WEAK TOPOLOGICAL PHASES IN THE PRESENCE OF INTERACTIONS

OMAR ANTOLÍN CAMARENA, ARUN DEBRAY, CAMERON KRULEWSKI, NATALIA PACHECO-TALLAJ, DANIEL SHEINBAUM, AND LUUK STEHOUWER

Abstract. We investigate the stability of weak symmetry-protected topological phases (SPTs) in the presence of short-range interactions, focusing on the tenfold way classification. Using Atiyah's Real KR-theory and Anderson-dualized bordism, we classify free and interacting weak phases across all Altland-Zirnbauer symmetry classes in low dimensions. Extending the free-to-interacting map of Freed-Hopkins, we mathematically compute how the behavior of free weak SPTs changes when interactions are introduced as well as predict intrinsically-interacting weak phases in certain classes. Our mathematical techniques involve T-duality and the James splitting of the torus. Our results provide a mathematical framework for understanding the persistence of weak SPTs under interactions, with potential implications for experimental and theoretical studies of these phases.

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0. Introduction

The tenfold way has been a successful paradigm in condensed matter physics for classifying symmetry-protected topological (SPT) phases of free fermions. By specifying whether a phase has time reversal symmetry, charge-conjugation symmetry, or chiral symmetry, one obtains one out of ten Altland-Zirnbauer classes. This tenfold classification unified many known examples of topological phases, like the quantum spin Hall effect phase (class AII) [KWB$^+$07], and predicted the existence of new topological phases, spurring a large body of experimental research. Some of these phases have been experimentally realized, such as the Chern insulator (class A) [CZF$^+$13].

Within a given Altland-Zirnbauer class, there are different kinds of phases, distinguished for example by the spatial symmetries of the phase. *Strong* phases, protected solely by internal symmetry, are robust even in the presence of symmetry-preserving strong disorder. On the other hand, *weak* phases, which are protected by a lattice translation symmetry, are a priori less robust in the presence of disorder. Weak topological phases can be formed from layers of lower-dimensional strong topological phases; as such, invariants of weak phases are often built from invariants of lower-dimensional strong phases [FKM07, CH20, LZYZ16]. Weak topological insulators exhibit various exotic phenomena. Their construction by stacking often results in anisotropic gapless edge modes [CH20, LZYZ16, YMF12, RIR$^+$13], while translation symmetry defects called dislocations can trap topological bound states [Ran10, HZ24, MEAS16, SMJZ14, XJG$^+$21]. Weak SPTs also have the potential for producing non-abelian anyons [MEAS16, HZ24] and hosting helical edge states [MBM12, ZYS$^+$23, ITT11, WW21, SSB16]. They have been studied experimentally, e.g. in [HTE17, LWC$^+$18, ZNK$^+$21].

The question of whether weak phases persist in the presence of disorder or short-range interactions is an active area of research [RKS12, CH20, BQ12, WZ12, Hug15]. It is predicted that classifications of free and interacting SPT phases in a given dimension and symmetry type are abelian groups, and that "turning on interactions" (considering a free fermion Hamiltonian up to deformations that allow interactions) should define a group homomorphism from the free phases to the interacting phases, which we call the *free-to-interacting map*. In this work, we mathematically model free-to-interacting maps between the KR-theory classification of free phases [Kit09] and the Anderson-dual bordism classification of interacting phases [FH21, FH20], and explore some of the physical consequences due to the existence of this map. Building on Freed-Hopkins' free-to-interacting map for strong phases for the ten Altland-Zirnbauer classes [FH21, §9.2], we propose a model for the free-to-interacting map for weak phases for all ten symmetry types and compute all the groups of translation-invariant free and interacting phases in low dimensions.

The K-theory framework for classifying free-fermion phases in the tenfold way was introduced contemporaneously by Kitaev [Kit09] and Ryu-Schnyder-Furusaki-Ludwig [RSFL10], and later expanded upon by Freed-Moore [FM13], Thiang [Thi16], Alldridge-Max-Zirnbauer [AMZ20], and others. This framework encodes the ten Altland-Zirnbauer classes in the two shifts of complex K-theory (A and AIII) and the eight shifts of real KO-theory (D, BDI, AI, CI, C, CII, AII, DIII) according to Bott periodicity. In §1.2, we present the K-theory framework in a way that makes this correspondence between K-theory and symmetry generators precise, using the language of Clifford algebras and group C^*-algebras. In

Definition 1.10, we give our mathematical model for the group of free fermionic phases with symmetry C^*-algebra A. In work of Stehouwer [Stea] to appear, this group is identified with a K-theory group:

Theorem 1.11 (Stehouwer [Stea])**.** *The group of A-symmetric free SPT phases is isomorphic to $KO_2(A)$.*

Corollary 1.18 shows that in the examples corresponding to the tenfold way, these K-theory groups recover the KR-groups of a point (strong phases) or the Brillouin torus (weak phases).

The interacting classification has a different mathematical flavor. Freed-Hopkins [FH21, FH20], inspired by previous work of Kapustin and collaborators [Kap14, KTTW15], classify (strong) interacting SPT phases in a two-step process:

(1) To each symmetry class s, Freed-Hopkins associate a family of Lie groups $H_d(s)$, one in each dimension d, which are the symmetry groups of Euclidean-signature spacetime corresponding to the symmetries described in the tenfold way.
(2) Then, they propose that interacting SPTs are classified by the deformation classes of their low-energy limits, which are reflection-positive invertible field theories on manifolds with $H_d(s)$-structure.

Freed-Hopkins-Teleman [FHT10] and Freed-Hopkins [FH21] compute these groups of invertible field theories in terms of bordism groups.

Freed-Hopkins also provide a mathematical model for the free-to-interacting map for strong phases in the tenfold way.

Ansatz (Freed-Hopkins [FH21, §9.2])**.** *Under the identifications above, the free-to-interacting map is the Anderson dual of the class s version of the Atiyah-Bott-Shapiro map.*

In [FH21, §9.3], Freed and Hopkins show their ansatz matches previous computations by other methods. In §1.4, we discuss Anderson duality and Atiyah-Bott-Shapiro maps, and state the Freed-Hopkins ansatz precisely in Definitions 1.53 and 1.56 and Ansatz 1.58.

In this paper, we generalize the above story to weak phases. We first propose a model for the classification of weak phases in the presence of interactions.

Ansatz 1.36. *The data of a discrete translation-invariant topological phase is equivalent to a family of phases parametrized by the spatial torus $\mathbf{T}^d$.*

This is inspired by work of Freed-Hopkins [FH20], in particular their Example 2.3. We also provide a new physical argument supporting this ansatz in §2.

To define our model for the weak free-to-interacting map, we make crucial use of two mathematical facts: T-duality and the James splitting. T-duality (§1.5) allows us to exchange the spatial and Brillouin tori, while James splitting (§1.6) is the topological avatar of the fact that weak invariants can be constructed from lower-dimensional strong invariants. See [MT16a, FR16] for prior work on T-duality in this context, and [MT16a, FR16, Xio18, Steb] for James splitting.

Ansatz 1.73. *Let $x \in KR^{s-2}(\overline{\mathbb{T}}^d)$ be a discrete translation-invariant free fermion theory in d dimensions and of real symmetry type s. The long-range effective theory of x is given by* *the image of x under the composition*

$$\mathrm{F2I}_{weak}\colon KR^{s-2}(\overline{\mathbb{T}}^d) \xrightarrow[(1.64a)]{\mathrm{T}_{\mathbb{R}}^{-1}} KO^{d-s-2}(\mathbf{T}^d) \xrightarrow[(1.53)]{\mathrm{F2I}_s} \mho^{d+2}_{H(s)}(\mathbf{T}^d). \tag{0.1}$$

We briefly explain this notation:

- $\mathrm{T}_{\mathbb{R}}^{-1}$ is a T-duality isomorphism,
- $\mathrm{F2I}_s$ is Freed-Hopkins' strong free-to-interacting map applied to the torus, and
- $\mho^{d+2}_{H(s)}$ is a certain generalized cohomology group classifying reflection-positive invertible field theories on $H(s)$-manifolds (see §1.3).

We also discuss a version of this ansatz for complex symmetry types in Ansatz 1.73.

In §3, we test our ansatz by computing $\mathrm{F2I}_{weak}$ in dimensions 1, 2, and 3 for all ten Altland-Zirnbauer types; the results are given in Corollaries 3.4, 3.7, 3.10, 3.13, 3.16, 3.19, 3.22, 3.25, 3.28 and 3.31. We compare these results with the literature; where weak interacting phases have been studied before, our computations agree with prior work. Some of our computations are in classes where interacting weak phases have not been studied before; it would be interesting to compare our predictions with other approaches.

Throughout this paper, we introduce concepts by first implementing them concretely in class AII, corresponding to time-reversal topological insulators. This in particular includes Appendix A, where we compute Freed-Hopkins' twisted Atiyah-Bott-Shapiro map $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to KO_2$; Freed-Hopkins showed this map is surjective [FH21, Corollary 9.93], and we give another proof which also allows us to explicitly calculate the map on a set of manifold generators for $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$. It would be interesting to try this for other Altland-Zirnbauer classes.

We intend this work to be an introduction to free-to-interacting maps as well as a full computation of the maps in the case of weak phases. Ongoing work of some of the authors addresses free-to-interacting maps for more general crystalline symmetries.

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1. The Ansatz for the Weak Free-to-Interacting Map

1.1. Fermionic symmetry groups.

Fermionic groups: In condensed matter, symmetry groups G_f of fermionic systems have the extra structure of what we will call a *fermionic group* [Ben88, Ste22]. This means that G_f comes equipped with a central element $(-1)^F \in G_f$ of order two and a homomorphism $\phi : G_f \to \mathbb{Z}_2$ labeling time-reversing elements such that $(-1)^F$ is time-preserving.

Symmetries form a superalgebra: A *superalgebra* is a $\mathbb{Z}_2$-graded algebra. Each Altland-Zirnbauer class specifies a set of symmetry operators, which generate a superalgebra over $\mathbb{R}$ or $\mathbb{C}$. The reader should be warned that the physical interpretation of the $\mathbb{Z}_2$-grading here is given by time-reversing versus time-preserving symmetries, as opposed to fermions versus bosons.

From superalgebra to fermionic group: The superalgebras we obtain by the Altland-Zirnbauer classification are *super division algebras*, meaning all homogeneous elements are invertible. There are exactly ten such superalgebras [Wal64]

$$D_i^{\mathbb{C}}, D_j^{\mathbb{R}} \quad i = 0, 1, \ j = 0, \dots, 7, \tag{1.1}$$

which can be constructed explicitly as certain Clifford algebras. Given a super division algebra A, the set $S(A)$ of norm-1 elements of A acquires the structure of a compact Lie group from the multiplication on A. The grading operator defines a homomorphism $\phi\colon S(A) \to \mathbb{Z}_2$, and $(-1)^F$ generates a central $\mathbb{Z}_2$ subgroup of $S(A)$, making $S(A)$ into a fermionic group.

Example 1.2. The Altland-Zirnbauer class AII, corresponding to topological insulators with a time-reversal symmetry, has a time-reversal symmetry squaring to $(-1)^F$ and a charge Q generating a U(1) symmetry corresponding to conservation of particle number. These symmetries are subject to a *spin-charge relation*: the -1 in this U(1) is equal to $(-1)^F$, and time-reversal acts on U(1) by complex conjugation.

The algebra generated by T and Q over $\mathbb{R}$ is isomorphic to the Clifford algebra

$$C\ell_{-2} \coloneqq \mathbb{R}\langle e_1, e_2\rangle/(e_1^2 = e_2^2 = -1, e_1e_2 + e_2e_1 = 0). \tag{1.3}$$

This isomorphism can be explicitly realized by sending $T \mapsto e_1$ and $e^{i\pi\theta} \in \mathrm{U}(1)$ to $\cos(\theta) + e_1e_2\sin(\theta)$ [Ste22, Example 4].

The second step is to find $S(C\ell_{-2})$, which by definition of the pin groups is equal to $\mathrm{Pin}^-(2)$. First consider the real superalgebra

$$A' \coloneqq \mathbb{C}[T]/(T^2 = -1, iT + Ti = 0), \tag{1.4}$$

where i is even and T is odd. There is an isomorphism $\phi\colon A' \to C\ell_{-2}$ of real superalgebras defined by setting $\phi(i) = e_1e_2$ and $\phi(T) = e_1$, then extending linearly to all of A'.

From the viewpoint of A', it is easier to find the homogeneous norm-one elements: the unit complex numbers, which generate a U(1) subgroup of $S(A')$, and the $\mathbb{Z}_4$ subgroup generated by T. The operator T acts on U(1) by complex conjugation, and $T^2 = -1$ is in U(1), so we see that

$$S(C\ell_{-2}) \cong S(A') \cong \frac{\mathrm{U}(1) \rtimes \mathbb{Z}_4^T}{\mathbb{Z}_2}. \tag{1.5}$$

The homomorphism ϕ is the unique one which is trivial when pulled back to U(1) and nontrivial when pulled back to $\mathbb{Z}_4^T$; $(-1)^F$ is the common central element.

1.2. *K*-theory classifications of free fermion phases.

The classification of SPT phases of complex free fermions can be connected to K-theory as follows [FM13]. For a symmetry group G, consider a one particle state space V, which furnishes a representation R of G. We want to understand the space of all gapped Hamiltonians H on V with symmetry G. After shifting the Fermi energy to zero, a gapped Hamiltonian is defined as a linear operator without kernel that intertwines R. This splits the representation $V = V_{valence} \oplus V_{conduction}$ into $\pm$-eigenspaces of H. Therefore H defines an element

$$V_{valence} - V_{conduction} \in K_G^0(\mathrm{pt}) \tag{1.6}$$

in the representation ring of G. If two Hamiltonians give different elements of $K_G^0(\mathrm{pt})$, a path between them must involve crossing the gap. Conversely, two different Hamiltonians with the same decomposition $V = V_{valence} \oplus V_{conduction}$ are in the same path component by spectral flattening. Therefore, the set of components π_0 essentially[1] equals $K_G^0(\mathrm{pt})$.

With enough care about the mathematical details, the above heuristic applies in various settings:

(1) When G contains time-reversal symmetries, they act anti-unitarily on V. We have to accommodate for this in the definition of the representation ring.
(2) Some symmetries have additional constraints relating to fermion parity, such as $T^2 = (-1)^F$ for a time-reversal symmetry. Since $(-1)^F$ acts by -1 on V, we have to enforce this relation in the representation ring.
(3) In positive spatial dimension d, it is reasonable to require $G = \mathbb{Z}^d$ to be the symmetry group of a lattice of atoms. Stable homotopy theory for noncompact G is still in development. Since the group algebra of an infinite group will not suffice for these purposes, we define $K_G^0(\mathrm{pt})$ to be the K-theory of the complex group C^*-algebra $C^*(G)$ of G.

As argued by [Thi16, Example 9.1-9.3], there is an isomorphism $K_{\mathbb{Z}^d}^0(\mathrm{pt}) \cong K^0(\mathbb{T}^d)$, closely related to Bloch's theorem. Here $K^0(\mathbb{T}^d)$ is the K-theory of the Brillouin zone torus. The isomorphism is given by a Fourier transform to momentum space, a special case of the Pontryagin duality isomorphisms of C^*-algebras

$$C^*(\mathbb{Z}^d) \cong C(\widehat{\mathbb{Z}^d}, \mathbb{C}) = C(\mathbb{T}^d, \mathbb{C}). \tag{1.7}$$

Here $C(X, \mathbb{C})$ denotes the ring of continuous functions on X and $\widehat{\mathbb{Z}^d} := \mathrm{Hom}(\mathbb{Z}^d, U(1)) = \mathbb{T}^d$ is the Pontryagin dual of $\mathbb{Z}^d$. Explicitly, a vector bundle E over $\mathbb{T}^d$ gives a $C^*(\mathbb{Z}^d)$-module $\Gamma(E)$ of continuous sections of E by mapping $\vec{n} \in \mathbb{Z}^d$ to the function $\mathbb{T}^d \to \mathbb{C}$ given by $e^{i\vec{n}\cdot\vec{k}}$. Here we used the common convention of identifying $\mathbb{T}^d$ with a quotient of the box $[-\pi, \pi]^d$ using the map $k \mapsto (\vec{n} \mapsto e^{i\vec{n}\cdot\vec{k}})$. We have therefore reproduced the fact [Kit09] that class A topological insulators in spatial dimension d are classified by $K^0(\mathbb{T}^d)$.

In order to address the question of which topological phases survive in the continuum limit, we redo the above argument for $G = \mathbb{R}^d$ the group of continuous translations. We again have the Fourier transformation isomorphism

$$C^*(\mathbb{R}^d) \cong C(\widehat{\mathbb{R}^d}, \mathbb{C}) = C(\mathbb{R}^d, \mathbb{C}), \tag{1.8}$$

so that

$$K_{\mathbb{R}^d}^0(\mathrm{pt}) = K_0(C^0(\mathbb{R}^d)) = \tilde{K}^0(S^d). \tag{1.9}$$

This agrees with the classification of strong class A topological insulators.

In the above discussion, we implicitly assumed our fermions are charged. In other words, we assumed the existence of a polarization giving the one particle spaces V and V^* of creation and annihilation operators, thus disallowing unpaired Majoranas. There is an analogous

[1]In the representation ring, we quotient out by additional relations such as $V - V = 0$ to ensure $K_G^0(\mathrm{pt})$ is a group. A priori, there is no physical justification for requiring this invertibility under stacking (which is given by direct sum since we are on a 1-particle space). However, here we restrict to invertible phases. Phases which are unstably nontrivial are called *fragile* phases [PWV18, EPW19].

discussion for neutral fermions, resulting in KO- instead of K-theory. This approach can be formulated in the Bogoliubov-de-Gennes formalism [AMZ20]. Even though most of the condensed matter literature does not use Majorana fermions, we will focus on this perspective, following our main references [Kit09] and [FH21].

