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Surprising Manifestations of Incompleteness

Mathematicians, scientists, and philosophers alike describe their respective quests for knowledge as akin to exploring some wild and unknown land, charting and taming the wilderness into new ideas and theories. Typically, there is no end to this imaginary landscape, and discovery can continue indefinitely as old questions are answered and new ones are posed. In mathematics, this scenario was shown to be merely a dream by a result called Gödel's Incompleteness Theorem, a result proven in 1931 that showed every significant formulation of mathematics contains statements that cannot be proven to be true or false. Gödel, the mathematician responsible for the theorem, even provided an example of such a statement whose truth could never be known within any formulation of number theory. Mathematics had been shown to be incomplete, and certain holes in mathematical understanding wouldn't be filled. In this respect, mathematics began to look like Kant's trascendental philosophy, in which absolute knowledge of objects is impossible, as it would lead to contrdictory proofs and metaphysical inconsistencies. Kant's incompleteness arose in vastly different contexts and ways; he could not demonstrate it in the rigorous way that Gödel did, and it was applied to metaphysics rather than mathematics. Nonetheless, there are striking analogies between these concepts: each revolutionized its respective field and forced a new understanding of rigorous, logical truth. Each demonstrated that logical reasoning could not answer every question completely. As demonstrated by the similarities between Kant and Gödel's discoveries about the limits of reason, incompleteness is a property common to logical systems of thought.

Gödel's Incompleteness: The Limits of Mathematical Logic

In order to understand the implications of Gödel's result one should understand its context and why it works. For it is one thing to understand that formal mathematics cannot prove every true statement – yet it is quite another to realize that this was rigorously proven using formal mathematics! Before Gödel, mathematics was viewed as a formal system of axioms from which a set of rules of inference led to true statements called theorems. Sometimes, paradoxes arose, but these were addressed by clarifying the axioms or definitions that led to them. These systems were often used to make statements about numbers, which led to the term number theory being used to describe a system of axioms sufficiently powerful to express mathematical statements. The most significant such axiomatic system was Russel & Whitehead's *Principia Mathematica*, which was intended to encapsulate all true statements in mathematics, and none of the false ones. It was carefully constructed to avoid the paradoxes that mixed different levels of logic that doomed similar past projects. Gödel's Theorem abruptly inserted itself into this framework, showing not just "that there were irreparable 'holes' in the [*Principia Mathematica*], but more generally, that no axiomatic system whatsoever could produce all number-theoretical truths, unless it were inconsistent!"[1] This result was not just counterintuitive, but also disruptive; it shattered mathematicians' hopes that some sort of absolute mathematical knowledge was possible.

For such a powerful statement, Gödel's Incompleteness Theorem can be proven relatively simply. Gödel first assigns a number to each symbol used in formal number theory: $\#(0) = 1$, $\#(1) = 2$, $\#(+) = 3$, etc. Only a small number of symbols need to be assigned: 0, 1, $+$, $\times$, $=$, $\wedge$, $\neg$, (,), $\forall$, and variables. Other symbols can be built from these, such as $\neq$ as $\neg =$. Then, every meaningful phrase w in number theory can be broken down into symbols $\langle w_1, \ldots, w_n \rangle$, and has a *Gödel number* given by $\ulcorner w \urcorner = \prod_{k=1}^{n} p_k^{\#(w_k)}$, where p_k is the k^{th} prime number.[2] (In practice, it is generally unnecessary to compute the actual Gödel number of a statement; it is usually sufficient to know that such a number exists.) Though the specific values chosen for each symbol are somewhat arbitrary (Hofstader uses an entirely different numbering system), Gödel had found a way to represent statements about number theory as statements within number theory, about which further statements could be made. For example, one can define the shorthand $\mathrm{Prf}(x, y)$ to be the statement that "x is the Gödel number encoding a proof for a theorem with Gödel number y." But

[1]Hofstader, Douglas R. *Gödel, Escher, Bach.* (New York: Vintage Books, 1980.), 18.
[2]Kennedy, Juliette. "Kurt Gödel." The Stanford Encyclopedia of Philosophy, Fall 2011 ed. <http://plato.stanford.edu/archives/fall2011/entries/goedel/>

since this is a statement about number theory, it has a Gödel number. Thus, it is possible to express that a statement is provable using number theory as a number: it is the Gödel number for the phrase $\exists x : \mathrm{Prf}(x, y)$, or that there is some natural number x that satisfies the relation $\mathrm{Prf}(x, y)$, so that the string encoded as x proves the theorem encoded as y. Using the notion of Prf, it is possible for numbers to refer to theorems and for statements within number theory to correspond to statements about number theory.

