TYPE IIA STRING THEORY AND TMF WITH LEVEL STRUCTURE

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Abstract. We look at a new string^h tangential structure first introduced by Devalapurkar and relate it to the $W_7 = 0$ condition of Diaconescu-Moore-Witten for type IIA string theory and M-theory. We show that a string^h structure on the target space automatically satisfies the $W_7 = 0$ condition and we also explain when the $W_7 = 0$ condition lifts to a string^h structure. Devalapurkar initially constructed $MString^h$ in such a way that it orients $tmf_1(3)$; we extend Devalapurkar's result, showing that $MString^h$ orients $tmf_1(n)$. We compute the homotopy groups of $MString^h$ in the dimensions relevant for physical applications, and apply them to anomaly cancellation applications for certain compactifications of type IIA string theory.

Contents

1. Introduction

Green-Schwartz in their seminal paper [GS84] showed that 10d heterotic string theory is free of perturbative anomalies if the target space manifold comes with a trivialization of the fractional first Pontryagin class $\frac{1}{2}p_1$. Such a condition arises due to the necessity to cancel the gauge anomaly using the 2-form B-field, and a target space that is a consistent background for heterotic string

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Date: November 13, 2024.

It is a pleasure to thank Sanath Devalapurkar for helpful discussions on the content of this paper, including bringing to our attention his definition of string^h and its application to $tmf_1(3)$. We also thank Theo Johnson-Freyd and Greg Moore for helpful conversations. The author MY is supported by the EPSRC Open Fellowship EP/X01276X/1.

theory is said to have a string structure. Later work by Witten [Wit87] showed that it was possible to assign a modular form to a manifold with a string structure by studying the partition function of the heterotic string worldsheet. More specifically he showed that the elliptic genus, a natural invariant in a generalized cohomology theory called elliptic cohomology, can be reconstructed from the index of the supercharge on the worldsheet. This gave the first hints at a link between string theory and elliptic cohomology.

Ando-Hopkins-Rezk [AHR10] then showed that the work of Witten can be interpreted in homotopy theory: Hopkins-Miller [HM14, Hop95, Hop02] constructed a highly structured ring spectrum TMF, called "topological modular forms," such that Witten's construction factors as a highly structured map of spectra

$$\sigma\colon \mathit{MString} \longrightarrow \mathit{TMF} \tag{1.1}$$

together with a comparison map from $\pi_*\mathit{TMF}$ to the ring of modular forms, which is an isomorphism after tensoring with $\mathbb{Z}[1/6]$. Hence, the Ando-Hopkins-Rezk orientation (1.1) makes the connection between heterotic string theory and elliptic cohomology precise. Subsequently, the Ando-Hopkins-Rezk map, as well as its physical incarnation due to Stolz-Teichner [ST11], have been used to prove anomaly cancellation results in heterotic string theory by Tachikawa, Yamashita, Yonekura, and Zhang [Tac22, TY23b, Yon22, TYY23, TY23a, TZ24].

One can reasonably wonder if there exist relationships between the other superstring theories and TMF, hence strengthening the string theory and elliptic cohomology correspondence. A natural first question to consider is if there is a different "string-like" structure which can be related to elliptic cohomology, and which enforces specific symmetry constraints on the target space.

The goal of this work is to explore this question in the context of type IIA string theory, and to address the homotopical questions this raises. We will propose a "string-like" structure that enforces the Diaconescu-Moore-Witten [DMW02] symmetry constraint given by $W_7(TM) = 0$, where $W_7(TM) \in H^7(M;\mathbb{Z})$ is an integral Stiefel-Whitney class. This constraint is needed to resolve a sign ambiguity in the partition function.

More specifically our solution for the "string-like" structure is a *string^h structure*, a variant of string structure recently defined by Devalapurkar [Dev22].

Definition 1.2 (Devalapurkar [Dev22])**.** Let $V \to X$ be a spin^c vector bundle with determinant line bundle $L \to X$. A *string^h structure* on V is the data of a trivialization of $\square_{ku}(\lambda(V \oplus L))$, where λ is the spin characteristic class with $2\lambda = p_1$ and $\square_{ku}\colon H^4(X;\mathbb{Z}) \to ku^7(X)$ is the Bockstein for the cofiber sequence $\Sigma^2 ku \xrightarrow{\beta} ku \to H\mathbb{Z}$.

We in fact give three definitions of string^h structures (Definitions 2.12, 2.13, and 2.15) and show they are equivalent in Theorem 2.17. Then we show that string^h structures answer the call we raised above.

- String^h structures are indeed twisted string structures, and if V has a string^h structure then $W_7(V)$ is canonically trivialized.
- Moreover, a spin^c structure and a trivialization of W_7 is a *good approximation* of a string^h structure in a range of dimensions relevant to string theory, as we explain further below.
- There is a map of spectra $\mathit{MString}^h[\frac{1}{n}] \to \mathit{tmf}_1(n)$, where the latter is the spectrum of *(connective) topological modular forms with level structure* for the subgroup $\Gamma_1(n) \subset \mathrm{SL}_2(\mathbb{Z})$.[1]

[1]For $n = 3$, this was first shown a different way by Devalapurkar [Dev22].

Main Results. The first of our results makes precise the first bullet above. Namely it facilitates the connection between string^h structures and the structure that corresponds to the $W_7 = 0$ condition.

Theorem 4.16. *If V is a string^h vector bundle, then V has a canonical spin^c structure and trivialization of $W_7(V)$.*

Theorems 4.21 and 4.23. *Let X be a spin^c manifold of dimension $n \le 8$. Every trivialization of $W_7(X)$ lifts to a string^h structure. If X is closed, this is also true in dimension 9.*

This means that for the purpose of studying compactifications of type IIA string theory in dimensions 8 and below, there is no loss of generality in upgrading the spin^c structure and trivialization of W_7 to a string^h structure. We will use this to study anomalies of these theories in Examples 4.27 and 4.28.

To prove those anomaly cancellation results, we need to calculate groups of reflection-positive invertible field theories on string^h manifolds (possibly with extra data), which by work of Freed-Hopkins [FH21b] and Grady [Gra23] reduces to computing string^h bordism groups of spaces. The germ of this calculation is a collection of string^h orientations of the spectra $tmf_1(n)$, the spectra of connective topological modular forms with level structure for $\Gamma_1(n) \subset \mathrm{SL}_2(\mathbb{Z})$, as constructed by Meier [Mei23], which we explain in §3.

Theorem (Devalapurkar [Dev22, Theorem 5])**.** *There is a map of E_∞-ring spectra $\sigma_D \colon MString^h_{(2)} \to tmf_1(3)_{(2)}$.*

We lift this to arbitrary n:

Theorem 3.7. *For all $n \ge 2$, there are maps of E_∞-ring spectra*

$$\sigma_1(n) \colon MString^h[1/n] \longrightarrow tmf_1(n). \tag{1.3}$$

In Theorem 3.22, we lift the induced map $MString^h[1/n] \to Tmf_1(n)$ on mixed Tmf to a map of $\mathbb{Z}/2$-equivariant ring spectra.

These theorems partially address an open question dating back to Hill-Lawson [HL16, §1]. It is not obvious whether $\sigma_D \simeq \sigma_1(3)$, and we would be interested in knowing whether this is the case.

Using Theorem 3.7, we computed string^h bordism groups in degrees relevant for physics applications.

Proposition 3.32. *For $n = 2, 3$, the map on homotopy groups $\sigma_1(n) \colon \Omega_*^{\mathrm{String}^h}[1/n] \to tmf_1(n)_*$ is surjective.*

Theorem 3.34.

(1) *In degrees 15 and below, $\Omega_*^{\mathrm{String}^h}$ is* additively *isomorphic to the graded ring*

$$\mathbb{Z}[x_2, x_4, x_6, x_8, y_8, x_{10}, x_{12}, y_{12}, x_{14}, \dots]/(\cdots) \tag{1.4}$$

where $|x_i| = |y_i| = i$ and all generators and relations not listed are in degrees 16 and above.

(2) *In degrees 7 and below, this isomorphism can be chosen to be a ring isomorphism. After inverting 2, the same is true in degrees 11 and below, and after inverting 6, the same is true in all degrees 15 and below.*

Similar to how a string structure on a manifold M induces a spin structure on its free loop space LM, we could wonder if string^h has the analogous property for spin^c structures on LM. Huang-Han-Duan [HHD21] showed that, because the groups Spin_n^c are not simply connected, there are multiple, inequivalent notions of a spin^c structure on the loop space of a manifold, parametrized by an integer k called the *level*. Unfortunately, a string^h structure does not induce any of these structures!

Theorem 2.60. *There is a closed string^h manifold M such that LM is not spin^c for* any *choice of level.*

Outline. The structure of the paper is as follows: In §2 we introduce three equivalent definitions of string^h and prove a number of properties of the spectrum $MString^h$. In §2.2 we explain the relationship between string^h and an already known spectrum called $MString^c$. In §3 we show how $MString^h$ orients $tmf_1(n)$ and compute its homotopy groups in degree less than 16. For those mainly interested in the relations to physics, §3 can be skipped and one can proceed to §4 where we review the Diaconescu-Moore-Witten anomaly. In §4.1 we summarize how the $W_7 = 0$ anomaly cancellation is related to string^h and vice versa. We then give examples in §4.2 of how the string^h structure can be used to more easily compute the anomalies of theories where a $W_7 = 0$ structure is equivalent to a string^h structure. In Appendix A, we prove Theorem A.1, which is a computation needed for the anomaly cancellation result in Example 4.28.

2. The String^h tangential structure

In this section we review work of Devalapurkar [Dev22] on the definitions and basic properties of string^h structures. We first give three definitions of string^h structures (Definitions 2.12, 2.13, and 2.15). String^h structures are analogues of spin^c structures, and we will frequently make this comparison to provide context for a definition or construction. We will then define a canonical string^h structure on a direct sum of string^h vector bundles in Proposition 2.27, so that $MString^h$ has an induced E_∞-ring structure (Theorem 2.41). We will then reprove a theorem of Devalapurkar [Dev22] that as E_∞-ring spectra, $MString^h \simeq MString \wedge MU$ (Theorem 2.43).

2.1. String^h structures.

As a lead up to the string^h definitions we start off with several equivalent ways to define spin^c structures.

- **Trivialization of a class:** A spin^c structure on an oriented vector bundle $V \to X$ is a trivialization of $\square_{\mathbb{Z}}(w_2(V))$, where $\square_{\mathbb{Z}}\colon H^2(X;\mathbb{Z}/2) \to H^3(X;\mathbb{Z})$ is the Bockstein.
- **Lift of a class:** A spin^c structure on V is a class $c_1 \in H^2(X;\mathbb{Z})$ and an identification of $c_1 \bmod 2 = w_2(V)$.
- **Twisted spin structure:** A spin^c structure on V is data of a complex line bundle L and a spin structure on $V \oplus L$. L is called the *determinant line bundle* of the spin^c structure.
- **Structure group:** A spin^c structure on V, where V has rank n, is a lift of the principal SO_n-bundle of frames $\mathcal{B}_{\mathrm{SO}}(V) \to X$ of V to a principal spin^c bundle $\mathcal{B}_{\mathrm{Spin}^c}(V) \to X$, i.e. a G-structure for $G = \mathrm{Spin}_n^c$ with its usual map to O_n.

We will give string^h analogues of each of the first three definitions: trivializing a class in Definition 2.12, lifting a class in Definition 2.13, and in terms of a twisted string structure in Definition 2.15. These definitions are equivalent, which we prove in Theorem 2.17.

Definition 2.1. For $n \geq 5$, Spin_n is a compact, simple, simply connected Lie group, so there is a canonical[2] isomorphism $\varphi\colon H^4(B\mathrm{Spin}_n;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$. We will let $\lambda := \varphi^{-1}(1)$.

[2]As $\mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$, we need to disambiguate 1 and -1. We choose the isomorphism $H^4(B\mathrm{Spin}_n;\mathbb{Z}) \to \mathbb{Z}$ to be the one such that the induced isomorphism $H^4(B\mathrm{Spin}_n;\mathbb{R}) \to \mathbb{R}$ sends the Chern-Weil class of the Killing form to a positive number.

As usual, λ defines a characteristic class of spin vector bundles by pulling back by the classifying map. This class is often denoted $\frac{1}{2}p_1$ (see Lemma 2.4).

Remark 2.2. For all $n \geq 5$, pulling back by the inclusion $\mathrm{Spin}_n \to \mathrm{Spin}_{n+1}$ sends $\lambda \mapsto \lambda$. Therefore we may define $\lambda \in H^4(B\mathrm{Spin}_n;\mathbb{Z})$ for $n < 5$ by pulling back from $B\mathrm{Spin}_5$, and by passing to the colimit over all $B\mathrm{Spin}_n$, we obtain $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z})$.

Lemma 2.3 ([Deb24, Lemma 1.6])**.** *Let V_1 and V_2 be two vector bundles over a topological space X each with a spin structure. Then $\lambda(V_1 \oplus V_2) = \lambda(V_1) + \lambda(V_2)$.*

Lemma 2.4. *If $p_1 \in H^4(B\mathrm{Spin}_n;\mathbb{Z})$ denotes the first Pontrjagin class, then $2\lambda = p_1$.*

If V is a real vector bundle, then $V \otimes \mathbb{C} = V \oplus V$ and $\lambda(V \otimes \mathbb{C}) = 2\lambda(V)$. In particular, if V is a spin vector bundle then $2\lambda(V) = p_1(V)$.

Definition 2.5 (Miller, Stolz-Teichner [ST04, §5])**.** A *string structure* on a spin vector bundle V is a trivialization of $\lambda(V)$.

The evocative name "string" for this structure is due to Haynes Miller; to our knowledge Stolz-Teichner were the first to use it in print. Previously, string structures were sometimes referred to as $\mathrm{O}\langle 8\rangle$*-structures* or $\langle 8\rangle$*-structures* (e.g. [Gia71]).

Definition 2.6. Let $V \to X$ be a vector bundle with spin^c structure $\mathfrak{s}$ and determinant line bundle L. We define

$$\lambda^c(V,\mathfrak{s}) \coloneqq \lambda(V \oplus L) \in H^4(X;\mathbb{Z}). \tag{2.7}$$

Often $\mathfrak{s}$ will be implicit, in which case we will write $\lambda^c(V)$ instead.

The class λ^c is called q_2 in [Dua18] and $\widehat{p}$ in [CN19, §2.7]. The classes p_c from [CY20, (3.10)] and $\frac{c_1^2 - p_1}{2}$ from [Dev22, Construction 2] both equal $c_1(L)^2 - \lambda^c$.

Remark 2.8. If the spin^c structure on V is induced from a complex structure, then [CN19, Lemma 2.39]

$$\lambda^c(V) = -c_2(V) - c_1(V)^2. \tag{2.9}$$

If the spin^c structure on V is induced from a spin structure, so that L is trivial, then $\lambda^c(V) = \lambda(V)$. Thus if V is both complex and spin, $\lambda(V) = -c_2(V)$.

Definition 2.10. Recall the Bott map $\beta\colon \Sigma^2 ku \to ku$ in connective K-theory. Its cofiber is the Postnikov 0-truncation $\tau_0\colon ku \to H\mathbb{Z}$, which is in particular a morphism of ring spectra. Thus, associated to the cofiber sequence

$$\Sigma^2 ku \xrightarrow{\beta} ku \xrightarrow{\tau_0} H\mathbb{Z} \tag{2.11}$$

there is a long exact sequence in cohomology; let $\square_{ku}\colon H^n(\text{–};\mathbb{Z}) \to ku^{n+3}(\text{–})$ denote the connecting morphism in this long exact sequence, which is called the *ku-theoretic Bockstein homomorphism.*

We will let $\square_{\mathbb{Z}}\colon H^n(\text{–};\mathbb{Z}/2) \to H^{n+1}(\text{–};\mathbb{Z})$ denote the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$.

Definition 2.12 (Devalapurkar [Dev22])**.** A *string^h structure* on a spin^c vector bundle $V \to X$ is a trivialization of $\square_{ku}(\lambda^c(V)) \in ku^7(X)$.

As always, we say a manifold M is string^h if TM is string^h.

Definition 2.13. A string^h structure on a spin^c vector bundle $V \to X$ is equivalent to a class $c_2^{ku}(V) \in ku^4(X)$ and data identifying $\tau_0(c_2^{ku}(V)) = \lambda^c(V)$.

Equivalence of these definitions follows immediately from the long exact sequence induced from (2.11). The third definition, which we give in Definition 2.15, is not as obviously equivalent.

Definition 2.14. Let $V \to X$ be a virtual vector bundle. A (X,V)-*twisted string structure* on a vector bundle $E \to M$ is data of a map $f\colon M \to X$ and a string structure on $E \oplus f^*V$.

Given an (X,V)-twisted string structure on E, the bundle $f^*V \to M$ is called the *ancillary bundle*.

Definition 2.15. Let $S \to B\mathrm{U}$ denote the tautological bundle. A string^h structure on a vector bundle $V \to X$ is a $(B\mathrm{U}, S)$-twisted string structure.

The data of a $(B\mathrm{U}, S)$-twisted string structure on a bundle $E \to M$ induces a spin^c structure on E as follows: the two-out-of-three data for spin^c structures implies spin^c structures on $E \oplus f^*(S)$ and on $f^*(S)$ induce one on E. But $E \oplus f^*(S)$ is string, hence spin, hence spin^c, and S is complex, hence spin^c, so E acquires a canonical spin^c structure.

Remark 2.16. Definition 2.14 is not the standard way to define twisted string structures, though it appears implicitly in [BDDM24, §3] and is inspired by a related definition of twisted spin structure due to Hason-Komargodski-Thorngren [HKT20, §4.1]. A more conventional definition, which goes back to Wang [Wan08, Definition 8.4], chooses $d \in H^4(X;\mathbb{Z})$ and defines an (X,d)-twisted string structure on a spin vector bundle $E \to M$ to be a map $f\colon M \to X$ and a trivialization of $\lambda(E) - f^*(d)$; see also Sati-Schreiber-Stasheff [SSS12, §2.2]. This definition cannot apply to our situation: by construction, any (X,d)-twisted string vector bundle is spin, but the tangent bundle to $\mathbb{CP}^2$ admits a string^h structure, where the ancillary bundle is $-T\mathbb{CP}^2$, and $T\mathbb{CP}^2$ is not spin.

Definition 2.14 is one way to remedy this definition, though there are twists of string bordism according to Wang's definition that Definition 2.14 cannot describe, including the twists studied in [Deb24, Deb]. To include these twists and string^h structures in a single framework, one can generalize Wang's definition as follows. Let SH be the *(restricted) supercohomology* spectrum[3] introduced by Freed [Fre08, §1] and Gu-Wen [GW14], which is uniquely specified up to homotopy equivalence by $\pi_0(\mathit{SH}) \cong \mathbb{Z}$, $\pi_{-2}(\mathit{SH}) \cong \mathbb{Z}/2$, and the k-invariant $\beta_{\mathbb{Z}} \circ \mathrm{Sq}^2\colon H^{-2}(-;\mathbb{Z}/2) \to H^1(-;\mathbb{Z})$. There is a class $\widetilde\lambda \in \mathit{SH}^4(B\mathrm{SO})$ which is additive in direct sums [Jen05, Corollary 4.9] and whose pullback to $B\mathrm{Spin}$ is the image of the usual $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z})$ under the connective cover map $H^k(-;\mathbb{Z}) \to \mathit{SH}^k(-)$ [Fre08, Proposition 1.9(i)]. A string structure on an oriented vector bundle $E \to M$ is precisely a trivialization of $\widetilde\lambda(E) \in \mathit{SH}^4(M)$ (see [JFT20, §1.4]), so given a space X and $\widetilde d \in \mathit{SH}^4(X)$, one can define an $(X,\widetilde d)$-*twisted string structure* on an oriented vector bundle $E \to M$ to be a map $f\colon M \to X$ and a trivialization of $\widetilde\lambda(E) - f^*(\widetilde d) \in \mathit{SH}^4(M)$. This or closely related definitions appear in [FHT10, JF20, TY23a, TY23b, DY23]; pulling back to $B\mathrm{Spin}$ recovers Wang's definition, but over $B\mathrm{SO}$ this is strictly more general.

[3]Different authors mean different things by "supercohomology;" we use "restricted" to contrast with *extended supercohomology* as introduced by Kapustin-Thorngren [KT17] and Wang-Gu [WG18, WG20]. See also [GJF19, §5.3, §5.4].

Let $r\colon BU \to BSO$ be the map defined by forgetting the complex structure; then a string^h structure is a $(BU, -r^*\widetilde{\lambda})$-twisted string structure, which follows from the equation $\widetilde{\lambda}(E \oplus F) = \widetilde{\lambda}(E) + \widetilde{\lambda}(F)$ as noted above.

Theorem 2.17. *Definitions 2.12, 2.13, and 2.15 are equivalent.*

Proof. Exactness of the Bockstein long exact sequence associated to (2.11) implies Definitions 2.12 and 2.13 are equivalent, so we will focus on relating Definitions 2.13 and 2.15.