The main difference in the new set-up will be that the complex one particle Hilbert space V is replaced by a real Hilbert space $\mathcal{M}$. The self-adjoint gapped Hamiltonian H is replaced with a skew-adjoint gapped operator Ξ on $\mathcal{M}$, which one should think of as $-iH$. Even though it is not possible to look at the positively imaginary and negatively imaginary eigenvalues of Ξ on $\mathcal{M}$, the operator does induce a complex structure $\Xi/|\Xi|$ on $\mathcal{M}$. Stably, the space of complex structures becomes a classifying space for KO^{-2}. Since by a spectral flattening procedure the space of such gapped skew-adjoint Ξ is homotopy equivalent to the space of complex structures, this hints towards a relationship between neutral phases and KO-theory. This discussion generalizes to arbitrary symmetry groups, taking into account that time-reversal symmetries should anti-commute with Ξ.

We can use the formalism of Karoubi triples [Thi16, DK70] to make this discussion mathematically precise: let A be the (real or complex) super C^*-algebra of symmetries, graded by time-reversal.[2] A *Karoubi triple* $(\mathcal{M}, \Xi_1, \Xi_2)$ consists of of a finitely generated (ungraded) A-module $\mathcal{M}$ and maps $\Xi_i : \mathcal{M} \to \mathcal{M}$ satisfying $\Xi^2 = -\mathrm{id}_{\mathcal{M}}$ and $\Xi_i a = (-1)^{|a|} a \Xi_i$ for all $a \in A$.[3] One can think of a Karoubi triple as a formal difference $[\Xi_1] - [\Xi_2]$ of Hamiltonians with A-symmetry. We now want to impose that $[\Xi_1] - [\Xi_2] = 0$ if Ξ_1 and Ξ_2 are in the same path component. So define a Karoubi triple to be *elementary* when Ξ_1 is in the same path component as Ξ_2 in the space of complex structures Ξ such that $\Xi a = (-1)^{|a|} a \Xi$ for all $a \in A$. Two Karoubi triples $(\mathcal{M}, \Xi_1, \Xi_2), (\mathcal{M}', \Xi'_1, \Xi'_2)$ are *isomorphic* if there exists an A-module isomorphism $\mathcal{M} \to \mathcal{M}'$ intertwining Ξ_i with Ξ'_i for $i = 1, 2$. Note that there is an obvious notion of direct sum $\oplus$ of Karoubi triples. We say two triples T_1, T_2 are *stably equivalent* if there exists an elementary triple T' such that $T_1 \oplus T'$ is isomorphic to $T_2 \oplus T'$. The set of Karoubi triples can be thought of as a stabilization of the space of A-symmetric Bogoliubov de-Gennes Hamiltonians Ξ.

Definition 1.10. The group of *A-symmetric free SPT phases* is the set of Karoubi triples modulo stable equivalence under $\oplus$.

If $A = C^*(\mathbb{Z}^d; F) \otimes C\ell_{-s}$, where $F = \mathbb{R}$, resp. $F = \mathbb{C}$, we will refer to A-symmetric free SPT phases as *discrete translation-invariant free SPT phases of real, resp. complex Altland-Zirnbauer class s.* Similarly, $A = C^*(\mathbb{R}^d; F) \otimes C\ell_{-s}$ gives *continuous translation-invariant free SPT phases.*[4]

The following theorem and remark will be proven in upcoming work of Stehouwer [Stea]:

[2]In this work, we will restrict to the case where A is the tensor product of a tenfold way symmetry as explained in §1.1 with the group C^*-algebra of the Lie group of translation symmetries, either discrete $\mathbb{Z}^d$ or continuous $\mathbb{R}^d$.

[3]There is an infinite dimensional version of the Karoubi description given here, which can be shown to be equivalent [Gom21, §4]. Using modules that are not finitely generated can be more suitable for physics, for example if we want to take the unbounded above valence band into account.

[4]Because $C\ell_{-s}$ is finite-dimensional, different notions of C^*-tensor product agree.

Theorem 1.11 ([Stea])**.** *The group of A-symmetric free SPT phases is isomorphic to $KO_2(A)$.*

Remark 1.12. Suppose A is a real super C^*-algebra containing a subalgebra $\mathbb{C}$, which is not necessarily in the center. We think of this subalgebra as generating charge. Suppose additionally that $A = A_+ \oplus A_-$ where $a_\pm \in A_\pm$ if and only if $a_\pm z = z^\pm a_\pm$ for all $a \in A_\pm$ and $z \in \mathbb{C}$. This defines a $\mathbb{Z}_2$-grading μ on A not necessarily equal to the $\mathbb{Z}_2$-grading ϕ given by time-reversal. Note that these this grading commutes with the other $\mathbb{Z}_2$-grading on A in the sense that the corresponding operators with eigenvalues ± 1 commute. Therefore, there is a product/diagonal $\mathbb{Z}_2$-grading c. Then $KO_0(A,c) \cong KO_2(A,\phi)$, where $K_i(A,\lambda)$ denotes the degree i K-theory of the algebra A with $\mathbb{Z}_2$-grading λ. This connects the description of Theorem 1.11 to the discussion of the beginning of this section and in particular to [FM13].

Example 1.13. Take $A = C^*(\mathbb{Z}^d;\mathbb{R}) \otimes C\ell_{-2}$ to be the tensor product of a d-dimensional discrete translation symmetry and the internal symmetry algebra of class AII (see Example 1.2). Using the fact that $K_i(A \otimes C\ell_{\pm 1}) \cong K_{i\mp 1}(A)$ [Kar68], we see that the group of A-symmetric free SPT phases is given by $KO_2(A) \cong KO_4(C^*(\mathbb{Z}^d;\mathbb{R}))$. We can now apply arguments as above to relate $\mathbb{Z}^d$ to the torus, but there is one important subtlety. Namely, the Fourier transform crucially uses the complex numbers through the factor $e^{i\vec{n}\cdot\vec{k}}$ and

$$\overline{e^{i\vec{n}\cdot\vec{k}}} = e^{i\vec{n}\cdot(-\vec{k})}. \tag{1.14}$$

So under the isomorphism $C^*(\mathbb{Z}^d;\mathbb{C}) \cong C(\mathbb{T}^d;\mathbb{C})$ of complex C^*-algebras, complex conjugation on the left hand side gets mapped to the operation mixing complex conjugation with the involution $k \mapsto -k$ on the Brillouin zone. Therefore the K-theory of $C^*(\mathbb{Z}^d;\mathbb{R})$ is not the KO-theory of the torus, but its KR-theory for this involution. We obtain that the classification of class AII topological insulators is given by

$$KO_4(C^*(\mathbb{Z}^d;\mathbb{R})) \cong KR_4(C(\mathbb{T}^d;\mathbb{C})) \cong KR^{-4}(\mathbb{T}^d). \tag{1.15}$$

Replacing $\mathbb{Z}^d$ by a continuous translation symmetry $\mathbb{R}^d$, we obtain similarly that

$$KO_4(C^*(\mathbb{R}^d;\mathbb{R})) \cong \widetilde{KR}^{-4}(S^d). \tag{1.16}$$

For example, consider the $d = 3$ time-reversal invariant insulator in class AII studied first in [FK07, FKM07]. We classify its phases using (1.15) as

$$K_2(C^*(\mathbb{Z}^3) \otimes C\ell_{+2}) \cong KR^{-4}(\mathbb{T}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3, \tag{1.17}$$

see [FM13, Theorem 11.14]. As observed in e.g. [Kit09] and [FR16, Theorem 3.35], one $\mathbb{Z}_2$ invariant encodes the strong phase detected by the Fu-Kane-Mele invariant. The $\mathbb{Z}$ invariant counts the number of Kramers pairs of electrons, one $\mathbb{Z}_2$ invariant encodes the strong phase, and the $(\mathbb{Z}_2)^3$ vector invariant encodes the weak topological phases: phases protected by the discrete translation symmetry. These phases may be viewed as quantum spin Hall phases living on each two-dimensional cross section of the three-dimensional material. Indeed, for a continuous translation symmetry, we obtain $\widetilde{KR}^{-4}(S^3) \cong \mathbb{Z}_2$ and only the first $\mathbb{Z}_2$ survives.

Example 1.13 generalizes to all tenfold classes to obtain the following corollary of Theorem 1.11:

Corollary 1.18. *Discrete (resp. continuous) translation-invariant free SPT phases of real Altland-Zirnbauer class s in spatial dimension d are classified by $KR^{s-2}(\mathbb{T}^d)$ (resp. $\widetilde{KR}^{s-2}(S^d)$). There is a similar statement for the two complex classes, replacing KR by complex K-theory.*

Remark 1.19. In our convention, class A weak SPT phases are classified by *unreduced* K-theory $K^0(\mathbb{T}^d)$. In the decomposition $K^0(\mathbb{T}^d) \cong K^0(\mathrm{pt}) \oplus \tilde{K}^0(\mathbb{T}^d)$, the first term corresponds to the 0-cell of the Brillouin zone. Physically, this $K^0(\mathrm{pt}) \cong \mathbb{Z}$-valued invariant is a comparison count of the number of bands below versus above the gap. An analogous argument applies to the other classes, where the invariant can also be $\mathbb{Z}/2$-valued or nonexistent depending on $KO^{s-2}(\mathrm{pt})$. This invariant is typically ignored in the condensed matter literature, but we would argue it should be included as a weak phase corresponding to a 0-dimensional strong phase.

1.3. **Bordism classifications of interacting phases.** As mentioned in the introduction, when we "turn on interactions" by regarding free fermion Hamiltonians in the context of all symmetry-protected gapped lattice Hamiltonians, (i.e. representatives of invertible topological phases) it is conjectured that deformation classes of invertible topological phases are classified by their low-energy behavior, captured by a reflection-positive invertible field theory (IFT). See [Fre19, FH21] for further discussion of this ansatz, which is supported by a strong body of computational evidence [FH21, Cam17, KT17, BC18, WG18, GJF19, FH20, ABK21, Deb21, BCHM22].

1.3.1. *The classification of reflection-positive invertible field theories.* Reflection-positive IFTs are defined at a mathematical level of rigor by Freed-Hopkins [FH21, §8] in the topological case and Grady-Pavlov [GP21, §5] in the nontopological case.[5] They are classified using generalized cohomology; before we give the classification in Theorem 1.24, we review some key definitions.

Let O denote the infinite orthogonal group $\mathrm{colim}_n\, \mathrm{O}(n)$, and let $\rho\colon H \to \mathrm{O}$ be a homomorphism of topological groups; this is equivalent to a collection of topological groups H_n for $n \geq 1$ and maps $i_n\colon H_n \to H_{n+1}$ and $\rho_n\colon H_{n+1} \to \mathrm{O}(n)$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n & \xrightarrow{i_n} & H_{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_{n+1}} & & \\ \cdots & \longrightarrow & \mathrm{O}(n) & \xrightarrow{-\oplus 1} & \mathrm{O}(n+1) & \longrightarrow & \cdots \end{array} \tag{1.20}$$

commutes. For example, $\rho_n\colon H_n \to \mathrm{O}(n)$ could be the inclusion $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$, or the spin group with the vector representation $\mathrm{Spin}(n) \to \mathrm{O}(n)$.

Definition 1.21. Given $\rho\colon H \to \mathrm{O}$ as above, let $\Omega_*^H(-)$ denote the generalized homology theory called *H-bordism*: $\Omega_n^H(X)$ is the abelian group of closed n-manifolds with an H-structure [Che66] and a map to X under disjoint union, modulo bordisms of such data.

For example, Ω_*^{SO} is the bordism theory of oriented manifolds.

[5] See [JF17, MS23, FHJF+24, CFH+24, Ste24] for more about reflection positivity in the noninvertible setting.

Definition 1.22. There is a duality on generalized homology and cohomology called *Anderson duality* [And69, Yos75]. Given a generalized homology theory E_*, the *Anderson dual of* E_* is the generalized cohomology theory $(I_{\mathbb{Z}}E)^*$ defined to satisfy the following universal property: for all spaces X, there is a natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}(E_{n-1}(X), \mathbb{Z}) \longrightarrow (I_{\mathbb{Z}}E)^n(X) \longrightarrow \operatorname{Hom}(E_n(X), \mathbb{Z}) \longrightarrow 0. \tag{1.23}$$

One can check that (1.23) actually uniquely characterizes a generalized cohomology theory $(I_{\mathbb{Z}}E)^*$. Moreover, because Hom and Ext are contravariant functors in their first argument, Anderson duality defines a contravariant functor on cohomology theories: given a natural transformation $E_*(-) \Rightarrow F_*(-)$, there is a natural transformation $(I_{\mathbb{Z}}E)^*(-) \Leftarrow (I_{\mathbb{Z}}F)^*(-)$.

The short exact sequence (1.23) splits, but *not* naturally,[6] implying an isomorphism from $(I_{\mathbb{Z}}E)^n(X)$ to the direct sum of the torsion subgroup of $E_{n-1}(X)$ with the free part of $E_n(X)$.

For more on $I_{\mathbb{Z}}$ and its appearance in this context, see Freed-Hopkins [FH21, §5.3, 5.4].

Theorem 1.24 (Freed-Hopkins [FH21, Theorem 1.1], Grady [Gra23, Theorem 1])**.** *Let $\mho_H^*$ denote the Anderson dual cohomology theory to Ω_*^H. Then there is a natural isomorphism from the abelian group of deformation classes of d-dimensional IFTs on manifolds with H-structure to $\mho_H^{d+2}$.*

As always, d is the spatial dimension of the theory.

1.3.2. *Spacetime symmetry groups for the tenfold way.* Theorem 1.24 leads us to use the cohomology theory $\mho_H^*$ to model interacting phases, but we need to determine H and its map to O for the ten collections of symmetries we are interested in. The reference [Ste22, §3.2 and 3.3] provides a unified way of doing this.

There is a construction of a spacetime structure group $H(G)$ from an internal symmetry group G indicated in [FH21]; see [Ste22] for a construction based on [Sto98]. Given a fermionic group $(G, \phi, (-1)^F)$, one first takes the central product

$$\widetilde{H} \coloneqq \frac{G \times \mathrm{Pin}^-}{\langle((-1)^F, -1)\rangle}, \tag{1.25}$$

where $-1 \in \mathrm{Pin}^-$ is the nontrivial element in the kernel of the map to O. There is a homomorphism $\widetilde{\phi}\colon \widetilde{H} \to \mathbb{Z}_2$ defined by sending $(g, B) \in G \times \mathrm{Pin}^-$ to $\phi(g) + \det(B)$, where $\det\colon \mathrm{Pin}^- \to \mathbb{Z}_2$ corresponds to the homomorphism taking the $\{\pm 1\}$-valued determinant of B under the canonical isomorphism $\{\pm 1\} \cong \mathbb{Z}_2$.

Finally, the tangential structure $H(G)$ associated to G is the group $\widetilde{\phi}^{-1}(0)$, with the map to O induced by the map on Pin^-. It is easy to show that the corresponding family of topological groups $H_d(G)$ is obtained by replacing Pin^- by $\mathrm{Pin}^-(d)$ in the above discussion.

Proposition 1.26. *Let G be a fermionic group with $\phi = 0$ trivial and let $i : G \to G$ be an involution. Define the two fermionic groups*

$$G_\pm \coloneqq \frac{G \rtimes \mathrm{Pin}^\pm(1)}{\mathbb{Z}_2^F}, \tag{1.27}$$

[6]This is a generalization of the unnatural splitting of the short exact sequence in the universal coefficient theorem [And69, §4].

where the semidirect product is defined using i *and* $\det\colon \mathrm{Pin}^{\pm}(1) \to \mathbb{Z}_2$*. The* ϕ *is defined by projection onto the second factor. Then there is an isomorphism of fermionic structure groups*

$$H_d(G_\pm) \cong \frac{G \rtimes \mathrm{Pin}^{\mp}(d)}{\mathbb{Z}_2^F}, \tag{1.28}$$

where the semidirect product is again defined using i *and* $\det : \mathrm{Pin}^{\mp}(d) \to \mathbb{Z}_2$

Proof. First some notation: denote the canonical odd element $T \in \mathrm{Pin}^{\pm}(1) \subset G_\pm$, so $T^2 = (\pm 1)^F$ and $gT = Ti(g)$ for $g \in G$ the elements with $\phi(g) = 0$. Given elements $x_1, x_2 \in \mathrm{Pin}^+(d)$, we define a new group structure (the 'graded opposite') by

$$x_1 * x_2 := \begin{cases} (-1)^F x_1 x_2 & \text{both odd,} \\ x_1 x_2 & \text{otherwise.} \end{cases} \tag{1.29}$$

Then $(\mathrm{Pin}^+(d), *) \cong \mathrm{Pin}^-(d)$ as fermionic groups.