The statement that Gödel used to prove his theorem is straightforward: it simply claims that it is not a theorem of *Principia Mathematica*. However, none of the techiques described thus far allow a statement to refer to itself. Gödel introduced a quine operator, which takes a formula with at least one free variable and returns the formula with its Gödel number plugged into the free variable. For example, if x is the statement $a \neq 0$ with Gödel number $\ulcorner x \urcorner$, then $\mathrm{Quine}(x)$ is the statement (or, equivalently, the Gödel number corresponding to the statement) that $\ulcorner x \urcorner \neq 0$. This happens to be a true statement, but it is easy to make $\mathrm{Quine}(x)$ false, such as when x corresponds to $a = 0$. With quining, self-reference is possible, so Gödel's statement can be explicitly constructed. First, let u represent the statement $\nexists a, b : \mathrm{Prf}(a, b) \wedge b = \mathrm{Quine}(c)$, or that for a given c, there is no number a which forms a proof-pair with c. Then, Gödel's statement G is just $\mathrm{Quine}(u)$, which substitutes $\ulcorner u \urcorner$ for c. Thus, G is the claim that no a is a proof pair with $\mathrm{Quine}(u)$, or that the formula with Gödel number $\ulcorner \mathrm{Quine}(u) \urcorner$ is not a theorem of number theory. But that is exactly G's Gödel number! Thus, G claims that it itself is not a theorem. Of course, this means that G cannot be proven within a given formulation of number theory, for if it could, then it would be false, and number theory would be inconsistent. If G is not a theorem, then there is no contradiction, and G is true. G is the undecidable proposition that proves Gödel's First Incompleteness Theorem.

A few related theorems follow. Gödel's Second Incompleteness Theorem demonstrates that a proof of a system's consistency can only exist if the system is inconsistent. Since all statements can be provem in an inconsistent system, consider the statement x which asserts that $0 \neq 0$. Thus, claiming that number theory is consistent means $\nexists a : \mathrm{Prf}(a, x)$ – that no number encodes for a string proving $0 \neq 0$. The formalization of the proof is much more difficult than that of the first theorem, but as long as the consistency of the system is assumed, this claim to consistency isn't a theorem.[3] More worryingly, Tarski's Theorem demonstrates that there is no formula $T(a)$ that is equivalent to stating that the formula with Gödel numbr a expresses a truth. If such a formula existed, it could be plugged into a statement t which says $\exists a : \neg T(a) \wedge b = \mathrm{Quine}(a)$, where b is a free variable. Then, however, Quine(t) asserts that Quine(t) is a false statement, which is absurd: if it were true, it would be false, and vice versa. Thus there is no way to express number-theoretical truth in number theory.[4]

Unsurprisingly, these theorems had a huge impact on the philosophy of mathematics and logic, refuting mathematicians' attempts to demonstrate completeness and consistency of any interesting mathematical system. A cornerstone of mathematical thought was that mathematical theorems derived in a logical way were as close as one could get to absolute truth, since they were obtained from reasoning in sensible ways about sensible axioms of numbers. For example, the objective of the *Principia Mathematica* was "to derive all of mathematics from logic, and, to be sure, without contradictions."[5] Gödel, however, showed that provability in general is weaker than truth, casting doubt on this perspective of mathematics as the inevitable path from logic to truth. Questioning the consistency of number theory also unnerved mathematicians who spent long hours determining whether conjectures were true or not, yet discovered that mathematics might not even be conistent, making many of their results meaningless. Yet various mathematical results have applications in physics and other sciences; do these conclusions about consistency and completeness generalize to the sciences based on mathematical logic synthesized with empirical data? The incompleteness of pure mathematics has little itself to do with philosophical questions about the nature of the world, but mathematical applications suggest the inability to know certain truths about the real world, and even to demonstrate that certain properties are unknowable. Quantum mechanics promised exactly this uncertainty at roughly the same time as Gödel, showing that at sufficiently small scales it is impossible to know absolute information about an object and that even observing an object's properties can change those properties. At some point, mathematicians and

[3] Hofstader, Douglas R. *Gödel, Escher, Bach.* (New York: Vintage Books, 1980.), 450.
[4] *Ibid.*, 580-81.
[5] *Ibid.*, 23.

scientists alike became frustrated, wondering if there was worth in a field where the ends of knowledge were being reached.

