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A TALE OF TWO PERIODICITIES

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Abstract. We provide a common explanation for two 4-periodic phenomena arising in the study of connective real K-theory: first, the family of twists of ko over $B\mathbb{Z}/2$, and second, the torsion in the Picard group of Q_1-local $\mathcal{A}(1)$-modules. These phenomena have been previously studied, as they lead to 4-periodic patterns in Smith homomorphisms, resp. in the ko-homology of $(B\mathbb{Z}/2)^n$. In this paper, we exhibit both of these 4-periodicities as consequences of an isomorphism of two extraspecial groups of order 32.

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0. Introduction

In topology, many occurrences of the number 8 have the same origin: Bott periodicity in KO-theory. Quite different-looking 8s, to do with quadratic forms, bordism groups, or Clifford algebras, can be identified by theorems in algebra, geometry, or topology [Ran].

In prior work joint with Devalapurkar, Liu, Stehouwer, and Thorngren [DDK+25, DDK+26, DKS26], we learned of two different 4-periodic phenomena related to connective ko-theory. As a parallel to the connections between all these 8s, we wondered whether these two 4-periodicities can be identified.

These two periodicities are:

- the 4-periodic family of twists of spin bordism and ko-theory over $B\mathbb{Z}/2$, as discussed in [KTTW15, TY19, HKT20, COSY20, WWZ20, DDK+25, DDK+26], and
- the 4-periodic pattern in the ko-homology of $B(\mathbb{Z}/2)^n$ arising from the Picard group of Q_1-local stably invertible $\mathcal{A}(1)$-modules, as studied by [Yu95, Bru14].

The former periodicity comes from the Thom isomorphism: both ko-theory and spin bordism have Thom isomorphisms for *spin* vector bundles, but either theory evaluated on the Thom space of a non-spin vector bundle can be identified with a *twisted* bordism or ko-group over the base space. Over $B\mathbb{Z}/2$, there are four inequivalent twists, corresponding to the trivial rank-zero bundle, $\sigma - 1$, $2\sigma - 2$, and $3\sigma - 3$, where $\sigma \to B\mathbb{Z}/2$ is the universal bundle. Once we take four copies of σ, we

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again have an untwisted Thom isomorphism: 4σ admits a spin structure. In quantum physics, this collection of four twisted structures corresponds to a set of four types of fermionic systems: those with a certain symmetry operator that is either unitary or antiunitary and which squares either to 1 or to -1; see e.g. the references above for details.

Meanwhile, one can discover the latter periodicity when computing ko-homology via the Adams spectral sequence. The input data to this computation is a module over the subalgebra $\mathcal{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ of the mod 2 Steenrod algebra. Yu [Yu95] showed that, after localizing at the collection of Q_1-Margolis homology isomorphisms (see Definition 2.1), the category of $\mathcal{A}(1)$-modules has exactly four isomorphism classes of stably invertible objects up to suspension.

A priori, these two 4-periodic phenomena do not have much in common. We began to wonder whether they were related in prior work of two of us and Stehouwer in [DKS26], where both phenomena played a role in computing twisted ko-theory of $(B\mathbb{Z}/2)^n$ and a corresponding summand of twisted spin bordism.

In this paper, we identify the common origin of both periodicities. Specifically, we define a commutative monoid $\mathbb{M}$ whose elements are certain extraspecial 2-groups and whose monoid operation, the *fermionic tensor product*, is a generalization of the central product (see Proposition 1.10). $\mathbb{M}$ maps to our two periodicities in the following ways.

- Taking twisted cohomology defines a homomorphism from $\mathbb{M}$ to the Picard group of Q_1-local $\mathcal{A}(1)$-modules.
- Passing to the *diagonal subgroup* (see Example 1.16) defines a twist of ko over $B\mathbb{Z}/2$.

We prove in Theorem 0.6 that the torsion $\mathbb{Z}/4$ subgroup of the group completion of $\mathbb{M}$ admits surjective maps to the corresponding $\mathbb{Z}/4$ subgroups in the $\mathbb{Z}/2$-twists of spin bordism and the stable Q_1-local Picard group, respectively.

Now we go into more detail about our main results. In Example 1.16, we define extraspecial 2-groups $E_{\ell,k}$ of order $2^{\ell+k+1}$ for all $\ell, k \geq 0$. Heuristically, $E_{\ell,k}$ is defined by taking ℓ copies of Pin_1^+ and k copies of Pin_1^-, then asking them to commute in the graded sense. Then (Definition 1.21) we define $\mathbb{M} \coloneqq \{E_{\ell,k} \mid \ell, k \geq 0\}$. The fermionic groups $E_{\ell,k}$ within $\mathbb{M}$ correspond, after taking a twisted group algebra, to the Clifford algebras $C\ell_{\ell,k}$. There is an isomorphism $E_{4,0} \cong E_{0,4}$ (see Proposition 1.20), corresponding to the isomorphism $C\ell_{0,4} \cong C\ell_{4,0}$. This yields an important relation in $\mathbb{M}$, which leads to our first main result.

Theorem 0.1. *The monoid homomorphism*

$$\mathbb{M} \xrightarrow[(3.7)]{\Phi} \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) \xrightarrow[(2.10)]{y} \mathbb{Z} \oplus \mathbb{Z}/4 \tag{0.2}$$

defined in Corollary 3.7 is the group completion.

The map Φ is the twisted cohomology map mentioned above. Theorem 2.10 includes that y is an isomorphism, so Φ on its own is also a group completion; we compose with y to make the 4-periodicity easier to see.

The $\mathbb{Z}/4$ torsion summand of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ constructed in Yu's theorem [Yu95] (Theorem 2.10) is somewhat mysterious, so it is nice that Theorem 0.1 gives a group-theoretical explanation for it.

In Corollary 1.28, we show that the map sending $E_{\ell,k}$ to the twisting data of the diagonal subgroup $\Delta_{\ell,k} \subset E_{\ell,k}$ defines a monoid homomorphism

$$\delta \colon \mathbb{M} \longrightarrow [B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}]. \tag{0.3}$$

Since $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ is an abelian group, by Theorem 0.1 there is a unique group homomorphism

$$\widetilde{\delta}\colon \mathbb{Z}\oplus\mathbb{Z}/4 \longrightarrow [B\mathbb{Z}/2, BO/B\mathrm{Spin}] \tag{0.4}$$

satisfying $\delta = \widetilde{\delta}\circ y\circ\Phi$. This suggests an obvious way to compare these two periodicities: if $t\colon \mathbb{Z}/4\hookrightarrow\mathbb{Z}\oplus\mathbb{Z}/4$ is the inclusion of the torsion subgroup, we could take the comparison map

$$\Psi\colon \mathbb{Z}/4 \xhookrightarrow{t} \mathbb{Z}\oplus\mathbb{Z}/4 \xrightarrow{\widetilde{\delta}} [B\mathbb{Z}/2, BO/B\mathrm{Spin}] \xrightarrow[(2.23)]{\cong} \mathbb{Z}/4. \tag{0.5}$$

However, we show in Proposition 4.3 that this map is *not* an isomorphism! Instead, it sends $1\mapsto 2$.

Nonetheless, these two 4-periodicities are still related, as the fermionic group $E_{1,0}$ maps to generators of both $\mathbb{Z}/4$s. In §4 we comment on why the comparison map Ψ does not work: there is an "off-by-one error" in that $\Phi(E_{1,0})$ is a suspension of the generator of the torsion subgroup of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$, and this is the obstruction to Ψ being an isomorphism.

Once we correct for this shift, the suspense is over, and we obtain a correspondence between these two periodicities. Specifically, in Lemma 2.12, we construct a projection homomorphism ς from $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ onto its torsion subgroup by suspending or desuspending a Q_1-locally invertible $\mathcal{A}(1)$-module until its image in the Picard group is torsion. This suffices to fix the "off-by-one" issue noted above.

Theorem 0.6. *Both maps $\widetilde{\delta}\circ y$ and ς in the diagram*

$$\begin{array}{ccc} & \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) & \\ {\scriptstyle \widetilde{\delta}\circ y}\swarrow\ {\scriptstyle (0.4,\,2.10)} & & \ {\scriptstyle \varsigma}\searrow\ {\scriptstyle (2.12)} \\ [B\mathbb{Z}/2, BO/B\mathrm{Spin}] & & \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])_{\mathrm{tors}} \\ \cong\big\downarrow{\scriptstyle (2.23)} & & {\scriptstyle (2.10)}\big\downarrow\cong \\ \mathbb{Z}/4 & & \mathbb{Z}/4 \end{array} \tag{0.7}$$

send $\Phi(E_{1,0})$ to a generator, hence are surjective.

Thus, after applying ς to desuspend $\Phi(E_{1,0})$, restricting from $E_{\ell,k}$ to its diagonal subgroup $\Delta_{\ell,k}$—i.e. applying the map $\widetilde{\delta}\circ y$ in (0.7)—identifies the two $\mathbb{Z}/4$s we study in this note, hence also the two 4-periodicities they define. As briefly discussed in [DKS26, Remark 5.46], our primary goal in this note is to understand whether these two four-periodic phenomena were related, and Theorem 0.6 offers a strong answer: in this sense, they're the *same* 4-periodicity.

It would be interesting to upgrade the correspondence in Theorem 0.6 into a canonically defined group isomorphism $\mathbb{Z}/4\xrightarrow{\cong}\mathbb{Z}/4$—one can use Theorem 0.6 to produce an isomorphism like this, but only after making choices.

Remark 0.8. The four periodicities in this paper yield physical interpretations. Just as Bott periodicity informs the classification of topological insulators and superconductors [Kit09], the four periodicities in this paper give rise to a periodic structure in the classification of more general, interacting versions of these fermionic quantum systems. This structure was originally observed by Queiroz–Khalaf–Stern [QKS16] and two of us, joint with Stehouwer, studied it mathematically in [DKS26].