Define the map

$$\psi\colon \frac{G \rtimes \mathrm{Pin}^{\mp}(d)}{\mathbb{Z}_2^F} \to H(G_\pm) \subseteq \frac{G_\pm \times \mathrm{Pin}^-(d)}{\mathbb{Z}_2^F} \tag{1.30a}$$

by

$$\psi(g \rtimes x) = \begin{cases} (g, x) & \det(x) = 0, \\ (gT, x) & \det(x) = 1. \end{cases} \tag{1.30b}$$

This is well-defined because we quotient by all common $\mathbb{Z}_2^F$ on both sides, and ψ lands in $H(G_\pm)$ because $\det(x) + \phi(T) = 0$ if $\det(x) = 0$, and $\det(x) + \phi(gT) = 0$ if $\det(x) = 1$. To check that this is a homomorphism, we let $g_1 \rtimes x_1,\ g_2 \rtimes x_2 \in G \rtimes \mathrm{Pin}^{\mp}(d)$ and have to show $\psi((g_1 \rtimes x_1)(g_2 \rtimes x_2)) = \psi(g_1 \rtimes x_1)\psi(g_2 \rtimes x_2)$. There are four cases depending on $\det x_1$ and $\det x_2$. The most nontrivial case is the one for which both are 1:

$$(g_1 \rtimes x_1)(g_1 \rtimes x_2) = g_1 i(g_2) \rtimes (\mp 1)^F x_1 x_2, \tag{1.31}$$

where we have used the product $*$ in case we are working in $\mathrm{Pin}^-(d)$ and the normal product of $\mathrm{Pin}^+(d)$ otherwise. This element is indeed mapped to

$$(g_1 T, x_1)(g_2 T, x_2) = (g_1 T g_2 T, x_1 x_2) = ((\pm 1)^F 1 g_1 i(g_2), x_1 x_2). \tag{1.32}$$

The other three cases are easier. It is not hard to see that ψ is a bijection. ☐

Example 1.33. We illustrate how to use Proposition 1.26 to determine the tangential structures for symmetry classes BDI and DIII, which are the cases $s = 1$ and $s = -1$ respectively. There are isomorphisms of fermionic groups $S(C\ell_{\pm 1}) \cong \mathrm{Pin}^{\pm}(1)$; $\mathrm{Pin}^+(1) \cong \mathbb{Z}_2^F \times \mathbb{Z}_2^T$ and $\mathrm{Pin}^-(1) \cong \mathbb{Z}_4^T$, with $\mathbb{Z}_2^F \subset \mathbb{Z}_4^T$ the unique order-two subgroup. Now apply Proposition 1.26 with $G = \mathbb{Z}_2$ and the involution $i = \mathrm{id}$: the semidirect product $G \rtimes \mathrm{Pin}^{\pm}(1)$ simplifies to a direct product, and then $\mathbb{Z}_2$ cancels the $\mathbb{Z}_2$ in the denominator, so in (1.27), $G_\pm = \mathrm{Pin}^{\pm}(1)$. In exactly the same way, $H_d(G_\pm)$ simplifies to $\mathrm{Pin}^{\mp}(d)$. Thus Proposition 1.26 reproduces a well-known fact in the physics literature: fermionic systems with a time-reversal symmetry T with $T^2 = 1$ correspond to putting pin$^-$ structures on spacetime, and with $T^2 = (-1)^F$ correspond to putting pin$^+$ structures on spacetime.

Example 1.34. We come back to class AII. In Example 1.2, we obtained the fermionic group $S(C\ell_{-2}) \cong (\mathbb{Z}_4^T \rtimes \mathrm{U}(1))/(\mathbb{Z}_2^F)$ from the symmetry algebra of this class. Using Proposition 1.26, we will compute the tangential structure group $H_d(S(C\ell_{-2}))$: there is an isomorphism $\mathbb{Z}_4^T \cong \mathrm{Pin}^-(1)$ of fermionic groups: both have underlying group isomorphic to $\mathbb{Z}_4$ with the map to O_1 nontrivial, and this characterizes $\mathbb{Z}_4^T$ up to isomorphism of fermionic groups. Therefore, we can apply Proposition 1.26 with $G = \mathrm{U}(1)$ and i equal to complex conjugation. Using that $G_- = S(C\ell_{-2})$, we conclude

$$H_d(S(C\ell_{-2})) \cong \frac{\mathrm{Pin}^+(d) \ltimes \mathrm{U}(1)}{\mathbb{Z}_2^F}. \tag{1.35}$$

Metlitski [Met15, §III.B] introduces this group in the context of invertible phases, and calls it $\mathrm{Pin}_{\tilde{c}}$. Its appearance in the tenfold way is due to Freed-Hopkins [FH21, (9.9)], who call this group $\mathrm{Pin}^{\tilde{c}+}(d)$. We will follow Freed-Hopkins' notation, as we will also need $\mathrm{Pin}^{\tilde{c}-}(d) \coloneqq (\mathrm{Pin}^-(d) \ltimes \mathrm{U}(1))/\mathbb{Z}_2^F$.

The other seven classes in the tenfold way can be worked out in a similar manner. We summarize the results of each step in table 1.[7]

s	AZ class	A	$S(A)$	$H^c(s) = H(S(A))$
0	A	$\mathbb{C}$	$\mathrm{U}(1)$	$H^c(0) = \mathrm{Spin}^c$
1	AIII	$C\ell_1 \otimes \mathbb{C}$	$\mathrm{U}(1) \times \mathbb{Z}_2$	$H^c(1) = \mathrm{Pin}^c$

s	AZ class	A	$S(A)$	$H(s) = H(S(A))$
-3	CII	$C\ell_{-3}$	$\mathrm{Pin}^-(3)$	$H(-3) = \mathrm{Pin}^{h-} \coloneqq \mathrm{Pin}^- \times_{\{\pm 1\}} \mathrm{SU}(2)$
-2	AII	$C\ell_{-2}$	$\mathrm{Pin}^-(2)$	$H(-2) = \mathrm{Pin}^{\tilde{c}+} \coloneqq \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathrm{U}(1)$
-1	DIII	$C\ell_{-1}$	$\mathrm{Pin}^-(1)$	$H(-1) = \mathrm{Pin}^+$
0	D	$\mathbb{R}$	$\mathrm{Spin}(1)$	$H(0) = \mathrm{Spin}$
1	BDI	$C\ell_1$	$\mathrm{Pin}^+(1)$	$H(1) = \mathrm{Pin}^-$
2	AI	$C\ell_2$	$\mathrm{Pin}^+(2)$	$H(2) = \mathrm{Pin}^{\tilde{c}-} \coloneqq \mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathrm{U}(1)$
3	CI	$C\ell_3$	$\mathrm{Pin}^+(3)$	$H(3) = \mathrm{Pin}^{h+} \coloneqq \mathrm{Pin}^+ \times_{\{\pm 1\}} \mathrm{SU}(2)$
4	C	$C\ell_4$	$\mathrm{Spin}(3)$	$H(4) = \mathrm{Spin}^h \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}(2)$

TABLE 1. Summary of the procedure outlined in §1.3.2 beginning with an Altland-Zirnbauer class (second column) and then building a super division algebra A (third column), a fermionic group $S(A)$ (fourth column), and a tangential structure $H(S(A))$ (fifth column). For the tangential structures, the maps to O are all trivial on U(1) and SU(2) and are the usual maps $\mathrm{Spin} \to \mathrm{SO} \to \mathrm{O}$ or $\mathrm{Pin}^\pm \to \mathrm{O}$ on the other factors; since $\{\pm 1\}$ is in the kernel of all of these maps, these maps descend across the quotient by $\{\pm 1\}$ to produce well-defined maps $H(s) \to \mathrm{O}$. First table: the two complex cases. Second table: the eight real cases. Tables adapted from [FH21, (9.24), (9.25)] and [Ste22, Table 1].

[7] These are not the only conventions for the superalgebras, fermionic groups, and spacetime tangential structures in the literature: see [Ste22] and the references therein.

1.3.3. *What changes for weak phases?* To the best of our knowledge, a mathematical model for the classification of weak phases with interactions has not been widely applied in the literature. In this subsubsection, we propose one in Ansatz 1.36, building from an ansatz of Freed-Hopkins [FH20, Ansatz 2.1] to obtain a homotopical model in Corollary 1.38 – families of invertible field theories over the spatial torus. In §2 we will further discuss and justify this ansatz from a physical point of view.

Ansatz 1.36. *The data of a discrete translation-invariant topological phase is equivalent to a family of phases parametrized by the spatial torus* $\mathbf{T}^d$*.*

See in particular [FH20, Example 2.3].

The appearance of the real (unit cell) torus $\mathbb{R}^d/\mathbb{Z}^d = \mathbf{T}^d$ is understood from the gauge theory point of view through the so-called crystalline equivalence principle [TE18], where if $\mathbb{Z}^d$ is a spatial symmetry group and a theory is defined on $\mathbb{R}^d$, there is a procedure for gauging the spatial symmetry and considering the emergent gauge theory on the quotient space $\mathbb{R}^d/\mathbb{Z}^d$. More generally, Freed and Hopkins propose an ansatz [FH20, Ansatz 3.3, Remark 2.6] that the invertible field theories with (spatial) dimension d on a compact d-dimensional manifold Y are classified by (a possibly twisted version of) $\mho_H^{d+2}(Y)$. They also consider stacks, and thus can obtain invertible field theories on any quotient $\mathbb{R}^d/G$ with G locally compact. In §2 we present a first-principles derivation in which the unit cell spatial torus $\mathbf{T}^d$ must appear in many-body interacting systems that have discrete translation symmetry, without the particular need to appeal to field theory. Nevertheless, both points of view can be combined when we employ the Freed-Hopkins ansatz for the spectrum classifying SPT phases.

Ansatz 1.37 (Freed-Hopkins [FH20, Ansatz 2.1, Remark 2.6])**.** *The classification of (interacting) invertible d-dimensional phases of symmetry type* $\rho\colon H \to \mathrm{O}$ *over a compact, stably framed manifold* Y *is naturally equivalent to the classification of d-dimensional reflection-positive IFTs of manifolds with an* H*-structure and a map to* Y*, i.e. the generalized cohomology group* $\mho_H^{d+2}(Y)$*.*

Freed-Hopkins' ansatz is more general than ours; we include only the special case[8] we need.

Corollary 1.38. *Assuming Ansatzes 1.36 and 1.37, deformation classes of invertible discrete translation-invariant topological phases in (spatial) dimension d and Altland-Zirnbauer class s are classified by d-dimensional reflection-positive IFTs on* $H(s)$*-manifolds with a map to* $\mathbf{T}^d$*, i.e. by the generalized cohomology group* $\mho_{H(s)}^{d+2}(\mathbf{T}^d)$*.*

We will discuss this further in §2.

1.4. **Freed-Hopkins' free-to-interacting map for strong phases.** Freed and Hopkins connect the K-theoretic classification of free theories to the invertible-field-theoretic classification of interacting theories using a *free-to-interacting map* ([FH21] (9.71)). The kernel of this map comprises the theories that are nontrivial under two-body nearest-neighbor interactions, but which may be trivialized using higher-order interactions: a famous example

[8]This case is studied for $s = 0$, $d = 2$ in their Ex. 2.3.

of such a theory is eight copies of the time-reversal symmetric Majorana chain studied by Fidkowski-Kitaev [FK10]. The cokernel of this map consists of "interaction-enabled" phases: interacting phases that have no free analog. For example, there is a class CI superconductor in $d = 3$ with an intrinsically interacting phase generating a $\mathbb{Z}_2$ interaction-enabled classification [WS14, §V.B]. Thus the free-to-interacting map allows one to mathematically study the physical questions of whether a free phase is robust to interactions and whether new phases arise in the interacting setting.

The free-to-interacting map is built out of two main ingredients. The first one, the Atiyah-Bott-Shapiro (ABS) orientation, provides a way to get from a bordism class to a K-theory class. Then, bordism is Anderson-dual to the interacting IFT classification (recall Definition 1.22), so to land in IFTs instead of bordism we implement this duality and use the Anderson self-duality of KO-theory.

1.4.1. *ABS Orientation.* We start with the ABS map in the real case. There is a classical ABS map from spin bordism Ω_*^{Spin} to the KO-theory of a point, first defined in [Ati68, §11]. Here, we follow Freed-Hopkins [FH21, §9.6.3], who use a model for KO-theory developed in [AS69] and follow [LM89, §II.7]. An element in Ω_n^{Spin} is represented by an n-dimensional spin manifold, while an element in $KO_n(\mathrm{pt})$ is (the equivalence class of) a $C\ell_n$-module equipped with a Clifford-linear Fredholm operator. Choose a spin manifold M with a Riemannian metric g, and let ∇ be the induced Levi-Civita connection on the *Dirac bundle*

$$\mathcal{S} \coloneqq P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} C\ell_n, \tag{1.39}$$

where P_{Spin} is the $\mathrm{Spin}(n)$-principal bundle associated to the spin structure on M. We obtain a Clifford-linear Dirac operator $\not{D}_M \colon C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$ by acting by the covariant derivative followed by Clifford multiplication $c\colon TM \times \mathcal{S} \to \mathcal{S}$:

$$\mathcal{S} \xrightarrow{\nabla} T^*M \otimes \mathcal{S} \overset{g}{\simeq} TM \otimes \mathcal{S} \hookrightarrow C\ell(TM) \otimes \mathcal{S} \to \mathcal{S}. \tag{1.40}$$

This operator extends to an operator on the appropriate Sobolev completion $\overline{C^\infty(\mathcal{S})}$ of $C^\infty(\mathcal{S})$. In local coordinates, for $s \in C^\infty(\mathcal{S})$, $\not{D}_M$ has the formula

$$\not{D}_M(s) = \sum e_j \cdot \nabla_{e_j}(s). \tag{1.41}$$

The ABS map

$$\begin{aligned} \mathrm{ABS}\colon \Omega_n^{\mathrm{Spin}} &\to KO_n \\ M &\mapsto (\overline{C^\infty(\mathcal{S})}, \not{D}_M) \end{aligned} \tag{1.42}$$

sends a spin manifold M to the Hilbert space $\overline{C^\infty(\mathcal{S})}$ equipped with the Dirac operator.

Freed and Hopkins [FH21, §9.2.2] develop its twisted generalizations

$$\mathrm{ABS}_s\colon \Omega_n^{H(s)} \to KO_{n+s} \tag{1.43}$$

by showing that an n-manifold M with $H(s)$-structure has a canonical twisted spinor bundle with a twisted $C\ell_{n+s}$-linear Dirac operator.[9]

[9] A construction of Stolz [Sto98, §9.3] overlaps with Freed-Hopkins' definition for $s = \pm 1$: the index theory is the same, but Stolz does not turn it into a map of spectra. See also [Sto88, Zha17, Fre24] for more on index theory on pin$^+$ and pin$^-$ manifolds.

Example 1.44 (Twisted ABS for class AII)**.** We go into the details of Freed-Hopkins' construction for the case $s = -2$: see [FH21, §9.2.2] for the proofs of these assertions.

In class AII, $H(s) = \mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathrm{U}_1$ (Table 1).

An element of $\Omega_n^{\mathrm{Pin}^{\tilde{c}+}}$ is represented by an n-manifold with $\mathrm{Pin}_n^{\tilde{c}+}$-structure, which is the same as a lift of the classifying map of the tangent bundle $M \xrightarrow{TM} B\mathrm{O}(n)$ to a map $M \to B\mathrm{Pin}_n^{\tilde{c}+}$. This gives us a $\mathrm{Pin}_n^{\tilde{c}+}$ principal bundle $P_{\mathrm{Pin}_n^{\tilde{c}+}} \to M$. The group $\mathrm{Pin}_n^{\tilde{c}+}$ embeds into the superalgebra $C\ell_n \otimes C\ell_{-2}$, as follows from [FH21, Lemma 9.27]. We thus have a *twisted Dirac bundle*

$$\mathcal{S}' := P_{\mathrm{Pin}_n^{\tilde{c}+}} \times_{\mathrm{Pin}_n^{\tilde{c}+}} (C\ell_n \otimes C\ell_{-2}) \to M. \tag{1.45}$$

We can define a Clifford multiplication map

$$c\colon TM \otimes \mathcal{S}' \to \mathcal{S}' \tag{1.46}$$

by using the Clifford multiplication $TM \otimes C\ell(TM) \to C\ell(TM)$ and tensoring with $C\ell_{-2}$. Now choose a Riemannian metric on M, and choose a connection ∇ on the principal $\mathrm{Pin}_n^{\tilde{c}+}$-bundle of frames $P_{\mathrm{Pin}^{\tilde{c}+}} \to M$ whose induced connection on the principal $\mathrm{O}(n)$-bundle of frames is the Levi-Civita connection. This induces a connection on the twisted Dirac bundle, which following tradition we also denote ∇. Now we can define a *twisted Clifford-linear Dirac operator* $\not{D}_M = e_i \cdot \nabla_{e_i}$ acting on sections of $\mathcal{S}'$ by taking the covariant derivative followed by Clifford multiplication. This acts $C\ell_n \otimes C\ell_{-2}$-linearly, so $(\overline{C^\infty(\mathcal{S}')}, \not{D}_M)$ gives an element of $KO_{n-2}(\mathrm{pt})$.

For example, on the $\mathrm{pin}^{\tilde{c}+}$ manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$, this twisted Dirac index evaluates to the generator of $KO_2(\mathrm{pt})$. We prove this in an indirect manner, using a Smith homomorphism, in Appendix A; it would be interesting to find an index-theoretic proof.

The twisted ABS map and twisted Dirac operators are discussed in full generality for all symmetry classes $H(s)$ in [FH21, §9.2.2, 9.2.3].

Just as for the real case, there is an ABS orientation landing in complex K-theory. The classical map

$$\mathrm{ABS}^c\colon \Omega_n^{\mathrm{Spin}^c} \to K_n \tag{1.47}$$

is from spin^c bordism to the K-homology of a point and sends a spin^c manifold M to the complex $\mathbb{C}\ell_n$-linear Dirac operator acting on smooth sections of the complex Dirac bundle $\mathcal{S} := P_{\mathrm{Spin}^c(n)} \times_{\mathrm{Spin}^c(n)} \mathbb{C}\ell_n$.

To incorporate the case $H^c(1) = \mathrm{Pin}^c$, Freed-Hopkins in [FH21] (9.44) develop the twisted generalization of this map:

$$\mathrm{ABS}_1^c\colon \Omega_n^{\mathrm{Pin}^c} \to K_{n+1}. \tag{1.48}$$

1.4.2. *Anderson Self-Duality of K-Theory.* The ABS orientation of the previous subsubsection defines a map from twisted spin bordism to KO-homology. However, the invertible field theories modelling interacting phases are classified by *Anderson-dual* twisted spin bordism (Theorem 1.24), while free theories are classified by *KO-cohomology* (Corollary 1.18). To reconcile these descriptions, we may apply Anderson duality (Definition 1.22) and exploit the Anderson self-duality of KO-theory and K-theory [And69, Theorem 4.16].[10]

Corollary 1.49. *There is an isomorphism of cohomology theories*

$$(I_{\mathbb{Z}}KO)^* \cong KO^{*-4}. \tag{1.50}$$

For complex K-theory, there is actually an isomorphism $I_{\mathbb{Z}}K^* \cong K^*$ with no shift. However, by Bott periodicity, $K^* \cong K^{*+2}$, so we may choose to insert a fourfold shift.