However, a dissenting line of thought claims Gödel's incompleteness does not destroy the goal of rigorously demonstrating the consistency of mathematics by claiming that Gödel's Second Incompleteness Theorem does not adequately generalize to the full notion of consistency. In the first theorem, the incompleteness result does not depend on the specific meaning of the Gödel string G; however, the proof of the second theorem depends on the string that is equivalent to consistency as interpreted by the system. In a sense, a system T "cannot prove its consistency only when there is a sentence both which T 'recognizes' as a consistency statement, and which T cannot prove."[6] Though Gödel's second theorem is rigorous, it only applies to systems of logic where certain conditions of provability are met. Thus, generalizing a mathematical statement to a philosophical statement is not necessarily valid, and many of the conclusions about consistency overstate the importance of the Second Incompleteness Theorem. From this viewpoint, the more general notion of consistency might yet be demonstrable, though Gödel's theorems still create issues with strict number theory.

Kant's Incompleteness: Antinomies in Metaphysics

Much of what can be said about Gödel and mathematics also applies to Kant and metaphysics. Kant's goal is to create a new theory of metaphysics, and in order to do this he illustrates several holes in previous theories. Specifically, he creates four antinomies, or pairs of mutually contradictory statements that appear to be equally true. The four antinomies are whether the world has a beginning in time and space or is eternal and infinite; whether matter can be infinitely divided; whether free will exists or Nature determines all actions; and whether God is necessary to explain the universe.[7] Each antinomy is formulated as a pair of opposing statements, e.g. the second, written as a thesis that "everything in the world is constituted out of the simple," and an antithesis

[6] Kennedy, Juliette. "Kurt Gödel." The Stanford Encyclopedia of Philosophy, Fall 2011 ed. <http://plato.stanford.edu/archives/fall2011/entries/goedel/>

[7] Kant, Immanuel, trans. James W. Ellington. *Prolegomena to Any Future Metaphysics.* Indianapolis: Hackett Publishing Company, 1977. pp. 74-75.

that "there is nothing simple, but everything is composite."[8] Kant considers these statements equally valid and equally provable, so while on first glance these might seem to suggest an analogue to Gödel's incompleteness in that a statement cannot be shown to be true or false, they actually are more akin to inconsistency, since Kant finds proofs for both sides of each antinomy.

Generally, Kant does quite different things with his concerns with metaphysics than Gödel does with mathematics; instead of using them to arrive at a weaker definition of provability and truth, he uses them to introduce specific solutions and build his new metaphysics. Kant does not lose faith in logical reasoning, but rather points out that specific assumptions underlie each antinomy. For example, "the idea of the world, the fact that it is taken to be a mind-independent object, acts as the underlying assumption motivating both parties to the [first two] antinomies."[9] Kant does not assume that the mind's perception of the world corresponds accurately to the world in itself, and thus can dispose of the antinomies relatively quickly. The boundedness or unboundedness of the universe is merely an idea that has no basis in experience, and so "the magnitude of the world…would therefore have to exist in the world apart from all experience."[10] As such, by removing the idea that the magnitude of the world exists independent of perception, Kant can resolve the first antinomy. The second antinomy likewise deals with the nature of representations; the division of an object actually only corresponds to the division of its perception in the observer's mind, so a representation can only be divided as finely as corresponds to one's experience.[11] Similarly, the third antinomy can be resolved by assigning free will to things in themselves, while appearances are naturally determined. Though it seems counterintuitive that these two properties could coexist, Kant believes that cause and effect are imposed on perceptions by the mind, and that they do not correspond to objective reality. Thus, determinism is a product of the mind as well, since it is based on causality. The final antinomy is also related to cause and effect: given that

[8] *Ibid.*

[9] Grier, Michelle. "Kant's Critique of Metaphysics." The Stanford Encyclopedia of Philosophy, Summer 2011 ed. <http://plato.stanford.edu/archives/sum2011/entries/kant-metaphysics>

[10] Kant, Immanuel, trans. James W. Ellington. *Prolegomena to Any Future Metaphysics*. Indianapolis: Hackett Publishing Company, 1977. p. 77.

[11] *Ibid.*

causality is a product of the mind, it is equally valid to say that there is no necessary cause for all objects or that there is a God who causes.[12] In this way, Kant's antinomies lead to his revolution in metaphysics, where he asserts that perceptions can never be completely accurate depictions of actual objects and supports this claim by demonstrating that it resolves the antinomies.

