.45 and Appendix B]; it has the interesting property that the Smith homomorphism cannot be defined using ordinary cohomology: one must take the Poincaré dual submanifold in *ko*-cohomology or in spin cobordism.

defined by taking the induced spin^h structure is an isomorphism after tensoring with $\mathbb{Z}[1/2]$.

Proof. We discussed the interpretation of p_* as taking the induced spin^h structure right after (4.10), so all that remains is the isomorphism away from 2. Tensor the long exact sequence (4.10) with $\mathbb{Z}[1/2]$; since $\mathbb{Z}[1/2]$ is a flat $\mathbb{Z}$-module, the resulting sequence is still exact. Then plug in Lemma 4.11 and conclude. $\square$

This finishes the proof of Theorem 4.4.

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