Outline. We begin in §1 with a brief introduction to fermionic groups. Though there are already good introductions to fermionic groups in [Ben88, Sto98, Ste22, DKS26], in this paper we need

a symmetric monoidal ∞-category of discrete fermionic groups, so take a more homotopical viewpoint than those references. We then introduce our fundamental examples $E_{\ell,k}$ (Example 1.16), use them to build the commutative monoid $\mathbb{M}$ (Definition 1.21) and the diagonal subgroups $\Delta_{\ell,k}$ (1.19), and show that taking the twisting data of the diagonal defines a monoid homomorphism $\mathbb{M} \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ (Corollary 1.28).

In §2.1, we introduce Q_k-local stable module categories of $\mathcal{A}(n)$-modules, then give the first of our two periodicities, which arises as the $\mathbb{Z}/4$ subgroup of the Q_1-local stable Picard group of $\mathcal{A}(1)$-modules. The main result in this subsection is Yu's calculation [Yu95, Theorem 2.1] (here Theorem 2.10) that this Picard group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/4$.

In §2.2, we compute $[B\mathbb{Z}/2, BO/B\mathrm{Spin}] \cong \mathbb{Z}/4$ (Lemma 2.23) and interpret it as giving a 4-periodic sequence of twists of real K-theory, hence a $\mathbb{Z}/8 \times \mathbb{Z}/4$-grading on the twisted KO-cohomology ring of $B\mathbb{Z}/2$.

In §3, we construct the monoid homomorphism $\Phi\colon \mathbb{M} \to \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ that links these two periodicities together.

Finally, in §4, we prove our main theorems relating these two 4-periodicities: Theorems 0.1 and 0.6 and Proposition 4.3.

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Usage of AI. One of us consulted Claude during literature review, but we did not use AI to write or prove the results in this paper.

1. Fermionic groups

Let A be an E_∞-algebra and let $\mathcal{T}op_A$ denote the slice ∞-category of spaces X with a map $f_X\colon X \to A$. Maps in this ∞-category are continuous maps of spaces that commute with the maps f_X to A.

Proposition 1.1 (Ando–Blumberg–Gepner [ABG18, Proposition 6.12])**.** *For A an E_∞-algebra in spaces, there is a natural symmetric monoidal ∞-category structure on $\mathcal{T}op_{/A}$ whose value on two objects $(X, f_X\colon X \to A)$ and $(Y, f_Y\colon Y \to A)$ is the Cartesian product and external sum*

$$(X \times Y, \mu \circ (f_X \times f_Y)\colon X \times Y \to A), \tag{1.2}$$

where $\mu\colon A \times A \to A$ is the product on A.

See Beardsley–Luecke–Morava [BLM26a, Remark 5.10] for a little more detail on the relationship between Ando–Blumberg–Gepner's construction and $\mathcal{T}op_{/A}$. Naturality in Proposition 1.1 is with respect to composition with maps of E_∞-spaces.

Definition 1.3. Give $B\mathrm{O}$ and $B\mathrm{Spin}$ the E_∞-structures arising from the direct sum of vector bundles, resp. of spin vector bundles. Then, let $B\mathrm{O}/B\mathrm{Spin}$ be the cofiber, in E_∞-spaces, of the forgetful map $B\mathrm{Spin} \to B\mathrm{O}$.

Lemma 1.4 ([DY23, Proposition 1.37 and Lemma 1.41])**.** *As spaces, there is a preferred homotopy equivalence $\phi\colon B\mathrm{O}/B\mathrm{Spin} \xrightarrow{\simeq} K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$, but this does not extend to an equivalence of E_1-spaces.*

Definition 1.5. The category $\mathcal{F}erm\mathcal{G}rp$ of *(discrete)*[1] *fermionic groups* is the full sub-∞-category of $\mathcal{T}op_{/(B\mathrm{O}/B\mathrm{Spin})}$ on the objects $(X, f_X\colon X \to B\mathrm{O}/B\mathrm{Spin})$ such that X is connected and 1-truncated.

The property of being connected and 1-truncated is preserved by Cartesian products, so the symmetric monoidal structure on $\mathcal{T}op_{/(B\mathrm{O}/B\mathrm{Spin})}$ restricts to a symmetric monoidal ∞-category structure on $\mathcal{F}erm\mathcal{G}rp$.

Remark 1.6. The name *fermionic group* comes from physics, where these objects may be used to describe the symmetries of quantum systems of fermions; e.g. electrons.

Now that we have a well-behaved category of fermionic groups, let's see how to actually work with them.

Definition 1.7. Let $X \to B\mathrm{O}/B\mathrm{Spin}$ be a fermionic group. We define the following invariants associated to X:

(1) The *bosonic group* $G_b \coloneqq \pi_1(X)$. Since X is connected and 1-truncated, $X \simeq BG_b$.
(2) The composition

$$\theta\colon BG_b \simeq X \to B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{O}/B\mathrm{SO} \simeq B\mathrm{O}_1, \tag{1.8}$$

induced by the forgetful map $B\mathrm{Spin} \to B\mathrm{SO}$, can be reinterpreted as a class in $H^1(BG_b; \mathbb{Z}/2) \cong \mathrm{Hom}(G_b, \mathbb{Z}/2)$, which we will also call θ.
(3) The map $B\mathrm{O}_1 \to B\mathrm{O}$ classifying the universal bundle induces a map $B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{SO}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2,2)$. Proceeding as in the previous bullet point, we obtain a class $\omega \in H^2(BG_b; \mathbb{Z}/2)$. This is equivalent data to an isomorphism class of extension of G_b by $\mathbb{Z}/2$, which we call G_f.

The pair (θ, ω) is called the *twisting data* of X.

Lemma 1.9. *The data (G_b, θ, ω) determine a fermionic group up to isomorphism in the homotopy category of $\mathcal{F}erm\mathcal{G}rp$.*

Proof. As discussed above, G_b determines X up to homotopy, since $X \simeq BG_b$. The homotopy equivalence $B\mathrm{O}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ mentioned above sends the map $f_X\colon X \to B\mathrm{O}/B\mathrm{Spin}$ to the twisting data $(\theta, \omega)\colon BG_b \to K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$, hence determines f_X up to homotopy equivalence. To finish, observe that in the homotopy category of $\mathcal{T}op_{/Z}$ for a space Z, an isomorphism is a homotopy equivalence of spaces that commutes, up to homotopy, with the maps down to Z. □

[1] We emphasize "discrete" because, in many cases in prior literature, one works with a category of fermionic topological groups.

We will generally refer to a fermionic group either as the data (G_b, θ, ω), or just as G_f when the rest of the data is clear from context. The monoidal product of G_f and H_f in $\mathcal{F}erm\mathcal{G}rp$, using the symmetric monoidal structure described above, will be denoted $G_f \hat{\times} H_f$.

Proposition 1.10. *Let G_f and H_f be fermionic groups with twisting data $(G_b, \theta_G, \omega_G)$ and $(H_b, \theta_H, \omega_H)$, respectively. Then $G_f \hat{\times} H_f$ may be characterized, up to isomorphism in* $\mathrm{Ho}(\mathcal{F}erm\mathcal{G}rp)$*, in either of the following ways.*

(1) *The fermionic group K_f whose bosonic group is $K_b \coloneqq G_b \times H_b$ and whose twisting data is*

$$(\theta_K, \omega_K) = (\theta_G + \theta_H, \omega_G + \theta_G \theta_H + \omega_H). \tag{1.11}$$

(2) *The* fermionic tensor product *of G_f and H_f as described by Benson* [Ben88, §7]*: as a set, the direct product $G_f \times H_f$, with multiplication*

$$(g_1 \hat{\otimes} h_1)(g_2 \hat{\otimes} h_2) = \begin{cases} (-1)^F g_1 g_2 \hat{\otimes} h_1 h_2, & \text{if } \theta(h_1) = 1 \text{ and } \theta(g_2) = 1 \\ g_1 g_2 \hat{\otimes} h_1 h_2, & \text{if } \theta(h_1) = 0 \text{ or } \theta(g_2) = 0. \end{cases} \tag{1.12}$$

Proof. Part (2) follows from (1) and [DKS26, Lemma 1.12], so we focus on part (1). The monoidal product on $\mathcal{F}erm\mathcal{G}rp$ is the Cartesian product $BG_b \times BH_b$, which is homotopy equivalent to $B(G_b \times H_b)$, so we may take $K_b \coloneqq G_b \times H_b$. To compute the twisting data of $G_f \hat{\times} H_f$, it suffices to understand how the E_∞-structure on $B\mathrm{O}/B\mathrm{Spin}$ behaves under the (non-E_1) homotopy equivalence $B\mathrm{O}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$, which sends a map $B\mathrm{O}/B\mathrm{Spin}$ to a pair of degree-1 and degree-2 $\mathbb{Z}/2$-cohomology classes. In [DY23, (1.42)] (see also [FH21, (10.9)–(10.11)]), this identification is shown to produce the formula (1.11). □

Remark 1.13. For a fixed finite group G_b, one can consider the set of isomorphism classes of fermionic groups G_f with bosonic group isomorphic to G_b. By Lemma 1.9, we may equivalently consider the collection of different twisting data over BG_b. As a set, this is $[BG_b, B\mathrm{O}/B\mathrm{Spin}] \cong H^1(BG_b; \mathbb{Z}/2) \times H^2(BG_b; \mathbb{Z}/2)$; using the grouplike E_∞-structure on $B\mathrm{O}/B\mathrm{Spin}$, this becomes an abelian group, a direct summand of the *Brauer group* of BG_b [DK70, BLM26a]. Explicitly, the product in this group structure on the set $H^1 \times H^2$ has the formula (1.11).

Warning: we will write elements of $[X, B\mathrm{O}/B\mathrm{Spin}]$ as elements of $H^1(X; \mathbb{Z}/2) \times H^2(X; \mathbb{Z}/2)$. However, the group operation (1.11) is *not* the product of the operations on H^1 and H^2.