Corollary 1.51. *There is an isomorphism of cohomology theories*

$$(I_{\mathbb{Z}}K)^* \cong K^{*-4}. \tag{1.52}$$

1.4.3. *Free-to-Interacting Maps.* We now have all of the ingredients we need to define the free-to-interacting maps.

Definition 1.53 (Freed-Hopkins [FH21, Conjecture 9.70])**.** Let $\mathrm{ABS}_s\colon \Omega_n^{H(s)} \to KO_{n+s}$ be the twisted ABS map (1.43). Applying Anderson duality (1.22) gives a map

$$I_{\mathbb{Z}}\mathrm{ABS}_s\colon I_{\mathbb{Z}}KO^{d+s-2}(-) \to I_{\mathbb{Z}}\Omega^{d+2}_{H(s)}(-) \tag{1.54a}$$

of cohomology theories. The right side is by definition $\mho^{d+2}_{H(s)}$. The left side, by Anderson self-duality of KO-theory (1.49), is identified with $KO^{d+s-2}(\mathrm{pt})$, so $I_{\mathbb{Z}}\mathrm{ABS}_s$ is a map of cohomology theories of the form

$$\mathrm{F2I}_s\colon KO^{d+s-2}(-) \xrightarrow{\cong} I_{\mathbb{Z}}KO^{d+s+2}(-) \xrightarrow{I_{\mathbb{Z}}\mathrm{ABS}_s} I_{\mathbb{Z}}\Omega^{d+2}_{H(s)}(-) = \mho^{d+2}_{H(s)}(-). \tag{1.54b}$$

The free to interacting map is the composition

$$\mathrm{F2I}_{s,strong} \coloneqq \mathrm{F2I}_s(\mathrm{pt})\colon KO^{d+s-2} \to \mho^{d+2}_{H(s)}. \tag{1.55}$$

The complex version of the free-to-interacting map is given by a similar composition, implicit in [FH21]. We define a natural transformation of cohomology theories $\mathrm{F2I}^c_s\colon K^{d+s+2}(-) \to \mho^{d+2}_{H^c(s)}(-)$ just as in Definition 1.53, then evaluate it on pt to define $\mathrm{F2I}^c_{s,strong}$.

Definition 1.56 (Freed-Hopkins [FH21])**.** Let s be a complex symmetry type. The free-to-interacting map for theories in spatial dimension d and of symmetry type s is the composition

$$\mathrm{F2I}^c_{s,strong} \coloneqq \mathrm{F2I}^c_s(\mathrm{pt})\colon K^{d+s-2} \xrightarrow{\cong} I_{\mathbb{Z}}K^{d+s+2} \xrightarrow{I_{\mathbb{Z}}\mathrm{ABS}^c_s} I_{\mathbb{Z}}\Omega^{d+2}_{H^c(s)} = \mho^{d+2}_{H^c(s)}, \tag{1.57}$$

where the first arrow is the Anderson self-duality of K-theory (1.51) and the second map is the Anderson dual of the twisted ABS map defined in (1.47), (1.48).

Ansatz 1.58 (Freed-Hopkins [FH21, §9.2.6])**.**

(1) Under the identifications in Corollary 1.18 and Theorem 1.24 identifying the groups of strong free fermion phases, resp. reflection positive IFTs in dimension d and real Altland-Zirnbauer class s with KO^{d+s-2}, resp. $\mho^{d+2}_{H(s)}$, the homomorphism assigning to a free fermion Hamiltonian its low-energy invertible field theory is $\mathrm{F2I}_{s,strong}$.

[10] Anderson's proof appears in unpublished lecture notes, and it is also discussed in Yosimura [Yos75, Theorem 4]. There are several proofs by a variety of different methods; for example, see Freed-Moore-Segal [FMS07, Proposition B.11], Heard-Stojanoska [HS14, Theorem 8.1], Ricka [Ric16, Corollary 5.8], and Hebestreit-Land-Nikolaus [HLN20, Example 2.8].

(2) *The above is true mutatis mutandis for complex Altland-Zirnbauer class s with $H^c(s)$ in place of $H(s)$, K in place of KO, and* $\mathrm{F2I}^c_{s,strong}$ *in place of* $\mathrm{F2I}_{s,strong}$.

Recall the motivation for free-to-interacting maps given in §1.4: knowing these maps allows us to determine both whether a free-fermion SPT phase is stable to interactions and whether there are interaction-enabled phases that one cannot represent using free-fermion models.

Example 1.59. Return to class AII in dimension $d = 3$. Let $x \in KO^{-1}(\mathrm{pt}) \cong \mathbb{Z}_2$ be a free theory with nontrivial Fu-Kane-Mele invariant, which is the generator of $\mathbb{Z}_2$. Such a theory models for example a conducting surface state of the 3d topological insulator BiSb [TFK08]. Its image under the free-to-interacting map is the deformation class of the $\mathrm{pin}^{\tilde{c}+}$ topological field theory whose partition function is described by Witten in [Wit16, §4.7]. In Theorem A.9, we show that, when evaluated on the generating manifolds $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, and $\mathbb{RP}^4$ of $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$, this invariant is nontrivial on the first but trivial on the second two.

Since a free theory with the nontrivial Fu-Kane-Mele invariant is sent to a nontrivial interacting theory generating a $\mathbb{Z}_2$ subgroup of the interacting theories, we see that this strong phase is robust to interactions [FH21]. That the this invariant survives the addition of interactions was observed in [Wit16, §4.7] and [FH21], and an interacting $\mathbb{Z}_2$ Fu-Kane-Mele index was recently developed in [BBR24].

We have accounted for a $\mathbb{Z}_2$ subgroup of $\mho^5_{\mathrm{Pin}^{\tilde{c}+}} \cong (\mathbb{Z}_2)^3$; the remaining six elements are not in the image of the free-to-interacting map and thus are interaction-enabled phases. There is a generating set of $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ given by the Fu-Kane-Mele theory described above, together with two theories whose partition functions are

$$X \longmapsto (-1)^{\int_X w_2(TX)^2}, \qquad X \longmapsto (-1)^{\int_X w_1(TX)^4}, \tag{1.60}$$

detected by $\mathbb{CP}^2$ and $\mathbb{RP}^4$ respectively. These theories are closely related to *classical Dijkgraaf-Witten theories* [FQ93, §1],[11] in that they are given by a classical action which integrates a characteristic class. Unlike the classical actions of Dijkgraaf-Witten theories, the classes w_1^4 and w_2^2 do not depend on anything stronger that the homotopy type of X—in particular, they are independent of the choice of $\mathrm{pin}^{\tilde{c}+}$ structure. One could thus think of these theories as "bosonic;" frequently such theories are set aside by researchers investigating fermionic SPTs.

1.4.4. *The free-to-interacting map constrains the spectrum of SPT phases.* The ansatz that interacting phases are classified in terms of invertible field theories – and therefore, thanks to Theorem 1.24, in terms of bordism – is not the only model for the classification of interacting phases. The purpose of this subsubsection is to point out that the existence of

[11] The reason we write "closely related" instead of "are" is a few key differences between these theories and classical Dijkgraaf-Witten theories: in the former, we integrated a mod 2 characteristic class, and in the latter, one integrates an $\mathbb{R}/\mathbb{Z}$-valued cohomology class ω. Secondly, Dijkgraaf-Witten theory has a background principal bundle for a finite group G, and requires ω to be a characteristic class of G-bundles. Integrating $\mathbb{R}/\mathbb{Z}$-cohomology classes requires an orientation, but integrating mod 2 cohomology classes does not, so the theories in (1.60) are defined on any compact 4-manifold. See [Deb20, You20, Kim22, GRY24] for more information on unoriented generalizations of Dijkgraaf-Witten theory. Sometimes, theories given by integrating a characteristic class of the tangent bundle are called *gravitational theories*.

the free-to-interacting map, and one of its basic properties, strongly constrains the possible models for the classification of interacting phases.

As we discuss further in §2.2, Kitaev proposed the ansatz that invertible phases have the structure of a spectrum D: that is, D determines a generalized cohomology theory D^*, and the classification of G-symmetric SPT phases is the (possibly twisted) cohomology group $D^*(BG)$.[12] The model we have followed, which uses invertible field theories, chooses D^* to be the Anderson dual of spin bordism. But there are two more common choices for fermionic phases: *restricted supercohomology* SH_1 as introduced by Freed [Fre08, §1] and Gu-Wen [GW14], and *extended supercohomology* SH_2 as defined by Kapustin-Thorngren [KT17] and Wang-Gu [WG18]. See also Gaiotto-Johnson-Freyd [GJF19, §5.3, 5.4].[13] We will not need to know much about these generalized cohomology theories—only that $SH_1^*(\mathrm{pt})$ and $SH_2^*(\mathrm{pt})$ are concentrated in degrees 0 through 3.

Kitaev's argument producing the structure of a spectrum of invertible phases applies equally well for both the free and the interacting classifications, and the argument is compatible with the free-to-interacting map between them. Therefore we hypothesize that Kitaev's conjecture extends: that *the free-to-interacting map refines to a map of spectra*—indeed, this is how Freed-Hopkins [FH21, §9.2] construct their free-to-interacting maps. This does not constrain the spectrum of SPT phases very much, though: there are nontrivial maps from the K-theory spectrum to both restricted and extended supercohomology.

We can do better with one more piece of information: assume there is a procedure on phases of free fermion theories which is analogous to the field-theoretic process of compactification. After taking a continuum limit, one ought to be able to formulate a topological phase of (spatial) dimension d on a closed d-manifold M, together with some additional structure such as a lattice, a twisted spin structure for fermionic SPT, etc.[14] By choosing M to be a product $M = N_1 \times N_2$, we can compactify on N_1 to pass from a d-dimensional phase formulated on M to a $(d - \dim(N_1))$-dimensional phase formulated on N_2. With some care applied to the tangential structure on N_1,[15] this procedure is expected to define a homomorphism from d-dimensional SPT phases to $(d-\dim(N_1))$-dimensional phases, and it is routinely applied in the condensed-matter theory literature, e.g. [HSHH17, Tan17, RL20]. As this procedure can be applied in the same ways to free and to interacting phases, we expect compactification to commute with the free-to-interacting map. Though this may not literally be compactification on free fermion phases, as it is not yet clear whether the process of putting the theory on a general manifold is possible before taking a continuum limit, we expect a homomorphism of this sort to exist for free fermion phases, and we will refer to this homomorphism as compactification.

Ansatz 1.61.

[12] It is predicted that there are two different versions of D, one for bosonic phases and the other for fermionic phases. In this paper we focus on the latter.

[13] Wang-Gu [WG20] introduce another variant that we will not use in this subsubsection; it does not suffer the K3 problem we discuss here, but our argument can be adapted to their version of supercohomology.

[14] For example, we might to be able to glue a lattice model on M together from a Hamiltonian description on contractible patches.

[15] See Schommer-Pries [SP18, §9] for a careful general analysis of the tangential structures needed to compactify; we just need a special case addressed by Yamashita-Yonekura [YY23, §7.3].

Physics version: *The free-to-interacting map commutes with the procedure of compactifying on closed spin manifolds.*

Math version: *The free-to-interacting map is a map of $MT\mathrm{Spin}$-module spectra.*

Here $MT\mathrm{Spin}$ is the spectrum whose associated generalized homology theory is spin bordism. The connection between the two versions of Ansatz 1.61 is discussed by Yamashita-Yonekura [YY23, §7.3]; see also Tachikawa-Yamashita [TY23, §2.2.6]. Freed-Hopkins' free-to-interacting maps satisfy Ansatz 1.61 [FH21, §10].

Proposition 1.62. *Assuming Ansatz 1.61, SH_1 and SH_2 cannot be the spectrum of fermionic phases.*

Proof. Let K denote the K3 surface, which is a closed spin 4-manifold whose spin bordism class generates Ω_4^{Spin} [Mil63]. The $MT\mathrm{Spin}$-module structure on KO is through the Atiyah-Bott-Shapiro map (1.42); since this is an isomorphism in degrees 0 through 7 [ABP67], compactifying a free fermion theory represented by a class $x \in KO^m$ on the K3 surface is the same thing as multiplying x by a generator a of KO^{-4}: the minus sign is an artifact of the switch from homology to cohomology. In particular, this map is an isomorphism $KO^4 \to KO^0$.

As noted above, there is no n such that $SH_i^n(\mathrm{pt})$ and $SH_i^{n-4}(\mathrm{pt})$ are both nonzero, for $i = 1$ or $i = 2$. Therefore in both restricted and extended supercohomology, compactifying on K3 is the zero map.

Suppose SH_i with either $i = 1$ or $i = 2$ models the spectrum of interacting fermionic SPTs, and let τ denote the twist of supercohomology over $B\mathbb{Z}_2$ corresponding to Altland-Zirnbauer class DIII. Then compatibility of the free-to-interacting maps with compactification means that the following diagram must commute:

$$\begin{array}{ccccc} \mathbb{Z} & & KO^4 & \xrightarrow{\mathrm{F2I}} & SH_i^{5+\tau}(B\mathbb{Z}_2) & & \\ \cong\big\downarrow & & \big\downarrow a & & \big\downarrow 0 & & \\ \mathbb{Z} & & KO^0 & \xrightarrow{\mathrm{F2I}} & SH_i^{1+\tau}(B\mathbb{Z}_2) & & \mathbb{Z}_2 \end{array} \qquad \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}_2 \tag{1.63}$$

For both SH_1 and SH_2, the classification of interacting class DIII phases in (spacetime) dimension 0 is $\mathbb{Z}_2$: see, e.g., Wang-Gu [WG20, §VII.E.2.d].[16] And the free-to-interacting map in dimension 0 in class DIII is well-known to be nonzero: see [FH21, §9.3.1] and the references therein. But this is not compatible with the compactification map $KO^4 \to KO^0$ being an isomorphism and the compactification map on supercohomology vanishing. □

We model interacting phases in class DIII with Anderson-dualized pin$^+$ bordism; therefore in (spacetime) dimension 4 we have $\mathbb{Z}_{16}$ [Gia73, §2], in dimension 0 we have $\mathbb{Z}_2$ (*ibid.*), and the free-to-interacting maps in these dimensions are surjective [FH21, Corollary 9.83]. Therefore we learn that compactifying on K3 is the unique surjective map $\mathbb{Z}_{16} \to \mathbb{Z}_2$, which is dual to the fact that the K3 surface represents $8 \in \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$ [KT90, Lemma 5.3].

[16] As remarked above, this version of supercohomology is not the same as SH_1 or SH_2, but all three agree in dimension 0.

1.5. **T-duality.** Whereas the real torus $\mathbf{T}^d$ appears in our Ansatz 1.36 for interacting weak phases, non-interacting fermionic topological phases are traditionally formulated over the crystalline momentum space torus $\mathbb{T}^d$. These tori behave differently, particularly when symmetries are included. However, T-duality, a construction that originated in string theory [Bus87], precisely relates these two tori in a manner that allows us to recast non-interacting results in terms of $\mathbf{T}^d$ and thus to define a free-to-interacting map. We note that T-duality has been employed many times to treat problems in non-interacting fermionic topological phases [MT15, MT16b, MT16a, HMT17, GT19]. Associated to a d-dimensional lattice Π are the unit cell or spatial torus $\mathbf{T}^d := \mathbb{R}^n/\Pi$ and the momentum space torus or Brillouin zone $\mathbb{T}^d := \mathrm{Hom}(\Pi, \mathrm{U}(1))$. The Brillouin zone has a $\mathbb{Z}_2$-action given by complex conjugation on $\mathrm{U}(1)$. The Fourier transform between position and momentum space has a K-theoretic analog in the T-duality isomorphisms[17]

$$\mathrm{T}_{\mathbb{R}}\colon KO^\bullet(\mathbf{T}^d) \xrightarrow{\sim} KR^{\bullet-d}(\mathbb{T}^d) \qquad \text{TRS squares to } 1 \tag{1.64a}$$

$$\mathrm{T}_{\mathbb{H}}\colon KSp^\bullet(\mathbf{T}^d) \xrightarrow{\sim} KQ^{\bullet-d}(\mathbb{T}^d) \qquad \text{TRS squares to } -1 \tag{1.64b}$$

$$\mathrm{T}_{\mathbb{C}}\colon K^\bullet(\mathbf{T}^d) \xrightarrow{\sim} K^{\bullet-d}(\mathbb{T}^d) \qquad \text{Chern insulators} \tag{1.64c}$$

which can be defined in terms of a pull-convolve-push construction for topological bundles called the *Fourier-Mukai transform*. These isomorphisms have been well studied in the condensed matter literature; see for instance [Kit09, HMT17, MT16a]. There is also a C^*-algebraic approach to this material: see [Ros15]. Here we review the perspective of [HMT17, MT16a].