For topological spaces X, there is an isomorphism $\rho\colon [X, BSU] \xrightarrow{\cong} ku^4(X)$, and the composition

$$[X, BSU] \xrightarrow{\rho} ku^4(X) \xrightarrow{\tau_0} H^4(X;\mathbb{Z}) \tag{2.18}$$

sends a map $f\colon X \to BSU$ to $f^*(c_2)$. Thus a class $x \in H^4(X;\mathbb{Z})$ lifts to $ku^4(X)$ if and only if x is the second Chern class of an SU-structured vector bundle.

Now we show a string^h structure in the sense of Definition 2.13 induces one in the sense of Definition 2.15. By assumption, we have lifted $\lambda^c(V)$ to a class $c_2^{ku}(V) \in ku^4(X)$, which as above is equivalent data to a vector bundle $\widetilde{E} \to X$ with SU-structure and an identification $c_2(\widetilde{E}) = \lambda^c(V)$. Let L be the determinant bundle of V and let $E \coloneqq \widetilde{E} \oplus L$; we will show $V \oplus E$ has a string structure, which means checking that we have data of trivializations

- $w_1(V \oplus E) = 0$,
- $w_2(V \oplus E) = 0$, and
- $\lambda(V \oplus E) = 0$.

Because $\widetilde{E}$ and L are oriented, E is oriented, and because V and E are oriented, $V \oplus E$ is oriented, and therefore $w_1(V \oplus E)$ is trivialized.

Since V is spin^c with determinant bundle L, we have data of a trivialization of $w_2(E) + w_2(L)$ coming from the spin structure on $V \oplus L$, and the SU-structure on $\widetilde{E}$ induces a spin structure on $\widetilde{E}$ (see, e.g., [Sto67]), hence also a trivialization of $w_2(\widetilde{E})$. The Whitney sum formula provides for us an identification

$$w_2(V \oplus E) = w_2(V) + w_2(L) + 0 = 0. \tag{2.19}$$

On to λ. As described above, $V \oplus L$ and $\widetilde{E}$ are spin, so we have an identification

$$\lambda(V \oplus E) = \lambda(V \oplus L \oplus \widetilde{E}) = \lambda(V \oplus L) + \lambda(\widetilde{E}) \tag{2.20}$$

using the Whitney sum formula in Lemma 2.3. By Remark 2.8, $\lambda(\widetilde{E}) = -c_2(\widetilde{E}) = -\lambda(V \oplus L)$, providing the required trivialization of $\lambda(V \oplus E)$ and therefore a string structure.

Finally, we will show that a string^h structure in the sense of Definition 2.15 induces a string^h structure in the sense of Definition 2.13. Let E denote the ancillary bundle. Because $V \oplus E$ is string, it is in particular spin, so $w_2(V) + w_2(E)$ is trivialized. We therefore have a spin^c structure on V with determinant bundle $L \coloneqq \mathrm{Det}(E)$, because $w_2(E) = w_2(\mathrm{Det}(E))$ canonically.

Let $\widetilde{E} \coloneqq E - \mathrm{Det}(E)$. Then we have canonical isomorphisms of complex line bundles

$$\mathrm{Det}(\widetilde{E}) \cong \mathrm{Det}(E) \otimes \mathrm{Det}(-\mathrm{Det}(E)) \cong \mathrm{Det}(E) \otimes (\mathrm{Det}(E))^\vee \cong \underline{\mathbb{C}}, \tag{2.21}$$

giving us data of an SU-structure on $\widetilde{E}$, and therefore a class $c_2^{ku} \in ku^4(X)$. We are done if we can show that $\tau_0(c_2^{ku}) = \lambda^c(V)$, i.e. that $c_2(\widetilde{E}) = \lambda(V \oplus L)$. As in the previous part of this proof, the string structure on $V \oplus E$ furnishes an identification $\lambda(V \oplus L) + \lambda(\widetilde{E}) = \lambda(V \oplus E) = 0$, so $\lambda(V \oplus L) = -\lambda(\widetilde{E})$; applying Remark 2.8 allows us to conclude $c_2(\widetilde{E}) = \lambda(V \oplus L) = \lambda^c(V)$. ☐

We now derive a few basic properties of string^h structures. We start by establishing in Proposition 2.27 the string^h analogue of the fact that a direct sum of spin^c bundles is also spin^c.

Lemma 2.22 (Whitney sum formula for λ^c)**.** *Let $V, W \to X$ be spin^c vector bundles. Then in $H^4(X;\mathbb{Z})$,*

$$\lambda^c(V \oplus W) = \lambda^c(V) + c_1(V)c_1(W) + \lambda^c(W). \tag{2.23}$$

Proof. By naturality, it suffices to prove this for V and W the tautological bundles over $B\mathrm{Spin}^c_{n_1}$, resp. $B\mathrm{Spin}^c_{n_2}$ for $n_1, n_2 \gg 0$. Using Duan's calculation [Dua18, Theorem D] of $H^*(B\mathrm{Spin}^c_n;\mathbb{Z})$ and the Künneth formula, we learn that $H^4(B\mathrm{Spin}^c_{n_1} \times B\mathrm{Spin}^c_{n_2};\mathbb{Z})$ lacks 2-torsion, so if we can show $2\lambda^c(V\oplus W) = 2\lambda^c(V) + 2c_1(V)c_1(W) + 2\lambda^c(W)$, that would suffice to prove the lemma. That is, we want to prove

$$p_1(V \oplus W \oplus (L_V \otimes L_W)) = p_1(V \oplus L_V) + 2c_1(V)c_1(W) + p_1(V \oplus L_W), \tag{2.24}$$

where L_V, L_W denote the determinant line bundles of V and W, respectively. Here we used the fact that the determinant line bundle for a direct sum of spin^c vector bundles is the tensor product of their determinant line bundles.

The first Pontrjagin class satisfies a Whitney sum formula $p_1(E\oplus F) = p_1(E)+p_1(F)$ if E and F are oriented [Bro82, Theorem 1.6] (see also [Tho62]), and using that formula, we can reduce (2.24): to prove the lemma, it suffices to prove that for complex line bundles L_1 and L_2,

$$p_1(L_1 \otimes L_2) = p_1(L_1) + 2c_1(L_1)c_1(L_2) + p_1(L_2). \tag{2.25}$$

For any rank-2 oriented real vector bundle E, $p_1(E) = e(E)^2$, and the Euler class of a complex line bundle is additive in tensor products, from which (2.25) follows, and then the lemma too. ☐

Lemma 2.26 (Conner-Floyd [CF66, §8])**.** *For $n \ge 1$ the classes $c_1, \dots, c_n \in H^*(B\mathrm{U}_n;\mathbb{Z})$ have canonical preimages $c_1^{MU}, \dots, c_n^{MU} \in MU^*(B\mathrm{U}_n)$. Therefore the same is true with MU replaced with any complex-oriented ring spectrum E with $\pi_0(E) \cong \mathbb{Z}$.*

These classes are called the *Conner-Floyd Chern classes.* To prove the part about E, it suffices to observe that the complex orientation and isomorphism $\pi_0(E) \cong \mathbb{Z}$ give rise to maps of E_∞-ring spectra $MU \to E \to H\mathbb{Z}$ whose composition is the usual ring map $MU \to H\mathbb{Z}$, so the lift from c_k to c_k^{MU} passes through some class c_k^E in E-cohomology.

Proposition 2.27. *If $V, W \to X$ are string^h vector bundles, there is a canonical string^h structure on $V \oplus W$ extending the usual direct-sum spin^c structure, characterized in the following equivalent ways.*

- **Lift of a class:** *The Whitney sum formula Lemma 2.22 implies that if $c_2^{ku}(V)$, resp. $c_2^{ku}(W)$ are lifts of $\lambda^c(V)$, resp. $\lambda^c(W)$ across τ_0, then $c_2^{ku}(V) + c_1^{ku}(V)c_1^{ku}(W) + c_2^{ku}(W)$ is a lift of $\lambda^c(V \oplus W)$ and thus defines a string^h structure on $V \oplus W$.*
- **Trivialization of a class:** *We will show the equality*

$$\square_{ku}(\lambda^c(V \oplus W)) = \square_{ku}(\lambda^c(V)) + \square_{ku}(\lambda^c(W)), \tag{2.28}$$

so the trivializations of $\square_{ku}(\lambda^c(V))$ and $\square_{ku}(\lambda^c(W))$ induce a trivialization of $\square_{ku}(\lambda^c(V \oplus W))$, hence a string^h structure on $V \oplus W$.

- **Twisted string structure:** *Let E, resp. F be the ancillary bundles to V, resp. W. Then $V \oplus E \oplus W \oplus F$ is a direct sum of two string vector bundles, hence acquires a string structure;*

switching E and W, we have produced a string^h structure on $V \oplus W$ with ancillary bundle $E \oplus F$.

Proof. The Whitney sum formula and linearity of the Bockstein do not quite prove (2.28): they tell us that

$$\square_{ku}(\lambda^c(V \oplus W)) = \square_{ku}(\lambda^c(V)) + \square_{ku}(c_1(V)c_1(W)) + \square_{ku}(\lambda^c(W)). \tag{2.29}$$

So we will show $\square_{ku}(c_1(V)c_1(W)) = 0$. It suffices to do this for the universal case, which is a class in $ku^7(B\mathrm{U}_1 \times B\mathrm{U}_1)$, and this is the zero group [BG10, Theorem 5.2.1]. Thus we have a canonical trivialization of $\square_{ku}(c_1(V)c_1(W))$, or equivalently a canonical lift to ku^4, namely $c_1^{ku}(V)c_1^{ku}(W)$. Therefore the trivialization and lifting pieces of the proposition are equivalent by using exactness like in the proof of Theorem 2.17.

Therefore we are done if we can show that under the process we described in the proof of Theorem 2.17, the direct-sum string structure on $V \oplus E \oplus W \oplus F$ produces the “obvious” lift of $\lambda^c(V \oplus W)$, namely

$$c_2^{ku}(V) + c_1^{ku}(V)c_1^{ku}(W) + c_2^{ku}(W). \tag{2.30}$$

Since $c_1(V)c_2(W)$ has a canonical lift we just need to show that the direct sum string structure provides lifts to $c_2(V)$ and $c_2(W)$. The lift we get from the direct-sum string structure is the ku^4-class corresponding to the virtual SU-structured vector bundle $E \ominus \mathrm{Det}(E) \oplus F \ominus \mathrm{Det}(F)$. Taking c_2 of this gives $c_2(E - \mathrm{Det}(E)) + c_2(F - \mathrm{Det}(F))$ but we have already proven in Theorem 2.17 that $c_2(E - \mathrm{Det}(E)) = c_2(\widetilde{E}) = \lambda^c(V)$ and the lift of this class is $c_2^{ku}(V)$; a similar statement holds for $c_2(F - \mathrm{Det}(F))$ where the lift is given by $c_2^{ku}(W)$. ☐

Proposition 2.31 (String implies string^h)**.** *If V is a vector bundle with a string structure, there is a canonical string^h structure on V characterized in the following equivalent ways.*

- **Trivialization of a class:** *Since V is string, $\lambda^c(V) = \lambda(V) = 0$, and $\square_{ku}(0)$ has a canonical trivialization.*
- **Lift of a class:** *There is a canonical lift of $0 \in H^4(X;\mathbb{Z})$ to ku-cohomology, namely $0 \in ku^4(X)$.*
- **Twisted string structure:** *If $E = 0$, the string structure on V induces a string structure on $V \oplus E$, so we obtain a string^h structure with ancillary bundle 0.*

Proof. This amounts to the assertion that if you take the 0 characteristic class or vector bundle and pass it through the identifications we constructed in the proof of Theorem 2.17, you still end up with 0, which is straightforward to verify. ☐

Proposition 2.32 (Complex implies string^h)**.** *If V is a complex vector bundle, there is a canonical string^h structure on V characterized in the following equivalent ways.*

- **Trivialization of a class:** *The ku-Bockstein of the universal class $\lambda^c \in H^4(B\mathrm{U};\mathbb{Z})$ lands in $ku^7(B\mathrm{U})$, which is the zero group* [BG10, Theorem 5.5.1].
- **Lift of a class:** *The map $\tau_0\colon ku^4(B\mathrm{U}) \to H^4(B\mathrm{U};\mathbb{Z})$ is surjective, and in the notation of* [BG10, Theorem 5.5.1], *the class $-c_2 - c_1^2$ is a preimage of λ^c.*
- **Twisted string structure:** *If $E = -V$, then $V \oplus E = 0$ has a canonical string structure, endowing V with a string^h structure with ancillary bundle $-V$.*

See [Dev22, Remark 7] for a fourth perpective on Proposition 2.32.

Proof. As usual, the equivalence of the first two perspectives follows from the long exact sequence coming from (2.11). To bring in the third perspective, recall from the proof of Theorem 2.17 that the preimage of λ^c in the second perspective is $-c_2^{ku} - (c_1^{ku})^2$ of the ancillary bundle (the c_2^{ku} came from the SU-bundle, and $(c_1^{ku})^2$ from the determinant line bundle), which matches the second perspective. ☐

Proposition 2.33. *The construction in Proposition 2.31 sends the canonical string structure on a direct sum of string vector bundles to the canonical string^h structure from Proposition 2.27. The same is true for Proposition 2.32 with "string" replaced with "complex."*

Proof. Both parts of this proposition follow quickly using the twisted string structure/ancillary bundle perspective. For example, if V and W are string, the canonical identification $V \oplus W \xrightarrow{\cong} V \oplus 0 \oplus W \oplus 0$ identifies the string^h structure on $V \oplus W$ from Proposition 2.27 with the string^h structure induced from the direct-sum string structure on $V \oplus W$. The proof for complex vector bundles is analogous. ☐

We denote by $B\mathrm{String}^h$ the space which classifies string^h bundles. We describe the properties of $B\mathrm{String}^h$ by first considering it in the context of Definitions 2.12 and 2.15. Starting with Definition 2.12, the maps $[X, \Sigma^7 ku]$ represent classes in $\Sigma^7 ku$, which through suspension is equivalent to $[\Sigma X, \Sigma^8 ku]$. By the loops-suspension adjunction, this gives $[X, \Omega B\mathrm{U}\langle 8\rangle]$, and we take $\Omega B\mathrm{U}\langle 8\rangle$ as the space that classifies $ku^7(X)$.

Definition 2.34 (Devalapurkar [Dev22, Construction 2])**.** The space $B\mathrm{String}^h$ is the fiber of the map

$$\square_{ku} \colon B\mathrm{Spin}^c \longrightarrow \Omega B\mathrm{U}\langle 8\rangle. \tag{2.35}$$

This space arises from conidering all those spin^c vector bundles such that λ^c is trivial in $ku^7(X)$.

The map $B\mathrm{String}^h \to B\mathrm{Spin}^c$ can also be deduced from Definition 2.15; for a manifold X a string structure on $TX \oplus \widetilde{E} \oplus L$ in particular gives a spin structure on $TX \oplus L$, since $\widetilde{E}$ is spin as it has an SU-structure. Therefore $w_2(TM) = w_2(L)$ and X is spin^c. The string structure on $TX \oplus \widetilde{E} \oplus L$ implies $\lambda(TX \oplus L) = -\lambda(\widetilde{E}) = c_2(\widetilde{E})$ where the last equality is due to Remark 2.8. Hence $B\mathrm{String}^h$ fits into the following pullback square:

$$\begin{array}{ccc} B\mathrm{String}^h & \longrightarrow & B\mathrm{SU} \\ \downarrow & \lrcorner & \downarrow c_2 \\ B\mathrm{String} & \xrightarrow{\lambda^c} & K(\mathbb{Z}, 4)\,. \end{array} \tag{2.36}$$

The diagram in Equation (2.36) then implies that a string^h structure on a spin^c vector bundle V with determinant bundle L is equivalent data to a $(B\mathrm{SU}, c_2)$-twisted string structure on $V \oplus L$.

By Proposition 2.31 we can relate $B\mathrm{String}^h$ to $B\mathrm{String}$, giving the following diagram whose rows are fiber sequences [Dev22, Lemma 3].

$$\begin{array}{ccccc} B\mathrm{String}^h & \longrightarrow & B\mathrm{Spin}^c & \longrightarrow & \Omega B\mathrm{U}\langle 8\rangle \\ \uparrow & & \uparrow & & \uparrow \square_{ku} \\ B\mathrm{String} & \longrightarrow & B\mathrm{Spin} & \xrightarrow{\lambda} & K(\mathbb{Z}, 4)\,. \end{array} \tag{2.37}$$

Lemma 2.38. *The low-degree homotopy groups of* $B\mathrm{String}^h$ *are*

$$\begin{aligned} &\pi_0(B\mathrm{String}^h) \cong 0 && \pi_5(B\mathrm{String}^h) \cong 0 \\ &\pi_1(B\mathrm{String}^h) \cong 0 && \pi_6(B\mathrm{String}^h) \cong \mathbb{Z} \\ &\pi_2(B\mathrm{String}^h) \cong \mathbb{Z} && \pi_7(B\mathrm{String}^h) \cong 0 \\ &\pi_3(B\mathrm{String}^h) \cong 0 && \pi_8(B\mathrm{String}^h) \cong \mathbb{Z}^2 \\ &\pi_4(B\mathrm{String}^h) \cong \mathbb{Z} && \pi_9(B\mathrm{String}^h) \cong \mathbb{Z}/2, \end{aligned} \tag{2.39}$$

and $\pi_{10}(B\mathrm{String}^h)$ *is an extension of* $\mathbb{Z}/2$ *by* $\mathbb{Z}$.

Proof. We apply the long exact sequence in homotopy groups for the top fiber sequence in (2.37). There is a homotopy equivalence $B\mathrm{Spin}^c \simeq B\mathrm{Spin} \times B\mathrm{U}_1$ (see [FH21b, §10] for this and similar

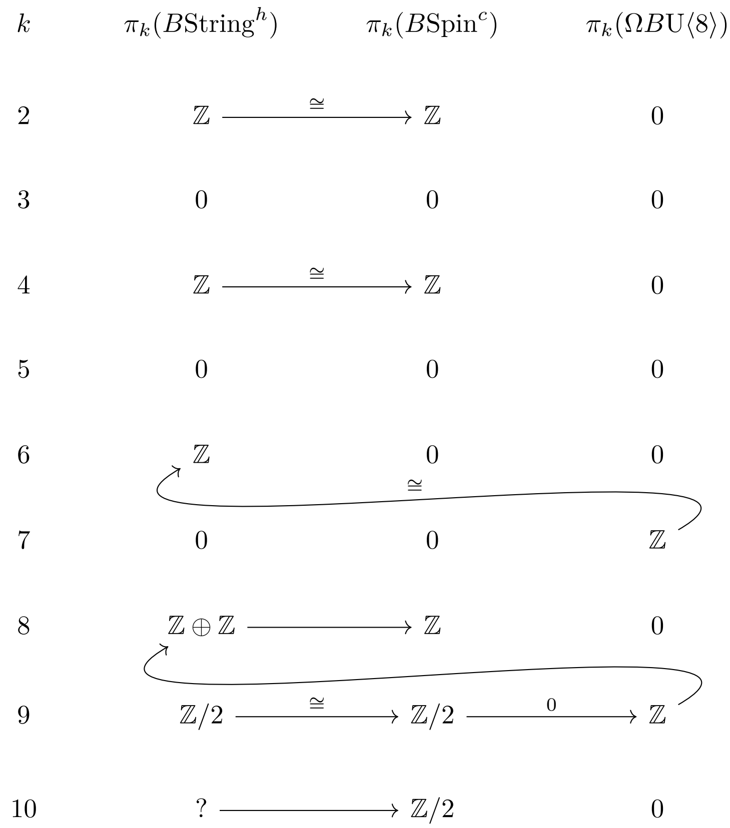

Figure 1. Homotopy Long Exact Sequence for computing the homotopy groups of $B\mathrm{String}^h$ in degrees up to 10.

equivalences), which allows us to compute $\pi_*(B\mathrm{Spin}^c)$. The homotopy groups of $\Omega B\mathrm{U}\langle 8\rangle$ come from Bott periodicity. Using these, we work out the long exact sequence on homotopy groups in Figure 1, which proves the claim. □

Let *MString*^{h} denote the Thom spectrum of the map $V\colon B\mathrm{String}^h \to B\mathrm{O}$; by the Pontrjagin-Thom theorem, the homotopy groups of this spectrum are isomorphic to the bordism groups of manifolds with string^h structures on their stable normal bundles.

Remark 2.40. Sometimes in this paper, we will consider manifolds with string^h structures on their stable tangent bundles, rather than stable normal bundles. A priori this is a different tangential structure classified by the Madsen-Tillmann spectrum $\mathit{MTString}^h$, the Thom spectrum of $-V\colon B\mathrm{String}^h \to B\mathrm{O}$, but one can show using Proposition 2.27 that a string^h structure on $V \to X$ is equivalent data to a string^h structure on $-V \to X$, like for orientations, spin structures, spin^c structures, etc. This furnishes a canonical equivalence $\mathit{MString}^h \simeq \mathit{MTString}^h$. We will therefore pass between tangential and normal string^h structures, and tangential and normal string^h bordism and Thom spectra, without comment, and likewise for spin, spin^c, string, and stably almost complex structures.