Example 1.14 ($\mathrm{Pin}_1^\pm$)**.** Recall that $\mathrm{Pin}_1^\pm$ is the group of units in the Clifford algebra on one generator e, with $e^2 = \pm 1$. Thus $\mathrm{Pin}_1^+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathrm{Pin}_1^- \cong \mathbb{Z}/4$. In both cases, quotienting by $\{\pm 1\}$ gives $\mathrm{O}_1 \cong \mathbb{Z}/2$, with e mapping to the generator. We thus obtain two fermionic groups $\mathrm{Pin}_1^\pm$ with bosonic part $(\mathrm{Pin}_1^\pm)_b \cong \mathbb{Z}/2$.

Write $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[x]$ with $|x| = 1$.

- We choose $\theta = x$ for both Pin_1^+ and Pin_1^-. This represents the fact that the quotient maps $\mathrm{Pin}_1^\pm \to \mathrm{O}_1$ are both surjective.
- The class ω is the extension class of $\mathbb{Z}/2 \to \mathrm{Pin}_1^\pm \to \mathbb{Z}/2$. The extension is split for Pin_1^+ and nonsplit for Pin_1^-, so $\omega = 0$ for Pin_1^+ and $\omega = x^2$ for Pin_1^-.

The twisting data for Pin_1^+ is thus $(x, 0)$ and for Pin_1^- is (x, x^2).

Example 1.15. The *Conway fermionic group* has $G_f = \mathrm{Co}_0$ and $G_b = \mathrm{Co}_1$, with θ trivial. Here Co_1 is the sporadic simple group discovered by Conway in [Con68, Con69], where it is called $\cdot 1$, and Co_0 is its double cover, called $\cdot 0$ by Conway (*ibid.*), which is the automorphism group of the

Leech lattice. See Johnson-Freyd–Treumann [JFT20, §6] and Albert–Okada–Tachikawa [AOT26] for applications of this fermionic group to questions in physics.

Example 1.16 ([DKS26, Definition 1.36])**.** Define $E_{\ell,k}$ to be the fermionic tensor product

$$E_{\ell,k} \coloneqq \underbrace{\mathrm{Pin}_1^+ \hat{\times} \cdots \hat{\times} \mathrm{Pin}_1^+}_{\ell \text{ copies}} \hat{\times} \underbrace{\mathrm{Pin}_1^- \hat{\times} \cdots \hat{\times} \mathrm{Pin}_1^-}_{k \text{ copies}}. \tag{1.17}$$

Thus $(E_{\ell,k})_b \cong (\mathbb{Z}/2)^{\ell+k}$, and

$$E_{\ell,k} \hat{\times} E_{\ell',k'} \cong E_{\ell+\ell',k+k'}. \tag{1.18}$$

See [DKS26, (1.40)] for the twisting data of $E_{\ell,k}$. Define the *diagonal subgroup* $\Delta_{\ell,k} \subset E_{\ell,k}$ to be the pullback

$$\begin{array}{ccc} \Delta_{\ell,k} & \longrightarrow & E_{\ell,k} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z}/2 & \xrightarrow{1\mapsto(1,\dots,1)} & (\mathbb{Z}/2)^{\ell+k}. \end{array} \tag{1.19}$$

$\Delta_{\ell,k}$ is a fermionic group with bosonic group $\mathbb{Z}/2$ and twisting data pulled back from that of $E_{\ell,k} \to (\mathbb{Z}/2)^{\ell+k}$.

The first examples are trivial: the inclusions $\Delta_{1,0} \to E_{1,0}$ and $\Delta_{0,1} \to E_{0,1}$ are isomorphisms. The inclusion $\Delta_{1,1} \to E_{1,1}$ is more interesting: one can show that there is an isomorphism $g\colon D_8 \to E_{1,1} = \mathrm{Pin}_1^+ \hat{\times} \mathrm{Pin}_1^-$ sending a $\pi/2$ rotation r to $(1, e_1)$ and a reflection s to $(e_1, 1)$. (Here we are using the description of the fermionic tensor product from part (2) of Proposition 1.10.) $\Delta_{1,1}$ is the preimages of $(0,0)$ and $(1,1)$ under $E_{1,1} \to (\mathbb{Z}/2)^2$, so $\Delta_{1,1} = \{\pm(1,1), \pm(e_1,e_1)\}$. Therefore $g^{-1}(\Delta_{1,1})$ is the order-4 group $\{1, r^2, s, sr^2\}$. In other words, $\Delta_{1,1}$ is "trivial": its bosonic group is $\mathbb{Z}/2$ and its twisting data is $(0,0)$.

Proposition 1.20. *For $\ell, k \geq 0$, there is an isomorphism of fermionic groups $\varphi_{\ell,k}\colon E_{\ell+4,k} \xrightarrow{\cong} E_{\ell,k+4}$; restricted to $\Delta_{\ell+4,k} \subset E_{\ell+4,k}$, $\varphi_{\ell,k}$ defines an isomorphism $\Delta_{\ell+4,k} \cong \Delta_{\ell,k+4}$.*

Except for the bit about the diagonal subgroups, Proposition 1.20 is proven in [DKS26, Corollary 2.31], which constructs an explicit isomorphism $\varphi_{0,0}$ for the case $(\ell,k) = (0,0)$, then takes the fermionic tensor product with $E_{\ell,k}$ to define $\varphi_{\ell,k}$. The reader can check directly that this choice of $\varphi_{\ell,k}$ restricts to an isomorphism of the diagonal subgroups.

Definition 1.21. Let $\mathbb{M}$ be the commutative monoid formed from the set

$$\{E_{\ell,k} \mid \ell, k \geq 0\}/\cong \tag{1.22}$$

equipped with the fermionic tensor product $\hat{\times}$.

Lemma 1.23. *The map $\varepsilon\colon \mathbb{N}^2 \to \mathbb{M}$ defined by $(\ell,k) \mapsto E_{\ell,k}$ induces an isomorphism*

$$\widetilde{\varepsilon}\colon \mathbb{N}^2/((4,0)-(0,4)) \xrightarrow{\cong} \mathbb{M}. \tag{1.24}$$

Proof. Proposition 1.20 shows that ϵ factors through $\mathbb{N}^2/((4,0)-(0,4))$, so we have a surjective monoid homomorphism $\widetilde{\varepsilon}$ as in the lemma statement, and we are done if we can show $\widetilde{\varepsilon}$ is injective.

Let $\psi\colon \mathbb{M} \to \mathbb{N} \times \mathbb{Z}/8$ send $E_{\ell,k} \mapsto (\ell+k, \ell-k \bmod 8)$. To see that this is a well-defined monoid homomorphism, we check that isomorphic fermionic groups $E_{\ell,k} \cong E_{\ell',k'}$ have equivalent images. Isomorphic groups must have identical cardinality, so we have $\ell + k = \ell' + k'$.

For an additional invariant, we can use the following. Given a fermionic group G_f, the *fermionic group algebra* [Ben88, §7] $\mathbb{R}^f[G_f]$ is defined to be the quotient

$$\mathbb{R}^f[G_f] \coloneqq \mathbb{R}[G_f]/(-1+(-1)^F), \tag{1.25}$$

where $(-1)^F$ is the unique nonzero element of the kernel of $G_f \to G_b$. The homomorphism $\theta\colon G_b \to \mathbb{Z}/2$ (see part (2) of Definition 1.7) gives $\mathbb{R}^f[G_f]$ the structure of a $\mathbb{Z}/2$-graded algebra over $\mathbb{R}$, and the Morita class as a $\mathbb{Z}/2$-graded algebra is an invariant of the isomorphism type of G_f.

Albuquerque–Majid [AM02] show that $\mathbb{R}^f[E_{\ell,k}]$ is isomorphic as a $\mathbb{Z}/2$-graded algebra to the Clifford algebra $C\ell_{\ell,k}$.[2] It is standard (see, e.g., [ABS64]) that $C\ell_{\ell,k}$ is $\mathbb{Z}/2$-graded Morita equivalent to $C\ell_{\ell',k'}$ if and only if $\ell - k \equiv \ell' - k' \bmod 8$. Thus $\ell - k \bmod 8$ is also an isomorphism invariant of $E_{\ell,k}$.

We want to show $\widetilde{\varepsilon}$ is injective, so we may as well show $\psi \circ \widetilde{\varepsilon}$ is injective. Suppose that $\ell, k, \ell', k' \in \mathbb{N}$ are such that $\ell + k = \ell' + k'$ and $\ell - k \equiv \ell' - k' \bmod 8$. Then there is some $x \in \mathbb{Z}$ such that $\ell - k = \ell' - k' + 8x$. Add this equation to $\ell + k = \ell' + k'$ to conclude that $2\ell = 2\ell' + 8x$ or $\ell = \ell' + 4x$, and thus $k = k' - 4x$. That is, (ℓ, k) and (ℓ', k') can be identified by applying the relation $(4,0) = (0,4)$ x times. □

Proposition 1.26. *For all $\ell, k, \ell', k' \geq 0$, the twisting data of $\Delta_{\ell+\ell',k+k'}$, as an element of the abelian group $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$, is the sum of the twisting data of $\Delta_{\ell,k}$ and the twisting data of $\Delta_{\ell',k'}$.*

Proof. Let $(\theta_1, \omega_1), (\theta_2, \omega_2) \in [BG_b, BO/B\mathrm{Spin}]$ be the twisting data of two fermionic groups $G_f^{(1)}, G_f^{(2)}$ whose bosonic groups are both isomorphic to G_b. The addition in the set of homotopy classes of maps to an E_∞-space A is defined as the pullback of the external sum (see Proposition 1.1) by the diagonal map.

Unspooling this, the sum of (θ_1, ω_1) and (θ_2, ω_2) (again, we are using the non-product group law (1.11)) is the pullback of the twisting data of $G_b \times G_b = (G_f^{(1)} \mathbin{\hat{\times}} G_f^{(2)})_b$ across the diagonal map $\Delta\colon G_b \to G_b \times G_b$.