The *Poincaré line bundle* $\mathcal{L}$ is the complex line bundle on $\mathbf{T}^d \times \mathbb{T}^d = \mathbb{R}^d/\Pi \times \mathrm{Hom}(\Pi, \mathrm{U}(1))$ obtained as the quotient of the trivial bundle $\mathbb{C} \times \mathbb{R}^n \times \mathbb{T}^d$ by the Π-action via characters

$$\pi \cdot (z, v, \chi) \sim (e^{2\pi i \chi(\pi)} z, v + \pi, \chi). \tag{1.65}$$

Bloch waves come from sections of the restrictions of $\mathcal{L}$ to different momentum cross-sections $\mathbf{T}^d \times \{\chi\} \subset \mathbf{T}^d \times \mathbb{T}^d$. The T-duality map can then be expressed as a pull-push along the correspondence

$$\mathbf{T}^d \xleftarrow{\;p\;} \mathbf{T}^d \times \mathbb{T}^d \xrightarrow{\;\widehat{p}\;} \mathbb{T}^d \tag{1.66}$$

twisted by the Poincaré line bundle

$$E \mapsto \widehat{p}_*(p^* E \otimes \mathcal{L}) \tag{1.67}$$

where the pushforward $\widehat{p}_*$ is, intuitively, "integrating out the $\mathbf{T}^d$ direction" and thus reduces the dimension by d.[18] To recover (1.64a), note that in the presence of TRS squaring to 1, the involution $k \mapsto -k$ lifts to an antilinear action of T on $\mathcal{L}$, so $p^*E \otimes \mathcal{L}$ is a Real bundle in

[17] In (1.64b), KSp is the K-theory of quaternionic bundles and KQ is a Real-equivariant version of KSp introduced by Dupont [Dup69].

[18] Rigorously, $\widehat{p}_*$ is fiber integration in the appropriate K-theory.

$KR(\mathbf{T}^d \times \mathbb{T}^d)$ and $\widehat{p}_*$ is the pushforward in KR-cohomology. Equations (1.64b) and (1.64c) follow from similar reasoning.[19]

1.6. **Splitting the generalized cohomology of tori.** The generalized cohomology of a torus $\mathbb{T}^d$ has a convenient description in terms of the generalized cohomology of spheres (interpreted as cells in a cellular decomposition of $\mathbb{T}^d$), using the fact that for spaces X, Y, there is a homotopy equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \tag{1.68}$$

For instance, $\Sigma T^2 \simeq \Sigma S^1 \vee \Sigma S^1 \vee \Sigma S^2$. If we iterate this equivalence over the d-fold product of circles $T^d = (S^1)^{\times d}$ and and use the suspension isomorphism for generalized cohomology, we get the following identity, where we've labelled the circle factors $\mathbb{T}^d \simeq S^1_1 \times \ldots \times S^1_d$.

Lemma 1.69 (James splitting of the torus)**.** *The generalized cohomology of the d-dimensional torus is*

$$\widetilde{E}_0(\mathbb{T}^d) \cong \bigoplus_{I \subset \{1,\ldots,n\}} \widetilde{E}_0(S^I) \cong \bigoplus_{n=1}^{d} \widetilde{E}_{-n}^{\oplus\binom{d}{n}} \tag{1.70}$$

we denote by $S^I := \wedge_{i\in I} S^1_i$ *the* $|I|$*-dimensional sphere factor indexed by* $I \subset \{1, \ldots, n\}$.

The James splitting interacts nicely with T-duality. Given a spatial torus $\mathbf{T}^d = S^1_1 \times \ldots \times S^1_d$, the Brillouin zone is $\mathbb{T}^d \simeq \widehat{S}^1_1 \times \ldots \times \widehat{S}^1_d$ where $\widehat{S}^1_i$ are the dual circles (1d "dual tori") to S^1_i. Then,

Lemma 1.71 (T-duality and James splitting)**.** *Under the James splitting isomorphisms* $K(\mathbf{T}^d) \simeq \bigoplus_I K(S^I)$ *and* $K(\mathbb{T}^d) \simeq \bigoplus_J K(\widehat{S}^J)$*, T-duality maps* $K(S^I)$ *to* $K(\widehat{S}^{I^c})$*, where* $K = KO,\ KR,\ KSp,\ KQ,\ KU$ *as appropriate, and* I^c *is the complement of* I.

We refer the reader to [MT16a, §6] for more details.

1.7. **Comparing strong and weak phases.** In the previous sections we have discussed how the generalized cohomology of the torus splits into several summands, some of which are strong, and some of which are weak. Here, we would like to emphasize that which cell—top or bottom—of the torus is associated to the strong phase depends on which torus we consider. On the real space torus $\mathbf{T}^d$, strong phases correspond to the summands in $KO^{d+s-2}(\mathrm{pt}) \subset KO^{d+s-2}(\mathbf{T}^d)$; i.e. the summands coming from the point, or 0-cell of the torus. On the dual torus, which forms the Brillouin zone, the strong phases instead come from the top cell, in the sense that if one crushes all cells except for the top one, the resulting space is $\mathbb{Z}_2$-equivariantly homeomorphic to $\bar{S}^d$, and the induced pullback map on phases is the inclusion of the strong phases in the group of weak phases. In summary, the inclusion of strong phases into the total classification including strong and weak phases interacts with T-duality in the way outlined in the following diagram.

[19] Another way to see this is that, because $\mathbf{T}^d$ has trivial $\mathbb{Z}_2$ action, $KO(\mathbf{T}^d) \simeq KR(\mathbf{T}^d)$ and $KSp(\mathbf{T}^d) \simeq KQ(\mathbf{T}^d)$ so the T-duality maps (1.64a) and (1.64b) seem like they change the K-theory type but in fact they are a Fourier-Mukai transform internal to KR, KQ respectively, where one of the sides has trivial $\mathbb{Z}_2$-action and thus reduces to an ordinary nonequivariant K-theory.

$$\begin{array}{ccc} \widetilde{KR}^{s-2}(\bar{S}^d) & \hookrightarrow & KR^{s-2}(\bar{\mathbb{T}}^d) \\ {\scriptstyle \mathrm{T}_\mathbb{R}}\uparrow\cong & & {\scriptstyle \mathrm{T}_\mathbb{R}}\uparrow\cong \\ \widetilde{KO}^{d+s-2}(S^0) & \hookrightarrow & KO^{d+s-2}(\mathbf{T}^d) \end{array} \tag{1.72}$$

Here the top horizontal arrow is induced by a $\mathbb{Z}_2$-equivariant collapse map $(\bar{\mathbb{T}}^d)_+ \to \bar{S}^d$ and the lower by the crush map $(\mathbf{T}^d)_+ \to S^0$. This, and the analogous complex K-theory diagram, commute by Lemma 1.71, as these maps pick out the top cell of $\mathbb{T}^d$ and the bottom cell of $\mathbf{T}^d$. These inclusions are split and so realize the strong phases as a direct summand of all phases.

1.8. **The ansatz for the free-to-interacting map for weak phases.** Now we have all the ingredients we need to state our main ansatz (Ansatz 1.73): a model for the free-to-interacting map in terms of generalized cohomology.

Fix a (spatial) dimension d and symmetry type s. In Corollary 1.18, we modeled the group of free weak phases as $KR^{s-2}(\overline{\mathbb{T}}^d)$ in the real case and $K^{s-2}(\mathbb{T}^d)$ in the complex case, and in Corollary 1.38 we modeled the group of interacting weak phases as $\mho^{d+2}_{H(s)}(\mathbf{T}^d)$ in the real case and $\mho^{d+2}_{H^c(s)}(\mathbf{T}^d)$ in the complex case.

For the moment restrict to real symmetry types. We need to get from the Brillouin torus to the spatial torus, so our first step is to use T-duality (1.64a) to get from $KR^{s-2}(\overline{\mathbb{T}}^d)$ to $KO^{d+s-2}(\mathbf{T}^d)$. After that, we simply apply the free-to-interacting map $\mathrm{F2I}_s\colon KO^{d-s-2}(\mathbf{T}^d) \to \mho^{d+2}_{H(s)}(\mathbf{T}^d)$ of Definition 1.53, evaluated on the spatial torus. For complex symmetry types, the story is completely analogous, using the T-duality isomorphism of (1.64c) and the free-to-interacting map from Definition 1.56.

Ansatz 1.73.

(1) Let $x \in KR^{s-2}(\overline{\mathbb{T}}^d)$ be a discrete translation-invariant free fermion theory in d dimensions and of real *symmetry type s. The long-range effective theory of x is given by the image of x under the composition*

$$\mathrm{F2I}_{weak}\colon KR^{s-2}(\overline{\mathbb{T}}^d) \xrightarrow[(1.64a)]{\mathrm{T}_\mathbb{R}^{-1}} KO^{d-s-2}(\mathbf{T}^d) \xrightarrow[(1.53)]{\mathrm{F2I}_s} \mho^{d+2}_{H(s)}(\mathbf{T}^d). \tag{1.74a}$$

(2) Let $x \in K^{s-2}(\mathbb{T}^d)$ be a discrete translation-invariant free fermion theory in d dimensions and of complex *symmetry type s. The long-range effective theory of x is given by the image of x under the composition*

$$\mathrm{F2I}^c_{weak}\colon K^{s-2}(\mathbb{T}^d) \xrightarrow[(1.64c)]{\mathrm{T}_\mathbb{C}^{-1}} K^{d-s-2}(\mathbf{T}^d) \xrightarrow[(1.56)]{\mathrm{F2I}^c_s} \mho^{d+2}_{H^c(s)}(\mathbf{T}^d). \tag{1.74b}$$

This ansatz has several consequences for free and interacting weak phases. The following consequence refines the observation that weak phases break up into a direct sum of strong phases, which is well-known in the physics literature (see e.g. [Xio18, §7.3 and Proposition F.8] and [GOP+20, §9]), and applies it to our free-to-interacting map. The decomposition can be subtle; T-duality is essential for a clear understanding of this phenomenon.

Lemma 1.75 (Weak phases are built from strong phases)**.** *Write* $\mathrm{F2I}^d_{weak}$ *for the weak free-to-interacting map in dimension d from Ansatz 1.73, and* $\mathrm{F2I}^d_{strong}$ *for the strong free-to-interacting map in dimension d from Definition 1.53. We have that*

$$\mathrm{F2I}^d_{weak} = \bigoplus_{k=0}^{d} \binom{d}{k} \mathrm{F2I}^{d-k}_{strong}. \tag{1.76}$$

The analogous statement is true for the complex free-to-interacting maps.

Proof. The James splitting Lemma 1.69 of the Brillouin zone is equivariant with respect to the involutions on $\bar{\mathbb{T}}^d$ [FM13, Theorem 11.8]. Therefore there is a $\mathbb{Z}_2$-equivariant stable equivalence $\bar{\mathbb{T}}^d \simeq_{\text{stably}} \bigvee_{I\subseteq[d]} \bar{S}^I$, under which the element $x \in KR^{s-2}(\bar{\mathbb{T}}^d)$ splits into elements $x_I \in \widetilde{KR}^{s-2}(\bar{S}^I) = KO^{s-2+|I|}(\mathrm{pt})$. Under T-duality, by Lemma 1.71, we get elements $\mathrm{Dual}(x_I) = \bar{x}_{I^c} \in KO^{s-2+d-|I^c|} = KO^{s-2+|I|}$. Each I thus gives us a strong free-to-interacting map

$$\mathrm{F2I}^I_{strong}\colon \widetilde{KR}^{s-2}(\bar{S}^{|I|}) \to \mho^{|I|+2}_{H(s)}. \tag{1.77}$$

□

As a result, the kernel and cokernel of $\mathrm{F2I}^d_{weak}$ can be computed from those of $\mathrm{F2I}^k_{strong}$ as k varies from 0 to the dimension:

$$\ker \mathrm{F2I}^d_{weak} = \bigoplus_{k=0}^{d} \binom{d}{k} \ker \mathrm{F2I}^{d-k}_{strong}\,, \qquad \operatorname{coker} \mathrm{F2I}^d_{weak} = \bigoplus_{k=0}^{d} \binom{d}{k} \operatorname{coker} \mathrm{F2I}^{d-k}_{strong}. \tag{1.78}$$

This corollary makes the statement that weak phases are built from strong phases of lower dimension precise within our framework. There are two physical consequences from this result. First, if the kth strong phase is robust to interactions, then so is kth component of the weak phase, and vice versa. Similarly, all interaction-enabled weak phases arise from interaction-enabled strong phases in lower dimensions—in our model, there are no interaction-enabled phases that do not arise from lower-dimensional phenomena. We give more examples in §3.

Example 1.79. Freed-Hopkins [FH21, Corollary 9.93] calculated that in class AII, the strong free-to-interacting map is always injective in low degrees, including up to spatial dimension 3. From Lemma 1.75, we conclude that weak phases of translation-invariant class AII insulators in dimensions up to three are always robust to interactions.

In $d = 3$ in particular, the QSH phases associated to the three planar surfaces of the insulator are robust to interactions. Meanwhile, there are two interaction-enabled phases associated to the top-dimensional cell, coming from $\operatorname{coker} \mathrm{F2I}^3_{strong}$, as discussed in Example 1.59. There is also an interaction-enabled phase coming from the zero cell, encoded in $\operatorname{coker} \mathrm{F2I}^0_{strong}$; see [WPS14] for a physics interpretation of this phase.

Remark 1.80. Weak phases are protected by discrete translation symmetries. However, their free and interacting classifications can also be used to study situations in which a certain form of crystalline disorder called a dislocation is present. Dislocations are localized disruptions in the crystalline order that can host topologically protected modes—for example, three-dimensional TIs can host one-dimensional helical modes [RZV09]. In [Ran10], Ran developed a criterion for when these protected modes could occur: if $\vec{B}$ is the Burgers vector of the dislocation, and $\vec{M}$ is a vector of $(d-1)$-dimensional indices, then helical modes can exist if $\vec{B}\cdot\vec{M}$ takes the possible nonzero value.

Our framework can generalize this condition to the interacting setting and to other symmetry types. Consider the group $\widetilde{\mho}_H^{d+2}(S^1\vee\cdots\vee S^1)$ classifying codimension-one weak indices for systems with symmetry type H. By the suspension isomorphism and wedge axiom, this group is isomorphic to $(\mho_H^{d+1})^{\oplus d}$, so we may consider its elements to be d-vectors $\vec{M}$ of d spacetime-dimensional invertible field theories. We may still consider the Burgers vector $\vec{B}$ to be a vector in the cubic lattice $\mathbb{Z}^d$. Then the generalized dislocation pairing $\vec{B}\cdot\vec{M}$ is valued in $\mho_H^{d+1}$. For example, the three-dimensional weak topological insulator and the helical modes condition of [Ran10] concerns the pairing of $\vec{M}\in(\mho^4_{\mathrm{Pin}^{\tilde{c}+}})^{\oplus 3}\cong(\mathbb{Z}_2)^3\subset\mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ with a vector $\vec{B}\in\mathbb{Z}^3$, which takes a binary value in $\mho^4_{\mathrm{Pin}^{\tilde{c}+}}\cong\mathbb{Z}_2$.

2. Physical Justification

Why would the spatial unit cell torus appear in the classification of SPT phases? We know it appears under T-duality in the free fermion classification, yet so far we have not rigorously derived it on the interacting side. In the literature, this has been justified for the group cohomology classification through the crystalline equivalence principle, where it appears as the classifying space of spatial translations $B\mathbb{Z}^d$ [TE18], thus requiring a rather strong physical statement in order to employ the topology of the unit cell. Here we provide a general derivation from first principles and a functional analysis perspective as to why, independently of the choice of cohomology theory that classifies SPT phases, the unit cell spatial torus arises in the classification of discrete translation invariant topological phases.

2.1. Physical Interpretation.

Let us consider the single particle Hilbert space in one dimension

$$\mathscr{H}_1 = L^2(\mathbb{R}). \tag{2.1}$$

There is a spatial decomposition of $\mathscr{H}_1$ (direct integral decomposition) using a unit cell $\mathbf{T}=B\mathbb{Z}$ [RS12]. Let

$$\mathscr{H}_1' = L^2([0,2\pi]) \tag{2.2}$$

where $[0,2\pi]$ is the unit cell (with boundary) and consider the direct integral over the periodic unit cell $\mathbf{T}$ [RS12]

$$\mathscr{V}_1 = \int_{\mathbf{T}}^{\oplus}\mathscr{H}_1'\frac{d\theta}{2\pi}. \tag{2.3}$$

There is a unitary equivalence $U\colon L^2(\mathbb{R})\to\mathscr{V}_1$ given by

$$(U\psi)_\theta(\tilde{x}) = \sum_{n=-\infty}^{\infty} e^{-n\theta}\psi(\tilde{x}+2\pi n). \tag{2.4}$$

This decomposition can be extended to arbitrary $\mathbb{R}^d$:

$$(U\psi)_{(\theta_1,\ldots,\theta_d)}(\tilde{x}) = \sum_{n\in\mathbb{Z}^d} e^{-\sum_{j=1}^d\theta_j n_j}\psi(\vec{\tilde{x}}+\sum_i^d n_i a_i), \tag{2.5}$$

where a_i are the chosen lattice generators that will be relevant according to the periodicity of the system's Hamiltonian $\mathcal{H}$.