We will show that $\mathit{MString}^h$ is also complex oriented, and what the fact that string structures leading to string^h structures implies at the level of $\mathit{MString}$ and $\mathit{MString}^h$. The following omnibus theorem summarizes the multiplicative properties of $\mathit{MString}^h$.

Theorem 2.41.

(1) *$\mathit{MString}^h$ has a canonical E_∞-ring structure whose induced map on bordism groups is the direct product $M, N \mapsto M \times N$ with string^h structure as in Proposition 2.27.*
(2) *$\mathit{MString}^h$ has a canonical E_∞-$\mathit{MString}$-algebra structure refining the construction in Proposition 2.31.*
(3) *$\mathit{MString}^h$ has a canonical E_∞-MU-algebra structure refining the construction in Proposition 2.32; in particular, $\mathit{MString}^h$ is a complex-oriented ring spectrum.*
(4) *Forgetting from a string^h structure to a spin^c structure refines to the data of a canonical E_∞-$\mathit{MString}^h$-algebra structure on MSpin^c.*
(5) *The algebra structures described above are compatible in the sense that the following diagram is commutative whether one starts from $MU\langle 6\rangle$, $\mathit{MString}$, or MU:*

$$\begin{array}{ccccc} MU\langle 6\rangle & \longrightarrow & \mathit{MString} & \longrightarrow & \mathit{MSpin} \\ \downarrow & & \downarrow & & \downarrow \\ MU & \longrightarrow & \mathit{MString}^h & \longrightarrow & \mathit{MSpin}^c, \end{array} \tag{2.42}$$

where $MU\langle 6\rangle$ is the Thom spectrum of the 5-connected cover $B\mathrm{U}\langle 6\rangle$ of $B\mathrm{U}$.[4]

Parts (1) and (3) are originally due to Devalapurkar [Dev22, Construction 2, Corollary 4], proven in a different way. The rest of Theorem 2.41 is implicit in [Dev22].

Proof. The string^h structure on a direct sum of string^h vector bundles that we introduced in Proposition 2.27 defines a lift of the direct-sum map $\oplus\colon B\mathrm{O} \times B\mathrm{O} \to B\mathrm{O}$ to a map $\oplus\colon B\mathrm{String}^h \times$

[4] $\mathrm{U}\langle 6\rangle$ is playing the role of "tangential structure that induces both a complex structure and a string structure," analogous to $\mathit{MSU} \simeq MU\langle 4\rangle$ for complex and spin structures. In both cases it is possible to do a little better than $MU\langle n\rangle$, in that $MU\langle n\rangle$ is not the final tangential structure that induces a complex structure and a spin/string structure with compatible orientations. For complex and spin bordism, this is Stong's "complex-spin" structure [Sto67] $M\Sigma$, equivalent to data of a stably almost complex structure and a class $\in H^2(-;\mathbb{Z})$ with $2x = c_1$, so that $w_2 = c_1 \bmod 2 = 0$ gives a spin structure. Thus following the same line of reasoning as in [SSS12, §2.2.2] or [Deb24, Remark 1.55], this is a $(B\mathrm{U}_1, \mathcal{O}(2))$-twisted SU-structure, so $M\Sigma \simeq \mathit{MSU} \wedge (B\mathrm{U}_1)^{\mathcal{O}(2)-2}$, and this factorization can be made compatible with the ring spectrum structures on both sides. For complex and string structures, we have this together with a trivialization of $\lambda = -c_2$, so this tangential structure is equivalent to a $(B\mathrm{U}_1, \mathcal{O}(2))$-twisted $\mathrm{U}\langle 6\rangle$-structure and its Thom spectrum is $MU\langle 6\rangle \wedge (B\mathrm{U}_1)^{\mathcal{O}(2)-2}$, once again as ring spectra. In this section, the statements expressing compatibility of MU and $\mathit{MString}$ over $\mathit{MString}^h$ and under $MU\langle 6\rangle$ are true and slightly stronger if we replace $MU\langle 6\rangle$ with $MU\langle 6\rangle \wedge (B\mathrm{U}_1)^{\mathcal{O}(2)-2}$.

$B\mathrm{String}^h \to B\mathrm{String}^h$ commuting with the forgetful maps to $B\mathrm{O}\times B\mathrm{O}$, resp. $B\mathrm{O}$. This direct-sum map defines the structure of an E_∞-space on $B\mathrm{String}^h$; since it commutes with the forgetful maps, we see that the forgetful map $B\mathrm{String}^h \to B\mathrm{O}$ is a map of E_∞-spaces, where $B\mathrm{O}$ has the direct-sum structure.

Now apply the Thom spectrum functor; it is a theorem of Lewis [LMSM86, §IX.7] that the Thomification of an E_∞-space with an E_∞-map to $B\mathrm{O}$ is naturally an E_∞-ring spectrum, and passing this through the Pontrjagin-Thom construction, one sees that, just like in unoriented bordism, the ring structure corresponds to the direct product on manifolds. This proves part (1).

The remaining four parts of the theorem also fall in a similar way: if $\xi_1\colon B_1 \to B\mathrm{O}$ and $\xi_2\colon B_2 \to B\mathrm{O}$ are E_∞-maps of spaces and $\eta\colon B_1 \to B_2$ is an E_∞-map commuting with the maps to $B\mathrm{O}$, then the Thomification of η is a map of ring spectra, making $M\xi_2$ into an $M\xi_1$-algebra. Thus:

- Proposition 2.33 implies parts (2) and (3).
- In Proposition 2.27, we saw that the forgetful map $B\mathrm{String}^h \to B\mathrm{Spin}^c$ is compatible with direct sums, which implies part (4).

For the last part of the theorem, it suffices to know that, given two $\mathrm{U}\langle 6\rangle$-structured vector bundles V and W, the direct-sum string^h structures on $V\oplus W$ given by forgetting to MU, then to $MString^h$, versus to $MString$ then to $MString^h$, coincide; and the analogous for the spin^c structure induced on a direct sum of string vector bundles. In both cases, this can be checked by computing the characteristic classes involved in the trivializations used to define the direct-sum string^h, resp. spin^c, structure. ☐

Concretely, all of this means that products of string, complex, string^h, spin, and spin^c manifolds are compatible with all of the forgetful maps between these structures.

Theorem 2.43 (Devalapurkar [Dev22])**.** *The composition*

$$MString \wedge MU \xrightarrow{(2.41),\,\#2\ and\ 3} MString^h \wedge MString^h \xrightarrow{\mu} MString^h \tag{2.44}$$

is an equivalence of E_∞-ring spectra.

Proof. A map $X \to B\mathrm{String}^h$ is, up to homotopy, equivalent data to a string^h rank-zero virtual vector bundle on X, so we will define maps to $B\mathrm{String}^h$ by writing down string^h vector bundles. To do so, we will use Definition 2.15: a string^h structure on a vector bundle $V \to M$ is equivalent data to a complex vector bundle $E \to M$ and a string structure on $V\oplus E$. We will therefore represent a map $M \to B\mathrm{String}^h$ as a pair (V,E) with E complex and $V\oplus E$ string. For instance:

- the map $B\mathrm{String} \to B\mathrm{String}^h$ giving a string vector bundle the string^h structure of Proposition 2.31 is $(V,0)\colon B\mathrm{String} \to B\mathrm{String}^h$, where $V \to B\mathrm{String}$ is the tautological bundle; and
- the map $B\mathrm{U} \to B\mathrm{String}^h$ giving a complex vector bundle the string^h structure from Proposition 2.32 is $(E,-E)\colon B\mathrm{U} \to B\mathrm{String}^h$, where $E \to B\mathrm{U}$ is the tautological bundle.

In a similar way we will represent a map to $B\mathrm{String}$ as a string vector bundle, a map to $B\mathrm{U}$ as a complex vector bundle, etc. Let $W \to B\mathrm{String}^h$ be the tautological bundle, with ancillary bumdle F; then with these notational conventions the maps

$$(W,F) \mapsto (W\oplus F, -F)\colon B\mathrm{String}^h \to B\mathrm{String}\times B\mathrm{U} \tag{2.45a}$$

$$(V,E) \mapsto (V\oplus E, -E)\colon B\mathrm{String}\times B\mathrm{U} \to B\mathrm{String}^h \tag{2.45b}$$

are homotopy inverses to each other; keeping track of the effect of these maps on W we have the following commutative diagram.

$$\begin{array}{ccc} B\mathrm{String}^h & \underset{(V\oplus E,-E)}{\overset{(W,F)\mapsto(W\oplus F,-F)}{\rightleftarrows}} & B\mathrm{String}\times B\mathrm{U} \\ & {\scriptstyle W}\searrow \quad \swarrow{\scriptstyle V\oplus E} & \\ & B\mathrm{O} & \end{array} \tag{2.46}$$

One can verify that each of the maps in this diagram is compatible with direct sums, so (2.46) is a commutative diagram of E_∞-spaces and E_∞-morphisms whose horizontal arrows are equivalences of E_∞-spaces. Thus $W\colon B\mathrm{String}^h\to B\mathrm{O}$ and $V\oplus E\colon B\mathrm{String}\times B\mathrm{U}\to B\mathrm{O}$ are homotopy equivalent as E_∞-spaces with E_∞-maps to $B\mathrm{O}$, so the Thom spectrum functor produces equivalent E_∞-ring spectra from them: $\mathit{MString}^h$ and $\mathit{MString}\wedge MU$.

Lastly, we need that the specific map in the theorem statement implements this equivalence. This follows because the map $(V,E)\mapsto(V\oplus E,-E)$ from (2.45b) factors, as an E_∞-map, as the composition of the maps putting canonical string^h structures on the string, resp. complex vector bundles V, resp. E, followed by direct sum. The Thom spectrum functor turns direct sum into multiplication, and now we are done. ☐

2.2. **Relation between string^h structures and spin^c structures on loop spaces.** Thus far we have seen in multiple ways that string^h structures are to string structures as spin^c structures are to spin structures. How far does the analogy go? In this subsection, we give an example where the analogy fails to hold: string structures are closely related to spin structures on loop spaces, but this is not true for string^h structures and spin^c structures on loop spaces – instead, spin^c structures on loop spaces are governed by a different structure called a string^c structure (see [HHD21]). We will review the definitions of string^c structures and show that string^c structures induce string^h structures, but not vice versa.

Remark 2.47. We studied this question with applications to string theory in mind. Witten [Wit88] showed at a physics level of rigor that the index of the supercharge in a (1+1)d nonlinear sigma model with target space a string manifold M equals Ochanine's elliptic genus [Och87]. This elliptic genus can be recovered as the S^1-equivariant index of a Dirac operator on LM, using the string structure on M to define a spin structure on LM. The results in this subsection suggest that string^c structures, rather than string^h structures are the right way to generalize this for the spin^c Dirac operator.

Definition 2.48 ([HHD21])**.** Let $k\in\mathbb{Z}$. A *strong string^c structure of level* $(2k+1)$ is a $(B\mathrm{U}_1, L^{\otimes(2k+1)})$-twisted string structure, where $L\to B\mathrm{U}_1$ denotes the tautological bundle.

Remark 2.49. The definition in [HHD21] is phrased differently, as a spin^c structure and a trivialization of a characteristic class, but one can show the two are equivalent by an argument similar to that of Theorem 2.17.

Remark 2.50. Definition 2.48 for $k=0$ was introduced earlier, by Chen-Han-Zhang [CHZ11, Definition 3.1], and is sometimes just called a string^c structure. On a spin^c vector bundle $V\to X$, this structure is obstructed by λ^c from Definition 2.6. Thus at least a priori this structure is stronger than a string^h structure, which only requires $\square_{ku}(\lambda^c)$ to vanish. In particular, a string^c structure induces a string^h structure.

See Sati [Sat11a, Sat11b] for some applications of this structure in physics.

Remark 2.51. There exist spin^c vector bundles $V \to X$ such that $\lambda^c(-V) \neq -\lambda^c(V)$, which means that a string^c structure on V is not equivalent data to a string^c structure on $-V$. Thus tangential and normal string^c structures are not equivalent. This in particular implies the Thom spectra classifying tangential and normal string^c structures do not have E_∞-ring spectrum structures corresponding on bordism groups to direct product.

We will compare Definition 2.48 with notions of spin^c structures on loop spaces. Before doing so, we recall from [McL92] the analogous story for string manifolds and spin structures on their loop spaces.

Let $\mathcal{G}$ be a *Fréchet Lie group* – that is, a group that is a Fréchet manifold, such that multiplication and inversion are smooth. Brylinski [Bry00, Proposition 1.6] showed that the group of Fréchet Lie group central extensions

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{G} \longrightarrow 1, \tag{2.52}$$

such that $\widetilde{\mathcal{G}} \to \mathcal{G}$ is a principal U_1-bundle, is naturally isomorphic to the *Segal-Mitchison cohomology group* [Seg70, Seg75] $H^2_{\mathrm{SM}}(\mathcal{G}; \mathrm{U}_1)$. If G is a (finite-dimensional) Lie group, then LG is a Fréchet Lie group, and if G is compact, then there is a canonical isomorphism due to Brylinski-McLaughlin [BM94]

$$H^2_{\mathrm{SM}}(LG; \mathrm{U}_1) \xrightarrow{\cong} H^4(BG; \mathbb{Z}). \tag{2.53}$$

See also [ADH21, Chapter 23].

In particular, if G is connected, simple, and simply connected, there is a canonical isomorphism $H^4(BG;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$. The central extension of LG classified by $1 \in \mathbb{Z}$ is denoted $\widehat{LG}$, and is the *universal central extension*: for any abelian Lie group A, any Fréchet Lie group central A-extension $\widetilde{\mathcal{G}} \to LG$ which is a principal A-bundle is isomorphic to an associated bundle

$$\widetilde{\mathcal{G}} \cong \widehat{LG} \times_{\mathrm{U}_1} A \tag{2.54}$$

for some Lie group homomorphism $\mathrm{U}_1 \to A$ [PS86, Chapter 4].

Now let $G = \mathrm{Spin}_n$, and assume $n \geq 5$ so that G is simple and simply connected, and we have the universal central extension $\widehat{L\mathrm{Spin}_n}$ of $L\mathrm{Spin}_n$ by U_1. For any spin manifold M, the frame bundle of LM lifts canonically to an $L\mathrm{Spin}_n$-bundle $LP \to LM$.

Definition 2.55 (McLaughlin [McL92, §1])**.** A *spin structure on* LM is a lift of LP to a principal $\widehat{L\mathrm{Spin}_n}$-bundle $\widehat{LP} \to LM$.

See also Killingback [Kil87] and Witten [Wit88, §3].

Theorem 2.56 (McLaughlin [McL92])**.** *If M has a string structure, then LM has a spin structure.*

Remark 2.57. Pilch-Warner [PW88, §3] showed the converse to Theorem 2.56 is not true, but versions of the converse with additional hypotheses do hold; see McLaughlin (*ibid.*), Kuribayashi [Kur96], Kuribayashi-Yamaguchi [KY98], Stolz-Teichner [ST04, ST05], Waldorf [Wal10, Wal12a, Wal12b, Wal15, Wal16a, Wal16b], Kottke-Melrose [KM13], Capotosti [Cap16], and Ludewig [Lud23]. See also Waldorf [Wal23] for an overview.

In a parallel manner, we would expect that for M a string^h manifold, there is a spin^c structure on LM. Since Spin^c_n is neither simple nor simply connected, the story is more complicated – there is not a universal central extension by U_1, and we will have to care about a $\mathbb{Z}$ worth of central extensions, corresponding to the level of the string^c structure in Definition 2.48.

Definition 2.58. Let $k\in\mathbb Z$. The Fréchet Lie group $\widehat{L_k\mathrm{Spin}_n^c}$ is the central extension of $L\mathrm{Spin}_n^c$ by $\mathrm U_1$ which, under the isomorphism $H^2_{\mathrm{SM}}(L\mathrm{Spin}_n^c;\mathrm U_1)\cong H^4(B\mathrm{Spin}_n^c;\mathbb Z)$ from (2.53), is identified with the class $\lambda^c-kc_1^2$.

Huang-Han-Duan [HHD21] define these groups in a different but equivalent way.

Naturality of (2.53) implies that the pullback of the central extension $\widehat{L_k\mathrm{Spin}_n^c}\to L\mathrm{Spin}_n^c$ along the inclusion $L\mathrm{Spin}_n\to L\mathrm{Spin}_n^c$ is the universal central extension of $L\mathrm{Spin}_n$. So even though we do not have a universal central extension in the spin^c setting, we favor these central extensions out of the $\mathbb Z^2$ of all possible central extensions.

For any spin^c manifold M, the frame bundle on LM canonically lifts to an $L\mathrm{Spin}_n^c$-bundle $LQ\to LM$.

Definition 2.59. Let M be a spin^c manifold. A *level* $(2k+1)$ *spin^c structure on the loop space* LM is a lift of $LQ\to LM$ to a principal $\widehat{L_k\mathrm{Spin}_n^c}$-bundle $\widehat{LQ}\to LM$.

Again, this definition is different but equivalent to Huang-Han-Duan's notion of a *weak string^c structure of level* $(2k+1)$ [HHD21, Definition 4.1].

Now we can refine our earlier question: if string^h is to string as spin^c is to spin, does the loop space of a string^h manifold have a level $2k+1$ spin^c structure for some k? We were surprised to obtain a negative answer.

Theorem 2.60. *There are closed string^h manifolds M such that LM is not spin^c for* any *choice of level.*

To prove this we will use a characteristic-class criterion for a loop space having a spin^c structure of a given level.

Lemma 2.61. *Let A be an E_1-space. Then there is a natural homotopy equivalence $LA\simeq A\times\Omega A$.*

Here ΩA is the space of loops in A based at the identity. See [Zil77, Agu81, Hai21] for proofs and generalizations of Lemma 2.61.

Definition 2.62. Recall that the Serre spectral sequence for the fibration $G\to EG\to BG$ defines a transgression map $\tau\colon H^4(BG;\mathbb Z)\to H^3(G;\mathbb Z)$.

(1) Let $c\coloneqq\tau(c_1)\in H^1(\mathrm{Spin}^c;\mathbb Z)$.
(2) Let $\mu^c\coloneqq\tau(\lambda^c)\in H^3(\mathrm{Spin}^c;\mathbb Z)$.

Lemma 2.61 implies a homotopy equivalence

$$BL\mathrm{Spin}^c\simeq B\mathrm{Spin}^c\times B\Omega\mathrm{Spin}^c\simeq B\mathrm{Spin}^c\times\mathrm{Spin}^c, \tag{2.63}$$

so the Künneth formula tells us that the classes c and μ^c, as well as their products with classes in $H^*(B\mathrm{Spin}^c;\mathbb Z)$, define integer-valued cohomology classes for $BL\mathrm{Spin}^c$, and therefore by pullback for $BL\mathrm{Spin}_n^c$ for all n.

Proposition 2.64 (Huang-Han-Duan [HHD21, Remark 4.3])**.** *Let M be a spin^c manifold. Then LM has a spin^c structure of level $2k+1$ if and only if $\mu^c(LM)-2kc(LM)c_1(LM)=0$ in $H^3(LM;\mathbb Z)$.*

Definition 2.65. The *loop transgression* map $\nu\colon H^*(M;\mathbb Z)\to H^{*-1}(LM;\mathbb Z)$ is the composition $\pi_!\circ\mathrm{ev}^*$, where $\mathrm{ev}\colon S^1\times LM\to M$ is the evaluation $(x,\gamma)\mapsto\gamma(x)$ and $\pi_!$ is integration over S^1.

This is different from the map τ from Definition 2.62!