Now let $G_b = \mathbb{Z}/2$, $G_f^{(1)} = \Delta_{\ell,k}$, and $G_f^{(2)} = \Delta_{\ell',k'}$. By definition, $\Delta_{\ell+\ell',k+k'}$ is the pullback

$$\begin{array}{ccc} \Delta_{\ell+\ell',k+k'} & \longrightarrow & E_{\ell+\ell',k+k'} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z}/2 & \xrightarrow[\Delta]{} & (\mathbb{Z}/2)^{\ell+\ell'+k+k'}. \end{array} \tag{1.27a}$$

By construction, this map factors through $\Delta_{\ell,k} \mathbin{\hat{\times}} \Delta_{\ell',k'}$:

$$\begin{array}{ccccccc} \Delta_{\ell+\ell',k+k'} & \longrightarrow & \Delta_{\ell,k} \mathbin{\hat{\times}} \Delta_{\ell',k'} & \longrightarrow & E_{\ell,k} \mathbin{\hat{\times}} E_{\ell',k'} & \xlongequal[(1.18)]{} & E_{\ell+\ell',k+k'} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \downarrow \\ \mathbb{Z}/2 & \xrightarrow[\Delta]{} & \mathbb{Z}/2 \times \mathbb{Z}/2 & \xrightarrow[(\Delta,\Delta)]{} & (\mathbb{Z}/2)^{\ell+k} \times (\mathbb{Z}/2)^{\ell'+k'} & = & (\mathbb{Z}/2)^{\ell+\ell'+k+k'} \end{array} \tag{1.27b}$$

Apply the general discussion above to the leftmost square in (1.27b) to see that, as we wanted to show, the twisting data of $\Delta_{\ell+\ell',k+k'}$ is the sum of the twisting data of $\Delta_{\ell,k}$ and $\Delta_{\ell',k'}$. □

[2] Our Clifford algebra conventions match [DKS26, Definition 1.47]: $C\ell_{\ell,k}$ is generated as an $\mathbb{R}$-algebra by $\ell + k$ odd, pairwise anticommuting elements of which ℓ square to 1 and k square to -1. In addition to Albuquerque–Majid [AM02] , see also Thiang [Thi15, Remark 6.4], Abłamowicz [Abł16], Ichikawa–Tachikawa [IT23, Lemma 2.7], and [DKS26, Corollary 1.67].

Corollary 1.28. *The map sending $E_{\ell,k}$ to the twisting data of $\Delta_{\ell,k}$ is a homomorphism of commutative monoids $\delta\colon \mathbb{M} \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$.*

Corollary 1.29. *If $\ell - k \equiv \ell' - k' \bmod 4$, there is an isomorphism of fermionic groups $\Delta_{\ell,k} \cong \Delta_{\ell',k'}$.*

Proof. By Proposition 1.20, the homomorphism $\mathbb{M} \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ from Corollary 1.28 identifies the images of $E_{4,0}$ and $E_{0,4}$. By Example 1.16, this homomorphism sends $E_{1,1} \mapsto 0$, hence the image of $E_{\ell,k}$ only depends on $\ell - k$. Combining these two facts, this homomorphism factors through the homomorphism $\mathbb{M} \to \mathbb{Z}/4$ sending $E_{\ell,k} \mapsto \ell - k \bmod 4$. □

Example 1.30. $E_{1,0}$ and $E_{0,1}$ are their own diagonal subgroups: by construction, $\Delta_{1,0} \to E_{1,0}$ is the pullback of the inclusion of the diagonal $\mathbb{Z}/2 \subset (E_{1,0})_b \cong \mathbb{Z}/2$, and the pullback of an isomorphism is an isomorphism. Likewise for $E_{0,1}$. Therefore $\delta(E_{1,0})$ and $\delta(E_{0,1})$ are just the twisting data of these two fermionic groups, which we calculated in Example 1.14: $\delta(E_{1,0}) = (x, 0)$ and $\delta(E_{0,1}) = (x, x^2)$ (in the notation of Remark 1.13).

Since the isomorphism type of $\Delta_{\ell,k}$ only depends on $\ell - k \bmod 4$ by Corollary 1.29, we can write down the four classes.

(0) When $\ell - k \equiv 0 \bmod 4$, $\Delta_{\ell,k} \cong \Delta_{1,1} \cong \mathbb{Z}/2 \times \{\pm 1\}$, with trivial twisting data, as we saw in Example 1.16.
(1) When $\ell - k \equiv 1 \bmod 4$, $\Delta_{\ell,k} \cong \Delta_{1,0} \cong \mathrm{Pin}_1^+$, as we saw in Example 1.16.
(2) When $\ell - k \equiv 2 \bmod 4$, one can show that $\Delta_{\ell,k} \cong \mathbb{Z}/4$ with $\theta = 0$, e.g. by computing it inside $E_{2,0}$.
(3) When $\ell - k \equiv 3 \bmod 4$, $\Delta_{\ell,k} \cong \Delta_{0,1} \cong \mathrm{Pin}_1^-$ as in Example 1.16.

2. The two periodicities

In this section, we introduce the two 4-periodicities that will be the main focus of this paper, as well as some background notions and definitions for them. These manifest as groups isomorphic to $\mathbb{Z}/4$, specifically the torsion subgroup of the Q_1-local stable Picard group of $\mathcal{A}(1)$-modules (Theorem 2.10) and (a nice subgroup of) the twists of KO-theory over $B\mathbb{Z}/2$ (Lemma 2.23), which we introduce in §2.1 and §2.2 respectively.

2.1. Torsion in the Picard group of Q_1-local $\mathcal{A}(1)$-modules.

Let $\mathcal{A}$ denote the *mod 2 Steenrod algebra*, the $\mathbb{Z}$-graded $\mathbb{Z}/2$-algebra of natural transformations $H^*(-;\mathbb{Z}/2) \to H^{*+k}(-;\mathbb{Z}/2)$ commuting with the suspension isomorphism. (The grading on $\mathcal{A}$ is such that a natural transformation increasing the degree of a cohomology class by k has degree k in $\mathcal{A}$.) $\mathcal{A}$ is generated by the *Steenrod squares* Sq^n in degree n for each $n \geq 1$, subject to relations called *Adem relations*. For any space or spectrum X, $H^*(X;\mathbb{Z}/2)$ is naturally a $\mathbb{Z}$-graded $\mathcal{A}$-module, essentially by definition of $\mathcal{A}$.

The Cartan formula is the comultiplication map in a cocommutative Hopf algebra structure on $\mathcal{A}$: see [Mil58, Corollary 1 and Appendix 1]. Thus the category of left $\mathcal{A}$-modules acquires a symmetric monoidal structure with respect to tensor product over $\mathbb{Z}/2$.

For $n \geq 0$, let $\mathcal{A}(n)$ denote the subalgebra of $\mathcal{A}$ generated by Sq^k with $k \leq 2^n$, which is a Hopf subalgebra. We will mostly focus on $\mathcal{A}(1)$ in this note. Whenever we refer to $\mathcal{A}$- or $\mathcal{A}(n)$-modules, we implicitly mean in the graded sense: that a degree-k element of $\mathcal{A}$ or $\mathcal{A}(n)$ acts on a homogeneous element x in a way that raises the degree of x by k.

Definition 2.1 ([AM71, Mar83])**.** Let $Q_0 \coloneqq \mathrm{Sq}^1$ and inductively define $Q_k \coloneqq [Q_{k-1}, \mathrm{Sq}^{2^k}]$. Thus $Q_k \in \mathcal{A}(n)$ for $k \leq n$. The Adem relations imply that $Q_k^2 = 0$, so for any $\mathcal{A}(n)$-module M and $k \leq n$, one may regard M as a chain complex with differential Q_k.

In this situation, the *Q_k-Margolis homology* of M, denoted $H_*(M; Q_k)$, is the homology of the above chain complex.

Since Q_k acts in a manner compatible with the grading, the $\mathbb{Z}$-grading of M induces a $\mathbb{Z}$-grading on $H_*(M; Q_k)$. For any $k \leq n$, Q_k-Margolis homology commutes with direct sums of $\mathcal{A}(n)$-modules by definition, and satisfies a Künneth formula with respect to tensor products.

Proposition 2.2 (Margolis [Mar83, Chapter 19, Proposition 17(b)])**.** *For $\mathcal{A}(n)$-modules M and N, and $k \leq n$, there is a natural isomorphism*

$$H_*(M \otimes N; Q_k) \cong H_*(M; Q_k) \otimes H_*(N; Q_k). \tag{2.3}$$

Moreover, $H_*(\mathcal{A}(n); Q_k) = 0$ [AM71, §4].

Definition 2.4 (Adams–Margolis [AM71, Definition 4.1], Margolis [Mar83, Chapter 23])**.** Let $n \in \mathbb{N}$, $k \leq n$, and $f \colon M \to N$ be a map of bounded-below $\mathcal{A}(n)$-modules.

(1) The map f is a *stable isomorphism* if its kernel and cokernel are free $\mathcal{A}(n)$-modules.
(2) The map f is a *Q_k-local stable isomorphism* if is an isomorphism on Q_k-Margolis homology.

By [Mar83, Chapter 14, Proposition 1], $f \colon M \to N$ is a stable isomorphism if and only if there are free $\mathcal{A}(n)$-modules L_1 and L_2 and an $\mathcal{A}(n)$-module isomorphism

$$\overline{f} \colon L_1 \oplus M \xrightarrow{\cong} L_2 \oplus N \tag{2.5}$$

such that $\overline{f}|_M = (0, f)$.