However, Kant is merely trading one form of incompleteness for another; though he resolves the antinomies, he does so in a way that prevents absolute knowledge of the world. Kant's metaphysics are consistent, as he has addressed their contradictions, but incomplete, since they cannot prove true statements about objects, but only their physical representations. Alternatively, one could define knowledge as corresponding to representations rather than actual objects, but this sacrifices absolute truth for completeness. Both formulations create a very Gödelian concept of knowledge as inevitably weaker than absolute truth, though one limits knowledge and the other removes truth from the equation. Though Kant begins from a very different set of ideas than Gödel, he concludes very similarly that knowledge or provability cannot be complete, and that there must exist some unprovable truths. However, Kant's work is in metaphysics, not mathematics, and so the consequences differ somewhat. Arguably, the former encompasses the latter, and so Kant anticipated Gödel, but it is clear from the major differences in Kant's and Gödel's methodologies of proof that Kant could not have guessed at the incompleteness of mathematics due to self-reference. Rather, Kant's incompleteness corresponds more to the sensible world and the nature of true statements about material objects.

Thus, while the notion of truth in the real world was only a tangential question for Gödelian incompleteness, it is central to Kant. Though the idea that observations and reality were distinct is as old as Plato, Kant was the first to formulate the distinction in terms of a filter through one's perceptions of objects. Cause and effect, for example, are useful for humans, and thus are built into human perception even they don't necessarily have meaning outside of it. Would all rational beings observe cause and effect on the world, or do others use different lenses to see the world? True logical reasoning depends on

[12] *Ibid.*, pp. 81-82.

statements of the form "A implies B," suggesting that reason is closely linked with cause and effect, and that any creature that cannot connect cause and effect must use a fundamentally different epistemology. However, that cause and effect so heavily influence reason creates another link between Kant and Gödel, though this time it is negative: under Kantian metaphysics, cause and effect have no correlation to reality, and so neither does logic. Thus, Gödel's reasoning does not correspond to the real world, even though it predicts similar things to Kant's!

Filling in the Holes

Another very important difference between Gödel's mathematical incompleteness and Kant's metaphysical incompleteness is how each deals with the inevitable gaps in knowledge that incompleteness creates. Mathematical incompleteness is a factor in relatively few theorems, so each can be dealt with on a case-by-case basis. For example, Euclid's parallel postulate, one of his five axioms of geometry, states that if two lines intersect a third at angles that sum to less than 90°, then they must intersect. This is a perfectly reasonable statement in ordinary geometry, but it was eventually shown to be independent of the other four more fundamental axioms. Thus, different interpretations of it are equally valid, such as claiming that one line could have multiple parallel lines that intersect at only one point or that a line could have no parallels through a given point. Thus, geometry split in three, with the new branches known as hyperbolic and elliptical gemoetry, respetctively.[13] Each of these interpretations is valid, and they have many things in common and various uses in the other branches of mathematics and physics.

Every undecidable statement in mathematics can cause a similar bifurcation, including Gödel's statement G. If one considers it to be true, number theory goes on much as before. However, it is equally valid, and considerably more interesting, to add $\neg G$ as an axiom. Since G was shown to be true, this appears nonsensical, but all that is required is a generalization of the natural numbers. Specifically, it is an axiom that some number forms a proof-pair with Quine(u). However, it was also shown that 0 is not that number, nor is 1, nor 2, etc. Thus, these generalized natural numbers do not fit anywhere on the number

[13] Hofstader, Douglas R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, 1980. 90-93.

line, and are "best visualized as... infinitely large integers."[14] Conventional properties of numbers, such as commutativity, still hold, but asking the size of a generalized natural number is a meaningless question, for assigning a size would place it among the naturals, which do not form proof-pairs with Quine(u). Interestingly, these generalized naturals display a property similar to Heisenburg's Uncertainty Principle: there are various ways to index them as ordered triples of integers (so one might look like $(1, -5, 7)$ in one system, but $(2, 2, 3)$ in another). In some schemes, it is simple to calculate the sum of two indexed numbers, and in others it is easy to compute their product. But no single scheme allows one to know both the sum and the product.[15] This is yet another manifestation of incompleteness, in a manner and place it was never expected from $\neg G$. Bifurcation around undecidable propositions leads to further mathematical discoveries and interesting results despite the question of incompleteness.