Proposition 2.66 (Huang-Han-Duan [HHD21, §2.4])**.** *In $H^*(BL\mathrm{Spin}^c;\mathbb{Z})$, $\nu(\lambda^c) = \mu^c$ and $\nu(c_1) = c$. Moreover, $\nu(xy) = \nu(x)y + (-1)^{|x|}\nu(y)x$.*

Corollary 2.67 (Huang-Han-Duan [HHD21, Theorem 5.1])**.** *If M is strong string^c of level $2k+1$, then LM has a spin^c structure of level $2k+1$.*

Proof. Since M is strong string^c of level $2k+1$, $\lambda^c(M) - kc_1(L)^2 = 0$, where $L \to M$ is the determinant line bundle of the associated spin^c structure. Thus $\nu(\lambda^c(M) - kc_1(L)^2) = 0$ in $H^3(LM;\mathbb{Z})$. By Proposition 2.66, this means $\mu^c(LM) - 2c(LM)c_1(LM) = 0$, which by Proposition 2.64 implies LM has a spin^c structure of level $2k+1$. ☐

Proof of Theorem 2.60. Let $M := \mathbb{CP}^m \times \mathbb{CP}^n$ for $m, n \geq 3$; since M is complex, then by Proposition 2.32 M has a string^h structure. We will show that there is no $k \in \mathbb{Z}$ such that $\mu^c(LM) - 2kc(LM)c_1(LM) = 0$, so that by Proposition 2.64 LM does not have a spin^c structure for any level. Let $x \in H^1(M;\mathbb{Z})$ be the first Chern class of the first projective space factor and y be the first Chern class of the second projective space factor, so that

$$p_1(M) = (m+1)x^2 + (n+1)y^2 \tag{2.68}$$

by the Whitney sum formula for p_1.[5] The determinant line bundle L for this complex structure satisfies $c_1(L) = c_1(M) = (m+1)x + (n+1)y$, so

$$\begin{aligned} p_1(M) - (2k+1)c_1(L)^2 &= (m+1)^2x^2 + (n+1)y^2 - (3k+1)((m+1)x + (n_1)y) \\ &= 2k(m+1)x^2 - 2k(n+1)y^2 - (4k+2)(m+1)(n+1)xy. \end{aligned} \tag{2.69}$$

Since $2(\lambda^c - kc_1^2) = p_1 - (2k+1)c_1^2$, this implies

$$\lambda^c(M) - kc_1(L)^2 = k(m+1)x^2 - k(n+1)y^2 - (2k+1)(m+1)(n+1)xy. \tag{2.70}$$

By Proposition 2.66, we want to show that the loop transgression of (2.70) does not vanish for any $k \in \mathbb{Z}$, as this will imply that LM does not have a spin^c structure of any level.

First assume $k \neq 0$. In this case, it suffices to pull back across the standard inclusion $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^m \times \mathbb{CP}^n$, which on cohomology sends $x \mapsto x$ and $y \mapsto 0$. That is, because the loop transgression map is natural, showing that $\nu(k(m+1)x^2) \neq 0$ in $H^3(L\mathbb{CP}^m;\mathbb{Z})$ implies that the transgression of (2.70) does not vanish.

We will compare the loop transgression maps on $\mathbb{CP}^m$ and $\mathbb{CP}^\infty$. Since $m \geq 3$, the inclusion $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^\infty$ is at least 7-connected. The natural isomorphism $\pi_k(X) \simeq \pi_{k-1}(\Omega X)$ thus tells us that $\Omega\mathbb{CP}^m \to \Omega B\mathrm{U}_1$ is at least 6-connected. For any space X, there is a natural fibration $\Omega X \to LX \to X$; combining these two connectedness estimates with the long exact sequence of the fibration, we learn $L\mathbb{CP}^m \to LB\mathrm{U}_1$ is also at least 6-connected. Naturality of the loop transgression map gives us a commutative diagram

$$\begin{array}{ccc} H^4(\mathbb{CP}^\infty;\mathbb{Z}) & \xrightarrow{\cong} & H^4(\mathbb{CP}^m;\mathbb{Z}) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ H^3(L\mathbb{CP}^\infty;\mathbb{Z}) & \xrightarrow{\cong} & H^3(L\mathbb{CP}^m;\mathbb{Z}), \end{array} \tag{2.71}$$

[5]As we noted during the proof of Lemma 2.22, the first Pontrjagin class satisfies the Whitney sum formula for oriented vector bundles.

and since the maps $\mathbb{CP}^m \to \mathbb{CP}^\infty$ and $L\mathbb{CP}^m \to L\mathbb{CP}^\infty$ are at least 6-connected, the maps on H^4 and H^3 are isomorphisms. Since $k \neq 0$, then to show $\nu(k(m+1)x^2) \neq 0$ in $H^3(L\mathbb{CP}^m;\mathbb{Z})$, it suffices to show that the transgression map $H^4(\mathbb{CP}^\infty;\mathbb{Z}) \to H^3(L\mathbb{CP}^\infty;\mathbb{Z})$ is injective. This we know: by Proposition 2.66 (then pulling back along $\mathrm{U}_1 \hookrightarrow \mathrm{Spin}^c$), $\nu(c_1^2) = 2cc_1 \neq 0$.

It remains to rule out $k = 0$. For this, let $\ell \coloneqq \min(m,n) \geq 2$ and consider the diagonal embedding $\Delta\colon \mathbb{CP}^\ell \hookrightarrow \mathbb{CP}^\ell \times \mathbb{CP}^\ell \hookrightarrow \mathbb{CP}^m \times \mathbb{CP}^n$. Under this map, both x and y pull back to $z \coloneqq c_1(\mathbb{CP}^\ell) \in H^2(\mathbb{CP}^\ell;\mathbb{Z})$, so setting $k = 0$ in (2.70) and pulling back, we want to show that $\nu(-(m+1)(n+1)z^2) \neq 0$ in $H^3(L\mathbb{CP}^\ell;\mathbb{Z})$. Since $\ell \geq 2$, the rest of this argument is the same as that of the previous paragraph. ☐

3. Orienting $tmf_1(n)$

In this section we produce string^h orientations of $tmf_1(n)$ in Theorem 3.7. We start by introducing *TMF* and *Tmf* with level structure, and from there introduce $tmf_1(n)$.

The spectrum of *(periodic) topological modular forms TMF* is the global sections of a sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the étale site of the moduli stack of elliptic curves $\mathcal{M}_{ell}$, that is $TMF = \mathcal{O}^{top}(\mathcal{M}_{ell})$. The homotopy ring $\pi_{2*}(TMF)$ (i.e. the same ring with degrees doubled) is rationally[6] isomorphic to the ring

$$\widetilde{\mathrm{MF}}[\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z}] \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm}]/(c_4^3 - c_6^2 - 1728\Delta), \quad |c_4| = 9, \quad |c_6| = 12, \quad |\Delta| = 12 \tag{3.1}$$

of weakly holomorphic integral modular forms. The homotopy groups of *TMF* are periodic with period 576.

The sheaf $\mathcal{O}^{top}$ extends to define a sheaf on the étale site of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$ of $\mathcal{M}_{ell}$, and the global sections of $\mathcal{O}^{top} \to \overline{\mathcal{M}}_{ell}$ are a spectrum *Tmf* which is neither periodic nor connective, called *non-periodic nonconnective topological modular forms* or *mixed Tmf*. The homotopy ring of *Tmf* is closely related to the ring of holomorphic integral modular forms

$$\mathrm{MF}(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)\,. \tag{3.2}$$

There is a map $\pi_{2*}(Tmf) \to \mathrm{MF}(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})$, and after inverting 6, this is an isomorphism *but only in nonnegative degrees*. Therefore one defines the connective cover $tmf \coloneqq \tau_{\geq 0} Tmf$, so that there is an isomorphism $\pi_{2*}(tmf) \cong \mathrm{MF}(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z}) \otimes \mathbb{Z}[1/6]$ in all degrees.

By considering moduli spaces with a little extra structure, one obtains interesting variants of *TMF* and *Tmf*.

Definition 3.3 (Hill-Lawson [HL16])**.** Let $n \geq 1$, $\mathcal{M}_1(n)$ denote the moduli stack of elliptic curves with a chosen point of order n, and $\overline{\mathcal{M}}_1(n)$ be the Deligne-Mumford compactification of $\mathcal{M}_1(n)$. The global sections of the pullback of $\mathcal{O}^{top}$ to $\mathcal{M}_1(n)[1/n]$, resp. to the log-étale site of $\overline{\mathcal{M}}_1(n)[1/n]$ are denoted $TMF_1(n)$, resp. $Tmf_1(n)$.

Hill-Lawson also define analogous series of spectra $TMF(n)$ and $TMF_0(n)$, and $Tmf(n)$ and $Tmf_0(n)$. Prior to their work, various examples of these families of spectra were introduced by Behrens [Beh06, Beh07], Mahowald-Rezk [MR09], and Stojanoska [Sto12].

Both $TMF_1(n)$ and $Tmf_1(n)$ are E_∞-ring spectra by construction (in fact, E_∞ *TMF*-, resp. *Tmf*-algebra spectra), and there is a ring spectrum map $Tmf_1(n) \to TMF_1(n)$. There is a rational isomorphism from $\pi_{2*}(TMF_1(n))$ to the ring $\mathrm{MF}(\Gamma_1(n),\mathbb{Z}[\frac{1}{n}])$ of weakly holomorphic modular forms for the congruence subgroup $\Gamma_1(n) \subset \mathrm{SL}_2(\mathbb{Z})$, also called *integral modular forms at level n*.

[6] In fact, these two rings are isomorphic after inverting 6.

However, this analogy does not continue to mixed $\mathit{Tmf}_1(n)$: the ring $\pi_*(\mathit{Tmf}_1(n)) \otimes \mathbb{Q}$ and the ring of holomorphic modular forms for $\Gamma_1(n)$ tensored with $\mathbb{Q}$ are not always isomorphic, even restricted to nonnegative degrees. This means that the connective cover of $\mathit{Tmf}_1(n)$ is not always the right analogue of tmf.

Fortunately, the discrepancy is not huge: the sole discrepancy is that $\pi_1(\mathit{Tmf}_1(n))$ may be nonzero, and frequently it is 0, including for all $n \leq 22$ (see, for example, [Mei22, Remark 3.14]). Meier [Mei23], following a general procedure of Lawson [Law15] to remove π_1, constructs for all $n \geq 2$ an E_∞-ring spectrum $\mathit{tmf}_1(n)$ with $\pi_1(\mathit{tmf}_1(n)) = 0$ and a map $\mathit{tmf}_1(n) \to \mathit{Tmf}_1(n)$ which is an isomorphism for $n = 0$ and $n \geq 2$, implying $\pi_{2*}(\mathit{tmf}_1(n))$ is rationally isomorphic to the ring of holomorphic modular forms of level n in all degrees. For this paper, $\mathit{tmf}_1(n)$ always refers to Meier's construction, whether or not this is the connective cover of $\mathit{Tmf}_1(n)$.

Remark 3.4. Lawson-Naumann [LN14] constructed $\mathit{tmf}_1(3)$ 2-locally as an E_∞-ring spectrum before Meier's work, and identified it with $BP\langle 2\rangle$; in this case, $\pi_1(\mathit{Tmf}_1(3))$ vanishes. See also Hill-Meier [HM17].

Since topological modular forms with level structure were first systematically studied by Hill-Lawson [HL16], it has been an open question to orient them by a Thom spectrum which is a better approximation than MU or $\mathit{MString}$: see, for example [HL16, §1]. Wilson [Wil15] provided some answers to this question, but does not answer it for $\mathit{tmf}_1(n)$. Recently, Devalapurkar [Dev22] answered this for $\mathit{tmf}_1(3)$ using forthcoming work of Hahn-Senger:

Theorem 3.5 (Devalapurkar [Dev22, Theorem 5])**.** *There is a map of E_∞-ring spectra $\sigma_D \colon \mathit{MString}^h_{(2)} \to \mathit{tmf}_1(3)_{(2)}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathit{MString}_{(2)} & \xrightarrow{\sigma} & \mathit{tmf}_{(2)} \\ \downarrow & & \downarrow \\ \mathit{MString}^h_{(2)} & \xrightarrow{\sigma_D} & \mathit{tmf}_1(3)_{(2)}. \end{array} \tag{3.6}$$

We will lift this to arbitrary n:

Theorem 3.7. *For all $n \geq 2$, there are maps of E_∞-ring spectra*

$$\sigma_1(n) \colon \mathit{MString}^h[1/n] \longrightarrow \mathit{tmf}_1(n) \tag{3.8}$$

such that the composition of $\sigma_1(n)$ with the complex orientation on $\mathit{MString}^h$ constructed in Theorem 2.41, Item 3, is the complex orientation of $\mathit{tmf}_1(n)$ constructed in Senger [Sen23, Theorem 1.7]*, and there is a commutative square*

$$\begin{array}{ccc} \mathit{MString}[1/n] & \xrightarrow{\sigma} & \mathit{tmf}[1/n] \\ \downarrow & & \downarrow \\ \mathit{MString}^h[1/n] & \xrightarrow{\sigma_1(n)} & \mathit{tmf}_1(n)\,. \end{array} \tag{3.9}$$

Remark 3.10. Our proof uses completely different methods than Devalapurkar's, and it would be interesting to know whether there is a 2-local equivalence $\sigma_D \simeq \sigma_1(3)$.

The E_∞-ring map $\mathit{MString} \to \mathit{MString}^h$ is the one from Theorem 2.41, and the map $\mathit{tmf}[1/n] \to \mathit{tmf}_1(n)$ is induced by the inclusion of the moduli stack of elliptic curves with a chosen point of order 3 into the moduli stack of all elliptic curves. We prove Theorem 3.7 by first showing it for neither-connective-nor-periodic $\mathit{Tmf}_1(n)$, then lifting to $\mathit{tmf}_1(n)$.

Proposition 3.11. *The analogue of Theorem 3.7, but with $\mathit{Tmf}_1(n)$ in place of $\mathit{tmf}_1(n)$, is true.*

Proof of Proposition 3.11. Throughout this proof we invert n.

We will repeatedly use the fact that if A, B, C, and D are E_∞-ring spectra, and $f\colon A \to C$ and $g\colon B \to D$ are E_∞-ring homomorphisms, then $f \wedge g\colon A \wedge B \to C \wedge D$ has a canonical E_∞-ring homomorphism structure.

Specifically, use this fact to daisy-chain together the following E_∞-ring maps:

(1) the E_∞ equivalence $\mathit{MString}^h \simeq \mathit{MString} \wedge \mathit{MU}$ we established in Theorem 2.43,
(2) the σ-orientation $\sigma\colon \mathit{MString} \to \mathit{tmf}$ constructed by Ando-Hopkins-Rezk [AHR10, Theorem 12.3],
(3) the complex orientation $M(n)\colon \mathit{MU} \to \mathit{tmf}_1(n) \to \mathit{Tmf}_1(n)$ due to Senger [Sen23, Theorem 1.7],[7]
(4) the unit map $A(n)\colon \mathit{tmf} \to \mathit{Tmf}_1(n)$ of the E_∞-tmf-algebra structure on $\mathit{Tmf}_1(n)$ obtained by Hill-Lawson [HL16, Theorem 6.1].

Thus, the following composition is a homomorphism of E_∞-ring spectra.

(3.12)
$$\mathit{MString}^h \underset{(1)}{\xrightarrow{\simeq}} \mathit{MString} \wedge \mathit{MU} \underset{(2,\,3)}{\xrightarrow{\sigma\wedge M(n)}} \mathit{tmf} \wedge \mathit{Tmf}_1(n) \underset{(4)}{\xrightarrow{A(n)\wedge \mathrm{id}}} \mathit{Tmf}_1(n) \wedge \mathit{Tmf}_1(n) \xrightarrow{\mu} \mathit{Tmf}_1(n),$$

where the final map is multiplication. □

Proposition 3.13. *Let R be a connective E_∞-ring spectrum with isomorphisms $\psi\colon \pi_0(R) \xrightarrow{\cong} \mathbb{Z}$ and $\pi_1(R) = 0$. Given a morphism $f\colon R \to \tau_{\ge 0}\mathit{Tmf}_1(n)$ of E_∞-ring spectra, there is a canonical lift of f to a map $\widetilde{f}\colon R \to \mathit{tmf}_1(n)$.*

Proof. Meier [Mei23, Proposition 2.9, Lemmas 2.10 and 2.11] constructs a pullback square of E_∞-ring spectra

$$\begin{array}{ccc} \mathit{tmf}_1(n) & \longrightarrow & H\pi_0(\mathit{Tmf}_1(n)) \\ \downarrow & \lrcorner & \downarrow \varphi \\ \tau_{\ge 0}\mathit{Tmf}_1(n) & \xrightarrow{\tau_{\le 1}} & \tau_{0:1}\mathit{Tmf}_1(n), \end{array} \tag{3.14}$$

and shows φ is the unique E_∞-ring map $H\pi_0(\mathit{Tmf}_1(n)) \to \tau_{0:1}\mathit{Tmf}_1(n)$ inducing a ring isomorphism on π_0. Therefore it suffices to produce E_∞-ring maps $a\colon R \to \tau_{\ge 0}\mathit{Tmf}_1(n)$ and $b\colon R \to H\pi_0(\mathit{Tmf}_1(n))$ and an identification of their compositions with the maps $\tau_{\le 1}$, resp. φ in (3.14):

$$\begin{array}{ccc} R & \xrightarrow{b} & H\pi_0(\mathit{Tmf}_1(n)) \\ {\scriptstyle a}\downarrow & & \downarrow \varphi \\ \tau_{\ge 0}\mathit{Tmf}_1(n) & \xrightarrow{\tau_{\le 1}} & \tau_{0:1}\mathit{Tmf}_1(n). \end{array} \tag{3.15}$$

Choose $a = f$ and let b be the composition

$$R \xrightarrow{\tau_{\ge 0}} H\pi_0(R) \xrightarrow{\psi} H\mathbb{Z} \xrightarrow{\mathbf{1}} H\pi_0\mathit{Tmf}_1(n), \tag{3.16}$$

where $\mathbf{1}$ is the unit. To provide an identification $\tau_{\le 1} \circ a \simeq \varphi \circ b$, first use that the target is 1-truncated, so that both compositions canonically factor through $\tau_{\le 1}R$. Since R is connective and

[7] Absmeier [Abs21, Theorem 1] uses different methods to construct E_∞-orientations $\mathit{MU}[\zeta_n, 1/n] \to \mathit{Tmf}_1(n)$, where ζ_n is a primitive n^{th} root of unity; we use Senger's orientation to avoid ζ_n.

$\pi_1(R) = 0$, the 0-truncation map $\tau_{\leq 1}R \to \tau_{\leq 0}R \simeq H\pi_0(R)$ is an equivalence of E_∞-ring spectra. Thus we without loss of generality replace R with $H\pi_0(R)$.

For both $\tau_{\leq 1} \circ a$ and $\varphi \circ b$, the induced map on π_0 is the localization $\mathbb{Z} \to \mathbb{Z}[1/n]$, using the specified isomorphism $\psi\colon \pi_0(R) \xrightarrow{\cong} \mathbb{Z}$ and Meier's identification [Mei23, Lemma 2.11] $\pi_0(\mathit{Tmf}_1(n)) \cong \mathbb{Z}[1/n]$. As the ring homomorphism $\mathbb{Z} \to \mathbb{Z}[1/n]$ is étale,[8] a theorem of Lurie [Lur17, Theorem 7.5.0.6] shows that this map on π_0 lifts uniquely to an E_∞-ring map $H\pi_0(R) \to \tau_{0:1}\mathit{Tmf}_1(n)$ with a contractible space of automorphisms. Thus $\tau_{\leq 1} \circ a$ and $\varphi \circ b$ are canonically equivalent up to contractible data and we may conclude. ☐

Now proving Theorem 3.7 amounts to showing $\mathit{MString}^h$ satisfies the hypotheses of Proposition 3.13.

Proof of Theorem 3.7. Thom spectra of rank-zero virtual vector bundles, such as $\mathit{MString}^h$, are connective. Connectivity provides a canonical lift of the E_∞-ring map $\mathit{MString}^h \to \mathit{Tmf}_1(n)$ constructed in Proposition 3.11 to an E_∞-ring map $\mathit{MString}^h \to \tau_{\geq 0}\mathit{Tmf}_1(n)$. Therefore to lift to $\mathit{tmf}_1(n)$, it suffices to show $\Omega_0^{\mathrm{String}^h} \cong \mathbb{Z}$ and $\Omega_1^{\mathrm{String}^h} \cong 0$, then invoke Proposition 3.13.

If M is a spin^c manifold of dimension 3 or below, $\lambda^c(M) = 0$, because it is an element of $H^4(M;\mathbb{Z}) \cong 0$. Therefore M admits a canonical string^h structure: lift $\lambda^c(M)$ to $0 \in ku^4(M)$. This implies that for $k \leq 2$, $\Omega_k^{\mathrm{String}^h} \to \Omega_k^{\mathrm{Spin}^c}$ is an isomorphism, and $\Omega_0^{\mathrm{Spin}^c} \cong \mathbb{Z}$ and $\Omega_1^{\mathrm{Spin}^c} \cong 0$. ☐

3.0.1. *Real-equivariance.* We briefly discuss a Real-equivariant generalization of Theorem 3.7, and as with everything, we begin with the spin^c story.

Complex conjugation defines a $\mathbb{Z}/2$-action on complex K-theory; the resulting $\mathbb{Z}/2$-equivariant spectrum is called *Real(-equivariant) K-theory* and denoted KR [Ati66]. The underlying spectrum of KR is KU, and the $\mathbb{Z}/2$-(homotopy) fixed point spectrum is KO. KR is *cofree* (see, e.g., [HZ20]), meaning that its structure as a $\mathbb{Z}/2$-E_∞-ring spectrum is induced from a $\mathbb{Z}/2$-action on the spectrum KU by E_∞-ring maps.

Landweber [Lan67, Lan68], Fujii [Fuj76], Araki [Ara79a, Ara79b], and Araki-Murayama [AM78] constructed a $\mathbb{Z}/2$-equivariant-ring spectrum MR whose underlying spectrum is MU with $\mathbb{Z}/2$-action by complex conjugation. MR also has an E_∞-structure: see Hill-Hopkins-Ravenel [HHR16, §B.12].