Proposition 2.6 (Wall [Wal62, Theorem 2])**.** *The following are equivalent for a map $f \colon M \to N$ of bounded-below $\mathcal{A}(1)$-modules.*

(1) *f is a stable isomorphism.*
(2) *f is both a Q_0-local and a Q_1-local stable isomorphism.*

For $n > 1$, Adams–Margolis [AM71, Theorem 4.2] gave a similar characterization of stable isomorphisms, but using Margolis homology for a more general class of operations than Q_k. Nevertheless, since $H_*(\mathcal{A}(n); Q_k) = 0$ for any $n \in \mathbb{N}$ and $k \leq n$, every stable isomorphism is a Q_k-local stable isomorphism.

For any $k \leq n$, it is immediate that the Q_k-local equivalences contain all $\mathcal{A}(n)$-module isomorphisms and satisfy a two-out-of-three property. Following [Lur17, Proposition 4.1.7.4], to obtain a good theory of localizations, we also need that Q_k-local equivalences are stable with respect to tensor products, which follows from the Künneth theorem (Proposition 2.2). Thus we conclude:

Theorem 2.7 (Ricka [Ric16, Proposition 2.6])**.** *There is a symmetric monoidal stable ∞-category $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}[Q_k^{-1}]$ which is the localization of the category of bounded-below $\mathcal{A}(n)$-modules at the Q_k-local equivalences, and the localization functor $L_k \colon \mathcal{M}od_{\mathcal{A}(n)} \to \mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}[Q_k^{-1}]$ is symmetric monoidal.*

It is also possible to construct the *stable module category* $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}$, the localization of the category of $\mathcal{A}(n)$-modules at the stable isomorphisms, as a symmetric monoidal ∞-category in a similar way. See Ricka [Ric16, Proposition 1.9] and Yang [Yan22, §3.5]. We will not need this so will not say much more about it.

Definition 2.8. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

(1) An object $x \in \mathcal{C}$ is *invertible* if there is another object $x^{-1} \in \mathcal{C}$ and an isomorphism $x \otimes x^{-1} \cong \mathbf{1}$.
(2) The *Picard group* of $\mathcal{C}$, denoted $\mathrm{Pic}(\mathcal{C})$, is the abelian group of isomorphism classes of invertible objects, with unit the isomorphism class of $\mathbf{1}$ and the group operation given by $\otimes$.

If $\mathcal{C}$ is a symmetric monoidal ∞-category, we define $\mathrm{Pic}(\mathcal{C})$ to be the Picard group of the homotopy category of $\mathcal{C}$.

See [HMS94, May01, vdMW21] for more background on Picard groups, and some example computations.

Example 2.9. The Picard groups of $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}$ are known for all n.

(1) Adams–Priddy [AP76, Theorem 3.7] constructed an isomorphism $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$; the unique nonzero torsion class is that of the *Joker* $\Sigma^{-2}(\mathcal{A}(1)/\mathrm{Sq}^3)$.
(2) Bhattacharya–Ricka [BR17, Theorem 4.6] showed that $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(2)}) \cong \mathbb{Z}^2$.
(3) Pan–Yan [PY26, Corollary 4.8] showed that for all $n \geq 3$, $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}) \cong \mathbb{Z}^2$.
(4) Gheorghe–Isaksen–Ricka [GIR18, Theorem 5.6] and Gao–Li [GL24, Theorem 4.3] prove that the Picard groups of the $\mathbb{C}$-, resp. $\mathbb{R}$-motivic analogues of $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}$ are both isomorphic to $\mathbb{Z}^4$.

We will be interested in the Q_1-local Picard group of $\mathcal{A}(1)$.

Theorem 2.10 (Yu [Yu95, Theorem 2.1])**.** *There is an isomorphism*

$$y\colon \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/4 \tag{2.11}$$

such that:

- $y^{-1}((1,0))$ *is the* Q_1*-localization of* $\Sigma\mathbb{Z}/2$*, and*
- $y^{-1}((0,1))$ *is* $T_1 \coloneqq \Sigma^{-1}H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2)$ *(which is* Q_1*-local).*

The isomorphism y *induces an isomorphism from the torsion summand of* $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ *to* $\mathbb{Z}/4$*; the module* T_i *in Figure 1 is invertible in* $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]$ *and* $y(T_i) = i \in \mathbb{Z}/4$*.*

See also Bruner [Bru14, Theorem 7.1 and Proposition 8.2], who gave another proof of Theorem 2.10 and also showed $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_0^{-1}]) \cong \mathbb{Z}$. As far as we know, these are the only values of k and n for which the isomorphism type of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(n)}[Q_k^{-1}])$ is known. In Figure 1, we draw Q_1-local $\mathcal{A}(1)$-modules T_1, T_2, T_3, and T_4 representing the four torsion classes in $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$.

The first of our two $\mathbb{Z}/4$s is the torsion subgroup of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$. It can be interpreted as a periodicity in the following way: if M is a bounded-below, Q_1-local $\mathcal{A}(1)$-module, then the isomorphism type of $M \otimes T_1^{\otimes \ell}$, modulo free $\mathcal{A}(1)$-module summands, is 4-periodic in ℓ. This fact was used to make computations of the ko-homology of $(B\mathbb{Z}/2)^n$ in [Yu95, Theorem A], and of certain twisted ko-homology groups over $(B\mathbb{Z}/2)^n$ in [DKS26, §5].

A corollary of Theorem 2.10 is that there is a preferred projection from $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ onto its torsion subgroup.

Lemma 2.12. *The function*

$$\varsigma\colon \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) \longrightarrow \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])_{\mathrm{tors}} \tag{2.13}$$

defined by suspending or desuspending a Q_1-locally invertible $\mathcal{A}(1)$-module until it is stably isomorphic to one of the modules T_i is a group homomorphism.

Proof. After applying the isomorphisms

$$\begin{aligned} y\colon \operatorname{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) &\xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/4 \\ y|_{\text{tors}}\colon \operatorname{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])_{\text{tors}} &\xrightarrow{\cong} \mathbb{Z}/4 \end{aligned} \tag{2.14}$$

from Theorem 2.10, ς is identified with the function $(a, b) \mapsto b$, which is a homomorphism. □

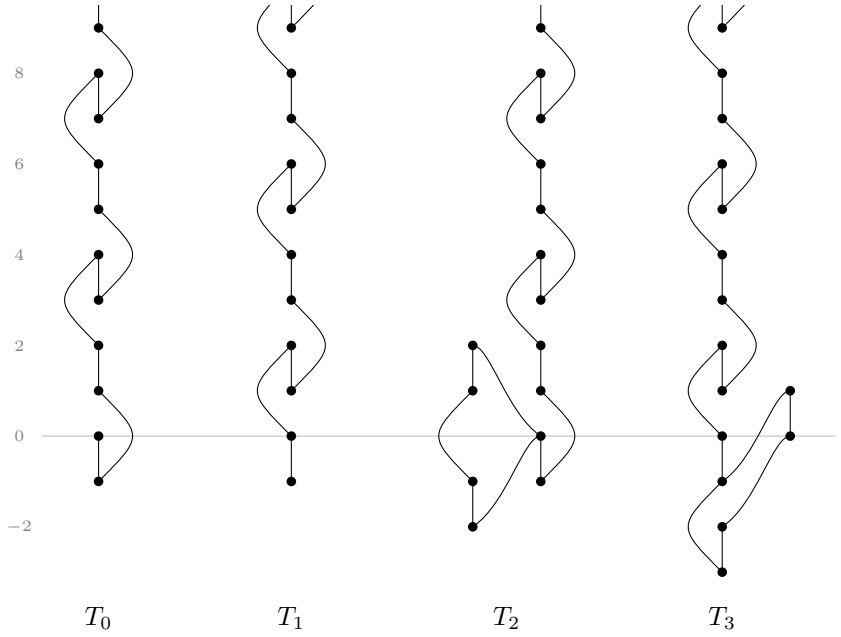


FIGURE 1. By Theorem 2.10, the torsion subgroup of $\operatorname{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$ is isomorphic to $\mathbb{Z}/4$. In this figure, the $\mathcal{A}(1)$-module T_i is in the class $i \in \mathbb{Z}/4 \cong \operatorname{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])_{\text{tors}}$. Each module T_i is the unique $\mathcal{A}(1)$-module with this property, up to isomorphism, which is Q_1-local and has no free $\mathcal{A}(1)$-module summands. Though we have but a finite amount of space, each T_i continues upward in the same 4-periodic pattern.

2.2. **Smith periodicity.** In this subsection, we discuss our second $\mathbb{Z}/4$, the group $[B\mathbb{Z}/2, BO/B\text{Spin}]$ (Lemma 2.23). We discuss how this applies to Picard-group-graded twisted KO-theory of $B\mathbb{Z}/2$ as well as a relation to families of Smith homomorphisms in bordism theory.

Notation 2.15. Recall the E_∞-space $BO/B\text{Spin}$ from Definition 1.3, and for an E_∞-ring spectrum R,[3] let $\text{GL}_1(R)$ be the ∞-group of units of R, as defined in May [May77, §III.2]. May–Quinn–Ray [May77, Lemma IV.2.6] construct an E_∞-map $T\colon BO/B\text{Spin} \to B\text{GL}_1(MT\text{Spin})$, where $MT\text{Spin}$ is the Thom spectrum whose homotopy groups are the spin bordism groups. Using the

[3]The construction works more generally for R an E_1-ring spectrum: see Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14a, ABG+14b]. We stick to only the level of generality we need.

Atiyah–Bott–Shapiro maps $MT\mathrm{Spin} \to ko \to KO$ [ABS64], which Joachim [Joa04] showed admit E_∞-structures, we then get E_∞-maps $T\colon B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{GL}_1(ko) \to B\mathrm{GL}_1(KO)$.[4]

Proposition 2.16. *$T\colon B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{GL}_1(ko)$ admits a section, and there is an equivalence of E_∞-spaces*

$$B\mathrm{GL}_1(ko) \simeq B\mathrm{O}/B\mathrm{Spin} \times B(B\mathrm{SO}_\otimes), \tag{2.17}$$

where $B\mathrm{SO}_\otimes$ is the space $B\mathrm{SO}$ with E_∞-structure induced from the tensor product of vector bundles.