Kant, however, does not use bifurcation, since his incompleteness is common and different for every observer. It would be far too complicated to track all such possibilities, so instead Kant considers each observer a self-contained system. No observer can know everything accurately, and the observations of different beings will disagree, but each observer's perception of reality is consistent, though not complete. The questions raised by the antinomies will never be answered, though a rational mind can understand Kant's explanation for why this is the case. In general, Kant believes philosophical questions are answered by moving to a new frame where they are meaningless, which is precisely how he resolved his antinomies, but this means the original questions cannot be answered. The difference between Kant's solutions to incompleteness and Gödel's are indicative of the differences in set-theoretical mathematics and metaphysics as a whole: Gödel found it is generally easy to define truth and provability given axioms and modes of reasoning but that only a few statements caused him the most trouble, but Kant could not know that anything observed about reality is necessarily absolutely true. Thus, it is possible for mathematicians to work around these holes, paying attention to which axioms are assumed so as to not fall into them, while philosophers have to divide truth into many more sections

[14] *Ibid.*, 454.
[15] *Ibid.*, 455.

corresponding to each observer. These two vastly different consequences of incompleteness are a surprising development given that somewhat similar logic underlies both systems, and amplifies the differences in philosophical and mathematical modes of thinking. Mathematical logic is necessarily more rigorous, though still not complete, and thus its limits are relatively well known relative to the philosophical reasoning that depends on incompleteness in reality.

Why, though, does this Kantian metaphysics apply to all philosophy? Kant was not the only philosopher to offer theories behind causality and the relation of the sensible and the real world. However, Kant, like Gödel, was dealing with the theory of theories. Gödel's meta-mathematics allowed him to make incompleteness not just a property of one axiomatic system, but of mathematical logic itself, granting it its enormous significance. Similarly, Kant's meta-metaphysics dealt with antinomies that had dogged many different and competing theories of metaphysics, and enabled him to establish metaphysical ideas that applied to not just one theory, but to the general concept of philosophical thinking. Thus, given that incompleteness is a crucial part of Kant's transcendental ideal, it clearly plays a fundamental role in philosophical logic as well as metaphysical logic. Other meta-theories of philosophy do exist and offer competition to Kant, though, but they fit into Kantian incompleteness as well. Those theories that allow for or predict incompleteness agree with and reinforce the idea that absolute truth cannot be reached, and those that don't stumble over Gödel's results and modern physics that formulaically limits what can be empirically observed and what can be known. Synthesizing Kantian incompleteness with Gödelan incompleteness, despite their different origins, illuminates the differences between mathematical and philosophical logic, from which very different ideas produce very similar results that have very different consequences. And if incompleteness can ocur in these two so logical fields, it must be present elsewhere, too, as logic in science and everyday life flows from that of mathematics and metaphysics.

Conclusion

Though Kant and Gödel established principles of incompleteness in very different ways within the fields of mathematics and metaphysics, respectively, their ideas share many basic principles which suggest that incompleteness is an inevitable consequence of human thought and logic. Physics, for example, underwent a change from primarily deterministic theories in the mid-nineteenth century to relativistic and quantum concepts that limited the amount of knowledge an observer could have about a sufficiently small experiment. Once again, incompleteness pops up at a certain point in the human logical thought process, though it does so in a different form. The consequences, too, are quite different; quantum mechanics can be experimentally tested, and could someday produce applied products such as dizzyingly powerful quantum computers. From this perspective it is not as necessary to work around incompleteness in the sciences as it is to embrace it and use it like any other technique or tool. The world is unexpectedly much more interesting in its incompleteness, as it leads to technologies that would not have been possible in classical mechanics and ideas that challenge the greatest thinkers.

Yet Gödel's theorems have no such applications, immediate or otherwise; set theory is an abstraction of an abstraction of an abstraction, useful primarily for reinforcing the foundations of mathematics. Kantian metaphysics is similar; one's perception of reality, though incomplete, is consistent, and thus an entire life can be lived from within one's mind, completely uncaring whether it corresponds to reality. Yet these results are important to more than just mathematicians and philosophers, for the notions of truth and knowledge of truth apply to everyday life. Knowing that logical reasoning cannot lead to all truths is important in understanding the basis of science, and understanding how different observers perceive different realities as per Kant or quantum physics allows one to challenge the conventional perception of things and ideas, and thus to learn more about them. But the most important aspect of incompleteness is the weakening of truth to something less attached to logic and reason; knowing the reasoning behind this could reshape an entire worldview in a way that does affect one's interactions with other observers. That incompleteness is present in all manner of logical thought leads to a rethinking of logic itself, and logic does pervade everyday life. While logic is still valid for most statements if one is careful, knowing its limitations can be a valuable and applied result of this abstract notion of incompleteness. Without incompleteness, one's conception of reason itself would be incomplete, and so

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