Definition 3.17 ([AM78, Ara79a])**.** A *Real-orientation* of a $\mathbb{Z}/2$-ring spectrum E is a homomorphism of $\mathbb{Z}/2$-ring spectra $\mathit{MR} \to E$.

Araki-Murayama [AM78, §7] proved KR is Real-oriented; the Real-orientation may be chosen to be a $\mathbb{Z}/2$-E_∞-ring map $\mathit{cf}_{\mathbb{R}}$.

Nonequivariantly, the complex orientation $\mathit{MU} \to \mathit{KU}$ constructed by Conner-Floyd [CF66, §5] factors through E_∞-ring maps $u\colon \mathit{MU} \to \mathit{MSpin}^c$ and $\widehat{A}\colon \mathit{MSpin}^c \to \mathit{KU}$; the former can be constructed similar to the methods we used in Theorem 2.41 and the latter is due to Joachim [Joa04]. Halladay-Kamel [HK24] recently generalized this to the Real-equivariant setting.

Theorem 3.18 (Halladay-Kamel [HK24])**.** *There is a $\mathbb{Z}/2$-E_∞-ring spectrum $\mathit{MSpin}^c_{\mathbb{R}}$ and $\mathbb{Z}/2$-E_∞-ring maps $u_{\mathbb{R}}\colon \mathit{MR} \to \mathit{MSpin}^c_{\mathbb{R}}$ and $\widehat{A}_{\mathbb{R}}\colon \mathit{MSpin}^c_{\mathbb{R}} \to \mathit{KR}$ such that*

(1) *the underlying spectrum of $\mathit{MSpin}^c_{\mathbb{R}}$ is MSpin^c,*
(2) *forgetting to underlying spectra, $\widehat{A}_{\mathbb{R}}$, resp. $u_{\mathbb{R}}$ restrict to $\widehat{A}$, resp. u, and*

[8] Meier [Mei23] works in a more general setting where a E_∞-ring spectrum R has $\pi_0 R$ an étale extension of $\mathbb{Z}_S$. The étale condition is not strictly necessary for the purposes of our proof.

(3) *the Real orientations* $\widehat{A}_{\mathbb{R}} \circ u_{\mathbb{R}}$ *and* $cf_{\mathbb{R}}$ *are equivalent.*

That is, Halladay-Kamel answer the question, "what is to KR as $MSpin^c$ is to KU?"

We prove an analogue of Theorem 3.18 for topological forms with level structure in Theorem 3.22. However, the version of the theorem stated there is slightly weaker than the naïve generalization of Theorem 3.7: the orientation lands in $Tmf_1(n)_{\mathbb{R}}$, rather than $tmf_1(n)_{\mathbb{R}}$, as aspects of the lifting argument from $\tau_{\geq 0} Tmf_1(n)$ to $tmf_1(n)$ are tricky to make equivariant. Moreover, we did not construct an E_∞ map, only a map of $\mathbb{Z}/2$-ring spectra. Ultimately this is because there is not yet a construction of a $\mathbb{Z}/2$-E_∞-ring map $MR \to tmf_1(n)_{\mathbb{R}}$ (see [Mei23, Sen23]). We predict that such an E_∞ refinement exists.

The following theorem is a combination of work of Hill-Meier [HM17] and Meier [Mei23].

Theorem 3.19. *There are* $\mathbb{Z}/2$-E_∞*-ring spectra* $tmf_1(n)_{\mathbb{R}}$, $Tmf_1(n)_{\mathbb{R}}$, *and* $TMF_1(n)_{\mathbb{R}}$ *whose underlying spectra are* $tmf_1(n)$, $Tmf_1(n)$, *and* $TMF_1(n)$ *respectively and whose* $\mathbb{Z}/2$*-fixed point spectra are* $tmf_0(n)$, $Tmf_0(n)$, *and* $TMF_0(n)$ *respectively. The* E_∞*-ring spectrum maps*

$$MU[1/n] \xrightarrow{M(n)} tmf_1(n) \longrightarrow Tmf_1(n) \longrightarrow TMF_1(n) \tag{3.20a}$$

lift to $\mathbb{Z}/2$*-ring maps,*

$$MR[1/n] \xrightarrow{M_{\mathbb{R}}(n)} tmf_1(n)_{\mathbb{R}} \longrightarrow Tmf_1(n)_{\mathbb{R}} \longrightarrow TMF_1(n)_{\mathbb{R}}, \tag{3.20b}$$

the last two of which are E_∞.

Definition 3.21. Let $MString^h_{\mathbb{R}} \coloneqq MR \wedge MString$, where $MString$ is given the cofree $\mathbb{Z}/2$-E_∞-ring structure arising from the trivial action.

Thus $MString^h_{\mathbb{R}}$ is a $\mathbb{Z}/2$-E_∞-ring spectrum whose underlying spectrum is $MString^h$.

Theorem 3.22. *For all* $n \geq 2$, *there is a map of* $\mathbb{Z}/2$*-ring spectra*

$$\sigma_1(n)_{\mathbb{R}} \colon MString^h_{\mathbb{R}}[1/n] \longrightarrow Tmf_1(n)_{\mathbb{R}} \tag{3.23}$$

which on underlying spectra is $\sigma_1(n)$ *composed with the usual map* $tmf_1(n) \to Tmf_1(n)$, *and such that the Real orientation* $M_{\mathbb{R}}(n) \colon MR[1/n] \to Tmf_1(n)$ *factors as a Real orientation* $v_{\mathbb{R}} \colon MR \to MString^h_{\mathbb{R}}$ *followed by the usual orientation* $MString^h \to Tmf_1(n)$. *On underlying spectra,* $v_{\mathbb{R}}$ *is* v.

Proof. The proof strategy is the same as for Proposition 3.11. To adapt that proof, we need the following data.

1. A refinement of the complex orientation $MU \to Tmf_1(n)$ to a Real orientation $MR[1/n] \to Tmf_1(n)_{\mathbb{R}}$, which is provided by Meier [Mei22, Theorem 3.6] (here Theorem 3.19).
2. An extension of the E_∞-ring map $\sigma \colon MString \to tmf$ to a map between the respective cofree $\mathbb{Z}/2$-E_∞-ring spectra associated to the trivial $\mathbb{Z}/2$-actions on $MString$ and tmf. By [BH15, §6.2.2], it suffices to show that σ is equivariant for the trivial $\mathbb{Z}/2$-actions on its domain and codomain, which is trivially true.
3. Lastly we need to refine the E_∞-ring map $tmf[1/n] \to Tmf_1(n)$ to a $\mathbb{Z}/2$-E_∞-ring map $tmf[1/n] \to Tmf_1(n)_{\mathbb{R}}$, where $tmf[1/n]$ is cofree, indued from the trivial $\mathbb{Z}/2$-action. Without loss of generality we may replace $tmf[1/n]$ by $Tmf[1/n]$, then precompose with the map $tmf[1/n] \to Tmf[1/n]$ (which refines to $\mathbb{Z}/2$-spectra in the same way as in the previous bullet point). Again by [BH15, §6.2.2], it suffices to show that $Tmf[1/n] \to Tmf_1(n)$ is $\mathbb{Z}/2$-equivariant for the trivial $\mathbb{Z}/2$-action on $Tmf[1/n]$ and the $\mathbb{Z}/2$-action on $Tmf_1(n)$ defined in [HM17, §4.1].

For this, we return to the moduli of elliptic curves. The $\mathbb{Z}/2$-action on $\mathit{Tmf}_1(n)$ is the map induced on global sections of $\mathcal{O}^{top}$ from a $\mathbb{Z}/2$-action on (the log-étale site of) $\overline{\mathcal{M}}_1(n)[1/n]$, the Deligne-Mumford compactification of the modulo stack of elliptic curves C with a chosen point x of order n (see Definition 3.3). This $\mathbb{Z}/2$-action sends $(C, x) \mapsto (C, -x)$; therefore the map $\overline{\mathcal{M}}_1(n)[1/n] \to \overline{\mathcal{M}}[1/n]$ forgetting x is $\mathbb{Z}/2$-equivariant with respect to the trivial action on $\overline{\mathcal{M}}[1/n]$. Taking sections of $\mathcal{O}^{top}$, we obtain the usual map $\mathit{Tmf}[1/n] \to \mathit{Tmf}_1(n)$, together with the fact that it is $\mathbb{Z}/2$-equivariant.

With these lifts in place, the construction of the map $\mathit{MString}^h_{\mathbb{R}} \to \mathit{Tmf}_1(n)_{\mathbb{R}}$ proceeds just as before. At the time of writing, the Real orientation of $\mathit{Tmf}_1(n)$ has not been refined to a $\mathbb{Z}/2$-E_∞-map (see [Sen23, Question 1.10]), so this construction is just a $\mathbb{Z}/2$-ring spectrum map. ☐

We would like to compare Theorem 3.22 with Halladay-Kamel's Real-equivariant lift of the Atiyah-Bott-Shapiro orientation. However, the constructions of $\mathit{MString}^h_{\mathbb{R}}$ and $\mathit{MSpin}^c_{\mathbb{R}}$ are difficult to relate, so we leave the comparison as a conjecture.

Lemma 3.24 (Hill-Meier [HM17])**.** *There is a $\mathbb{Z}/2$-E_∞-ring map $\Lambda\colon (\mathit{Tmf}_1(3)_{\mathbb{R}})_{(2)} \to KR_{(2)}$.*

Hill-Meier do not explicitly state Lemma 3.24 in this form, but they provide all the pieces, so we show how to assemble those pieces into a proof.

Proof. As noted above, KR is cofree, and $\mathit{Tmf}_1(3)_{\mathbb{R}}$ is also cofree [HM17, §4.1]. Hill-Meier (*ibid.*, §4.2), using a theorem of Hill-Lawson [HL16, Theorem 6.2], show that there is an E_∞-map of nonequivariant spectra $\widetilde{\Lambda}\colon \mathit{Tmf}_1(3)_{(2)} \to KU_{(2)}$ which is equivariant for the $\mathbb{Z}/2$-actions on $\mathit{Tmf}_1(3)_{(2)}$ and $KU_{(2)}$. Blumberg-Hill [BH15, §6.2.2] (see also [HM17, Theorem 2.4]) show that if $\phi\colon R \to S$ is a $\mathbb{Z}/2$-equivariant map of E_∞-ring spectra with respect to $\mathbb{Z}/2$-actions by E_∞-ring maps on R and S, then ϕ upgrades to a $\mathbb{Z}/2$-E_∞-ring map on the cofree $\mathbb{Z}/2$-E_∞-ring spectra built from R and S. Applying this to $\widetilde{\Lambda}$, we conclude. ☐

Question 3.25. *Throughout this question, implicitly 2-localize.*

Does there exist a $\mathbb{Z}/2$-E_∞-ring map $\mathit{MString}^h_{\mathbb{R}} \to \mathit{MSpin}^c_{\mathbb{R}}$ which on underlying spectra is the map $\mathit{MString}^h \to \mathit{MSpin}^c$ of Theorem 2.41 and such that the following diagram commutes?

$$\begin{array}{ccccc} MR & \xrightarrow{v_{\mathbb{R}}} & \mathit{MString}^h_{\mathbb{R}} & \xrightarrow{\sigma_1(n)_{\mathbb{R}}} & \mathit{Tmf}_1(3)_{\mathbb{R}} \\ & \searrow^{u_{\mathbb{R}}} & \downarrow{\scriptstyle \exists?} & & \downarrow{\scriptstyle \Lambda} \\ & & \mathit{MSpin}^c_{\mathbb{R}} & \xrightarrow{\widehat{A}_{\mathbb{R}}} & KR \end{array} \tag{3.26}$$

If such a map exists, it would also be nice to describe it geometrically, e.g. in terms of characteristic classes of $\mathbb{Z}/2$-equivariant vector bundles.

Another potential benefit of Real-equivariance would arise by taking fixed points. Halladay-Kamel [HK24, §4] study the fixed-point spectrum $(\mathit{MSpin}^c_{\mathbb{R}})^{\mathbb{Z}/2}$; it appears to be an unwieldy object, but it admits an E_∞-ring map $\widetilde{u}\colon \mathit{MSpin} \to (\mathit{MSpin}^c_{\mathbb{R}})^{\mathbb{Z}/2}$, so that one can form the composition

$$\mathit{MSpin} \xrightarrow{\widetilde{u}} (\mathit{MSpin}^c_{\mathbb{R}})^{\mathbb{Z}/2} \xrightarrow{(\widehat{A}_{\mathbb{R}})^{\mathbb{Z}/2}} (KR)^{\mathbb{Z}/2} \simeq KO, \tag{3.27}$$

and (*ibid.*, Corollary 6.14) this recovers the usual Atiyah-Bott-Shapiro orientation.

Because $(\mathit{Tmf}_1(n)_{\mathbb{R}})^{\mathbb{Z}/2} \simeq \mathit{Tmf}_0(n)$ [HM17, §4], one could try to generalize Halladay-Kamel's approach to orient $\mathit{Tmf}_0(n)$. Ultimately because $MR^{\mathbb{Z}/2}$ is complicated (though understood: see [HK01, GM17]), we expect $(\mathit{MString}^h_{\mathbb{R}})^{\mathbb{Z}/2}$ to not be easy to work with.

Proposition 3.28. *Let $\xi\colon B \to B\mathrm{O}$ be a tangential structure with two-out-of-three data such that there is an E_∞-ring map $\psi\colon M\xi \to (\mathit{MString}^h_{\mathbb R})^{\mathbb Z/2}$. Then by forming a composition analogous to (3.27), there is a canonical orientation $M\xi[1/n] \to \mathit{Tmf}_0(n)$.*

The most naïve generalization of Halladay-Kamel's construction leads to $\xi = \mathrm{String}$, and the String-orientation of $\mathit{Tmf}_0(n)$ is not new information. It would be interesting to understand whether a more careful use of Proposition 3.28 could be used to construct an orientation of $\mathit{Tmf}_0(n)$ by some nontrivial *MString*-algebra Thom spectrum, analogously to the String^h-orientation of $\mathit{tmf}_1(n)$. We note that Wilson [Wil15, Corollary 4.16] has produced an orientation $\mathit{MSpin}\langle w_4\rangle[1/3] \to \mathit{tmf}_0(3)$ (hence also to $\mathit{Tmf}_0(3)$), where $\mathrm{Spin}\langle w_4\rangle$ is the tangential structure which is a spin structure and a trivialization of the Stiefel-Whitney class w_4; we do not know whether Wilson's orientation factors through $(\mathit{MString}^h_{\mathbb R})^{\mathbb Z/2}[1/3]$ in Proposition 3.28.

3.1. Low-degree homotopy groups of $\mathit{MString}^h$.

In this subsection, we compute low-dimensional string^h bordism groups, and also calculate the effects of some of the orientations $\sigma_1(n)$ of the previous section on homotopy groups.

We will recall a few facts about the Brown-Peterson spectrum BP since many of the results in the remainder of this section utilize BP and its siblings $BP\langle n\rangle$ in their proofs. BP is obtained by localizing MU at a prime p, and then $MU_{(p)}$ is expressible as a wedge sum of suspensions of BP [BP66]. This direct-sum decomposition does not play especially well with the multiplication on MU.

Theorem 3.29 (Basterra-Mandell [BM13, Theorem 1.1])**.** *The wedge-sum decomposition of $MU_{(p)}$ into a sum of shifts of BP may be chosen so that the maps $MU_{(p)} \leftrightarrows BP$ splitting off the lowest-degree summand are E_4-algebra maps.*

Corollary 3.30. *Given an E_∞-ring homomorphism $f\colon MU \to E$, where E is p-local, we can precompose with the E_4-ring homomorphism $BP \to MU_{(p)}$ to obtain an E_4-ring homomorphism $\widetilde f\colon BP \to E$.*

It is not known whether one can strengthen this result to an E_n-splitting for some $n > 4$; Lawson [Law18, Remark 4.4.7] and Senger [Sen24, Theorem 1.3] have shown that it is not possible to do so for $n \geq 2(p+3)$.

The homotopy rings of BP and $BP\langle n\rangle$ are

$$\begin{aligned} BP_* &\cong \mathbb Z_{(p)}[v_1, v_2, \dots] \\ BP\langle n\rangle_* &\cong \mathbb Z_{(p)}[v_1, v_2, \dots, v_n]. \end{aligned} \tag{3.31}$$

In both cases $|v_i| = 2(p^i - 1)$. The map $BP \to BP\langle n\rangle$ sends $v_i \mapsto v_i$ for $1 \leq i \leq n$ and sends $v_i \mapsto 0$ for $i > n$.

Now we introduce the main results of this subsection. Our first result generalizes work of Hopkins-Mahowald (unpublished) and Devalapurkar [Dev19], who showed that the Ando-Hopkins-Rezk orientation $\sigma\colon \mathit{MString} \to \mathit{tmf}$ is surjective on homotopy groups.

Proposition 3.32. *The following maps are surjective on homotopy groups.*

$$\begin{aligned} \sigma_1(3)&\colon \mathit{MString}^h[1/3] \longrightarrow \mathit{tmf}_1(3) \\ \sigma_1(2)&\colon \mathit{MString}^h[1/2] \longrightarrow \mathit{tmf}_1(2). \end{aligned} \tag{3.33}$$

We will prove this as a consequence of Corollaries 3.47 and 3.53.

Proof. We begin with $\sigma_1(3)$. In Corollary 3.53, we will establish that the map $\widetilde{\sigma}_1(3)\colon BP \wedge MString \to tmf_1(3)_{(2)}$ obtained from Corollary 3.30 is 7-connected. Lawson-Naumann [LN12, Theorem 1.1] show that the generators of the homotopy ring $(tmf_1(3)_{(2)})_*$ are in degrees less than 7, so since $\widetilde{\sigma}_1(3)$ is a map of ring spectra, it is surjective on homotopy groups in all degrees.

Since $\sigma_1(3)$ factors through $\widetilde{\sigma}_1(3)$, we conclude that after localizing at 2, $\sigma_1(3)$ is surjective on homotopy groups. To finish, we argue we lift from $\mathbb{Z}_{(2)}$ to $\mathbb{Z}[1/3]$ by observing that $tmf_1(3)_*$ is a polynomial ring over $\mathbb{Z}[1/3]$ on two generators in degrees 2 and 6, and we already observed that these generators are in the image of $\sigma_1(3)$ up to multiplication by a number prime to 6. Thus it suffices to show that $\sigma_1(3)[1/6]$ hits the images of these generators in $tmf_1(3)[1/6]_*$, which can easily be checked with the Atiyah-Hirebruch spectral sequence because $MString^h \simeq MU \wedge MString$ and $tmf_1(3)$ lack torsion once 6 is inverted.

The argument for $tmf_1(2)$ is essentially the same, except using Corollary 3.47, which says that $\widetilde{\sigma}_1(2)$ is 11-connected, and the fact that $tmf_1(2)_*$ is a polynomial ring over $\mathbb{Z}[1/2]$ with generators in degrees below 9.[9] □

It would be interesting to generalize this to $tmf_1(n)$ for $n > 3$.

Theorem 3.34. *There is an isomorphism of graded abelian groups*

$$\Omega_*^{\mathrm{String}^h} \xrightarrow{\cong} \mathbb{Z}[x_2, x_4, x_6, x_8, y_8, x_{10}, x_{12}, y_{12}, x_{14}, \dots]/(\cdots) \tag{3.35}$$

where $|x_i| = |y_i| = i$ *and all generators and relations not listed are in degrees* 16 *and above. In degrees* 7 *and below, this map is a ring homomorphism.*

In degrees 8 through 15, (3.35) is an isomorphism after inverting 6.

We will prove Theorem 3.34 by first showing $\Omega_k^{\mathrm{String}^h}$ lacks p-torsion for $k \le 15$ for all primes p; then the theorem reduces to a rational calculation. As we will explain, the generators can be seen from the map from MU_* and $MString_*$:

- The map from MU_* hits x_2 , x_4, x_6, x_{12} and x_{14} for which the first three generators have descriptions as $\mathbb{CP}^1$, $\mathbb{CP}^2$, and a Milnor hypersurface $H_{22} \amalg \mathbb{CP}^2$.
- The map from $MString_*$ rationally hits y_8 and y_{12}.

Proposition 3.36. *Localized at a prime* $p \ge 5$, $(\Omega_*^{\mathrm{String}^h})_{(p)}$ *is torsion-free.*

Proof. With this assumption on p in place, both $MString$ and MU split as sums of shifts of BP.[10] Therefore

$$MString^h_{(p)} \simeq \bigvee_i \Sigma^{n_i} BP \wedge BP. \tag{3.37}$$

Quillen [Qui69] showed $\pi_*(BP \wedge BP)$ is torsion-free (see also [Wil82, Theorem 3.11]), so $\pi_*(MString^h)_{(p)}$ is also torsion-free. □

Lemma 3.38. *Suppose that* $\pi_*(BP \wedge tmf)$ *lacks* p*-torsion in degrees* 15 *and below. Then* $\Omega_*^{\mathrm{String}^h}$ *also lacks* p*-torsion in degrees* 15 *and below.*

Proof. Brown-Peterson showed that $MU_{(p)}$ is a wedge sum of copies of $\Sigma^{n_i} BP$ for $n_i \ge 0$ [BP66]. Thus it suffices to prove that $\pi_*(BP \wedge MString)$ lacks p-torsion in degrees 15 and below. Since the

[9] This fact appears in Hill [Hil07], where it is attributed to Hopkins-Mahowald (unpublished) and Behrens [Beh06].