Let us now consider N particles in d-dimensions:

$$\mathscr{H}_N = \bigotimes_{i=1}^{N} L^2(\mathbb{R}_i^d) \approx L^2(\mathbb{R}^{dN}). \tag{2.6}$$

Let us now consider the diagonal action of $\mathbb{Z}^d$ on $\mathbb{R}^{dN}$ given by

$$(\vec{x}_1, ..., \vec{x}_n) \mapsto (\vec{x}_1 + \vec{a}, ..., \vec{x}_n + \vec{a}) \tag{2.7}$$

with $\vec{a} \in \mathbb{Z}^d$. Consider the quotient by the action $\mathbb{R}^{dN}/\mathbb{Z}^d$. This is homeomorphic to $\mathbf{T}^d \times \mathbb{R}^{d(N-1)}$; however, we have yet to choose a fundamental region for this action. We choose a convenient fundamental region $R(N,d)$ that is better described with the following coordinates:

$$\vec{y}_1 = \frac{1}{N}\sum_{i=1}^{N} \vec{x}_i, \tag{2.8}$$

$$\vec{y}_j = \vec{x}_j - \vec{x}_1 \ \forall j \neq 1. \tag{2.9}$$

Thus, we choose the fundamental region $R(N,d)$ in which our diagonal action simply becomes translation in the $\vec{y}_1$- direction. With this choice of fundamental region, we consider the Hilbert space $L^2(R(N,d)) = \mathscr{V}_N'$ and we can construct a direct integral Hilbert space $\mathscr{V}_N = \int_{T^d}^{\oplus} \mathscr{V}_N' d\theta$. Furthermore, there is a unitary equivalence $U\colon L^2(\mathbb{R}^{dN}) \to \mathscr{V}_N$ given by essentially the same formula

$$(U\psi)_{(\theta_1,...,\theta_d)}(\vec{\tilde{y}}_1, \vec{y}_2, ..., \vec{y}_N) = \sum_{m\in\mathbb{Z}^d} e^{-\sum_{j=1}^{d}\theta_j m_j} \psi\left(\vec{\tilde{y}}_1 + \sum_i^d m_i a_i, ..., \vec{y}_N\right). \tag{2.10}$$

Notice that exchanging the x_i's is equivalent to exchanging the $\vec{y}_i$'s whenever $i \neq 1$. However, exchanging $\vec{x}_1$ with $\vec{x}_j$ is equivalent to $\vec{y}_1 \mapsto \vec{y}_1$ and $\vec{y}_j \mapsto -\vec{y}_j$. Thus, we have the added bonus that the above decomposition works equally well for bosonic or fermionic wave functions. The above construction is well defined for every finite N and thus we can rewrite Fock space $\mathscr{F}(L^2(\mathbb{R}^d))$ as

$$\mathscr{V}_\infty = \bigoplus_{N\geq 0} \int_{\mathbf{T}^d}^{\oplus} \mathscr{V}_N' \frac{d\theta_1 ... d\theta_d}{(2\pi)^d}. \tag{2.11}$$

A system has discrete translation symmetry if for every finite N, its Hamiltonian $\mathcal{H}_N$ commutes with a representation of a lattice Λ in $\mathscr{F}(L^2(\mathbb{R}^d))$, i.e. that the Hamiltonian diagonalizes in the direct integral decomposition

$$U\mathcal{H}_N U^{-1} = \int_{\mathbf{T}^d}^{\oplus} \mathcal{H}_N(\theta_1, ..., \theta_d) \frac{d\theta_1 ... d\theta_d}{(2\pi)^d} \tag{2.12}$$

where the fiber Hamiltonian $\mathcal{H}(\theta_1, ..., \theta_d)$ must satisfy certain quasi-periodic boundary conditions on $[0, 2\pi]^d$. Thus our many-body Hamiltonian can be formally viewed as a map

$$\mathcal{H}\colon \mathbf{T}^d \to \mathcal{L}_d := \mathcal{L}\left(\bigoplus_{N\geq 0} L^2(R(N,d))\right) \tag{2.13}$$

to some subspace of the self-adjoint operators on $\bigoplus_{N\geq 0} L^2(R(N,d))$. (It is actually a section on the operators on a "Fock bundle" over $\mathbf{T}^d$ but for now this is not relevant.)

The standard definition of a topological phase is as an equivalence class of systems under adiabatic evolution, which generally is generally interpreted as homotopy classes of maps [Kit09] between the Hamiltonians of different systems. For systems that have discrete translation symmetry, this is equivalent to

$$[\mathbf{T}^d, \mathcal{L}_d]. \tag{2.14}$$

A similar but slightly simpler analysis can be made for systems with full translation symmetry. Above, the torus arose as the quotient $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$; in the case of symmetry under the full translation group $\mathbb{R}^d$ we'd have instead just a point $\mathrm{pt} = \mathbb{R}^d/\mathbb{R}^d$. The end result would be that phases in dimension d protection by continuous translation symmetry are given by

$$[\mathrm{pt}, \mathcal{L}_d^{strong}], \tag{2.15}$$

where $\mathcal{L}_d^{strong}$ is a space of Hamiltonians with full translation symmetry.

For Hamiltonians with a unique ground state and short range interactions, there is a notion of symmetry protected topological phase (SPT), which must satisfy certain properties under stacking. This led Kitaev to conjecture that for SPT phases, the spaces of Hamiltonians $\{\mathcal{L}_d^{strong}\}$ form a spectrum; a modest extension of Kitaev's conjecture would be to conjecture that the $\{\mathcal{L}_d\}$ also form a spectrum. If one assumes these conjectures, then eq. (2.14) and eq. (2.15) become d-dimensional cohomology groups of the torus and of the point for the generalized cohomology theories given by the respective spectra. How are the two spectra, Kitaev's $\{\mathcal{L}_d^{strong}\}$, and $\{\mathcal{L}_d\}$ related? We can look for inspiration in the setting of free fermion systems, where, as we explain below in §2.2, the two spectra are the same: $\Sigma^{s-2}KO$ (where the parameter s specifies the symmetry class). It seems reasonable to further extend Kitaev's conjecture adding that the analogous phenomenon happens for SPT phases: the spectrum for weak and strong phases are equivalent—though this is not so easy to see from this analytical description in terms of Hamiltonians. Notice that if one assumes this extension of Kitaev's conjecture, then the James splitting (Lemma 1.69) implies that weak phases are built out of lower dimensional strong phases as in Lemma 1.75.

There are now quite a few proposals for what Kitaev's spectrum should be: see [CGL+13, KTTW15, GJF19, Xio18] among others. Here we have been considering the ansatz proposed by Freed-Hopkins [FH21, FH20] based on invertible TFTs as a low-energy limit. However, we have to interpret this ansatz carefully. On the one hand, the original conjecture in [FH21] is that *strong* SPT phases are classified by the group $\mho_H^{d+2}(\mathrm{pt})$. Our proposed extension of Kitaev's conjecture then says that weak SPT phases should then by classified by $\mho_H^{d+2}(\mathbf{T}^d)$, where the correct physical interpretation of $\mathbf{T}^d$ is the spatial torus.

2.2. **Kitaev's conjecture for free fermions and T-duality.** The different proposals for classifying SPT phases in $d+1$ dimensions generally satisfy Kitaev's conjecture [Kit13a, Kit15] that SPT phases form a spectrum in the sense of algebraic topology, i.e. there is always a map

$$\Omega SPT_{d+1}(G) \longrightarrow SPT_d(G), \tag{2.16}$$

which is often a homotopy equivalence. Any of the tenfold way classifications for type $H(s)$ in d dimensions can be essentially written in the form $KR^{s-2}(\mathbb{T}^d)$ (or $K^{s-2}(\mathbb{T}^d)$ in the complex cases) [Kit09, FM13] so that phases of free fermion systems in dimension d are classified by the group

$$FF_d(\mathbb{Z}^d, H(s)) \cong KR^{s-2}(\mathbb{T}^d). \tag{2.17}$$

As we can see, this a priori seems to contradict Kitaev's conjecture since the degree of the K-theory group, $s-2$, does not depend at all on the dimension of the system. However, if we use the T-duality isomorphism (1.64a) we have

$$KR^{s-2}(\mathbb{T}^d) \cong KO^{d+s-2}(\mathbf{T}^d), \tag{2.18}$$

which is the correct instantiation of Kitaev's conjecture when we have discrete translation invariance and weak phases. We can go on to see this satisfies Kitaev's original formulation of the conjecture for strong free fermion SPT phases by mapping to the bottom cell, i.e.

$$FF_d^{strong}(\mathbb{Z}^d, H(s)) \cong KO^{d+s-2}(\mathrm{pt}). \tag{2.19}$$

So the spectrum $\Sigma^{s-2}KO$ satisfies the conjecture. Hence T-duality plays an important role in Kitaev's conjecture for free fermions.

Remark 2.20. We can interpret Kitaev's proposal for strong phases on the interacting side field-theoretically as follows, compare [GJF19, §3.2]. An element of $\mho^{d+1}_{H(s)}(X)$ is a d-space-dimensional invertible field theory of symmetry type s equipped with a background field valued in X. In particular, the suspension isomorphism $\mho^{d+1}_{H(s)}(S^k) \cong \mho^{d+1}_{H(s)} \oplus \mho^{d+1-k}_{H(s)}$ can be interpreted as follows. Given a d-dimensional invertible quantum field theory Z with background field valued in S^k, this gives a $(d-k)$-dimensional invertible quantum field theory by sending[20]

$$N^{d-k} \mapsto Z(N \times S^k \xrightarrow{pr} S^k). \tag{2.21}$$

The factor $\mho^{d+1}_{H(s)}$ simply corresponds to elements of $\mho^{d+1}_{H(s)}(S^k)$ which do not depend on the S^k-valued background field.

We can in this way also reinterpret elements of $\mho^{d+2}_{H(s)}(\mathbf{T}^d)$ as $(d+1)$-dimensional field theories with target $\mathbf{T}^d$. By taking appropriate cells of $\mathbf{T}^d$, we can interpret the lower-dimensional terms in James splitting. Specifically, if $\mathbf{T}^k \subseteq \mathbf{T}^d$ is a subtorus corresponding to a subset of $\{1, \ldots, d\}$ of size k, we can define a map $\mho^{d+2}_{H(s)}(\mathbf{T}^d) \to \mho^{d+2-k}_{H(s)}$ as

$$N^{d-k} \mapsto Z(N \times \mathbf{T}^k \to \mathbf{T}^d) \tag{2.22}$$

where the map to $\mathbf{T}^d$ is induced by the inclusion $\mathbf{T}^k \subseteq \mathbf{T}^d$.

3. Examples: the Tenfold Way

In this section, we apply Lemma 1.75 to compute the groups of phases for weak topological insulators and superconductors in spatial dimensions 1, 2, and 3 with symmetry types according to the tenfold way.

[20]If N is an $H(s)$-manifold, we use the stably framed structure on S^k arising from the standard isomorphism $TS^k \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{k+1}$ to make $N \times S^k$ into an $H(s)$-manifold.

For illustrative purposes, we will discuss Class AII in detail. Class AII includes some of the first weak phases studied in the literature, the weak topological insulators (WTIs) of [FK07] and [FKM07]. We focus on dimension $3+1$. As reviewed in Example 1.13, free phases of band insulators are given by the group $KR^{-4}(\bar{\mathbb{T}}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}$. Under T-duality, this group is isomorphic to the real KO-theory group $KO^{-1}(\mathbf{T}^3)$ on the spatial torus. Using the James splitting (Lemma 1.69) we obtain an alternative computation of this group as

$$KO^{-1}(\mathbf{T}^3) \cong KO^{-1}(\mathrm{pt}) \oplus 3KO^{-2}(\mathrm{pt}) \oplus 3KO^{-3}(\mathrm{pt}) \oplus KO^{-4}(\mathrm{pt}) \tag{3.1}$$

$$= \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3 \oplus \mathbb{Z}. \tag{3.2}$$

Here blue summands are the strong phases, regarded as a subgroup of the group of weak phases (see §1.7). The red summands are captured by the invariant that counts the number of valence bands. This is generally not an interesting invariant and is often excluded in the physics literature; see Remark 1.19 for details. With respect to the James splitting, strong phases (blue summands) correspond to the bottom cell of the spatial torus, and the valence-band-counting invariant (red summands) corresponds to the top cell.[21]

In (3.1), the strong phase in the $\mathbb{Z}_2$ summand $KO^{-1}(\mathrm{pt})$ is detected by the Fu-Kane-Mele invariant [FKM07], while the $\mathbb{Z}$ summand counts the number of valence bands. The remaining $(\mathbb{Z}_2)^3$ coming from $3KO^{-2}(\mathrm{pt})$ comprises the weak phases, which may be viewed as quantum spin Hall (QSH) phases localized to two-dimensional surfaces of the three-dimensional material.

We compute the interacting classification using the James splitting as well. We have

$$\begin{aligned} \mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^3) &\cong \mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus 3\mho^4_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus 3\mho^3_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \oplus \mho^2_{\mathrm{Pin}^{\tilde{c}+}}(\mathrm{pt}) \\ &\cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3 \oplus (\mathbb{Z}_2)^3. \end{aligned} \tag{3.3}$$

Once again the colors illustrate which phases come from the top and bottom cells of $\mathbf{T}^d$. The first triplet of $\mathbb{Z}_2$'s comes from the interacting weak phases, while the second triplet of $\mathbb{Z}_2$'s comes from the interacting strong phases in 2d. That the weak $(\mathbb{Z}_2)^3$ injects into $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^3)$ corroborates the expectation that these weak phases are stable under interactions [Zou18, §III.A], [LQZ12].

In $d=3$, there is also a $(\mathbb{Z}_2)^2$ classification of interaction-enabled phases. These phases are all strong; i.e. they arise from $\mho^5_{\mathrm{Pin}^{\tilde{c}+}}$ applied to the bottom cell of the spatial torus. These interaction-enabled phases were originally found in the physics literature in [WPS14] and connected to bordism theory in [Met15].

Corollary 3.4 includes the classification for all three relevant dimensions.

[21]As T-duality incorporates Poincaré duality, the description is opposite for the Brillouin torus: the strong phases correspond to the top cell of $\overline{\mathbb{T}}^d$, and the valence-band-counting invariant corresponds to the bottom cell, see §1.7.

Corollary 3.4 (Symmetry class AII, $s = -2$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type AII is:*

(3.5)

d	$\ker(\mathrm{F2I}) \longrightarrow$	$KO^{d-4}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^{\tilde{c}+}}(\mathbf{T}^d)$	$\longrightarrow \mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0
3	0	$\mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2^2$

Literature Note 3.6. The classification of these free weak phases has been studied from many perspectives in the literature: see, for example, De Nittis-Gomi [DNG15, DNG18b, DNG18c, DNG23a, DNG23b] , Fiorenza-Monaco-Panati [FMP16], and Kaufmann-Li-Wehefritz-Kaufmann [KLWK16a, KLWK16b, KLWK16c, KLWK24]. $\mathrm{Pin}^{\tilde{c}+}$ bordism groups in these dimensions were first computed by Freed-Hopkins [FH21, Theorem 9.87].

We continue with the seven other real symmetry types and the two complex symmetry types.

Corollary 3.7 (Symmetry class D, $s = 0$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type D is:*

(3.8)

d	$\ker(\mathrm{F2I}) \longrightarrow$	$KO^{d-2}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Spin}}(\mathbf{T}^d)$	$\longrightarrow \mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^2$	0
3	0	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	0

Literature Note 3.9. Hughes [Hug15, 6m29s] gives a classification of the weak free phases in these dimensions, modulo the band-counting $\mathbb{Z}_2$ subgroup. Freed-Hopkins [FH20, Example 2.3] study class D phases on a torus in dimension 2, and observe that the free-to-interacting map is an isomorphism in that dimension; this example is also studied by Ran [Ran10]. The spin bordism groups used in Corollary 3.7 were first computed by Milnor [Mil63].

Corollary 3.10 (Symmetry class BDI, $s = 1$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type BDI is:*

(3.11)

d	$\ker(\mathrm{F2I}) \to$	$KO^{d-1}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^-}(\mathbf{T}^d)$	$\to \mathrm{coker}\,(\mathrm{F2I})$
1	$8\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	0
2	$(8\mathbb{Z})^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8^2 \oplus \mathbb{Z}_2$	0
3	$(8\mathbb{Z})^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8^3 \oplus \mathbb{Z}_2$	0

Literature Note 3.12. Corollary 3.10 is in agreement with work of Xiao-Kawabata-Luo-Ohtsuki-Shindou [XKL+23], who study 3d class BDI weak phases and conclude that the three $\mathbb{Z}$-valued invariants of weak topological phases remain nontrivial in the presence of interactions.

The pin$^-$ bordism groups used in this computation were first computed by Anderson-Brown-Peterson [ABP69]. The Majorana chain with its time-reversal symmetry is a 1-dimensional strong phase in class BDI, generating the $\mathbb{Z}$ summand of free phases and the $\mathbb{Z}_8$ summand of interacting phases in $d = 1$ [Kit01, FK10, FK11, TPB11]; this phase thus defines higher-dimensional weak phases and so contributes to the kernel of the free-to-interacting map in all higher degrees. There are no interaction-enabled phases in dimensions 6 and below.

Corollary 3.13 (Symmetry class AI, $s = 2$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type AI is:*

(3.14)

d	$\ker(\mathrm{F2I})$ $\to$	$KO^d(\mathbf{T}^d)$ $\xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^{\tilde c-}}(\mathbf{T}^d)$ $\to$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
3	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^4$	$\mathbb{Z}_2^4$

Literature Note 3.15*.* The pin$^{\tilde c-}$ bordism groups used in this computation were computed by Freed-Hopkins [FH21, Theorem 9.87]. Like in Corollary 3.10, the interaction-enabled weak phases in dimensions 2 and 3 are a consequence of the interaction-enabled *strong* phase in dimension 1 in this class; this strong phase appears in Freed-Hopkins [FH21, Corollary 9.95] (they use spacetime dimension, so call that phase 2-dimensional). De Nittis-Gomi [DNG14, DNG16] classify the free weak phases in class AI using Real-equivariant vector bundles.

Corollary 3.16 (Symmetry class CI, $s = 3$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type CI is:*

(3.17)

d	$\ker(\mathrm{F2I})$ $\to$	$KO^{d+1}(\mathbf{T}^d)$ $\xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^{h+}}(\mathbf{T}^d)$ $\longrightarrow$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
3	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2^3$

Literature Note 3.18*.* The pin^{h+} bordism groups appearing in this computation were computed by Freed-Hopkins [FH21, Theorem 9.97]; they use the notation G^+ for Pin^{h+}. Just as in Corollary 3.13, the interaction-enabled strong phase in dimension 1, first studied by Freed-Hopkins [FH21, Corollary 9.101], gives rise to interaction-enabled weak phases in higher dimensions.

Corollary 3.19 (Symmetry class C, $s = 4$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type C is:*

(3.20)

d	$\ker(\mathrm{F2I}) \to$	$KO^{d+2}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Spin}^h}(\mathbf{T}^d) \to$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
3	0	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$

Literature Note 3.21. Both a free weak phase and an interaction-enabled weak phase contribute to the classification in $d = 3$. The spin^h bordism groups used in the computation in Corollary 3.19 were first computed by Freed-Hopkins [FH21, Theorem 9.97], though they use the notation G^0 for Spin^h.