Madsen–Snaith–Tornehave [MST77, Lemma 1.13] prove $B\mathrm{SL}_1(ko) \simeq B\mathrm{SO}/B\mathrm{Spin} \times B(B\mathrm{SO}_\otimes)$, and the proof of Proposition 2.16 is completely analogous—in particular, Proposition 2.16 is not a new result. See also Antieau–Gepner–Gómez [AGG14, §1], who prove a version of Proposition 2.16 but do not discuss E_∞-structures.

We care about $B\mathrm{GL}_1(R)$ because it grades twisted R-cohomology, as follows.

Definition 2.18 (Ando–Blumberg–Gepner [ABG18, Definition 1.2])**.** Let R be an E_∞-ring spectrum and $f\colon X \to B\mathrm{GL}_1(R)$ be a map. The *f-twisted R-cohomology groups* of X are

$$R^{*-f}(X) \coloneqq \pi_{-*}\mathrm{Map}_R(M^R f, X), \tag{2.19}$$

where $M^R f$ denotes the R-module Thom spectrum of f, defined in [ABG+14a, Definition 2.20].

Donovan–Karoubi [DK70] defined twisted KO-groups associated to a space X, $a \in H^1(X;\mathbb{Z}/2)$, and $b \in H^2(X;\mathbb{Z}/2)$. See also [Kar68].

Proposition 2.20 (Hebestreit–Sagave [HJ20])**.** *Donovan–Karoubi's twisted KO-theory groups are naturally isomorphic to the twisted KO-cohomology groups, in the Ando–Blumberg–Gepner sense, of the map*

$$X \xrightarrow{(a,b)} K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \xrightarrow[(1.4)]{\phi^{-1}} B\mathrm{O}/B\mathrm{Spin} \xrightarrow{T} B\mathrm{GL}_1(KO). \tag{2.21}$$

Since KO is an E_∞-ring spectrum, the KO-cohomology groups of a space X assemble into a $\mathbb{Z}$-graded ring, and in fact a $\mathbb{Z}/8$-graded ring thanks to Bott periodicity. For a fixed twist (a,b), the twisted KO-groups of X do not in general admit a graded ring structure, only a $KO^*(X)$-module structure.[5] This is similar to the structure present (or absent) in cohomology with local coefficients.

On the other hand, if one considers all twists at once, it is possible to multiply: given maps $f_1, f_2\colon X \to B\mathrm{GL}_1(KO)$ and classes $x_i \in KO^{n_i - f_i}(X)$, there is a product $x_1x_2 \in KO^{n_1+n_2-f_1-f_2}(X)$, such that [DK70]

$$KO^\bullet(X) \coloneqq \bigoplus_{f \in [X, B\mathrm{GL}_1(KO)]} KO^{*-f}(X) \tag{2.22}$$

has the data of a $\mathbb{Z}/8 \times [X, B\mathrm{GL}_1(KO)]$-graded ring. The subring in gradings $\mathbb{Z}/8 \times \{0\}$ is the untwisted KO-cohomology of X with its usual ring structure. This is a cousin of the $\mathbb{Z} \times H^1(X;\mathbb{Z}/2)$-grading on ordinary cohomology with local coefficients: see [Čad99, Las05, Wen24, BW25, DYY25] for computations using this graded ring structure.

[4] By definition, the space $B\mathrm{GL}_1(R)$ only depends on $\Omega^\infty(R)$, so the map $B\mathrm{GL}_1(ko) \to B\mathrm{GL}_1(KO)$ is in fact an equivalence of E_∞-spaces.

[5] However, in *K- and KO-homology*, there can be ring structures induced by the Pontrjagin product: see Douglas [Dou06, §2.2] and Freed–Hopkins–Teleman [FHT11, §3.6].

By Proposition 2.16, the $[X, B\mathrm{GL}_1(KO)]$-grading splits into two pieces: a grading by $[X, BO/B\mathrm{Spin}]$ and one by $[X, B(B\mathrm{SO}_\otimes)]$. The latter is in general quite mysterious,[6] so we will focus on the former.

Lemma 2.23. $[B\mathbb{Z}/2, BO/B\mathrm{Spin}] \cong \mathbb{Z}/4$.

Thus the twisted KO-cohomology groups of $B\mathbb{Z}/2$ form a $\mathbb{Z}/8 \times \mathbb{Z}/4$-graded ring. It would be interesting to compute this graded ring. Lemma 2.23 was stated without proof in [DY23, §1.2.3] and [LY24, Prop. C.37 (1)].

Proof. By Lemma 1.4, there is a homotopy equivalence $BO/B\mathrm{Spin} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$, so there is a bijection *that is not necessarily a group isomorphism*

$$[B\mathbb{Z}/2, BO/B\mathrm{Spin}] \approx [B\mathbb{Z}/2, K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)] = H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \times H^2(B\mathbb{Z}/2; \mathbb{Z}/2). \tag{2.24}$$

Thus $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ is a group with four elements, so to finish the proof, it suffices to exhibit an element $x \in [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ with $2x \neq 0$. For this, we use the fact that, since the map $q\colon BO \to BO/B\mathrm{Spin}$ is E_∞ and the composition

$$BO \to BO/B\mathrm{Spin} \xrightarrow[(1.4)]{\phi} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \tag{2.25}$$

sends $V \mapsto (w_1(V), w_2(V))$ (see [DY23, (1.42)], and the related [DK70, Lemma 7]), we can add classes in $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ by passing through ϕ and using the Whitney sum formula: given (a_1, b_1) and (a_2, b_2) in $H^1(X; \mathbb{Z}/2) \times H^2(X; \mathbb{Z}/2)$, their sum as elements of $[X, BO/B\mathrm{Spin}]$ is

$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + a_1 a_2 + b_2). \tag{2.26}$$

Thus if $x \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the unique nonzero class, the sum of the twists $(x, 0)$ and $(x, 0)$ is $(0, x^2)$, which is nonzero. □

Remark 2.27. One can show that $[B\mathbb{Z}/2, B(B\mathrm{SO}_\otimes)] \cong \mathbb{Z}/2$. Thus we can consider a non-null-homotopic map $f\colon B\mathbb{Z}/2 \to B(B\mathrm{SO}_\otimes)$ and then map to $B\mathrm{GL}_1(ko)$. Such a map is an example of a "higher twist" of ko, and it would be interesting to better understand how it interacts with the $\mathbb{Z}/4$ we discuss in this section, either from the point of view of twisted KO-theory or from that of twisted spin structures. Brooks [Bro20] makes some related computations of higher twists of complex K-theory.

One can compute $[B\mathbb{Z}/2, B(B\mathrm{SO}_\otimes)]$ in a few ways. Since $B\mathbb{Z}/2$ is 2-local, we may 2-localize $B(B\mathrm{SO}_\otimes)$; then, Adams–Priddy [AP76, Corollary 1.4] identify $[-, B(B\mathrm{SO}_\otimes)_{(2)}]$ with $\tau_{\geq 2} ko_{(2)}$, where $\tau_{\geq 2}$ denotes the Postnikov 1-connected cover. So we want to understand $(\tau_{\geq 2} ko)^1(B\mathbb{Z}/2)$; here are two ways forward.

- The cofiber sequence $\tau_{\geq 2} ko \to ko \to \tau_{\leq 1} ko$ induces a long exact sequence in the cohomology of $B\mathbb{Z}/2$. Bruner–Greenlees [BG10, §A.9] calculate $ko^*(B\mathbb{Z}/2)$, and one can attack $\tau_{\leq 1} ko^*(B\mathbb{Z}/2)$ with the Atiyah–Hirzebruch spectral sequence.
- Alternatively, since $\tau_{\geq 2} ko$ is a ko-module, there is a natural isomorphism

$$\tau_{\geq 2} ko^n(X) \coloneqq \pi_{-n} \mathrm{Map}(X, \tau_{\geq 2} ko) \cong \pi_{-n} \mathrm{Map}_{ko}(ko \wedge X, \tau_{\geq 2} ko). \tag{2.28}$$

[6] See Dadarlat–Pennig [DP15] and Pennig [Pen16] for geometric models of the analogue of $[X, B(B\mathrm{SO}_\otimes)]$ for complex K-theory.

Having rephrased this question internal to the category of ko-module spectra, we can finish it with Baker–Lazarev's ko-module Adams spectral sequence [BL01].[7] See [Bak20, DY23, Kur25, DKS26, DYY26] for some examples of computations with this spectral sequence over ko; in particular, using standard techniques in $\mathcal{A}(1)$-modules one can show the spectral sequence computing $[B\mathbb{Z}/2, B(B\mathrm{SO}_{\otimes})]$ collapses to imply the claim.

By contrast, Gómez [Góm10] showed that if G is *any* compact Lie group, there are no higher twists of complex K-theory over BG.

The above story using maps to $B\mathrm{GL}_1(R)$ to define a graded ring structure on twisted R-cohomology is closely related to the extended gradings on (equivariant) parametrized cohomology discussed in work of Costenoble, Hudson, Tilson, and Waner [CW92, CW16, Cos17, CHT22, CH23, Cos23, CHT24, CH24, CH25a, CH25b, CH25c, CH26] and Beaudry–Lewis–May–Petersen–Tatum [BLM+25, BLM+26b], as well as to the *permutation twisted cohomology* of a finite group as introduced by Balmer–Gallauer [BG25, §10] and also studied by Gómez and Miller [Mil25, GM26a, GM26b].

We finish this subsection with a more geometric description of the isomorphism in Lemma 2.23. Recall that a vector bundle admits a spin structure if its first and second Stiefel–Whitney classes vanish. Twists of spin structure quantify exactly how a vector bundle may *fail* to be spin.

The following definition, in the special case $a = 0$, is due to B.L. Wang [Wan08, Definition 8.2]. It is also closely related to Kreck's definition of a normal 1-smoothing [Kre99, §2].