[10] For MU this is a theorem of Brown-Peterson [BP66, Theorem 1.3]; for $MString$ one combines Brown-Peterson's theorem with a calculation due to Giambalvo [Gia69, Corollary 1].

Ando-Hopkins-Rezk orientation $MString \to tmf$ is 15-connected [Hil09, Theorem 2.1] and BP is connective, then there is a 15-connected map $MString \wedge BP \to tmf \wedge BP$, so if there is no p-torsion in $\pi_*(BP \wedge tmf)$ in degrees 15 and below, the same is true for $MString \wedge BP$, and therefore also for $MString^h$. ☐

Proposition 3.39. *In degrees* 15 *and below,* $\Omega_*^{\mathrm{String}^h}$ *lacks* 3*-torsion.*

Proof. By Lemma 3.38, it suffices to show $tmf_k(BP)$ lacks 3-torsion for $k \le 15$. We will prove this using the Baker-Lazarev Adams spectral sequence [BL01]; for 3-local tmf-homology specifically, this spectral sequence was developed by Henriques and Hill (see [Hil07, DFHH14]), building on work of Behrens [Beh06] and Hopkins-Mahowald (unpublished).

For a spectrum X, this spectral sequence has the signature

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^{tmf}}(H^*(X;\mathbb{Z}/3), \mathbb{Z}/3) \Longrightarrow tmf_{t-s}(X)_3^\wedge, \tag{3.40}$$

where

$$\mathcal{A}^{tmf} \coloneqq \mathbb{Z}/3\langle \beta, \mathcal{P}^1\rangle/(\beta^2, (\mathcal{P}^1)^3, \beta(\mathcal{P}^1)^2\beta - (\beta\mathcal{P}^1)^2 - (\mathcal{P}^1\beta)^2), \tag{3.41}$$

with a $\mathbb{Z}$-grading specified on $\mathcal{A}^{tmf}$ by $|\beta| = 1$ and $|\mathcal{P}^1| = 4$. The $\mathcal{A}^{tmf}$-action on $H^*(X;\mathbb{Z}/3)$ is specified by having β act as the Bockstein for the short exact sequence $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 0$ and $\mathcal{P}^1$ act as the first Steenrod power.[11] See [Hil07, Hil09, BR21, DY23, BDDM24] for additional computations with this spectral sequence.

Recall that $H^*(BP;\mathbb{Z}/3) \cong \mathbb{Z}/3[\mathcal{P}^1, \mathcal{P}^2, \dots]$. Therefore $\beta \in \mathcal{A}^{tmf}$ acts trivially on $H^*(BP;\mathbb{Z}/3)$, and we can determine the $\mathcal{P}^1$-action using the Adem relations, which in this case simplify to

$$\mathcal{P}^1\mathcal{P}^n = (n+1)\mathcal{P}^{n+1} \tag{3.42}$$

for $n \ge 0$. Thus, if we define the $\mathcal{A}^{tmf}$-module $N_3 \coloneqq \mathcal{A}^{tmf}/(\beta)$, so that $N_3 \cong \mathbb{Z}/3[\mathcal{P}^1]/((\mathcal{P}^1)^3)$, then there is an isomorphism

$$H^*(BP;\mathbb{Z}/3) \cong N_3 \oplus \Sigma^{12}N_3 \oplus P, \tag{3.43}$$

where P is concentrated in degrees 16 and above, so is irrelevant for us. In (3.43), N_3 is spanned by $\{1, \mathcal{P}^1, \mathcal{P}^2\}$ and $\Sigma^{12}N_3$ is spanned by $\{\mathcal{P}^3, \mathcal{P}^4, \mathcal{P}^5\}$.

To describe $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3, \mathbb{Z}/3)$, it will be helpful to recall $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3, \mathbb{Z}/3)$, computed by Henriques-Hill [Hil07, DFHH14]. We do not actually need all of this structure; all we use is that there is a class $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3, \mathbb{Z}/3)$ which is not nilpotent. An h_0-action on the E_∞-page lifts to multiplication by 3.

$\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3)$ is given in degrees 15 and below in [BDDM24, Figure 2], which is strictly speaking all we need, but we can give a fully general calculation without much more effort.

Lemma 3.44. *Inside* $\mathcal{A}^{tmf}$*, let* $x_1 \coloneqq \beta$*,* $x_5 \coloneqq \mathcal{P}^1\beta - \beta\mathcal{P}^1$*, and* $x_9 \coloneqq (\mathcal{P}^1)^2\beta - \beta(\mathcal{P}^1)^2$*, so that* $|x_i| = i$ *for* $i = 1, 5, 9$*.*

1. *The algebra* $\mathcal{B} \coloneqq \langle x_1, x_5, x_9\rangle \subset \mathcal{A}^{tmf}$ *is an exterior algebra on the classes* x_1*,* x_5*, and* x_9*.*
2. *There is an isomorphism of* $\mathcal{A}^{tmf}$*-modules* $N_3 \cong \mathcal{A}^{tmf} \otimes_{\mathcal{B}} \mathbb{Z}/3$*.*

Proof. To verify (1), use the relations in (3.41) to verify that $x_1^2 = 0$, $x_5^2 = 0$, and $x_9^2 = 0$ and that there are no additional relations between any of x_1, x_5, and x_9. For (2), one first shows that for any $u \in \mathcal{A}^{tmf}$, $\beta u \otimes 1 = u \otimes \beta(1) = 0$ in $\mathcal{A}^{tmf} \otimes_{\mathcal{B}} \mathbb{Z}/3$, since we can use $x_1 \in \mathcal{B}$ to move β across

[11] In other words, we have specified the $\mathcal{A}^{tmf}$-action by defining an algebra homomorphism from $\mathcal{A}^{tmf}$ to the mod 3 Steenrod algebra. This homomorphism is **not** injective! See Henriques [DFHH14, §13.3].

the tensor product. However, for degree reasons, it is impossible to do the same to make $\mathcal{P}^1 \otimes 1$ or $(\mathcal{P}^1)^2 \otimes 1$ equal to 0, so this tensor product is isomorphic to N_3. ☐

Corollary 3.45. *There is an isomorphism of* $\mathrm{Ext}_{\mathcal{A}^{tmf}}(\mathbb{Z}/3, \mathbb{Z}/3)$*-modules*

$$\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3, \mathbb{Z}/3) \cong \mathrm{Ext}_{\mathcal{B}}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3[h_0, y_1, y_2] \tag{3.46}$$

with $h_0 \in \mathrm{Ext}^{1,1}$, $y_1 \in \mathrm{Ext}^{1,5}$, *and* $y_2 \in \mathrm{Ext}^{1,9}$.

We draw these Ext groups in Figure 2, right.

Proof. The first isomorphism in (3.46) is Shapiro's lemma [Eck53]. The second isomorphism in (3.46) is the standard calculation of Ext of an exterior algebra using Koszul duality, e.g. as in [Pri70]. ☐

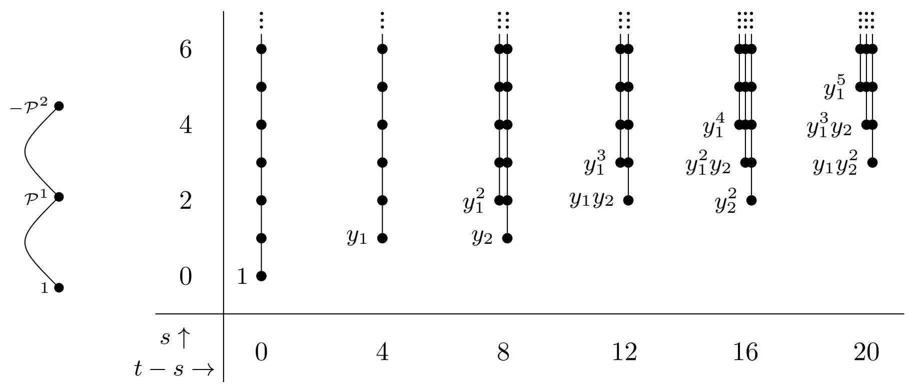

FIGURE 2. Left: the $\mathcal{A}^{tmf}$-module N_3. Right: $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3)$, which we calculated in Corollary 3.45 (see also [BDDM24, Figure 2]). The vertical lines denote h_0-actions. We use these Ext groups in the proof of Proposition 3.39.

For our application to $tmf_*(BP)$, the takeaway is that in even topological degrees, $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3)$ is 0 or a free $\mathbb{Z}/3[h_0]$-module (i.e. a direct sum of "h_0-towers"), and in odd topological degrees, $\mathrm{Ext}_{\mathcal{A}^{tmf}}(N_3)$ vanishes. Therefore by (3.43), at least in degrees 15 and below, the same is true for the E_2-page of the Baker-Lazarev Adams spectral sequence for $tmf_*(BP)$. Since differentials lower topological degree by one, the spectral sequence collapses at E_2 in degrees 16 and below, and h_0 lifts to multiplication by 3, so each $\mathbb{Z}/3[h_0]$-module contributes a free summand to $tmf_*(BP)^\wedge_3$. Therefore there is nothing on the E_∞-page that can contribute 3-torsion to $tmf_*(BP)$ in degrees 15 and below. ☐

Since the orientation $\sigma_1(n) : MString^h \to tmf_1(n)$ factors through $MString \wedge BP$, we have the following result.[12]

Corollary 3.47. *The* E_4*-orientation* $\widetilde{\sigma}_1(2)\colon MString \wedge BP \to (tmf_1(2))_{(3)}$ *obtained by smashing the* E_4*-orientation* $BP \to tmf_1(2)_{(3)}$ *from Corollary 3.30 with the string orientation of* $tmf_1(2)$ *is* 11*-connected.*

[12] There is a homotopy equivalence $tmf_0(2) \simeq tmf_1(2)$; we mostly refer to this object as $tmf_1(2)$ to streamline our notation, but it is often called $tmf_1(2)$ in the literature.

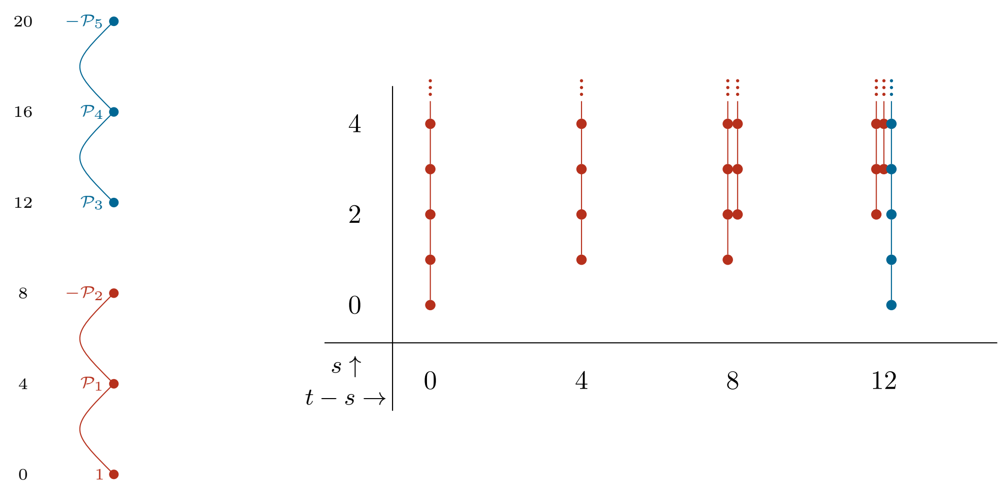

FIGURE 3. Left: The $\mathcal{A}^{tmf}$-module structure on $H^*(BP;\mathbb{Z}/3)$ in degrees 16 and below. Right: The E_2-page of the Adams spectral sequence computing the nontrivial homotopy groups in degrees less than 16.

Proof. By construction $\widetilde{\sigma}_1(2)$ factors as a composition of $MString \wedge BP \to tmf \wedge BP$, which is 15-connected, followed by $\overline{\sigma}_1(2)\colon BP \wedge tmf \to tmf_1(2)$, and both are E_4-ring spectrum maps. Therefore it suffices to show $\overline{\sigma}_1(2)$ is 11-connected.

On homotopy groups, $\overline{\sigma}_1(2)$ is a ring homomorphism, so it sends $1 \mapsto 1$ and therefore is an isomorphism on π_0. Because $\overline{\sigma}_1(2)$ is an isomorphism on π_0, then $\overline{\sigma}_1(2)^*\colon H^0_{tmf}(tmf_1(2)) \to H^0_{tmf}(tmf \wedge BP)$ is also an isomorphism. By (3.43), $H^0_{tmf}(tmf \wedge BP)$ splits as $\mathcal{A}^{tmf}$-modules as the sum of N_3 and an 11-connected summand, and by [Mat16, Theorem 4.13], there is a $tmf_{(3)}$-module equivalence $(tmf_1(2))_{(3)} \simeq tmf_{(3)} \wedge Y$ for a spectrum Y with $H^*(Y;\mathbb{Z}/3) \cong N_3$, so $H^*_{tmf}(tmf_1(2)) \cong N_3$. Thus since $\overline{\sigma}_1(2)^*$ is an $\mathcal{A}^{tmf}$-module map which is an isomorphism on H^0, it must map the N_3 summand in $H^*_{tmf}(tmf_1(2))$ isomorphically onto the N_3 summand from $tmf \wedge BP$. Then by the Baker-Lazarev Adams spectral sequence, the map is 11-connective. ☐

Remark 3.48. Because $\pi_2(MString^h) \cong \mathbb{Z}$ but $\pi_2(tmf_1(2)) \cong \mathbb{Z}[1/2]$, $\sigma_1(2)$ is not 11-connected. But Corollary 3.47 implies that localized at 3, $\sigma_1(2)$ is surjective on homotopy, because $\pi_*(MString^h)_{(3)}$ surjects onto $\pi_*(BP \wedge tmf)$ and $\overline{\sigma}_1(n)$ hits the generators of the homotopy ring of $tmf_1(2)$.

Lemma 3.49. *There is a 2-local equivalence of tmf-module spectra*

$$BP \wedge tmf \xrightarrow{\simeq} \bigvee_{i=0}^{\infty} \Sigma^{n_i} tmf_1(3), \tag{3.50}$$

for some natural numbers n_i.

Proof. Maps of tmf-modules $BP \wedge tmf \to \Sigma^{n_i} tmf_1(3)$ are equivalent to maps of spectra $BP \to \Sigma^{n_i} tmf_1(3)$, so to win, we need to produce maps of spectra $f_i\colon BP \to \Sigma^{n_i} tmf_1(3)$ such that the sum of the induced maps of tmf-modules is an isomorphism on mod 2 cohomology. In other words, we need to produce classes in $tmf_1(3)^*(BP)$.

Set up the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(BP; tmf_1(3)^q(\mathrm{pt})) \Longrightarrow tmf_1(3)^{p+q}(BP); \tag{3.51}$$

since $H^*(BP)$ and $tmf_1(3)^*$ are concentrated in even degrees, this spectral sequence collapses. Boardman in [Boa99, Theorem 12.4] shows that the cohomological Atiyah-Hirzebruch spectral sequence is conditionally convergent when applied to a connective spectrum such as BP, and is strongly connective when it collapses (*ibid.*, Remark after Theorem 7.1). Therefore (3.51) converges strongly, so classes on the E_∞-page lift to maps $BP \to tmf_1(3)$, implying it is possible to produce enough such maps to get the isomorphism on mod 2 cohomology. ☐

Corollary 3.52. *In degrees* 15 *and below,* $\Omega_*^{\mathrm{String}^h}$ *lacks* 2*-torsion.*

Proof. By Lemma 3.38, it suffices to show $\pi_*(tmf \wedge BP)$ lacks 2-torsion in degrees 15 and below, which follows from Lemma 3.49 and the fact that $\pi_*(tmf_1(3))$ is torsion-free, as we saw from its relation to $BP\langle 2\rangle$. ☐

Corollary 3.53. *The map* $\overline{\sigma}_1(3)\colon BP \wedge MString \to tmf_1(3)_{(2)}$ *is* 7*-connected.*

Proof. The proof is almost exactly the same as that of Corollary 3.47. What makes that proof work is that the quotients of $H^*_{tmf}(tmf \wedge BP)$ and $H^*_{tmf}(tmf_1(2))$ by all classes in degrees 12 and above are isomorphic, cyclic $\mathcal{A}^{tmf}$-modules on a generator in degree 0, so that we could lift an evidently 0-connected map to an isomorphism on H^*_{tmf} in degrees 11 and below.

Therefore it suffices to show that $H^*_{tmf}(BP \wedge tmf)$ and $H^*_{tmf}(tmf_1(3))$ (this time with $\mathbb{Z}/2$ coefficients, not $\mathbb{Z}/3$ coefficients like in the previous paragraph), quotiented by all classes in degrees 8 and above, are isomorphic cyclic $\mathcal{A}(2)$-modules on a generator in degree 0. For $tmf_1(3)$, there is a tmf-module equivalence $tmf_1(3)_{(2)} \simeq tmf \wedge DA(1)$ for a spectrum $DA(1)$ with $H^*(DA(1);\mathbb{Z}/2) \cong \mathcal{A}(2)/(\mathrm{Sq}^1)$ [Mat16, Theorem 1.2]. For $BP \wedge tmf$, we need to compute $H^*(BP;\mathbb{Z}/2)$ as an $\mathcal{A}(2)$-module. Brown-Peterson [BP66, Corollary 1.2] show that $H^*(BP;\mathbb{Z}/2) \cong \mathcal{A}/(\mathrm{Sq}^1)$, so we are done by the observation that the inclusion $\mathcal{A}(2) \to \mathcal{A}$ is 7-connected, so that $\mathcal{A}(2)/(\mathrm{Sq}^1) \to \mathcal{A}/(\mathrm{Sq}^1)$ is 7-connected (since Sq^8 is the lowest-degree Steenrod square not contained in $\mathcal{A}(2)$). ☐

Proof of Theorem 3.34. By Propositions 3.36 and 3.39 and Corollary 3.52, $\Omega_*^{\mathrm{String}^h}$ is torsion-free in degrees 16 and below, so to determine the homotopy as a graded abelian group in those degrees we may rationalize: if $E_{\mathbb{Q}} \coloneqq E \wedge H\mathbb{Q}$, then $\pi_*(E_{\mathbb{Q}}) \cong \pi_*(E) \otimes \mathbb{Q}$, so we have

$$\pi_*(MString^h \wedge \mathbb{Q}) \cong \pi_*(MString \wedge MU \wedge \mathbb{Q}) = (MU_{\mathbb{Q}})_*(MString). \tag{3.54}$$

We can calculate the $MU_{\mathbb{Q}}$-homology of $MString$ using the Atiyah-Hirzebruch spectral sequence, which always collapses over $\mathbb{Q}$. Thus we obtain a ring isomorphism

$$\pi_*(MString \wedge MU \wedge \mathbb{Q}) \cong \pi_*(MU) \otimes \mathbb{Q} \otimes \pi_*(MString). \tag{3.55}$$

There are isomorphisms $\pi_*(MU)_{\mathbb{Q}} \cong \mathbb{Q}[x_{2i} : i \geq 1]$ with $|x_{2i}| = 2i$ and $\pi_*(MString) \otimes \mathbb{Q} \cong \mathbb{Q}[y_{4i} : i \geq 2]$ with $|y_{4i}| = 4i$; in degrees 15 and below we thus have the generators in the theorem statement.

To probe the multiplicative structure we notice that since $MString^h \to tmf_1(3)$ is surjective on homotopy groups, that implies after localizing at $p = 2$ then $BP \wedge MString \to tmf_1(3)$ is an isomorphism on homotopy groups in degrees 7 and below and that the rings are the same in those degrees. Localizing at $p = 3$ we get an isomorphism on homotopy groups of $MString^h$ with $tmf_1(2)$ in degrees 11 and below by Corollary 3.47. Finally, localizing at $p \geq 5$ we see that the bordism ring is polynomial in generators as both BP_* and $MString_*$ are polynomial (the latter after p-localizing). ☐

3.2. **Does $\sigma_1(n)$ split?** Anderson-Brown-Peterson [ABP67] showed that the Atiyah-Bott-Shapiro maps [ABS64] $\mathit{MSpin} \to \mathit{ko}$ and $\mathit{MSpin}^c \to \mathit{ku}$ admit 2-local sections $\mathit{ko} \to \mathit{MSpin}$, resp. $\mathit{ku} \to \mathit{MSpin}^c$, and used these sections, along with higher-degree analogues, to effectively determine spin and spin^c bordism. The analogous question for the Ando-Hopkins-Rezk orientation [AHR10] $\sigma\colon \mathit{MString} \to \mathit{tmf}$ is a longstanding open question in homotopy theory, discussed for example in [MG95, MH02, MR09, Lau04, Lau16, LO16, LO18, LS19, Dev19, Abs21, Dev24]. It therefore seems reasonable to ask:

Question 3.56. *Let $p = 2$ or 3 and $p \nmid n$. Does the map $\sigma_1(n)\colon \mathit{MString}^h_{(p)} \to \mathit{tmf}_1(n)_{(p)}$ have a section? What about Devalapurkar's orientation σ_D?*

One could also ask this question localized at a large prime (i.e. $p \geq 5$), where it is much easier, as both $\mathit{MString}^h$ and $\mathit{tmf}_1(n)$ for many n are known to decompose into sums of shifts of $BP \wedge BP$, resp. BP (see the proof of Proposition 3.36, resp. [Mei23, §5]). Thus we focus on the harder primes. We think for $p = 2$, $n = 3$, and for $p = 3$, $n = 2$, Question 3.56 has an affirmative answer.