Corollary 3.22 (Symmetry class CII, $s = -3$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type CII is:*

(3.23)

d	$\ker(\mathrm{F2I}) \to$	$KO^{d+3}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^{h-}}(\mathbf{T}^d) \to$	$\mathrm{coker}\,(\mathrm{F2I})$
1	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
2	$(2\mathbb{Z})^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	0
3	$(2\mathbb{Z})^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^2$

Literature Note 3.24. Just as in Corollary 3.10, the generator of the group of one-dimensional strong phases becoming torsion in the interacting classification ([FH21, Corollary 9.103]) gives rise to weak phases in the kernel of the free-to-interacting map in higher dimensions. In $d = 3$ there are also two interaction-enabled phases. Xiao-Kawabata-Luo-Ohtsuki-Shindou [XKL+23] briefly discuss 3d weak interacting phases in class CII: they claim that the three $\mathbb{Z}$-valued topological indices of free phases remain nontrivial in the presence of interactions, which our computations verify.

The pin^{h-} bordism groups that we used in this computation are computed by Freed-Hopkins [FH21, Theorem 9.97]; they write G^- for Pin^{h-}.

Corollary 3.25 (Symmetry class DIII, $s = -1$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type DIII is:*

(3.26)

d	$\ker(\mathrm{F2I}) \to$	$KO^{d-3}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^+}(\mathbf{T}^d) \to$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
2	0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2$	0
3	$16\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2^6$	0

Literature Note 3.27. The pin$^+$ bordism groups used in this computation were computed by Giambalvo [Gia73, §2]. De Nittis-Gomi [DNG22] classify the free weak phases in this class using equivariant cohomology. The weak phases arising from $d = 1$ are stable to interactions. The strong phase in $d = 3$ breaks from generating a $\mathbb{Z}$ of free phases to a $\mathbb{Z}_{16}$ of interacting phases; this has been argued in many different ways: see for example [Kit13b, FCV13, MFCV14, WS14, YX14, Kit15, KTTW15, TY16, Wit16, SSR17, WG20, FH21].

Corollary 3.28 (Symmetry class A, $s = 0$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type A is:*

(3.29)

d	$\ker(\mathrm{F2I}) \to$	$K^d(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Spin}^c}(\mathbf{T}^d) \to$	$\mathrm{coker}\,(\mathrm{F2I})$
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}^2$	$\mathbb{Z}$
3	0	$\mathbb{Z} \oplus \mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}^6$	$\mathbb{Z}^3$

Literature Note 3.30. Varjas-de Juan-Lu [VdJL17, §II] observe that the Hall conductivity, a $\mathbb{Z}$-valued invariant of weak free class D phases in 3d, remains a well-defined, $\mathbb{Z}$-valued invariant of interacting systems, which is consistent with our computations. The $\mathbb{Z}$ summand in $d = 2$ corresponding to the integer quantum Hall effect (a strong phase) is stable under interactions and contributes to weak phases in $d = 3$. There are also interaction-enabled phases in $d = 2$, which contribute to weak interaction-enabled phases in the $d = 3$ Chern insulator. The calculation of spin^c bordism groups is attributed to Anderson-Brown-Peterson [ABP67]; see Bahri-Gilkey [BG87a, §1] for an explicit description.

Corollary 3.31 (Symmetry class AIII, $s = 1$)**.** *The free-to-interacting map for the groups of weak phases in Altland-Zirnbauer type A is:*

(3.32)

d	$\ker(\mathrm{F2I}) \longrightarrow$	$K^{d-1}(\mathbf{T}^d) \xrightarrow{\mathrm{F2I}}$	$\mho^{d+2}_{\mathrm{Pin}^c}(\mathbf{T}^d) \longrightarrow$	$\mathrm{coker}\,(\mathrm{F2I})$
1	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_4$	0
2	$(4\mathbb{Z})^2$	$\mathbb{Z}^2$	$\mathbb{Z}_4^2$	0
3	$8\mathbb{Z} \oplus (4\mathbb{Z})^3$	$\mathbb{Z} \oplus \mathbb{Z}^3$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^3$	$\mathbb{Z}_2$

Literature Note 3.33. Corollary 3.31 is in agreement with work of Xiao-Kawabata-Luo-Ohtsuki-Shindou [XKL+23], who discuss how the weak $\mathbb{Z}$-valued indices in 3d remain nontrivial in the presence of interactions. See also Claes-Hughes [CH20], who study the behavior of these indices under disorder. De Nittis-Gomi [DNG18a] study free weak phases in this class in terms of objects called *chiral vector bundles.*

Like in Corollaries 3.10 and 3.22, the nontrivial kernel of the one-dimensional strong free-to-interacting map ([FH21, Corollary 9.91]) produces phases in the kernel of the weak free-to-interacting map in higher dimensions. The generator of the group of one-dimensional strong phases is the class of the Su-Schrieffer-Heeger model [SSH79]. In three dimensions there is an additional free phase that breaks down, as well as an interaction-enabled phase. Pin^c bordism groups were first computed by Bahri-Gilkey [BG87a, BG87b].

Appendix A. Calculation of the twisted ABS map $\Omega_4^{\mathrm{Pin}^{\tilde c+}} \to \mathbb{Z}_2$

Our goal in this appendix is to explicitly calculate Freed-Hopkins' twisted Atiyah-Bott-Shapiro map in dimension 4 in class $s = -2$. We use this calculation in Example 1.59.

Freed-Hopkins' original calculation of this free-to-interacting map in [FH21, §10] is purely homotopy-theoretical, coming from an Adams spectral sequence computation. We make a more concrete and less technical calculation, which additionally results in an explicit understanding of the manifold generators of the relevant bordism groups. Specifically, we use the long exact sequence associated to a *Smith homomorphism*, following a general theory worked out in [DDK⁺24]. See [HS13, DDHM23, DL23, DYY23, Deb24, DK24, DNT24] for more examples of calculations applying this technique.

Remark A.1. Freed-Hopkins' original definition of the twisted Atiyah-Bott-Shapiro map is index-theoretic, as we review in §1.4.1. It therefore seems reasonable that there should be a description of the map $\Omega_4^{\mathrm{Pin}^{\tilde c+}} \to KO_2 \cong \mathbb{Z}_2$ as a mod 2 index of the Dirac operator from Example 1.44. We would be interested in learning whether it is possible to prove Theorem A.9 by calculating this mod 2 index on a generating set for $\Omega_4^{\mathrm{Pin}^{\tilde c+}}$.

Definition A.2 (Hason-Komargodski-Thorngren [HKT20, §4.1])**.** Let $V \to X$ be a virtual vector bundle. An (X, V)*-twisted spin structure* on a vector bundle $E \to M$ is data of a map $f\colon M \to X$ and a spin structure on $E \oplus f^*(V)$.

Given a fermionic group G_f, there is often a virtual vector bundle $V \to BG_b$ such that the tangential structure associated to G as defined in §1.3.2 is equivalent to a (BG_b, V)-twisted spin structure. This occurs for all fermionic groups we consider in this paper; that it is not true in general follows from work of Gunawardena-Kahn-Thomas [GKT89]. We will need the following three examples.

Lemma A.3 (Freed-Hopkins [FH21, §10])**.**

(1) $\mathrm{Pin}^{\tilde c+}$ structures are naturally equivalent to $(B\mathrm{O}(2), -V)$-twisted spin structures, where $V \to B\mathrm{O}(2)$ is the tautological bundle.
(2) Pin^+ structures are naturally equivalent to $(B\mathrm{O}(1), -\sigma)$-twisted spin structures, where $\sigma \to B\mathrm{O}(1)$ is the tautological bundle.
(3) Spin^c structures are naturally equivalent to $(B\mathrm{U}(1), -L)$-twisted spin structures, where $L \to B\mathrm{U}(1)$ is the tautological complex line bundle.

Remark A.4 (Alternate characterizations)*.* Stolz [Sto88, §6] showed that pin$^+$ structures are naturally equivalent to $(B\mathrm{O}(1), 3\sigma)$-twisted spin structures, and it is implicit in Stong [Sto68, Chapter XI] that spin^c structures are $(B\mathrm{U}(1), L)$-twisted spin structures.

The pullback of $V \to B\mathrm{O}(2)$ along the standard inclusion $B\mathrm{O}(1) \to B\mathrm{O}(2)$ is isomorphic to $\sigma \oplus \underline{\mathbb{R}}$, which means that a pin$^+$ structure on a vector bundle $E \to X$ induces a pin$^{\tilde c+}$ structure: a spin structure on $E - f^*\sigma$, where f is some map $X \to B\mathrm{O}(1)$, is equivalent data to a spin structure on $E - f^*\sigma \oplus \underline{\mathbb{R}}$.

Similarly, the pullback of $V \to B\mathrm{O}(2)$ along the map $B\mathrm{U}(1) \to B\mathrm{O}(2)$ induced by the inclusion $\mathrm{U}(1) \cong \mathrm{SO}(2) \hookrightarrow \mathrm{O}(2)$ is L. Thus, analogously to the way a pin$^+$ structure defines a pin$^{\tilde c+}$ structure, a spin^c structure also induces a pin$^{\tilde c+}$ structure. In particular, complex manifolds have canonical pin$^{\tilde c+}$ structures induced from their canonical spin^c structures.

Definition A.5. It follows from Lemma A.3, part (3), that a spin^c structure on a manifold M is equivalent data to a complex line bundle $L \to M$ and a spin structure on $TM \oplus L$. If M is an almost complex manifold, this is equivalent to the condition $c_1(TM \oplus L) \bmod 2 = 0$, i.e. $c_1(M) \equiv c_1(L) \bmod 2$ by the Whitney sum formula. Since $c_1(M) = c_1(\mathrm{Det}_{\mathbb{C}}(TM))$, we can also use the determinant bundle to characterize spin^c structures.

There is a canonical isomorphism $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ sending $\mathcal{O}(m) \mapsto m$, and $\mathrm{Det}_{\mathbb{C}}(T\mathbb{CP}^n) \cong \mathcal{O}(-(n+1))$, so a spin^c structure on $\mathbb{CP}^n$ is equivalent data to an integer m such that $m \equiv n+1 \bmod 2$: then $TM \oplus \mathcal{O}(m)$ admits a spin structure, and its spin structures are a torsor over $H^1(\mathbb{CP}^n;\mathbb{Z}/2) = 0$, so this spin structure is unique. We let $(\mathbb{CP}^n, m)$ denote the spin^c manifold $\mathbb{CP}^n$ with the spin^c structure defined by $\mathcal{O}(m)$ in this way.

Thus the spin^c structure on $\mathbb{CP}^n$ induced by its complex structure is $(\mathbb{CP}^n, -(n+1))$. If we refer to $\mathbb{CP}^n$ as a spin^c manifold without clarifying, we mean this structure.

Lemma A.6 (Freed-Hopkins [FH21, Theorem 9.87])**.** *There is an isomorphism* $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \cong \mathbb{Z}_2^3$.

Proposition A.7. *The bordism classes of the following three manifolds are linearly independent in* $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$*, and therefore form a basis.*

(1) $\mathbb{RP}^4$*, with pin*$^{\tilde{c}+}$ *structure induced from either of its two pin*$^+$ *structures.*
(2) $\mathbb{CP}^2$*, with pin*$^{\tilde{c}+}$ *structure induced from the spin*^{c} *structure* $(\mathbb{CP}^2, -1)$*.*
(3) $\mathbb{CP}^1 \times \mathbb{CP}^1$*, with pin*$^{\tilde{c}+}$ *structure induced from its complex structure.*

Proof. The fact that the bordism classes of $\mathbb{RP}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are linearly independent in $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}}$ is shown in [DYY23, Proposition A.25]. Thus it suffices to find a bordism invariant $\xi\colon \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to \mathbb{Z}_2$ which vanishes on $\mathbb{RP}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, but is nonzero on $(\mathbb{CP}^2, -1)$.

Given a pin$^{\tilde{c}+}$ manifold X, let $E \to X$ denote the rank-2 vector bundle associated to the pin$^{\tilde{c}+}$ structure: by Lemma A.3, part (1), a pin$^{\tilde{c}+}$ structure is a $(B\mathrm{O}(2), -V)$-twisted spin structure, so we have a map $f\colon X \to B\mathrm{O}(2)$, and $E := f^*(V)$. By a standard argument due to Pontryagin [Pon47] (see Milnor-Stasheff [MS74, Theorem 4.9]), $\xi\colon (X,E) \mapsto \int_X w_2(E)^2$ is a bordism invariant $\Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to \mathbb{Z}_2$.

If the pin$^{\tilde{c}+}$ structure on X is induced from a pin$^+$ structure, then as discussed above the pullback map of E factors through $B\mathrm{O}(1)$ and therefore $E \cong L \oplus \underline{\mathbb{R}}$ for some real line bundle L. Thus in this case $w_2(E) = 0$, so $\xi(\mathbb{RP}^4) = 0$.

To show $\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$, we use that since the pin$^{\tilde{c}+}$ structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is induced from its complex structure,

$$\begin{aligned}
\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) &= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} w_2(\mathrm{Det}_{\mathbb{C}}(T(\mathbb{CP}^1 \times \mathbb{CP}^1)) \\
&= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} c_1(\mathrm{Det}_{\mathbb{C}}(T(\mathbb{CP}^1 \times \mathbb{CP}^1)) \bmod 2 \\
&= \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} c_1(\mathbb{CP}^1 \times \mathbb{CP}^1) \bmod 2.
\end{aligned} \tag{A.8}$$

Since $\mathbb{CP}^1 \cong S^2$ has a spin structure, so does $\mathbb{CP}^1 \times \mathbb{CP}^1$, and therefore its first Chern class is even, so $\xi(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$.

For $(\mathbb{CP}^2, -1)$, $E = \mathcal{O}(-1)$, which has odd Chern class, so $w_2(E) \neq 0$. Since $H^*(\mathbb{CP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^2)$ with $|a| = 2$, then as soon as we know $w_2(E) \neq 0$ we see $w_2(E)^2$ is the unique nonzero element of $H^4(\mathbb{CP}^2;\mathbb{Z}_2)$, so $\xi(\mathbb{CP}^2, -1) = 1$. ∎

Now that we know a set of generators, we can state the main theorem of this appendix, which is the calculation of the twisted Atiyah-Bott-Shapiro map on these generators.

Theorem A.9. *The twisted Atiyah-Bott-Shapiro map* $\mathrm{ABS}_{-2}\colon \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \to KO_2 \cong \mathbb{Z}_2$ *sends* $[\mathbb{RP}^4] \mapsto 0$, $[\mathbb{CP}^2, -1] \mapsto 0$, *and* $[\mathbb{CP}^1 \times \mathbb{CP}^1] \mapsto 1$.

The key fact that enables us to get at ABS_{-2} is:

Proposition A.10 (Freed-Hopkins [FH21, Proposition 10.27])**.** *For* $-3 \le s \le -1$, *the twisted Atiyah-Bott-Shapiro map* ABS_s *factors as*

$$\Omega_n^{H(s)} \xrightarrow{\mathrm{sm}_V} \Omega_{n+s}^{\mathrm{Spin}}(B\mathrm{O}(-s)) \xrightarrow{c} \Omega_{n+s}^{\mathrm{Spin}} \xrightarrow{\mathrm{ABS}_0} KO_{n+s}, \tag{A.11}$$

where $V \to B\mathrm{O}(-s)$ *is the tautological bundle,* sm_V *is the* Smith homomorphism *defined by taking a manifold representative of the Poincaré dual of the Euler class of* V*, and* c *is the map forgetting the data of the map to* $B\mathrm{O}(-s)$.

Remark A.12. Freed-Hopkins do not define their map in exactly this way. Instead of sm_V, they use the map defined by the zero section of the tautological bundle; see [DDK+24, Proposition 3.17] for a proof identifying this with the Smith homomorphism. Instead of $\mathrm{ABS}_0 \circ c$, they tensor ABS_0 with a map to KO-theory corresponding to the trivial line bundle on $B\mathrm{O}(-s)$, but the trivial line bundle pulls back from the point so we may use the forgetful map c.

Remark A.13. The Euler class mentioned in Proposition A.10 is not the usual Euler class, but an analogue in the spin cobordism generalized cohomology theory. This Euler class has subtle behavior and can be tricky to calculate: see [DDK+24, Appendix B]. For this reason, we will for the most part calculate the Smith homomorphism indirectly in this section.

The Smith homomorphisms in Proposition A.10 can be fit into long exact sequences which often can be explicitly computed. Focusing again on $s = -2$, we have:

Proposition A.14. *There is a long exact sequence*

$$\cdots \to \Omega_4^{\mathrm{Pin}^+} \xrightarrow{i} \Omega_4^{\mathrm{Pin}^{\tilde{c}+}} \xrightarrow{\mathrm{sm}_V} \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\delta} \Omega_3^{\mathrm{Pin}^+} \to \cdots \tag{A.15}$$

where i *is the map on bordism corresponding to the induced pin*$^{\tilde{c}+}$ *structure on a pin*$^+$ *manifold described above and* δ *applied to the bordism class of a spin manifold* Σ *and a rank-*2 *vector bundle* $E \to \Sigma$ *is the bordism class of the sphere bundle* $S(E)$ *with a certain pin*$^+$ *structure.*

Proof. Let $E \to X$ be a virtual vector bundle and $F \to X$ be a vector bundle of rank r. Let $p\colon S(F) \to X$ be the sphere bundle of F. Introduce the following three tangential structures:

(1) a ξ-structure is a $(S(F), p^*(E))$-twisted spin structure,
(2) a η-structure is an (X, E)-twisted spin structure, and
(3) a ζ-structure is an $(X, E \oplus F)$-twisted spin structure.