Definition 2.29. Let X be a space, $a \in H^1(X;\mathbb{Z}/2)$, and $b \in H^2(X;\mathbb{Z}/2)$. An (X,a,b)-*twisted spin structure* on a vector bundle $E \to M$ is the data of a map $f\colon M \to X$ and trivializations of $w_1(E) + f^*(a)$ and $w_2(E) + f^*(a)^2 + f^*(b)$.

By a *trivialization* of a cohomology class, we mean a null-homotopy of the representing map to an Eilenberg–Mac Lane space. Also, for (X,a,b) fixed, the (X,a,b)-twisted spin structures on a vector bundle $E \to M$ define a homotopy type; we consider two (X,a,b)-twisted spin structures to be equivalent if they belong to the same path component.

Using Lemma 1.4, we will identify (a,b) with a homotopy class of maps $X \to B\mathrm{O}/B\mathrm{Spin}$.

Lemma 2.30. *Let $V \to X$ be a vector bundle, $a \in H^1(X;\mathbb{Z}/2)$, and $b \in H^2(X;\mathbb{Z}/2)$. Let M be a closed manifold whose tangent bundle has an (X,a,b)-twisted spin structure induced by a map $f\colon M \to X$. If $N \subset M$ is a closed submanifold whose mod 2 homology class is Poincaré dual to the mod 2 Euler class of $f^*(V)$, then N has a canonical twisted spin structure for the sum*

$$(a,b) \oplus q(V - \mathrm{rank}(V)) \in [X, B\mathrm{O}/B\mathrm{Spin}]. \tag{2.31}$$

See also [DYY25, Lemma A.2]. Here $q\colon B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}$ is the quotient map. Note that we subtracted off the rank of V since $B\mathrm{O}$ classifies rank-zero virtual vector bundles.

Proof. Since the fundamental class of N is Poincaré dual to $e(V)$, there is an isomorphism $TM|_N \cong TN \oplus f^*V|_N$. Then use the Whitney sum formula. □

Lemma 2.30 organizes different kinds of twisted spin structure over the same space X into families related by iteratively passing from M to N.

[7] Boardmann [Boa99, Theorem 15.6] showed that half-plane Adams spectral sequences such as (2.28) are conditionally convergent, since ko and $B\mathbb{Z}/2$ are finite type, and thus one must in general be careful with convergence. However, this particular spectral sequence collapses, so it strongly converges (*ibid.*, §7) and everything is OK.

Example 2.32. Lemma 2.23 implies that for twists of spin structure over $B\mathbb{Z}/2$, we obtain a 4-periodic family. Specifically, if $\sigma \to B\mathbb{Z}/2$ denotes the tautological bundle, we obtain the following four structures. Under the isomorphism $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}] \xrightarrow{\cong} \mathbb{Z}/4$ from Lemma 2.23, entry (i) in the list below maps to $i \in \mathbb{Z}/4$.

(0) We begin with the trivial twist, where $a = 0$ and $b = 0$. A $(B\mathbb{Z}/2, 0, 0)$-twisted spin structure on $E \to M$ is the data of a spin structure on E and a principal $\mathbb{Z}/2$-bundle $P \to M$. This is sometimes called a spin$\times\mathbb{Z}/2$ structure.
(1) The next entry in the family is a $(B\mathbb{Z}/2, x, 0)$-twisted spin structure, where $x \coloneqq w_1(\sigma)$. By Peterson [Pet68, §7], this is equivalent to a pin$^-$ structure.
(2) Applying (2.26), our next twist is $(B\mathbb{Z}/2, 0, x^2)$. By Campbell [Cam17, §7.9], this kind of twisted spin structure is a G-structure for $G = \mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$.
(3) The next twist is $(B\mathbb{Z}/2, x, x^2)$-twisted spin structure, which by Stolz [Sto88, §8] is a pin$^+$ structure.

If one applies (2.26) once more, one gets back to the trivial twist, illustrating that this family is 4-periodic. See [DDK+26, Example 7.8] and references therein. In particular, this family admits a physical interpretation in terms of dimensional reductions of quantum field theories; see e.g. [KTTW15, TY19, HKT20, COSY20, WWZ20, DDK+25, DDK+26].

Remark 2.33. The periodicity of twists induced by a cyclic subgroup of $[X, B\mathrm{O}/B\mathrm{Spin}]$ lifts to a periodicity of iterated *Smith homomorphisms*, which are roughly speaking the effects on bordism classes of the construction described in Lemma 2.30. This is the context in which this fourfold periodicity for $X = B\mathbb{Z}/2$ is studied in the references cited above. See [DDK+26, §6] for more information.

Although we discuss them here in terms of the dual to the *cohomology* Euler class, these homomorphisms are more naturally understood as representing the Euler class in *cobordism*; see e.g. [DDK+26, Appendix B].

3. From fermionic groups to Q_1-local $\mathcal{A}(1)$-modules

To prove Theorem 0.1, we need to pass from fermionic groups to $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$. We will do this by the following chain of symmetric monoidal functors, whose notation we will explain and/or recall below:

$$\Phi\colon \mathcal{F}erm\mathcal{G}rp \xrightarrow{\mathbf{T}} \mathcal{T}op^{ft}/_{B\mathrm{GL}_1(ko)} \xrightarrow{M} \mathcal{M}od_{ko}^{ft} \xrightarrow{H^*_{ko}} \mathcal{M}od_{\mathcal{A}(1)}^{ft} \xrightarrow[(2.7)]{L_1} \mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]. \tag{3.1}$$

(1) Recall from Definition 1.5 that $\mathcal{F}erm\mathcal{G}rp$ is a full subcategory of $\mathcal{T}op/_{B\mathrm{O}/B\mathrm{Spin}}$. The functor $\mathbf{T}$ composes the map $BG \to B\mathrm{O}/B\mathrm{Spin}$ with the E_∞-map $T\colon B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{GL}_1(ko)$ from Equation (2.15). By Proposition 1.1, the E_∞-structure on T induces a symmetric monoidal structure on $\mathbf{T}$.

    The superscript *ft* means finite type: since we have restricted to finite fermionic groups, their classifying spaces are finite type.
(2) The functor M is Ando–Blumberg–Gepner–Hopkins–Rezk's Thom spectrum functor [ABG+14a, Definition 2.20], which by Ando–Blumberg–Gepner [ABG18, Theorem 1.6] is symmetric monoidal. Moreover, the Thom isomorphism theorem (specifically its incarnation in [DY23, Lemma 2.24]) can be used to show that M sends finite-type spaces over $B\mathrm{GL}_1(ko)$ to finite-type ko-modules.

(3) For a ko-module M, let $H^*_{ko}(M) \coloneqq \pi_{-*}\mathrm{Map}_{ko}(M, H\mathbb{Z}/2)$ [EKMM97, Definition IV.1.7], where $H\mathbb{Z}/2$ carries the E_∞-ko-algebra structure given by Postnikov truncation $ko \to H\mathbb{Z}$ followed by reduction mod 2. This is naturally a module over $H^*_{ko}(H\mathbb{Z}/2)$, which Baker [Bak20, Theorem 5.1] shows has a unique isomorphism to $\mathcal{A}(1)$. Like for ordinary cohomology, H^*_{ko} of a finite type ko-module is a finite-type $\mathcal{A}(1)$-module, which means that for all $n \in \mathbb{Z}$, the subspace of homogeneous degree-n elements is finite-dimensional.

We will show below in Lemma 3.4 that H^*_{ko} is a symmetric monoidal functor via the *cross product* map $H^*_{ko}(M) \otimes H^*_{ko}(N) \to H^*_{ko}(M \wedge_{ko} N)$.

(4) Finally, L_1 is the Q_1-localization functor from Theorem 2.7, which Ricka [Ric16, Proposition 2.6] showed is symmetric monoidal. This needs no finiteness assumptions, so we implicitly compose with the forgetful functor from finite type $\mathcal{A}(1)$-modules to all $\mathcal{A}(1)$-modules, which is symmetric monoidal.

Thus Φ is a symmetric monoidal functor of ∞-categories, once we fulfill our promise about the cross product.

Definition 3.2. Let $\mu\colon H\mathbb{Z}/2 \wedge_{ko} H\mathbb{Z}/2 \to H\mathbb{Z}/2$ be the multiplication map in the E_∞-ko-algebra structure on $H\mathbb{Z}/2$ we discussed above. For ko-module maps $\phi_1\colon M \to \Sigma^{t_1} H\mathbb{Z}/2$ and $\phi_2\colon N \to \Sigma^{t_2} H\mathbb{Z}/2$, define their *cross product* $t_1 \times t_2$ to be the composition

$$M \wedge_{ko} N \xrightarrow{\phi_1 \wedge \phi_2} \Sigma^{t_1} H\mathbb{Z}/2 \wedge_{ko} \Sigma^{t_2} H\mathbb{Z}/2 \xrightarrow{\simeq} \Sigma^{t_1+t_2} H\mathbb{Z}/2 \wedge_{ko} H\mathbb{Z}/2 \xrightarrow{\mu} \Sigma^{t_1+t_2} H\mathbb{Z}/2. \tag{3.3a}$$

Passing to homotopy groups of mapping spectra, we get a map

$$\times\colon H^*_{ko}(M) \otimes H^*_{ko}(N) \to H^*_{ko}(M \wedge_{ko} N), \tag{3.3b}$$

which we also call the cross product.

Lemma 3.4. *The functor $H^*_{ko}\colon \mathcal{M}od^{ft}_{ko} \to \mathcal{M}od^{ft}_{\mathcal{A}(1)}$ is symmetric monoidal via the cross product $\times\colon H^*_{ko}(M) \otimes H^*_{ko}(N) \to H^*_{ko}(M \wedge_{ko} N)$. In particular, the cross product (3.3b) is an isomorphism for all finite type ko-modules M and N.*

The proof of Lemma 3.4 is identical to the usual proof of the standard fact that, by the Künneth formula, mod 2 cohomology is a symmetric monoidal functor out of the ∞-category of finite type spaces. Absent the finite type hypothesis, $H^*(-;\mathbb{Z}/2)$ and H^*_{ko} are only lax symmetric monoidal.