Question 3.56 passes a few basic checks.

Proposition 3.57 (Devalapurkar)**.** *If there is a section $s\colon \mathit{tmf}_{(2)} \to \mathit{MString}_{(2)}$ of σ, then there is a section $s'\colon \mathit{tmf}_1(3)_{(2)} \to \mathit{MString}^h_{(2)}$ of $\sigma_1(3)$.*

Proof. This is immediate since $MU \wedge \mathit{tmf}$ would split off from $\mathit{MString}^h$, and $\mathit{tmf}_1(3)$ itself splits off of $MU \wedge \mathit{tmf}$ by Lemma 3.49. ☐

In addition, an affirmative answer to Question 3.56 would imply that $\sigma_1(n)$ is surjective on homotopy after p-completion. For $(p, n) \in \{(2, 3), (3, 2)\}$, we proved homotopy surjectivity unconditionally in Proposition 3.32.

Proposition 3.58. *There is no 2-local section of $\sigma_1(3)$ that is a map of BP-module spectra, where $\mathit{MString}^h_{(2)}$ acquires its BP-module structure from the E_4-map $BP \to MU_{(2)} \to \mathit{MString}^h_{(2)}$ and $\mathit{tmf}_1(3)_{(2)}$ acquires its BP-module structure from the equivalence $\mathit{tmf}_1(3)_{(2)} \simeq BP\langle 2\rangle$.*

Proof. Suppose such a section existed and rationalize. That would imply the existence of a section of

$$\sigma_1(3)_*\colon \pi_*(\mathit{MString}^h) \otimes \mathbb{Q} \longrightarrow \mathit{tmf}_1(3)_* \otimes \mathbb{Q} \tag{3.59}$$

which is linear with respect to $BP_* \otimes \mathbb{Q} \cong \mathbb{Q}[v_1, v_2, \dots]$. Since

$$\pi_*(\mathit{MString}^h) \otimes \mathbb{Q} \cong \pi_*(\mathit{MString}) \otimes \mathbb{Q} \otimes \pi_*(MU) \tag{3.60}$$

and $\pi_*(MU) \otimes \mathbb{Q}$ is a free $\pi_*(BP) \otimes_{\mathbb{Z}_{(2)}} \mathbb{Q}$-module and $\pi_*(\mathit{MString}) \otimes \mathbb{Q}$ is a polynomial algebra, v_3 acts injectively on $\pi_*(\mathit{MString}^h \otimes \mathbb{Q})$. However, since $\mathit{tmf}_1(3)_{(2)}$ acquired its BP-module structure by being a form of $BP\langle 2\rangle$, v_3 acts as zero on $\mathit{tmf}_1(3)_* \otimes \mathbb{Q}$. A section equivariant for the action of v_3 cannot carry a zero action to an injective action. ☐

Ultimately the "problem" causing the negative result in Proposition 3.58 is that the equivalence

$$BP \wedge \mathit{tmf} \simeq \bigvee_{i \in 0}^{\infty} \Sigma^{n_i} \mathit{tmf}_1(3)_{(2)} \tag{3.61}$$

from Lemma 3.49 is not an equivalence of BP-modules.

4. String^h and the Diaconescu-Moore-Witten Anomaly

We will now explain the formalism for string^h structures has a natural place in type IIA string theory by understanding its relationship with the Diaconescu-Moore-Witten anomaly. Let X be a 10-dimensional manifold which serves as the target space for type IIA string theory. The intimate way in which type IIA and M-theory are related means that the same anomaly also manifests in M-theory on $Y = X \times S^1$, and is in a sense where it originates. In particular, the partition function for the RR-fluxes in type IIA string theory can be matched with the corresponding partition function computed in M -theory, and we review how an anomaly arises in M-theory by looking at a certain part of its partition function. A priori the way in which the anomalies arise in M-theory and type IIA are different, but it was shown in [DMW02], that the anomaly cancellation information is equivalent. Therefore whatever tangential structure that the anomaly cancellation defines is shared by both M-theory and type IIA target space.

M-theory consists of two types of branes: the M2 and M5 branes. On the M2 brane there is an associated 3-form field C with field strength $G = dC$. The topological quantization of G is given by choosing any element $\alpha \in H^4(X;\mathbb{Z})$ and letting G_α be the "mode" of G contributing to the topological sector labeled by α. From this we can form the partition function of M-theory by considering the contributions from all α.

Definition 4.1 (Diaconescu-Moore-Witten [DMW00, §5])**.** Let X be a closed spin 10-manifold and $\alpha \in H^4(X;\mathbb{Z})$. Assume that there is a class $x \in ku^4(X)$ such that $\tau_0(x) = \alpha$; as we observed in the proof of Theorem 2.17, x may be represented by an SU-bundle, and because $B\mathrm{SU}_5 \to B\mathrm{SU}$ is 11-connected, x may be represented by a rank-5 complex vector bundle $E \to X$ with SU-structure. Then define

$$f(\alpha) \coloneqq q(E \otimes \overline{E}) + ((\mathrm{Ind}(\Lambda^2(E)) + \mathrm{Ind}(E)) \bmod 2) \in \mathbb{Z}/2, \tag{4.2}$$

where $\mathrm{Ind}(V)$ denotes the index of the Dirac operator coupled to V and q denotes the mod 2 index of this Dirac operator.

Definition 4.3. Let X be a closed spin 10-manifold. The partition function of the topological sector of M-theory on X is given by the following sum over $\alpha \in H^4(X;\mathbb{Z})$ for G:[13]

$$\mathcal{Z}_M \sim \sum_{\alpha \in H^4(X;\mathbb{Z})} (-1)^{f(\alpha)} \exp\left(-|G_\alpha|^2\right), \tag{4.4}$$

where $|G_\alpha|^2 = \int G_\alpha \wedge \star G_\alpha$, and we are assuming the ku^4 lifts chosen in Definition 4.1 in the definition of f exist.

Even though the topological sector is not the full partition function of M-theory, it can already give hints at anomalies, in particular by studying the way the ambiguity of (4.4) with respect to the existence of lifts across τ_0.

Definition 4.5. The k^{th} *integral Stiefel-Whitney class* $W_k \in H^k(B\mathrm{O};\mathbb{Z})$ is $W_k \coloneqq \square_{\mathbb{Z}}(w_{k-1})$.

Thus, for example, a spin^c structure on an oriented vector bundle V is equivalent data to a trivialization of $W_3(V)$.

[13] We write $\sim$ rather than $=$ because of some prefactors that are gauge-invariant and thus not relevant for the present discussion. See [DMW00, §3] for more on these terms.

Proposition 4.6 (Diaconescu-Moore-Witten [DMW02])**.** *With X as above, given data of a trivialization of $W_7(X)$ the quantity $f(\alpha)$ from Definition 4.1 is well-defined for all $\alpha \in H^4(X;\mathbb{Z})$: each α has a ku-cohomology lift, and the quantity* (4.2) *does not depend on the choice of lift.*

Proposition 4.6 has the physics consequence that the partition function of the topological sector of type IIA string theory on X, which in general suffers a sign ambiguity, is well-defined when $W_7(X)$ is trivialized. More heuristically speaking, in order for X to be a valid background of type IIA string theory, either we must have $W_7(X) = 0$ or wrap branes within submanifolds of X. In this paper we will consider the first option.

Remark 4.7. It is possible to generalize this story to the case when X is merely spin^c. In this case, the relation to M-theory is slightly changed: if $L \to X$ is the determinant line bundle of the spin^c structure, then the total space of the unit sphere bundle $S(L)$ is a closed 11-manifold with a canonical spin structure induced from the spin^c structure on X, and one thinks of type IIA string theory on X as a "twisted compactification" of M-theory on $S(L)$. In this setting, there is a generalization of Proposition 4.6 implying that on a closed spin^c 10-manifold X, the data of a trivialization of $W_7(X)$ suffices to resolve the sign ambiguities in the partition function of the topological sector of type IIA string theory on X.

Definition 4.8. The *Diaconescu-Moore-Witten anomaly cancellation condition* is the requirement $W_7(X) = 0$. A *Diaconescu-Moore-Witten (DMW) structure* on a vector bundle V is a spin^c structure and a trivialization of $W_7(V)$.

See [FSS20, SS23] for more on how generalized cohomology theories can be applied to M-theory from the DMW anomaly, and [FH21a] for more on the tangential structure of M-theory when time-reversal is taken into account.

The following serves as a sketch of the argument given in [DMW00] for their anomaly cancellation condition. We start by unpacking the conditions on the function $f(\alpha)$. In particular $f(\alpha)$ satisfies the property that

$$f(\alpha + \alpha') = f(\alpha) + f(\alpha') + \int_X \alpha \, \mathrm{Sq}^2(\alpha' \bmod 2)\,, \tag{4.9}$$

and $(X, \alpha) \mapsto f(\alpha)$ is a bordism invariant in $\mathrm{Hom}(\Omega_{10}^{\mathrm{Spin}}(K(\mathbb{Z}, 4)), \mathbb{Z}/2)$. Suppose $\gamma \in H^4(X;\mathbb{Z})$ is torsion. Then, while $|G_\alpha|^2$ is invariant under $\alpha \to \alpha + \gamma$, $f(\alpha)$ is often not invariant. Consider a specific transformation $\alpha \to \alpha + 2\gamma$; then,

$$f(\alpha + 2\gamma) = f(\alpha) + f(2\gamma) + \int_X \alpha \, \mathrm{Sq}^2(2\gamma \bmod 2), \tag{4.10}$$

but $2\gamma \bmod 2 = 0$ so we only need to consider the new term $f(2\gamma)$. Expanding again, we see

$$f(2\gamma) = f(\gamma) + f(\gamma) + \int_X \gamma \, \mathrm{Sq}^2(\gamma \bmod 2)\,. \tag{4.11}$$

Stong [Sto86] shows that $\int_X \gamma \mathrm{Sq}^2(\gamma) = \mathrm{Sq}^4(\mathrm{Sq}^2(\gamma))$; combining this with the Wu formula, Diaconescu-Moore-Witten [DMW00, §6] show that

$$\int_X \gamma \mathrm{Sq}^2(\gamma \bmod 2) = \int_X \gamma \mathrm{Sq}^2(\lambda(X) \bmod 2). \tag{4.12}$$

The effect of (4.10) in the partition function is given by $(-1)^{f(\alpha)+f(2\gamma)}$. As $2f(\gamma) = 0$, we only have to worry about $\int_X \gamma \mathrm{Sq}^2(\gamma \bmod 2)$.

Lemma 4.13 ([DMW00, §6])**.** *Let* $\langle -, - \rangle\colon \mathrm{Tors}(H^4(X;\mathbb{Z})) \otimes \mathrm{Tors}(H^7(X;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}$ *denote the torsion pairing on a closed spin* 4*-manifold* X*. Then for all* $\gamma \in \mathrm{Tors}(H^4(X;\mathbb{Z}))$*,*

$$\frac{1}{2}\int_X \gamma \cdot \mathrm{Sq}^2(\lambda(X) \bmod 2)) = \langle \gamma, \square_{\mathbb{Z}}(\mathrm{Sq}^2(\lambda(X)))\rangle. \tag{4.14}$$

The equality (4.14) takes place in $\mathbb{Q}/\mathbb{Z}$: for the left-hand side, the integral is an element of $\mathbb{Z}/2$, so dividing by 2 we obtain an element of $(\frac{1}{2}\mathbb{Z})/\mathbb{Z}$, which is a subgroup of $\mathbb{Q}/\mathbb{Z}$.

Finally, a direct calculation with the Wu formula and the relation $\lambda(X) \bmod 2 = w_4(X)$ shows

$$\square_{\mathbb{Z}}(\mathrm{Sq}^2(\lambda(X) \bmod 2)) = W_7(X), \tag{4.15}$$

so the DMW condition $W_7(X) = 0$ fixes the sign of the partition function unambiguously.[14]

4.1. **Relating Diaconescu-Moore-Witten anomaly cancellation with string^h.** We will show how a string^h structure induces the Diaconescu-Moore-Witten anomaly cancellation.

Theorem 4.16. *Let* $V \to X$ *be a string*^{h} *vector bundle. Then* $W_7(V)$ *admits a canonical trivialization.*

Thus the Diaconescu-Moore-Witten anomaly cancellation condition (Definition 4.8) is automatically satisfied on string^h 10-manifolds.

Proof. We use the characterization of string^h structures from Definition 2.12: that we have trivialized $\square_{ku}(\lambda^c(V))$.To relate Definition 2.12 to the $W_7(X) = 0$ anomaly cancellation condition of Diaconescu-Moore-Witten we observe the following commutative diagram, whose rows are cofiber sequences:

$$\begin{array}{ccccccc} \Sigma^2 ku & \xrightarrow{\beta} & ku & \xrightarrow{\tau_0} & H\mathbb{Z} & \xrightarrow{\square_{ku}} & \Sigma^3 ku \\ \downarrow{\scriptstyle \tau_{\le 2}} & & \downarrow{\scriptstyle \tau_{\le 2}} & & \| & & \downarrow{\scriptstyle \tau_0} \\ \Sigma^2 H\mathbb{Z} & \longrightarrow & \tau_{\le 2} ku & \longrightarrow & H\mathbb{Z} & \xrightarrow{\square_{\mathbb{Z}}\mathrm{Sq}^2} & \Sigma^3 H\mathbb{Z}\,. \end{array} \tag{4.17}$$

The top map builds ku as an extension with $\square_{ku}$ as the k-invariant, β the Bott map, and the map down is the truncation map. This builds $\tau_{\le 2} ku$ also as an extension with k-invariant $\square_{\mathbb{Z}}\mathrm{Sq}^2$. Extending the left most square gives a map of cofiber sequences

$$\begin{array}{ccccc} \Sigma^4 ku & \xrightarrow{=} & \Sigma^4 ku & & \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta^2} & & \\ \Sigma^2 ku & \xrightarrow{\beta} & ku & \xrightarrow{\tau_0} & H\mathbb{Z} \\ \downarrow{\scriptstyle \tau_{\le 2}} & & \downarrow & & \downarrow \\ \Sigma^2 H\mathbb{Z} & \longrightarrow & \tau_{\le 2} ku & \longrightarrow & H\mathbb{Z} \end{array} \tag{4.18}$$

This implies the map $H\mathbb{Z} \to H\mathbb{Z}$ is the identity by the third isomorphism theorem and right most square commutes. On cohomology, the rightmost commuting square gives

$$\begin{array}{ccc} H^4(X;\mathbb{Z}) & \xrightarrow{\square_{ku}} & ku^7(X) \\ \downarrow & & \downarrow{\scriptstyle \tau_0} \\ H^4(X;\mathbb{Z}) & \xrightarrow{\square_{\mathbb{Z}}\mathrm{Sq}^2} & H^7(X;\mathbb{Z})\,, \end{array} \tag{4.19}$$

[14] In conversation with Moore, it became known to the authors that the theory could be consistent even if the W_7 anomaly is non-trivial. If the partition function vanishes, that does not inherently mean that the theory itself is invalid. We plan to return to theories with nontrivial values of this anomaly in future work.

where q is the restriction to cohomology. For a string^h vector bundle V, we we compute $\square_{\mathbb{Z}}\mathrm{Sq}^2(\lambda^c(V))$. We first take the mod 2 reduction of λ^c which is $w_4(V\oplus L)$, where $L\to X$ is the determinant line bundle of V. Applying the Whitney sum formula gives $w_4(V\oplus L)=w_4(V)+w_2(V)^2$, upon using the fact that $w_2(TX)=w_2(L)$. The action by Sq^2 is obtained by the Wu formula, for which we get $\mathrm{Sq}^2(w_4(B)+w_2(B)^2)=w_2(V)w_4(V)+w_6(V)$. Applying $\square_{\mathbb{Z}}$ then implies $\square_{\mathbb{Z}}\mathrm{Sq}^2(\lambda^c(V))=W_7(V)$. The square commuting means $\tau_0\square_{ku}=\square_{\mathbb{Z}}\mathrm{Sq}^2$ and that if $\tau_0\square_{ku}(\lambda^c(V))=0$ then $\square_{\mathbb{Z}}\mathrm{Sq}^2(\lambda^c(V))=0$. Therefore, if we have a string^h structure on V, then $W_7(V)$ is canonically trivialized. □

Remark 4.20. Let us take the target space of type IIA string theory to have a string^h structure. As a first level consistency check, we recall that a string^h structure induces a spin^c structure. This is consistent with the fact that the target space of type IIA string theory has a spin^c structure. We exhibit the spin^c structure on type IIA by observing the transformation of the gravitino field Ψ in the low energy supergravity. This is a fermion that is charged under a U_1-gauge symmetry, where the U_1-bundle arises from dimensionally reducing away the M-theory circle. A gauge transformation $\Psi\to e^{2\pi i q}\Psi$ reflects the spin^c structure if q is half integral, and it was shown in [DLP98, BEM04] that this is the case.

While a string^h structure always induces a $W_7=0$ condition, one can also ask the reverse question, which is slightly more nontrivial. If we want to study type IIA on manifolds with a string^h structure, we should see how strong of an assumption the string^h structure is.

Theorem 4.21. *Let $V\to X$ be a vector bundle with DMW structure. Then there is a class $\rho(V)\in H^9(X;\mathbb{Z})$ which vanishes if V is string^h. If X is a manifold of dimension 10 or below with DMW structure, $\rho(X)$ is the complete obstruction to lifting the DMW structure to a string^h structure.*

Lemma 4.22. *Let $B\mathrm{Spin}^c\langle W_7\rangle$ be the fiber of the map $B\mathrm{Spin}^c\xrightarrow{W_7}K(\mathbb{Z},7)$. The homotopy groups up to degree 10 of $B\mathrm{Spin}^c\langle W_7\rangle$ are given by:*

$$\pi_*(B\mathrm{Spin}^c\langle W_7\rangle)=\{0,\,0,\,\mathbb{Z},\,0,\,\mathbb{Z},\,0,\mathbb{Z},\,0,\mathbb{Z},\,\mathbb{Z}/2,\,\mathbb{Z}/2,\ldots\}.$$

Proof. This follows immediately from studying the homotopy long exact sequence for $B\mathrm{Spin}^c\langle W_7\rangle\to B\mathrm{Spin}^c\to K(\mathbb{Z},7)$, and using the homotopy groups in Figure 1. □

Proof of Proposition 4.21. Let the space F be the fiber of the map $f: B\mathrm{String}^h\to B\mathrm{Spin}^c\langle W_7\rangle$. For a map $X\to B\mathrm{Spin}^c\langle W_7\rangle$, we want to quantify the first obstruction to lifting against the map f. This will be given by a cohomology class $H^{n+1}(X;\pi_n(F))$. The long exact sequence in homotopy groups for the fiber sequence $F\to B\mathrm{String}^h\to B\mathrm{Spin}^c\langle W_7\rangle$ is given in Figure 4. We see that the only homotopy group that contributes to the obstruction for manifolds X in the degrees we are considering is $\pi_8(F)=\mathbb{Z}$. Therefore the obstruction class is in $H^9(X;\mathbb{Z})$. □

If an obstruction class in H^n trivializes, the choices of trivializations live in a torsor for H^{n-1}. We summarize the implications below:

- For spin^c manifolds in dimension ≤ 5, the $W_7=0$ condition is always satisfied and for dimension 6 the $W_7=0$ condition is not trivialized uniquely as $H^6(X,\mathbb{Z})$ is not necessarily trivial.
- For spin^c manifolds X that are dimension 7 and below, the obstruction for a $W_7=0$ condition to lift to a string^h structure vanishes, and there are no choices of trivializations. In dimension 8, the obstruction vanishes but not canonically.