Then [DDK+24, Corollary 5.8] there is a long exact sequence

$$\cdots \to \Omega_n^{\xi} \xrightarrow{p_*} \Omega_n^{\eta} \xrightarrow{\mathrm{sm}_F} \Omega_{n-r}^{\zeta} \xrightarrow{\delta} \Omega_{n-1}^{\xi} \to \cdots, \tag{A.16}$$

called the *Smith long exact sequence*, where sm_F is the Smith homomorphism associated to F and δ is induced by taking the sphere bundle of the pullback of F with a certain ξ-structure.

Let $X = B\mathrm{O}(2)$ and $V \to B\mathrm{O}(2)$ denote the tautological bundle. Then let $E = -V$ and $F = V$, so that a ζ-structure is a spin structure with a map to $B\mathrm{O}(2)$ and, by Lemma A.3, a η-structure is equivalent to a $\mathrm{pin}^{\tilde{c}+}$ structure.

There is a homotopy equivalence $S(V) \simeq B\mathrm{O}(1)$ such that the bundle map $p\colon S(V) \to B\mathrm{O}(2)$ is identified with $i\colon B\mathrm{O}(1) \to B\mathrm{O}(2)$,[22] so a ξ-structure is a $(B\mathrm{O}(1), -i^*(V))$-twisted spin structure; as noted above, this is equivalent to a $(B\mathrm{O}(1), -\sigma)$-twisted spin structure and therefore by Lemma A.3 a pin^+ structure. This finishes the identification of this Smith long exact sequence with the one in the theorem statement. ☐

Corollary A.17. *For any closed* pin^+ *4-manifold* X*,* $\mathrm{ABS}_{-2}(X) = 0$*. In particular,* $\mathrm{ABS}_{-2}(\mathbb{RP}^4) = 0$*.*

Proof. Exactness of (A.15) implies $\mathrm{sm}_V \circ i = 0$, so $\mathrm{sm}_V(X) = 0$; Proposition A.10 tells us that ABS_{-2} factors through sm_V, so $\mathrm{ABS}_{-2}(X) = 0$ too. ☐

That's one-third of Theorem A.9 right there!

For $(\mathbb{CP}^2, -1)$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ we have to perform a more detailed analysis: $\mathbb{CP}^2$ has no pin^+ structure (as that combined with an orientation would be a spin structure, but $\mathbb{CP}^2$ is not spin), and though $\mathbb{CP}^1 \times \mathbb{CP}^1$ has a pin^+ structure, that structure does not induce the $\mathrm{pin}^{\tilde{c}+}$ structure we use in this section.

Definition A.18. Define maps $\varphi_1, \varphi_2, \varphi_3\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$ as follows on a closed spin 2-manifold Σ with rank-2 vector bundle $E \to \Sigma$.

(1) $\varphi_1 = \mathrm{ABS}_0 \circ c$, as in Proposition A.10.
(2) φ_2 is the composition

$$\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\det} \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(1)) \xrightarrow{\mathrm{sm}_\sigma} \Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2, \tag{A.19}$$

where det is induced from the determinant map $\mathrm{O}(2) \to \mathrm{O}(1)$, sm_σ is the Smith homomorphism introduced by Anderson-Brown-Peterson [ABP69], which takes a manifold representative of the Poincaré dual of the Euler class of the principal $\mathrm{O}(1)$-bundle, and the isomorphism $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$ was established by (*ibid.*, Theorem 5.1).

(3) $\varphi_3(\Sigma, E) = \int_\Sigma w_2(E)$.

Proposition A.20. *The following map is an isomorphism:*

$$\boldsymbol{\varphi} \coloneqq (\varphi_1, \varphi_2, \varphi_3)\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \xrightarrow{\cong} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{A.21}$$

The bordism classes of the following manifolds form the basis for $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$ *dual to* $(\varphi_1, \varphi_2, \varphi_3)$*.*

- $(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2)$*, where* S^1_{nb} *refers to the nonbounding spin structure on the circle.* $\boldsymbol{\varphi}(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = (1, 0, 0)$*.*

[22] This is a standard result; one non-original reference is [DDK+24, Example 7.57].

- $(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}})$, *where* S^1_b *refers to the bounding spin structure on the circle and* $\sigma_R \to S^1_{nb} \times S^1_b$[23] *is the pullback of the Möbius bundle* $\sigma \to S^1$ *by the projection onto the second factor of* $S^1_{nb} \times S^1_b$. $\varphi(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = (0, 1, 0)$.
- $(\mathbb{CP}^1, \mathcal{O}(1))$: $\boldsymbol{\varphi}(\mathbb{CP}^1, \mathcal{O}(1)) = (0, 0, 1)$.

Proof. Mitchell-Priddy [MP84, Theorem C] show that, modulo odd-primary torsion, for any generalized homology theory h_*, there is a natural map $\psi_1 \colon h_*(B\mathrm{O}(2)) \to h_*(B\mathrm{SO}(3))$ and an isomorphism

$$(c, \psi_1, \psi_2, \det) \colon h_n(B\mathrm{O}(2)) \xrightarrow{\cong} h_n(\mathrm{pt}) \oplus \widetilde{h}_n(B\mathrm{SO}(3)) \oplus h_n(L(2)) \oplus \widetilde{h}_n(B\mathrm{O}(1)) \tag{A.22}$$

for a certain spectrum $L(2)$ and map $\psi_2 \colon B\mathrm{O}(2) \to L(2)$. Bayen [Bay94, §3.5.3, §3.6.3] shows $\Omega_k^{\mathrm{Spin}}(L(2))$ vanishes in degrees 3 and below, so we will not need to worry about this factor. Wan-Wang [WW19, §5.5.3] show $\widetilde{\Omega}_2^{\mathrm{Spin}}(B\mathrm{SO}(3))) \cong \mathbb{Z}_2$, and Anderson-Brown-Peterson [ABP69] show $\widetilde{\Omega}_2^{\mathrm{Spin}}(B\mathrm{O}(1)) \cong \mathbb{Z}_2$. The additional hypothesis on odd-primary torsion can be removed: Randal-Williams [RW08, §5.1] shows that the odd-primary torsion in $\widetilde{h}_*(B\mathrm{O}(2))$ coincides with that of a 3-connected space, meaning there can be none in degree 2. Thus we have the abstract isomorphism (A.21) and the fact that φ_1 and φ_2 are linearly independent, but we still need to address φ_3. The integral of a stable characteristic class is a bordism invariant by an argument of Pontryagin [Pon47] (see also Milnor-Stasheff [MS74, Theorem 4.9]), so φ_3 indeed defines a map $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$; we need to show this map is linearly independent of φ_1 and φ_2. To do so, we will calculate $\boldsymbol{\varphi}$ on the three surfaces in the theorem statement.

- φ_2 and φ_3 by definition vanish on trivial bundles, so $\boldsymbol{\varphi}(S^1_{nb} \times S^1_{nb}, \underline{\mathbb{R}}^2) = (?, 0, 0)$; for the value of φ_1 observe that $\mathrm{ABS}_0 \circ c$ is the Arf invariant, which equals 1 on $S^1_{nb} \times S^1_{nb}$.
- For $(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}})$, we have $S^1_{nb} \times S^1_b = \partial(S^1_{nb} \times D^2)$, so c kills this manifold and therefore φ_1 does too. For φ_2, $\mathrm{Det}(\sigma_R \oplus \underline{\mathbb{R}}) \cong \sigma_R$. This bundle is trivializable when restricted to $S^1_{nb} \times \{x\} \subset S^1_{nb} \times S^1_b$ for any $x \in S^1_b$, which means S^1_{nb} is Poincaré dual to the Euler class of σ_R and therefore $\mathrm{sm}_\sigma(S^1_{nb} \times S^1_b, \sigma_R) = S^1_{nb}$, whose class in $\Omega_1^{\mathrm{Pin}^-}$ is nonzero [KT90, Theorem 2.1]. Thus $\varphi_2(S^1_{nb} \times S^1_b, \sigma_R \oplus \underline{\mathbb{R}}) = 1$. For φ_3, $w_2(\sigma_R \oplus \underline{\mathbb{R}}) = w_2(\sigma_R) = 0$, because σ_R is a line bundle.
- Finally, $(\mathbb{CP}^1, \mathcal{O}(1))$: since $\mathbb{CP}^1 \cong S^2$ is simply connected, it has a unique spin structure, which bounds D^3 and therefore has trivial Arf invariant, so $\varphi_1(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. Since $\mathcal{O}(1))$ is complex, it is oriented, so its real determinant bundle vanishes, and therefore $\varphi_2(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. Since $c_1(\mathcal{O}(1)) \mapsto 1$ under the isomorphism $H^2(\mathbb{CP}^1; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ defined by the orientation induced by the complex structure, $w_2(\mathcal{O}(1)) = c_1(\mathcal{O}(1)) \bmod 2$ is the nonzero element of $H^2(\mathbb{CP}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and therefore $\int_{\mathbb{CP}^1} w_2(\mathcal{O}(1)) = 1$.

Thus we have shown that the bordism classes of these three surfaces are linearly independent, and dual to the three invariants in $\boldsymbol{\varphi}$, as promised. ∎

Recall the map δ from Proposition A.14.

[23] The "R" in σ_R refers to the **R**ight-hand factor of S^1.

Lemma A.23. *There is a (necessarily unique) isomorphism* $q\colon \Omega_3^{\mathrm{Pin}^+} \xrightarrow{\cong} \mathbb{Z}_2$*, and the composition* $q\circ\delta\colon \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2)) \to \mathbb{Z}_2$ *equals* φ_2*.*

Proof. The calculation $\Omega_3^{\mathrm{Pin}^+} \cong \mathbb{Z}_2$ is due to Giambalvo [Gia73, §2]. To identify $q\circ\delta = \varphi_2$, it suffices by Proposition A.20 to show $\delta(S^1_{nb}\times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$, $\delta(S^1_{nb}\times S^1_b, \sigma_R\oplus\underline{\mathbb{R}}) = 1$, and $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$.

First observe that sm_V is not surjective: its domain and codomain are both sets of size 8 (Lemma A.6, resp. Proposition A.20) but $\mathrm{sm}_V(\mathbb{RP}^4) = 0$ (Corollary A.17). Since sm_V is not surjective, exactness of (A.15) implies $\delta \neq 0$. Therefore if we can show $\delta(S^1_{nb}\times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$ and $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$, then it must follow that $\delta(S^1_{nb}\times S^1_b, \sigma_R\oplus\underline{\mathbb{R}}) = 1$.

For $S^1_{nb}\times S^1_{nb}$, the vector bundle is trivial, so the total space of its sphere bundle is T^3 with some pin$^+$ structure. Since T^3 is orientable, this pin$^+$ structure is induced from a spin structure, but $\Omega_3^{\mathrm{Spin}} = 0$ [Mil63] so T^3 bounds some compact spin 4-manifold X. This is also a pin$^+$ null-bordism of T^3, so $\delta(S^1_{nb}\times S^1_{nb}, \underline{\mathbb{R}}^2) = 0$.

For $\mathbb{CP}^1$, the sphere bundle map $S(\mathcal{O}(1)) \to \mathbb{CP}^1$ is one definition of the Hopf fibration, so the total space is S^3. The argument in the previous paragraph shows that any pin$^+$ structure on any closed, orientable 3-manifold is null-bordant, so $\delta(\mathbb{CP}^1, \mathcal{O}(1)) = 0$. □

Thus, by exactness of (A.15), $\varphi_2\circ\mathrm{sm}_V = 0$.

Lemma A.24. $\boldsymbol{\varphi}\circ\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1) = (1,0,0)$.

Proof. By Lemma A.23, $\varphi_2\circ\mathrm{sm}_V = 0$.

To compute $\varphi_3\circ\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1)$, let $i\colon \Sigma \hookrightarrow \mathbb{CP}^1\times\mathbb{CP}^1$ be a manifold representative for the Poincaré dual of the Euler class of the principal $\mathrm{O}(2)$-bundle $V \to \mathbb{CP}^1\times\mathbb{CP}^1$ associated to the pin$^{\tilde{c}+}$ structure. If $\nu \to \Sigma$ denotes the normal bundle to the embedding i, then there is an isomorphism $i^*(V)\cong\nu$ and $\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1)$ is by definition the class of $(\Sigma, i^*(V))$ in $\Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$. Apply the Whitney sum formula to the decomposition $T\Sigma\oplus\nu \cong i^*(T(\mathbb{CP}^1\times\mathbb{CP}^1))$ to deduce

$$w_1(\Sigma) + w_1(\nu) = i^*(w_1(\mathbb{CP}^1\times\mathbb{CP}^1)) \tag{A.25a}$$

$$w_2(\Sigma) + w_1(\Sigma)w_1(\nu) + w_2(\nu) = i^*(w_2(\mathbb{CP}^1\times\mathbb{CP}^1)). \tag{A.25b}$$

Since $\mathbb{CP}^1\cong S^2$, it has a spin structure, so $\mathbb{CP}^1\times\mathbb{CP}^1$ does as well, and therefore $w_i(\mathbb{CP}^1\times\mathbb{CP}^1) = 0$ for $i = 1,2$. Thus (A.25a) simplifies to $w_1(\Sigma) = w_1(\nu)$, and so (A.25b) simplifies to $w_2(\Sigma) + w_1(\Sigma)^2 + w_2(\nu) = 0$. The Wu formula implies $w_2(\Sigma) + w_1(\Sigma)^2 = 0$ because the Wu class $v_2 = w_2 + w_1^2$ vanishes on closed 2-manifolds such as Σ, so we have calculated that $w_2(\nu) = 0$ and therefore

$$\varphi_3(\mathbb{CP}^1\times\mathbb{CP}^1) := \int_\Sigma w_2(\nu) = 0. \tag{A.26}$$

Since $\varphi_i\circ\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1) = 0$ for $i = 2,3$, to show $\varphi_1\circ\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1) = 1$ is the same as showing $\boldsymbol{\varphi}(\mathbb{CP}^1\times\mathbb{CP}^1) \neq 0$. Since $\boldsymbol{\varphi}$ is an isomorphism (Proposition A.20), this is the same as showing $\mathrm{sm}_V(\mathbb{CP}^1\times\mathbb{CP}^1) \neq 0$, which by exactness of (A.15) is equivalent to showing $[\mathbb{CP}^1\times\mathbb{CP}^1] \notin \mathrm{Im}(i)$. The domain of i, $\Omega_4^{\mathrm{Pin}^+}$, is a cyclic group [Gia73, §2] and by construction $[\mathbb{RP}^4] \in \mathrm{Im}(i)$, so since the bordism classes of $\mathbb{RP}^4$ and $\mathbb{CP}^1\times\mathbb{CP}^1$

are linearly independent (Proposition A.7), $[\mathbb{CP}^1 \times \mathbb{CP}^1]$ cannot also be in $\mathrm{Im}(i)$. Thus $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^1 \times \mathbb{CP}^1) = 1$. □

Lemma A.27. $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = 0$.

Proof. One characterization of $\mathrm{sm}_V(\mathbb{CP}^2, -1)$ is that it is the bordism class of the zero set of any section of $V \to \mathbb{CP}^2$ which is transverse to the zero section [DDK+24, Definition 3.7]. For the $\mathrm{pin}^{\tilde{c}+}$ structure $(\mathbb{CP}^2, -1)$, $V = \mathcal{O}(-1)$, and the zero set of any such section is isotopic to the standard embedding $\mathbb{CP}^1 \to \mathbb{CP}^2$. Thus $\mathrm{sm}_V(\mathbb{CP}^2, -1) \in \Omega_2^{\mathrm{Spin}}(B\mathrm{O}(2))$ is the bordism class of $\mathbb{CP}^1$ with some spin structure and some rank-2 vector bundle. The map φ_1 forgets the vector bundle, so $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) \in \Omega_2^{\mathrm{Spin}}$ is the bordism class of $\mathbb{CP}^1 \cong S^2$ with some spin structure. Since S^2 is simply connected, it has a unique spin structure, which therefore is the spin structure appearing at the boundary $S^2 \cong \partial D^3$, where D^3 is given its canonical (also unique) spin structure. Therefore $[\mathbb{CP}^1] = 0$ in Ω_2^{Spin} and therefore $\varphi_1 \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = 0$. □

Remark A.28. It is possible to show $\boldsymbol{\varphi} \circ \mathrm{sm}_V(\mathbb{CP}^2, -1) = (0, 0, 1)$ similarly to the proof of Lemma A.24.

Since the first component of $\boldsymbol{\varphi} \circ \mathrm{sm}_V$ is ABS_{-2}, Corollary A.17 and Lemmas A.24 and A.27 finish the proof of Theorem A.9.

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Instituto de Matemáticas, UNAM, Circuito exterior, Ciudad Universitaria, 04510 Ciudad de México, CDMEX, Mexico

Email address: omar@matem.unam.mx
URL: https://www.matem.unam.mx/~omar/

Department of Mathematics, The University of Kentucky, 719 Patterson Office Tower, Lexington, KY 40506, USA

Email address: a.debray@uky.edu
URL: https://adebray.github.io/

Massachusetts Institute of Technology, Department of Mathematics, Simons Building (Building 2) 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Email address: camkru@mit.edu
URL: https://cakrulewski.github.io/

Massachusetts Institute of Technology, Department of Mathematics, Simons Building (Building 2) 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Email address: nataliap@mit.edu

Department of Physics, CINVESTAV, Av. Instituto Politécnico Nacional 2508, Col. San Pedro Zacatenco, Delegación Gustavo A. Madero, 07360 Ciudad de México, CDMEX, Mexico

Email address: dshein@ciencias.unam.mx
URL: https://d-shein.github.io/

Department of Mathematics & Statistics, Dalhousie University, 6316 Coburg Road, PO Box 15000, Halifax, NS, B3H 4R2, Canada

Email address: luukstehouwer@gmail.com
URL: https://sites.google.com/view/luuk-stehouwer