Lemma 3.5. *In $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]$, $\Phi(E_{1,0}) \cong \Sigma T_1$ and $\Phi(E_{0,1}) \cong \Sigma T_0$.*

Proof. After unwinding the definition, $\Phi(E_{1,0})$ is (the Q_1-localization of) H^*_{ko} of the Thom spectrum of the map $T \circ (x, 0)\colon B\mathbb{Z}/2 \to B\mathrm{O}/B\mathrm{Spin} \to B\mathrm{GL}_1(ko)$, where $x \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ is the generator. If $\sigma \to B\mathbb{Z}/2$ denotes the tautological line bundle, then $(x, 0) = (w_1(\sigma), w_2(\sigma))$, so $M\mathbf{T}(E_{1,0}) \simeq ko \wedge (B\mathbb{Z}/2)^{\sigma-1}$, as follows by combining [DY23, Example 1.6 and Lemma 1.8]. For any spectrum X, there is a natural $\mathcal{A}(1)$-module isomorphism $H^*_{ko}(X) \cong H^*(X;\mathbb{Z}/2)$ [DKS26, Lemma 5.16], where the $\mathcal{A}(1)$-module structure on the latter is the restriction of the usual $\mathcal{A}$-module structure. Finally, Campbell [Cam17, Example 6.4] shows $H^*((B\mathbb{Z}/2)^{\sigma-1});\mathbb{Z}/2) \cong \Sigma T_1$. As this is already Q_1-local by [Yu95, Chapter 2], $L_1(\Sigma T_1) \cong \Sigma T_1$. As we saw in Figure 1, T_1 is an invertible object of $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]$, and invertibility is closed under suspension (since $(\Sigma M)^{-1} = \Sigma^{-1} M^{-1}$).

For $E_{0,1}$, the story is very similar, but with (x, x^2) in place of $(x, 0)$ as the map to $B\mathrm{O}/B\mathrm{Spin}$, $-\sigma$ in place of σ, and $(B\mathbb{Z}/2)^{1-\sigma}$ in place of $(B\mathbb{Z}/2)^{\sigma-1}$. Campbell [Cam17, Example 6.6] shows $H^*((B\mathbb{Z}/2)^{1-\sigma};\mathbb{Z}/2) \cong \Sigma T_0$, and ΣT_0 is Q_1-local [Yu95, Chapter 2], so the Q_1-localization of ΣT_0 is itself, and it is an invertible object of $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]$, as displayed in Figure 1. □

Corollary 3.6. *For all $\ell, k \in \mathbb{N}$, $\Phi(E_{\ell,k})$ is invertible.*

Proof. By Lemma 3.5, $\Phi(E_{1,0}) \cong \Sigma T_1$ and $\Phi(E_{0,1}) \cong \Sigma T_0$, and by Theorem 2.10, both of these are invertible in $\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]$. Since Φ is symmetric monoidal, $\Phi(E_{\ell,k})$ is a tensor product of copies of $\Phi(E_{1,0})$ and $\Phi(E_{0,1})$, and a tensor product of invertible objects is invertible. □

Corollary 3.7. *Φ restricts to a homomorphism of commutative monoids*

$$\Phi \colon \mathbb{M} \longrightarrow \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]). \tag{3.8}$$

We will frequently compose Φ with the isomorphism $y \colon \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}]) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/4$ from Theorem 2.10.

4. Proofs of the main theorems

Our first main result, Theorem 0.1, is that the map $y \circ \Phi \colon \mathbb{M} \to \mathbb{Z} \oplus \mathbb{Z}/4$ from Corollary 3.7 is a group completion. That is, we will show that any map $\mathbb{M}$ to a group G factors uniquely through Φ.

Proof of Theorem 0.1. Using the monoid isomorphism $\widetilde{\varepsilon} \colon \mathbb{N}^2/((4,0)-(0,4)) \to \mathbb{M}$ from Lemma 1.23, it suffices to show that the following map is a group completion:

$$\begin{aligned} y \circ \Phi \circ \widetilde{\varepsilon} \colon \mathbb{N}^2/((4,0)-(0,4)) &\longrightarrow \mathbb{Z} \oplus \mathbb{Z}/4 \\ (1,0) &\longmapsto (1,1) \\ (0,1) &\longmapsto (1,0). \end{aligned} \tag{4.1}$$

Let G be a group and $f \colon \mathbb{N}^2/((4,0)-(0,4)) \to G$ be a monoid homomorphism. We may without loss of generality assume G is abelian and write elements of G additively. Define

$$\begin{aligned} \widetilde{f} \colon \mathbb{Z} \oplus \mathbb{Z}/4 &\longrightarrow G \\ (a,b) &\longmapsto (a-b)f(0,1) + bf(1,0). \end{aligned} \tag{4.2}$$

Since $f(0,4) = f(4,0)$ by the relation in the domain of f, $\widetilde{f}(a,b)$ only depends on $b \bmod 4$, so indeed $\widetilde{f}$ is well-defined as a homomorphism of abelian groups.

Since $\widetilde{f}(1,0) = f(0,1)$ and $\widetilde{f}(1,1) = f(1,0)$, $f = \widetilde{f} \circ y \circ \Phi \circ \widetilde{\varepsilon}$. To show this factorization is unique, suppose $\widetilde{f}'$ is another map with $\widetilde{f}' \circ y \circ \Phi \circ \widetilde{\varepsilon} = f$. Thus $\widetilde{f}'(1,0) = f(0,1) = \widetilde{f}(1,0)$ and $\widetilde{f}'(1,1) = f(1,0) = \widetilde{f}(1,1)$. That is, $\widetilde{f}$ and $\widetilde{f}'$ agree on a generating set $\{(1,0),(1,1)\}$ of $\mathbb{Z} \oplus \mathbb{Z}/4$, so $\widetilde{f} = \widetilde{f}'$. □

Finally, we compare the two $\mathbb{Z}/4$s that we introduced in §2.1 and §2.2, which gave rise to our two periodicities. We do this in two ways. First, in Proposition 4.3, we compute the "obvious" map $\mathbb{Z}/4 \cong \mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])_{\mathrm{tors}} \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$, namely the map induced from $\delta \colon \mathbb{M} \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ by the universal property of the group completion. This map turns out to be $2 \colon \mathbb{Z}/4 \to \mathbb{Z}/4$—not an isomorphism. After discussing what went wrong, we prove Theorem 0.6, offering a different way to identify these two $\mathbb{Z}/4$s as generated by the fermionic group $E_{1,0}$.

Proposition 4.3. *The map $\Psi \colon \mathbb{Z}/4 \to \mathbb{Z}/4$ defined in (0.5) is multiplication by 2.*

Proof. Recall that $t \colon \mathbb{Z}/4 \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}/4$ is the inclusion of the torsion subgroup and $\widetilde{\delta} \colon \mathbb{Z} \oplus \mathbb{Z}/4 \to [B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ is the unique map such that $\delta = \widetilde{\delta} \circ y \circ \Phi$, as guaranteed by Theorem 0.1. We calculate $\widetilde{\delta} \circ t$ on the generator of $\mathbb{Z}/4$, which is the module T_1 by Theorem 2.10: we have $\Phi(E_{1,0}) - \Phi(E_{0,1}) = T_1$, so

$$\widetilde{\delta}(0,1) = \delta(E_{1,0}) - \delta(E_{0,1}). \tag{4.4}$$

By Example 1.30, $\delta(E_{1,0}) = (x, 0)$ and $\delta(E_{1,1}) = (x, x^2)$. As we saw in Example 2.32, these correspond to 1, resp. 3 in $\mathbb{Z}/4$ under Lemma 2.23, so we conclude that $\widetilde{\delta}(0,1) = 1-3 = 2 \in \mathbb{Z}/4$. □

One might have expected Proposition 4.3 to be an isomorphism. The issue stems from the different requirements for normalization between the Picard groupoid and the group of twists. In the former, torsion elements might not begin in degree zero, but in the latter, twists are given by rank zero virtual bundles. In particular, by Lemma 3.5, $\Phi(E_{1,0})$ is not a torsion element of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$. Instead, it is the *suspension* of a torsion element. Generally, one does not think of M and ΣM as vastly different modules. But this means that $(0,1) \in \mathbb{Z} \oplus \mathbb{Z}/4$ is not represented just by $E_{1,0}$: now, the suspension matters, and we have to represent $(0,1)$ by the formal difference $E_{1,0} - E_{0,1}$.

One would want to modify Ψ to fix this, but the setup is very rigid: there is a unique $\mathbb{Z}/4$ subgroup of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$, and the group completion is essentially unique. So, while Ψ does not reflect how suspension shouldn't change the $\mathbb{Z}/4$ periodicity of Q_1-locally invertible $\mathcal{A}(1)$-modules, there is not a good way to modify Ψ. This is why we formulate Theorem 0.6 as a correspondence.

Proof of Theorem 0.6. We compute each side of the diagram. On the left, we have $\widetilde{\delta}(y(\Phi(E_{1,0}))) = \delta(E_{1,0})$ by definition of $\widetilde{\delta}$. By Example 1.30, $\delta(E_{1,0}) = (x, 0)$, and by Example 2.32, this is a generator of $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$.

On the right, by definition ς suspends or desuspends a module until it is stably equivalent to T_0, T_1, T_2, or T_3 (see Lemma 2.12). By Lemma 3.5, $\Phi(E_{1,0}) \cong \Sigma T_1$, so $\varsigma(\Phi(E_{1,0})) = T_1$, which by Theorem 2.10 generates the torsion subgroup of $\mathrm{Pic}(\mathcal{S}t\mathcal{M}od_{\mathcal{A}(1)}[Q_1^{-1}])$. □

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