$*$	$\pi_*(F)$		$\pi_*(B\mathrm{String}^h)$		$\pi_*(B\mathrm{Spin}^c\langle W_7\rangle)$
2	0		$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}$
3	0		0		0
4	0		$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}$
5	0		0		0
6	0		$\mathbb{Z}$		$\mathbb{Z}$
7	0		0		$\mathbb{Z}$
8	$\mathbb{Z}$	$\hookrightarrow$	$\mathbb{Z}\oplus\mathbb{Z}$	$\twoheadrightarrow$	$\mathbb{Z}$
			0		
9	0		$\mathbb{Z}/2$	$\xrightarrow{\cong}$	$\mathbb{Z}/2$
10	$\mathbb{Z}$	$\longrightarrow$	$?$	$\twoheadrightarrow$	$\mathbb{Z}/2$

Figure 4. Homotopy Long Exact Sequence for computing the homotopy groups of $B\mathrm{String}^h$ in degrees up to 10.

- For spin^c manifolds in dimension 9 and 10, it is unclear whether a W_7 condition lifts to a string^h structure. It was claimed that $\square_{\mathbb{Z}}(\mathrm{Sq}^2(a \bmod 2)) = 0$ gives a lift of $a \in H^4(X;\mathbb{Z})$ to K-theory and hence a string^h structure on X. We see here that it is not a priori clear that one obtains a K-theory lift in dimension 10, but it is possible in dimension 9. It is still possible that the obstruction vanishes on closed DMW 10-manifolds.

Theorem 4.23. *Let M be a closed 9 dimensional spin^c manifold. Every DMW structure on M lifts to a string^h structure.*

Lemma 4.24. *If M is a closed, R-oriented n-manifold, for R a ring spectrum, then the map $\Sigma^n R \to R \wedge M_+$ induced by the top cell of M splits off as a direct summand.*

Proof. The bottom cell of any space always splits off stably, by the inclusion of the basepoint followed by the crush map. If M is R-oriented, Atiyah-Poincaré duality identifies $R \wedge M_+$ with its $(\dim(M))$-shifted (R-module) Spanier-Whitehead dual $R \wedge \Sigma^{-\dim(M)} D(M)_+$; the top cell of $D(M)$ corresponds to the bottom cell of M, hence splits off, and therefore the top cell of $R \wedge M_+$ does as well. $\square$

As a result, the $p = 9$ column of the $ku^*(M)$ spectral sequence of a closed spin^c 9-manifold M splits off as a direct sum, which prohibits any differentials or extensions involving this column.

Proof of Proposition 4.23. Let $\lambda^c \in H^4(M;\mathbb{Z})$ such that $\square_{\mathbb{Z}}\mathrm{Sq}^2(\lambda^c) = 0$. If λ^c exists on the E_∞ page of the $ku^*(M)$ spectral sequence then λ^c has a K-theory lift to $ku^4(M)$. The homotopy groups of ku start in degree 0 given by $\mathbb{Z}$, and by Bott periodicity are $\mathbb{Z}$ in each negative even degree. The class λ^c appears in bidegree $(p,q) = (4,0)$ The only differentials that λ^c can admit, and within the range of degrees that we are considering, are d_3 and d_5:

- The d_3 differential is given by $\square_{\mathbb{Z}} \circ \mathrm{Sq}^2 \circ \mathrm{mod}\, 2$, which maps λ^c to $W_7(M)$. But since this class is trivial by assumption, d_3 vanishes.
- The d_5 differential maps λ^c to bidegree $(9,-4)$, but by Lemma 4.24 the class in $(9,-4)$ must split off as a direct sum and therefore cannot be killed by a differential, hence d_5 vanishes.

Thus, λ^c survives to $E_\infty^{4,0}$ which means it has a ku-cohomology lift, and M has a string^h structure. $\square$

4.2. **Applications of string^h for type IIA compactifications.** Consider a compactification of type IIA string theory down to dimension $d < 10$. We know that imposing Diaconescu-Moore-Witten's anomaly cancellation condition $W_7 = 0$ resolves a sign ambiguity in the partition function, but it is a priori possible that the compactified theory has an anomaly α of some other provenance. This anomaly is a unitary $(d+1)$-dimensional invertible field theory of manifolds equipped with a DMW structure and possibly a map to a space X (e.g. $X = BG$ if we have a background gauge field for the group G), so by work of Freed-Hopkins [FH21b] and Grady [Gra23], α is classified in terms of the bordism groups $\Omega_k^{\mathrm{Spin}^c\langle W_7\rangle}(X)$ for $k = d+1, d+2$.

There is a standard procedure to compute bordism groups of manifolds with a trivialized characteristic class such as DMW-structures (see, for example, [BDDM24, §3.3.2]): first, use the Serre spectral sequence to study $H^*(B\mathrm{Spin}^c\langle W_7\rangle)$, then use that cohomology as input to the Adams or Atiyah-Hirzebruch spectral sequence. This is thus quite a bit more complicated than just computing spin^c bordism.

In this subsection, we will use string^h bordism to simplify the bordism computations underlying anomaly cancellation of these IIA compactifications. Specifically, we will lift from DMW-structures to string^h structures, and show that in dimensions $d \le 8$, this loses no information about the anomaly. We will also see that the Atiyah-Hirzebruch and Adams spectral sequences for string^h bordism are relatively straightforward after our work in the previous section. See [DY24, DY23, Tac22, TY23b] for more examples of anomaly cancellations in compactifications of supergravity and heterotic string theory. We highlight here how using string^h affects the computations in different dimensions:

- If M is a manifold of dimension 5 or below, every spin^c structure on M lifts uniquely to a string^h structure. Therefore in dimensions 5 and below, lifting to a string^h structure does not buy us anything new over computing with spin^c.
- For manifolds in dimensions $6 \le d \le 9$, spin^c and DMW structures are not equivalent, and every DMW structure lifts uniquely to a string^h structure. Therefore given a d-dimensional field theory on manifolds with a DMW structure (and perhaps also some background fields), the anomaly is trivializable as an IFT of DMW manifolds if and only if it is trivializable as a string^h theory. Because string^h bordism is easier to compute than DMW bordism, as we will see in a few examples below, this can assist in anomaly cancellation computations.

- In dimension 10, because we do not know whether every DMW structure lifts to a string^h structure, we do not know whether restricting to string^h manifolds loses information with regards to anomaly cancellation.

The real highlight of when string^h leads to simplifications is when we are concerned with anomalies of a compactified theory that has some Lie group global symmetry. Using the change of rings from $\mathcal{A}$ to $\mathcal{E}(2)$ that is afforded to us by the orientation $\sigma_1(3)$, the Adams spectral sequence can be used to compute these bordism groups in low degrees.

Remark 4.25. If we suppose a naïveness to string^h as well as the $W_7 = 0$ condition and only considered spin^c structures for target space manifolds, then we will start to see the difference after degree 6. After this degree is when string^h bordism begins to have more free summands than in spin^c and thus there could potentially be more perturbative anomalies to check.

Example 4.26. Consider any theory that arises as a compactification of type IIA in dimension 9 or below, with a G symmetry where G is a Lie group of the type $\mathrm{U}_n, \mathrm{SU}_n$, or Sp_n for $n > 1$. The bordism groups relevant for anomaly cancellation will be $\Omega_*^{\mathrm{Spin}^c\langle W_7\rangle}(BG)$, and by the above discussion $\Omega_*^{\mathrm{String}^h}$ maps surjectively onto the DMW bordism groups in dimensions $* \leq 9$. Since the homology is sparse and concentrated in even degree the Atiyah-Hirzebruch spectral sequence that computes $\Omega_*^{\mathrm{String}^h}(BG)$ therefore collapses on the E_2-page. For applications to anomalies of theories with G-symmetry, the Anderson-dual $(I_{\mathbb{Z}}\Omega^{\mathrm{String}^h})^*(BG)$ is free and concentrated in odd degrees. This implies that any theory with a string^h structure and a G global symmetry has no global anomalies to cancel, and once the perturbative anomalies are cancelled then the theory is anomaly free.

Example 4.27. Consider any theory that arises as a compactification of type IIA in dimension 9 or below, with a U_1 global symmetry. If one is interested in computing $\Omega_*^{\mathrm{Spin}^c\langle W_7\rangle}(B\mathrm{U}_1)$, the situation is much more complicated from the point of view of the Atiyah-Hirzebruch spectral sequence because the low degree homology classes for $B\mathrm{U}_1$ are more nontrivial. However, this is where being able to lift to string^h pays off. Since the homology of $B\mathrm{U}_1$ is in even degrees, and by Theorem 3.34 the homotopy groups of *MString*^{h} are also concentrated in even degrees and there is no torsion. Therefore the Atiyah-Hirzebruch spectral sequence for $\Omega_*^{\mathrm{String}^h}(B\mathrm{U}_1)$ collapses on the E_2 page and the anomalies share the same properties as in Example 4.26.

Example 4.28. Let G by a connected, simple, simply connected Lie group. We explain how the bordism groups $\Omega_*^{\mathrm{Spin}^c\langle W_7\rangle}(BG)$ can be computed using the lift to string^h. Since the homology of G begins in degree 4, and $\Omega_*^{\mathrm{String}^h} \to \Omega_*^{\mathrm{Spin}^c}$ is an isomorphism in degrees < 6 we see by the Atiyah-Hirzebruch spectral sequence that the first place where the two groups $\Omega_*^{\mathrm{String}^h}(BG)$ and $\Omega_*^{\mathrm{Spin}^c}(BG)$ can potentially begin to differ is in bi-degree $(p, q) = (4, 6)$. Therefore $\Omega_*^{\mathrm{String}^h}(BG) \to \Omega_*^{\mathrm{Spin}^c}(BG)$ is an isomorphism in degrees ≤ 9. In Theorem A.1, we prove that the groups $\Omega_*^{\mathrm{Spin}^c}(BG)$ are torsion free in degrees 9 and below. The anomalies for these symmetries therefore share the same features as in Example 4.26. In higher degrees we predict it will be easier to use the Adams spectral sequence and changing rings to $\mathcal{E}(2)$, since the complications with using Atiyah-Hirzebruch spectral sequence build up very quickly in this high of degree. We leave the details to future work.

Appendix A.

The purpose of this appendix is to prove the following theorem.

Theorem A.1. *Let G be a connected, simple, simply connected Lie group. Then $ku_*(BG)$ is torsion-free in degrees* 10 *and below.*

This is an ingredient in our anomaly cancellation result in Examples 4.27 and 4.28; however, it requires different techniques than we used in that section, so we have siloed it off here.

Lemma A.2.

(1) *Theorem A.1 is true for $G = \mathrm{Sp}_n$, SU_n, and (for $n \le 6$) Spin_n.*
(2) *If we localize at a prime $p > 5$, the theorem is true for all G in the statement of Theorem A.1. Localized at $p = 5$, the theorem is true for all such G except perhaps E_8, and localized at $p = 3$, the theorem is true for all such G except perhaps F_4, E_6, E_7, and E_8.*

Proof. For part (1), let G be Sp_n, SU_n, or (for $n \le 6$) Spin_n and set up the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(BG; ku_q) \Longrightarrow ku_{p+q}(BG). \tag{A.3}$$

For these choices of G, $H_*(BG;\mathbb{Z})$ is torsion-free and concentrated in even degrees. Since ku_* is also torsion-free and concentrated in even degrees, the spectral sequence collapses to imply the first part of the lemma statement.

The proof of part (2) is similar except for using the $ku_{(p)}$-homology Atiyah-Hirzebruch spectral sequence, whose input is the $\mathbb{Z}_{(p)}$-homology of BG. Assume $p \ge 7$, or $p = 5$ and $G \ne E_8$, or $p = 3$ and $G \notin \{F_4, E_6, E_7, E_8\}$. Borel [Bor61, Théorèmes B et 2.5] shows that for these choices of G and p, $H^*(BG;\mathbb{Z})$ lacks p-torsion and is concentrated in even degrees, so the Atiyah-Hirzebruch spectral sequence collapses as in the previous paragraph. ☐

The Atiyah-Hirzebruch-style proof of Lemma A.2 does not generalize nicely to the remaining cases of Theorem A.1, so we use the Adams spectral sequence. Choose a prime p and let $\mathcal{A}$ denote the p-primary Steenrod algebra, the $\mathbb{Z}$-graded noncommutative $\mathbb{Z}/p$-algebra consisting of natural transformations $H^*(-;\mathbb{Z}/p) \to H^{*+n}(-;\mathbb{Z}/p)$ that commute with the suspension functor. Then the Adams spectral sequence has signature

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \pi^s_{t-s}(X)^\wedge_p, \tag{A.4}$$

where π_*^s denotes stable homotopy groups and $(-)^\wedge_p$ denotes p-completion.

Definition A.5. Let $Q_i \in \mathcal{A}$ denote the i^{th} Milnor primitive; thus Q_0 is the Bockstein operator for $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ and Q_1 is the commutator of Q_0 and Sq^2 (if $p = 2$) or $\mathcal{P}^1$ (if $p > 2$).

Let $\mathcal{E}(1) \coloneqq \langle Q_0, Q_1\rangle \subset \mathcal{A}$; the Adem relations imply $\mathcal{E}(1)$ is an exterior algebra on Q_0 and Q_1.

Theorem A.6 (Adams [Ada74, §16])**.** *For each prime p, there is a spectrum ℓ with the following properties.*

(1) *$ku_{(p)} \simeq \ell \vee \Sigma^2\ell \vee \cdots \vee \Sigma^{2(p-2)}\ell$, or if $p = 2$, $ku_{(2)} \simeq \ell$.*
(2) *There is an $\mathcal{A}$-module isomorphism $H^*(\ell;\mathbb{Z}/p) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/p$.*

The first part of Theorem A.6 implies that, if we can prove $\ell_*(BG)$ is torsion-free in degrees 10 and below, then we have proven Theorem A.1. The second part of Theorem A.6 allows us to simplify the Adams spectral sequence calculating ℓ-homology: following the same reasoning as in the proof of Corollary 3.45, if one plugs in $X = \ell \wedge Y$ to (A.4), the spectral sequence simplifies to

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(H^*(Y;\mathbb{Z}/p), \mathbb{Z}/p) \Longrightarrow \ell_{t-s}(Y)^\wedge_p. \tag{A.7}$$

The ℓ_*-module structure on $\ell_*(Y)$ manifests in this spectral sequence through the action of the algebra $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ on the E_2-page of (A.7). Explicitly, this algebra is [BG03, §2.1]

$$\mathrm{Ext}^{*,*}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[h_0, v_1] \tag{A.8}$$

with $h_0 \in \mathrm{Ext}^{1,1}$ and $v_1 \in \mathrm{Ext}^{1,2p-2}$. The action of h_0 on the E_∞-page of (A.7) lifts to detect multiplication by p on $\ell_*(Y)$.

On the E_2-page of (A.7), an h_0-*tower* is a free $\mathbb{Z}/p[h_0]$-module of rank 1.

Lemma A.9. *Suppose Y is a CW complex with finitely many cells in each dimension and the E_2-page of the spectral sequence (A.7) for Y consists solely of h_0-towers in even $(t-s)$-degrees as long as $t - s \leq N$. Then in degrees $k \leq N - 1$, $\ell_k(Y)$ is torsion-free.*

For any connected Lie group G, there is a choice of BG which is a CW complex with finitely many cells in each dimension, so this hypothesis does not worry us.

Proof. By assumption, $t - s$ is even for all nonzero classes on the E_2-page with $t - s \leq N$; since Adams differentials change the parity of $t - s$, this forces all differentials in that range to vanish. Then, all extension questions for multiplication by p in that range are resolved by the h_0-action: since the E_∞-page for $t-s \leq N-1$ is a direct sum of h_0-towers, there can be no hidden extensions by p in this range. Since Y has finitely many cells in each dimension, $\ell_k(Y)$ is a finitely generated $\mathbb{Z}_{(p)}$-module for each k, so we conclude that, in the range claimed, $\ell_k(Y)$ is a free $\mathbb{Z}_{(p)}$-module of finite rank. $\square$

Remark A.10. Because $|v_1|$ is even, (A.8) implies $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ consists of h_0-towers in even degrees.

Lemma A.11 (Adams-Priddy [AP76, §3])**.** *Up to multiplication by a unit in $(\mathbb{Z}/p)^\times$, there is a unique nonzero $\mathcal{E}(1)$-module map $f\colon \Sigma^{-1}\mathcal{E}(1) \to \Sigma^{-1}\mathbb{Z}/p$. Given any such map, let $\hat{\mathcal{O}} \coloneqq \ker(f)$; the isomorphism type of $\hat{\mathcal{O}}$ does not depend on f. Moreover, there is an $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p)$-equivariant isomorphism $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\hat{\mathcal{O}}, \mathbb{Z}/p) \cong \mathrm{Ext}^{s+1,t+1}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p)$, so $\mathrm{Ext}_{\mathcal{E}(1)}(\hat{\mathcal{O}}, \mathbb{Z}/p)$ consists of h_0-towers in even degrees.*

For any $\mathcal{E}(1)$-module M, let $M_{\geq d}$ denote the $\mathcal{E}(1)$-submodule generated by homogeneous classes in degrees d and greater.

Definition A.12. We say that two $\mathcal{E}(1)$-modules M and N are *isomorphic up to degree d*, denoted $M \cong_{<d} N$, if there is an $\mathcal{E}(1)$-module isomorphism $M/M_{\geq d} \xrightarrow{\cong} N/N_{\geq d}$.

Proposition A.13. *Let Y be a CW complex with finitely many cells in each dimension. Suppose that $H^*(Y; \mathbb{Z}/p)$ is isomorphic up to degree d to a direct sum of copies of $\Sigma^{2m_i}\mathbb{Z}/p$ and $\Sigma^{2n_j}\hat{\mathcal{O}}$ for various m_i, n_j. Then for $k \leq d-2$, $\ell_*(Y)$ has no p-torsion.*

Proof. Use the long exact sequence in Ext associated to the short exact sequence $0 \to M_{\geq n} \to M \to M/M_{\geq n} \to 0$ to show the map $M \to M/M_{\geq d}$ induces an isomorphism in Ext in degrees $d-2$ and below. Therefore for the purpose of calculating $\ell_*(Y)$ in degrees $d-2$ and below, we may replace $H^*(Y; \mathbb{Z}/p)$ with a sum of shifts of $\mathbb{Z}/p$ and $\hat{\mathcal{O}}$ by even degrees. The result then follows from Lemma A.9 and the observations in Remark A.10 and Lemma A.11 that $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ and $\mathrm{Ext}_{\mathcal{E}(1)}(\hat{\mathcal{O}}, \mathbb{Z}/p)$ consist of h_0-towers in even degrees. $\square$

Thus to prove Theorem A.1, it would suffice to prove the following assertions.

Proposition A.14. *The following are isomorphisms of $\mathcal{E}(1)$-modules up to degree* 12*.*

- *For $G = G_2$, F_4, E_6, E_7, E_8, and* Spin_7*,* $\widetilde{H}^*(BG;\mathbb{Z}/2) \cong_{<12} \Sigma^4\mathcal{O} \oplus \Sigma^8\mathbb{Z}/2$.
- *For $G = \mathrm{Spin}_8$ and Spin_9,* $\widetilde{H}^*(BG;\mathbb{Z}/2) \cong_{<12} \Sigma^4\mathcal{O} \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^8\mathbb{Z}/2$.
- *For $n \geq 10$,* $\widetilde{H}^*(B\mathrm{Spin}_n;\mathbb{Z}/2) \cong_{<12} \Sigma^4\mathcal{O} \oplus \Sigma^8\mathbb{Z}/2 \oplus \Sigma^8\mathcal{O}$.
- *For $G = F_4, E_6, E_7$, and E_8,* $\widetilde{H}^*(BG;\mathbb{Z}/3) \cong_{<12} \Sigma^4\mathcal{O} \oplus \Sigma^8\mathbb{Z}/3$.
- $\widetilde{H}^*(BE_8;\mathbb{Z}/5) \cong_{<12} \Sigma^4\mathbb{Z}/5$.

Proof. In [LY22], the authors compute the $\mathcal{A}(1)$-module structure on $H^*(BG;\mathbb{Z}/2)$ in the degrees we need for $G = \mathrm{Spin}_n$, G_2, F_4, E_6, E_7, and E_8, from which the $\mathcal{E}(1)$-module structures in the theorem statement follow.

The assertion for $H^*(BF_4;\mathbb{Z}/3)$ follows from the products and Steenrod operations given by Toda [Tod73, Theorems I, II, III]. The cohomology $H^*(BE_6;\mathbb{Z}/3)$, can be computed from [Bor61, Théorème 2.3] where the author computes $H^*(E_6;\mathbb{Z}/3)$. The cohomology $H^*(BE_7;\mathbb{Z}/3)$ can be computed from [Ara61, Theorem 8] and $H^*(BE_8;\mathbb{Z}/3)$ can be computed from [Ara61, Theorem 9], where the author computes $H^*(E_7;\mathbb{Z}/3)$ and $H^*(E_8;\mathbb{Z}/3)$ respectively together with Steenrod powers on the generators; the Kudo transgression theorem [Kud56] determines the Steenrod powers in the cohomology of the classifying spaces. The assertion for BE_8 at $p = 5$ follows from Borel [Bor61, Théorème 2.3] calculating $H^*(E_8;\mathbb{Z}/5)$ together with a quick transgression argument. □

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